

Last time Calc I Review

This time Integration by parts

Motivation

using Calc I knowledge (FTC) we know the antiderivative of  $x$  is  $\frac{1}{2}x^2 + C$

$$\cos x \quad \sin x$$

It is natural to ask what is the antiderivative of  $\ln x$ ?

$$\tan x ?$$

To solve this problem we need a new tool.

Recall product rule

$$(uv)' = u'v + uv'$$

if we integrate on both sides w.r.t.  $x$ .

$$\begin{aligned}\Rightarrow \int uv \, dx &= \int uv' \, dx + \int u'v \, dx \\ &= \int u \, dv + \int v \, du\end{aligned}$$

rearrange

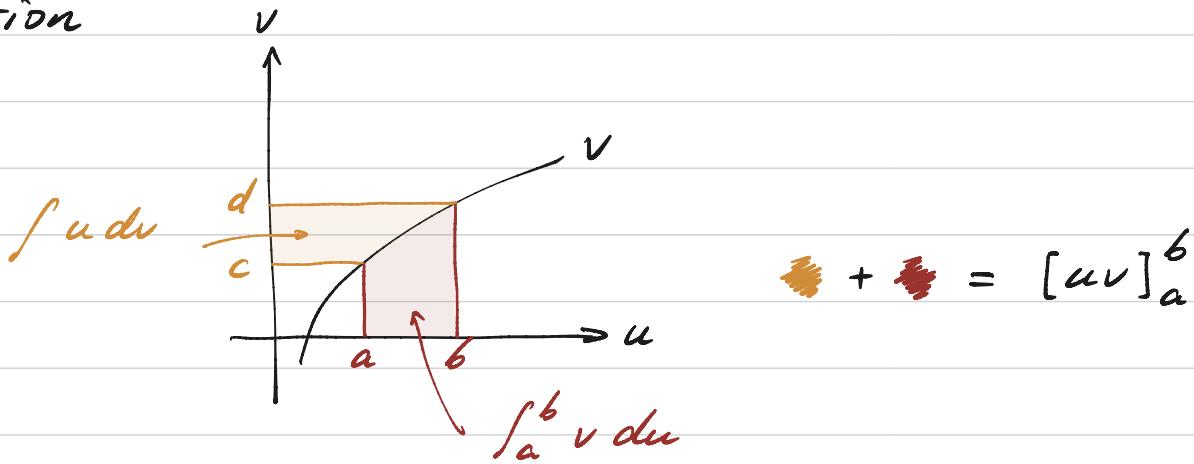
$$\Rightarrow \boxed{\int u \, dv = uv - \int v \, du}$$

Example 1 compute  $\int \ln x \, dx$

take  $u = \ln x$   $v = x$

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \, d\ln x \\ &= x \ln x - \int x \frac{1}{x} \, dx \\ &= x \ln x - x + C\end{aligned}$$

Intuition



N.B. There's a list of choices called LIATE rule  
tells which function to choose as  $u$  first  
However in general there's no easy way to tell  
immediately which function to take as  $u$  or  $v$ .

L  $\ln$   $\log$

I  $\arcsin$   $\arccos$   $\arctan$  ...

A algebraic  $x$   $1+x^2$

T  $\sin$   $\cos$  ...

E  $e^x$

Example 2  $\int \arctan x \, dx$

$$\begin{aligned}
 &= x \arctan x - \int x \, d \arctan x \\
 &= x \arctan x - \int \frac{x}{1+x^2} \, dx \\
 &\quad \text{use substitution law} \\
 &= \frac{1}{2} \int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2} \ln(1+x^2) + \tilde{C} \\
 &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C
 \end{aligned}$$

Example 3  $\int x e^x \, dx = \int \frac{x}{u} \frac{de^x}{v}$

$$\begin{aligned}
 &= x e^x - \int e^x \, dx \\
 &= x e^x - e^x + C
 \end{aligned}$$

Step

1. Choose function  $u$ .

2. The choice of  $v$  depends on  $u$ .

That is, in order to match the integration by parts formula, say we are computing  $\int f(x) dx$

Then we claim  $dv = \frac{f(x)}{u(x)} dx$ , so that

$$\int f \, dx = \int u \cdot \frac{f}{u} \, dx = \int u \, dv$$

Example 4

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{1+x^2}} dx &= \int x^2 \underbrace{\frac{x}{\sqrt{1+x^2}} dx}_{\frac{1}{2} d(x^2+1)} \\
 &= \frac{\frac{1}{2} d(x^2+1)}{\sqrt{1+x^2}} = d\sqrt{1+x^2} \\
 &= \int \underbrace{\frac{x^2}{u} du}_{\sqrt{v}} d\sqrt{1+x^2} \\
 &= x^2 \sqrt{1+x^2} - \int \underbrace{\sqrt{1+x^2} dx^2}_{d(x^2+1)} = d(x^2+1) \\
 &= x^2 \sqrt{1+x^2} - \frac{2}{3} (1+x^2)^{\frac{3}{2}} + C
 \end{aligned}$$

NB. There's more than one way to solve Ex 4.

Example 4'

$$\int \frac{x^3}{\sqrt{1+x^2}} dx$$

substitution rule  $u = 1+x^2 \quad du = 2x dx$   
 $x^3 dx = x^2 \cdot (x dx) = \frac{1}{2} (u-1) du$

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{1+x^2}} dx &= \int \frac{\frac{1}{2}(u-1) du}{\sqrt{u}} \\
 &= \frac{1}{2} \int \underbrace{\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}}_{u^{\frac{1}{2}} - u^{-\frac{1}{2}}} du \\
 &= u^{\frac{1}{2}} - u^{-\frac{1}{2}} \\
 &= \frac{1}{3} u^{\frac{3}{2}} - u^{\frac{1}{2}} + C \\
 &= \frac{1}{3} (1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + C
 \end{aligned}$$

Last time Integration by parts

$$\int u \, du = uv - \int v \, du$$

This time: we focus on a particular type of integral

*integers*

$$\int \sin^m x \cos^n x \, dx \quad \otimes$$

Tools to solve  $\otimes$

- (A) substitution rule
- (B) trig identity
- (C) double angle formulae

$$\cos^2 x + \sin^2 x = 1$$

$$\sin(2x) = 2 \sin x \cdot \cos x$$

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x &= 2 \cos^2 x - 1 \\ &&= 1 - 2 \sin^2 x \end{aligned}$$

! Remember these equations

Note that (D) implies

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

We'll start by looking at the following examples.

Example 1.  $\int \sin^3 x \cos^2 x \, dx$

$$= \int \underbrace{\sin^2 x}_{1-\cos^2 x} \cos^2 x \underbrace{\sin x \, dx}_{\text{substitution } u=\cos x}$$

$$= \int (1-u^2) u^2 (-du) \quad du = -\sin x \, dx$$

$$= \int u^4 - u^2 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

In general, if m or n is odd we can use substitution rule  $u = \cos x, \sin x$ .

Example 2  $\int \cos^2 x \sin^2 x \, dx$

$$= \int \frac{1+\cos 2x}{2} \cdot \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{4} \int 1 - \cos^2 2x \, dx$$

$$= \frac{1}{4} \int 1 - \frac{1+\cos 4x}{2} \, dx$$

$$= \frac{1}{8} \int 1 - \cos 4x \, dx$$

$$= \frac{1}{8} x - \frac{1}{32} \sin 4x + C$$

If m and n both are even we can use trig identity and double angle formulae.

Similarly we can compute

$$\int \tan^m x \sec^n x \, dx$$

Tools to solve  $\oplus$

(A) substitution rule

(B) trig identity  $\sec^2 x = 1 + \tan^2 x$  !

(C) double angle formulae

Recall

$$(\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$

Example 3. 
$$\begin{aligned} & \int \tan x \sec^4 x \, dx \\ &= \int \tan x \sec^2 x \sec^2 x \, dx \\ &= \int \tan x (1 + \tan^2 x) \sec^2 x \, dx \\ &= \int u (1 + u^2) \, du \quad \swarrow u = \tan x \\ &= \frac{1}{2} u^2 + \frac{1}{4} u^4 + C \\ &= \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x + C \end{aligned}$$

Another way to compute this

$$\text{Take } u = \sec^2 x$$

$$du = 2 \sec^2 x \cdot \tan x \, dx$$

$$\text{Then } \int \tan x \sec^4 x \, dx$$

$$= \int u \cdot \frac{1}{2} du$$

$$= \frac{u^2}{4} + \tilde{C}$$

$$= \frac{\sec^4 x}{4} + \tilde{C}$$

$$\left. \begin{aligned} &= \frac{1}{4} (1 + \tan^2 x)^2 + \tilde{C} \\ &= \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + \frac{1}{4} + \tilde{C} \end{aligned} \right\} = C$$

\* Here I'm checking these two method gives the same solution. You don't need to write these in homework.

In this section we consider the contains square roots of the form

$$\sqrt{a^2 - x^2} \quad \sqrt{x^2 + a^2} \quad \sqrt{x^2 - a^2}$$

We will make trig substitutions

$$\sqrt{a^2 - x^2} \quad \sqrt{x^2 + a^2} \quad \sqrt{x^2 - a^2}$$

$$x = a \sin \theta \quad a \tan \theta \quad a \sec \theta$$

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \left[0, \frac{\pi}{2}\right) \text{ or } \left[\pi, \frac{3\pi}{2}\right)$$

$$\cos x \geq 0 \quad \sec x \geq 0 \quad \tan x \geq 0$$

Note that we can use trig identities to remove square roots.

$$\begin{aligned} x &= a \sin \theta \\ \text{e.g. } \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= \sqrt{a^2 \cos^2 \theta} \\ &= |a \cos \theta| \end{aligned}$$

N.B. We have to specify range of  $\theta$  so that we can remove  $| \cdot |$ .

Example 1.  $\int \sqrt{9-x^2} dx$

Take  $x = 3 \sin \theta \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$= \int \sqrt{9-(3 \sin \theta)^2} d(3 \sin \theta)$$

$$= \int |3 \cos \theta| \cdot 3 \cos \theta d\theta \quad \text{Note that } \cos \theta > 0 \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$= 9 \int \cos^2 \theta d\theta$$

$$= 9 \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{9}{2} \int 1 + \cos 2\theta d\theta$$

$$= \frac{9}{2} \theta + \frac{9}{4} \underline{\sin 2\theta} + C$$

$$2 \sin \theta \cos \theta$$

$$= \frac{9}{2} (\theta + \sin \theta \cos \theta) + C$$

$$= \frac{9}{2} \left( \arcsin \frac{x}{3} + \frac{x}{3} \cdot \sqrt{1 - \frac{x^2}{3^2}} \right) + C$$

$$= \frac{9}{2} \arcsin \frac{x}{3} + \frac{x \sqrt{9-x^2}}{2} + C$$

$$\text{Example 2} \quad \int \frac{1}{x^2 \sqrt{x^2 + 4}} dx$$

Take  $x = 2 \tan \theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= \int \frac{d(2 \tan \theta)}{4 \tan^2 \theta \cdot \sqrt{4 \tan^2 \theta + 4}}$$

$$= \int \frac{2 \sec^2 \theta d\theta}{4 \tan^2 \theta \sqrt{4 \sec^2 \theta}}$$

$$= \frac{1}{4} \int \frac{\sec^2 \theta d\theta}{\tan^2 \theta \cdot \sec \theta}$$

$$= \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta$$

$$\frac{1}{\cos \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{\cos \theta}{\sin^2 \theta}$$

$$= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$

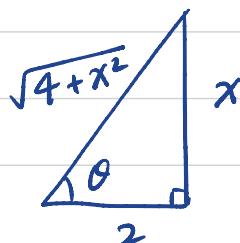
$$\frac{d(\sin \theta)}{\sin^2 \theta}$$

Recall  $\tan \theta = \frac{x}{2}$

$$= -\frac{1}{4} \frac{1}{\sin \theta} + C$$

$$\Rightarrow \sin \theta = \frac{x}{\sqrt{4+x^2}}$$

$$= -\frac{\sqrt{4+x^2}}{4x} + C$$



$$\text{Example 3. } \int \frac{x}{\sqrt{3-2x-x^2}} dx$$

$$= -(x^2+2x+1-1) + 3$$

$$= 4 - (x+1)^2$$

Take  $u = x+1$

$$= \int \frac{u-1}{\sqrt{4-u^2}} d(u-1)$$

Take  $u = 2\sin\theta$

$$= \int \frac{2\sin\theta - 1}{\sqrt{4-4\sin^2\theta}} d(2\sin\theta) \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$= \int \frac{2\sin\theta - 1}{|2\cos\theta|} (2\cos\theta) d\theta$$

$$= \int 2\sin\theta - 1 d\theta$$

$$= -2\cos\theta - \theta + C$$

$$= -\sqrt{4-u^2} - \arcsin \frac{u}{2} + C$$

$$= -\sqrt{3-2x-x^2} - \arcsin \left( \frac{x+1}{2} \right) + C$$

Last time integral containing  $\int$   
This time integrating rational functions

Defining rational functions

$$R(x) = \frac{P(x)}{Q(x)} \quad \text{where } P, Q \text{ are polynomials}$$

$$\text{e.g. } \frac{1}{x+1}, \quad \frac{2x+1}{x^2+4x+3}, \quad \frac{x^4-2}{(x-1)(x^2+1)}$$

If  $\deg P < \deg Q$ , we call  $R$  a proper rational function

$$\text{e.g. } \frac{1}{x^4-1} \quad \frac{x}{x^2+2} \quad \text{proper}$$

$$\frac{x^2}{x^2+2} \quad \frac{x^4}{(x-1)^2} \quad \text{improper}$$

How to solve  $\int R(x) dx$  ?

Rewrite  $R(x)$  as sum of simpler rational functions. Then use substitution rule.

Example 1.  $\frac{x}{x+4} = \frac{x+4-4}{x+4} = 1 - \frac{4}{x+4}$

improper proper

$$\int \frac{x}{x+4} dx = \int 1 - \frac{4}{x+4} dx$$

$$= x - 4 \ln|x+4| + C$$

$$\text{Example 2. } \frac{1}{x^2 - 4} = \frac{1}{\underline{(x-2)} \underline{(x+2)}}$$

To decompose write

$$\frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$$

$$\text{Numerator gives } (A+B)x + 2(A-B) = 1 \\ \Rightarrow A = -B = \frac{1}{4}$$

$$\begin{aligned}\int \frac{1}{x^2-4} dx &= \frac{1}{4} \int \frac{1}{x-2} - \frac{1}{x+2} dx \\&= \frac{1}{4} (\ln|x-2| - \ln|x+2|) + C \\&= \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + C\end{aligned}$$

If  $Q$  is a product of distinct linear factors

$$Q = (a_1x + b_1) \cdots (a_nx + b_n)$$

take

$$R = \frac{A_1}{ax+b_1} + \cdots + \frac{A_n}{ax+b_n}$$

$$\text{Example 3. } \frac{5x^2+2}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}$$

$$\Rightarrow Ax^2+2Ax+2A+Bx^2+Cx = 5x^2+2$$

$$\Rightarrow \begin{cases} A+B=5 \\ 2A+C=0 \\ 2A=2 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=4 \\ C=-2 \end{cases}$$

$$\begin{aligned} \int \frac{5x^2+2}{x(x^2+2x+2)} dx &= \int \frac{1}{x} + \frac{4x-2}{x^2+2x+2} dx \\ &= \ln|x| + 2 \int \frac{2x-1}{(x+1)^2+1} dx \\ &= \ln|x| + 2 \int \frac{2(u-1)-1}{u^2+1} du \quad u=x+1 \\ &= \ln|x| + 4 \int \frac{u}{u^2-1} du - 6 \int \frac{1}{u^2+1} du \\ &= \ln|x| + 2 \int \frac{du}{u^2-1} - 6 \arctan u + C \\ &= \ln|x| + 2 \ln|u-1| - 6 \arctan u + C \\ &= 3 \ln|x| - 6 \arctan(x+1) + C \end{aligned}$$

If Q contains distinct irreducible quadratic factors, take  $\frac{Ax+B}{ax^2+bx+c}$  for the quadratic term.

$$\text{Example 4} \quad \frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{(x-1)^2(x+1)}$$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\Rightarrow 4x = A(x+1)(x-1) + B(x-1) + C(x-1)^2$$

$$= (A+C)x^2 + (B-2C)x + (-A+B+C)$$

$$\Rightarrow A=1 \quad B=2 \quad C=-1$$

$$\int \frac{4x}{x^3 - x^2 - x + 1} dx = \int \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} dx$$

$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$$= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

If Q contains repeated linear factor say  $(ax+b)^r$   
take  $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots + \frac{A_r}{(ax+b)^r}$

Similarly, for repeated quadratic factor  
 $(ax^2 + bx + c)^r$

take  $\frac{A_1x + B_1}{ax^2 + bx + c} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$

Last time  $\int \frac{P(x)}{Q(x)} dx$

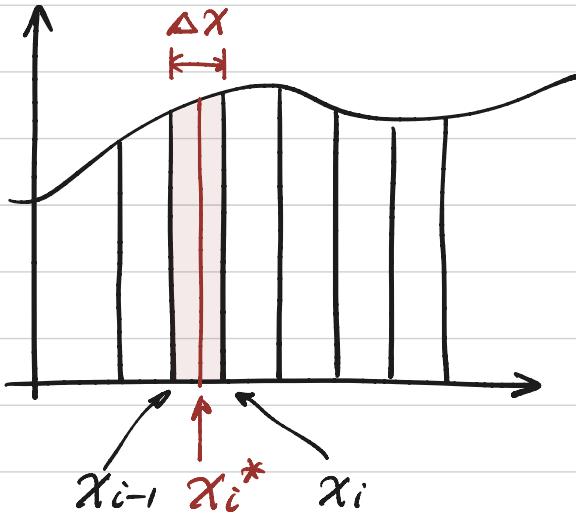
This time: Approximate integrals.

### Motivation

In general, it is difficult to compute the antiderivative of a function and apply FTC. We then seek for an approximate value of the integral.

Recall in Calculus I, the integral is defined as limits of Riemann sums.

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \cdot \Delta x$$



If instead of taking  $n \rightarrow \infty$ , we sum over a finite number of intervals,

$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(x_i^*) \cdot \Delta x$$

For the finite sum above

- if  $x_i^* = x_{i-1}$  left endpoint approx.
- if  $x_i^* = x_i$  right endpoint approx.
- if  $x_i^* = \frac{x_{i-1} + x_i}{2}$  midpoint approx.

We usually use midpoint approx. Let's write the above formula explicitly

Midpoint rule

$$\int_a^b f(x) dx \approx M_n = \Delta x (f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n))$$

where  $\bar{x}_i$  = midpoints  $\Delta x = \frac{b-a}{n}$

Another way to approximate the integral is the

Trapezoidal rule

$$\int_a^b f(x) dx \approx T_n$$

$$= \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n))$$

where  $\Delta x = \frac{b-a}{n}$

Note that  $T_n =$

$$\left( \underbrace{\frac{f(x_0) + f(x_1)}{2}} + \underbrace{\frac{f(x_1) + f(x_2)}{2}} + \dots + \underbrace{\frac{f(x_{n-1}) + f(x_n)}{2}} \right) \Delta x$$

area of trapezoid

Error of approx.

$$E_M = \int_a^b f(x) dx - M_n. \quad E_T = \int_a^b f(x) dx - T_n.$$

Error bounds: suppose  $|f''(x)| \leq K$  for  $a \leq x \leq b$  then

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \quad |E_T| \leq \frac{K(b-a)^3}{12n^2}$$

Similar to trapezoidal rule another rule to approximate integrals is

Simpson's rule

$$\begin{aligned} \int_a^b f(x) dx &\approx S_n \\ &= \frac{\Delta x}{3} \left( f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots \right. \\ &\quad \left. + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n) \right) \end{aligned}$$

$$\text{where } \Delta x = \frac{b-a}{n}$$

Error  $E_S = \int_a^b f(x) dx - S_n$

Error bound: suppose  $|f^{(4)}(x)| \leq K$  for  $a \leq x \leq b$  then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

How to use these approx. in practice?

A naive example

Take  $f(x) = x^2$ ,  $1 \leq x \leq 4$  and consider to find  $n \geq 1$   
s.t.  $|E_M| < 0.1$

$$\text{Given } |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

here  $K = 2$  as  $f''(x) \equiv 2$

$$b-a = 4-1 = 3$$

$$\Rightarrow \frac{2 \cdot 3^3}{24n^2} = \frac{54}{24n^2} \leq 0.1$$

$$\Rightarrow n \geq \sqrt{\frac{54}{24}} \approx 4.7$$

So the smallest  $n$  to take is 5.