# Spring 2021 MATH231 Section CDQ Discussion

WF 9-10am

This document can be found here or on my website. I will continue update it until the end of semester.

## Contact

- TA for section CDQ: Xinran Yu
- Email: xinran4@illinois.edu. Please included MATH231 in your email subject.
- Office hour: Wed 10-11am[1]

## Zoom

- Please use your cameras and microphones in breakout rooms.
- Interrupt me/using the "Raise Hand" feature on Zoom to ask questions.
- You can call me into your breakout room/return to the main room to ask for help.
- It is also possible for me to join your breakout rooms randomly to check if you have any questions.

## Worksheet

- Worksheet can be found on Moodle under Groupwork folder.
- Ask for hints when you get stuck on a problem.

## Submission

- Submit on Moodle under Groupwork folder.
- **1 submission per group**. Once a file is uploaded, everyone in the same group will be able to see/edit the file. [2]
- Group remains the same until each midterm.
- 1st worksheet of the week is due on **Thursday** at **8AM** CST. [3]
- 2nd worksheet of the week is due on **Saturday** at **8AM** CST.
- Worksheet solutions available at 12:30PM CST on the due date.

## Grading

Worksheets are graded with 2, 1 or 0.

2 - the worksheet uploaded is satisfactory

1 - the worksheet uploaded is unsatisfactory and needs improvement. Your TA will comment on what should be improved for next time.

0 - the worksheet was not uploaded

[1] Office hour is run for all students in MATH231

[2] Groups are assigned randomly by Moodle

[3] Central Standard Time

# Contents

# Worksheet 1

Recall

**Theorem 1.1** (Fundamental Theorem of Calculus)**.** *Ref p.26*

*Part 1* *If $f(x)$ is **continuous** over an interval $[a, b]$, and the function $F(x)$ is defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \quad x \in [a, b],$$

*then $F'(x) = f(x)$ over $[a, b]$.*

*Part 2* *If $f(x)$ is **continuous** over an interval $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$ i.e. $F'(x) = f(x)$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(a) - F(b).$$

**Example 1.2.** Let

$$g(x) = \int_a^{b(x)} f(t)\,\mathrm{d}t.$$

Apply chain rule and FTC

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_c^{b(x)} f(t)\,\mathrm{d}t = b'(x) \cdot f(b(x)).$$

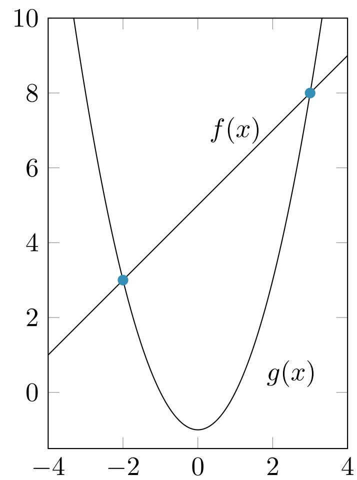
# Worksheet 2

Recall

- **Substitution rule/Change of variable**: let $u = g(x)$, then

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u. \tag{Q1-3}$$

- **Compute area between curves.** (Q4-7)
  - Draw the graph.
  - Find intersection points by solving $f(x) = g(x)$, say they are $x = a$ and $x = b$.
  - Area $= \int_a^b f(x) - g(x)\,\mathrm{d}x$.



# Worksheet 3

**Volume of a solid of revolution**: Slices of volume are circles. Ref

$$\text{Vol } = \int_a^b \pi f(x)^2 \, \mathrm{d}x.$$

Q3. Slices are squares/triangles.

$$\text{Vol } = \int_a^b \text{ Area of slices } \mathrm{d}x.$$

E.g.

$$\text{Vol } = \int_a^b f(x)^2 \, \mathrm{d}x.$$

# Worksheet 4

**Volume by cylindrical shells**:

$$\text{Vol } = \int_a^b 2\pi r \cdot f(x)\, \mathrm{d}x.$$

Rotation about $y$-axis: $r = x$.
Rotation about the vertical line $x = a$: $r = |a - x|$.

# Worksheet 5

Recall

- Since $\sin x$ is oscillating between -1 and 1, $\lim_{x\to\infty} \sin x$ does not exists.
- we can use L'Hopital's Rule to compute indeterminate forms "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$".

**Theorem 5.1** (L'Hopital's Rule)**.**
*Assumptions:*

$$\begin{aligned} &\boldsymbol{f(x) \to 0} \qquad \textit{as } x \to a, \\ &\boldsymbol{g(x) \to 0}, \\ &\boldsymbol{g'(x) \neq 0}. \end{aligned}$$

*Conclusion:*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

**Warning: check the assumptions before applying L'Hopital's Rule.**

# Worksheet 6

**Integration by parts**:

$$(uv)' = u'v + uv' \implies \int \boldsymbol{u}\,\mathbf{d}\boldsymbol{v} = \boldsymbol{uv} - \int \boldsymbol{v}\,\mathbf{d}\boldsymbol{u}.$$

Choose $u$ based on which of these comes first, (search "integration by parts what to choose as $u$"):

(1) **L**ogarithmic functions: $\ln x$

(2) **I**nverse trigonometric functions: $\arcsin x$

(3) **A**lgebraic functions: $x$

(4) **T**rigonometric functions: $\sin x$

(5) **E**xponential functions: $e^x$

# Worksheet 7

# Worksheet 8

Recall: $1 + \tan^2 x = \sec^2 x$.

# Worksheet 9

**Substitution rule/Change of variable**: let $u = g(x)$, then

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u.$$

**Integration by parts**:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

# Worksheet 10

**Partial fractions decomposition**: Find $A, B$ such that

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} - \frac{B}{x-b}.$$

The following table is from this website

| **Factor in denominator** | **Term in partial fraction decomposition** |
|---|---|
| $ax+b$ | $\frac{A}{ax+b}$ |
| $(ax+b)^k$ | $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k},\ k=1,2,3,\ldots$ |
| $ax^2+bx+c$ | $\frac{Ax+B}{ax^2+bx+c}$ |
| $\left(ax^2+bx+c\right)^k$ | $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k},\ k=1,2,3,\ldots$ |

**Typo in solution**

WS10 Q3.

$$\frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{E}{x-1} + \frac{F}{(x-1)^{\mathbf{2}}} + \frac{G}{(x-1)^{\mathbf{3}}}.$$

# Worksheet 11

**Improper integrals**: There are two types of improper integrals $\int_a^b f(x)\,\mathrm{d}x$:

(1) $a$ or $b$ (or both) infinite, e.g $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$.

(2) The function $f(x)$ blows up in the interval $[a, b]$, e.g $\int_0^1 \ln x\,\mathrm{d}x$.

To compute improper integrals, e.g.:

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,\mathrm{d}x.$$

# Worksheet 12

**Simpson's rule**: Let $x_i$'s be equally spaced points,

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)).$$

Coefficient is 4 for odd $i$, $i \neq 0, n$; coefficient is 2 for even $i$, $i \neq 0, n$.

If you want to use the formula $\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\arctan(\frac{x}{a}) + C$, for complex $a$. Recall how we obtain this formula. We do trig substitution, $x = a\tan\theta$, and use the identity $1+\tan^2\theta = \sec^2\theta$. This identity is still valid for complex $\theta$. So with $x = i\tan\theta$, $\theta = \arctan(-ix)$,

$$
\begin{aligned}
I = \int \frac{1}{x^2-1}\,\mathrm{d}x &= \int \frac{1}{x^2+i^2}\,\mathrm{d}x \\
&= \int \frac{1}{i^2\tan^2\theta + i^2} i\sec^2\theta\,\mathrm{d}\theta
\end{aligned}
$$

(Here we need the derivative of $\tan\theta$, but you can check this is $\sec^2\theta$ in the complex case)

$$
= -i\int \frac{1}{\sec^2\theta}\sec^2\theta\,\mathrm{d}\theta = -i\theta + C = -i\arctan(-ix) + C.
$$

The above checked our formula is valid for complex $a$. Hence, we can plug in the value of arctan

$$
\arctan t = \frac{1}{i}\ln\sqrt{\frac{1+it}{1-it}} = \frac{i}{2}\left[\ln(1-it) - \ln(1+it)\right].
$$

So

$$
\arctan(-ix) = \frac{1}{i}\ln\sqrt{\frac{1+x}{1-x}} = \frac{i}{2}\left[\ln(1-x) - \ln(1+x)\right].
$$

$$
I = -i\cdot\frac{i}{2}\left[\ln(1-x) - \ln(1+x)\right] + C = \frac{1}{2}\left[\ln(1-x) - \ln(1+x)\right] + D.
$$

Note that $D$ should be a real constant as $I$ is real.

# Worksheet 13

Let $\mathrm{d}s$ be the arclength differential.

**Arc length**:

$$L = \int \mathrm{d}s = \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

**Surface area**:

$$A = \int 2\pi x\,\mathrm{d}s = \int 2\pi x \cdot \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

## Worksheet 14

$$\text{Force} = \int \rho g \cdot \text{depth}(y) \cdot \text{width}(y)\, \mathrm{d}y.$$

# Worksheet 15

**Integral test**: If $f(x)$ is **continuous, positive and decreasing** on $[N,\infty)$. Then

$$\int_N^\infty f(x)\,\mathrm{d}x \text{ converges } \implies \sum_{n=N}^\infty f(n) \text{ converges;}$$
$$\int_N^\infty f(x)\,\mathrm{d}x \text{ diverges } \implies \sum_{n=N}^\infty f(n) \text{ diverges.}$$

**Warning: check the assumptions before applying integral test.**

# Worksheet 16

Geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 \cdots = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } |r| \geq 1 \end{cases}$$

Integral test: If $f(x)$ is continuous, positive and decreasing on $[N, \infty)$. Then

$$\int_N^\infty f(x)\,\mathrm{d}x \text{ converges} \implies \sum_{n=N}^{\infty} f(n) \text{ converges;}$$

$$\int_N^\infty f(x)\,\mathrm{d}x \text{ diverges} \implies \sum_{n=N}^{\infty} f(n) \text{ diverges.}$$

**Error estimate**: Assume $\sum_{n=1}^{\infty} a_n$ converges

$$S = \overbrace{a_1 + a_2 + \cdots + a_n}^{\text{partial sum } S_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{\text{reminders } R_n}.$$

Note that the first term in $R_n$ is $a_{n+1}$,

$$\int_{n+1}^\infty f(x)\,\mathrm{d}x < R_n < \int_n^\infty f(x)\,\mathrm{d}x.$$

# Worksheet 17

**Alternating series test**: Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all $n$. If

- $\lim\limits_{n\to\infty} b_n = 0$;
- $\{b_n\}$ is a decreasing sequence the series,

then $\sum_n a_n$ is convergent.

**Ratio test**: Let $L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- $L < 1$, $\sum_n a_n$ convergent;
- $L > 1$, $\sum_n a_n$ divergent;
- $L = 1$ no conclusion.

# Worksheet 18

**Comparison test**: If $a_n$, $b_n > 0$ and $a_n \leq b_n$ for all large $n$ then

- $\sum b_n$ converges, then $a_n$ also converges;
- $\sum a_n$ diverges, then $b_n$ also diverges.

**Limit comparison test**: Given $\sum a_n$, $\sum b_n$, with $a_n$, $b_n > 0$ If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = C$ for some $C \neq 0$, $C \neq \infty$. Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

# Worksheet 19

**Ratio test**: Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

**Root test**: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

# Worksheet 20

**Radius of convergence**: For a power series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

where $a$ and $c_n$ are numbers. The **radius of convergence** is a number $R \in [0, \infty]$ s.t.

- the series converges if $|x-a| < R$;
- the series diverges if $|x-a| > R$.

To find $R$, compute $L = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$. Apply ratio test and check the boundary case.

Recall

**Ratio test**: Let $L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

# Worksheet 21

**Power series expansion**: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $|x| < R$ then we can differentiate and integrate $f(x)$:

- $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$;
- $\int f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C$.

# Worksheet 22

**Taylor series**:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

**Maclaurin series**: taking Taylor Series about $x = 0$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \end{aligned}$$

**Examples to remember**:

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1 \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$

## Worksheet 23

## Worksheet 24

# Worksheet 25

Match the graph of the parametric equations with the parametric curves:

- Check the range of $x$ and $y$
- Find specific points which lies on the graph
- Other property: oscillation, symmetry...