Ch 11. Infinite sequence and series

Def. An <u>sequence</u> is an infinite list of numbers written in a definite order.

Notation: $\{a_1, a_2, \cdots, a_n, \cdots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples $\{1, 2, 3, 4, \cdots\}$
$\{7, 1, 8, 2, 8, \cdots\}$

Some sequences can be defined by giving a formula for the n-th term a_n

Examples 1. $a_n = \left(\frac{1}{2}\right)^n$ $\quad \{a_n\} = \{\frac{1}{2}, \frac{1}{4}, \cdots\}$

2. $a_n = (-1)^n$ $\quad \{a_n\} = \{-1, 1, -1, 1, \cdots\}$

3. $a_n = \frac{n}{n+1}$ $\quad \{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$

Some sequences may not have a simple / explicit defining equation

Examples 1. a_n = the digit in the n-th decimal place of π

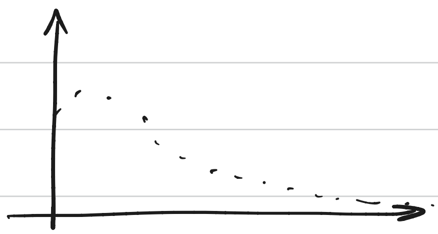
2. The Fibonacci sequence

$$a_1 = 1, \quad a_2 = 1, \quad a_n = a_{n-1} + a_{n-2}$$

$$\{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}$$

A sequence "is" a function f that only takes values on natural numbers. So we will study properties such as graph and convergency.

Example



$$\lim_{n\to\infty} a_n = 0$$

$L < \infty$

Def. A sequence <u>has limit L</u> if for any ε there is an N s.t. if $n > N$ then

$$|a_n - L| < \varepsilon$$

We say $\{a_n\}$ <u>converges</u> to L.

Intuition

Def. $\lim\limits_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N s.t. if $n > N$ then $a_n > M$.

Examples

1. $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{1+\frac{1}{n}} = 1$

2. $\lim\limits_{n\to\infty} \dfrac{1}{n^r} = \begin{cases} 0 & \text{if } r>0 \\ \infty & \text{if } r<0 \end{cases}$

3. $\lim\limits_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \\ \text{DNE} & \text{if } r<1 \end{cases}$

Limit law for sequences

if $\{a_n\}$. $\{b_n\}$ are convergent sequences then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n \qquad c \text{ const.}$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$$

$$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \qquad p>0 \quad a_n>0$$

Squeeze Theorem

$$b_n \le a_n \le c_n \Rightarrow \lim_{n\to\infty} a_n = L$$

$$b_n \to L, \quad c_n \to L$$

<u>Thm</u> If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

If f is continuous

$$\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} f(a_n) = f(L)$$

Example 1.

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$$

$$= \sin 0 = 0$$

Example 2. $\lim_{n\to\infty} \frac{\ln(n+2)}{\ln(1+4n)}$

L'Hopital

$$\lim_{x\to\infty} \frac{\ln(x+2)}{\ln(1+4x)} = \lim_{x\to\infty} \frac{\frac{1}{x+2}}{\frac{1}{1+4x}\cdot 4}$$

$$= \lim_{x\to\infty} \frac{4x+1}{4(x+2)} = 1$$

Example 3.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n}$$

$$= e^{\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)}$$

$$= e$$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\cdot -\frac{1}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

Last time: sequence
This time: series.

Def. We call $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ a <u>series</u> and

$$s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$$ the <u>partial sums.</u>

Note that $\{s_N\}$ is itself a sequence. So it make sense to talk about if $\{s_N\}$ converges or not.

Def. The series $\sum a_n$ is called <u>convergent</u> if its partial sum is convergent. Otherwise $\sum a_n$ is called <u>divergent.</u>

Example 1. (geometric series)
Consider $a_n = r^n$ r: common ratio.

$$a_0 = r^0 = 1 \qquad s_0 = a_0 = 1$$
$$a_1 = r^1 = r \qquad s_1 = a_0 + a_1 = 1 + r$$
$$a_2 = r^2 = r^2 \qquad s_2 = a_0 + a_1 + a_2 = 1 + r + r^2$$
$$\vdots \qquad \vdots$$

$$s_N = \underbrace{1 + r + r^2 + \cdots + r^N}_{\text{we are interested in this sum.}}$$

Let $R_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \cdots + r^N$

$$rR_N = \qquad\qquad r + r^2 + \cdots + r^N + r^{N+1}$$

$$\Rightarrow \quad R_N - rR_N = 1 - r^{N+1}$$

$$\Rightarrow \quad R_N = \frac{1 - r^{N+1}}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \lim_{N \to \infty} R_N$$

$$= \begin{cases} \dfrac{1}{1-r} & \text{if } -1 < r < 1 \qquad \text{conv.} \\ \infty & \text{if } r \geq 1 \\ \text{DNE} & \text{if } r \leq -1 \end{cases} \qquad \text{div.}$$

Example 2. Compute $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ using the above

$$\sum_{n=1}^{\infty} 2^{2n} 6^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 6 \cdot 6^{-n}$$

$$= 6 \cdot \sum_{n=1}^{\infty} \underbrace{\left(\frac{4}{6}\right)}_{\substack{\shortparallel \\ \frac{2}{3}}}{}^{n} = 6 \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}}$$

$$= 6 \cdot 2 = 12$$