

Last time Calc I Review

This time Integration by parts

Motivation

using Calc I knowledge (FTC) we know the antiderivative of  $x$  is  $\frac{1}{2}x^2 + C$

$$\cos x \quad \sin x$$

It is natural to ask what is the antiderivative of  $\ln x$ ?

$$\tan x ?$$

To solve this problem we need a new tool.

Recall product rule

$$(uv)' = u'v + uv'$$

if we integrate on both sides w.r.t.  $x$ .

$$\begin{aligned}\Rightarrow \int uv \, dx &= \int uv' \, dx + \int u'v \, dx \\ &= \int u \, dv + \int v \, du\end{aligned}$$

rearrange

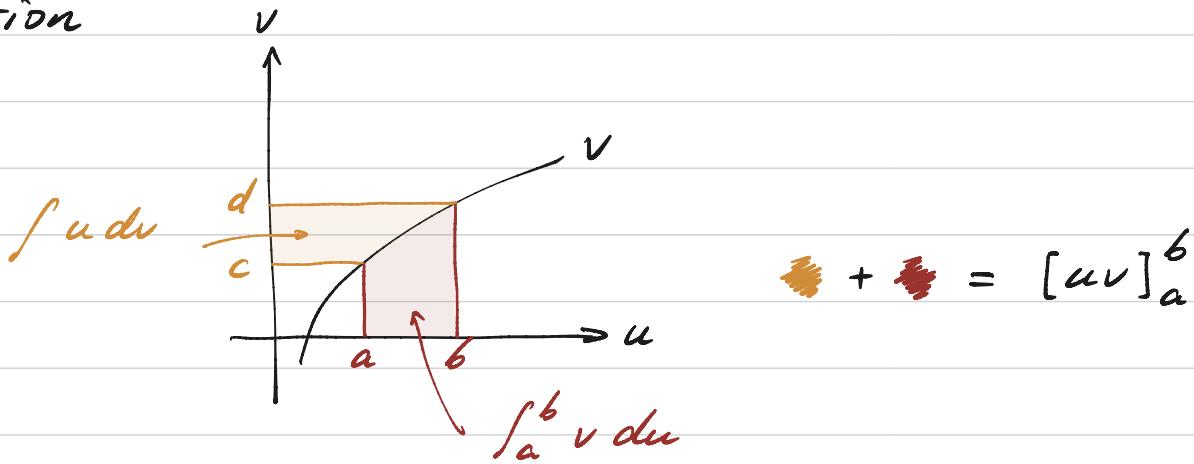
$$\Rightarrow \boxed{\int u \, dv = uv - \int v \, du}$$

Example 1 compute  $\int \ln x \, dx$

take  $u = \ln x$   $v = x$

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \, d\ln x \\ &= x \ln x - \int x \frac{1}{x} \, dx \\ &= x \ln x - x + C\end{aligned}$$

Intuition



N.B. There's a list of choices called LIATE rule  
tells which function to choose as  $u$  first  
However in general there's no easy way to tell  
immediately which function to take as  $u$  or  $v$ .

L  $\ln$   $\log$

I  $\arcsin$   $\arccos$   $\arctan$  ...

A algebraic  $x$   $1+x^2$

T  $\sin$   $\cos$  ...

E  $e^x$

Example 2  $\int \arctan x \, dx$

$$\begin{aligned}
 &= x \arctan x - \int x \, d \arctan x \\
 &= x \arctan x - \int \frac{x}{1+x^2} \, dx \\
 &\quad \text{use substitution law} \\
 &= \frac{1}{2} \int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2} \ln(1+x^2) + \tilde{C} \\
 &= x \arctan x - \frac{1}{2} \ln(1+x^2) + C
 \end{aligned}$$

Example 3  $\int x e^x \, dx = \int \frac{x}{u} \frac{de^x}{v}$

$$\begin{aligned}
 &= x e^x - \int e^x \, dx \\
 &= x e^x - e^x + C
 \end{aligned}$$

Step

1. Choose function  $u$ .

2. The choice of  $v$  depends on  $u$ .

That is, in order to match the integration by parts formula, say we are computing  $\int f(x) dx$

Then we claim  $dv = \frac{f(x)}{u(x)} dx$ , so that

$$\int f \, dx = \int u \cdot \frac{f}{u} \, dx = \int u \, dv$$

Example 4

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{1+x^2}} dx &= \int x^2 \frac{x}{\sqrt{1+x^2}} dx \\
 &= \frac{\frac{1}{2} d(x^2+1)}{\sqrt{1+x^2}} = d\sqrt{1+x^2} \\
 &= \int \frac{x^2}{u} d\sqrt{1+x^2} \\
 &= x^2 \sqrt{1+x^2} - \int \sqrt{1+x^2} \frac{dx^2}{u} = d(x^2+1) \\
 &= x^2 \sqrt{1+x^2} - \frac{2}{3} (1+x^2)^{\frac{3}{2}} + C
 \end{aligned}$$

NB. There's more than one way to solve Ex 4.

Example 4'

$$\int \frac{x^3}{\sqrt{1+x^2}} dx$$

substitution rule  $u = 1+x^2 \quad du = 2x dx$   
 $x^3 dx = x^2 \cdot (x dx) = \frac{1}{2} (u-1) du$

$$\begin{aligned}
 \int \frac{x^3}{\sqrt{1+x^2}} dx &= \int \frac{\frac{1}{2}(u-1) du}{\sqrt{u}} \\
 &= \frac{1}{2} \int \underbrace{\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}}_{u^{\frac{1}{2}} - u^{-\frac{1}{2}}} du \\
 &= \frac{1}{3} u^{\frac{3}{2}} - u^{\frac{1}{2}} + C \\
 &= \frac{1}{3} (1+x^2)^{\frac{3}{2}} - (1+x^2)^{\frac{1}{2}} + C
 \end{aligned}$$

Last time Integration by parts

$$\int u \, du = uv - \int v \, du$$

This time: we focus on a particular type of integral

*integers*

$$\int \sin^m x \cos^n x \, dx \quad \otimes$$

Tools to solve  $\otimes$

- (A) substitution rule
- (B) trig identity
- (C) double angle formulae

$$\cos^2 x + \sin^2 x = 1$$

$$\sin(2x) = 2 \sin x \cdot \cos x$$

$$\begin{aligned} \cos(2x) &= \cos^2 x - \sin^2 x &= 2 \cos^2 x - 1 \\ &&= 1 - 2 \sin^2 x \end{aligned}$$

! Remember these equations

Note that (D) implies

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

We'll start by looking at the following examples.

Example 1.  $\int \sin^3 x \cos^2 x \, dx$

$$= \int \underbrace{\sin^2 x}_{1-\cos^2 x} \cos^2 x \underbrace{\sin x \, dx}_{\text{substitution } u = \cos x}$$

$$= \int (1-u^2) u^2 (-du) \quad du = -\sin x \, dx$$

$$= \int u^4 - u^2 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

In general, if m or n is odd we can use substitution rule  $u = \cos x, \sin x$ .

Example 2  $\int \cos^2 x \sin^2 x \, dx$

$$= \int \frac{1+\cos 2x}{2} \cdot \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{4} \int 1 - \cos^2 2x \, dx$$

$$= \frac{1}{4} \int 1 - \frac{1+\cos 4x}{2} \, dx$$

$$= \frac{1}{8} \int 1 - \cos 4x \, dx$$

$$= \frac{1}{8} x - \frac{1}{32} \sin 4x + C$$

If m and n both are even we can use trig identity and double angle formulae.

Similarly we can compute

$$\int \tan^m x \sec^n x \, dx$$

Tools to solve  $\oplus$

(A) substitution rule

(B) trig identity  $\sec^2 x = 1 + \tan^2 x$  !

(C) double angle formulae

Recall

$$(\tan x)' = \sec^2 x \quad (\sec x)' = \sec x \tan x$$

Example 3. 
$$\begin{aligned} & \int \tan x \sec^4 x \, dx \\ &= \int \tan x \sec^2 x \sec^2 x \, dx \\ &= \int \tan x (1 + \tan^2 x) \sec^2 x \, dx \\ &= \int u (1 + u^2) \, du \quad \swarrow u = \tan x \\ &= \frac{1}{2} u^2 + \frac{1}{4} u^4 + C \\ &= \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x + C \end{aligned}$$

Another way to compute this

$$\text{Take } u = \sec^2 x$$

$$du = 2 \sec^2 x \cdot \tan x \, dx$$

$$\text{Then } \int \tan x \sec^4 x \, dx$$

$$= \int u \cdot \frac{1}{2} du$$

$$= \frac{u^2}{4} + \tilde{C}$$

$$= \frac{\sec^4 x}{4} + \tilde{C}$$

$$\left. \begin{aligned} &= \frac{1}{4} (1 + \tan^2 x)^2 + \tilde{C} \\ &= \frac{\tan^4 x}{4} + \frac{\tan^2 x}{2} + \frac{1}{4} + \tilde{C} \end{aligned} \right\} = C$$

\* Here I'm checking these two method gives the same solution. You don't need to write these in homework.