MATH541 Functional Analysis, Spring 2021

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Warning: I'm typing the notes slowly. Given that lecture recordings are not uploaded regularly, you can expect no updates for weeks.

Contents

1 Baire's Categroy Theorem 20210125

Ref: A Course in Functional Analysis, John B. Conway, 1985

1. Metric space
2. Chicago suburb distance $\mathbb{R}^b$ compact = closed and bounded no longer true
3. Cauchy sequence, completeness
4. Open, closed ball
5. Nowhere dense set, dense set, closure, interior.

$$Y \text{ is nowhere dense} \iff \bar{Y}^C \text{ is open and dense.}$$

Theorem 1.1 (Baire's theorem)**.** *In a complete metric space, the countable union of nowhere dense sets is again nowhere dense.*

Lemma 1.2. *The intersection of open dense sets is again open dense.*

Using the above lemma + induction to prove Baire's theorem.

Dense, nowhere dense, somewhere dense. Stack Exchange Theorem in notes: countable intersection of open dense is dense, then countable union does not have interior points. Need X complete metric space, so that the limit point is in X.

2 Baire's Categroy Theorem Cont. 20210127

Last time: open set, closed sets, theorem: let (X, d) be a complete metric space, O_n open dense, then $\cap_n O_n$ is dense.

1. intuition dense set $\cong$, taking away a countable set of points
2. proof idea completeness $\rightarrow$ geometric series.
3. Use Baire's theorem to show no function $f : [0, 1] \rightarrow \mathbb{R}$ continuous exactly at $\mathbb{Q}$

4. proof hard works is to find complete metric space and makes the theorem work

5. Normed space. A normed space is complete if absolute convergent sequences are convergent. Banach space.

6. isometry

7. $\|f(x)\|_{C(K)} = \sup_{k \in K} |f(k)|$.

Question 2.1. Let $C_b(\mathbb{R})$ be the set of continuous and bounded function. Is $C_b(\mathbb{R}) = C(K)$ for some compact K? — Yes.

Want to do: Start with Banach space, create new ones.

Lemma 2.2. *Let $T : X \to Y$ be a linear map between* ***normed spaces****. TFRE*

1. *T is continuous.*
2. *T is continuous at 0.*
3. *$\|T\|_{op} = \sup_{\|x\| \leq 1} \|Tx\|$ is finite.*
4. *T is Lipschitz.*

Homogeneity, duality

Lemma 2.3. *Let X be a normed sapce and Y be Banach. Then the vector space $L(X, Y)$ with the norm $\| \, \|_{op}$ becomes a Banach space.*

$L(\text{normed}, \text{Banach})$ is Banach.

Corollary 2.4. *$X^* = L(X, \mathbb{C})$ is Banach.*

3 Basic Banach Space Theory 20210129

proof of Lemma 2.3. Step 1. (T_n) Cauchy implies $(T_n(x_k))$ Cauchy.

Step 2. Let $f(x) := \lim T_n(x)$. Prove $\limsup \|T_n(x) - f(x)\| = 0$.

$$\|T_n - T\| = \|T_n - \lim T_m\| = \lim \|T_n - T_m\|$$
$$\leq \limsup_{m,n \geq N} \|T_n - T_m\| < \epsilon$$

$\|T_n - T\| < \epsilon$ implies $\|T_n(x) - T(x)\| < \epsilon$, and so $\limsup \|T_n(x) - f(x)\| = 0$.

Step 3. f is bounded, and $T_n \to f$. □

Corollary 3.1. *X Banach, then $L(X, X) = L(X)$ is Banach algebra.*

Definition 3.2. A **Banach algebra** is a Banach space $(\mathcal{A}), \| \|$ together with a product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, with $\|ab\| \leq \|a\|\|b\|$.

1. closed subset of Banach is Banach.
2. $K(X, Y) := \{T : X \to Y \mid \overline{T(B_X)} \text{ compact}\}$ is closed .
3. In finite dimension, linear bounded T is compact.

Definition 3.3 (Totally bounded)**.**

$$\forall \epsilon, \exists N \text{ s.t. } Y \subset \bigcup_{j=1}^{N} B(x_j, \epsilon)$$

This is equivalent to relatively compact. Ref

Theorem 3.4.

$$K(H, H)^{**} = B(H, H)$$

(We'll this theorem later.)

Theorem 3.5.

$$\exists \iota : X \to X^{**};\ \iota(x)(f) = f(x),\ \textit{with } f : X \to \mathbb{K}$$

1. *ι is an isometry.*
2. *$\overline{\iota(x)}$ is the completion of X.*

Part 1 follows from Hahn-Banach.

Definition 3.6. (X, d) is a metric space. A **completion** (Y, d') is given by

1. $\iota : X \to Y$ is an isometry.
2. $\iota(X)$ is dense.
3. (Y, d') is complete.

Completion is unique.

4 Basic Banach Space Theory Cont. 20210201

Completion problem: see Theorem 3.5

proof of Theorem 3.5.

Claim 4.1. $\|\iota(x)\|_{X^{**}} \leq \|x\|_X$.

Note that

$$\begin{aligned}
\|\iota(x)\|_{X^{**}} &= \sup_{\|f(x)\|_{X^*}\le 1} |\iota(x)(f)| && \text{(by definition)}\\
&= \sup_{\|f(x)\|_{X^*}\le 1} |f(x)| && (\iota \text{ inclusion})\\
&\le \sup_{\|f\|_{X^*}\le 1} \|x\| \le \|x\|.
\end{aligned}$$

By definition $\|f\|_{X^*} \le 1 \iff |f(x)| \le \|x\|$. □

For a normed space the completion achieves in X^{**}.

Banach space

Lemma 4.2. *$C_b(x, x_0)$ is a Banach space.*

$$C_b(x, x_0) = \{f : X \to \mathbb{R} \mid f \textit{ continuous and } \exists C, |f(x)| \le Cd(x, x_0)\}.$$

Norm: $\|f\| = \sup_x \frac{|f(x)|}{d(x,x_0)}$.

An embedding isometry $\iota : X \to C_b(X)^*; i(x)(f) = f(x)$. Hint: use evaluation map

$$\sup_{\|f\|\le 1} |f(x) - f(x_0)| = d(x, x_0).$$

Distance attaining function is $f(x) = d(x, x_0)$, where $x \ne x_0$.

Theorem 4.3 (Hahn-Banach Extension)**.** *Given a vector space X, a sublinear map $q : X \to \mathbb{R}$ s.t.*

$$q(x + y) \le q(x) + q(y) \textit{ (subadditive) and } q(sx) = sq(x),\ s > 0.$$

Let $Y \subset X$ and $f : Y \to \mathbb{R}$ linear, with $f \le q$, then $\exists F : X \to \mathbb{R}$ linear $F \le q$ and $F|_Y = f$.

warning This theorem is completely algebraic. There is no topology.

Lemma 4.4. *We can always add an extra dimension.*

Proof. Step 1. $Y \subset X = \{y + tx_0 \mid t \in \mathbb{R}\}$. Candidates for F (extend 1-dim): $F(y + tx_0) = F(y) + tF(x_0) = f(y) + ta_0$ for some a_0. What is a_0? trick

$$\begin{aligned} F(y+tx_0) &\le q(y+tx_0) & \implies \quad & f(y)+ta_0 \le q(y+tx_0) \\ F(y-tx_0) &\le q(y-tx_0) & & f(y)-sa_0 \le q(y-sx_0) \end{aligned}$$

$$\implies \quad \begin{aligned} a_0 &\le \frac{q(y+tx_0)-f(y)}{t}, t>0 \\ a_0 &\ge \frac{f(y)-q(y-sx_0)}{s}, s>0 \end{aligned} \quad \implies \quad \begin{aligned} a_0 &\le \inf \frac{q(y+tx_0)-f(y)}{t}, t>0 \\ a_0 &\ge \sup \frac{f(y)-q(y-sx_0)}{s}, s>0 \end{aligned}$$

Check the sup is less than inf:

$$\begin{aligned} \frac{f(y)-q(y-sx_0)}{s} &\le \frac{q(z+tx_0)-f(z)}{t} \\ \iff \quad f(y)t - q(y-sx_0)t &\le q(z+tx_0)s - f(z)s \\ f(y)t + f(z)s &\le q(z+tx_0)s + q(y-sx_0)t \\ f(yt+sz) &\le q(yt + tsx_0 - tsx_0 + sz) \\ &\le q(yt - stx_0) + q(tsx_0 + sz) \\ &\le tq(y-sx_0) + sq(tx_0+z) \end{aligned}$$

This exactly fits the assumption, so we can pick $a_0 = \sup \frac{f(y)-q(y-sx_0)}{s}$.

Step 2. Use Zorn's lemma. Consider

$$\mathcal{L} = \{(Z, F) \mid Y \subset Z, F \le q \text{ on } Z, F|_Y = f\}.$$

Order on the set: $(Z_1, F_1) \le (Z_2, F_2)$ if $Z_1 \subset Z_2$ and $F_2|Z_1 = F_1$. Every chain has an upper bound $Z_\infty = \cup Z_i, F = \cup F_i$. Hence there exists a maximal element $(Z_{\max}, F_{\max}) \in \mathcal{L}$.

Claim 4.5. $Z_{max} = X$.

If not, $\exists x_0 \notin Z_{\max}$ apply lemma to $F_{\max}$, $Z_{\max} + \mathbb{R}x_0$ admits $F'_{\max}$. Contradiction. $\square$

Remark 4.6. Hahn-Banach is also true for $\mathbb{C}$.

5 Hahn-Banach Theorem 20210203

Lemma 5.1. *Take C convex, $0 \in C$. The **Minkowski functional***

$$q_C(x) = \inf\{\lambda \mid x \in \lambda C\}$$

is sublinear.

Proof. $x, y \in V$. Let $\epsilon > 0$, choose λ, μ s.t. $x \in \lambda C, y \in \mu C$.

$$q_C(x) \leq \lambda \leq (1+\epsilon)\, q_C(x)$$
$$q_C(y) \leq \mu \leq (1+\epsilon)\, q_C(y).$$

Then $z = \frac{\lambda}{\lambda+\mu}\frac{x}{\lambda} + \frac{\mu}{\lambda+\mu}\frac{y}{\mu} \in C$. Therefore $x + y = (\lambda+\mu)\left(\frac{x}{\lambda+\mu} + \frac{y}{\lambda+\mu}\right)$. So

$$q_C(x+y) \;\leq\; \lambda + \mu \;\leq\; (1+\epsilon)\,(q_C(x) + q_C(y)).$$

Send $\epsilon \to 0$. $\square$

Corollary 5.2. *Let C, D be nonempty convex sets $C \cap D = \varnothing$. There there exists $f : V \to \mathbb{R}$ s.t. $f(x) \leq f(y)$ for all $x \in C, y \in D$.*

Proof. Take $x_0 \in C, y_0 \in D$. trick Shifting trick: let

$$B := C - D - (x_0 - y_0),$$

where $C - D := \{x - y \mid x \in C, y \in D\}$. Since $x - y \neq 0$, $y_0 - x_0 \notin B$. Let $Y = \mathbb{R}(y_0 - x_0)$.

Claim 5.3. $q_B(x_0 - y_0) \geq 1$.

Define $f(t(y_0 - x_0)) = t$, then $f \leq q_B$. Hahn-Banach extension gives $F : V \to \mathbb{R}$, with $F \leq q$ and $F(y_0 - x_0) = 1$. Note that $q_B(x - y - (x_0 - y_0)) \leq 1$ implies

$$\begin{aligned}
&F(x - y - (x_0 - y_0)) \leq 1 \\
\implies &F(x - y) - F(x_0 - y_0) \leq 1 \\
&\qquad\qquad F(x) \leq F(y) + 1 - F(y_0 - x_0) = F(y)
\end{aligned}$$

□

Theorem 5.4. *For X a normed space and $q(x) = \|x\|$, X subset of complex vector space, $\forall x$ with unit norm, $\exists$ a complex linear functional $f \leq \|\cdot\|$ with $|f(x)| = 1$.*

Proof. Consider X as a real normed space. Take x_0 in X and let $Y = \mathbb{R}x_0 + i\mathbb{R}x_0$, $\|x_0\|$. Define $f(zx_0) = \mathrm{Re}(z)$. Note that $f \leq q$ as

$$f(zx_0) = \mathrm{Re}(z) \leq |z| = \|zx_0\| \leq (zx_0).$$

Then $\exists F : X \to \mathbb{R}$ with $F(x) \leq \|x\|$ real linear and $F(x_0) = 1$.

Fabrication: want to define $G(x) = F(x) - iF(ix)$. If G is complex linear and $F = \mathrm{Re}\, G$, $G(x) = \mathrm{Re}\, G(x) + \mathrm{Im}\, G(x) = F(x) - \mathrm{Re}(iG(x))$.

Claim 5.5.

1. *$G(x) = F(x) - iF(ix)$ is complex linear*
2. *$|G(x)| \leq \|x\|$*

□

6 Hahn-Banach Theorem Cont. 20210205

Theorem 6.1 (Complex version Hahn-Banach)**.** *Let X be a complex vector space. If $f : Y \to \mathbb{C}$ is a complex linear functional on a complex linear subspace $Y \subset X$,* *and $q : X \to [0,\infty]$ a sublinear function and $q(zx) = q(x)$, $|z| = 1$ (semi-norm). If $|f| \le q$, then there exists $F : X \to \mathbb{C}$, such that $|F| \le q, F|_Y = f$*

Proof. Apply the real Hahn-Banach to $\tilde{f} = \operatorname{Re} f$. $\tilde{F} : X \to \mathbb{R}$. Define a new F by

$$\boxed{F(x) = \tilde{F}(x) - i\tilde{F}(ix).}$$

Check F is complex linear. $\square$

Hahn-Banach separation.

Lemma 6.2. *Let C be a convex set and q_C is a Minkowski functional*

1. *$x \in C$ then $q_C(x) \le 1$*
2. *$x \notin C$ then $q_C(x) \ge 1$.*

$$\{x \mid q_C(x) < 1\} \subset X \subset \{x \mid q_C(x) \le 1\}.$$

And the inclusions are strict.

Proof. $q_C(y) = \inf\{\lambda \mid \frac{y}{\lambda} \in C\}$. For part 1, $x \in C$ so $q_C(x) \le \lambda = 1$.

For part 2, assume $q_C(x) < 1$, then $\exists \lambda < 1$ such that $\frac{x}{\lambda} \in C$. This (together with convexity) implies

$$x = (1-\lambda) \cdot 0 + \lambda \cdot \frac{x}{\lambda} \in C,$$

contradiction. $\square$

C may or may not contain the boundary.

1. Topology
2. filter
3. continuous

Definition 6.3. A **filter** on a set X is a subset $\mathcal{F} \subset 2^X$ such that

1. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
2. If $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

It is **nontrivial** if $\forall A \in \mathcal{F}$, $A \neq \emptyset$.

Definition 6.4. A **neighbourhood filter** is a collection $(\mathcal{F}_X)_{x \in X}$ of filters.

Remark 6.5.

1. (Topology $\Rightarrow$ Filter)
   Given topology τ, $\mathcal{F}_X$ is generated by the non-empty open sets.
   $$\mathcal{F}_X = \{A \subset X \mid \exists O \text{ open }, x \in O \subset A\}.$$
   Neighbourhood filter.
2. (Filter $\Rightarrow$ Topology)
   Given a filter $\mathcal{F}_X$, define O is open iff $\forall x \in O$, $O \in \mathcal{F}_X$. intuition A topology can equivalently be defined by open sets or neighbourhood filters.

Lemma 6.6. $(\tau^{\mathcal{F}})^{\tau} = \tau$.

Definition 6.7. f is **continuous** at x if $\forall B \in \mathcal{F}_{f(x)}, f^{-1}(B) \in \mathcal{F}_X$.

Recall: If $f : X \to Y$ continuous and $K \subset X$ compact, then $f(K)$ compact

Definition 6.8. A space $(X, +, \cdot, \tau)$ is a **topological vector spaces** if

1. $(X, +, \cdot)$ is a vector space
2. $+ : X \times X \to X$ continuous
   $\cdot : \mathbb{K} \times X \to X$ continuous

Example 6.9. 1. $\mathbb{R}^2$ with the Chicago railway metric is not a topological vector space. $+$ not continuous.

2. Let (Ω, Σ, μ) be a measure space. Define
$$L_0(\Omega, \Sigma, \mu) = \{f : \Omega \to \mathbb{R} \mid f \text{ measurable}, \mu(|f| > \epsilon) \to 0 \text{ as } \epsilon \to \infty\}.$$
Define
$$d(f, 0) := \inf\{\epsilon \mid \mu(|f| > \epsilon) < \epsilon\},\ d(f, g) = d(f - g, 0).$$
This is a translation invariant metric. Hence a translation invariant topological vector space.

7 Vector space 20210208

1. Topological space
2. Topological vector space $(X, +, \cdot, \tau)$, in particular, the translation map $T_x : X \to X; y \mapsto T_x(y) = x + y$ is a homeomorphism
3. Application to Hahn-Banach
4. Tychonoff's theorem

Motivational lemma

Lemma 7.1. *Let X be a topological vector space, $f : X \to \mathbb{R}$ be a linear nonzero continuous map, then the image of an open convex set is open.*

Proof. If f is linear and O is convex then $f(O)$ is convex. Convex sets of $\mathbb{R}$ is intervals.

Assume $f(O) = (a, b]$ or $[a, b]$. That is there is a $x \in O$, $f(x) = \sup_{y \in O} f(y)$, then $f(x) = b$. Since $f(x_0) \neq 0$ with $f(x_0) = 1$, $(f \neq 0)$, we consider $x(t) = x + tx_0$. Then

O open implies there is a t_0, for all $|t| < t_0$, $x + tx_0 \in O$ (translation is continuous). But now

$$f(x + tx_0) = f(x) + tf(x_0) = b + t \cdot 1 > b.$$

Contradiction. $\square$

later Extension is continuous.

Theorem 7.2 (Tychonoff). *For each $j \in J$, let X_j be a topological space. If each X_j is compact, then $X = \prod_{j \in J} X_j$ is compact in the product topology.*

Clarification: $x = (x_i)_{i \in I}$, O is a neighborhood of x if there are i_j, O_j such that $O = \{(y_i) \mid y_{i_j} \in O_{i_j}\}$.

Example 7.3. Let X_i be a metric space, the index set $I = \mathbb{N}$. Now the following defines a distance of the product topology

$$d((x_n),(y_n)) = \sum_{n \geq 0} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)}.$$

$$\{(y_n) \mid d((x_n),(y_n)) < \epsilon\} \quad \supset \quad \{(y_n) \mid \mathrm{dist}(x_j, y_j) < \frac{\epsilon}{2}, j = 1, \cdots n\}.$$

Proof. Assume $d(x_j, y_j) \leq \frac{\epsilon}{2}$ for all j. Then

$$\begin{aligned} d((x_n),(y_n)) &= \sum_{n \geq 0} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} \\ &\leq \sum_{n=1}^{m} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} + \sum_{n > m} 2^{-m} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon \end{aligned}$$

(choose large m so that the second term is less than $\frac{\epsilon}{2}$). *Any continuity condition only depends on finitely many terms* $\square$

1. non-trivial filter, filter converges to x, ultra filter

intuition Filter is the analogue of sequence converging to something. They want to being small.

Definition 7.4. We say that **a filter $\mathcal{F}$ converges to** x if $\mathcal{F} \supset \mathcal{N}_x$.

Every neighbourhood is contained in the filter.

Definition 7.5. A maximal non-trivial filter is called a **ultra filter**.

Remark 7.6. Let $\mathcal{U}$ be an ultra filter then for every $A \subset X$, either $A \in \mathcal{U}$ or $A^C \in \mathcal{U}$.

Proof. Fix $A \subset X$.

Case 1 $A \in \mathcal{U}$ done.

Case 2 $A \notin \mathcal{U}$ then $A^C \in \mathcal{U}$. (Prove by contradiction, assume $A^C \notin \mathcal{U}$) Define $\tilde{\mathcal{U}}$ to be the smallest filter which contains A^C and elements in $\mathcal{U}$. (Show $\tilde{\mathcal{U}}$ is again a filter). Indeed this new filter $\tilde{\mathcal{U}}$ is closed by superset. Need to show if $\tilde{A}, \tilde{B} \in \tilde{\mathcal{U}}$ implies $\tilde{A} \cap \tilde{B} \in \tilde{\mathcal{U}}$.

- $\tilde{A}, \tilde{B} \in \mathcal{U}$ done.
- $\tilde{A}, \tilde{B} \supset A^C$ done.
- $\tilde{A} \in \mathcal{U}, \tilde{B} \supset A^C$. We know $\tilde{B} \supset A^C$ implies $\tilde{B}^C \subset A$, and we know $\tilde{A} \neq A$, so $\tilde{A} \cap \tilde{B} = \emptyset$.

Then $\tilde{\mathcal{U}}$ is a filter, contradicting to the fact $\mathcal{U}$ is an ultra filter.

$\square$

Corollary 7.7. *Every ultra filter on an interval converges.*

Lemma 7.8. *(X, τ) is compact iff every ultra filter converges.*

Proof. Ref.

($\Rightarrow$) Let (X, τ) be compact and $\mathcal{U}$ be an ultra filter. Assume $\mathcal{U}$ does not converge to any point. Then $\forall x \in X$, $\mathcal{N}_x \not\subset \mathcal{U}$. Then every point has a neighbourhood O_x which is not in $\mathcal{U}$.

Take the open cover $\cup_x O_x$ of X, O_x as above. By compactness, there is a finite subcover $O_{x_1} \cup \cdots \cup O_{x_n}$. Since $\mathcal{U}$ is an ultra filter, $O_{x_i}^C \in \mathcal{U}$, and the finite intersection of $O_{x_i}^C$'s is in $\mathcal{U}$. But

$$\left(\bigcap_{i=1}^{n} O_{x_i}^C \right)^C = \bigcup_{i=1}^{n} O_{x_i} = X$$

implies $\cap_{i=1}^n O_{x_i}^C = \emptyset \in \mathcal{U}$, contradiction.

($\Leftarrow$) Let $X \subset \cup_x O_x$, O_x open. Assume that $X \not\subset \cup_{i=1}^n O_{x_i}$ for any finite subset of indices. Then $\cap_{i=1}^n O_{x_i}^C \neq \emptyset$. Define

$$\mathcal{F} = \{A \mid \exists i_1, \cdots, i_n \text{ s.t. } \bigcap_{i=1}^{n} O_{x_i}^C \subset A\}.$$

This is a filter, let $\mathcal{U}$ be the ultra filter contains $\mathcal{F}$. Then $\mathcal{U}$ converges, say to some $x_0 \in X$, then $\mathcal{N}_{x_0} \subset \mathcal{U}$. Then there is a neighbourhood of x_0 which is contained in $\mathcal{U}$, and then $O_x^C \in \mathcal{F} \subset \mathcal{U}$. But $O_x \cap O_x^C = \emptyset$, contradiction. $\square$

proof of Theorem 7.2. Ref.

Let $X = (\prod_i X_i, \tau_i)$, $\mathcal{F}$ be an ultra filter. Let $\pi_i : X \to X_i$ be the projection to the i-th term. Note that $\pi_i(\mathcal{F})$ is also an ultra filter, so it converges to some $x_i \in X_i$. Then $\mathcal{F}$ converges to $(x_i)_{i \in I}$.

Claim 7.9. *Let $x = (x_i)_{i \in I}$, if $O \in \mathcal{N}_x$ then $O \in \mathcal{U}$.*

This means $O \supset O_{i_1} \times \cdots \times O_{i_n} \times X_{j_1} \times X_{j_1} \times \cdots$. Now $\pi_{i_k}^{-1}(O_{i_k}) = W_k$ open and belongs to $\mathcal{U}$, as $O_{i_k} \in \mathcal{U}$. Hence, the finite intersection of W_k's is in $\mathcal{U}$. Then $O \in \mathcal{U}$. $\square$

8 Locally Convex Topological Vector Spaces (LCTVS) 20210210

Recall

1. Topological vector spaces $(X, +, \cdot)$
2. Tychonoff theorem
3. intuition An ultra filter is a generalisation of sequence converging to a point.

Definition 8.1. A topological vector space is called **locally convex** if $\forall x$, $\forall O \in \mathcal{N}_x$, $\exists W$ convex such that $x \in W \subset O$.

Example 8.2.

1. Let X is a normed space, $\mathcal{N}_x = \{O \mid \exists x > 0, \operatorname{int}(B_r) + x \subset O\}$.
2. Let $X = C^\infty(\mathbb{R})$, K a compact subset, with semi-norm $\|f\|_{K,n} = \sup_{x \in K} \sup_{1 \le i \le n} |f^{(i)}(x)|$. (This is a semi-norm because $\operatorname{supp} f$ can be in K^C) The resulting topology is locally convex.

Example 8.3 (Non-examples)**.**

1. $L_0 = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ measurable}\}$, with
$$d(f, 0) = \inf\{\epsilon \mid \mu(|f| > \epsilon) < \epsilon\}.$$
2. $\|f\|_p = (E|f|^p)^{l/p}$ with $0 < p < 1$. $B_p = \{f \mid \|f\|_p < 1\}$. (Cannot put in a convex set if it is infinite dimension). The first example is when $p \to 0$. (E is expectation?)

Theorem 8.4. *Let (X, τ) be a topological space, the following are equivalent.*

1. *X is a Locally convex topological vector spaces (LCTVS)*
2. *$\exists (q_i)_{i \in I}$ of semi-norms on X*
3. *$O \ni N_0$ iff $\exists i$, $\exists r$ s.t. $\{x \mid q_i(x) < r\} \subset O$.*

intuition The topology is determined by many different shaped balls. Open iff contain one of the balls.

Proof of Theorem 8.4

($\Leftarrow$) Take a point $x \in X$ and O is an open neighbourhood of x. Define a translation map $T_{-x} : X \to X$, by $T_{-x}(y) = y - x$. Note that T_{-x} is a homeomorphism, so $T_{-x}(O) =: W$ is an open neighbourhood of 0. By (iii), $\exists i$ s.t. $\{y \mid q_i(y) < 1\} \subset W$. Define $V = x + \tilde{W} = \{\tilde{y} \mid q_i(\tilde{y} - x < 1)\} \subset O$.

Definition 8.5. A set $W \ni 0$ is called **absolutely convex** if

$$\sum_{j=1}^{n} |\lambda_j| \leq 1 \implies \sum_{j=1}^{n} \lambda_j x_j \in W.$$

Definition 8.6. A set W is **balanced** if $|z| = 1$, $zw = w$ for all $w \in W$.

Remark 8.7. W is absolutely convex if W is convex and balanced.

($\Rightarrow$) Prove existance of seminorms. Take $\mathbb{K} = \mathbb{R}$ let O be open and $\exists W \subset O$ containing 0 and convex. Since $- : X \to X$; $-x \mapsto x$ is continuous, we know $(-)^{-1}(W) \supset V$ is convex, $V \in \mathcal{N}_x$ (Aside: $W \cap -W$ is convex and balanced).

Define $q_V(x) = \inf\{\lambda \mid \frac{x}{\lambda} \in V\}$

Lemma 8.8. *q_V is a semi-norm.*
That is, $q_V(\lambda x) = |\lambda| q_V(x)$ and subadditive $q_V(x+y) \leq q_V(x) + q_V(y)$.

Then

$$\frac{1}{4} \subset \{y \mid q_V(y) < \frac{1}{2}\}(\text{ball of some semi-norm}) \subset V.$$

For every neighbourhood of 0 can choose a semi-norm

For $\mathbb{K} = \mathbb{C}$. Want for any set O, find a W which is convex and contained in $\cap_{|z|=1} zO$ (in a intersection of rotations). $(\cap_{|z|=1} zO)^C = \cup_{|z|=1}(zO)^C$.

Question: Is $B = \cup_{|z|=1}(zO)^C$ closed? – Yes. Let $T = \{z \mid |z| = 1\}$. The map $T \times X \to X$; $(z, x) \mapsto zx$ is continuous and T is compact.

Lemma 8.9. *B is closed. (A compact translation of a closed set is closed.)*

Proof. Let A be an ordered index set, $x_\alpha \in B$, $x_\alpha \to x$ meaning for a neighbourhood O of x, $\exists \alpha_0$, $\forall \alpha > \alpha_0$, $x \in O$. $\square$

Then $0 \notin B$, and $\exists W \subset \cap_{|z|=1} zO)^C$ convex and $\cap_{|z|=1} zw$ is balanced convex set.

9 Hahn-Banach Separation Theorem 20210212

Lemma 9.1. *Let X, Y be locally convex topological vector spaces. A linear map $T : X \to Y$ is continuous if and only if T is continuous at 0.*

Propersition 9.2. *Let X be a locally convex topological space and $f : X \to \mathbb{R}$ be a linear and continuous map. Let W be an open convex neighbourhood of 0. Then either $f(W) = \{0\}$ or $f(W)$ is open.*

Theorem 9.3 (Hahn-Banach Separation Theorem)**.** *Let C be a* ***convex nonempty*** *subset in a topological space X and* $\boldsymbol{x \notin C}$*, then*

1. *there exists a linear map $f : X \to \mathbb{R}$ such that $f(y) \leq f(x)$, $\forall y \in C$,*
2. *if in addition X is a locally convex topological vector space and C is open, then f is continuous, nontrivial and $f(y) < f(x)$, $\forall y \in C$.*

Proof. (1) Let $x_0 \in C$, then $\tilde{C} = C - \{x_0\}$ contains 0, by Lemma 5.1, the Minkowski functional $q_{\tilde{C}} = \inf\{\lambda \mid y \in \lambda\tilde{C}\}$ is sublinear. Let $V = \mathbb{R}(x - x_0)$ and define $f(t(x - x_0)) = t$, which is linear. Then $x - x_0 \notin C - \{x_0\}$. By Lemma 6.2, $y \in C$ implies $q_{\tilde{C}}(y - x_0) \leq 1$. Therefore

$$f(y - x_0) \leq f(x - x_0) = 1 \implies f(y) \leq f(x).$$

(2) Now if C is open then $\tilde{C} = C - \{x_0\}$ is open (here we only require a topological space, we don't actually need locally convexity). Consider $g : X \times X \to X$, $g(x, y) = x - y$. This map is continuous, $0 \in \tilde{C}$.

There exists V_1, V_2 neighbourhoods of 0, such that $V_1 - V_2 \subset \tilde{C}$. Define $V = V_1 \cap V_2$ (V is a neighbourhood of 0). Then $0 \in V - V \subset \tilde{C}$. By previous part $f|_{\tilde{C}} \leq 1$. Check the following later Hence

$$f(V - V) \subset f(\tilde{C}) \subset \{y \mid f(y) \leq r\}.$$

Then for all $y = a - b \in V - V$, $f(y) \leq 1$ and $-y = b - a \in V - V$ so $f(-y) \leq 1$. This means f is bounded. Hence f is continuous at 0. By previous Lemma, f is continuous and $f(\tilde{C})$ is open (image of open convex set is open). Then $f(y - x_0) < 1$ for all $y \in C$. □

Theorem 9.4. *Let C, D be* ***nonempty convex*** *sets. If $C \cap D = \emptyset$, then there is a linear functional f on X such that $f(x) < f(y)$, for all $x \in C, y \in D$.*

Proof. trick Consider $\tilde{C} = C - D = \{x - y \mid x \in C, y \in D\}$. Note that $\tilde{C}$ is open if either C or D is open, and $0 \notin \tilde{C}$. Now shift the set, i.e. let $\tilde{D} = \tilde{C} - \{(x_0 - y_0)\}$. Apply previous theorem $0 \notin \tilde{C}$, so there exists a $f \neq 0$ and continuous, $f(z) < f(0)$, for all $z \in \tilde{C} = C - D$. Say $z = x - y$, for $x \in C$ and $y \in D$. Then $f(x) < f(y)$. □

Theorem 9.5. *Let C be a* ***closed convex*** *set and D be a* ***compact convex*** *set in a locally convex topological vector space. Then there exists a continuous nontrivial f and $r < s$ such that $f(x) < r < s < f(y)$ for all $x \in D$ and $y \in C$.*

Proof. Assume C is closed and D is compact. C^C is open, $D \cap C = \emptyset$. For any $x \in D$ there is a W_x convex such that $(x + W_x) \cap C = \emptyset$.

Consider the open sets $x + \frac{W_x}{2}$, their union $\cup(x + \frac{W_x}{2})$ gives an open cover of D. Then there is a finite subcover $D \subset \cup_i (x_i + \frac{W_{x_i}}{2})$. Take $W = \cap_i \frac{W_{x_i}}{2}$, and let $y = d + w \in D + W$. Then there exists an x_j such that $d = x_j + \frac{W_{x_j}}{2}$. Therefore,

$$y = d + w \in x_j + \frac{W_{x_j}}{2} + W \subset x_j + \frac{W_{x_j}}{2} + \frac{W_{x_j}}{2} \not\subset C.$$

(Convexity implies $\frac{W_{x_j}}{2} + \frac{W_{x_j}}{2} \subset W_{x_j}$.) analogue Triangle inequality on metric spaces.

Hence we have a strict separation between $D + W$ and C, and we can find a nontrivial continuous f such that $f(x) < f(d + w)$, where $x \in C$, $d \in D$ and $w \in W$. Note that $f(D)$ is compact as D is compact, so $f(D)$ is a closed interval $[a, b]$. Then

$$f(D + W) = f(D) + f(W) = [a, b] + (-\alpha, \beta)$$

($f(W)$ is a neighbourhood of 0 check). So for all $x \in C$,

$$f(x) \leq a - \alpha < a \leq \inf\{f(y) \mid y \in D\}.$$

□

Example 9.6.

1. Let X be a normed space, and $C = \{x \mid \|x\| \leq 1\} = \bar{B_X}$. Take x_0 such that $\|x_0\| > 1$, then $D = \{x_0\}$ compact. There exists f such that $f(x) \leq 1$, $\|f\| \leq 1$ and $f(x_0) > 1$.
2. Take a ball B_X and a triangle D.

Next, we want to make the separation line unique.

10 Weak Topology 20210215

Definition 10.1. Let X be a Banach space and $Y \subset X^*$ a subspace. Then $\boldsymbol{\sigma(X, Y)}$**-topology** is the coarsest topology making all the functional $y \in Y$ continuous. This means the semi-norms defining this topology are given by

$$q_{y_1,\cdots,y_n}(x) = \max_{i=1,\cdots,n} |y_i(x)|.$$

Every locally convex space is given by semi-norms. Semi-norms are indexed by finite subsets of Y.

Theorem 10.2. *The dual space of* $(X, \sigma(X, Y))$ *is* Y *(as a set). That is,*

$$\boxed{(X, \sigma(X, Y))^* = Y.}$$

Note the two spaces only equal as a set, not necessarily as a topological space. Because Y on the LHS can be taken as a algebraic dual without topological assumptions, whereas Y on the RHS is a topological vector space (may with its own norm).

Remark 10.3. Let X be a locally convex topological vector space and Y a Banach space or locally convex topological vector space, then $L(X, Y)$ is also a locally convex topological vector space.

Proof. Step 1: $Y \subset (X, \sigma(X, Y))^*$.

Claim 10.4. *For every $y \in Y$, $f_y(x) = y(x)$ is continuous with respect to the new topology.*

It suffice to show f is continuous at 0: $\forall \epsilon$, $\exists O \in \sigma(X, Y)$ containing 0, such that if $x \in O$, then $|f(x)| < \epsilon$. $(f(0) = 0)$. In this new topology open neighbourhood means there exists a semi-norm in system such that $O \supset \{x \mid q(x) < \delta\}$, i.e there exists some $B_q(\delta) \subset O$. This is equivalent to say $|f(x)| \leq C \cdot q(x)$, for some semi-norm q. compare In Banach space we don't have a choice of the norm, so we require $|f(x)| \leq C \cdot \|x\|$.

In our case, the semi-norm $q_y(x) = |y(x)|$ does the job, because $|f_y(x)| = |y(x)| = q_y(x)$. More generally, the semi-norm is given by $q_y(x) = \max_j |y_j(x)|$.

Step 2: $(X, \sigma(X, Y))^* \subset Y$.

Let $f : X \to \mathbb{K}$ be continuous. By definition there exists a q such that $|f(x)| \leq q(x)$ and $q(x) = \max_j |y_j(x)|$. Fix $y_1, \cdots, y_n$ and define a map

$$\begin{aligned} \phi : X &\longrightarrow \mathbb{K}^n \\ x &\longmapsto (y_1(x), \cdots, y_n(x)). \end{aligned}$$

Then $\phi(X) \subset \mathbb{K}^n$ is a subspace. Denote $Z = \phi(X)$, then $z = (y_1(x), \cdots, y_n(x))$. Consider the map

$$\begin{aligned} \psi : Z &\longrightarrow \mathbb{K} \\ z &\longmapsto f(x). \end{aligned}$$

This map is well-defined, linear, and $|\psi(z)| \leq \max_j |z_j| = \|z\|_\infty$. By Hahn-Banach, there exists $\tilde{\psi} : l_\infty^m \to \mathbb{K}$, such that $\tilde{\psi}|_Z(z) = \psi(z)$ and $\|\tilde{\psi}\| = \|\psi\| \leq \|z\|_\infty$. Note that $\tilde{\psi}(z) \in (l_\infty^m)^* = l_1^m$. This means there exists $\alpha_1, \cdots, \alpha_n$ such that $\tilde{\psi}(z) = \sum_j \alpha_j z_j$. This means

$$f(x) = \psi(\phi(x)) = \tilde{\psi}(\phi(x)) = \sum_j \alpha_j \phi_j(x) = f_y(x),$$

where $y = \sum_j \alpha_j y_j$. $\square$

Example 10.5. Let X be a space and take $Y = X^*$. Then

- $\sigma(X, X^*)$ is called the **weak topology** of X and $(X, \sigma(X, X^*)) = X^*$,
- $\sigma(X^*, X)$ is called the **weak* topology** of X^* and $(X^*, \sigma(X^*, X)) = X$.

11 Weak Topology cont. 20210219

Theorem 11.1 (Goldstine). *Let X be a Banach space, then the image of the closed unit ball $B_X \subset X$ under the canonical embedding ι into the closed unit ball $B_{X^{**}}$ of the bidual space X^{**} is weak*-dense.*

$$\overline{B_X}^{\sigma(X^{**}, X^*)} = B_{X^{**}}$$

intuition The unit ball with weak*-topology is compact. In finite dimension, close + bounded = compact. Generalisations of finite dimension.

Proof. Recall that X^{**} is a locally convex topological vector space with respect to $\sigma(X^{**}, X^*)$-topology. This topology is given by the semi-norm $q(x^{**}) = \max_j |x^{**}(x_j^*)|$, with $x_1^*, \cdots, x_j^* \in X^*$.

The canonical embedding $\iota : X \to X^{**}$, is an isometry (Hahn-Banach Theorem) and $\iota|_{B_X} : B_X \to B_{X^{**}}$. We want to show the closure $\overline{\iota(B_X)}$ with respect to the $\sigma(X^{**}, X^*)$ topology satisfies $\overline{\iota(B_X)} = B_X^{**}$. Prove by contradiction.

Assume that $x^{**} \notin \overline{\iota(B_X)}$, with $\|x^{**}\|_{X^{**}} \leq 1$. Note that $\overline{\iota(B_X)}$ is closed, compact and convex. By Hahn-Banach separation (Theorem 9.5), there exists a nontrivial continuous map $f : X^{**} \to \mathbb{R}$ so that $|f(\iota(x))| \leq 1 < s < |f(x^{**})|$ for all $x \in B_X$. On one hand we have

$$\|f\|_{X^{**}} = \sup_{\|x\| \leq 1} |f(x^{**})| = \sup_{\|x\| \leq 1} |f(\iota(x))| \leq 1.$$

Then by definition,

$$|x^{**}(f)| \leq \|x^{**}\|_{X^{**}} \cdot \|f\|_{X^{**}} \leq 1.$$

On the other hand we have $|x^{**}(f)| = f(x^{**}) > 1$. Contradiction. $\square$

Example 11.2. Let $X = C_0 = \{(x_n) \mid \lim_n x_n = 0\}$, with $\|(x_n)\| = \sup_n \|x_n\|$. Then $X^* = l_1$ because

$$\begin{aligned}\|y_n\|_1 = \sum_n y_n &= \sup_k \sum_{i=1}^{k} |y_k| \\ &= \sup_k < y, \epsilon_1, \cdots, \epsilon_k, 0, \cdots, 0 > .\end{aligned}$$

where $\epsilon_i = \mathrm{sgn}(y_i)$ and $< y, z >= \sum y_n z_n$. And $X^{**} = l_\infty = \{(x_n) \mid \sup_n |x_n| < \infty\}$.

What is $\sigma(l_\infty, l_1)$-topology? The answer is pointwise convergence on bounded set. Consider bounded sequences x^α ($\|x^\alpha\| \leq C$). Then $x^\alpha \to x \in l_\infty$ iff for all $y \in l_1$, $x^\alpha(y) \to x(y)$.

For bounded sets $\|x^\alpha\| \leq 1$, $\forall \alpha$,

$$x^\alpha \to x \quad \Longleftrightarrow \quad x_n^\alpha \to x_n, \forall n.$$

($\Rightarrow$) Take $y_n = (0, \cdots, 1, \cdots, 0) \in l_1$.

($\Leftarrow$) Let $y \in l_1$ and $\epsilon > 0$ then there exists n_0 such that $\sum\limits_{n > n_0} |y_n| < \frac{\epsilon}{2}$. There exists α_0 such that any $\alpha > \alpha_0$, $|x_n^\alpha - x_n| < \frac{\epsilon}{2}$ for all $n > n_0$. We need

$$|x^\alpha(y) - x(y)| \leq < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Let $y^N = (y_1, \cdots, y_N, 0, \cdots, 0)$, $y^N \to y$ in $\sigma(l_\infty, l_1)$ because we can use pointwise convergence.

12 Alaoglu's Theorem 20210222

Alaoglu's Theorem says that the closed unit ball in X^* is compact in the weak*-topology.

Theorem 12.1 (Alaoglu)**.** *Given a topological vector space X, and let $B_{X^*} = \{x^* \in X^* \mid \|x^*\| \leq 1\}$ be the closed unit ball in X^*. Then B_{X^*} is compact in X^* with respect to the weak*- topology on X^*.*

Proof. Ref. or see Conway p.134

Let the set $D_x = \{z \in \mathbb{K} \mid |z| \leq 1\}$. Consider the product $D := \prod_{x \in B_x} D_x$. Since D_x is compact in $\mathbb{K}$, Tychonov's theorem says that D compact in the product topology. Elements in D are functionals, given by $\mu \in K$, $\mu(x) = \mu_x \in D \subset \mathbb{C}$, although they need not to be linear.

The inclusion

$$\iota : B_{X^*} \subset \prod_{x \in B_x} D =: K$$

is given by

$$\iota(x^*)(x) = x^*(x).$$

Note that $\iota(B_{X^*}) \subset K$. Indeed, if $\|x\| \leq 1$ and $\|x^*\| \leq 1$, then $|x^*(x)| \leq 1 \in D$.

Claim 12.2. *$\iota(B_{X^*})$ is closed. Hence, $\iota(B_{X^*})$ is a compact subspace of K.*

Proof of the claim. Take a net (x^*_α) in B_{X^*} which converges to $f \in D$ pointwisely. So $f(x) = \lim_{\alpha \to \infty} x^*_\alpha(x)$. In particular $|f(x)| \leq 1$ for all $\|x\| \leq 1$. (Need to show f is in the range. We can not take $\mathbb{N}$ as index set, instead replacing $\mathbb{N}$ by a partially ordered set. Usually the index set is given by the neighbourhood basis of f. Let $O_i \in \mathcal{N}_f$, $i = 1, 2$, then $O_1 \cap O_2 \in \mathcal{N}_f$ and $O_1 \cap O_2 \geq O_i$.)

For $x \in X$, define $F(x) = \beta^{-1} f(\beta x)$ for some β such that $\|\beta x\| \leq 1$ (check this is well defined). Then F agrees with f on B_X. We claim that F is linear. Take $x_i \in X$, $i = 1, 2$. Consider $y = \frac{x_1 + x_2}{\|x_1\| + \|x_2\|}$. If we take $\lambda = \frac{\|x_1\|}{\|x_1\| + \|x_2\|}$, then by convexity

$y = \lambda \frac{x_1}{\|x_1\|} + (1-\lambda)\frac{x_2}{\|x_2\|} \in B_X$. Then

$$\begin{aligned} f(y) &= \lim_\alpha x_\alpha^*(y) = \lim_\alpha x_\alpha^* \left(\frac{x_1}{\|x_1\| + \|x_2\|} \right) + x_\alpha^* \left(\frac{x_2}{\|x_1\| + \|x_2\|} \right) \\ &= f\left(\frac{x_1}{\|x_1\| + \|x_2\|} \right) + f\left(\frac{x_2}{\|x_1\| + \|x_2\|} \right). \end{aligned}$$

So

$$\begin{aligned} F(x_1 + x_2) &= f(y) \cdot (\|x_1\| + \|x_2\|) \\ &= \left(f\left(\frac{x_1}{\|x_1\| + \|x_2\|} \right) + f\left(\frac{x_2}{\|x_1\| + \|x_2\|} \right) \right) \cdot (\|x_1\| + \|x_2\|) \\ &= F(x_1) + F(x_2). \end{aligned}$$

We have a linear functional $F \in X^*$ satisfying $|F(x)| \leq 1$ when $\|x\| \leq 1$. This means $\|F\|_{X^*} \leq 1$. So $F \in B_{X^*}$ $\square$

Definition 12.3. A Banach space is **reflexive** if $X^{**} = X$.

Goal: to show X is reflexive iff X^* is reflexive.

Propersition 12.4. *A closed subspace of a reflexive Banach space is reflexive.*

Proof. The following diagram is commutative. (Check)

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^{**} \\ {\scriptstyle j}\big\uparrow & & \big\uparrow{\scriptstyle j^{**}} \\ Y & \xrightarrow{\iota_Y} & Y^{**} \end{array}$$

Step 1. $Y^{**} = Y$. Take an element $y^{**} \in Y^{**}$, note that

$$j^{**}(y^{**})(x^*) = y^{**} \circ j^*(x^*) = y^{**}(x^* \circ j) = x^*|_Y \in Y^*.$$

So we can apply y^{**} to this element, and define $\phi(x^*) = y^{**}(x^*|_Y)$

Lemma 12.5. *If $T : Y \to X$ is isometric, then $T^{**} : Y^{**} \to X^{**}$ is also isometric.*

The above lemma says Y^{**} embeds isometrically into X^{**} (we will prove this later). If in addition, $X^{**} = X$, we deduce that for every y^{**} there exists an $x \in X$ such that

$$y^{**}(x^*|_Y) = x^*(x).$$

We want to show $x \in Y$. We claim that $y^{**} \in Y$, otherwise by Hahn-Banach separation there exists x^* such that $x^*(y^{**}) = 1$ and $x^*|_Y = 0$. The last equation says $x^*(x) = y^{**}(x^*|_Y) = x^*|_Y = 0$. A contradiction (as $y^{**} \in Y^{**} \subset X^{**} = X$). $\square$

Lemma 12.6. *If $T : X \to Y$ is isometric then $T^* : Y^* \to X^*$ sends closed unit ball to closed unit ball.*

Proof. Note that $T^*(B_{Y^*}) \subset B_{X^*}$. Indeed,

$$\begin{aligned} \|T^*\| &= \sup_{\|y^*\| \le 1} \|T^*(y^*)\| = \sup_{\|y^*\| \le 1} \|y^* \circ T\| \\ &= \sup_{\|y^*\| \le 1, |x| \le 1} |y^* \circ T(x)| = \sup_{\|y^*\| \le 1, |x| \le 1} |y^*(x)| \le 1. \end{aligned}$$

So $|T^*(y^*)| \le \|T^*\| \|y^*\| \le 1$.

To show T^* is onto, take $x^* \in B_{X^*}$. Can define $f(Tx) = x^*(x)$, $\|f\| \le 1$. By Hahn-Banach there exists y^* such that $y^*(Tx) = f(Tx) = x^*(x)$. $T^*(y) = x^*$. $\square$

Lemma 12.7. *If $T : Y \to X$ is a surjection, then $T^* : X^* \to Y^*$ is an isometry.*

Proof of the Lemma 12.5. The previous two lemma gives the result. $\square$

13 Reflexive Spaces 20210224

Theorem 13.1. *X is reflexive $\iff$ X^* is reflexive.*

Proof. ($\Rightarrow$) Assume that $X = X^{**}$. Then B_{X^*} is closed in $\sigma(X^*, X^{**}) = \sigma(X^*, X)$. Take an element x^{***} in $B_{X^{***}}$, there exists a sequence $x^*_\alpha \to x^{***}$ in $\sigma(X^{***}, X^{**})$ topology. Since B_{x^*} is closed in $\sigma(X^*, X)$, there is an x^* such that $x^*_\alpha \to x^*$. This means $x^{***} = x^*$.

($\Leftarrow$) If X^* is reflexive then X^{**} is reflexive, but $X \subset X^{**}$ as a closed subspace. $\square$

Remark 13.2. X is reflexive iff B_{X^*} is $\sigma(X^*, X^{**})$ closed.

Definition 13.3. A Banach space is called **uniformly convex**, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| > \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Lemma 13.4. *Take (x_n) a sequence with*

$$\limsup_n \|x_n\| \leq 1 \quad \textit{and} \quad \lim_n \inf_{m>n} \left\| \frac{x_n + x_m}{2} \right\| = 1.$$

Then (x_n) is Cauchy.

Proof. Let $\epsilon > 0$. Since $\limsup_n \|x_n\| \leq 1$, we can choose $\epsilon_0 > 0$, $\exists n_0$ such that $\|x_n\| \leq 1 + \epsilon_0$, for all $n > n_0$. So $\|\frac{x_n}{1+\epsilon_0}\| \leq 1$, for all $n > n_0$. Then

$$\left\| \frac{x_n + x_m}{2(1+\epsilon_0)} \right\| = \left\| \frac{x_n + x_m}{2} \right\| \cdot \frac{1}{1+\epsilon_0} \geq \frac{1}{(1+\epsilon_0)^2},$$

for all $n > n_0$.

Taking $\frac{1}{(1+\epsilon_0)^2} = 1 - \delta$. Using uniform convexity (contrapositive), we have $\forall n$, $\exists m$

$$\left\| \frac{x_n - x_m}{2(1+\epsilon_0)} \right\| < \epsilon.$$

Conclusion: Above shows $\forall \epsilon$, $\exists n_0$, $\forall n > n_0$, $\exists m$, such that $\|x_n - x_m\| < 2\epsilon(1 + \epsilon_0)$.

We use this for $\epsilon = 2^{-k}$, then there exists a converging subsequence x_{n_k} such that $\|x_{n_k} - x_{n_{k+1}}\| \leq 2^{-k}$. $\square$

Theorem 13.5 (Milman-Pettis)**.** *Uniformly convex Banach spaces are reflexive.*

Proof. See: Ref.

Let $x^{**} \in B_{X^{**}}$, $\|x^{**}\| = 1$. Then by definition of $\|x^{**}\|$, for all n, there exists $x_n^* \in B_{X^*}$, such that $x^{**}(x_n^*) \geq 1 - \frac{1}{n}$. Since $B_X \subset B_{X^{**}}$ is dense in $\sigma(X^{**}, X^*)$. Let $q_n(y) = |x_n^*(y)|$. There exists (x_k) in B_X such that

$$|q_n(x^{**} - x_k)| = |x_n^*(x_k) - x^{**}(x_n^*)| \leq \frac{1}{2k}, \text{ for } n = 1, \cdots, k.$$

In particular, apply the above to $n = k$, then

$$|x_k^*(x_k) - x^{**}(x_k^*)| \leq \frac{1}{2k} \implies -\frac{1}{2k} + x^{**}(x_k^*) \leq x_k^*(x_k).$$

Recall $x^{**}(x_k^*) \geq 1 - \frac{1}{k}$. So $1 - \frac{3}{2k} \leq x_n^*(x_k) \leq 1$ (RHS because x_n^* is in the unit ball).

Then take $m > k$, we have

$$2 - \frac{6}{2k} \leq 1 - \frac{3}{2k} + 1 - \frac{3}{2m} \leq x_k^*(x_k) + x_m^*(x_m) \leq x_k^*(x_k + x_m) \leq \|x_k + x_m\| \leq 2. \quad (1)$$

Taking $\liminf$ on both sides we get $\liminf \|\frac{x_k + x_m}{2}\| = 1$, and $\limsup \|x_k\| \leq 1$. By the above lemma (x_n) is Cauchy.

Remark 13.6. Assume there are two sequences x_n, $\tilde{x}_n$ satisfies the property (1), then then $\lim x_n = \lim \tilde{x}_n$.

Now if (y_n^*) is another family using the above construction, then there exists $(\tilde{x}_n)$ in B_X such that

$$|y_n^*(\tilde{x}_k) - x^{**}(x_n^*)| \leq \frac{1}{2k}.$$

Then $x^*(x_k) \to x^{**}(x)$ and $y^*(\tilde{x}_k) \to x^{**}(y)$ implies $x^{**} = \lim x_n = \lim \tilde{x}_n$ in $\sigma(X^{**}, X^*)$. $\square$

14 Reflexive Spaces cont. 20210226

Real analysis: $L_p(\Omega, \Sigma, \mu) = \{[f] \mid f : \Omega \to \mathbb{K}, f \text{ measurable}, \int |f|^p \, \mathrm{d}\mu < \infty\}$, where Ω is a set, Σ is a σ-algebra and μ is a σ-additive measure. Recall

- Simple functions $f = \sum_{j=1}^{n} \alpha_j 1_{E_j}$ are dense.

- $\|f\|_p = \sup\limits_{\|g\|_{p'} \le 1} |\int fg \, \mathrm{d}\mu|$.

  Use Hölder inequality, say $\|f\|_p = 1$, then $g = \mathrm{sgn}(f) \cdot |f|^{p/p'}$.

Corollary 14.1. *If $1 \le p \le \infty$, then $L_{p'}$ embeds isometrically into L_p^*,*

$$\begin{aligned} \iota_{p'} : L_{p'} &\to L_p^* \\ g &\mapsto \left(\iota_{p'}(g) : f \mapsto \iota_{p'}(g)(f) = \int fg \, \mathrm{d}\mu \right) \end{aligned}$$

and $\|f\|_p = \|\iota_{p'}(g) : L_p \to \mathbb{K}\|$.

Theorem 14.2. *Let $1 < p < \infty$ and assume L_p is reflexive. Then $L_{p'}^* = L_p$.*

(Here we check isometric isomorphism, there are two type of isomorphisms for Banach spaces, see more here)

Proof. Let $\varphi : L_{p'} \to \mathbb{K}$ with $\|\varphi\|_{L_{p'}^*} = 1$. Recall $L_{p'} \hookrightarrow L_p^*$ is an isometry. By Hahn-Banach extension, there exists a $\hat{\varphi} : L_p^* \to \mathbb{K}$, with $\hat{\varphi}|_{L_{p'}} = \varphi$.

$$\begin{tikzcd} L_{p'} \arrow[r, hook, "\iota_{p'}"] \arrow[d, "\varphi"'] & L_p^* \arrow[dl, dashed, "\exists\hat{\varphi}"] \\ \mathbb{K} & \end{tikzcd}$$

To show $\iota_{p'}$ is surjective, take $\eta \in L_p^*$. If we can find $g \in L_{p'}$ such that $\int fg \, \mathrm{d}\mu = \eta(f)$, then $\iota_{p'}(g) = \eta$ and we are done. Such a g exists by commutativity and reflexivity

$$\varphi(g) = \hat{\varphi}(\iota_{p'}(g)) = \iota_{p'}(g)(f) = \int fg \, \mathrm{d}\mu \quad \implies \quad \iota_{p'}(g)(f) = \eta(f).$$

□

Example 14.3 (Discrete case)**.** Let $\Omega = I$, $\Sigma = 2^I$, μ be the counting measure. If $I = \mathbb{N}$, then

$$L_p(\mathbb{N}, \Sigma, \mu) = \ell_p = \{(x_n) \mid \sum_n |x_n|^p < \infty\}.$$

What is the f defining the functional $\varphi : \ell(\mathbb{N}) \to \mathbb{K}$? Well, f is given by a sequence $(y_n) = ((0, 0, \cdots, \frac{1}{n}, \cdots, 0, 0))$. One can show that the

$$\|y_n\|_{p'} = \sup_n \left(\sum_{k=1}^n |y_k|^{p'} \right)^{\frac{1}{p'}} = \sup\{\varphi(x_n) \mid \|x_n\| \leq 1\}.$$

Prove using Hölder.

Theorem 14.4. *$\ell_p^* = \ell_{p'}$ for $1 < p < \infty$.*

Remark 14.5. For $I = \mathbb{N}$, let $c_0 = \{(x_n) \in \ell_\infty \mid \lim x_n = 0\}$. Then $c_0^* = \ell^1$, $c_0^{**} = \ell_1^* = \ell_\infty$.

Corollary 14.6. *$B_{\ell_1} \subset B_{\ell_\infty^*}$ is $\sigma(\ell_\infty^*, \ell_\infty)$-dense.*

This means for any $\varphi \in \ell_\infty^*$, for any $f_i \in \ell_\infty$, there exists $g \in \ell_1$, with $\|g\|_{\ell_1} \leq \|\varphi\|$, such that

$$|\varphi(f_i) - f_j(g)| \leq \epsilon \text{ i.e. arbitrarily closed.}$$

Or there exists a net $(g_\alpha) \in \ell_1$ with $\|g_\alpha\|_{\ell_1} \leq \|\varphi\|$, such that

$$\varphi(f) = \lim_\alpha f(g_\alpha) = \lim_\alpha \sum_{n \in \mathbb{N}} f(n) g_\alpha(n).$$

Remark 14.7. Let $\varphi : \ell_\infty \to \mathbb{K}$, and assume $\varphi(1) = 1$. TFAE

- $\|\varphi\| = 1$

- $\forall g \geq 0$, $\varphi(g) \geq 0$.

We call this **positive functionals**.

Define the **state space** $S(\ell_\infty) = \{\varphi \mid \varphi(1) = 1, \|\varphi\| = 1\}$. Then discrete probability measures are dense in the state space. Indeed if $\varphi(1) = 1$ and $\|\varphi\| = 1$, then there is $g_\alpha \in \ell_1$ with $g_\alpha(1) = 1$, $\|g_\alpha\| \leq 1$ and $g_\alpha(f) \to \varphi(f)$. That is $g_\alpha \to \varphi$ in $\sigma(\ell_\infty^*, \ell_\infty)$.

Lemma 14.8. *$\|g_\alpha\|_{\ell_1} = 1$ and $\sum_n g_\alpha(n) = 1$ implies $g_\alpha \leq 0$.*

This means g_α are discrete probability measures because $\varphi(f) = \lim_\alpha \sum_{n\in\mathbb{N}} f(n) g_\alpha(n)$ exists.

Theorem 14.9. *Let be $\varphi : C(K) \to \mathbb{C}$ be such that $\varphi(1) = 1$ and $\|\varphi\| = 1$. Then there exists a net $(x_j)_{j=1}^{n(\alpha)} (\lambda_j^\alpha)_{j=1}^{n(\alpha)}$, where $\sum \lambda_j^\alpha = 1$ such that*

$$\varphi(f) = \lim_\alpha \sum_{j=1}^{n(\alpha)} f(x_j^\alpha) \cdot \lambda_j^\alpha.$$

Proof. The Banach space $C(K)$ embeds into the Banach space $\ell_\infty(K)$ (view this as a discrete index set, no topology) isometrically via $\iota(f)(k) = f(k)$.

$$\begin{tikzcd} C(K) \arrow[r, hook, "\iota"] \arrow[d, "\varphi"'] & \ell_\infty(K) \arrow[dl, dashed, "\exists\hat{\varphi}"] \\ \mathbb{K} & \end{tikzcd}$$

As previous seen, $\hat{\varphi}$ exists by Hahn-Banach extension. Also have $\hat{\varphi}(1) = 1$, $\|\hat{\varphi}\| = 1$ and then $\hat{\varphi} \in S(\ell_\infty(K))$. By previous remark, and also the fact that every function in ℓ_1 is support on a countable number of points

$$\hat{\varphi}(F) = \lim_{\alpha} \sum_{(t_j)} F((t_j^{\alpha})) \cdot \lambda_j^{\alpha}$$

where $\sum_{\alpha} \lambda_j^{\alpha} = 1$. Can replace LHS of this equation by $\lim_{\alpha} \lim_M \sum_{j=1}^{M} \lambda_j^{\alpha,M} \cdot F(t_j^{\alpha})$ with $\sum_{j=1}^{M} \lambda_j^{\alpha,M} = 1$ (technical detail skipped). But

$$F = \iota(f) = \lim_{\alpha'} \sum_{j=1}^{M(\alpha')} \lambda_j^{\alpha'} \cdot f(t_j^{\alpha'}).$$

□

Consider $C[0,1]$. It is separable (admits a countable dense subset), whereas $\ell_{\infty}(\mathbb{N})$ is non-separable.

Corollary 14.10. *If $\varphi : C[0,1] \to \mathbb{C}$, with $\varphi(1) = 1$ and $\|\varphi\| = 1$. Then there exists a sequence $(t_j^n)(\lambda_j^n)$, where $\sum \lambda_j^n = 1$ such that*

$$\varphi(f) = \lim_{\alpha} \sum_{j=1}^{M(n)} f(x_j^n) \cdot \lambda_j^n.$$

15 Riesz-Thorin Theorem 20210301

Theorem 15.1 (Riesz-Thorin)**.** *Let A be a linear operator and let $1 \le p_0, p_1, q_0, q_1 \le \infty$ where $p_0 \ne p_1$ and $q_0 \ne q_1$. Suppose $A : L_{p_0} \to L_{q_0}$ is bounded and $A : L_{p_1} \to L_{q_1}$ is bounded. Let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

where $\theta \in (0,1)$. Then

$$\|A\|_{L_p \to L_q} \le \|A\|_{L_{p_0} \to L_{q_0}}^{1-\theta} \cdot \|A\|_{L_{p_1} \to L_{q_1}}^{\theta}.$$

If we call $\|A\|_{L_{p_0} \to L_{q_0}}^{1-\theta} = M_0$ and $\|A\|_{L_{p_1} \to L_{q_1}}^{\theta} = M_1$, then $\|A\|_{L_p \to L_q} \le M_0^{1-\theta} \cdot M_1^{\theta}$.

In our case (finite dimensional), the same matrix makes sense. In general, $A : L_{p_0} \cap L_{p_1} \to L_{q_0} + L_{q_1}$.

We will use the following lemma to prove Riesz-Thorin Theorem.

Lemma 15.2 (Hadamard's Three-Line Theorem)**.** *Suppose $f(z)$ is bounded and continuous function on $0 \le \mathrm{Re}(z) \le 1$ and analytic in the interior. Denote*

$$M_\theta = \sup_{y \in \mathbb{R}} |f(\theta + iy)|.$$

Then $M_\theta \le M_0^{1-\theta} M_1^{\theta}$ for $\theta \in (0,1)$.

If we control the function on boundary then we control the function in the interior.

Example 15.3. Map from a strip to a disk. Let $f(z) = \sum a_n z^n$ be an analytic function, $a_0 = f(0) = \frac{1}{2\pi i} \int \frac{f(z)}{z} \, \mathrm{d}z$. Then

$$|a_0| \le \int |f(z)| \, \mathrm{d}z = \frac{1}{2\pi i} \int f(e^{i\theta}) \, \mathrm{d}\theta \le \sup |f(e^{i\theta})|.$$

Proof. Ref.

Recall $L_p \hookrightarrow L_{p'}^*$ isometrically. So

$$\|A\|_{L_p \to L_q} = \sup \Big\{ \sum_{kj} y_j \cdot A_{jk} \cdot x_k \;\Big|\; \sum |x_i|^p \le 1, \sum |y_j|^{q'} \le 1 \Big\}.$$

Assume $\sum |x_i|^p = 1$ and $\sum |y_j|^{q'} = 1$. Define a function

$$x_k(z) = \frac{x_k}{|x_k|} |x_k|^{p \cdot \left(\frac{1-z}{p_0} + \frac{z}{p_1} \right)} \quad \text{and} \quad y_j(z) = \frac{y_j}{|y_j|} |y_j|^{q' \cdot \left(\frac{1-z}{{q_0}'} + \frac{z}{{q_1}'} \right)}.$$

Then $F(z) = \sum_{jk} y_j(z) \cdot A_{jk} \cdot x_k(z)$ is also analytic. Take $0 \le \mathrm{Re}(z) \le 1$ and define $G(z) = M_0^{z-1} M_1^{-z} F(z)$.

Claim 15.4. *$|G(it)| \le 1$ and $|G(1 + it)| \le 1$.*

Take $z = it$, then

$$G(it) = \sum_{jk} \frac{x_k}{|x_k|}|x_k|^{p\cdot\left(\frac{1-it}{p_0}+\frac{it}{p_1}\right)} \cdot A_{jk} \cdot \frac{y_j}{|y_j|}|y_j|^{q'\cdot\left(\frac{1-it}{q_0{}'}+\frac{it}{q_1{}'}\right)}$$

$$= \sum_{jk} \alpha_k |x_k|^{\frac{p}{p_0}} \cdot A_{jk} \cdot \beta_j |y_j|^{\frac{q'}{q_0{}'}}$$

$$= \|A\|_{L_{p_0}\to L_{q_0}} \cdot \left\| \sum_{jk} \alpha_k |x_k|^{\frac{p}{p_0}} \right\|^{\frac{p_0}{p_0}} \cdot \left\| \sum_{jk} \beta_j |y_j|^{\frac{q'}{q_0{}'}} \right\|^{\frac{q_0{}'}{q_0{}'}} \leq 1,$$

where $|\alpha_k|, |\beta_k| = 1$ (?). Similarly for $G(1+it)$.

The Three-Line Lemma gives $|G(\theta)| \leq 1$. Note that

$$G(\theta) = M_0^{\theta-1}M_1^{-\theta} \sum_{jk} \frac{x_k}{|x_k|}|x_k|^{p\cdot\left(\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\right)} \cdot A_{jk} \cdot \frac{y_j}{|y_j|}|y_j|^{q'\cdot\left(\frac{1-\theta}{q_0{}'}+\frac{\theta}{q_1{}'}\right)}$$

$$= M_0^{\theta-1}M_1^{-\theta} \sum_{jk} \frac{x_k}{|x_k|}|x_k|^{p\cdot\frac{1}{p}} \cdot A_{jk} \cdot \frac{y_j}{|y_j|}|y_j|^{q'\cdot\frac{1}{q'}}$$

$$= M_0^{\theta-1}M_1^{-\theta} \sum_{jk} x_k \cdot A_{jk} \cdot y_j.$$

This implies $|\sum_{jk} x_k \cdot A_{jk} \cdot y_j| \leq M_0^{1-\theta}M_1^{\theta}$. □

Corollary 15.5. *Assume x, y are complex numbers and $r \leq s \leq r'$ then*

$$(|x+y|^r + |x-y|^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}}(|x|^s + |y|^s)^{\frac{1}{s}}.$$

Example 15.6. When $r = 2$, $x, y \in \mathbb{R}$, then we get the parallelogram law

$$(|x+y|^2 + |x-y|^2)^{\frac{1}{2}} = (x^2 + 2xy + y^2 + x^2 - 2xy + y^2)^{\frac{1}{2}} = \sqrt{2}(x^2+y^2)^{\frac{1}{2}}.$$

Proof. Take the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then

$$A\begin{pmatrix} x \\ y \end{pmatrix} = (|x+y|^r + |x-y|^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}}(|x|^s + |y|^s)^{\frac{1}{s}}$$

For the case $s \geq 2$,

$$\|A\|_{L^2_\infty \to L^2_\infty} = \sup \left\{ \max(|x+y|, |x-y|) \,\middle|\, |x| \leq 1,\ |y| \leq 1 \right\} \leq 2.$$

$$\|A\|_{L^2_2 \to L^2_2} \leq (|x+y|^2 + |x-y|^2)^{\frac{1}{2}} = \sqrt{2}(x^2+y^2)^{\frac{1}{2}} \leq \sqrt{2}.$$

Using Riesz-Thorin Theorem we obtain

$$\|A\|_{L^2_s \to L^2_s} \leq 2^{1-\theta} \sqrt{2}^{\,\theta} = 2^{1-\frac{\theta}{2}} = 2^{1-\frac{1}{s}},$$

with the last step given by $\frac{1}{s} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$.

For $1 \leq s \leq 2$, we note that $r \leq s \leq r'$ implies $s' \leq r$. It suffices to consider $r = s'$. Again Riesz-Thorin Theorem gives

$$\|A\|_{L_s \to L_s} \leq \|A\|^{1-\theta}_{L_1 \to L_\infty} \cdot \|A\|^{\theta}_{L_2 \to L_2} \leq 1^{1-\theta} \cdot \sqrt{2}^{\,\theta} = 2^{\frac{1}{s'}} = 2^{1-\frac{1}{s}},$$

with $\frac{1}{s} = \frac{1-\theta}{1} + \frac{\theta}{2}$ and $\frac{1}{s'} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$. Note that

$$\|A\|_{L_1 \to L_\infty} = \max_{jk} |A_{jk}|.$$

□

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Clarkson's inequality

Minkowski $L_p(\ell_q) \subset \ell_q(L_p)$ if $p \leq q$

$\ell_p(L_q) \subset L_q(\ell_p)$ if $p \leq q$

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