Last time Calc I Review
This time Integration by parts

Motivation

using Calc I knowledge (FTC) we know the antiderivative of x is $\frac{1}{2}x^2+C$
$\cos x$ $\sin x$

It is natural to ask what is the antiderivative of $\ln x$?
$\tan x$?

To solve this problem we need a new tool.

Recall product rule

$$(uv)' = u'v + uv'$$

if we integrate on both sides w.r.t. x.

$$\Rightarrow \int uv\, dx = \int uv'dx + \int u'v\, dx$$
$$= \int u\, dv + \int v\, du$$

rearrange

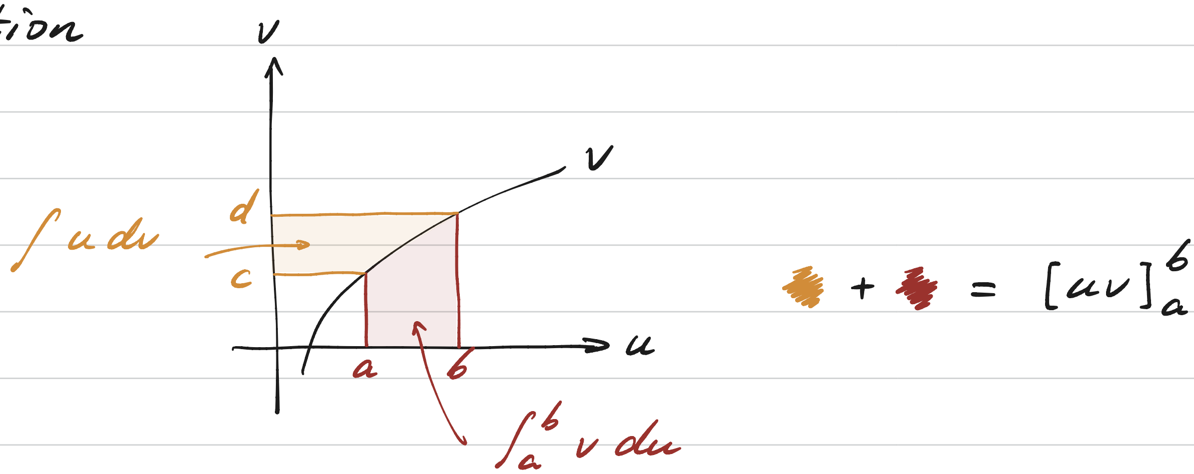
$$\Rightarrow \boxed{\int u\, dv = uv - \int v\, du}$$

Example 1 compute $\int \ln x \, dx$

take $\quad u = \ln x \qquad v = x$

$$\int \ln x \, dx = x \ln x - \int x \, d\ln x$$
$$= x \ln x - \int \underbrace{x \tfrac{1}{x}}_{=1} \, dx$$
$$= x \ln x - x + C$$

Intuition



NB. There's a list of choices called LIATE rule
tells which function to choose as u first
However in general there's no easy way to tell
immediately which function to take as u or v.

L	ln log
I	arcsin arccos arctan ···
A	algebraic x. $1+x^2$
T	sin cos ···
E	e^x

Example 2 $\int \underbrace{\arctan x}_{u} \, d\underbrace{x}_{v}$

$$= x \arctan x - \int x \, d \arctan x$$

$$= x \arctan x - \underbrace{\int \frac{x}{1+x^2} \, dx}_{\text{use substitution law}}$$

$$= \frac{1}{2} \int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2} \ln(1+x^2) + \tilde{C}$$

$$= x \arctan x - \frac{1}{2} \ln(1+x^2) + C$$

Example 3 $\int x e^x \, dx = \int \underbrace{x}_{u} \, d\underbrace{e^x}_{v}$

$$= x e^x - \int e^x \, dx$$

$$= x e^x - e^x + C$$

Step

1. Choose functon u.
2. The choice of v depends on u.

That is, in order to match the integration by parts formula, say we are computing $\int f(x) dx$

Then we claim $dv = \frac{f(x)}{u(x)} dx$, so that

$$\int f \, dx = \int u \cdot \frac{f}{u} \, dx = \int u \, dv$$

Example 4 $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int x^2 \underbrace{\frac{x}{\sqrt{1+x^2}}\,dx}$

$$= \frac{\frac{1}{2}\,d(x^2+1)}{\sqrt{1+x^2}} = d\sqrt{1+x^2}$$

$$= \int \underbrace{x^2}_{u}\, d\underbrace{\sqrt{1+x^2}}_{v}$$

$$= x^2\sqrt{1+x^2} - \int \sqrt{1+x^2}\, \underbrace{dx^2}_{= d(x^2+1)}$$

$$= x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

NB. There's more than one way to solve Ex 4.

Example 4' $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

substitution rule $\quad u = 1+x^2 \qquad du = 2x\,dx$

$$x^3\,dx = x^2\cdot(x\,dx) = \frac{1}{2}(u-1)\,du$$

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{\frac{1}{2}(u-1)\,du}{\sqrt{u}}$$

$$= \frac{1}{2}\int \underbrace{\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}}_{= u^{\frac{1}{2}} - u^{-\frac{1}{2}}}\,du$$

$$= \frac{1}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}} + C$$

$$= \frac{1}{3}(1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + C$$

Last time Integration by parts

$$\int u\,dv = uv - \int v\,du$$

This time: we focus on a perticular type of integral

integers

$$\int \sin^m x \cos^n x \quad dx \qquad \circledast$$

Tools to solve $\circledast$

(A) substitution rule

(B) trig identity

(C) double angle formulae

$$\cos^2 x + \sin^2 x = 1$$

$$\sin(2x) = 2\sin x \cdot \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

! Remember these equations

Note that (D) implies

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

We'll start by looking at the following examples.

Example 1. $\int \sin^3 x \cos^2 x \, dx$

$$= \int \underbrace{\sin^2 x}_{1-\cos^2 x} \cos^2 x \underbrace{\sin x \, dx}_{\text{substitution}}$$

$u = \cos x$

$du = -\sin x \, dx$

$$= \int (1-u^2)\, u^2 \, (-du)$$

$$= \int u^4 - u^2 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

In general, if m or n is odd we can use substitution rule $u = \cos x, \sin x$.

Example 2 $\int \cos^2 x \sin^2 x \, dx$

$$= \int \frac{1+\cos 2x}{2} \cdot \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{4} \int 1 - \cos^2 2x \, dx$$

$$= \frac{1}{4} \int 1 - \frac{1+\cos 4x}{2} \, dx$$

$$= \frac{1}{8} \int 1 - \cos 4x \, dx$$

$$= \frac{1}{8} x - \frac{1}{32} \sin 4x + C$$

If m and n both are even we can use trig identity and double angle formulae.

Similarly we can compute

$$\int \tan^m x \sec^n x \, dx$$

Tools to solve ⊛

(A) substitution rule

(B) trig identity $\sec^2 x = 1 + \tan^2 x$!

(C) double angle formulae

Recall

$$(\tan x)' = \sec^2 x \qquad (\sec x)' = \sec x \tan x$$

Example 3. $\int \tan x \sec^4 x \, dx$

$$= \int \tan x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan x \, (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int u(1+u^2) \, du$$

$u = \tan x$

$$= \frac{1}{2} u^2 + \frac{1}{4} u^4 + C$$

$$= \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x + C$$

Another way to compute this

Take $u = \sec^2 x$

$du = 2\sec^2 x \cdot \tan x \, dx$

Then $\int \tan x \sec^4 x \, dx$

$= \int u \cdot \frac{1}{2} \, du$

$= \frac{u^2}{4} + \tilde{C}$

$= \frac{\sec^4 x}{4} + \tilde{C}$

(*) $= \frac{1}{4}(1+\tan^2 x)^2 + \tilde{C}$

$= \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + \underbrace{\frac{1}{4} + \tilde{C}}_{=C}$

(*) Here I'm checking these two method gives the same solution. You don't need to write these in homework.

In this section we consider the contains sequare roots of the form

$$\sqrt{a^2-x^2} \qquad \sqrt{x^2+a^2} \qquad \sqrt{x^2-a^2}$$

We will make trig substitutions

	$\sqrt{a^2-x^2}$	$\sqrt{x^2+a^2}$	$\sqrt{x^2-a^2}$
$x =$	$a\sin\theta$	$a\tan\theta$	$a\sec\theta$
$\theta \in$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$\left[0, \frac{\pi}{2}\right)$ or $\left[\pi, \frac{3\pi}{2}\right)$
	$\cos x \geq 0$	$\sec x \geq 0$	$\tan x \geq 0$

Note that we can use trig identities to remove square roots.

e.g. $$\sqrt{a^2-x^2} \overset{x = a\sin\theta}{=} \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = |a\cos\theta|$$

NB. We have to specify range of θ so that we can remove $|\ |$.

Example 1. $\int \sqrt{9-x^2}\,dx$

Take $x = 3\sin\theta$ $\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= \int \sqrt{9-(3\sin\theta)^2}\, d(3\sin\theta)$$

$$= \int \underbrace{|3\cos\theta|}_{=3\cos\theta} \cdot 3\cos\theta\, d\theta$$

Note that $\cos\theta > 0$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$= 9 \int \cos^2\theta\, d\theta$$

$$= 9 \int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2} \int 1 + \cos 2\theta\, d\theta$$

$$= \frac{9}{2}\theta + \frac{9}{4}\underbrace{\sin 2\theta}_{2\sin\theta\cos\theta} + C$$

$$= \frac{9}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{9}{2}\left(\arcsin\frac{x}{3} + \frac{x}{3}\cdot\sqrt{1-\frac{x^2}{3^2}}\right) + C$$

$$= \frac{9}{2}\arcsin\frac{x}{3} + \frac{x\sqrt{9-x^2}}{2} + C$$

Example 2 $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Take $x = 2\tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$= \int \frac{d(2\tan\theta)}{4\tan^2\theta\cdot\sqrt{4\tan^2\theta+4}}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{4\sec^2\theta}}$$

$$= \frac{1}{4}\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\cdot\sec\theta}$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$\frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} = \frac{\cos\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

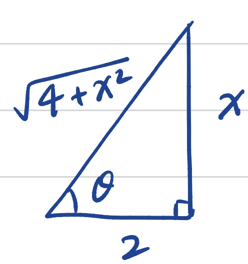
$$\frac{d(\sin\theta)}{\sin^2\theta}$$

$$= -\frac{1}{4}\,\frac{1}{\sin\theta} + C$$

$$= -\frac{\sqrt{4+x^2}}{4x} + C$$

Recall $\tan\theta = \frac{x}{2}$

$\Rightarrow \sin\theta = \frac{x}{\sqrt{4+x^2}}$



Example 3. $\displaystyle\int \frac{x}{\sqrt{\underbrace{3-2x-x^2}}}\,dx$

$$= -(x^2+2x+1-1)+3$$
$$= 4-(x+1)^2$$

Take $u = x+1$

$$= \int \frac{u-1}{\sqrt{4-u^2}}\,d(u-1)$$

Take $u = 2\sin\theta$

$$= \int \frac{2\sin\theta - 1}{\sqrt{4-4\sin^2\theta}}\,d(2\sin\theta) \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$= \int \frac{2\sin\theta - 1}{|2\cos\theta|}(2\cos\theta)\,d\theta$$

$$= \int 2\sin\theta - 1 \; d\theta$$

$$= -2\cos\theta - \theta + C$$

$$= -\sqrt{4-u^2} - \arcsin\frac{u}{2} + C$$

$$= -\sqrt{3-2x-x^2} - \arcsin\left(\frac{x+1}{2}\right) + C$$

Last time integral containing $\sqrt{\ }$
This time integrating rational functions

Defining rational functions

$$R(x) = \frac{P(x)}{Q(x)} \quad \text{where } P, Q \text{ are polynomials}$$

e.g. $\frac{1}{x+1}$, $\frac{2x+1}{x^2+4x+3}$, $\frac{x^4-2}{(x-1)(x^2+1)}$

If $\deg P < \deg Q$, we call R a proper rational function

e.g. $\frac{1}{x^4-1}$ $\frac{x}{x^2+2}$ proper

$\frac{x^2}{x^2+2}$ $\frac{x^4}{(x-1)^2}$ improper

How to solve $\int R(x)\,dx$?

Rewrite $R(x)$ as sum of simpler rational functions. Then use substitution rule.

Example 1. $\underbrace{\dfrac{x}{x+4}}_{\text{improper}} = \dfrac{x+4-4}{x+4} = 1 - \underbrace{\dfrac{4}{x+4}}_{\text{proper}}$

$$\int \frac{x}{x+4}\,dx = \int 1 - \frac{4}{x+4}\,dx$$
$$= x - 4\ln|x+4| + C$$

Example 2. $\dfrac{1}{x^2-4} = \dfrac{1}{(x-2)(x+2)}$

To decompoce write

$$\frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2) + B(x-2)}{(x-2)(x+2)}$$

Numerator gives $(A+B)x + 2(A-B) = 1$

$$\Rightarrow A = -B = \frac{1}{4}$$

$$\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\int \frac{1}{x-2} - \frac{1}{x+2}\,dx$$
$$= \frac{1}{4}\left(\ln|x-2| - \ln|x+2|\right) + C$$
$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

If Q is a product of distinct linear factors

$$Q = (a_1x+b_1)\cdots(a_nx+b_n)$$

take

$$R = \frac{A_1}{a_1x+b_1} + \cdots + \frac{A_n}{a_nx+b_n}$$

Example 3. $\dfrac{5x^2+2}{x(x^2+2x+2)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+2x+2}$

$$\Rightarrow Ax^2 + 2Ax + 2A + Bx^2 + Cx = 5x^2+2$$

$$\Rightarrow \begin{cases} A+B = 5 \\ 2A+C = 0 \\ 2A = 2 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 4 \\ C = -2 \end{cases}$$

$$\int \frac{5x^2+2}{x(x^2+2x+2)}\,dx = \int \frac{1}{x} + \frac{4x-2}{x^2+2x+2}\,dx$$

$$= \ln|x| + 2\int \frac{2x-1}{(x+1)^2+1}\,dx$$

$u = x+1$

$$= \ln|x| + 2\int \frac{2(u-1)-1}{u^2+1}\,du$$

$$= \ln|x| + 4\int \frac{u}{u^2-1}\,du - 6\int \frac{1}{u^2+1}\,du$$

$$= \ln|x| + 2\int \frac{d(u^2)}{u^2-1} - 6\arctan u + C$$

$$= \ln|x| + 2\ln|u-1| - 6\arctan u + C$$

$$= 3\ln|x| - 6\arctan(x+1) + C$$

If Q contains distinct irreducible quadratic factors, take $\dfrac{Ax+B}{ax^2+bx+c}$ for the quadratic term.

Example 4 $\quad \dfrac{4x}{x^3-x^2-x+1} = \dfrac{4x}{(x-1)^2(x+1)}$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\Rightarrow \quad 4x = A(x+1)(x-1) + B(x-1) + C(x-1)^2$$
$$= (A+C)x^2 + (B-2C)x + (-A+B+C)$$

$$\Rightarrow \quad A = 1 \qquad B = 2 \qquad C = -1$$

$$\int \frac{4x}{x^3-x^2-x+1}\,dx = \int \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\,dx$$
$$= \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$
$$= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

If Q contains repeated linear factor say $(ax+b)^r$

take $\quad \dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_r}{(ax+b)^r}$

Similarly, for repeated quadratic factor $(ax^2+bx+c)^r$

take $\quad \dfrac{A_1x+B_1}{ax^2+bx+c} + \cdots + \dfrac{A_rx+B_r}{(ax^2+bx+c)^r}$

Last time $\int \frac{P(x)}{Q(x)}\, dx$

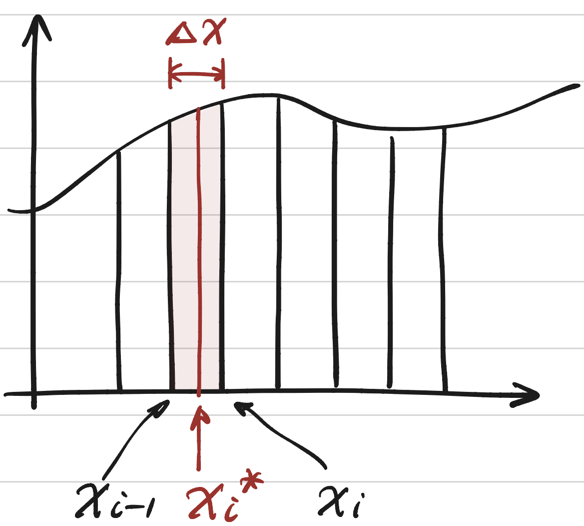
This time: Approximate integrals.

Motivation

In general, it is difficult to compute the antiderivative of a function and apply FTC. We then seek for an approximate value of the integral.

Recall in Calculus I, the integral is defined as limits of Riemann sums.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot \Delta x.$$



If instead of taking $n\to\infty$, we sum over a finite number of intervals,

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\cdot \Delta x$$

For the finite sum above

- if $x_i^* = x_{i-1}$ left endpoint approx.
- if $x_i^* = x_i$ right endpoint approx.
- if $x_i^* = \frac{x_{i-1}+x_i}{2}$ midpoint approx.

We usually use midpoint approx. Let's write the above formula explicitly

Midpoint rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left(f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right)$$

where $\bar{x}_i$ = midpoints $\quad \Delta x = \frac{b-a}{n}$

Another way to approximate the integral is the

Trapezoidal rule

$$\int_a^b f(x)\,dx \approx T_n$$

$$= \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

where $\Delta x = \frac{b-a}{n}$

Note that $T_n =$

$$\left(\underbrace{\frac{f(x_0)+f(x_1)}{2}} + \frac{f(x_1)+f(x_2)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right)\underline{\Delta x}$$

area of trapezoid

Error of approx.

$$E_M = \int_a^b f(x)\,dx - M_n. \qquad E_T = \int_a^b f(x)\,dx - T_n.$$

Error bounds: suppose $|f''(x)| \le K$ for $a \le x \le b$ then

$$|E_M| \le \frac{K(b-a)^3}{24n^2} \qquad |E_T| \le \frac{K(b-a)^3}{12n^2}$$

Similar to trapezoidal rule another rule to approximate integrals is

Simpson's rule

$$\int_a^b f(x)\,dx \approx S_n$$

$$= \frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\Big)$$

where $\Delta x = \frac{b-a}{n}$

Error $\quad E_S = \int_a^b f(x)\,dx - S_n$

Error bound: suppose $|f^{(4)}(x)| \le K$ for $a \le x \le b$ then

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

How to use these approx. in practice?

<u>A naive example</u>

Take $f(x) = x^2$, $1 \leq x \leq 4$ and consider to find $n \geq 1$ s.t. $|E_M| < 0.1$

$$\text{Given } |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

here $K = 2$ as $f''(x) \equiv 2$

$b - a = 4 - 1 = 3$

$$\Rightarrow \quad \frac{2 \cdot 3^3}{24n^2} = \frac{54}{24n^2} \leq 0.1$$

$$\Rightarrow \quad n \geq \sqrt{\frac{54}{24}} \approx 4.7$$

So the smallest n to take is 5.

Improper integrals.

So far we dealt with $\int_a^b f(x)\,dx$ for

- $x \in [a,b]$ a finite interval
- f piecewise continuous, finite

These are called <u>proper integrals</u> (note that this has nothing to do with proper $\frac{P(x)}{Q(x)}$).

In this section we are going to study <u>improper integrals</u>.

Type I

$$\int_a^\infty f(x)\,dx \overset{\text{def}}{:=} \lim_{t\to\infty} \int_a^t f(x)\,dx$$

$$\int_{-\infty}^b f(x)\,dx := \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

$$= \lim_{t\to-\infty} \int_t^a \quad + \lim_{t\to\infty} \int_a^t$$

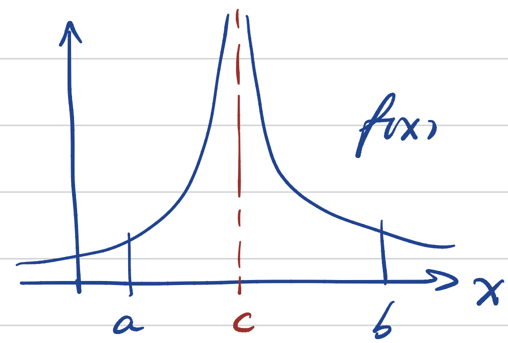
An improper integral is <u>convergent</u> if the above limit exists, otherwise it is <u>divergent</u>

Type II if $f \to \infty$ at some point $c \in [a,b]$

$$\int_a^c f(x)\,dx := \lim_{t\to c^-}\int_a^t f(x)\,dx$$

$$\int_c^b f(x)\,dx := \lim_{t\to c^+}\int_t^a f(x)\,dx$$

$$\int_a^b f(x)\,dx := \int_a^c + \int_c^b$$



Example 1. $\int_0^\infty e^{-x}\,dx$

$$= \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty} -e^{-t}-1 = -1$$

Example 2 $\int_1^\infty \frac{1}{x^p}\,dx \qquad p>1$

$$= \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx$$

$$= \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \lim_{t\to\infty}\frac{1}{p-1}\left(\frac{1}{t^{p-1}}-1\right)$$

$$= \frac{1}{p-1}$$

Note that

- this diverges if $p<1$
- We can replace 1 with any real number a.

Example 3. $\int_0^1 \frac{1}{x-1}\,dx$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^-} \left(\ln|t-1| - \ln|-1| \right)$$

$$= \lim_{t\to 1^-} \ln|t-1|$$

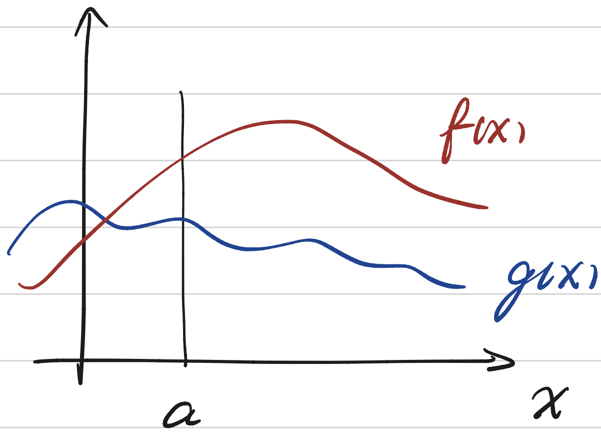
$$= -\infty \qquad \text{diverges.}$$

Comparison test for improper integrals.

Suppose $f(x)$, $g(x)$ are continuous, nonnegative and $f(x) \geq g(x)$ for $x \geq a$.

$$\int_a^\infty f(x)\,dx \text{ conv.} \Rightarrow \int_a^\infty g(x)\,dx \text{ conv.}$$

$$\int_a^\infty g(x)\,dx \text{ div.} \Rightarrow \int_a^\infty f(x)\,dx \text{ div.}$$



To remember

$$\int_a^\infty \frac{1}{x^p}\,dx \quad \begin{array}{ll} \text{conv.} & p>1 \\ \text{div} & p \leq 1 \end{array}$$

Example 4. Show the integral I is divergent

$$I = \int_1^\infty \frac{1+e^{-x}}{x}\,dx$$

Since $\underbrace{\frac{1+e^{-x}}{x}}_{f} > \underbrace{\frac{1}{x}}_{g}$ (as $e^{-x} > 0$) and

$$\int_1^\infty \frac{1}{x}\,dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \Big[\ln|x|\Big]_1^t$$

$$= \lim_{t\to\infty} \ln t = \infty \quad \text{diverges.}$$

By comparison test I diverges.

More examples on comparison test

1. $I = \int_1^\infty \frac{1}{\sqrt{x^6+1}}\,dx$ conv.

Note that for $1 \le x < \infty$

$$0 \le x^6 \le x^6+1$$

$$\Rightarrow 0 \le \sqrt{x^6} \le \sqrt{x^6+1}$$

$$\Rightarrow \underbrace{\frac{1}{\sqrt{x^6+1}}}_{g} \le \frac{1}{\sqrt{x^6}} = \underbrace{\frac{1}{x^3}}_{f}$$

Hence $\int_1^\infty \frac{1}{x^3}\,dx$ conv. $\Rightarrow$ I conv.

2. $I = \int_2^\infty \frac{\cos^2(x)}{x^2}\,dx$ conv.

Note that $0 \le \cos^2(x) \le 1$ for all x

$$\Rightarrow 0 \le \underbrace{\frac{\cos^2(x)}{x^2}}_{g} \le \underbrace{\frac{1}{x^2}}_{f}$$

Hence $\int_2^\infty \frac{1}{x^2}\,dx$ conv. $\Rightarrow$ I conv.

3. $I = \int_3^\infty \frac{1}{x - e^{-x}}\, dx$ div.

Since $0 < e^{-x} < x$ for $x > 3$

$\Rightarrow 0 < x - e^{-x} < x < \infty$

$\Rightarrow 0 < \underbrace{\frac{1}{x}}_{g} \le \underbrace{\frac{1}{x - e^{-x}}}_{f}$

Hence $\int_3^\infty \frac{1}{x}\, dx$ div $\Rightarrow$ I div