3. Proof of Theorem 1

Theorem 1. *Matrix $\boldsymbol{K}$ can be calculated recursively in the form of $\boldsymbol{K}_{k+1} = \begin{bmatrix} \boldsymbol{G}_{1,k+1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_{2,k+1} \end{bmatrix} \boldsymbol{K}_k$, where $\boldsymbol{G}_{1,k+1}$ and $\boldsymbol{G}_{2,k+1}$ are adaptively determined.*

Proof:

$$\boldsymbol{K}_{k+1} = \boldsymbol{B}_{k+1}^{-\frac{1}{2}} \;=\; \left(\alpha_{k+1}\boldsymbol{B}_k + (1-\alpha_{k+1})\,\Delta\boldsymbol{B}_{k+1}\right)^{-\frac{1}{2}} \tag{S4}$$

As $\boldsymbol{B}$ is a block diagonal matrix with $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B}_2 \end{bmatrix}$, $\boldsymbol{K}$ can be written in the form of $\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K}_2 \end{bmatrix}$. We mainly introduce the recursion of $\boldsymbol{K}_1$, and $\boldsymbol{K}_2$ can be updated similarly.

$$\begin{aligned}
\boldsymbol{K}_{1,k+1} =& \boldsymbol{B}_{1,k+1}^{-\frac{1}{2}} \\
=&\; \left(\alpha_{k+1}\boldsymbol{B}_{1,k} + (1-\alpha_{k+1})\,\Delta\boldsymbol{B}_{1,k+1}\right)^{-\frac{1}{2}} \\
=& \left(\left(\sqrt{\alpha_{k+1}}\boldsymbol{B}_{1,k}^{\frac{1}{2}}\right)\left(\boldsymbol{I} + \frac{1-\alpha_{k+1}}{\alpha_{k+1}}\boldsymbol{B}_{1,k}^{-\frac{1}{2}}\Delta\boldsymbol{B}_{1,k+1}\left(\boldsymbol{B}_{1,k}^{-\frac{1}{2}}\right)^T\right)\left(\sqrt{\alpha_{k+1}}\boldsymbol{B}_{1,k}^{\frac{1}{2}}\right)^T\right)^{-\frac{1}{2}} \\
=& \frac{1}{\sqrt{\alpha_{k+1}}}\left(\boldsymbol{I} + \frac{1-\alpha_{k+1}}{\alpha_{k+1}}\boldsymbol{B}_{1,k}^{-\frac{1}{2}}\Delta\boldsymbol{B}_{1,k+1}\left(\boldsymbol{B}_{1,k}^{-\frac{1}{2}}\right)^T\right)^{-\frac{1}{2}}\boldsymbol{B}_{1,k}^{-\frac{1}{2}} \\
=& \frac{1}{\sqrt{\alpha_{k+1}}}\left(\boldsymbol{I} + \frac{1-\alpha_{k+1}}{\alpha_{k+1}}\boldsymbol{K}_{1,k}\Delta\boldsymbol{B}_{1,k+1}\boldsymbol{K}_{1,k}^T\right)^{-\frac{1}{2}}\boldsymbol{K}_{1,k} \\
=& \frac{1}{\sqrt{\alpha_{k+1}}}\tilde{\boldsymbol{K}}_{1,k+1}^{-\frac{1}{2}}\boldsymbol{K}_{1,k}
\end{aligned} \tag{S5}$$

where $\tilde{\boldsymbol{K}}_{1,k+1} = \boldsymbol{I} + \frac{1-\alpha_{k+1}}{\alpha_{k+1}}\boldsymbol{K}_{1,k}\Delta\boldsymbol{B}_{1,k+1}\boldsymbol{K}_{1,k}^T$. $\Delta\boldsymbol{B}_{1,k+1}$ is symmetric and the rank is no more than 2. Thus, it can be reformulated into

$$\begin{aligned}
\Delta\boldsymbol{B}_{1,k+1} =& \begin{bmatrix}\boldsymbol{q}_{1,k+1} & \boldsymbol{q}_{2,k+1}\end{bmatrix}\begin{bmatrix}\beta_{1,k+1} & \\ & \beta_{2,k+1}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_{1,k+1} & \boldsymbol{q}_{2,k+1}\end{bmatrix}^T \\
=& \boldsymbol{Q}_{1,k+1}\boldsymbol{\Xi}_{1,k+1}\boldsymbol{Q}_{1,k+1}^T
\end{aligned}$$

where $\beta_{1,k+1}$ and $\beta_{2,k+1}$ are non-zero eigenvalues if $\mathrm{rank}(\Delta\boldsymbol{B}_{1,\mathrm{k+1}}) = 2$. $\boldsymbol{q}_{1,k+1}$ and $\boldsymbol{q}_{2,k+1}$ are the eigenvectors. If the rank is 1, $\beta_{1,k+1}$ or $\beta_{2,k+1}$ is 0. Thus,

$$\begin{aligned}
\tilde{\boldsymbol{K}}_{1,k+1} =& \boldsymbol{I} + \frac{1-\alpha_{k+1}}{\alpha_{k+1}}\boldsymbol{K}_{1,k}\boldsymbol{Q}_{1,k+1}\boldsymbol{\Xi}_{1,k+1}\boldsymbol{Q}_{1,k+1}^T\boldsymbol{K}_{1,k}^T \\
=& \boldsymbol{I} + \frac{1-\alpha_{k+1}}{\alpha_{k+1}}\left(\boldsymbol{K}_{1,k}\boldsymbol{Q}_{1,k+1}\right)\boldsymbol{\Xi}_{1,k+1}\left(\boldsymbol{K}_{1,k}\boldsymbol{Q}_{1,k+1}\right)^T \\
=& \boldsymbol{I} + \tilde{\boldsymbol{Q}}_{1,k+1}\tilde{\boldsymbol{\Xi}}_{1,k+1}\tilde{\boldsymbol{Q}}_{1,k+1}^T
\end{aligned} \tag{S6}$$

where $\tilde{\boldsymbol{Q}}_{1,k+1} = \boldsymbol{K}_{1,k}\boldsymbol{Q}_{1,k+1} \in \mathbb{R}^{m_1\times 2}$, $\tilde{\boldsymbol{\Xi}}_{1,k+1} = \frac{1-\alpha_{k+1}}{\alpha_{k+1}}\boldsymbol{\Xi}_{1,k+1}$ and $\mathrm{rank}(\tilde{\boldsymbol{\Xi}}_{1,k+1}) \leqslant 2$. As $\boldsymbol{B}_{1,k+1}$ is positive definite, $\boldsymbol{I} + \tilde{\boldsymbol{Q}}_{1,k+1}\tilde{\boldsymbol{\Xi}}_{1,k+1}\tilde{\boldsymbol{Q}}_{1,k+1}^T$ is also positive definite. For brevity, let $\tilde{\boldsymbol{Q}}_{1,k+1} = \begin{bmatrix}\tilde{\boldsymbol{q}}_{1,k+1} & \tilde{\boldsymbol{q}}_{2,k+1}\end{bmatrix}$, $\tilde{\boldsymbol{\Xi}}_{1,k+1} = \begin{bmatrix}\tilde{\beta}_{1,k+1} & \\ & \tilde{\beta}_{2,k+1}\end{bmatrix}$. We select the calculation way of $\tilde{\boldsymbol{K}}_{1,k+1}^{-\frac{1}{2}}$ based on the rank of $\tilde{\boldsymbol{\Xi}}_{1,k+1}$.

1) $\mathrm{rank}(\tilde{\boldsymbol{\Xi}}_{1,\mathrm{k+1}}) = 1$.

Let $\beta_{1,k+1} \neq 0$ and $\beta_{2,k+1} = 0$, here $\tilde{\boldsymbol{\Xi}}_{1,k+1} = \tilde{\beta}_{1,k+1}$, $\tilde{\boldsymbol{Q}}_{1,k+1} = \tilde{\boldsymbol{q}}_{1,k+1}$. Then, according to [1]

$$
\begin{aligned}
\tilde{\boldsymbol{K}}_{1,k+1}^{-\frac{1}{2}} &= \left(\boldsymbol{I} + \tilde{\boldsymbol{Q}}_{1,k+1}\tilde{\boldsymbol{\Xi}}_{1,k+1}\tilde{\boldsymbol{Q}}_{1,k+1}^{T}\right)^{-\frac{1}{2}} \\
&= \left(\boldsymbol{I} + \tilde{\beta}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T}\right)^{-\frac{1}{2}} \\
&= \boldsymbol{I} + \frac{\tilde{\boldsymbol{q}}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T}}{\tilde{\boldsymbol{q}}_{1,k+1}^{T}\tilde{\boldsymbol{q}}_{1,k+1}}\left(\frac{1}{\sqrt{1+\tilde{\beta}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T}\tilde{\boldsymbol{q}}_{1,k+1}}} - 1\right) \\
&= \boldsymbol{I} + \gamma_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T}
\end{aligned}
$$

where $\gamma_{1,k+1} = \frac{1}{\tilde{\boldsymbol{q}}_{1,k+1}^{T}\tilde{\boldsymbol{q}}_{1,k+1}}\left(\frac{1}{\sqrt{1+\tilde{\beta}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T}\tilde{\boldsymbol{q}}_{1,k+1}}} - 1\right)$. Thus, the recursion of $\boldsymbol{K}_1$ is

$$
\boldsymbol{K}_{1,k+1} = \frac{1}{\sqrt{\alpha_{k+1}}}\left(\boldsymbol{I} + \gamma_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T}\right)\boldsymbol{K}_{1,k} \tag{S7}
$$

Here, $\boldsymbol{G}_{1,k+1} = \frac{1}{\sqrt{\alpha_{k+1}}}\left(\boldsymbol{I} + \gamma_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T}\right)$.

2) $\mathrm{rank}(\tilde{\boldsymbol{\Xi}}_{1,\mathrm{k+1}}) = 2$.

The formula (S6) can be further reformulated into

$$
\begin{aligned}
\boldsymbol{I} + \tilde{\boldsymbol{Q}}_{1,k+1}\tilde{\boldsymbol{\Xi}}_{1,k+1}\tilde{\boldsymbol{Q}}_{1,k+1}^{T} &= \boldsymbol{I} + \tilde{\beta}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T} + \tilde{\beta}_{2,k+1}\tilde{\boldsymbol{q}}_{2,k+1}\tilde{\boldsymbol{q}}_{2,k+1}^{T} \\
&= \check{\boldsymbol{Q}}_{1,k+1}\check{\boldsymbol{\Lambda}}_{1,k+1}\check{\boldsymbol{Q}}_{1,k+1}^{T}
\end{aligned}
$$

where $\check{\boldsymbol{Q}}_{1,k+1}$ is the eigen matrix with $\check{\boldsymbol{Q}}_{1,k+1}^{T}\check{\boldsymbol{Q}}_{1,k+1} = \boldsymbol{I}$, $\check{\boldsymbol{\Lambda}}_{1,k+1}$ contains the eigenvalues. (S6) is realized by two successive rank-1 modification with first-order perturbation (FOP) [2]. Thus, (S5) is further calculated:

$$
\boldsymbol{K}_{1,k+1} = \frac{1}{\sqrt{\alpha_{k+1}}}\check{\boldsymbol{\Lambda}}_{1,k+1}^{-\frac{1}{2}}\check{\boldsymbol{Q}}_{1,k+1}^{T}\boldsymbol{K}_{1,k} \tag{S8}
$$

Here, $\boldsymbol{G}_{1,k+1} = \frac{1}{\sqrt{\alpha_{k+1}}}\check{\boldsymbol{\Lambda}}_{1,k+1}^{-\frac{1}{2}}\check{\boldsymbol{Q}}_{1,k+1}^{T}$.

In conclusion, $\boldsymbol{K}_{1,k+1}$ can be calculated adaptively in the form of $\boldsymbol{K}_{1,k+1} = \boldsymbol{G}_{1,k+1}\boldsymbol{K}_{1,k}$, where

$$
\boldsymbol{G}_{1,k+1} = \begin{cases} \frac{1}{\sqrt{\alpha_{k+1}}}\left(\boldsymbol{I} + \gamma_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}\tilde{\boldsymbol{q}}_{1,k+1}^{T}\right), & \mathrm{rank}\left(\Delta\boldsymbol{B}_{1,\mathrm{k+1}}\right) = 1 \\ \frac{1}{\sqrt{\alpha_{k+1}}}\check{\boldsymbol{\Lambda}}_{1,k+1}^{-\frac{1}{2}}\check{\boldsymbol{Q}}_{1,k+1}^{T}, & \mathrm{rank}\left(\Delta\boldsymbol{B}_{1,\mathrm{k+1}}\right) = 2 \end{cases}
$$

As $\mathrm{rank}(\Delta\boldsymbol{B}_{2,k+1}) \leq 2$ and $\boldsymbol{K}_{2,k+1} = \boldsymbol{G}_{2,k+1}\boldsymbol{K}_{2,k}$, $\boldsymbol{G}_{2,k+1}$ can be calculated recursively in the similar way. Thus, $\boldsymbol{K}_{k+1} = \begin{bmatrix} \boldsymbol{G}_{1,k+1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_{2,k+1} \end{bmatrix}\boldsymbol{K}_k$, where $\boldsymbol{G}_{1,k+1}$ and $\boldsymbol{G}_{2,k+1}$ are determined adaptively.