

Midterm Solutions

MATH231

Spring 2022

1 Midterm 1

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Points indicated in each step are the max points at that step. For example, in Q1: if you reached $x \arctan x - \frac{1}{2} \ln |u| + C$, but forgot to substitute back you x , I'll remove no more than 2 points.

Q1–Q4. Evaluate the following integrals. You may use any method other than the hint.

1. $\int \arctan x \, dx$

$$\begin{aligned} \int \arctan x \, dx &= x \arctan x - \int x \, d(\arctan x) \\ &= x \arctan(x) - \int \frac{x}{1+x^2} \, dx && \text{(IBP, 4pt)} \\ &= x \arctan x - \frac{1}{2} \int \frac{1}{u} \, du && \text{(Substitution } u = 1+x^2) \\ &= x \arctan x - \frac{1}{2} \ln |u| + C && \text{(2pt)} \\ &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C. && \text{(2pt)} \end{aligned}$$

2. $\int \frac{1}{\sqrt{x^2-2x}} \, dx$

$$\begin{aligned} \int \frac{1}{\sqrt{x^2-2x}} \, dx &= \int \frac{1}{\sqrt{u^2-1}} \, du \\ &\quad \text{(Completing the square and substitution } u = x-1, \text{ 2pt)} \\ &= \int \frac{1}{\sqrt{\sec^2 \theta - 1}} \, d(\sec \theta) \\ &\quad \text{(Substitution } u = \sec \theta, \text{ either } 0 < \theta < \frac{\pi}{2} \text{ or } \pi < \theta < \frac{3\pi}{2} \text{ see comment i below)} \\ &= \int \frac{\sec \theta \tan \theta}{|\tan \theta|} \, d\theta && \text{(Absolute value removed since } \tan \theta > 0, \text{ 3pt)} \\ &= \int \sec \theta \, d\theta && \text{(1pt)} \\ &= \ln |\sec \theta + \tan \theta| + C = \ln |u + \sqrt{u^2-1}| + C \\ &= \ln |x-1 + \sqrt{x^2-2x}| + C. && \text{(2pt, see comment ii below)} \end{aligned}$$

Comment:

- i. Strictly speaking, since that square root is the the denominator, we are not allowed to have $\tan \theta = 0$. So $\theta = 0$ (respectively $\theta = \pi$) is excluded from the range. However, I will not remove points if you write $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$
- ii. You need Calc I knowledge to compute $\int \sec \theta \, d\theta$. I didn't intend to test you on this, though we go through how to integrate this in a problem session. Hence, removing 2 points is reasonable if you reach that integral but didn't compute it.

3. $\int \frac{5x}{(x-2)(x+3)} dx$

To decompose, set $\frac{5x}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} = \frac{(A+B)x + (3A-2B)}{(x-2)(x+3)}$. (3pt)

Solve for A, B get $A = 2, B = 3$. (2pt)

$$\begin{aligned} \int \frac{5x}{x^2+x-6} dx &= \int \frac{2}{x-2} + \frac{3}{x+3} dx \\ &= 2 \int \frac{1}{x-2} d(x-2) + 3 \int \frac{1}{x+3} d(x+3) \\ &= 2 \cdot \ln|x-2| + 3 \cdot \ln|x+3| + C. \end{aligned} \quad (3pt)$$

Absolute value is needed.

4. $\int 16 \sin^2 x \cos^4 x dx$

Solution 1.

$$\begin{aligned} \int 16 \sin^2 x \cos^4 x dx &= 16 \int (\sin x \cos x)^2 \cdot \cos^2 x dx \\ &= 16 \int \frac{\sin^2(2x)}{4} \cdot \frac{1 + \cos(2x)}{2} dx \quad (3pt) \\ &= 2 \left(\int \sin^2(2x) dx + \int \sin^2(2x) \cdot \cos(2x) dx \right) \\ &= 2 \left(\int \frac{1 - \cos(4x)}{2} dx + \int \frac{1}{2} \sin^2(2x) d(\sin(2x)) \right) \\ &= \int 1 - \cos(4x) dx + \int \sin^2(2x) d(\sin(2x)) \\ &= x - \frac{\sin(4x)}{4} + \frac{\sin^3(2x)}{3} + C \quad (7pt) \end{aligned}$$

Solution 2. I actually didn't expect you to solve it this way. You'll see why in a second.

$$\begin{aligned} \int 16 \sin^2 x \cos^4 x dx &= \int 16 \sin^2 x \cos^2 x \cos^2 x dx \\ &= 16 \int \frac{1 - \cos(2x)}{2} \cdot \frac{1 + \cos(2x)}{2} \cdot \frac{1 + \cos(2x)}{2} dx \quad (3pt) \\ &= 2 \int \left(1 - \cos^2(2x) \right) \cdot \left(1 + \cos(2x) \right) dx \end{aligned}$$

(there are different ways to expand this expression, e.g. combine the first two terms)

$$\begin{aligned} &= 2 \int \left(1 - \frac{1 + \cos(4x)}{2} \right) \cdot \left(1 + \cos(2x) \right) dx \\ &= \int \left(1 - \cos(4x) \right) \cdot \left(1 + \cos(2x) \right) dx \\ &= \int 1 + \cos(2x) - \cos(4x) - \cos(4x) \cos(2x) dx \quad (5pt) \end{aligned}$$

Now you'll have to use some knowledge from a pre-calculus course. (I'm not assuming you memorize this by heart, so I only leave 2 points for the rest of the computation.) One of the product formula for trig functions says

$$\cos(u) \cos(v) = \frac{\cos(u+v) + \cos(u-v)}{2}.$$

Therefore

$$\begin{aligned}\int \cos(4x) \cos(2x) \, dx &= \frac{1}{2} \int \cos(6x) + \cos(2x) \, dx \\ &= \frac{1}{12} \sin(6x) + \frac{1}{4} \sin(2x) + \tilde{C}.\end{aligned}$$

Final answer:

$$\begin{aligned}I &= x + \frac{1}{2} \sin(2x) - \frac{1}{4} \sin(4x) - \frac{1}{12} \sin(6x) - \frac{1}{4} \sin(2x) + C \\ &= x - \frac{1}{4} \sin(4x) + \frac{1}{4} \sin(2x) - \frac{1}{12} \sin(6x) + C\end{aligned}$$

You can check that $\sin^3 \theta = \frac{3 \sin \theta - \sin(3\theta)}{4}$. So these two approaches give you the same answer.

Q5–Q6. Improper integrals.

This problem aims to test your specific knowledge of improper integrals. If you didn't use the definition in Q5 or the comparison test in Q6 at all, the maximum number of points you can get is half of the total points assigned to that part.

5. **Use the definition** to show that $\int_e^\infty \frac{1}{x\sqrt{\ln x}} \, dx$ diverges.

$$\int_e^\infty \frac{1}{x\sqrt{\ln x}} \, dx = \lim_{t \rightarrow \infty} \int_e^t \frac{1}{x\sqrt{\ln x}} \, dx \quad (3\text{pt})$$

$$= \lim_{t \rightarrow \infty} \int_e^t \frac{1}{\sqrt{\ln x}} \, d(\ln x) \quad (3\text{pt})$$

$$= \lim_{t \rightarrow \infty} 2\sqrt{\ln x} \Big|_e^t$$

$$s = \lim_{t \rightarrow \infty} 2\sqrt{\ln t} - 2. \quad (\text{diverges, 2pt})$$

6. **Use the comparison test** to show $\int_\pi^\infty \frac{x \sin^2 x + 1}{x^4} \, dx$ converges.

Use integral law

$$\int_\pi^\infty \frac{x \sin^2 x + 1}{x^4} \, dx = \int_\pi^\infty \frac{\sin^2 x}{x^3} \, dx + \int_\pi^\infty \frac{1}{x^4} \, dx =: I + II. \quad (2\text{pt})$$

Part II converges because $p = 4 > 1$. (2pt)

Comparison test for I : Note that $0 \leq \sin^2 x \leq 1$. (1pt)

$$\frac{\sin^2 x}{x^3} \leq \frac{1}{x^3} \implies I \text{ converges.} \quad (\text{converges by } p\text{-test, 3pt})$$