MATH231 CALCULUS II

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Contents

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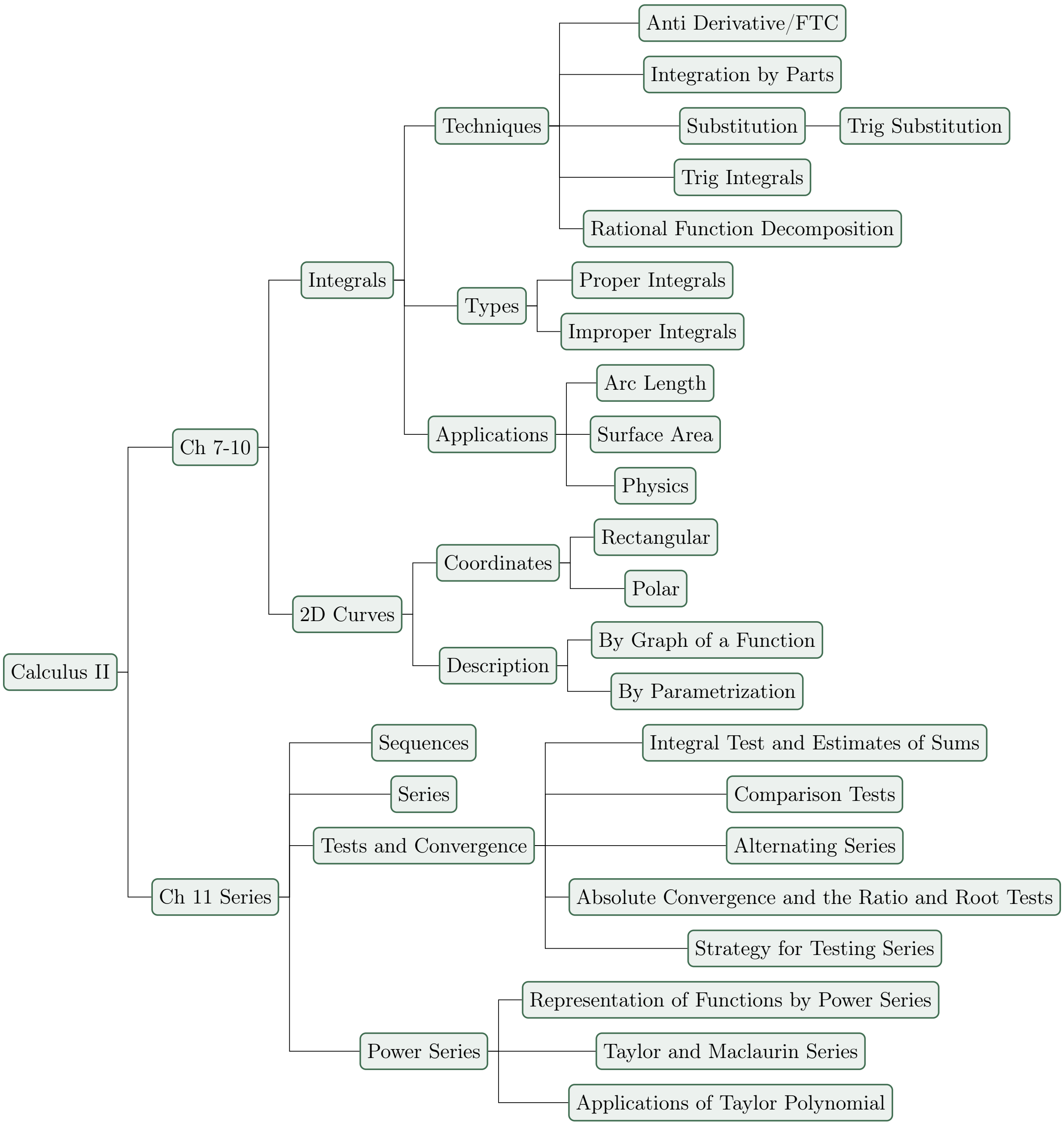
NB:

- §[number] in the margin refers to the section covered in the book
  James Stewart, *Calculus: Early Transcendentals*, 8th edition, 2016.
- "Typo22" means there is a typo in the handwritten notes from Spring 2022.

An overview of Calculus II



I. Calculus I Review

- Examples can be found in the assignment HW0.
- Make sure you understand how to take *derivatives* and compute *limits* and the basic *integrals*, as these concepts are essential for Calculus II.

1. Limits, Derivatives, and Integrals

1. **Limits Laws.**

(i) $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
(ii) $\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
(iii) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$

2. **L'Hôpital's Rule.** When $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, we may apply L'Hôpital's Rule

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Example 1.1 ("Counterexample" to L'Hôpital's Rule)**.**

$$\lim_{x\to 0} \frac{x+\sin x}{x} = DNE.$$

L'Hôpital's Rule does not apply because $\lim_{x\to 0} x+\sin x = 1 \neq 0$. •

3. **Derivatives.**

Definition 1.2.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Derivative Rules

- Product Rule: $\dfrac{d}{dx}[f(x)g(x))] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$,
- Chain Rule: $\dfrac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$.

Derivatives of elementary functions

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x,$$
$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad \frac{d}{dx}\tan x = \sec^2 x.$$

2. Integration

1. **Definition.** Reversing the process of differentiation:

Definition 2.1.
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(a + \Delta x \cdot i) \cdot \Delta x.$$

2. **Integration Laws.**

- $\displaystyle\int (f(x) \pm g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x$,
- $\displaystyle\int c \cdot f(x)\,\mathrm{d}x = c \cdot \int f(x)\,\mathrm{d}x$.

3. **Fundamental Theorem of Calculus.**

Theorem 2.2 (Fundamental Theorem of Calculus (FTC))**.** *(i) If f is continuous on $[a, b]$, then the function F defined by*
$$F(x) = \int_a^x f(t)\,dt$$
is continuous on $[a, b]$ and differentiable on (a, b), and $F'(x) = f(x)$ for all $x \in (a, b)$.
(ii) If f is continuous on $[a, b]$ and F is an antiderivative of f on $[a, b]$, then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

4. **Substitution Rule.** Let $u = g(x)$, then:
$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,du.$$

II. Chapter 7

1. Integration by Parts

[1] §7.1

- **New method for integrals**: Integration by parts provides a technique to evaluate integrals of *products* of functions.
- **Inverse of the product rule**: If $\dfrac{d}{\mathrm{d}x}(uv) = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$, then

$$\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u \tag{IBP}$$

1. **Motivation.** Using the *Fundamental Theorem of Calculus*, we know the antiderivative of basic functions such as x and e^x:

$$\int x \,\mathrm{d}x = \frac{x^2}{2} + C,$$
$$\int e^x \,\mathrm{d}x = e^x + C.$$

However, for functions like $\ln x$ or $\tan x$, a new tool is required: **Integration by Parts**.

2. **Formula Derivation.** **Key idea: product rule for differentiation**

Proof. Let u, v be functions of x. Recall that the product rule for differentiation formula says

$$(uv)' = u'v + uv'.$$

Integrating both sides with respect to x gives

$$\int uv' \,\mathrm{d}x = uv - \int u'v \,\mathrm{d}x.$$

Rearranging gives the formula for integration by parts:

$$\int u \,\mathrm{d}v = uv - \int v \,\mathrm{d}u.$$

□

3. **Example.**

Example 1.1 (Evaluate $\int x \ln x \,\mathrm{d}x$)**.** Take $u = \ln x$ and $dv = x \,\mathrm{d}x$. Then:

$$\begin{aligned}\int x \ln x \,\mathrm{d}x &= x \ln x - \int x \cdot \frac{1}{x} \,\mathrm{d}x \\ &= x \ln x - \int 1 \,\mathrm{d}x \\ &= x \ln x - x + C.\end{aligned}$$

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4. **How to Apply IBP.**

(i) Identify u and $\mathrm{d}v$,
(ii) Compute $\mathrm{d}u$ and v,
(iii) Substitute the above into the IBP formula and evaluate.

Choosing u and v (LIATE Rule): When applying integration by parts, the choice of u and $\mathrm{d}v$ can be guided by the LIATE rule. Take u to be the function that appears earlier in the list.

- **L**ogarithmic functions $\ln x, \log_a x$
- **I**nverse trigonometric functions $\arcsin x, \arctan x$ etc.
- **A**lgebraic functions x^a
- **T**rigonometric functions $\sin x, \cos x$ etc.
- **E**xponential functions e^x, a^x

However, in general, there is no easy way to immediately determine which function to choose as u. In practice, you won't need to remember this rule, as the computation becomes second nature.

5. **More Examples.**

Example 1.2 (Evaluate $\int \arctan x \,\mathrm{d}x$)**.** Let $u = \arctan x$ and $dv = \mathrm{d}x$. Then:

$$\mathrm{d}u = \frac{1}{1+x^2}\,\mathrm{d}x, \quad v = x$$

Substitute into the integration by parts formula:

$$\begin{aligned}\int \arctan x \,\mathrm{d}x &= x \arctan x - \int x \cdot \frac{1}{1+x^2}\,\mathrm{d}x \\ &= x \arctan x - \frac{1}{2}\ln\left(1+x^2\right) + C.\end{aligned}$$

The last step is done by substituting $w = 1 + x^2$:

$$\int x \cdot \frac{1}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}w}{w} = \ln w + C.$$

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As in the example above, there are situations where both *integration by parts* and *substitution* are needed.

Here's another example.

Example 1.3 (Evaluate $\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x$)**.** Take $u = x^2$ and $dv = \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$, so:

$$\mathrm{d}u = 2x\,\mathrm{d}x, \quad v = \sqrt{1+x^2}.$$

Substitute into the integration by parts formula:

$$\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x = \int x^2 \cdot \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2}\,\mathrm{d}x.$$

To compute $\int 2x\sqrt{1+x^2}\ \mathrm{d}x$: Using substitution, let $w = 1 + x^2$, then $\mathrm{d}w = 2x\ \mathrm{d}x$. We have

$$\int 2x\sqrt{1+x^2}\ \mathrm{d}x = \int \sqrt{w}\ \mathrm{d}w = \frac{2}{3}w^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C.$$

So the original integral is:

$$\int \frac{x^3}{\sqrt{1+x^2}}\ \mathrm{d}x = x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2} + C.$$

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You may notices that there is no need to apply the IBP at all. Here's another way to solve the same problem.

Example 1.4 (The same problem with substitution)**.** Using substitution rule, we let $u = 1 + x^2$ so that $\mathrm{d}u = 2x\ \mathrm{d}x$. Then

$$\begin{aligned}
\int \frac{x^3}{\sqrt{1+x^2}}\ \mathrm{d}x &= \int \frac{x^2 \cdot x}{\sqrt{u}}\ \mathrm{d}x = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\ \mathrm{d}u \\
&= \frac{1}{2}\int u^{1/2}\ \mathrm{d}u - \frac{1}{2}\int u^{-1/2}\ \mathrm{d}u \\
&= \frac{1}{3}u^{3/2} - u^{1/2} + C \\
&= \frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C.
\end{aligned}$$

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2. Trigonometric Integrals

[2] §7.2

- **Particular type of integral**:

$$\int \sin^n x \cos^m x\ \mathrm{d}x.$$

- **Tools to use**:
  - Trig formulae and identities (to reduce the powers of $\sin x$ or $\cos x$)
  - Substitution rule
  - Integration by parts
- **To memorize**:

$$\sin^2 x + \cos^2 x = 1,$$

$$\sin(2x) = 2\sin x\cos x, \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

  The other formulae can be derived from the above.

In this section, we are interested in solving integrals of the form:

$$\int \sin^n x \cos^m x\ \mathrm{d}x$$

where n, m are integers. (You will see in the homework that n and m could be noninteger).

We first recall the trigonometric formulae and identities.

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\sin x = \pm\sqrt{\frac{1-\cos(2x)}{2}} \qquad \cos x = \pm\sqrt{\frac{1+\cos(2x)}{2}}$$

It suffices to remember the equations in red; the rest can be derived from them. (Try it).

1. **First Example of Trigonometric Integral.** Before discussing the general approach, let's first look at an example to motivate the method and the overall strategy.

Example 2.1 (Evaluate $\displaystyle\int \sin^3 x \cos^2 x \; \mathrm{d}x$)**.** Apply the substitution rule. Let $u = \cos x$, then $\mathrm{d}u = -\sin x \; \mathrm{d}x$. We get

$$\begin{aligned}
\int \sin^3 x \cos^2 x \; \mathrm{d}x &= \int \sin^2 x \cos^2 x \cdot \sin x \; \mathrm{d}x && \text{(substitution } u = \cos x\text{)} \\
&= \int (1-u^2)u^2(-\,\mathrm{d}u) \\
&= -\int (u^4 - u^2) \; \mathrm{d}u \\
&= -\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C && \text{(Constant } C \text{ indefinite integral)} \\
&= -\left(\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}\right) + C.
\end{aligned}$$

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2. **General Strategy.**

(i) Use trig formulae to reduce the powers of $\sin x$ and $\cos x$'s.
(ii) If n or m is odd, substitution rule is needed. E.g. n is odd, rewrite

$$\int \sin^n x \cos^m x \; \mathrm{d}x = \int \sin^{n-1} x \cos^m x \; \sin x \; \mathrm{d}x.$$

Then take $u = \cos x$ so that $\mathrm{d}u = -\sin x \; \mathrm{d}x$.
(iii) Be careful with the sign!

3. **More Examples.** In the next example, you will see that the half-angle formulae are particularly useful when dealing with even powers of sine and cosine.

Example 2.2 (Evaluate $\int \sin^2 x \cos^2 x \, \mathrm{d}x$)**.** Use $\sin^2 x = \dfrac{1-\cos(2x)}{2}$ and $\cos^2 x = \dfrac{1+\cos(2x)}{2}$ to lowering the order of $\cos x$'s, we get:

$$\begin{aligned}
\int \sin^2 x \cos^2 x \, \mathrm{d}x &= \int \frac{1+\cos(2x)}{2} \cdot \frac{1-\cos(2x)}{2} \, \mathrm{d}x \\
&= \frac{1}{4} \int (1 \cos^2(2x)) \, \mathrm{d}x \\
&= \frac{1}{4} \int 1 - \frac{1+\cos(4x)}{2} \, \mathrm{d}x \\
&= \frac{1}{8} x - \frac{1}{32} \sin(4x) + C.
\end{aligned}$$

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Similarly we can compute

$$\int \tan^n x \sec^m x \, \mathrm{d}x.$$

Recall that $(\tan x)' = \sec^2 x$ and $(\sec x)' = \sec x \tan x$.

Example 2.3 (Evaluate $\int \tan x \sec^4 x \, \mathrm{d}x$)**.**

$$\begin{aligned}
\int \tan x \sec^4 x \, \mathrm{d}x = \int \tan x \sec^2 x \sec^2 x \, \mathrm{d}x &= \int \tan x (1 + \tan^2 x) \cdot \sec^2 x \, \mathrm{d}x \\
&\qquad\qquad (\text{substitution } u = \tan x) \\
&= \int u(1+u^2) \, \mathrm{d}u \\
&= \frac{u^2}{2} + \frac{u^4}{4} + C \qquad (\text{Constant } C \text{ due to indefinite integral}) \\
&= \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C.
\end{aligned}$$

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Example 2.4 (Another way to compute $\int \tan x \sec^4 x \, \mathrm{d}x$)**.** Take $u = \sec x$ then $\mathrm{d}u = 2\sec^2 x \tan x \, \mathrm{d}x$. We have

$$\int \tan x \sec^4 x \, \mathrm{d}x = \int \tan x \sec^2 x \sec^2 x \, \mathrm{d}x = \int \frac{1}{2} \, \mathrm{d}u = \frac{u^2}{4} + C' = \frac{\sec^4 x}{4} + C'.$$

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Note that the two method gives the SAME answer. Here's why

$$\frac{\sec^4 x}{4} + C' = \frac{(1+\tan x)^2}{4} + C' = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + \frac{1}{4} + C'.$$

The constant C and C' satisfies the relation: $C = \dfrac{1}{4} + C'$.

4. **Beyond Calculus II.** Why do we study $\int \sin^n x \cos^m x \, \mathrm{d}x$?

Integrals of the this form are studied for their broad applications in mathematics, physics, and engineering. These integrals appear in Fourier analysis, wave mechanics, and signal processing, where sine and cosine functions serve as fundamental building blocks.

3. Trigonometric substitution

[3] §7.3

- **Particular type of integral**: integral involving square root of quadric polynomials.
- **Tools to use**: Trig substitutions (the idea comes from the trig identities)

	x	Range of θ	$\mathrm{d}x$	$\sqrt{\cdots}$ becomes
$\sqrt{a^2-x^2}$	$a\sin(\theta)$	$-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$	$-a\cos(\theta)\ \mathrm{d}\theta$	$a\cos(\theta)$
$\sqrt{x^2+a^2}$	$a\tan(\theta)$	$-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$	$a\sec^2(\theta)\ \mathrm{d}\theta$	$a\sec(\theta)$
$\sqrt{x^2-a^2}$	$a\sec(\theta)$	$0 \leq \theta \leq \dfrac{\pi}{2}$ or $\pi \leq \theta < -\dfrac{\pi}{2}$	$a\sec(\theta)\tan(\theta)\ \mathrm{d}\theta$	$a\tan(\theta)$

1. **Trig Substitution Rule.** In this section, we consider integrals containing square roots of the form

$$\sqrt{a^2-x^2} \qquad \sqrt{x^2+a^2} \qquad \sqrt{x^2-a^2}.$$

We use trigonometric substitutions:

	x	Range of θ	$\mathrm{d}x$	$\sqrt{\cdots}$ becomes
$\sqrt{a^2-x^2}$	$a\sin(\theta)$	$-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$	$-a\cos(\theta)\ \mathrm{d}\theta$	$a\cos(\theta)$
$\sqrt{x^2+a^2}$	$a\tan(\theta)$	$-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$	$a\sec^2(\theta)\ \mathrm{d}\theta$	$a\sec(\theta)$
$\sqrt{x^2-a^2}$	$a\sec(\theta)$	$0 \leq \theta \leq \dfrac{\pi}{2}$ or $\pi \leq \theta < -\dfrac{\pi}{2}$	$a\sec(\theta)\tan(\theta)\ \mathrm{d}\theta$	$a\tan(\theta)$

Note that we use trigonometric identities to simplify the square root expressions.

For example:

$$\begin{aligned} x = a\sin\theta \implies \sqrt{a^2-x^2} &= \sqrt{a^2-a^2\sin^2\theta} \\ &= \sqrt{a^2\cos^2\theta} \\ &= |a\cos\theta|. \end{aligned}$$

Warning: We have to specify the range of θ so that we can get rid of $|\ |$.

2. **Examples.**

Example 3.1 (Evaluate $\int \sqrt{9-x^2}\,\mathrm{d}x$). Take $x = 3\sin\theta$, with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then

$$\begin{aligned}
\int \sqrt{9-x^2}\,\mathrm{d}x &= \int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,\mathrm{d}\theta = \int |3\cos\theta|\cdot 3\cos\theta\,\mathrm{d}\theta \\
&= \int 9\cos^2\theta\,\mathrm{d}\theta = \int 9\frac{1+\cos\theta}{2}\,\mathrm{d}\theta = \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \\
&= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C = \frac{9}{2}\arcsin\frac{x}{3} + \frac{9}{2}\frac{x}{3}\sqrt{1-\frac{x^2}{3^2}} + C \\
&= \frac{9}{2}\arcsin\frac{x}{3} + \frac{x\sqrt{9-x^2}}{2} + C.
\end{aligned}$$

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Example 3.2 (Evaluate $\int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x$). Take $x = 2\tan\theta$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x &= \int \frac{1}{(2\tan\theta)^2\sqrt{4\tan^2\theta+4}}(2\sec^2\theta)\,\mathrm{d}\theta \\
&= \int \frac{2\sec^2\theta}{4\tan^2\theta\cdot|2\sec\theta|}\,\mathrm{d}\theta = \int \frac{\sec\theta}{4\tan^2\theta}\,\mathrm{d}\theta \\
&= \int \frac{\cos\theta}{4\sin^2\theta}\,\mathrm{d}\theta && \text{(Using substitution: } u=\sin\theta,\ \mathrm{d}u=\cos\theta\,\mathrm{d}\theta) \\
&= \int \frac{1}{4u^2}\,\mathrm{d}u = -\frac{1}{4u} + C && \text{(Back-substitution)} \\
&= -\frac{1}{4\sin\theta} + C = -\frac{\sqrt{4+x^2}}{x} + C.
\end{aligned}$$

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Example 3.3 (Evaluate $\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$). Complete the square: $3-2x-x^2 = -(x^2+2x+1-1)+3 = -(x+1)^2+4$. Take $u = x+1$

$$\begin{aligned}
\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x &= \int \frac{u-1}{\sqrt{4-u^2}}\,\mathrm{d}u = \int \frac{2\sin\theta-1}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,\mathrm{d}\theta \\
&= \int \frac{2\sin\theta-1}{|2\cos\theta|}2\cos\theta\,\mathrm{d}\theta && \text{(Need } -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}) \\
&= \int 2\sin\theta - 1 d\theta = -2\cos\theta - \theta + C. \\
&= -\sqrt{4-u^2} - \arcsin\frac{u}{2} + C && \text{(Back-substitution)} \\
&= -\sqrt{3-2x-x^2} - \arcsin\frac{x+1}{2} + C.
\end{aligned}$$

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3. **Motivation.** Why do we study these type of integrals?

Because they frequently arise in problems related to *arc length* and *surface area* calculations. These integrals help us model and solve real-world geometric problems, such as determining the length of a curve or the area of a surface of revolution. Their importance will become evident as we explore further in Chapter 8.

[4] §7.4

- **Particular type of integral**: integral involving fractional functions.
- **Tools to use**: decomposing *improper* fractional functions into *proper* fractions. (See below for detail).

1. **Rational Functions.**

Definition 4.1. A ***rational function*** is a function of the form $R(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

If $\deg P < \deg Q$, it is *proper*; otherwise, it is *improper*.

Example 4.2.

$$R_1(x) = \frac{1}{x+1}, \qquad R_2(x) = \frac{2x+1}{(x+1)^2}, \qquad R_3(x) = \frac{x^3-3}{(x-7)(x+5)}.$$

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In this section we solve integral of the type $\int R(x)\,\mathrm{d}x$. The strategy is to rewrite $R(x)$ as a sum of simpler national functions. Then use substitution rule.

2. **Examples of integrals of rational functions.**

Example 4.3 (Evaluate $\int \frac{x}{x+4}\,\mathrm{d}x$)**.** Note that

$$\frac{x}{x+4} = \frac{x+4-4}{x+4} = 1 - \frac{4}{x+4}.$$

So

$$\int \frac{x}{x+4}\,\mathrm{d}x = \int 1 - \frac{4}{x+4}\,\mathrm{d}x = (x+4) - 4\ln|x+4| + C.$$

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If Q is a product of distinct linear factors,

$$Q = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n),$$

we take

$$R = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_n}{a_nx+b_n}.$$

Example 4.4 (Evaluate $\int \frac{1}{x^2-4}\,\mathrm{d}x$)**.** The denominator factors as $x^2-4 = (x-2)(x+2)$. Using partial fraction decomposition:

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}, \quad \text{where } A, B \text{ are constants.}$$

The numerator gives

$$(A+B)x + 2(A-B) = 1 \implies A = -B = \frac{1}{4}.$$

Plug this back into the integral, we have

$$\begin{aligned}\int \frac{1}{x^2-4}\,\mathrm{d}x &= \int \left(\frac{1}{4(x-2)} - \frac{1}{4(x+2)}\right)\,\mathrm{d}x \\ &= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C \\ &= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C.\end{aligned}$$

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If Q contains distinct irreducible quadratic factors, take the corresponding quadratic form

$$R = (\text{fraction with linear terms}) + \cdots + \frac{Ax+B}{ax^2+bx+c}.$$

Example 4.5 (Evaluate $\int \frac{5x^2+2}{x(x^2+2x+2)}\,\mathrm{d}x$)**.** To decomposition the fraction, we set

$$\frac{5x^2+2}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}, \quad \text{where } A, B, C \text{ are constants.}$$

The numerator gives

$$Ax^2 + 2ax + 2a + Bx^2 + Cx = 5x^2 + 1 \implies \begin{cases} A+B=5 \\ 2A+C=0 \\ 2A=2 \end{cases} \implies \begin{cases} A=1 \\ B=4 \\ C=-2 \end{cases}.$$

Plug this back into the integral, we have

$$\begin{aligned}\int \frac{5x^2+2}{x(x^2+2x+2)}\,\mathrm{d}x &= \int \frac{1}{x}\,\mathrm{d}x + \int \frac{4x-2}{x^2+2x+2}\,\mathrm{d}x \\ &= \ln|x| + \int \frac{4x-2}{x^2+2x+2}\,\mathrm{d}x = \ln|x| + \int \frac{4x-2}{(x+1)^2+1}\,\mathrm{d}x \\ &\qquad\qquad\qquad\qquad\qquad\qquad (\text{Substitution: } u = x+1) \\ &= \ln|x| + \int \frac{4u-6}{u^2+1}\,\mathrm{d}u = \ln|x| + \int \frac{4u}{u^2+1}\,\mathrm{d}u - \int \frac{6}{u^2+1}\,\mathrm{d}u + C \\ &= \ln|x| + 2\int \frac{1}{w+1}\,\mathrm{d}w - 6\arctan u + C \\ &= \ln|x| + 2\ln|w| - 6\arctan u + C \\ &= \ln|x| + 2\ln|u^2+1| - 6\arctan(x+1) + C \\ &= \ln|x| + 2\ln|x^2+2x+2| - 6\arctan(x+1) + C.\end{aligned}$$

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[5] Typo22

If Q contains a repeated linear factor, say $(ax+b)^r$, include terms of the form:
$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}.$$
Similarly, for a repeated quadratic factor, say $(ax^2+bx+c)^r$, include terms of the form:
$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Example 4.6 (Evaluate $\displaystyle\int \frac{4x}{x^3-x^2-x+1}\,\mathrm{d}x$)**.** To simplify, factorize the denominator:
$$x^3-x^2-x+1 = (x-1)^2(x+1).$$
To decomposition the fraction, we set
$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}, \quad \text{where } A, B, C \text{ are constants.}$$
The numerator gives
$$A(x+1)(x-1) + B(x-1) + C(x-1)^2 = (A+C)x^2 + (B-2C)x + (-A+B+C) = 4x,$$
so
$$A = 1, B = 2, C = -1.$$
Thus, the integral becomes:
$$\begin{aligned}\int \frac{4x}{x^3-x^2-x+1}\,\mathrm{d}x &= \int \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\,\mathrm{d}x \\ &= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C.\end{aligned}$$

•

5. Approximate Integration

- **Approximating integral**
- **Tools to use**: Midpoint Rule, Trapezoidal Rule, and Simpson's Rule.
- No need to memorize the statements. Know how to use them.

1. **Motivation.** In general, it is difficult to compute the antiderivative of a function and apply the Fundamental Theorem of Calculus, even with techniques we learned so far. Therefore, we seek an *approximate* value of the integral.

Recall from Calculus I, the integral is defined as limit of Riemann sums

Definition 5.1.
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x.$$

Since we are interested in an approximate value of the integral, instead of taking $n \to \infty$, we sum over a finite number of intervals:

$$\int_a^b f(x)\, \mathrm{d}x \approx \sum_{i=1}^{n} f(\xi_i)\Delta x.$$

For the finite sum above:

- If $\xi_i = a + \Delta x \cdot (i-1)$, it is a *left* endpoint approximation.
- If $\xi_i = a + \Delta x \cdot i$, it is a *right* endpoint approximation.
- If $\xi_i == a + \Delta x \cdot \dfrac{2i-1}{2}$ is the *midpoint*, it is a midpoint approximation.

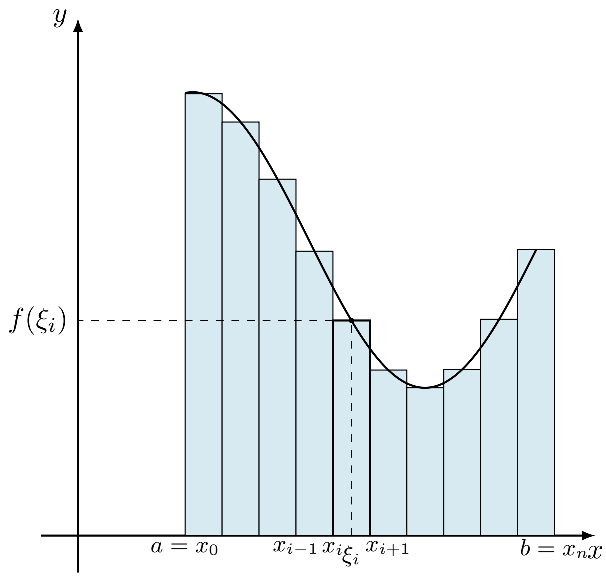

FIGURE 1. Riemann sum

2. **The Midpoint, Trapezoidal and Simpson's Rules.** We usually use the midpoint approximation. The formula is explicitly written as:

Theorem 5.2 (Midpoint rule)**.**

$$\begin{aligned}\int_a^b f(x)\, \mathrm{d}x \approx M_n &= \left(f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})\right)\Delta x \\ &= \sum_{i=1}^{n} f\Big(a + \Delta x \cdot \frac{2i-1}{2}\Big) \cdot \Delta x,\end{aligned}$$

where $\overline{x_i}$ are the midpoints and Δx is the width of each subinterval.

Another way to approximate the integral is the trapezoidal rule:

Theorem 5.3 (Trapezoidal rule).

$$\int_a^b f(x)\,\mathrm{d}x \approx T_n = \frac{\Delta x}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right]$$
$$= \frac{\Delta x}{2}\left(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\right).$$

Note that

$$T_n = \left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right)\Delta x.$$

Each term $\dfrac{f(x_{i-1})+f(x_i)}{2}\Delta x$ is the area of one trapezoid.

Similar to trapezoidal rule, another rule to approxiamte the integral is

Theorem 5.4 (Simpson's Rule).

$$\int_a^b f(x)\,\mathrm{d}x \approx S_n = \frac{\Delta x}{3}\left[f(a) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(b)\right].$$

3. **Error of Approximation.**

$$E_M = \int_a^b f(x)\,\mathrm{d}x - M_n, \qquad E_T = \int_a^b f(x)\,\mathrm{d}x - T_n \qquad E_S = \int_a^b f(x)\,\mathrm{d}x - S_n.$$

Error bounds: For $a \le x \le b$, suppose $|f''(x)| \le K$ for the trapezoidal rule and suppose $|f^{(4)}(x)| \le K$ for Simpson's rule, then:

$$|E_M| \le \frac{K(b-a)^3}{24n^2}, \qquad |E_T| \le \frac{K(b-a)^3}{12n^2} \qquad |E_S| \le \frac{K(b-a)^5}{180n^4}.$$

4. **Example.**

Example 5.5. Let $f(x) = x^2$ on the interval $[1,4]$. Determine the number of subintervals n required such that the error E_M in the Midpoint Rule approximation satisfies

$$|E_M| < 0.1.$$

Solution. The error bound for the Midpoint Rule is given by:

$$|E_M| \le \frac{K(b-a)^3}{24n^2}$$

where $a = 1, b = 4$ and $K = \max_{c\in[1,4]} |f''(c)| = 2$.

Substitute into the error formula we have:

$$|E_M| \le \frac{2\cdot(4-1)^3}{24n^2} = \frac{9}{4n^2} < 0.1 \implies n \ge \sqrt{\frac{9}{0.4}} \approx 4.74.$$

Since n must be an integer to ensure $|E_M| < 0.1$, the smallest number n is 5. •

6. Improper Integrals

- **Improper integrals**: deal with unbounded intervals or functions.
- **Tools to use**: taking limit of a proper integral. E.g.
  - Type I: $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x$
  - Type II: $\displaystyle\int_a^c f(x)\,\mathrm{d}x = \lim_{t\to c}\int_a^t f(x)\,\mathrm{d}x$
- **Comparison test**: (A) continuous, (B) nonnegative, (C) $f(x) \le g(x)$ then

$$\int_a^\infty f(x)\,\mathrm{d}x \text{ converges} \quad \Longrightarrow \quad \int_a^\infty g(x)\,\mathrm{d}x \text{ converges}$$
$$\int_a^\infty f(x)\,\mathrm{d}x \text{ diverges} \quad \Longleftarrow \quad \int_a^\infty g(x)\,\mathrm{d}x \text{ diverges}$$

- **To memorize**:

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x \qquad \begin{cases} \text{converges} & \text{p} > 1 \\ \text{diverges} & \text{p} \le 1 \end{cases}$$

So far, we dealt with $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ for:

- $f(x)$ being piecewise continuous.
- $x \in [a, b]$, a finite interval.

These are called ***proper integrals*** (note that this has nothing to do with proper fractional functions $R(x) = \dfrac{P(x)}{Q(x)}$).

In this section we are going to study ***improper integrals***.

1. **Definition of Improper Integrals.** Improper integrals involve limits where either the interval is infinite or the integrand has a discontinuity. We define two types of improper integrals.

Definition 6.1 (Type I improper integral)**.**

$$\begin{aligned}
\int_a^\infty f(x)\,\mathrm{d}x &:= \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x, \\
\int_{-\infty}^b f(x)\,\mathrm{d}x &:= \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x, \\
\int_{-\infty}^\infty f(x)\,\mathrm{d}x &:= \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x \\
&= \lim_{t\to-\infty}\int_t^a f(x)\,\mathrm{d}x + \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x.
\end{aligned}$$

Definition 6.2. An improper integral is ***convergent*** if the above limit exists, otherwise it is ***divergent***.

Definition 6.3 (Type II improper integral)**.** If $f(x)$ has a discontinuity at some point $c \in [a,b]$, we define

$$\begin{aligned}\int_a^c f(x)\,\mathrm{d}x &:= \lim_{t\to c}\int_a^t f(x)\,\mathrm{d}x,\\ \int_c^b f(x)\,\mathrm{d}x &:= \lim_{t\to c}\int_t^b f(x)\,\mathrm{d}x,\\ \int_a^b f(x)\,\mathrm{d}x &:= \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x\\ &\qquad = \lim_{t\to c}\int_a^t f(x)\,\mathrm{d}x + \lim_{t\to c}\int_t^b f(x)\,\mathrm{d}x.\end{aligned}$$

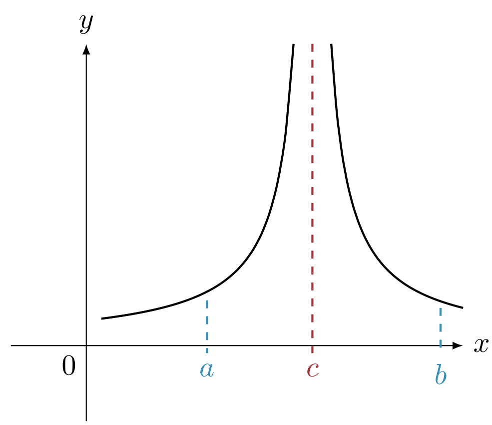

FIGURE 2. Type II indefinite integral

Example 6.4 (Evaluate $\int_0^\infty e^{-x}\,\mathrm{d}x$)**.**

$$\int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-e^{-t}+e^0\right) = -1.$$

•

Example 6.5 (Evaluate $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x, p>1$)**.**

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\frac{1}{p-1}\left(\frac{1}{t^{p-1}}-1\right) = \frac{1}{p-1}.$$

Note that this integral diverges if $p > 1$. •

Example 6.6 (Evaluate $\int_0^1 \frac{1}{x-1}\,\mathrm{d}x$)**.**

$$\int_0^1 \frac{1}{x-1}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,\mathrm{d}x = \lim_{t\to 1^-}\left[\ln|x-1|\right]_0^t = \lim_{t\to 1^-}\ln|t-1| = -\infty.$$

•

2. **Comparison Test for Improper Integrals.** Comparison tests can establish convergence or divergence of improper integrals. Suppose $f(x)$ and $g(x)$ are **continuous, nonnegative** functions and $f(x) \leq g(x)$ for $x \geq a$. Then

$$\int_a^\infty f(x)\,\mathrm{d}x \text{ converges} \quad \Longrightarrow \quad \int_a^\infty g(x)\,\mathrm{d}x \text{ converges}$$

$$\int_a^\infty f(x)\,\mathrm{d}x \text{ diverges} \quad \Longleftarrow \quad \int_a^\infty g(x)\,\mathrm{d}x \text{ diverges}$$

Example 6.7 (Example to remember)**.**

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x \qquad \begin{cases} \text{converges} & \text{p} > 1 \\ \text{diverges} & \text{p} \leq 1 \end{cases}$$

•

Example 6.8 (Show that $I = \displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,\mathrm{d}x$ diverges)**.** We take $f(x) = \dfrac{1+e^{-x}}{x}$ and $g(x) = \dfrac{1}{x}$. Since $f(x) \geq g(x)$ for all $x \geq 1$, and

$$\begin{aligned} \int_1^\infty g(x)\,\mathrm{d}x = \int_1^\infty \frac{1}{x}\,\mathrm{d}x &= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,\mathrm{d}x \\ &= \lim_{t\to\infty} [\ln x]_1^t = \lim_{t\to\infty} (\ln t - \ln 1) = \infty. \qquad \text{(The same computation as Example 6.6)} \end{aligned}$$

By the comparison test, $I = \displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,\mathrm{d}x$ also diverges.

•

Example 6.9 (Apply the comparison test to $I = \displaystyle\int_1^\infty \frac{1}{\sqrt{x^6+1}}\,\mathrm{d}x$)**.** Let $g(x) = \frac{1}{\sqrt{x^6+1}}$ and compare it to $f(x) = \frac{1}{\sqrt{x^6}}$.

Note that for $x \geq 1$,

$$\sqrt{x^6} \leq \sqrt{x^6+1} \quad \Longrightarrow \quad \frac{1}{\sqrt{x^6+1}} \leq \frac{1}{\sqrt{x^6}}.$$

Moreover, $p = 3 > 1$ so the integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^6}}\,\mathrm{d}x = \int_1^\infty x^{-3}\,\mathrm{d}x$ converges.

By comparison test, the integral I also converges. •

Example 6.10 (Apply the comparison test to $I = \displaystyle\int_2^\infty \frac{\cos^2 x}{x^2}\,\mathrm{d}x$)**.** Let $g(x) = \frac{\cos^2 x}{x^2}$ and compare it to $f(x) = \frac{1}{x^2}$.

Note that for $0 \leq \cos^2 x \leq 1$ for all x, so

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}.$$

Moreover, $p = 2 > 1$ so the integral $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^2}}$ converges. By comparison test, the integral I also converges. •

Example 6.11 (Apply the comparison test to $I = \displaystyle\int_3^\infty \frac{1}{x - e^{-x}}\,\mathrm{d}x$)**.** Let $f(x) = \frac{1}{x-e^{-x}}$ and compare it to $g(x) = \frac{1}{x}$.

Note that for $0 < e^{-x} < x$ $x > 3$, so

$$0 < x - e^{-x} \leq x < \infty \qquad \Longrightarrow \qquad 0 < \frac{1}{x} < \frac{1}{x - e^{-x}}.$$

Moreover, $p = 1$ so the integral $\displaystyle\int_3^\infty \frac{1}{x}$ diverges. By comparison test, the integral I also diverges. $\bullet$

III. Chapter 8

In this chapter, we will explore the applications of the techniques we have learned so far. We will apply integration methods to problems involving arc length, surface area, and other geometric quantities.

1. Arc Length

Let's first consider how to compute arc length

1. **Derivation of the Arc Length Formula.**

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|, \text{ where } |P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x.$$

Suppose $y = f(x)$, when taking limits as $\Delta x \to 0$, the expression $\dfrac{\Delta y}{\Delta x} \to f'$. This suggest

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

The arc length L of a curve $y = f(x)$ for $x \in [a, b]$, where f is continuous, is given by:

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

Similarly, if the curve is parameterized as $x = g(y)$ for $y \in [c, d]$, then the arc length is:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\,\mathrm{d}y.$$

Example 1.1. Let $y = e^x$ for $x \in [0, 2]$

$$L = \int_0^2 \sqrt{1 + (e^x)^2}\,\mathrm{d}x.$$

Alternatively, using $x = \ln y$, we rewrite the integral:

$$L = \int_1^{e^2} \sqrt{1 + \frac{1}{y^2}}\,\mathrm{d}y.$$

•

Example 1.2. Let $y^2 + x^2 = 1$ (Unit Circle) Using symmetry, take the upper half of the circle $(y \geq 0)$. Then

$$y = \sqrt{1 - x^2}, \quad -1 \leq x \leq 1,$$

and

$$L = \int_{-1}^{1} \sqrt{1 + \left(-\frac{x}{\sqrt{1 - x^2}}\right)^2}\,\mathrm{d}x = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\,\mathrm{d}x = \cdots = \pi.$$

•

2. **Arc Length Function.** Given a curve $y = f(x)$, the arc length function $s(x)$ from $x = a$ to $x = b$ is:

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, \mathrm{d}t.$$

Example 1.3. Let $f(x) = x^2 - \dfrac{\ln x}{8}$ for $x \in [1, \infty)$. Then

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)^2}\, \mathrm{d}x = \int_1^x \sqrt{1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2}}\, \mathrm{d}t \\ &= \int_1^x 2t + \frac{1}{8t}\, \mathrm{d}t = t^2 + \frac{\ln t}{8}\Big|_1^x = x^2 + \frac{\ln x}{8} - 1. \end{aligned}$$

•

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2. Surface Area of Revolution

1. **Derivation of the Surface Area of Revolution formula.** A surface of revolution is formed by rotating a curve about a line (e.g. the x- or y-axis).

To derive the area, recall the surface area of a cylinder is $2\pi Rl$.

If we take infinitesimal line segments $\mathrm{d}s$, the small piece is approximately a cylinder.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x)\, \mathrm{d}s = \int_a^b 2\pi f(x)\, \mathrm{d}s.$$

This expression needs to be rewritten in terms of x to make it computable.

Recall from last section $s(x) = \displaystyle\int_a^x \sqrt{1 + (f'(t))^2}\, \mathrm{d}t$. This tells us

$$ds = \sqrt{1 + (f'(t))^2}\, \mathrm{d}x.$$

We can rewrite the surface areas as follows.

For a curve $y = f(x)$ rotated about the x-axis, the surface area A is:

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, \mathrm{d}x.$$

If the curve $x = g(y)$ is rotated about the y-axis, then:

$$A = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, \mathrm{d}y.$$

2. **Examples.**

Example 2.1. Let $y = \sqrt{9 - x^2}$ for $x \in [-2, 2]$. Rotating about the x-axis, compute the surface area:

$$\begin{aligned}
\mathrm{d}s = \sqrt{1 + (y')^2}\,\mathrm{d}x &= \sqrt{1 + \left(-\frac{x}{\sqrt{9 - x^2}}\right)^2}\,\mathrm{d}x = \sqrt{1 + \frac{x^2}{9 - x^2}}\,\mathrm{d}x \\
&= \sqrt{\frac{9 - x^2 + x^2}{9 - x^2}}\,\mathrm{d}x = \frac{3}{\sqrt{9 - x^2}}\,\mathrm{d}x.
\end{aligned}$$

$$\begin{aligned}
A &= \int_{-2}^{2} 2\pi f(x)\,\mathrm{d}s = \int_{-2}^{2} 2\pi\sqrt{9 - x^2} \cdot \frac{3}{\sqrt{9 - x^2}}\,\mathrm{d}x \\
&= \int_{-2}^{2} 6\pi\,\mathrm{d}x = 6\pi(2 - (-2)) = 24\pi.
\end{aligned}$$

•

Example 2.2. Let $y = e^x$ for $x \in [0, 2]$. Rotating about the y-axis, set up the surface area of revolution

$$A = \int_0^2 2\pi(20 - e^x)\sqrt{1 + (e^x)^2}\,\mathrm{d}x.$$

•

3. Applications

1. **Hydrostatic Pressure and Force.** The force F exerted by a fluid on a submerged plate is given by

$$F = mg = \rho g A d$$

where ρ is the fluid density, g is gravitational acceleration, A is the surface area and d is the depth/width.

Example 3.1. Compute the force on one end of a submerged cylinder with radius 3 and depth 10. Here we have $\rho = \rho(y)$ and $d = 7 - y$ is a constant. Since the infinitesimal area ΔA is given by

$$\Delta A = 2\sqrt{9 - y_i^2}\Delta y,$$

taking limits as $\Delta y \to 0$, we have d

$$\mathrm{d}A = 2\sqrt{9 - y^2}\,\mathrm{d}y.$$

Substitute into the force, we have

$$F = \int_{-3}^{3} \rho g(7 - y)\,\mathrm{d}A = \int_{-3}^{3} (7 - y)\rho g\sqrt{9 - y^2}\,\mathrm{d}y.$$

•

2. **Moments and Center of Mass.** For a lamina with density ρ, the total mass of the lamina is:

$$M = \int \rho(x, y)\,\mathrm{d}A.$$

The moment about the x-axis is:

$$M_x = \rho\int_a^b f(x) \cdot \frac{f(x)}{2}\,\mathrm{d}x.$$

The moment about the y-axis is:

$$M_y = \rho \int_a^b x f(x)\,\mathrm{d}x.$$

The center of mass $(\overline{x}, \overline{y}) = (\frac{M_y}{M}, \frac{M_x}{M})$. (Notice the swap in x and y).

Example 3.2. Find the center of mass of a semicircular plate, suppose ρ is a constant:

$$\overline{y} = \frac{1}{\rho A} \cdot \rho \int_{-r}^{r} \frac{1}{2} f(x)^2\,\mathrm{d}x = \frac{1}{\frac{1}{2}\pi r^2} \int_{-r}^{r} \frac{1}{2}(r^2 - x^2)\,\mathrm{d}x \qquad \text{(Use symmetry)}$$

$$= \frac{2}{\pi r^2} \int_0^r r^2 - x^2\,\mathrm{d}x = \frac{2}{\pi r^2} \left[r^2 x - \frac{x^3}{3} \right]_0^r = \frac{4r}{3\pi}.$$

•

IV. Chapter 10

In this chapter, we introduce an alternative method for representing curves on the 2D plane. Within this framework, we will also explore the reformulated expressions for arc length and the surface area of a revolution.

1. Parametric Equations and Calculus

1. **Parametrization.**

- **New Concept**: Parametrization
- **Calculus of Parametric Curves**:

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

$$L = \int_{t_1}^{t_2} \mathrm{d}s,$$

$$S = \int_{t_1}^{t_2} \mathrm{d}A = \int_{t_1}^{t_2} 2\pi R\,\mathrm{d}s.$$

Consider a particle moving along a curve as follows:

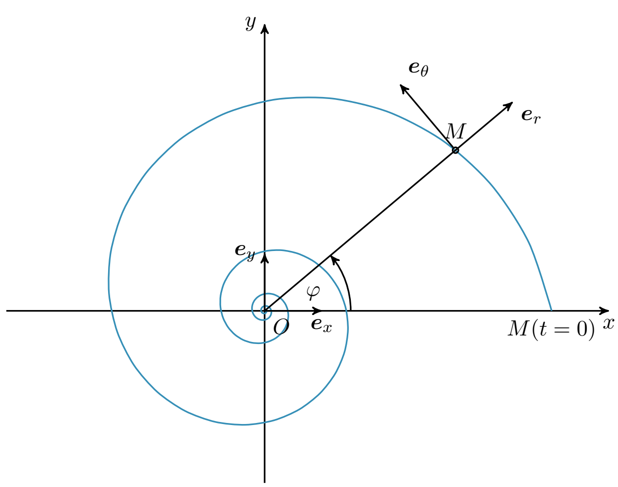

Figure 3. A plot of the Golden Spiral.

The curve cannot be expressed as $y = f(x)$ because it fails the vertical line test. However, if we introduce a parameter t representing the angle, then the Golden Spiral can be represented in polar coordinates as

$$r(t) = ae^{bt}, \text{ where } b = \frac{\ln(\phi)}{\pi} \text{ and } \phi = \frac{1+\sqrt{5}}{2} \text{ is the golden ratio.}$$

Definition 1.1. The system of equations

$$x = f(t),$$
$$y = g(t).$$

is called a ***parametric equation/parametrization***, and the resulting curve $\mathscr{C}$ is called a ***parametric curve***. We call t a ***parameter***.

Example 1.2 (Unit Circle)**.**

$$x = \cos t, 0 \leq t < 2\pi$$
$$y = \sin t.$$

Note that $\cos^2 t + \sin^2 t = 1$, which implies $x^2 + y^2 = 1$. Thus, these equations parametrize a unit circle. •

Note that parameterization is not unique

Example 1.3 (Circle with Opposite Orientation)**.**

$$x = \sin 2t, , 0 \leq t < \pi$$
$$y = \cos 2t.$$

These parametrize the same circle as in Example 1.1, but with opposite orientation. •

Example 1.4 (Circle of Radius r Centered at (a, b))**.** The equation of the circle is

$$(x - a)^2 + (y - b)^2 = r^2.$$

The corresponding parametrization is given by:

$$x = a + r\cos t, 0 \leq t < 2\pi$$
$$y = b + r\sin t.$$

Here r represents the scaling and a, b represents the translation. •

Example 1.5 (Parabola)**.** The curve $x = 6 - 4y^2$ can be parametrized directly as:

$$x = 6 - 4t^2, t \in \mathbb{R}$$
$$y = t.$$

•

2. **Calculus with Parametrization.** Calculus techniques can be applied to analyze parametrized curves. For a curve parametrized as $x = f(t)$ and $y = g(t)$, we can compute the following.

- **Tangent slope:** when $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$, the chain rule says $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Remark 1.6. *(i)* $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$ *is required to take the quotient.*

(ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \frac{\mathrm{d}y}{\mathrm{d}t} \neq 0$ *corresponds to the vertical line* $y = ct$.

(iii) $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0, \frac{\mathrm{d}y}{\mathrm{d}t} = 0$ *corresponds to the horizontal line* $x = ct$.

We now introduce the substitution $x = x(t),\ \mathrm{d}x = \dfrac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t$:

- **Infinitesimal line element**

$$\mathrm{d}s = \sqrt{(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \sqrt{(1 + \left(\frac{\mathrm{d}y/\ \mathrm{d}t}{\mathrm{d}x/\ \mathrm{d}t}\right)^2}\ \frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t.$$

- **Arc length:**

$$L = \int_a^b \sqrt{(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t.$$

- **Surface area (for revolution):**

$$S = \int_a^b 2\pi R\ \mathrm{d}s, \quad \text{where}\ \ \mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t.$$

3. **Example.**

Example 1.7. Consider the parametrization

$$\begin{aligned} x &= \cos^2 t, 0 \le t \le \frac{\pi}{4}. \\ y &= \sin^2 t. \end{aligned}$$

The *infinitesimal line element* is given by

$$\begin{aligned} \mathrm{d}s &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ \mathrm{d}t = \sqrt{(-2\cos t \sin t)^2 + (2\sin t\cos t)^2}\ \mathrm{d}t \\ &= \sqrt{4\cos^2 t \sin^2 t + 4\sin^2 t\cos^2 t}\ \mathrm{d}t = \sqrt{8\sin^2 t\cos^2 t}\ \mathrm{d}t \\ &= \sqrt{2}\sin(2t)\ \mathrm{d}t. \end{aligned}$$

Hence the *arc length* is given by

$$L = \int\ \mathrm{d}s = \int_0^{\frac{\pi}{4}} \sqrt{2}\sin(2t)\ \mathrm{d}t = -\frac{\sqrt{2}}{2}\cos(2t)\Big|_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}.$$

•

The *surface area* is given by

$$\begin{aligned} A &= \int_0^{\frac{\pi}{4}} 2\pi \cdot \sin^2(t) \cdot \sqrt{2}\sin(2t)\ \mathrm{d}t = 2\sqrt{2}\pi \int_0^{\frac{\pi}{4}} \sin^2(t)\cdot\sin(2t)\ \mathrm{d}t \\ &= 2\sqrt{2}\pi \int_0^{\frac{\pi}{4}} \frac{1-\cos(2t)}{2}\cdot\sin(2t)\ \mathrm{d}t \\ &= \sqrt{2}\pi \int_0^{\frac{\pi}{4}} \sin(2t)\ \mathrm{d}t + \sqrt{2}\pi \int_0^{\frac{\pi}{4}} \frac{1}{2}\sin(4t)\ \mathrm{d}t \\ &= \sqrt{2}\pi \left[-\frac{\cos(2t)}{2}\right]_0^{\frac{\pi}{4}} - \sqrt{2}\pi\left[-\frac{\cos(4t)}{8}\right]_0^{\frac{\pi}{4}} \\ &= \sqrt{2}\pi\left(\frac{1}{2} - \left(\frac{1}{8}+\frac{1}{8}\right)\right) = \sqrt{2}\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi\sqrt{2}}{4}. \end{aligned}$$

2. Polar Coordinates

In this section, we will explore an alternative method for representing points on the Euclidean plane.

- **New Concept**: Polar Coordinates
- **Calculus with Polar Coordinates**:

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$
$$L = \int_{\theta_1}^{\theta_2} \mathrm{d}s,$$
$$S = \int_{\theta_1}^{\theta_2} \mathrm{d}A = \int_{\theta_1}^{\theta_2} 2\pi R\,\mathrm{d}s.$$

1. **Curves in Polar Coordinates.** In polar coordinates, a point (r, θ) is represented as:

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

Conversely:

$$r = \sqrt{x^2 + y^2},$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Example 2.1. Converting $(1, 1)$ to Polar Coordinates Given $(x, y) = (1, 1)$:

$$r = \sqrt{1^2 + 1^2} = \sqrt{2},$$
$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

Thus, the polar coordinates are $(\sqrt{2}, \pi/4)$. •

Some curves, such as circles or spirals, can be expressed as simple functions in terms of polar coordinates

$$F(r, \theta) = 0.$$

We will explore how to compute arc length and surface area using polar coordinates.

2. **Examples.**

Example 2.2 (circle centered at the origin)**.** In rectangular coordinates, a circle of radius R centered at the origin is given by $x^2 + y^2 = R^2$. In polar coordinates, this is given by $r = R, \theta \in [0, 2\pi]$.

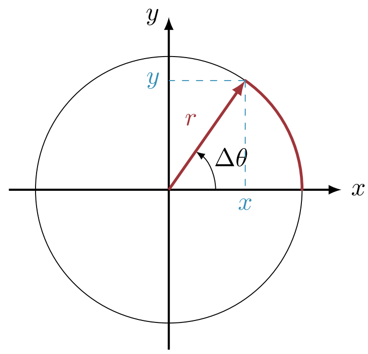

FIGURE 4. Circle of radius r

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Example 2.3. Consider a circle centered at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$, then $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$. We convert this into polar coordinates by plug in $x = r\cos\theta, y = r\sin\theta$:

$$
\begin{aligned}
x^2 + (y - \frac{1}{2})^2 = \frac{1}{4} &\iff r^2\cos^2\theta + (r\sin\theta - \frac{1}{2})^2 = \frac{1}{4} \\
&\iff r^2\cos^2\theta + r^2\sin^2\theta - r\sin\theta + \frac{1}{4} = \frac{1}{4} \\
&\iff r^2 - r\sin\theta = 0.
\end{aligned}
$$

Since $r > 0$, this equation is equivalent to $r = \sin\theta$.

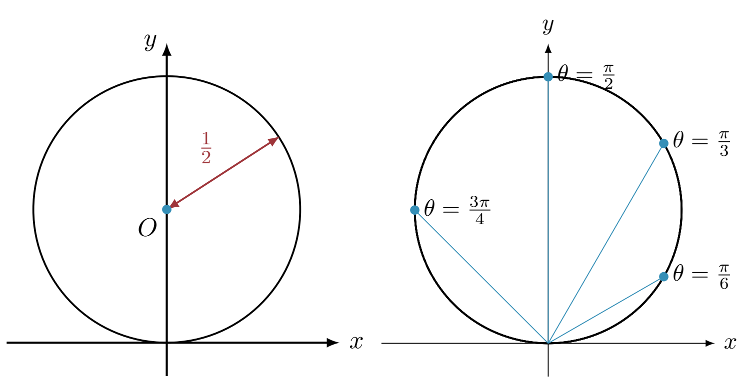

FIGURE 5. Circle of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$

Note that with the points winding around the *full* circle once when $\theta \in [0, \pi]$. •

Example 2.4 (cardioid)**.** The polar curve $r = 1 + \sin\theta$ gives a cardioid.

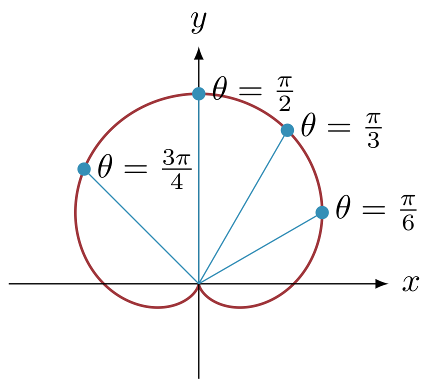

FIGURE 6. Cardioid $r = 1 + \sin\theta$

•

3. TANGENT

Now consider a polar curve of the form $r = f(\theta)$. Then,

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta.$$

The derivative of the parametrization with respect to θ is given by

$$\frac{dx}{d\theta} = f'\cos\theta - f\sin\theta, \quad \frac{dy}{d\theta} = f'\sin\theta + f\cos\theta.$$

We can compute its tangent by the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'\cos\theta + f\sin\theta}{f'\sin\theta - f\cos\theta}$$

Example 3.1. Let $r = 1 + \sin\theta$. Compute $\frac{dy}{dx}$.

$$x = (1+\sin\theta)\cos\theta, \quad y = (1+\sin\theta)\sin\theta.$$

Differentiating,

$$\frac{dx}{d\theta} = \cos\theta \cdot \sin\theta - (1+\sin\theta)\cos\theta, \quad \frac{dy}{d\theta} = \cos\theta \cdot \cos\theta - (1+\sin\theta)\sin\theta.$$

Thus,

$$\frac{dy}{dx} = \frac{\cos\theta + 2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta - \sin\theta} = \frac{\cos\theta + \sin(2\theta)}{\cos(2\theta) - \sin\theta}.$$

Note that:

$$\lim_{\theta\to\frac{3\pi}{2}^-} \frac{dy}{dx} = \lim_{\theta\to\frac{3\pi}{2}^-} \frac{\cos\theta + \sin(2\theta)}{\cos(2\theta) - \sin\theta} = \lim_{\theta\to\frac{3\pi}{2}^-} \frac{-\cos\theta + 2\cos(2\theta)}{-2\sin(2\theta) - \cos\theta} = -\infty. \qquad \text{(L'H)}$$

This means the tangent blows up at $\frac{3\pi}{2}$. •

4. Area

For $r = f(\theta)$, the area of a sector is approximately

$$\Delta A \approx \frac{1}{2} r^2 \Delta\theta.$$

Using a Riemann sum,

$$A \approx \sum_{i=1}^{n} \frac{1}{2} \left[f(\xi_i)\right]^2 \Delta\theta \qquad \Longrightarrow \qquad A = \int_{\theta_1}^{\theta_2} \frac{1}{2} \left[f(\xi)\right]^2 \, \mathrm{d}\theta.$$

Example 4.1. Find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2 \, \mathrm{d}\theta = \frac{1}{2}\int_{0}^{\pi/4} \cos^2(2\theta) \, \mathrm{d}\theta && \text{(Integrand is an even function)} \\ &= \int_0^{\pi/4} \frac{1+\cos(4\theta)}{2} \, \mathrm{d}\theta = \frac{1}{2}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{4}. \end{aligned}$$

[7]
Typo22

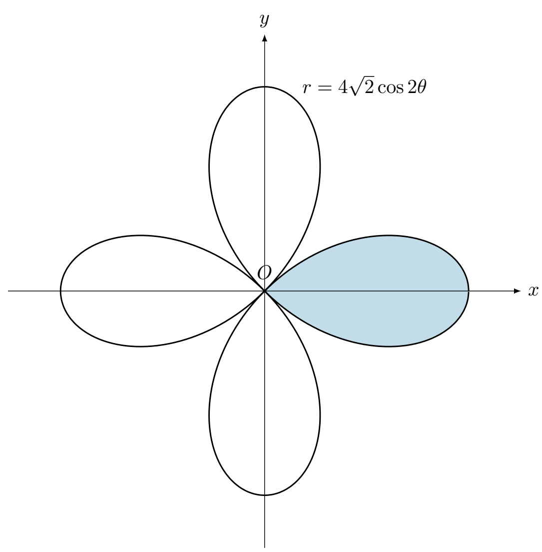

Figure 7. Four-leaf $r = \cos(2\theta)$

5. Arc Length

We compute the infinitesimal line element $\mathrm{d}s$ as follows:

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta.$$

Note that

$$\begin{aligned}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= (r')^2\cos^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta\\ &\quad + (r')^2\sin^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta\\ &= (r')^2 + r^2, \qquad \text{where } r' = f'(\theta).\end{aligned}$$

So

$$\mathrm{d}s = \sqrt{\Big(f(\theta)\Big)^2 + \Big(f'(\theta)\Big)^2}\,\mathrm{d}\theta.$$

The arc length of a polar curve is given by:

$$L = \int \mathrm{d}s = \int_{\theta_1}^{\theta_2} \sqrt{\Big(f(\theta)\Big)^2 + \Big(f'(\theta)\Big)^2}\,\mathrm{d}\theta.$$

Example 5.1. Find the arc length of $r = \theta$, $0 \le \theta \le 1$.

$$L = \int_0^1 \sqrt{\theta^2+1}\,\mathrm{d}\theta.$$

Using the substitution $\theta = \tan x$, $d\theta = \sec^2 x\,\mathrm{d}x$, we have

$$\begin{aligned}L &= \int_0^{\pi/4} \sec x \sec^2 x\,\mathrm{d}x && \text{(IBP with } u = \sec x,\ v = \tan x)\\ &= \sec x\tan x - \int_0^{\pi/4} \tan x\cdot\tan x\sec x\,\mathrm{d}x = \sec x\tan x - \int_0^{\pi/4}\tan^2 x\sec x\,\mathrm{d}x\\ &= \sec x\tan x - \int_0^{\pi/4}(\sec^2 x - 1)\sec x\,\mathrm{d}x = \sec x\tan x - L + \int_0^{\pi/4}\sec x\,\mathrm{d}x\end{aligned}$$

This implies

$$\begin{aligned}2L &= \sec x\tan x + \int_0^{\pi/4}\sec x\,\mathrm{d}x\\ &= \sec x\tan x + \ln|\sec x + \tan x|\Big|_0^{\pi/4} && \text{(From Calculus I)}\\ &= \frac{1}{2}\Big(\sqrt{2} + \ln\Big(1+\sqrt{2}\Big)\Big).\end{aligned}$$

•

V. Chapter 11: Infinite Sequences and Series

In this chapter, we introduce sequences and series. We will focus on how to test for convergence using tools like the integral test, comparison tests, and the ratio and root tests. We will also discuss alternating and absolutely convergent series, along with strategies for analyzing them. Finally, we will explore power series and Taylor and Maclaurin series, and how to use them to approximate functions.

1. Sequences

- **New Concept**: Sequence, limit of sequence, sequence converges/diverges
- **Example to memorize**:
  - $$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} 0 & \text{if } p > 0 \\ \infty & \text{if } p < 0 \end{cases}$$
  - $$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DNE} & \text{if } r < -1 \end{cases}$$

Definition 1.1. A ***sequence*** is an infinite list of members written in a definite order. We denote the sequence as $\{a_1, a_2, \dots, a_n, \dots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example 1.2 (sequences)**.**

(i) $\{1, 2, 3, 9, \dots\}$.
(ii) $\{7, 1, 8, 2, 8, \dots\}$.

•

Some sequences can be defined by giving a formula for the n-th term a_n.

Example 1.3 (sequences given by formulae)**.**

(i) $a_n = \frac{1}{n}$, $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.
(ii) $a_n = (-1)^{n-1}$, $\{a_n\} = \{-1, 1, -1, 1, \dots\}$.
(iii) $a_n = \frac{1}{3^n}$, $\{a_n\} = \{\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots\}$.

•

Some sequences may not have a simple/explicit defining equation.

Example 1.4 (sequences without explicit formulae)**.**

(i) $a_n =$ the digit in the n-th decimal place of π
(ii) The Fibonacci sequence: $a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$
$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$

Remark 1.5. *A sequence can be thought of as a function f defined only on the natural numbers. Therefore, we can examine properties such as the graph and convergence. For example,*

$$\lim_{n\to\infty} a_n = 0.$$

Definition 1.6. A sequence has ***limit*** L if for any $\epsilon > 0$, there is an N such that if $n > N$, then $|a_n - L| < \epsilon$. (We write this as

$$\forall \epsilon > 0, \exists N \text{ s.t. if } n > N, \text{ then } |a_n - L| < \epsilon.$$

We say a_n ***converges*** to L and denote it as $\lim_{n\to\infty} = L$.

Remark 1.7 (Intuition)**.** $\lim_{n\to\infty} a_n = \infty$ *means that for every positive number M, there is an integer N such that if $n > N$, then $a_n > M$.*

[8] figure needed

Example 1.8 (limit of a sequence)**.**

(i) $\lim_{n\to\infty} \frac{n}{n+1} = 0 = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$

$$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} 0 & \text{if } p > 0 \\ \infty & \text{if } p < 0 \end{cases}$$

(ii)

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DNE} & \text{if } r < -1 \end{cases}$$

1. **Limit Laws for Sequences.**

- **Tools for evaluate limits**: limit law, squeeze theorem,
- **Continuous function commutes with limit**:

$$f \text{ continuous} \implies \lim_{n\to\infty} f(a_n) = f\Big(\lim_{n\to\infty} a_n\Big).$$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then

(i) $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.
(ii) $\lim_{n\to\infty} (ca_n) = c \lim_{n\to\infty} a_n$, c constant.
(iii) $\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$.
(v) $\lim_{n\to\infty} (a_n)^p = \Big(\lim_{n\to\infty} a_n\Big)^p$.

Theorem 1.9 (Squeeze Theorem)**.** *If $b_n \leq a_n \leq c_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} a_n = L$.*

Theorem 1.10.

(i) If $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

(ii) Continuous function commutes with limit If f *is* ***continuous****, then* $\lim_{n\to\infty} a_n = L$ *implies* $\lim_{n\to\infty} f(a_n) = f(L)$.

Example 1.11. (i) $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty}\frac{1}{n}\right) = \sin(0) = 0$.

(ii) $\lim_{n\to\infty} \frac{\ln(n+2)}{\ln(1+4n)}$. Note that this is same as

$$\lim_{x\to\infty} \frac{\ln(x+2)}{\ln(1+4x)} = \lim_{x\to\infty} \frac{\frac{1}{x+2}}{\frac{4}{1+4x}} = \lim_{x\to\infty} \frac{1+4x}{4(x+2)} = 1. \qquad \text{(L'Hôpital's rule)}$$

(iii)

$$\lim_{x\to\infty} \left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty} e^{\left(1+\frac{1}{n}\right)^n} = e^{\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n} = e.$$

(Can apply L'Hôpital's rule to compute the limit)

•

2. Series

- **New concept**: series, partial sum, converges/diverges
- **Examples to memorize**:
  - Geometric series
$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1, \\ \infty & \text{if } r \geq 1 \\ \text{DNE} & \text{if } r \leq -1. \end{cases}$$
  - Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- **Tools to study series**: Series laws, converges $\sum a_n \implies a_n \to 0$.

Definition 2.1. We call $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ a ***series***, and

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \ldots + a_N$$

the ***partial sum***.

Remark 2.2. *Note that* S_N *is itself a sequence. So it makes sense to talk about whether* S_n *converges or not.*

Definition 2.3. The series $\sum a_n$ is called ***convergent*** if its partial sum is convergent. Otherwise, $\sum a_n$ is called ***divergent***.

Example 2.4 (Geometric Series)**.** Consider $a_n = r^n$, where r is the common ratio.

$$\begin{aligned} a_0 &= 1, & S_0 &= a_0 = 1, \\ a_1 &= r, & S_1 &= a_0 + a_1 = 1 + r, \\ a_2 &= r^2, & S_2 &= a_0 + a_1 + a_2 = 1 + r + r^2. \\ & & &\vdots \\ & & S_N &= 1 + r + r^2 + \ldots + r^N. \end{aligned}$$

Let $R_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \ldots + r^N$, then

$$\begin{aligned} rR_N &= r + r^2 + \ldots + r^{N+1}, \\ R_N - rR_N &= 1 - r^{N+1}, \\ R_N &= \frac{1 - r^{N+1}}{1 - r}, \quad \text{for } r \neq 1. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1, \\ \infty & \text{if } r \geq 1 \\ \text{DNE} & \text{if } r \leq -1. \end{cases}$$

Also, note that $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ because $\sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r^n$. Thus, the starting point matters. •

[9] Typo22

Example 2.5. Compute $\sum_{n=1}^{\infty} 2^{2n} \cdot 6^{1-n}$ using the formula from the previous example.

$$\begin{aligned} \sum_{n=1}^{\infty} 2^2 \cdot 6^{1-n} &= \sum_{n=1}^{\infty} 4^n \cdot 6 \cdot \left(\frac{1}{6}\right)^n = 6 \cdot \sum_{n=1}^{\infty} \left(\frac{4}{6}\right)^n = 6 \cdot \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \qquad (\text{Here } r = \frac{2}{3}) \\ &= 6 \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 6 \cdot 2 = 12. \end{aligned}$$

•

Example 2.6 (Harmonic Series)**.** $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$

Partial sums:

$$\begin{aligned}
S_2 &= 1+\frac{1}{2},\\
S_4 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\\
&> 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1+\frac{2}{2},\\
S_8 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\\
&> 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=1+\frac{3}{2},\\
S_{2^n} &= 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2^n}>1+\frac{n}{2}\xrightarrow{n\to\infty}\infty.
\end{aligned}$$

Hence, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. •

Example 2.7 (Telescope series)**.** Check that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$

We note that:

$$\begin{aligned}
\frac{1}{n(n+1)} &= \frac{1}{n}-\frac{1}{n+1},\\
S_N &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\
&= 1-\frac{1}{n+1}\xrightarrow{n\to\infty}1.
\end{aligned}$$

Hence, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converges to 1. •

Theorem 2.8. *If* $\displaystyle\sum_{n=1}^{\infty}a_n$ *and* $\displaystyle\sum_{n=1}^{\infty}b_n$ *converge, and c is a constant, then*

(i) $\displaystyle\sum_{n=1}^{\infty}a_n\pm b_n=\sum_{n=1}^{\infty}a_n\pm\sum_{n=1}^{\infty}b_n.$

(ii) $\displaystyle\sum_{n=1}^{\infty}ca_n=c\sum_{n=1}^{\infty}a_n.$

Example 2.9. Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\frac{1}{2^n}.$

We have

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{1-\frac{1}{2}}-1=2-1=1\quad\text{and}\quad\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$$

So the original series converges to $3\cdot1+1=4$. •

Theorem 2.10. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. By definition, we know $\lim_{n\to\infty} a_n = S_n = L$ for some real number L. This implies

$$\begin{aligned} &\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = L \\ \implies &\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0. \end{aligned}$$

$\square$

Corollary 2.11. *If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Example 2.12 (diverges series)**.**

(i) $\sum_{n=1}^{\infty} (-1)^n$.

(ii) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$.

(iii) $\sum_{n=1}^{\infty} \frac{n}{n+1}$.

3. Integral Test and Estimates

- **New tool for testing convergency**: **The Integral Test** f positive, continuous, decreasing for $x \geq 1$, and let $a_n = f(n)$. Then:

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\, \mathrm{d}x \text{ converges.}$$

We have been computing the exact value of a series so far for some special cases. However, in general, this is quite difficult. In those cases, we are interested in finding an estimate.

1. **The Integral Test.**

Theorem 3.1. *Suppose* $f(x) > 0$ *is a continuous and decreasing function for* $x \geq 1$, *and let* $a_n = f(n)$. *Then:*

$$\sum_{n=1}^{\infty} a_n \textit{ converges} \iff \int_1^{\infty} f(x)\, \mathrm{d}x \textit{ converges.}$$

[10] Image needed

Moreover:

$$\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\, \mathrm{d}x.$$

The error of this estimate is given by

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n.$$

We have

$$\int_{N+1}^{\infty} f(x)\,\mathrm{d}x \le R_N \le \int_{N}^{\infty} f(x)\,\mathrm{d}x.$$

Example 3.2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Let $f(x) = \dfrac{1}{x^2}$. For $x \ge 1$, $f(x)$ is continuous, positive, and decreasing. Then $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ converges implies $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. •

Example 3.3.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \le 1. \end{cases}$$

Recall $p > 1$, $\int_1^{\infty} \dfrac{1}{x^p}\,\mathrm{d}x$ converges. For $p \le 1$, it diverges. Apply the integral test. •

Example 3.4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges •

Let $f(x) = \dfrac{1}{x^2+1} > 0$. For $x \le 1$, we check f is continuous and decreasing:

$$f'(x) = -(x^2+1)^{-2} \cdot 2x < 0, x \ge 1.$$

Apply the integral test as follows:

$$\int_0^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \lim_{t\to\infty} \left(\arctan x|_1^t \right) = \lim_{t\to\infty} \left(\arctan t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \le \infty.$$

So the series converges.

4. Comparison Tests

- **New tool for testing convergency**:
  - **The Comparison Test**: $0 \le a_n \le b_n$ for all $n \ge N$, then
$$\sum b_n \text{ converges} \quad \Longrightarrow \quad \sum a_n \text{ converges},$$
$$\sum a_n \text{ diverges} \quad \Longrightarrow \quad \sum b_n \text{ diverges}.$$
  - **The Limit Comparison Test**: $0 \le a_n \le b_n$ for all $n \ge N$, and
$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ where } 0 < c < \infty.$$
Then
$$\sum b_n \text{ converges} \quad \Longleftrightarrow \quad \sum a_n \text{ converges}.$$

The idea of the comparison test for sequences is similar to that for integrals.

1. **The Comparison Test.**

Theorem 4.1. *Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms, and $a_n \le b_n$ for all $n \ge N$. Then:*

- *If $\sum b_n$ converges, then $\sum a_n$ converges.*
- *If $\sum a_n$ diverges, then $\sum b_n$ diverges.*

[11] revise

Example 4.2. Show that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges.

Note that $2n^2+4n+3 \le 2n^2$ for $n \le 1$. This implies
$$a_n := \frac{5}{2n^2+4n+3} \le \frac{5}{2n^2} =: b_n.$$

Therefore,
$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges}.$$

•

Example 4.3. Show that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

Note that $\ln n > 1$ for $n < e$. This implies
$$a_n := \frac{\ln n}{n} \le \frac{1}{n} =: b_n, n \ge 3.$$

Therefore,
$$\sum_{n=3}^{\infty} \frac{1}{n} \text{ diverges.} \quad \Longrightarrow \quad \sum_{n=3}^{\infty} a_n \text{ converges.} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

•

2. **The Limit Comparison Test.**

Theorem 4.4. *Suppose* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms, and:*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ where } 0 < c < \infty.$$

Then $\sum a_n$ *converges if and only if* $\sum b_n$ *converges.*

Example 4.5. Show that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

Take $b_n = \dfrac{1}{2^n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1.$$

Apply the limit comparison test, we conclude that $\sum b_n$ converges implies $\sum \dfrac{1}{2^n - 1}$ converges. •

Example 4.6. Show that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ diverges.

Take $b_n = \dfrac{2n^2}{n^{5/2}} = \dfrac{2}{\sqrt{n}}$ (this is the dominant part). Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{2}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{1/2}}{2\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{2}{2} = 1.$$

Apply the limit comparison test, we conclude that $\sum b_n$ diverges implies $\sum a_n$ diverges. •

5. Alternating Series

- **New Concept**: alternating series
- **The Alternating Series Test**:

  a_n positive, decreasing, limit goes to zero, then $\implies \displaystyle\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

- **Example to memorize**: The alternating harmonic series $\sum \dfrac{(-1)^{n+1}}{n}$ converges.

So far, we have studied series with positive terms. In this section, we will study series whose terms are alternating series, such as

- $\sum \dfrac{(-1)^{n+1}}{n}$ (alternating harmonic series).
- $\sum (-1)^n a_n$, where $a_n > 0$ and terms alternate in sign.

1. **The Alternating Series Test.** The following theorem tells us how to determine if an alternating series converges or diverges.

Theorem 5.1. *Given an alternating series* $\sum_{n=0}^{\infty}(-1)^n a_n$, *if*

(i) $a_n > 0$,
(ii) (decreasing a_n*)* $a_{n+1} \le a_n$ *for all* n,
(iii) $\lim_{n\to\infty} a_n = 0$,

then $\sum_{n=0}^{\infty}(-1)^n a_n$ *converges.*

[12] to complete

Proof. $\square$

Moreover, from the above proof, we see that if $\lim_{n\to\infty} a_n$ diverges, the series also diverges. So the **diverges test still holds**.

Example 5.2 (Alternating Harmonic Series)**.** Show that $\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}$ converges. We first check that the alternating series test applies:

- $a_n = \frac{1}{n} > 0.$
- $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ for all n.
- $\lim\limits_{n\to\infty} a_n = 0.$

Thus, the alternating series test tells us that the series converges. •

Example 5.3. Show that $\sum_{n=0}^{\infty}\frac{(-1)^n n^2}{n^3+1}$ converges. We first check that the alternating series test applies:

- $a_n = \frac{n^2}{n^3+1} > 0.$
- $a_{n+1} < a_n$ for $n \le 2$ because the function $f(x) = \frac{x^2}{x^3+1}$ is decreasing (not obvious, we compute the derivative):
$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0, \text{ where } x > \sqrt[3]{2}.$$
- $$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2}{n^3+1} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0.$$

Apply the alternating series test to $\sum_{n=2}^{\infty}(-1)^n a_n$ (because we need $n \le 2$), we conclude that $\sum_{n=2}^{\infty}(-1)^n a_n$ converges. So $\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + \sum_{n=2}^{\infty}(-1)^n a_n$ also converges. •

2. **Estimating alternating series.**

Theorem 5.4 (Alternating series estimation)**.** *Given* $\sum_{n=0}^{\infty}(-1)^n a_n$, $a_n > 0$ *satisfying*

- $a_n > 0$.
- $a_{n+1} \leq \frac{1}{n} = a_n$ *for all* n.
- $\lim_{n\to\infty} a_n = 0$.

Then $|R_n| = |S - S_n| \leq a_{n+1}$.

Proof. ∎ **[13]** revise

Last time: alternating series test. This time: absolute convergence and more tests.

6. Absolute Convergence

- **New Concept**: absolute convergence and conditional convergence.
- **New tool for testing convergency**: **The Ratio/Root Test**:

$$L_{ratio} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, \qquad L_{root} = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

Then:
  - If $L < 1$, the series converges absolutely.
  - If $L > 1$, the series diverges.
  - If $L = 1$, the test is inconclusive.

Definition 6.1. A series $\sum a_n$ is called ***absolutely convergent*** if the series of absolute values $\sum |a_n|$ converges.

Definition 6.2. A series $\sum a_n$ is called ***conditionally convergent*** if it converges but is not absolutely convergent.

Note that absolute convergence is stronger than convergence

$$\text{If } \sum |a_n| \text{ converges, then } \sum a_n \text{ converges.}$$

Proof. Observe that:

$$-a_n \leq |a_n| \leq a_n \quad \Longrightarrow \quad 0 \leq a_n + |a_n| \leq 2|a_n|.$$

We call $A_n = a_n + |a_n|$, $B_n = 2|a_n|$. By the comparison test, $\sum B_n$ converges implies $\sum |A_n|$ converges. Then

$$\sum a_n = \sum A_n - \sum |a_n| < \infty.$$

∎

1. **Examples.**

Example 6.3. The series $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent because:

- $\sum \frac{1}{n}$ diverges (harmonic series).
- $\sum \frac{(-1)^{n+1}}{n}$ converges by the alternating series test.

•

Example 6.4. The series $\sum \frac{(-1)^{n+1}}{n^2}$ is conditionally convergent because:

- $\sum \frac{1}{n^2}$ converges by the p-series test with $p = 2 > 1$.
- $\sum \frac{(-1)^{n+1}}{n^2}$ converges by the alternating series test.

•

2. **Ratio Test and Root Test.**

Theorem 6.5 (Ratio Test). *Given a series $\sum a_n$, let:*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then:

- *If $L < 1$, the series converges absolutely.*
- *If $L > 1$, the series diverges.*
- *If $L = 1$, the test is inconclusive.*

Theorem 6.6 (Root Test). *Given a series $\sum a_n$, let:*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then:

- *If $L < 1$, the series converges absolutely.*
- *If $L > 1$, the series diverges.*
- *If $L = 1$, the test is inconclusive.*

Remark 6.7. *(i) Note that we have absolute convergence.*
(ii) $L = 1$ case examples

$$\sum_{n=1}^{\infty} \frac{1}{n} \textit{ diverges and } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \textit{ converges.}$$

But in both cases L (for the ratio test) is given by

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

(iii) Prototype for both tests are the geometric series:

- $L_{ratio} = \lim_{n\to\infty} \left|\dfrac{r^{n+1}}{r^n}\right| = \lim_{n\to\infty} |r| = |r|$.
- $L_{root} = \lim_{n\to\infty} \sqrt[n]{|r|^n} = \lim_{n\to\infty} |r| = |r|$.

Recall that $|r| < 1$ corresponds to convergent series; and that $|r| > 1$ corresponds to divergent series.

3. **Examples.**

Example 6.8. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^2}{(2(n+1)-1)!}}{\dfrac{n^2}{(2n-1)!}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n)n^2} = 0 < 1.$$

Hence the series converges absolutely by ratio test. •

Example 6.9. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}}{(n+1)^2+1}}{\dfrac{(-1)^n}{n^2+1}} \right| = \lim_{n\to\infty} \frac{n^2+1}{2n^2+2n+2} = 1.$$

The ratio test has no conclusion. •

Instead, one can use the alternating series test to conclude that this series converges and the comparison test (with $A_n = \dfrac{1}{n^2+1} \leq B_n = \dfrac{1}{n^2}$) for absolute convergence.

Example 6.10. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$

$$L = \lim_{n\to\infty} \left| \sqrt[n]{\left(\frac{3n+1}{4-2n}\right)^{2n}} \right| = \lim_{n\to\infty} \left(\frac{3n+1}{4-2n}\right)^2 = \lim_{n\to\infty} \frac{9n^2+6n+1}{4n^2-16n+16} = \frac{9}{4} > 1.$$

Hence the series diverges absolutely by root test. • [14] Typo22

Example 6.11. $\displaystyle\sum_{n=4}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^2}{(2(n+1)-1)!}}{\dfrac{n^2}{(2n-1)!}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n)n^2} = 0 < 1.$$

Hence the series converges absolutely by ratio test. Hence the series converges absolutely by root test. •

For strategy of choosing converges tests, see "Supplementary Resources" on course webpage.

7. Power Series

Definition 7.1. A ***power series centered at*** a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Here, x is a variable, and c_n are coefficients.

Example 7.2. Take $a = 0$, then

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

This is a polynomial with infinitely many terms. Moreover if $c_n = 1$ for all n, then

$$f(x) = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n.$$

This is a geometric series, we know it converges when $|x| < 1$. •

The above example shows that a power series may converge for some values of x and diverge for others. We use convergence tests to determine this.

Example 7.3. When does $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$ converges?

Using ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right| = \lim_{n\to\infty} \frac{|x-3|}{1+\frac{1}{n}} = |x-3|.$$

Hence the series converges absolutely when $|x-3| < 1$ (i.e. $2 < x < 4$) and $|x-3| > 1$ (i.e. $x < 2$ or $x > 4$) diverges by ratio test.

Now we analysis the boundary cases:

- When $x = 2$, $\sum a_n = \frac{(-1)^n}{n}$ converges.
- When $x = 4$, $\sum a_n = \frac{1}{n}$ diverges.

Conclusion: the series converges when $x \in [2, 4)$. •

Theorem 7.4. *For a power series* $\sum c_n(x-a)^n$*, there are three possibilities:*

(i) The series converges only at $x = a$.
(ii) The series converges for all x.
(iii) There exists $R > 0$ *such that the series converges for* $|x-a| < R$ *and diverges for* $|x-a| > R$.

Definition 7.5. The number R is called the ***radius of convergence***. The ***interval of convergence*** is the interval that consists of all values of x for which the power series converges.

Example 7.6. For the series $\displaystyle\sum \frac{(x-3)^n}{n}$, the radius of convergence is $R = 2$, and the interval of convergence is $[2, 4)]$. •

Example 7.7. Compute the radius of converges and integral of converges for $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^n}$.

Using ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+1}}}{\frac{n(x+2)^n}{3^n}} \right| = \lim_{n\to\infty} \frac{|x+3|}{3\left(1+\frac{1}{n}\right)} = \frac{|x+2|}{3}.$$

The series converges when $\frac{|x+2|}{3} < 1$, so the radius of converges is $R = 3$.

Now we analysis the boundary cases:

- When $x = -5$, $\sum a_n = \frac{(-1)^n n}{3}$ diverges.
- When $x = 1$, $\sum a_n = \frac{n}{3}$ diverges.

So the interval of convergence is $x \in (-5, 1)$. •

8. Functions as Power Series

In this section, we will learn how to represent some functions as power series. An application of this technique is the approximation of certain integrals that do not have elementary antiderivatives.

We start by discussing how to find the power series representation through substitution, integration, and differentiation.

Recall we have seen that

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n, \quad \text{for } |u| < 1.$$

Example 8.1. Find the power series for $\frac{1}{1+x^2}$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^2 = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad \text{for } |x| < 1.$$

Take $u = (-x^2)$, then $|u| = |-x^2| = x^2 < 1$. So we have $|x| < 1$. •

Example 8.2. Find the power series for $\frac{1}{2+x}$.

$$\frac{1}{2+x} = \frac{1}{2}\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\frac{1}{1-\left(-\frac{x}{2}\right)}$$

(If $|x| < 2$, then $|u| = \left|\frac{x}{2}\right| < 1$, we may use the Equation of $\frac{1}{1-u}$.)

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x)^n.$$

•

1. Term-by-Term Differentiation and Integration.

Theorem 8.3. *If* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is differentiable within* $(a-R, a+R)$.

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} nc_n \frac{(x-a)^{n+1}}{n+1}.$$

Proof. One can prove this by computing the differentiation:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \quad \text{for } |x-a| < R.$$

and the integration:

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}, \quad \text{for } |x-a| < R.$$

□

2. Examples.

Example 8.4.

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}x}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{\mathrm{d}x}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1} \text{ when } |x| < 1.$$

•

Example 8.5. Recall by the Fundamental Theorem of Calculus,

$$\ln(1+x) - \ln(1+0) = \int_0^x \frac{1}{1+t}\,\mathrm{d}t.$$

This implies (note that $\ln(1+0) = 0$) for $|x| < 1$,

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1-(-t)}\,\mathrm{d}t = \int_0^x \sum_{n=0}^{\infty}(-t)^n\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty}\int_0^x (-1)^n t^n\,\mathrm{d}t = \sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{n+1}}{n+1}\right]_{t=0}^{x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}.\end{aligned}$$

Thus:

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}, \quad \text{for } |x| < 1.$$

•

Example 8.6. Another solution for solving $\ln(1+x)$.

$$\ln(1+x) = \int \sum_{n=0}^{\infty} (-1)^n x^n \, \mathrm{d}x \qquad \text{(Take } u = -t\text{, need } |u| = |-t| < 1\text{, i.e. } |t| < 1\text{)}$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^n \, \mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \int x^n \, \mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C, \quad \text{when } |x| < 1.$$

To determine C, take $x = 0$, we have

$$\ln(1+0) = 0 = C.$$

•

Example 8.7 ($\arctan(x)$)**.** the Fundamental Theorem of Calculus,

$$\arctan(x) - \arctan(0) = \int_0^x \frac{1}{1+t^2} \, \mathrm{d}t.$$

This implies (note that $\arctan(0) = 0$) for $|x| < 1$,

$$\begin{aligned} \arctan(x) &= \int_0^x \frac{1}{1+t^2} \, \mathrm{d}t = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n \, \mathrm{d}t \\ &= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n} \, \mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{2n+1}}{2n+1} \right]_{t=0}^{x} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \text{when } |x| < 1. \end{aligned}$$

Thus:

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \text{for } |x| < 1.$$

•

Example 8.8. Another solution for solving $\arctan(x)$.

$$\begin{aligned} \arctan(x) &= \int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty} (-x^2)^n \, \mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} \, \mathrm{d}x \\ &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \, \mathrm{d}x + C, \quad \text{when } |x| < 1. \end{aligned}$$

To determine C, take $x = 0$, we have

$$\arctan(0) = 0 = C.$$

•

9. Taylor and Maclaurin Series

Theorem 9.1. *Suppose the function $f(x)$ has a power series representation at a given by:*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n, \quad |x-a| < R.$$

Then $c_n = \frac{f^{(n)}(a)}{n!}$.

Proof. We compute derivatives:

$$f'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1},$$
$$f''(x) = \sum_{n=2}^{\infty} c_n n(n-1)(x-a)^{n-2}.$$

Taking $x = a$ yields

$$\begin{aligned} f'(a) &= c_1, && \text{(C_1 is the only non-vanishing term)} \\ f''(a) &= 2!c_2, \\ &\vdots \\ f^{(n)}(a) &= n!c_n. \end{aligned}$$

□

Definition 9.2. We define the ***Taylor series*** of f centered at a as:

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R.$$

When $a = 0$, this is called the ***Maclaurin series***:

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Example 9.3 ($f(x) = e^x$ at $a = 0$)**.** The derivatives of $f(x)$ are given by

$$f^{(n)}(x) = e^x \quad \text{for all } n.$$

So $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. We compute the radius of convergence:

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = 0 < 1.$$

The radius of convergence is ∞. •

Example 9.4 ($f(x) = \sin(x)$ at $a = 0$)**.** The derivatives of $f(x)$ are given by

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x).$$

Higher order derivatives repeat. So

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(Note that sin is an odd function). The radius of convergence $R = \infty$, as

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0 < 1.$$

•

Example 9.5 ($f(x) = \cos(x)$ at $a = 0$)**.** Check that

$$f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f'''(x) = \sin(x), \quad f^{(4)}(x) = \cos(x).$$

So

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{even function.}$$

The radius of convergence is again $R = \infty$. •

10. Application

1. **Estimating Integrals.** Let's consider a particular integral:

Example 10.1. Compute $\int_0^\infty e^{-x^2}\,\mathrm{d}x$.

Step 1. Get the Maclaurin series of $\int_0^\infty e^{-x^2}\,\mathrm{d}x$.. Recall the Maclaurin series expansion for e^{-x^2}:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

Integrating term by term:

$$\begin{aligned}\int e^{-x^2}\,\mathrm{d}x &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} + C.\end{aligned}$$

Evaluate at $x = 0$ and $x = 1$ gives

$$\int_0^1 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1^{2n+1}}{2n+1} + C - \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{0^{2n+1}}{2n+1} + C \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}.$$

To estimate the integral, we use the first five terms:

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475.$$

Using the alternating series estimation theorem, the error is bounded by:

$$|R| < |a_6| = \frac{1}{(2 \cdot 6 + 1)!} < 0.001.$$

•

2. **Approximating Functions.** We denote the N-th degree Taylor polynomial of f at a as:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

For example, $N = 1$

$$T_1(x) = f(a) + f'(a)(x-a).$$

This is the tangent line of the function f.

The error in this approximation is given by:

$$R_N(x) = f(x) - T_N(x).$$

Taylor's inequality states:

$$|R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!},$$

where M is an upper bound on $|f^{(N+1)}(x)|$ for x in the interval of interest.

Example 10.2. Let $f(x) = \sqrt[3]{x}$ with $N = 2$ at $a = 8$.

We compute:

$$f(x) = \sqrt[3]{x}, \quad f'(x) = \frac{1}{3}x^{-2/3}, \quad f''(x) = -\frac{2}{9}x^{-5/3}.$$
$$f(8) = \sqrt[3]{8} = 2, \quad f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}, \quad f''(8) = -\frac{2}{9} \cdot 8^{-5/3} = -\frac{2}{9} \cdot \frac{1}{32} = -\frac{1}{144}.$$

Then the second-degree Taylor polynomial is:

$$T_2(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

To estimate the error, we use Taylor's inequality:

$$|R_2(x)| \leq \frac{M|x-8|^3}{3!},$$

where M is an upper bound on $|f^{(3)}(x)| = \left|\frac{10}{27}x^{-8/3}\right|$ for x in the interval of interest.

For x near $a = 8$, the maximum value of $|f^{(3)}(x)|$ occurs at $x = 7$, so

$$|f^{(3)}(x)| \leq |f^{(3)}(7)| \leq \frac{10}{27} \cdot 7^{-8/3} < 0.0021.$$

Finally, the error is bounded by:

$$|R_2(x)| < \frac{M}{3!} < 0.0004.$$

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11. List of Common Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty.$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty.$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad |x| < \infty.$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1.$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$