Exam Solutions

MATH231

Spring 2022

Midterm 1

Points indicated in each step are the max points at that step. For example, in Q1: if you reached $x\arctan x - \frac{1}{2}\ln|u| + C$, but forgot to substitute back you x, I'll remove no more than 2 points.

Q1–Q4. Evaluate the following integrals. You may use any method other than the hint.

1. $\displaystyle\int \arctan x \, \mathrm{d}x$

$$
\begin{aligned}
\int \arctan x \, \mathrm{d}x &= x\arctan x - \int x \, \mathrm{d}(\arctan x) \\
&= x\arctan(x) - \int \frac{x}{1+x^2}\, \mathrm{d}x && \text{(IBP, 4pt)} \\
&= x\arctan x - \frac{1}{2}\int \frac{1}{u}\, \mathrm{d}u && \text{(Substitution } u = 1+x^2\text{)} \\
&= x\arctan x - \frac{1}{2}\ln|u| + C && \text{(2pt)} \\
&= x\arctan x - \frac{1}{2}\ln(1+x^2) + C. && \text{(2pt)}
\end{aligned}
$$

2. $\displaystyle\int \frac{1}{\sqrt{x^2-2x}}\, \mathrm{d}x$

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2-2x}}\, \mathrm{d}x &= \int \frac{1}{\sqrt{u^2-1}}\, \mathrm{d}u \\
&\qquad \text{(Completing the square and substitution } u = x-1\text{, 2pt)} \\
&= \int \frac{1}{\sqrt{\sec^2\theta - 1}}\, \mathrm{d}(\sec\theta) \\
&\qquad \text{(Substitution } u = \sec\theta\text{, either } 0 < \theta < \frac{\pi}{2} \text{ or } \pi < \theta < \frac{3\pi}{2} \text{ see comment i below)} \\
&= \int \frac{\sec\theta\tan\theta}{|\tan\theta|}\, \mathrm{d}\theta && \text{(Absolute value removed since } \tan\theta > 0\text{, 3pt)} \\
&= \int \sec\theta \, \mathrm{d}\theta && \text{(1pt)} \\
&= \ln|\sec\theta + \tan\theta| + C = \ln|u + \sqrt{u^2-1}| + C \\
&= \ln|x - 1 + \sqrt{x^2-2x}| + C. && \text{(2pt, see comment ii below)}
\end{aligned}
$$

Comment:

i. Strictly speaking, since that square root is the the denominator, we are not allowed to have $\tan\theta = 0$. So $\theta = 0$ (respectively $\theta = \pi$) is excluded from the range. However, I will not remove points if you write $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

ii. You need Calc I knowledge to compute $\displaystyle\int \sec\theta \, \mathrm{d}\theta$. I didn't intend to test you on this, though we go through how to integrate this in a problem session. Hence, removing 2 points is reasonable if you reach that integral but didn't compute it.

3. $\displaystyle\int \frac{5x}{(x-2)(x+3)}\,\mathrm{d}x$

To decompose, set $\dfrac{5x}{(x-2)(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{x+3} = \dfrac{(A+B)x + (3A-2B)}{(x-2)(x+3)}$. (3pt)

Solve for A, B get $A = 2$, $B = 3$. (2pt)

$$\begin{aligned}\int \frac{5x}{x^2+x-6}\,\mathrm{d}x &= \int \frac{2}{x-2} + \frac{3}{x+3}\,\mathrm{d}x \\ &= 2\int \frac{1}{x-2}\,\mathrm{d}(x-2) + 3\int \frac{1}{x+3}\,\mathrm{d}(x+3) \\ &= 2\cdot \ln|x-2| + 3\cdot \ln|x+3| + C. \qquad (3\text{pt})\end{aligned}$$

Absolute value is needed.

4. $\displaystyle\int 16\sin^2 x\cos^4 x\,\mathrm{d}x$

Solution 1.

$$\begin{aligned}\int 16\sin^2 x\cos^4 x\,\mathrm{d}x &= 16\int (\sin x\cos x)^2\cdot\cos^2 x\,\mathrm{d}x \\ &= 16\int \frac{\sin^2(2x)}{4}\cdot\frac{1+\cos(2x)}{2}\,\mathrm{d}x \qquad (3\text{pt}) \\ &= 2\left(\int \sin^2(2x)\,\mathrm{d}x + \int \sin^2(2x)\cdot\cos(2x)\,\mathrm{d}x\right) \\ &= 2\left(\int \frac{1-\cos(4x)}{2}\,\mathrm{d}x + \int \frac{1}{2}\sin^2(2x)\,\mathrm{d}(\sin(2x))\right) \\ &= \int 1-\cos(4x)\,\mathrm{d}x + \int \sin^2(2x)\,\mathrm{d}(\sin(2x)) \\ &= x - \frac{\sin(4x)}{4} + \frac{\sin^3(2x)}{3} + C \qquad (7\text{pt})\end{aligned}$$

Solution 2. I actually didn't expect you to solve it this way. You'll see why in a second.

$$\begin{aligned}\int 16\sin^2 x\cos^4 x\,\mathrm{d}x &= \int 16\sin^2 x\cos^2 x\cos^2 x\,\mathrm{d}x \\ &= 16\int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\,\mathrm{d}x \qquad (3\text{pt}) \\ &= 2\int \Big(1-\cos^2(2x)\Big)\cdot\Big(1+\cos(2x)\Big)\,\mathrm{d}x\end{aligned}$$

(there are different ways to expand this expression, e.g. combine the first two terms)

$$\begin{aligned}&= 2\int \left(1-\frac{1+\cos(4x)}{2}\right)\cdot\Big(1+\cos(2x)\Big)\,\mathrm{d}x \\ &= \int \Big(1-\cos(4x)\Big)\cdot\Big(1+\cos(2x)\Big)\,\mathrm{d}x \\ &= \int 1+\cos(2x)-\cos(4x)-\cos(4x)\cos(2x)\,\mathrm{d}x \qquad (5\text{pt})\end{aligned}$$

Now you'll have to use some knowledge from a pre-calculus course. (I'm not assuming you memorize this by heart, so I only leave 2 points for the rest of the computation.) One of the product formula for trig functions says

$$\cos(u)\cos(v) = \frac{\cos(u+v)+\cos(u-v)}{2}.$$

Therefore

$$\begin{aligned}\int \cos(4x)\cos(2x)\,\mathrm{d}x &= \frac{1}{2}\int \cos(6x)+\cos(2x)\,\mathrm{d}x\\ &= \frac{1}{12}\sin(6x)+\frac{1}{4}\sin(2x)+\tilde{C}.\end{aligned}$$

Final answer:

$$\begin{aligned}I &= x+\frac{1}{2}\sin(2x)-\frac{1}{4}\sin(4x)-\frac{1}{12}\sin(6x)-\frac{1}{4}\sin(2x)+C\\ &= x-\frac{1}{4}\sin(4x)+\frac{1}{4}\sin(2x)-\frac{1}{12}\sin(6x)+C\end{aligned}$$

You can check that $\sin^3\theta = \dfrac{3\sin\theta-\sin(3\theta)}{4}$. So these two approaches give you the same answer.

Q5–Q6. Improper integrals.

> This problem aims to test your specific knowledge of improper integrals. If you didn't use the definition in Q5 or the comparison test in Q6 at all, the maximum number of points you can get is half of the total points assigned to that part.

5. **Use the definition** to show that $\displaystyle\int_e^\infty \frac{1}{x\sqrt{\ln x}}\,\mathrm{d}x$ diverges.

$$\begin{aligned}\int_e^\infty \frac{1}{x\sqrt{\ln x}} &= \lim_{t\to\infty}\int_e^t \frac{1}{x\sqrt{\ln x}}\,\mathrm{d}x && \text{(3pt)}\\ &= \lim_{t\to\infty}\int_e^t \frac{1}{\sqrt{\ln x}}\,\mathrm{d}(\ln x) && \text{(3pt)}\\ &= \lim_{t\to\infty} 2\sqrt{\ln x}\,\Big|_e^t\\ s &= \lim_{t\to\infty} 2\sqrt{\ln t}-2. && \text{(diverges, 2pt)}\end{aligned}$$

6. **Use the comparison test** to show $\displaystyle\int_\pi^\infty \frac{x\sin^2 x+1}{x^4}\,\mathrm{d}x$ converges.

Use integral law

$$\int_\pi^\infty \frac{x\sin^2 x+1}{x^4} = \int_\pi^\infty \frac{\sin^2 x}{x^3}\,\mathrm{d}x + \int_\pi^\infty \frac{1}{x^4}\,\mathrm{d}x =: I+II. \qquad \text{(2pt)}$$

Part II converges because $p=4>1$. (2pt)
Comparison test for I: Note that $0\le \sin^2 x\le 1$. (1pt)

$$\frac{\sin^2 x}{x^3}\le \frac{1}{x^3} \implies I \text{ converges.} \qquad \text{(converges by } p\text{-test, 3pt)}$$

Practice midterm 2

2. Compute the surface area generated by rotating the curve $y = \sin\sqrt{x}$ about the x-axis, for $0 \le x \le \pi^2$.

   The infinitesimal line element is given by $\mathrm{d}s = \sqrt{1+(y')^2}\,\mathrm{d}x = \sqrt{1+\frac{\sin^2\sqrt{x}}{4x}}\,\mathrm{d}x$. Hence, the area integral is

$$A = \int_0^{\pi^2} 2\pi R\,\mathrm{d}s = \int_0^{\pi^2} 2\pi\sin\sqrt{x}\cdot\sqrt{1+\frac{\sin^2\sqrt{x}}{4x}}\,\mathrm{d}x \qquad (R = y)$$

   This integral doesn't seem to be solvable. Even if it is solvable, that wouldn't be something you will be asked to solve during an 45-minute exam.

6. Use the integral test to determine if $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n}$ converges or not.

   Consider $f(x) = \dfrac{\ln x}{x}$, $x \ge 1$. The function is continuous and positive. It is decreasing for $x > e$, since $\ln x > 1$ implies

$$f'(x) = \frac{1-\ln x}{x^2} < 0.$$

   So we can apply the integral test for $n \ge 3$. Consider

$$\int_3^{\infty}\frac{\ln x}{x}\,\mathrm{d}x = \lim_{t\to\infty}\int_{\ln 3}^{t} u\,\mathrm{d}u = \lim_{t\to\infty}\frac{1}{2}u^2\Big|_{\ln 3}^{t} = \infty.$$

   By integral test $\sum_{n=3}^{\infty}\frac{\ln n}{n}$ diverges, and so $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ also diverges.

Midterm 2

Similar to midterm 1, points indicated in each step are the max points at that step.

Q1–Q2. Compute the arc length or the surface area given by revolution for the following.

For Q1 and Q2, you don't have to write down the formula for ds in a separate line. 4 points is given if you correctly set up the integral. the

1. Arc length of the curve $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$ for $1 \leq x \leq e$.

$$\begin{aligned} \mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x = \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\, \mathrm{d}x \\ &= \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\, \mathrm{d}x = \left(\frac{x}{2} + \frac{1}{2x}\right) \mathrm{d}x. && \text{(3pt)} \end{aligned}$$

$$\begin{aligned} L &= \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right) \mathrm{d}x && \text{(2pt)} \\ &= \frac{1}{2}\int_1^e x + \frac{1}{x}\, \mathrm{d}x = \frac{1}{2}\left(\frac{x^2}{2} + \ln x\right)\Bigg|_1^e \\ &= \frac{1}{4} \cdot (e^2 - 1) + \frac{1}{2} \cdot (\ln e - \ln 1) = \frac{e^2 + 1}{4}. && \text{(5pt)} \end{aligned}$$

2. Surface area generated by rotating the curve $y = \sqrt{r^2 - x^2}$ about the x-axis, where $r > 0$ and $-r \leq x \leq r$.

$$\begin{aligned} \mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x = \sqrt{1 + \left(\frac{-2x}{2\sqrt{r^2 - x^2}}\right)^2}\, \mathrm{d}x \\ &= \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, \mathrm{d}x = \sqrt{\frac{r^2}{r^2 - x^2}}\, \mathrm{d}x. && \text{(3pt)} \end{aligned}$$

$$\begin{aligned} S &= \int 2\pi \cdot y\, \mathrm{d}s = \int_{-r}^{r} 2\pi \cdot \sqrt{r^2 - x^2} \cdot \sqrt{\frac{r^2}{r^2 - x^2}}\, \mathrm{d}x && \text{(3pt)} \\ &= \int_{-r}^{r} 2\pi \cdot r\, \mathrm{d}x \\ &= 2\pi r \cdot x|_{-r}^{r} = 4\pi r^2. && \text{(4pt)} \end{aligned}$$

Q3–Q6. Sequences and divergence tests.

3. Compute $\sum_{n=0}^{\infty} \sqrt{5}^{1-2n} \cdot 2^{n+2}$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \sqrt{5}^{1-2n} \cdot 2^{n+2} &= \sum_{n=0}^{\infty} \sqrt{5} \left(\sqrt{5}^2\right)^{-n} \cdot 2^n \cdot 2^2 \\
&= 4\sqrt{5} \cdot \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n && \text{(3pt)} \\
&= 4\sqrt{5} \cdot \frac{1}{1-\frac{2}{5}} = 4\sqrt{5} \cdot \frac{5}{3} = \frac{20\sqrt{5}}{3}. && \text{(3pt)}
\end{aligned}$$

4. Use the divergence test to determine if $\sum_{n=1}^{\infty} \left(1+\frac{1}{2n}\right)^n$ converges or not.

Q4: 2pt assigned for applying the divergent test; 6pt for computing the limit of the sequence.

The series diverges because $\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^n = \sqrt{e} \neq 0$. (2pt)

To compute this limit:

Method 1: Using the fact that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$. Let $m = 2n$ then (2pt)

$$\begin{aligned}
\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^n &= \lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{m/2} \\
&= \left(\lim_{m\to\infty} \left(1+\frac{1}{m}\right)^m\right)^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}. && \text{(4pt)}
\end{aligned}$$

Method 2: Compute the limit directly

$$\begin{aligned}
\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^n &= \lim_{n\to\infty} \exp\left(\ln\left(1+\frac{1}{2n}\right)^n\right) && \text{(exp and ln functions are inverses.)} \\
&= \exp\left(\lim_{n\to\infty} n \ln\left(1+\frac{1}{2n}\right)\right) = \exp\left(\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{2n}\right)}{\frac{1}{n}}\right).
\end{aligned}$$

To compute the limit inside exponential, apply L'Hopital's rule (x is used because we need the function to be differentiable, but n is discrete)

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{1}{1+\frac{1}{2x}} \cdot \frac{1}{2}\frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{2}}{1+\frac{1}{2x}} = \frac{1}{2}. \qquad \text{(4pt)}$$

So the final answer for the limit is $\sqrt{e} \neq 0$. (2pt)

5. Use the integral test to determine if $\sum_{n=1}^{\infty} n e^{-n^2}$ converges or not.

Let $f(x) = xe^{-x^2}$. Note that f is a positive and continuous function defined on $[1, \infty)$. Moreover f is decreasing because (2pt)

$$f'(x) = e^{-x^2}(1 - 2x^2) \quad \Longrightarrow \quad f'(x) < 0 \text{ when } x \geq 1. \qquad (2\text{pt})$$

The above argument guarantees that we can apply the integral test. Now compute the improper integral:

$$\begin{aligned}\int_1^{\infty} xe^{-x^2}\,\mathrm{d}x &= \int_1^{\infty} \frac{1}{2} e^{-x^2}\,\mathrm{d}x^2 = -\frac{1}{2}\lim_{t\to\infty} e^{-x^2}\Big|_1^t \\ &= -\frac{1}{2}\lim_{t\to\infty} e^{-t^2} - e^{-1} = \frac{1}{2e} < \infty. \qquad (4\text{pt})\end{aligned}$$

The series converges by integral test. (2pt)

6. Use the comparison test to determine if $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^6 + n}}$ converges or not.

Note that $\frac{n}{\sqrt{n^6}}$ dominates the sequence. Note that for $n \geq 1$,

$$\begin{aligned} n^3 = \sqrt{n^6} < \sqrt{n^6 + n} &\Longrightarrow \frac{1}{\sqrt{n^6+n}} < \frac{1}{n^3} \\ &\Longrightarrow \frac{n}{\sqrt{n^6+n}} < \frac{1}{n^2}. \qquad (4\text{pt})\end{aligned}$$

Apply comparison test with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (we know this converges), we conclude the series converges. (2pt)

Midterm 3

1. Use <u>the limit comparison test</u> to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^4+3n-1}}$ converges or diverges.

   Denote $a_n = \dfrac{\sqrt{n}}{\sqrt{n^4+3n-1}}$ and $b_n = \dfrac{\sqrt{n}}{\sqrt{n^4}} = \dfrac{\sqrt{n}}{n^2}$. Then (3pt)

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n^4+3n-1}}}{\frac{\sqrt{n}}{\sqrt{n^4}}} = \lim_{n\to\infty} \frac{n^2}{\sqrt{n^4+3n-1}} \\ &= \lim_{n\to\infty} \sqrt{\frac{1}{1+3/n^3-1/n^4}} = 1.\end{aligned}$$

(3pt)

   The limit is finite and not zero, so comparison test applies. Hence

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ converges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

(2pt)

Q2–Q4. Determine if the alternating series is <u>absolutely convergent</u>, <u>conditionally convergent</u> or <u>diverges</u>. You may use any method.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n \cdot n^2}{(2n)!}$

   Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}(n+1)^2}{[2(n+1)]!}}{\frac{(-3)^n n^2}{(2n)!}} \right| = \lim_{n\to\infty} \frac{3(n+1)^2}{n^2(2n+2)(2n+1)} = 0 < 1.$$

   By the ratio test, the series converges absolutely. (8pt)

3. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

   Root test:

$$L = \lim_{n\to\infty} \left| \frac{(-1)^n}{(\ln n)^n} \right|^{1/n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1.$$

   By the root test, the series converges absolutely. (6pt)

4. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$

Consider $|a_n| = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$. Take $b_n = \dfrac{1}{\sqrt{4n}+\sqrt{n}} = \dfrac{1}{3}\cdot\dfrac{1}{\sqrt{n}}$, [1] then

$$0 < b_n < |a_n|, \text{ and } \sum b_n \text{ is divergent as } p = \frac{1}{2}.$$

By comparison test the series of absolute value diverges. Hence the alternating series $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge absolutely. (4pt)

Now using the alternating series test to check conditional convergence:

- $|a_n| \geq 0$.
- $\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$.
- $\sqrt{(n+1)+1}+\sqrt{n+1} > \sqrt{n+1}+\sqrt{n}$ for all n implies $|a_n| > |a_{n+1}|$ for all n. (4pt)

By alternating series test, the original series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges conditionally. (2pt)

5. Determine the radius of convergence R and interval of convergence I for the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{(-4)^n\sqrt{n}}$.

Compute the limit

$$L = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(-4)^{n+1}\sqrt{n+1}}}{\frac{(x-2)^n}{(-4)^n\sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)\sqrt{n}}{4\sqrt{n+1}} \right| = \frac{|x-2|}{4}.$$

The inequality $L < 1$ gives $|x-2| < 4$. Hence $R = 4$ and the series converges when $-2 < x < 6$. (3pt)

Now consider the boundary cases: $x = -2$ and $x = 8$:

- When $x = -2$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^n}{(-4)^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is divergent. (3pt)
- When $x = 6$, we have $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(-4)^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Apply alternating series test, $a_n = \frac{1}{\sqrt{n}}$ is positive, decreasing and its limit is zero. Hence the alternating series is convergent. (4pt)

This implies $I = (-2, 6]$.

[1] there are more than one way to choose b_n, e.g. $\dfrac{1}{\sqrt{2n}+\sqrt{n}}$ or $\dfrac{1}{2\sqrt{n+1}}$

6. Write $f(x) = \ln(x+5)$ as a power series <u>centered at 0</u>.

There are two possible approaches.

Solution 1.

$$\ln(x+5) - \ln(5+0) = \int_0^x \frac{1}{5+t}\,\mathrm{d}t \qquad \text{(FTC, 2pt)}$$

$$\ln(x+5) = \frac{1}{5}\int_0^x \frac{1}{1-(-\frac{t}{5})}\,\mathrm{d}t + \ln 5 \qquad \text{(substitute } u = -\tfrac{t}{5}\text{, 2pt)}$$

$$= \frac{1}{5}\int_0^x \sum_{n=0}^{\infty}\left(-\frac{t}{5}\right)^n \mathrm{d}x$$

$$= \sum_{n=0}^{\infty} \frac{1}{5}\left(-\frac{1}{5}\right)^n \int_0^x t^n\,\mathrm{d}t + \ln 5$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}}\frac{x^{n+1}}{n+1} + \ln 5. \qquad \text{(2pt)}$$

Note that substitution changes the range whereas integration does not. Here we used $u = -\frac{x}{5}$. Then $|u| = \left|-\frac{x}{5}\right| < 1$ gives $|x| < 5$. So the above expression is valid only for $x \in (-5, 5)$.

Final answer: $\ln(x+5) = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}}\frac{x^{n+1}}{n+1} + \ln 5$ when $|x| < 5$. (2pt)

Solution 2. We have seen $\ln(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}u^n}{n}$, for $|u| < 1$ in class. Hence

$$\ln(x+5) = \ln\left[5\cdot\left(1+\frac{x}{5}\right)\right] \qquad \text{(substitute } u = \frac{x}{5}\text{)}$$

$$= \ln 5 + \ln(1+u)$$

$$= \ln 5 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}u^n}{n} \qquad \text{(Note that } n \text{ start from 1 here, and } |u| < 1.\text{)}$$

$$= \ln 5 + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{5^n\cdot n}$$

$$\text{or} \quad = \ln 5 + \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{5^{n+1}\cdot(n+1)} \qquad \text{when } |x| < 5. \text{ (Note that } n \text{ start from 0 here.)}$$

Final Exam

Q1–Q3. Compute the following integrals. You may use any method.

1. $\int (\ln x)^2 \, \mathrm{d}x$ [6pt]

   See Homework 1: 3rd question in Q1. Here we have a definite integral, so the final answer is
$$(e - 2e + 2e) - 2 = e - 2.$$

2. $\int (x-1)^2 \sqrt{1-(x-1)^2} \, \mathrm{d}x$ [10pt]

   Compare to Homework 1: 4th question in Q3.

   Trig substitution $x - 1 = \sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then the indefinite integral becomes:
$$\begin{aligned} 4\int \sin^2\theta\sqrt{1-\sin^2\theta}\, \mathrm{d}(\sin\theta) &= 4\int \sin^2\theta\cos^2\theta \, \mathrm{d}\theta = 2\int \sin^2(2\theta)\, \mathrm{d}\theta \\ &= \int 1 - \cos(4\theta)\, \mathrm{d}\theta = \theta - \frac{1}{4}\sin(4\theta). \end{aligned}$$
   Note that when $x = \sin\theta = 0$, $\theta = 0$ and $x = \sin\theta = 1$, $\theta = \frac{\pi}{2}$. Hence the definite integral is
$$\frac{\pi}{2} - \frac{1}{4}\sin(2\pi) - 0 + \frac{1}{4}\sin 0 = \frac{\pi}{2}.$$

3. $\int_0^1 \frac{1}{(1+x)\sqrt{x}} \, \mathrm{d}x$ (This is an improper integral.) [8pt]
$$\begin{aligned} \int_0^1 \frac{1}{(1+x)\sqrt{x}}\, \mathrm{d}x &= \lim_{t\to 0^+}\int_t^1 \frac{1}{(1+x)\sqrt{x}}\, \mathrm{d}x && \text{(improper integral)} \\ &= \lim_{t\to 0^+}\int_{\sqrt{t}}^1 \frac{2}{1+u^2}\, \mathrm{d}u && (u = \sqrt{x},\ \mathrm{d}u = \mathrm{d}\sqrt{x} = \frac{1}{2\sqrt{x}\, \mathrm{d}x}) \\ &= \lim_{t\to 0^+} 2\arctan u\Big|_{\sqrt{t}}^1 = 2\arctan 1 - \lim_{t\to 0^+} 2\arctan\sqrt{t} = \frac{\pi}{2}. \end{aligned}$$

Q4–Q5. Compute arc length or surface area.

4. Compute the arc length of the curve $x = \sin^3 t, y = 1 - \cos^3 t$ for $0 \le t \le \pi$. [10pt]
$$\begin{aligned} L &= \int_0^\pi \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t = \int_0^\pi \sqrt{(3\sin^2 t\cos t)^2 + (-3\cos^2 t\sin t)^2}\, \mathrm{d}t \qquad \text{(4pt)} \\ &= \int_0^\pi \sqrt{9\left(\sin^4 t\cos^2 t + \cos^4 t\sin^2 t\right)}\, \mathrm{d}t = 3\int_0^\pi \sqrt{\sin^2 t\cos^2 t\left(\sin^2 t + \cos^2 t\right)}\, \mathrm{d}t \\ &= 3\int_0^\pi \sin t\cos t\, \mathrm{d}t = \frac{3}{2}\int_0^\pi \sin(2t) = -\frac{3}{4}\cos 2t\Big|_0^\pi = \frac{3}{2}. \end{aligned}$$

5. Compute the surface area of the curve $y = \sqrt{1 - \frac{x^2}{2}}$ rotating about the x-axis, where $0 \le x \le 1$. [10pt]

$$\begin{aligned} \mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(\frac{-x}{2\sqrt{1-\frac{x^2}{2}}}\right)^2}\,\mathrm{d}x \\ &= \sqrt{1 + \frac{x^2}{4-2x^2}}\,\mathrm{d}x = \sqrt{\frac{4-x^2}{4-2x^2}}\,\mathrm{d}x. \qquad (3\text{pt}) \end{aligned}$$

$$\begin{aligned} S &= \int 2\pi \cdot y\,\mathrm{d}s = \int_0^1 2\pi \cdot \sqrt{1 - \frac{x^2}{2}} \cdot \sqrt{\frac{4-x^2}{4-2x^2}}\,\mathrm{d}x \qquad (3\text{pt}) \\ &= \int_0^1 \pi \cdot \sqrt{4-2x^2} \cdot \sqrt{\frac{4-x^2}{4-2x^2}}\,\mathrm{d}x \\ &= \pi \int_0^1 \sqrt{4-x^2}\,\mathrm{d}x \end{aligned}$$

There are two solution for the integral above.

Solution 1. Use geometry. This compute the region bounded by $x^2 + y^2 = 4$, $y = 0$, $x = 0$ and $x = 1$. So the value equals to the area of a sector plus a triangle, which is $\frac{\pi}{3} + \frac{1}{2}$.

Solution 2. Compute directly. Using trig substitution $x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ to compute this integral

$$\begin{aligned} \int_0^1 \sqrt{4-x^2}\,\mathrm{d}x &= \int_0^1 \sqrt{4 - 4\sin^2\theta}\,\mathrm{d}(2\sin\theta) \\ &= \int_{\theta_0}^{\theta_1} 4\cos^2\theta\,\mathrm{d}\theta \qquad \text{(We will determine } \theta_0 \text{ and } \theta_1 \text{ later.)} \\ &= 2\int_{\theta_0}^{\theta_1} 1 + \cos(2\theta)\,\mathrm{d}\theta \\ &= (2\theta + \sin(2\theta))\Big|_{\theta_0}^{\theta_1}. \end{aligned}$$

Note that when $x = 2\sin\theta = 0$, $\theta_0 = 0$; when $x = 2\sin\theta = 1$, $\theta_0 = \frac{\pi}{6}$. So the above is $\frac{\pi}{3} + \frac{1}{2}$. Final answer $\pi \cdot \left(\frac{\pi}{3} + \frac{1}{2}\right)$. (4pt)

Q6-Q8. Determine if the series is convergent or diverges. You may use any method.

6. $\sum_{n=0}^{\infty} 4^{1-\frac{n}{2}} 3^{n+2}$ [6pt]

Compare to Homework 4: 1st question in Q2. Check that $r = \frac{3}{2}$ so the geometric series diverges.

7. $\displaystyle\sum_{n=0}^{\infty} \frac{3n^2 - 7n}{n^4 - 2n + 1}$ [6pt]

Compare to Homework 5: 1st question in Q2. Limit comparison test with $b_n = \dfrac{1}{n^2}$. Then (2pt)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{3n^2-7n}{n^4-2n+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{3n^4 - 7n^3}{n^4 - 2n + 1} = \lim_{n\to\infty} = 3. \qquad (2\text{pt})$$

The limit is finite and not zero, so comparison test applies. Hence

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} \frac{3n^2 - 7n}{n^4 - 2n + 1} \text{ converges.} \qquad (2\text{pt})$$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(3n-1)!}{(2n)! \cdot 3^n}$ [6pt]

Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{(3(n+1)-1)!}{(2(n+1))! \cdot 3^{n+1}}}{\frac{(3n-1)!}{(2n)! \cdot 3^n}} \right| = \lim_{n\to\infty} \frac{(3n+2)(3n+1)n}{(2n+2)(2n+1)} = \infty > 1.$$

By the ratio test, the series diverges.

Q9. Determine if the alternating series is absolutely convergent, conditionally convergent or diverges. You may use any method.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(1 + \frac{1}{n}\right)^{n^2}}$ [8pt]

Root test:

$$L = \lim_{n\to\infty} \left(\left| \frac{(-1)^n}{\left(1+\frac{1}{n}\right)^{n^2}} \right| \right)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

By the root test, the series converges absolutely.

Q10. Determine the radius of convergence R and interval of convergence I for the following series.

10. $\displaystyle\sum_{n=2}^{\infty} \frac{(x+7)^n}{3^n \ln n}$ [10pt]

See Homework 6: 2nd question in Q1. Here the coefficients are changed. You should replace $(x+2)$ by $(x+7)$ and 2^n by 3^n.

Q11–Q12. Write the following function as a power series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$.

11. Find the <u>Maclaurin series</u> of $f(x) = \int_0^x \frac{2}{(t-1)(t-3)}\,\mathrm{d}t$.

One possible solution is:

$$\begin{aligned}
\int_0^x \frac{2}{(t-1)(t-3)}\,\mathrm{d}t &= \int_0^x \frac{1}{1-t} - \frac{1}{3-t}\,\mathrm{d}t \\
&= \int_0^x \frac{1}{1-t} - \frac{1}{3}\cdot\frac{1}{1-\frac{t}{3}}\,\mathrm{d}t \\
&= \int_0^x \sum_{n=0}^{\infty} t^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{t}{3}\right)^n \mathrm{d}t \qquad \text{(Substitution } u = \tfrac{t}{3} \text{ so } |t| < 3.\text{)} \\
&= \sum_{n=0}^{\infty}\int_0^x t^n\,\mathrm{d}t - \frac{1}{3^{n+1}}\sum_{n=0}^{\infty}\int_0^x t^n\,\mathrm{d}t \quad \text{(term by term integration)} \\
&= \sum_{n=0}^{\infty}\left(1 - \frac{1}{3^{n+1}}\right)\frac{x^{n+1}}{n+1}. \qquad \text{(vaild when } |x| < 3\text{)}
\end{aligned}$$

There are other ways to solve the problem.

12. Find the Taylor series of $f(x) = \cos x$ <u>centered at π</u>. [10pt]

See Homework 6: 1st question in Q3. Here sin is replaced by cos.