Preliminary to Ch7 of <u>Hamilton's Ricci flow</u>

- Ric flow $\quad \partial_t g = -2\,Ric \qquad$ (RF)

- normalized $\quad \partial_t g = -2\,Ric + \frac{2}{n} r g \qquad$ (NRF)

- Kulkani – Nomizu product

$$(\omega \circledwedge \eta)_{ijkl} = \omega_{ik}\eta_{jl} + \omega_{jl}\eta_{ik} - \omega_{il}\eta_{jk} - \omega_{jk}\eta_{il}$$

- decomposition of Rm

$$Rm = \frac{R}{2n(n-1)} g^2 + \frac{1}{n-2} \mathring{Ric}\, g + W$$

$$|g^2| = 8n(n-1)$$

Schur's lemma $\quad n \neq 2$

$$Rm = \frac{R}{2n(n-1)} g^2 . \quad \Rightarrow \quad R \text{ const.}$$

pf. $\quad Ric = \frac{1}{n} R g$

2nd Bianchi id $\quad d\,\mathrm{scal} = 2\,\mathrm{div}(Ric)$

$$LHS = dR$$

$$RHS = 2 g^{ij} \nabla_i (Ric_{jk}) dx^k$$

$$= 2 g^{ij} g_{jk} \nabla_i \left(\frac{R}{n}\right) dx^k = \frac{2}{n} dR$$

if $n > 2 \longrightarrow dR = 0 \Rightarrow R$ const.

Ch 5 Perelman's no collapsing

Einstein - Hilbert

$$E(g) = \int R_g \, d\mu$$

$$\frac{dE}{ds} = \int_M \langle \partial_s g, \tfrac{1}{2} R g - Ric \rangle \, d\mu$$

gradient flow of E. not parabolic

$$\partial_t g = 2 \nabla E(g) = \underbrace{Rg}_{} - 2Ric$$

want to get rid of this. (comes from $\partial_s d\mu$)
then RHS becomes RHS of Ric flow

To get rid of $\partial_s d\mu$.

- define $\mathcal{E}(g) = \int_M R e^{-f} d\mu$

$$\frac{d\mathcal{E}}{ds} = -\int_M \langle \partial_s g, Ric \rangle e^{-f} d\mu$$

$$+ \int_M \left(-\Delta \operatorname{tr}_g(\partial_s g) + \nabla_i \nabla_j \partial_s g_{ij}\right) e^{-f} d\mu$$

get rid of part of this by considering
$\int_M |\nabla f|^2 e^{-f} d\mu$ w/ $\partial_s (e^{-f} d\mu) = 0$

- Take $\mathcal{F}(g, f) = \int_M \left(R_g + |\nabla f|^2\right) e^{-f} \, d\mu$

$$= \int_M \left(R_g + 2\Delta f - |\nabla f|^2\right) e^{-f} \, d\mu$$

then $\frac{d}{ds}\mathcal{F} = -\int_M \langle \partial_s g, \, Ric + \nabla\nabla f \rangle e^{-f} \, d\mu.$

$\longrightarrow$ gradient flow of $\mathcal{F}$ is

$$\begin{cases} \partial_t g = -2\left(Ric + \nabla\nabla f\right) \\ \partial_t f = -R - \Delta f \end{cases} \qquad (\star)$$

- monotonicity formula

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |Ric + \nabla\nabla f|^2 e^{-f} \, d\mu \geq 0$$

- pullback via diffeo $(\star)$ becomes

$$\begin{cases} \partial_t \tilde{g} = -2 Ric_{\tilde{g}} \\ \partial_t \tilde{f} = -R_{\tilde{g}} - \Delta \tilde{f} + |\nabla \tilde{f}|^2 \end{cases}$$

$$\tilde{g} = \varphi^* g \qquad \tilde{f} = \varphi^* f.$$

- same monotonicity formula

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 0 \iff \partial_t g = \mathcal{L}_{\nabla f}\, g$$

Def Perelman's Entropy $\qquad \mathcal{F}$ + scaling

$\mathcal{W}(g, f, \tau)$

$$= \int_M \left(\tau \left(R + |\nabla f|^2 \right) + (f - n) \right) (4\pi\tau)^{-n/2} e^{-f} \, d\mu$$

scaling factor $\tau > 0$

Consider
$$\begin{cases} \partial_t g = -2Ric \\ \partial_t f = -\Delta f - R + |\nabla f|^2 + \frac{n}{2\tau} \\ \partial_t \tau = -1 \end{cases}$$

- entropy monotonicity

$$\frac{d}{dt} \mathcal{W} = 2\tau \int_M \left| Ric + \nabla\nabla f - \frac{1}{2\tau} g \right|^2 u \, d\mu \geq 0$$

$u = (4\pi\tau)^{-n/2} e^{-f}$

Def μ-invariant $\qquad$ monotonicity w.r.t time

$$\mu(g, \tau) = \inf \left\{ \mathcal{W} : f \in C^\infty, \int_M (4\pi\tau)^{-n/2} e^{-f} d\mu = 1 \right\} > \infty.$$

§ 5.4.3

logarithmic Sobolev ineq. $\Rightarrow$ $\mathcal{W}$ bdd below

Def κ-noncollapsed below the scale ρ

- $\rho \in (0,\infty]$ $\kappa > 0$ if $\forall\, B(x,r)$, $r<\rho$
- $|Rm(y)| \leq r^{-2}$ $\forall\, y \in B(x,r)$

$\triangleright$ $$\frac{Vol\, B(x,r)}{r^n} \geq \kappa$$

Thm 5.35 (Perelman: no local collapsing)

- $g(t)$, $t \in [0,T)$ RF sol. on closed M^n
- $T<\infty$

$\triangleright$ $\forall\, \rho \in (0,\infty)$, $\exists\, \kappa = \kappa(g(0), T, \rho) > 0$ s.t. $g(t)$ is κ-noncollapsed below the scale ρ for all $t \in [0,T)$.

Remark 5.36. *Perelman's entropy monotonicity formula rules out local collapse for finite time solutions of the Ricci flow on closed manifolds. The idea of the proof is that if a metric g is κ-collapsed at a point p on a distance scale r for κ small and r bounded, then $\mathcal{W}\left(g,f,r^2\right)$ is large and negative, on the order of $\log\kappa$ for f concentrated in a ball of radius r centered at p. This contradicts the monotonicity formula.* of μ

Ch 6. compactness thm

Thm 6.35

- $\{(M_i^n, g_i(t), O_i)\}_{i\in\mathbb{N}}$

  base pt $t=0$

  $t \in (\alpha, \omega)$

complete, pointed sol. of RF s.t.

- $|Rm(g_i(t))|_{g_i(t)} \le C$ on $M_i^n \times (\alpha, \omega)$ for some $C < \infty$.

- $\operatorname{inj}_{g_i(0)}(O_i) \ge \delta > 0$

$\triangleright$ $\exists$ subseq. $\xrightarrow{C^k} (M_\infty^n, g_\infty(t), O_\infty)$

a completed, pointed sol. of RF

$|Rm(g_\infty)|_{g_\infty} \le C$ on $M_i^n \times (\alpha, \omega)$

pf use Arzela - Ascoli

§ 7.1 Spherical space form

<u>Thm</u> (7.2)

$$- \quad \left|\frac{1}{n-2}\overset{\circ}{Ric}\, g\right|^2 + |W|^2 < \frac{2\varepsilon_n R^2}{n(n-1)}$$

$$\varepsilon_n = \frac{1}{5},\ \frac{1}{10},\ \frac{2}{(n-2)(n+1)} \quad \text{for } n = 4, 5, \geq 6$$

the ineq implies $\quad |Rm|^2 < (4\varepsilon_n + |g^2|^2)\dfrac{R^2}{2n(n-1)}$

▷ unique solution to IVP $\begin{cases} (NRF) & \text{for } t \in [0,\infty) \\ g(0) = g_0 \end{cases}$

▷ $t \to \infty \quad g(t) \to g_\infty$
- converges exp fast in C^k norm
- $scal(g_\infty) \equiv$ const.

▷ $M \cong$ spherical space form

Pinching estimate

const sec. curv. Rm

- $\overset{\circ}{Rm} = Rm - \dfrac{2R}{n(n-1)} Id_{\Lambda^2}$ $\qquad Id_{\Lambda^2} = \frac{1}{4} g^2$.

estimate how far is g from h:

$sec_h \equiv$ const > 0

Prop 7.4

- Pinching $\longrightarrow$ $|\nabla Rm|$ estimate

  - $(M^n, g(t))$ $\quad n \geq 3.$

    closed $\quad scal_{g(t)} > 0$ $\qquad$ Ric flow sol.

  - $|\mathring{R}m| \leq K R^{1-\varepsilon}$ $\qquad K < \infty,\ \varepsilon > 0$

$\triangleright$ $\forall \eta > 0,\ \theta > 0,\ \exists\, C = C(g_0, \eta, \theta) < \infty$ s.t.

if $R(\bar{x}, \bar{t}) \geq C$, $\quad R(\bar{x}, \bar{t}) \geq \eta \cdot \max_{M^3 \times [0, \bar{t}]} R$

$\Rightarrow \quad |\nabla Rm|(\bar{x}, \bar{t}) \leq \theta R^{3/2}(\bar{x}, \bar{t})$

Ric scales as g^{-1}, $|\nabla Rm| \sim g^{-3/2}$

pf by contradiction

$|\mathring{Rm}|$ bdd + $R>0$ $\Rightarrow$ $|Rm| < CR$

if Prop false, then $\exists\, \eta>0,\ \theta>0$ s.t. $\forall\, C_i \to \infty$
$\exists\, (x_i, t_i)$ s.t.

$$R(x_i, t_i) \geq \max\left\{ C_i,\ \eta \cdot \max_{M\times[0,t_i]} R \right\}$$

and $|\nabla Rm|(x_i, t_i) \geq \theta\, R^{3/2}(x_i, t_i)$

Perelman no local collapsing (5.41)
+ compactness thm (6.35)

$\longrightarrow$ $\exists\, g_i(t)$ dilated sol.

$$\underset{\displaystyle R(x_i,t_i)\, g(t_i + R(x_i,t_i)^{-1}\, t)}{\shortparallel} \longrightarrow (M^n_\infty, g_\infty)$$

complete ancient sol.

Rm_∞ bdd $\qquad t \in (-\infty, \omega) \quad \omega > 0$
$R_\infty > 0$

$|\mathring{Rm}_\infty| = 0$ on $M^n_\infty \times (-\infty, \omega)$

$$\Rightarrow \quad Rm(g_\infty) = \frac{2R_\infty}{n(n-1)}\, Id_{\Lambda^2}.$$

$\hookrightarrow$ $R(g_\infty) \equiv$ const. $\qquad \nabla Rm(g_\infty) = 0$

but $\lightning$ $|\nabla Rm(g_\infty)|(x_\infty, 0) > \theta\, R(g_\infty)^{3/2}(x_\infty, 0) > 0$

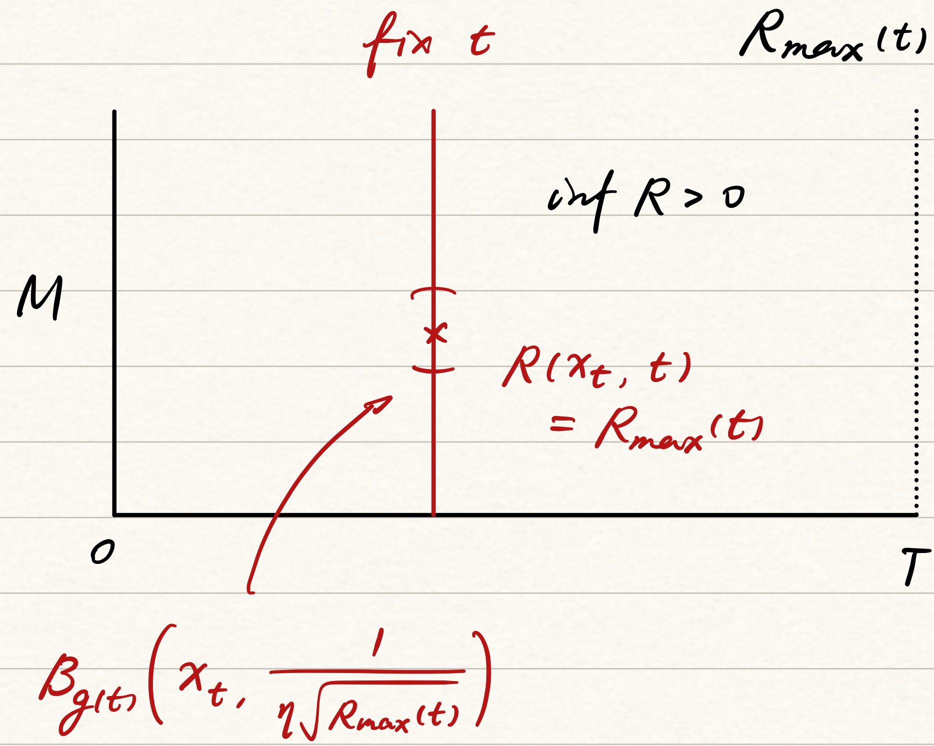
pf sketch of Thm 7.2

Step 1. Global pinching of R, $\displaystyle\lim_{t\to T}\frac{R_{max}}{R_{min}} = 1$

$$\inf_{M\times[0,T]} R > 0 \qquad \lim_{t\to T} R_{max}(t) = \infty$$

$$|\nabla R(x,t)| \leq C\, R_{max}(t)^{\frac{3}{2}-\delta}$$

$$R_{max}(t) \xrightarrow{t\to T} \infty$$



on that ball

$$R_{max}(t) - R(x,t) \leq \overbrace{\frac{1}{\eta\sqrt{R_{max}(t)}}}^{r} \cdot \max|\nabla R(t)|$$

$$\leq \frac{C}{\eta} R_{max}(t)^{1-\delta}.$$

$$\Rightarrow \quad R(x,t) \geq R_{max}(t)\left(1 - \frac{C}{\eta} R_{max}(t)^{-\delta}\right).$$

Claim: the above ball is all of M.

pf. it follows by applying Thm 1.127

Theorem 1.127 (Bonnet-Myers). *If* (M^n, g) *is a complete Riemannian manifold with* $\mathrm{Rc} \geq (n-1)K$, *where* $K > 0$, *then* $\operatorname{diam}(g) \leq \pi/\sqrt{K}$. *In particular,* M^n *is compact and* $\pi_1(M) < \infty$.

and Prop 7.4.

Step 2. estimate for NRF.

closed $(M, g(0))$, $Ric > 0$ $\exists\, C, \delta$ s.t.

NRF

$$|\widetilde{Rm} - \widetilde{\overset{\circ}{Rm}}| < C e^{-\delta \tilde{t}}$$

pf. scal invariant + Step 1 $\Rightarrow \lim\limits_{\tilde{t} \to \tilde{T}} \dfrac{\tilde{R}_{max}}{\tilde{R}_{min}} = 1$

decay estimate of $\tilde{f} = \dfrac{|\widetilde{Rm} - \widetilde{\overset{\circ}{Rm}}|^2}{\tilde{R}^2}$

$\dfrac{\partial \tilde{f}}{\partial t} \leq \tilde{\Delta} \tilde{f} + \cdots$ and use max principle

Step 3. $|\tilde{\nabla}^k \widetilde{Rm}| \leq C e^{-\delta \tilde{t}}$

pf. Start with estimating RF.

$$\frac{d}{dt}\int |\nabla^k Rm|^2 d\mu + 2\int_M |\nabla^{k+1} Rm|^2 d\mu$$

$$\leq \left(\int |\nabla^l Rm|^{\frac{2k}{l}} d\mu\right)^{\frac{l}{2k}} \cdot$$

$$\left(\int |\nabla^{k-l} Rm|^{\frac{2k}{k-l}} d\mu\right)^{\frac{k-l}{2k}} \cdot \left(\int |\nabla^k Rm|^2 d\mu\right)^{\frac{1}{2}}$$

works for $n \geq 4$, the arguement of Lemma 3.37 applies

Ex 7.2

$$F_\delta = \frac{|\overset{\circ}{Rm}|^2}{R^{2-\delta}}$$

compute $\frac{\partial F_\delta}{\partial t}$ and get a curvature term X

$$X = -2R\left(\overset{\circ}{\mathrm{Rm}}\right)_{ijk\ell}\left(B_{ijk\ell}+B_{ikj\ell}\right)+\left(|\mathrm{Rc}|^2-2\frac{R^2}{n(n-1)}\right)\left|\overset{\circ}{\mathrm{Rm}}\right|^2$$
$$+4\frac{R^2}{n(n-1)}\left(\overset{\circ}{\mathrm{Rm}}\right)_{pijq}\left(\overset{\circ}{\mathrm{Rm}}\right)_{qijp}.$$

X vanishes for const. sec. curv metric.

$$\frac{\partial F_\delta}{\partial t} = \Delta F_\delta + \frac{2(1-\delta)}{R}\langle\nabla R, \nabla F_\delta\rangle - \frac{2}{R^{4-\delta}}\left|R\nabla_i R_{jk\ell m} - \nabla_i R \cdot R_{jk\ell m}\right|^2$$
$$(7.7)\qquad -\frac{\delta(1-\delta)}{R^{4-\delta}}\left|\overset{\circ}{\mathrm{Rm}}\right|^2|\nabla R|^2 + \frac{2}{R^{3-\delta}}\left(\delta|\mathrm{Rc}|^2\left|\overset{\circ}{\mathrm{Rm}}\right|^2 - 2X\right),$$

7.11. $\delta > 0$ small + (7.9)

$$|\overset{\circ}{Rm}|^2 = \left|\frac{1}{n-2}\overset{\circ}{Ric}\otimes g\right|^2 + |W|^2 \le (1-\delta)^2\frac{2\varepsilon_n R^2}{n(n-1)} \quad (\star)$$

▷ $X \ge \frac{\delta}{n}R^2|\overset{\circ}{Rm}|^2$ $(\star^\ddagger)$ (7.11)

if $(\star)$ holds at $t=0$,
▷ then $(\star^\ddagger)$ holds for all $t \ge 0$
▷ $\exists K < \infty$, $\varepsilon > 0$ s.t. $|\overset{\circ}{Rm}| \le KR^{1-\varepsilon}$ (7.12)

pf.
lower bound of X $\Rightarrow$ upper bound of $\frac{\partial F_\delta}{\partial t}$

if $(\star^{\neq})$ holds, get (7.12) by maximum principle

use contradiction to check this

$\exists$ $t_0 \in (0,T)$ s.t. $(\star^{=})$ holds. RHS > 0

but upper bound of $\frac{\partial F}{\partial t}$ + $(\star^{=})$

$$\Rightarrow 0 \leq \frac{\partial F}{\partial t} \leq -\frac{4\delta}{nR} |\mathring{Rm}|^2 < 0$$