数列极限的概念

求面积

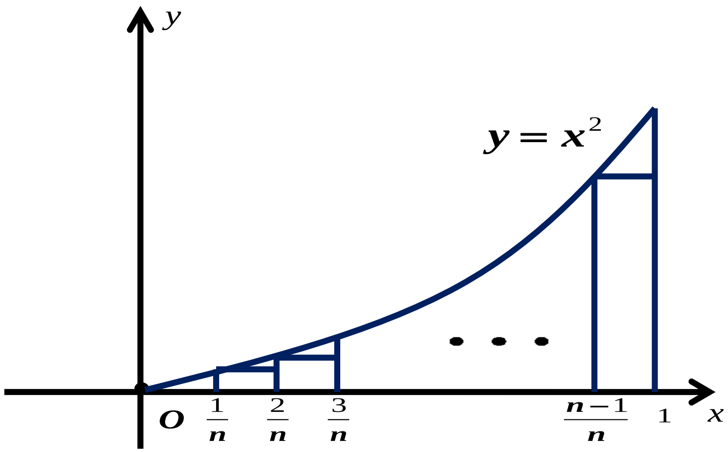
$$S_n \approx \frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2\right]$$

$$=\frac{1^2+2^2+\cdots+(n-1)^2}{n^3}$$

$$=\frac{(n-1)n(2n-1)}{6n^3}$$

$$=\frac{2n^2-3n+1}{6n^2}$$

$$=\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}\to\frac{1}{3}$$



数列极限的概念

设$\{x_n\}$为一数列，A为一实数，对$\forall \varepsilon > 0, \exists N$，使得当$n > N$时，有$|x_n - A| < \varepsilon$或$A - \varepsilon < x_n < A + \varepsilon$. 则称数列$\{x_n\}$在$n$趋向于无穷大时有极限或收敛，$A$为其极限值（或说$x_n$收敛于$A$），记为$\lim\limits_{n\to\infty} x_n = A$或$x_n \to A\,(n \to \infty)$.

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1) ε：度量数列与极限值的距离.

2) N：与ε相关，标示这变化的进程.

数列极限定义的证明题

$$\lim_{n\to\infty} x_n = A \Leftrightarrow \forall \varepsilon > 0, \exists N, s.t. \text{当} n > N \text{时}, \ \text{有} |x_n - A| < \varepsilon.$$

注：只能判断是否是极限，不能用来求极限

例1. 证明$\lim\limits_{n\to\infty}\dfrac{1}{n}=0.$

分析：找N,当$n > N$时,$\left|\dfrac{1}{n}-0\right| < \varepsilon \Rightarrow \dfrac{1}{n} < \varepsilon \Rightarrow n > \dfrac{1}{\varepsilon}.$

证明：$\forall \varepsilon > 0$，取$N = \left[\dfrac{1}{\varepsilon}\right]+1$,当$n > N$时,

$$n > \left[\frac{1}{\varepsilon}\right]+1 > \frac{1}{\varepsilon}$$

即$\left|\dfrac{1}{n}-0\right| < \varepsilon.$

数列极限定义的证明题

例2. 证明$\lim\limits_{n\to\infty}\sqrt[n]{a}=1.(a>1)$

分析：找N,当$n>N$时,$\left|\sqrt[n]{a}-1\right|<\varepsilon \Rightarrow \sqrt[n]{a}<\varepsilon+1 \Rightarrow \ln a^{\frac{1}{n}}<\ln(\varepsilon+1)$

$$\Rightarrow \frac{1}{n}\ln a<\ln(\varepsilon+1) \Rightarrow n>\frac{\ln a}{\ln(\varepsilon+1)}.$$

证明：$\forall\varepsilon>0$，取$N=\left[\dfrac{\ln a}{\ln(\varepsilon+1)}\right]+1$,当$n>N$时,

$$n>\left[\frac{\ln a}{\ln(\varepsilon+1)}\right]+1>\frac{\ln a}{\ln(\varepsilon+1)} \Rightarrow \frac{1}{n}\ln a<\ln(\varepsilon+1)$$

$$\Rightarrow \ln a^{\frac{1}{n}}<\ln(\varepsilon+1) \quad \Rightarrow \sqrt[n]{a}<\varepsilon+1$$

即有$\left|\sqrt[n]{a}-1\right|<\varepsilon.$

例. 证明$\lim\limits_{n\to\infty}\sqrt[n]{n}=1.$

分析：找N,当$n>N$时,$\left|\sqrt[n]{n}-1\right|<\varepsilon \Rightarrow \sqrt[n]{n}<\varepsilon+1$

$$\Rightarrow \sqrt[n]{n}=\sqrt[n]{n\cdot 1\cdots 1}\le \frac{n+(n-1)}{n}=2-\frac{1}{n}\quad <\frac{1}{n}+2\quad <\varepsilon+1 \Rightarrow \frac{1}{n}<\varepsilon-1$$

$$\Rightarrow \sqrt[n]{n}=\sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot 1\cdots 1}\le \frac{2\sqrt{n}+n-2}{n}=\frac{2}{\sqrt{n}}+1-\frac{2}{n}<\frac{2}{\sqrt{n}}+1<\varepsilon+1$$

$$\Rightarrow n>\frac{4}{\varepsilon^2}$$

证明：$\forall\varepsilon>0$, 取$N=\left[\dfrac{4}{\varepsilon^2}\right]+1$,当$n>N$时,

$$n>\left[\frac{4}{\varepsilon^2}\right]+1>\frac{4}{\varepsilon^2}\quad \Rightarrow \frac{2}{\sqrt{n}}+1<\varepsilon+1 \Rightarrow \frac{2\sqrt{n}+n-2}{n}=\frac{2}{\sqrt{n}}+1-\frac{2}{n}<\varepsilon+1$$

$$\Rightarrow \sqrt[n]{n}=\sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot 1\cdots 1}\le \frac{2\sqrt{n}+n-2}{n}<\varepsilon+1$$

即有$\left|\sqrt[n]{n}-1\right|<\varepsilon.$