

# Homework Solutions

MATH231

Spring 2022

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## Homework 0

### 1. Calculating Limits

- $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2}$

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 3)}{x - 2} = \lim_{x \rightarrow 2} (x + 3) = 5.$$

- $\lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h}$

Consider  $x$  as a “constant”

$$\lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 - 2xh - h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh - h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

- $\lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{x^2 + x} \right)$

$$\lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{x^2 + x} \right) = \lim_{x \rightarrow 0} \frac{x^2 + x - x}{x(x^2 + x)} = \lim_{x \rightarrow 0} \frac{x^2}{x^2(x + 1)} = \lim_{x \rightarrow 0} \frac{1}{x + 1} = 1.$$

### 2. The Chain Rule

- $\frac{d}{dx} \ln(x + \sin x)$

$$\frac{d}{dx} \ln(x + \sin x) = (1 + \cos x) \frac{1}{x + \sin x}.$$

- $\frac{d}{dx} \cos(x^2 e^x)$

$$\frac{d}{dx} \cos(x^2 e^x) = -(2xe^x + x^2 e^x) \sin(x^2 e^x).$$

### 3. Implicit Differentiation: Solve for $\frac{dy}{dx}$ for the following implicit function.

- $x^2 + y^2 = r^2$ , where  $r$  is a constant

Differentiate on both sides w.r.t.  $x$  gives  $2x + 2yy' = 0$ . Hence  $y' = -\frac{x}{y}$ .

- $\frac{x + y}{x - y} = x$

The above is equivalent to  $x + y = x^2 - xy$ . Differentiate on both sides w.r.t.  $x$  gives  $1 + y' = 2x - y - xy' \iff (1 + x)y' = 2x - y - 1$ . Hence  $y' = \frac{2x - y - 1}{x + 1}$ .

4. Linear Approximations and Differentials: Find the Taylor polynomials of degree two approximating the given function centered at the given point.

- $f(x) = \sin(2x)$  at  $a = \frac{\pi}{2}$

$$f' = 2 \cos(2x), f'' = -4 \sin(2x). \text{ So } f \sim -2\left(x - \frac{\pi}{2}\right).$$

- $f(x) = e^x$  at  $a = 1$

$$f' = f'' = e^x. \text{ So } f \sim e + e(x - 1) + \frac{e}{2}(x - 1)^2.$$

5. Mean Value Theorem: Determine if the Mean Value Theorem can be applied to the following function on the the given closed interval.

Both intervals are closed. It suffices to check that these functions are continuous on the given interval. One can do this by computing the derivative exists.

- $f(x) = 3 + \sqrt{x}, x \in [0, 4]$

$$\text{Here } f' = \frac{1}{2\sqrt{x}}.$$

- $f(x) = \frac{x}{1+x}, x \in [1, 3]$

$$\text{Here } f' = \frac{1}{(1+x)^2}.$$

6. L'Hospital's Rule

- $\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

Check that as  $x \rightarrow 2$ ,  $x^3 - 7x^2 + 10x \rightarrow 0$  and  $x^2 + x - 6 \rightarrow 0$  so L'Hospital's rule applies. Then

$$\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x \rightarrow 2} \frac{3x^2 - 14x + 10}{2x + 1} = -\frac{6}{5}.$$

The last step uses division property of limits.

- $\lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{x}}$

As exp and ln are continuous functions

$$\lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{x}} = \exp \left[ \ln \left( \lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{x}} \right) \right] = \exp \left( \lim_{x \rightarrow \infty} \frac{\ln(e^x + x)}{x} \right).$$

Check that as  $x \rightarrow \infty$ ,  $e^x + x \rightarrow \infty$  so L'Hospital's rule applies.

$$\begin{aligned} RHS &= \exp \left( \lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x} \right) \\ &= \exp \left( \lim_{x \rightarrow \infty} 1 + \frac{1 - x}{e^x + x} \right) \quad (\text{Check that L'Hospital's rule applies}) \\ &= \exp \left( 1 + \lim_{x \rightarrow \infty} \frac{-1}{e^x + 1} \right) = e. \end{aligned}$$

- $\lim_{x \rightarrow \infty} x \ln \left( 1 + \frac{3}{x} \right)$

$$\lim_{x \rightarrow \infty} x \ln \left( 1 + \frac{3}{x} \right) = \lim_{x \rightarrow \infty} \frac{\ln \left( 1 + \frac{3}{x} \right)}{\frac{1}{x}}.$$

Check that as  $x \rightarrow \infty$ ,  $\ln \left( 1 + \frac{3}{x} \right), \frac{1}{x} \rightarrow 0$  so L'Hospital's rule applies.

$$RHS = \lim_{x \rightarrow \infty} \frac{\frac{-3/x^2}{1+3/x}}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{3}{1 + \frac{3}{x}} = 3.$$

7. The Fundamental Theorem of Calculus: Find the derivative of the following

- $\int_1^x \frac{1}{t^3 + 1} dt$

Apply FTC

$$\frac{d}{dx} \int_1^x \frac{1}{t^3 + 1} dt = \frac{1}{x^3 + 1}.$$

- $\int_1^{\sqrt{x}} \sin t dt$

Let  $u(x) = \sqrt{x}$ . Apply chain rule and FTC

$$\frac{d}{dx} \int_1^x \frac{1}{t^3 + 1} dt = \sin u(x) \cdot \frac{du}{dx} = \sin u(x) \cdot \frac{1}{2\sqrt{x}} = \frac{\sin \sqrt{x}}{2\sqrt{x}}.$$

- $\int_x^{2x} t^3 dt$

Using subtraction property of integral,

$$\int_x^{2x} t^3 dt = \int_0^{2x} t^3 dt - \int_0^x t^3 dt.$$

Apply FTC to each term

$$\frac{d}{dx} \int_x^{2x} t^3 dt = 16x^3 - x^3 = 15x^3.$$

## 8. Substitution Rule

- $\int_{\frac{1}{2}}^0 \frac{x}{\sqrt{1-4x^2}} \, dx$

Take  $u = 1 - 4x^2$ , then  $du = -8x \, dx$  and  $dx = -\frac{1}{8} \, du$ .

$$\int_{\frac{1}{2}}^0 \frac{x}{\sqrt{1-4x^2}} \, dx = \int_0^1 -\frac{1}{8\sqrt{u}} \, du = -\frac{1}{4} \sqrt{u} \Big|_0^1 = -\frac{1}{4}.$$

- $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)} \, dx$

Take  $u = \sin(\pi x)$ , then  $du = \pi \cos(\pi x) \, dx$  and  $dx = \frac{1}{\pi} \, du$ .

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)} \, dx = \int_{\frac{\sqrt{2}}{2}}^1 \frac{1}{\pi u^2} \, du = -\frac{1}{u} \Big|_{\frac{\sqrt{2}}{2}}^1 = \frac{\sqrt{2}-1}{\pi}.$$

- $\int_0^1 x e^{4x^2+3} \, dx$

Take  $u = 4x^2 + 3$ , then  $du = 8x \, dx$

$$\int_0^1 x e^{4x^2+3} \, dx = \frac{1}{8} \int_3^7 e^u \, du = \frac{1}{8} \sqrt{u} \Big|_3^7 = \frac{e^7 - e^3}{8}.$$

## Homework 1

The following solutions provide a possible way to solve the problems. Any other reasonable solution is accepted.

1. Integration by parts (Note that the following integrals are indefinite. You need to add constants to your final answer.) You may also need to use substitution rule.

•  $\int \frac{\ln x}{x^2} dx$  [3pt]

Take  $u = \ln x, v = -\frac{1}{x}$ . Then

$$\int \frac{\ln x}{x^2} dx = \int \ln x d\left(-\frac{1}{x}\right) = -\frac{1}{x} \ln x - \int -\frac{1}{x} d(\ln x) = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

•  $\int x^2 \sin x dx$  [4pt]

Take  $u = x^2, v = -\cos x$ . Then

$$\int x^2 \sin x dx = -x^2 \cos x - \int -\cos x d(x^2) = -x^2 \cos x + 2 \int x \cos x dx.$$

To evaluate  $\int x \cos x dx$  we use integration by parts again with  $u = x, v = \sin x$ .

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + \tilde{C}.$$

Final answer:  $-x^2 \cos x + 2x \sin x + 2 \cos x + C$ .

•  $\int (\ln x)^2 dx$  [4pt]

Take  $u = (\ln x)^2, v = x$ . Then

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int x d(\ln x)^2 = x(\ln x)^2 - \int \ln x dx.$$

To evaluate  $\int \ln x dx$  we use integration by parts again, with  $u = \ln x, v = x$ .

$$\int \ln x dx = x \ln x - \int x d(\ln x) = x \ln x - \int 1 dx = x \ln x - x + C.$$

Final answer:  $x(\ln x)^2 - 2x \ln x + 2x + C$ .

•  $\int \arccos x dx$  [4pt]

Take  $u = \arccos x, v = x$ . Then

$$\int \arccos x dx = x \arccos x - \int x d(\arccos x) = x \arccos x - \int -\frac{x}{\sqrt{1-x^2}} dx.$$

To evaluate  $\int \frac{x}{\sqrt{1-x^2}} dx$  we use substitution rule with  $u = 1 - x^2$ ,  $du = -2x dx$ .

$$\int \frac{x}{\sqrt{1-x^2}} dx = - \int \frac{1}{2\sqrt{u}} du = -\sqrt{u} + C.$$

Final answer:  $x \arccos x - \sqrt{1-x^2} + C$ .

•  $\int e^{\sqrt{x}} dx$  [4pt]

Using substitution rule with  $t = \sqrt{x}$  we obtain

$$\int e^{\sqrt{x}} dx = 2 \int t e^t dt$$

Integration by parts: take  $u = t, v = e^t$ . Then

$$RHS = 2 \int t d(e^t) = 2 \left( t e^t - \int e^t dt \right) = 2(t e^t - e^t + \tilde{C}) = 2(\sqrt{x} - 1)e^{\sqrt{x}} + C.$$

2. Trigonometric integration: Evaluate the following integral of the form  $\int \sin^n x \cos^m x dx$ .

You need specify the values for  $\theta$ , so that you can get rid of absolute values.

•  $\int \sin^2 x \cos^3 x dx$  [3pt]

Note that

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x \cos^2 x \cdot \cos x dx = \int \sin^2 x (1 - \sin^2 x) \cdot \cos x dx.$$

Apply substitution rule with  $u = \sin x$ ,  $du = \cos x dx$ . So

$$RHS = \int u^2(1-u^2) du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

•  $\int \cos^4 x dx$  [4pt]

Note that

$$\begin{aligned} \cos^4 x &= \cos^2 x \cos^2 x = \frac{1}{4}(1 + \cos(2x))^2 = \frac{1}{4}(1 + 2\cos(2x) + \cos^2(2x)) \\ &= \frac{1}{4} \left( 1 + 2\cos(2x) + \frac{1}{2}(1 + \cos(4x)) \right) = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x). \end{aligned}$$

Hence

$$\begin{aligned} \int \cos^4 x dx &= \int \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x) dx \\ &= \frac{3}{8} \int 1 dx + \frac{1}{2} \int \cos(2x) dx + \frac{1}{8} \int \cos(4x) dx \\ &= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C. \end{aligned}$$

3. Trigonometric substitution

- $\int \frac{x^2}{\sqrt{9-x^2}} dx$  [6pt]

Let  $x = 3 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ , then  $dx = 3 \cos \theta d\theta$

$$\begin{aligned} \int \frac{x^2}{\sqrt{9-x^2}} dx &= \int \frac{9 \sin^2 \theta}{\sqrt{9-9 \sin^2 \theta}} \cdot 3 \cos \theta d\theta = \int \frac{27 \sin^2 \theta \cos \theta}{3 \cos \theta} d\theta \\ &= \int 9 \sin^2 \theta d\theta = \int \frac{9(1-\cos 2\theta)}{2} d\theta = \frac{9}{2} \theta - \frac{9}{4} \sin 2\theta + C \\ &= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C = \frac{9}{2} \left( \arcsin \left( \frac{x}{3} \right) - \frac{x}{3} \sqrt{1 - \left( \frac{x}{3} \right)^2} \right) + C \\ &= \frac{9}{2} \arcsin \left( \frac{x}{3} \right) - \frac{x \sqrt{9-x^2}}{2} + C. \end{aligned}$$

- $\int \frac{1}{\sqrt{25+x^2}} dx$  [6pt]

Let  $x = 5 \tan \theta$ ,  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ , then  $dx = 5 \sec^2 \theta d\theta$

$$\begin{aligned} \int \frac{1}{\sqrt{25+x^2}} dx &= \int \frac{5 \sec^2 \theta}{\sqrt{25+25 \tan^2 \theta}} d\theta = \int \frac{5 \sec^2 \theta}{\sqrt{25 \sec^2 \theta}} d\theta \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{25+x^2}}{5} + \frac{x}{5} \right| + C. \end{aligned}$$

- $\int \frac{1}{\sqrt{x^2+2x}} dx$  [6pt]

Note that

$$\int \frac{1}{\sqrt{x^2+2x}} dx = \int \frac{1}{\sqrt{(x+1)^2-1}} dx.$$

Let  $x+1 = \sec \theta$ ,  $0 \leq \theta < \frac{\pi}{2}$ , then  $dx = \sec \theta \tan \theta d\theta$

$$\begin{aligned} RHS &= \int \frac{\sec \theta \tan \theta}{\sqrt{\sec^2 \theta - 1}} d\theta = \int \frac{\sec \theta \tan \theta}{\tan \theta} dx \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C \\ &= \ln |x+1 + \sqrt{(x+1)^2-1}| + C \\ &= \ln |x+1 + \sqrt{x^2+2x}| + C. \end{aligned}$$



- $\int (x-2)^3 \sqrt{5+4x-x^2} \, dx$  [6pt]

Note that

$$\int (x-2)^3 \sqrt{5+4x-x^2} \, dx = \int (x-2)^3 \sqrt{9-(x-2)^2} \, dx.$$

Let  $x-2 = 3 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ , then  $dx = 3 \cos \theta \, d\theta$

$$RHS = \int (3 \sin \theta)^3 \cdot \sqrt{9-9 \sin^2 \theta} \cdot 3 \cos \theta \, d\theta = 3^5 \int \sin^3 \theta \cos^2 \theta \, d\theta.$$

To solve  $\int \sin^3 \theta \cos^2 \theta \, d\theta$ , apply substitution rule with  $u = \cos \theta$ ,  $du = -\sin \theta \, d\theta$ .  
So

$$\begin{aligned} \int \sin^3 \theta \cos^2 \theta \, d\theta &= \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta \, d\theta = - \int (1 - u^2) u^2 \, du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + \tilde{C} = \frac{\cos^5 \theta}{5} - \frac{\cos^3 \theta}{3} + \tilde{C} \end{aligned}$$

Final answer:

$$\begin{aligned} &3^5 \left[ \frac{1}{5} \left( \frac{\sqrt{9-(x-2)^2}}{3} \right)^5 - \frac{1}{3} \left( \frac{\sqrt{9-(x-2)^2}}{3} \right)^3 \right] + C \\ &= 3^5 \left[ \frac{1}{5} \left( \frac{\sqrt{5+4x-x^2}}{3} \right)^5 - \frac{1}{3} \left( \frac{\sqrt{5+4x-x^2}}{3} \right)^3 \right] + C. \end{aligned}$$

There's another way to do this problem. We will discuss that in problem session.

## Homework 2

Due: Friday, Feb 11, by the end of the class

### 1. Partial Fractions

•  $\int \frac{2x+5}{x^2+4x+8} dx$  [5pt]

We have

$$\int \frac{2x+5}{x^2+4x+8} dx = \int \frac{2x+4}{x^2+4x+8} dx + \int \frac{1}{x^2+4x+8} dx =: I + II.$$

Solve for  $I$ : substitution rule with  $u = x^2 + 4x + 8$ ,  $du = (2x + 4) dx$ . Then

$$I = \int \frac{1}{u} du = \ln |u| + C_1 = \ln |x^2 + 4x + 8| + C_1.$$

Solve for  $II$ : the second step uses substitution rule with  $u = \frac{x+2}{2}$ ,  $du = \frac{1}{2} dx$ .

$$\begin{aligned} II &= \int \frac{1}{(x+2)^2+4} dx = \frac{1}{2} \int \frac{1}{u^2+1} du \\ &= \frac{1}{2} \arctan u + C_2 = \frac{1}{2} \arctan \frac{x+2}{2} + C_2. \end{aligned}$$

Final answer:  $\ln |x^2 + 4x + 8| + \frac{1}{2} \arctan \frac{x+2}{2} + C.$

•  $\int \frac{2x^2 - x + 4}{(x^2 + 4)(x - 1)} dx$  [5pt]

Update: To decompose that rational function, set

$$\frac{2x^2 - x + 4}{(x^2 + 4)(x - 1)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x - 1} = \frac{(Ax + B)(x - 1) + C(x^2 + 4)}{(x^2 + 4)(x - 1)}.$$

Solving for  $A, B, C$  by comparing the coefficients gives  $A = 1, B = 0, C = 1$ .

We have

$$\int \frac{2x^2 - x + 4}{(x^2 + 4)(x - 1)} dx = \int \frac{x}{x^2 + 4} dx + \int \frac{1}{x - 1} dx =: I + II.$$

Solve for  $I$ : substitution rule with  $u = x^2 + 4$ ,  $du = 2x dx$ . Then

$$I = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C_1 = \frac{1}{2} \ln |x^2 + 4| + C_1.$$

Solve for  $II$ :

$$II = \ln |x - 1| + C_2.$$

Final answer:  $\frac{1}{2} \ln |x^2 + 4| + \ln |x - 1| + C.$

- $\int \frac{x}{x^4 + 2x^2 + 2} dx$  [4pt]

Substituting  $u = x^2$ ,  $du = 2x dx$  gives

$$\begin{aligned} \int \frac{x}{x^4 + 2x^2 + 2} dx &= \int \frac{1}{(u+1)^2 + 1} du = \frac{1}{2} \arctan(u+1) + C \\ &= \frac{1}{2} \arctan(x^2 + 1) + C. \end{aligned}$$

- $\int \ln(x^2 + 1) dx$  [4pt]

Integration by parts with  $u = \ln(x^2 + 1)$ ,  $v = x$  gives

$$\begin{aligned} \int \ln(x^2 + 1) dx &= x \ln(x^2 + 1) - \int \frac{2x^2}{x^2 + 1} dx \\ &= x \ln(x^2 + 1) - \int \frac{2x^2 + 2 - 2}{x^2 + 1} dx \\ &= x \ln(x^2 + 1) - 2 \int 1 dx + \int \frac{1}{x^2 + 1} dx \\ &= x \ln(x^2 + 1) - 2x + 2 \arctan x + C. \end{aligned}$$

- $\int \frac{1}{\sqrt{x} + x\sqrt{x}} dx$  [4pt]

Note that

$$\int \frac{1}{\sqrt{x} + x\sqrt{x}} dx = \int \frac{1}{1+x} \cdot \frac{1}{\sqrt{x}} dx.$$

Substituting  $u = \sqrt{x}$ ,  $du = \frac{1}{2\sqrt{x}} dx$  gives

$$RHS = 2 \int \frac{1}{1+u^2} du = 2 \arctan u + C = 2 \arctan(\sqrt{x}) + C.$$

- $\int \frac{1}{x + \sqrt[3]{x}} dx$  [4pt]

Note that

$$\int \frac{1}{x + \sqrt[3]{x}} dx = \int \frac{1}{x^{2/3} + 1} \cdot \frac{1}{\sqrt[3]{x}} dx.$$

Substituting  $u = x^{2/3}$ ,  $du = \frac{2}{3\sqrt[3]{x}} dx$  gives

$$RHS = \frac{3}{2} \int \frac{1}{u+1} du = \frac{3}{2} \ln|u+1| + C = \frac{3}{2} \ln|x^{2/3} + 1| + C.$$

## 2. Approximate Integration

- Use the Midpoint Rule with  $n = 5$  to approximate  $\int_0^{10} x^2 \, dx$ . [3pt]

The width of each subinterval is 2. Compute the value of  $f$  and substituting into the formula gives

$$\int_0^{10} x^2 \, dx \approx 2(1^2 + 3^2 + 5^2 + 7^2 + 9^2) = 330.$$

- Use the Trapezoidal Rule with  $n = 6$  to approximate  $\int_0^{\pi} \sin^2 x \, dx$ . [3pt]

The width of each subinterval is  $\frac{\pi - 0}{n} = \frac{\pi}{6}$ . Compute the value of  $f$  and substituting into the formula gives

$$\int_0^{\pi} \sin^2 x \, dx \approx \frac{\pi}{6 \cdot 2} \left[ 0 + 2 \cdot \frac{1}{4} + 2 \cdot \frac{3}{4} + 2 \cdot 1 + 2 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4} + 0 \right] = \frac{\pi}{2}.$$

## 3. Improper Integrals: compute the following integrals or show that it diverges.

- $\int_1^{\infty} \frac{1}{\sqrt{x}} \, dx$  [3pt]

$$\int_1^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{\sqrt{x}} \, dx = \lim_{t \rightarrow \infty} 2\sqrt{x} \Big|_1^t = \lim_{t \rightarrow \infty} (2\sqrt{t} - 2).$$

The limit goes to infinity, hence the integral diverges.

- $\int_1^{\infty} \frac{1}{1+x^2} \, dx$  [3pt]

$$\begin{aligned} \int_1^{\infty} \frac{1}{1+x^2} \, dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{1+x^2} \, dx = \lim_{t \rightarrow \infty} \arctan x \Big|_1^t \\ &= \lim_{t \rightarrow \infty} (\arctan t - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \end{aligned}$$

So the integral converges.

- $\int_{\pi}^{\infty} \sin x \, dx$  [3pt]

$$\int_{\pi}^{\infty} \sin x \, dx = \lim_{t \rightarrow \infty} \int_{\pi}^t \sin x \, dx = \lim_{t \rightarrow \infty} (-\cos x) \Big|_{\pi}^t = \lim_{t \rightarrow \infty} (\cos \pi - \cos t).$$

The limit does not exist, hence the integral diverges.

- $\int_e^\infty \frac{1}{x \ln x} \, dx$  [4pt]

Note that

$$\int_e^\infty \frac{1}{x \ln x} \, dx = \int_1^\infty \frac{1}{u} \, du = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{u} \, du = \lim_{t \rightarrow \infty} \ln u \Big|_1^t = \lim_{t \rightarrow \infty} (\ln t - \ln 1).$$

The first step uses substitution rule with  $u = \ln x$ ,  $du = \frac{1}{x} \, dx$ . The limit goes to infinity, hence the integral diverges.

- $\int_{-\infty}^\infty x e^{-x^2} \, dx$  [5pt]

Note that

$$\int_{-\infty}^\infty x e^{-x^2} \, dx = \int_{-\infty}^0 x e^{-x^2} \, dx + \int_0^\infty x e^{-x^2} \, dx =: I + II.$$

Let's compute  $II$

$$II = \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} \, dx = \lim_{t \rightarrow \infty} \left( -\frac{e^{-x^2}}{2} \right) \Big|_0^t = \lim_{t \rightarrow \infty} \left( \frac{1}{2} - \frac{e^{-t^2}}{2} \right) = \frac{1}{2}.$$

Since  $x e^{-x^2}$  is an odd function  $I = -II = -\frac{1}{2}$ . Hence the original integral converges to 0.