Introduction to Lovelock Metrics

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Abstract

Lovelock metrics serve as one possible extension of Einstein's gravitational theory into higher dimensions. An essential feature of Lovelock metrics lies in their ability to incorporate non-linear dependencies on second-order derivatives of the metric.

Several properties associated with Einstein metrics find their extensions within the realm of Lovelock metrics. For instance, these metrics are critical to the generalized Einstein-Hilbert action, providing a way to derive the Lovelock tensors. Furthermore, a generalization of the DeTurck trick enables us to apply of elliptic regularities within a modified harmonic gauge. Thus, it is demonstrated that asymptotically hyperbolic Lovelock metrics exhibit a comparable behavior within the collar neighborhood of the boundary.

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1 Introduction

The aim of these notes is to provide an introduction to the Lovelock metrics, an extension of the well-established Einstein metric. The focus is on presenting fundamental aspects concerning Lovelock metrics, thereby establishing a basic understanding of their properties and implications.

Let us begin by considering a compact Riemannian manifold M. The well-known Einstein field equations in vacuum boils down to the finding the tensors A^{ij} that adhere to the following properties, as originally outlined by Lovelock [Lov71],

(a) A^{ij} is symmetric;

(b) A^{ij} is a polynomial in terms of the metric tensor g and its first two coordinate derivatives
$$A^{ij} = A^{ij}(g_{ab}; g_{ab,c}, g_{ab,cd});$$

(c) A^{ij} is divergence free, reflecting the conservation of energy and momentum in a gravitational field;

(d) A^{ij} is linear in the second derivatives of g, and the field equations in vacuum takes the form $A^{ij} = 0$.

The resulting tensor A is a linear combination of the Einstein tensor E and metric g. This leads to the well-known Einstein field equation in dimension 4.

Remark 1.1.

1. We emphasize that the properties outlined in (a)-(d) do not see the dimension, offering a consistent framework regardless of the manifold's dimension. On the other hand, it is natural to consider spaces with higher dimensions in certain fields of physics. For instance, some quantum gravity models operate within 10 or 11 dimensions. So a more comprehensive framework that can accommodate the complexities of these additional dimensions is welcomed.

2. Lovelock theory focuses on second-order derivatives of the metric. Nevertheless, higher derivative gravity is also considered in literature. These higher derivative theories, however, lay beyond the scope of our purpose.

If we allow dependency of higher-dimensional analogues of the Ricci curvature, and avoid the introduction of higher derivatives that could lead to instabilities. That is, we modify assertion (d) to a nonlinear dependence of $\partial^2 g$. Then the resulting A^{ij} takes the form

$$\sum_{q\geq 0} a_q \delta^{ii_1\cdots i_{2q}}_{jj_1\cdots i_{2q}} R^{j_1 j_2}_{i_1 i_2} \cdots R^{j_{2q-1}j_{2q}}_{i_{2q-1}i_{2q}}.$$

As a convention, when $q=0$, we have $a_0\delta^i_j$ and when $2q>m$, $\delta=0$.

Remark 1.2. In the context of four dimensions, relaxing assertion (d) yields no novel insights. It can be demonstrated that A collapses into linear combinations of the Ricci curvature Ric and the metric g. The least dimension required here is 6.

This suggests us to define the Lovelock tensor as a generalization of the Einstein tensor to be

$$F_g(\alpha,\beta) = \sum \alpha_q \left(\mathrm{Ric}^{(2q)}_g - \lambda^{(2q)} g\right) + \beta_q \left(\mathrm{scal}^{(2q)}_g - (n+1)\lambda^{(2q)}\right) g,$$

where $\mathrm{Ric}^{(2q)}_g$ is the generalization of Ricci curvature

$$\mathrm{Ric}^{(2q)}_g = \mathscr{C}^{2q-1}_g(R^q_g), \text{ equivalently } \mathrm{Ric}^{(2q)}_{ij} = \delta^{i_1\cdots i_{2q}}_{ij_2\cdots j_{2q}} R^{j_2}_{i_1 i_2 j} R^{j_3 j_4}_{i_3 i_4} \cdots R^{j_{2q-1}j_{2q}}_{i_{2q-1}i_{2q}}.$$

and $\mathrm{scal}^{(2q)}$ is its trace.

Definition 1.3 (Lovelock metric)**.** A *Lovelock metric* g is such that $F_g(\alpha,\beta)=0$ when $\beta_q = -\frac{\alpha_q}{2q}$. (In this case we abbreviate $F_g(\alpha) = F_g(\alpha,\beta)$.)

Example 1.4.

- *Einstein metric.* If $\alpha = (1,0,\cdots,0)$, then $F_g(\alpha) = E$.
- *Lovelok-4.* $\mathrm{scal}^{(2)}_g = \mathrm{scal}_g$, $\mathrm{scal}^{(4)}_g = 6(|R|^2_g - 4|\mathrm{Ric}|^2_g + \mathrm{scal}^2_g)$.
- *Constant sectional curvature metric.* For a hyperbolic metric h, $\mathrm{Ric}^{(2q)}_h = \lambda^{(2q)}h, \mathrm{scal}^{(2q)}_h = (n+1)\lambda^{(2q)}$ and $E^{(2q)}_h = (1-(n+1)/2q)\lambda^{(2q)}h$.
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2 Properties of Lovelock metrics

Let us denote $h_{2q} = \frac{\mathrm{scal}_g^{(2q)}}{(2q)!}$. [1] Here we list several properties in Lovelock metrics.

1. *Chern-Gauss-Bonnet* [AW43]. For a even dimensional closed Riemannian manifold M, say $\dim M = 2k$,

$$\chi(M) = \frac{1}{(2\pi)^k k!}\int_M h_{2k}\, d\mu_g.$$

As an example, in dimension 4,

$$h_4 = \frac{1}{4}\left(|\mathrm{Rm}|^2 - 4|\mathrm{Ric}|^2 + |\,\mathrm{scal}\,|^2\right)$$
$$\chi(M) = \frac{1}{32\pi^2}\int_M |\mathrm{Rm}|^2 - 4|\mathrm{Ric}|^2 + |\,\mathrm{scal}\,|^2\, d\mu_g = \frac{1}{8\pi^2}\int_M h_4\, d\mu_g.$$

2. *Variational property* [Lab08]. The generalized Einstein-Hilbert action

$$H_{2q} : g \mapsto \int_M h_{2q}\, d\mu_g$$

is differentiable and its first variation is

$$H_{2q}'(h) = \frac{1}{2}\langle E^{(2q)}, h\rangle,$$

where $E^{(2q)} = \frac{1}{(2q)!\,q!}\mathrm{scal}_g^{(2q)} - \frac{1}{(2q-1)!}\mathrm{Ric}_g^{(2q)}$. Restricting to unit volume metrics, the critical points of the above functional are Lovelock metrics.

Einstein	Lovelock
$E = \mathrm{Ric} + ng = 0$	$F_g(\alpha,\beta) = 0$ with $\beta_q = -\frac{\alpha_q}{2q}$
E diffeomorphism invariant	$F_g(\alpha,\beta)$ diffeomorphism invariant
$\mathrm{div}\, G_g(\mathrm{Ric}_g) = 0$	$\mathrm{div}\, G_g^{(2q)}(\mathrm{Ric}_g^{(2q)}) = 0$
Einstein g critical points of $\int_M \mathrm{scal}$	Lovelock g critical points of $\int_M \mathrm{scal}^{(2q)}$
formal phg expansion near boundary	formal phg expansion near boundary
For $h - g_{S^n} \in C^{2,\alpha}$, $\exists$ Einstein filling (B^{n+1}, g)	For $h - g_{S^n} \in C^{2,\alpha}$, $\exists$ Lovelock filling (B^{n+1}, g)

Table 1: Properties of Einstein and Lovelock metrics

[1] Note that the Lipschitz-Killing curvature $\tilde{\ell}_{2q}$ in [Alb20] is related to h_{2q} by $\tilde{\ell}_{2q} = h_{2q}/q!$.

3. *Yamabe problem.* The Yamabe problem aims to find conformal metrics with constant scalar curvature. While the generalized version of the problem, which involves seeking a conformal metric with a constant scalar-$2q$ curvature, remains open, there are recent developments of the closely related σ_k-Yamabe problem. This involves k-admissible metrics where the role of the scalar curvature is replaced by $\sigma_k(P)$,[2] and the Yamabe invariant replaced by the k-maximal volume [GV04] or the k-Yamabe constant [STjW05].

4. *Metric expansion* [Alb20]. Let us bring in the conformal geometry framework. Suppose X^{n+1} be a Riemannian manifold with boundary M and let x denotes its boundary defining function. A conformally compact Lovelock metric $g = \frac{dx^2+h}{x^2}$ is asymptotically hyperbolic, hence near the boundary

$$\mathrm{Rm}_g = -\frac{1}{2}g^2 + O(x^{-3}).$$

Then the conformal infinity h has a formal polyhomogeneous expansion in a neighborhood of M as follows

$$h_x = \begin{cases} h_0 + h_2 x^2 + (\text{even powers}) + h_{n-1}x^{n-1} + h_n x^n + \cdots & n \text{ odd} \\ h_0 + h_2 x^2 + (\text{even powers}) + h_{n,1}\log(x)x^{n-1} + h_n x^n + \cdots & n \text{ even.} \end{cases}$$

5. *Ambient obstruction* [GL91]. Failure of having a smooth expansion can be measured by the obstruction tensor $O = x^{2-n}\,\mathrm{tf}(F_g(\alpha,\beta))|_{x=0}$.

3 The DeTurck trick for Lovelock tensors

Similar to the Einstein tensors, Lovelock tensors are generally not elliptic. The DeTurck's trick serves as a powerful tool for handling Einstein tensors. The method simplifies Einstein tensors by introducing an auxiliary metric to manage the challenging terms that hinder ellipticity. Application of the DeTurck trick can be extended to Lovelock tensors, as elaborated in [Alb20].

The motivation behind this lies in the expression of Ricci curvature in harmonic coordinates. In particular, we have

$$\begin{matrix} \mathrm{Ric}_g = -g^{-1}(\partial^2 \owedge g) + \text{l.o.t} \\ g^{ij}\Gamma_{ij}{}^{k} = 0 \end{matrix} \implies (\mathrm{Ric}_g)_{jl} = -g^{ik}\,\partial^2_{ik}\,g_{jl} + \text{l.o.t.}$$

[2] Here $\sigma_k(P)$ means the k-th order symmetric function in the eigenvalues of the Schouten tensor P. For a locally conformally flat metric, $\mathrm{scal}_g^{(2q)} = \sigma_{2q}(g^{-1}P)$.

The right hand side is of Laplace type.

Now, let's consider modified harmonic coordinates defined by $g^{ij}\Pi_{ij}{}^{k} = 0$, where $\Pi_{ij}{}^{k}$ represents the Christoffel symbol of an auxiliary metric t. This new gauge condition exhibits a similar effect, enabling us to express the Ricci curvature as an operator similar to Laplace of g. As a result, we can define the modified Einstein tensor

$$Q(g,t) = \mathrm{Ric}_g + ng - \delta_g^*\Big(gt^{-1}\delta_g G_g(t)\Big).[3].$$

This approach can be extended to an asymptotically hyperbolic Lovelock tensor, where the linearizations of Ricci-$(2q)$ and scalar-$(2q)$ curvatures are well understood. Specifically, the first variation of a Lovelock tensor is, up to higher-order terms in x, a linear combination of Laplace type operators, along with a term involving the gauge condition:

$$\begin{aligned}
&\Big(DF_{(\alpha,\beta)}(g)\Big)(r)\\
=&-\sum_q \frac{\lambda^{(2q)}}{2n(n-1)}\left[q(n-1)\Big(\alpha_q+(n+1)\beta_q\Big)(\Delta_g+2n)(ug)+(n-2q+1)\alpha_q(\Delta_g-2)(r_0)\right]\\
&+(c_1\delta_g^*+c_2 g\delta_g)\delta_g G_g(r)+O(x^{N+1}).
\end{aligned}$$

To extract a Laplace type operator, we define the modified Lovelock tensor

$$Q_{(\alpha,\beta)}(g,t) = F_{(\alpha,\beta)}(g) - \Phi_{(\alpha,\beta)}(g,t), \tag{1}$$

where t is an auxiliary metric and

$$\begin{aligned}
&\Phi_{(\alpha,\beta)}(g,t)\\
=&(c_1\delta_g^*+c_2 g\delta_g)\Big(gt^{-1}B_g(t)\Big)\\
=&-\sum_q \frac{\lambda^{(2q)}}{n(n-1)}\left[\alpha_q(n-2q+1)\delta_g^*-\Big(\alpha_q(q-1)+\beta_q(n-1)q\Big)g\delta_g\right]\Big(gt^{-1}\delta_g G_g(t)\Big).
\end{aligned}$$

The use of the DeTurck trick holds crucial significance in analyzing metric regularity of conformally compact Einstein metrics. Applying the DeTurck trick, we obtain a gauge-modified Einstein equation that is elliptic. Consequently, the usual elliptic

[3] G_g is the gravatational opeator and $\delta_g = -\operatorname{div}_g$

regularity argument can be applied. One expect a comparable line of reasoning can be extended to the examination of Lovelock metrics, though the inherent nonlinearity embedded within the Lovelock tensors the proof more challenging.

Implementing the help of implicit function theorem, one can generalize existence result of Einstein filling near hyperbolic ball. That is, given a conformal infinity h near the standard metric on S^n, there is corresponding Lovelock filling.

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