Homework Solutions

MATH231

Spring 2022

1 Homework 0 **2**

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Due: Friday, Jan 21 (no need to turn in)

1. Calculating Limits

   - $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

     $$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2}(x+3) = 5.$$

   - $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

     Consider x as a "constant"

     $$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2-2xh-h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh-h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

   - $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right)$

     $$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right) = \lim_{x\to 0} \frac{x^2+x-x}{x(x^2+x)} = \lim_{x\to 0} \frac{x^2}{x^2(x+1)} = \lim_{x\to 0} \frac{1}{x+1} = 1.$$

2. The Chain Rule

   - $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x+\sin x)$

     $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x+\sin x) = (1+\cos x)\frac{1}{x+\sin x}.$

   - $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x^2e^x)$

     $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x^2e^x) = -(2xe^x + x^2e^x)\sin(x^2e^x).$

3. Implicit Differentiation: Solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following implicit function.

   - $x^2+y^2=r^2$, where r is a constant

     Differentiate on both sides w.r.t. x gives $2x+2yy'=0$. Hence $y' = -\frac{x}{y}$.

   - $\frac{x+y}{x-y} = x$

The above is equivalent to $x + y = x^2 - xy$. Differentiate on both sides w.r.t. x gives $1 + y' = 2x - y - xy' \iff (1+x)y' = 2x - y - 1$. Hence $y' = \dfrac{2x - y - 1}{x+1}$.

4. Linear Approximations and Differentials: Find the Taylor polynomials of degree two approximating the given function centered at the given point.

   - $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

     $f' = 2\cos(2x)$, $f'' = -4\sin(2x)$. So $f \sim -2\Big(x - \dfrac{\pi}{2}\Big)$.

   - $f(x) = e^x$ at $a = 1$

     $f' = f'' = e^x$. So $f \sim e + e(x-1) + \dfrac{e}{2}(x-1)^2$.

5. Mean Value Theorem: Determine if the Mean Value Theorem can be applied to the following function on the the given closed interval.

   Both intervals are closed. It suffices to check that these functions are continuous on the given interval. One can do this by computing the derivative exists.

   - $f(x) = 3 + \sqrt{x}, x \in [0, 4]$

     Here $f' = \frac{1}{2\sqrt{x}}$.

   - $f(x) = \dfrac{x}{1+x}, x \in [1, 3]$

     Here $f' = \frac{1}{(1+x)^2}$.

6. L'Hospital's Rule

   - $\lim\limits_{x \to 2} \dfrac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

     Check that as $x \to 2$, $x^3 - 7x^2 + 10x \to 0$ and $x^2 + x - 6 \to 0$ so L'Hospital's rule applies. Then

     $$\lim_{x \to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x \to 2} \frac{3x^2 - 14x + 10}{2x + 1} = -\frac{6}{5}.$$

     The last step uses division property of limits.

   - $\lim\limits_{x \to \infty} (e^x + x)^{\frac{1}{x}}$

     As exp and ln are continuous functions

     $$\lim_{x \to \infty} (e^x + x)^{\frac{1}{x}} = \exp\left[\ln\left(\lim_{x \to \infty} (e^x + x)^{\frac{1}{x}}\right)\right] = \exp\left(\lim_{x \to \infty} \frac{\ln(e^x + x)}{x}\right).$$

Check that as $x \to \infty$, $e^x + x \to \infty$ so L'Hospital's rule applies.

$$\begin{aligned} RHS &= \exp\left(\lim_{x\to\infty} \frac{e^x+1}{e^x+x}\right) \\ &= \exp\left(\lim_{x\to\infty} 1 + \frac{1-x}{e^x+x}\right) && \text{(Check that L'Hospital's rule applies)} \\ &= \exp\left(1 + \lim_{x\to\infty} \frac{-1}{e^x+1}\right) = e. \end{aligned}$$

- $\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right)$

$$\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}.$$

Check that as $x \to \infty$, $\ln\left(1+\frac{3}{x}\right), \frac{1}{x} \to 0$ so L'Hospital's rule applies.

$$RHS = \lim_{x\to\infty} \frac{\frac{-3/x^2}{1+3/x}}{-1/x^2} = \lim_{x\to\infty} \frac{3}{1+\frac{3}{x}} = 3.$$

7. The Fundamental Theorem of Calculus: Find the derivative of the following

- $\displaystyle\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t$

  Apply FTC

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t = \frac{1}{x^3+1}.$$

- $\displaystyle\int_1^{\sqrt{x}} \sin t\,\mathrm{d}t$

  Let $u(x) = \sqrt{x}$. Apply chain rule and FTC

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t = \sin u(x)\cdot\frac{\mathrm{d}u}{\mathrm{d}x} = \sin u(x)\cdot\frac{1}{2\sqrt{x}} = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

- $\displaystyle\int_x^{2x} t^3\,\mathrm{d}t$

  Using subtraction property of integral,

$$\int_x^{2x} t^3\,\mathrm{d}t = \int_0^{2x} t^3\,\mathrm{d}t - \int_0^x t^3\,\mathrm{d}t.$$

  Apply FTC to each term

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{2x} t^3\,\mathrm{d}t = 16x^3 - x^3 = 15x^3.$$

8. Substitution Rule

- $\displaystyle\int_{\frac{1}{2}}^{0} \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$

  Take $u = 1-4x^2$, then $\mathrm{d}u = -8x\,\mathrm{d}x$ and $\mathrm{d}x = -\frac{1}{8}\,\mathrm{d}u$.

$$\int_{\frac{1}{2}}^{0} \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x = \int_{0}^{1} -\frac{1}{8\sqrt{u}}\,\mathrm{d}u = -\frac{1}{4}\sqrt{u}\Big|_{0}^{1} = -\frac{1}{4}.$$

- $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)}\,\mathrm{d}x$

  Take $u = \sin(\pi x)$, then $\mathrm{d}u = \pi\cos(\pi x)\,\mathrm{d}x$ and $\mathrm{d}x = -\frac{1}{8}\,\mathrm{d}u$.

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)}\,\mathrm{d}x = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{1}{\pi u^2}\,\mathrm{d}u = -\frac{1}{u}\Big|_{\frac{\sqrt{2}}{2}}^{1} = \frac{\sqrt{2}-1}{\pi}.$$

- $\displaystyle\int_{0}^{1} xe^{4x^2+3}\,\mathrm{d}x$

  Take $u = 4x^2+3$, then $\mathrm{d}u = 8x\,\mathrm{d}x$

$$\int_{0}^{1} xe^{4x^2+3}\,\mathrm{d}x = \frac{1}{8}\int_{3}^{7} e^{u}\,\mathrm{d}u = \frac{1}{8}\sqrt{u}\Big|_{3}^{7} = \frac{e^7-e^3}{8}.$$