

# Spring 2021 MATH231 Section CDQ Discussion

WF 9-10am

This document can be found [here](#) or on [my website](#). I will continue update it until the end of semester.

## Contact

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- Office hour: Wed 10-11am<sup>1</sup>

## Zoom

- Please use your cameras and microphones in breakout rooms.
- Interrupt me/using the “Raise Hand” feature on Zoom to ask questions.
- You can call me into your breakout room/return to the main room to ask for help.
- It is also possible for me to join your breakout rooms randomly to check if you have any questions.

## Worksheet

- Worksheet can be found on [Moodle](#) under Groupwork folder.
- Ask for hints when you get stuck on a problem.

## Submission

- Submit on [Moodle](#) under Groupwork folder.
- **1 submission per group.** Once a file is uploaded, everyone in the same group will be able to see/edit the file. <sup>2</sup>
- Group remains the same until each midterm.
- 1st worksheet of the week is due on **Thursday** at **8AM** CST. <sup>3</sup>
- 2nd worksheet of the week is due on **Saturday** at **8AM** CST.
- Worksheet solutions available at 12:30PM CST on the due date.

## Grading

Worksheets are graded with 2, 1 or 0.

2 - the worksheet uploaded is satisfactory

1 - the worksheet uploaded is unsatisfactory and needs improvement. Your TA will comment on what should be improved for next time.

0 - the worksheet was not uploaded

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<sup>1</sup>Office hour is run for all students in MATH231

<sup>2</sup>Groups are assigned randomly by Moodle

<sup>3</sup>Central Standard Time

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## Worksheet 1

Recall

**Theorem 1.1** (Fundamental Theorem of Calculus). [Ref p.26](#)

*Part 1* If  $f(x)$  is **continuous** over an interval  $[a, b]$ , and the function  $F(x)$  is defined by

$$F(x) = \int_a^x f(t) \, dt, \quad x \in [a, b]$$

then  $F'(x) = f(x)$  over  $[a, b]$ .

*Part 2* If  $f(x)$  is **continuous** over an interval  $[a, b]$ , and  $F(x)$  is any antiderivative of  $f(x)$  i.e.  $F'(x) = f(x)$ , then

$$\int_a^b f(x) \, dx = F(a) - F(b).$$

**Example 1.2.** Let

$$g(x) = \int_a^{b(x)} f(t) \, dt$$

Apply chain rule and FTC

$$g'(x) = \frac{d}{dx} \int_c^{b(x)} f(t) \, dt = b'(x) \cdot f(b(x)).$$

## Worksheet 2

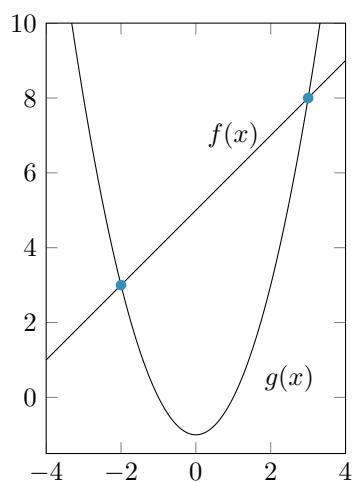
Recall

- **Substitution rule/Change of variable:** let  $u = g(x)$ , then

$$\int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du. \quad (\text{Q1-3})$$

- **Compute area between curves.** (Q4-7)

- Draw the graph.
- Find intersection points by solving  $f(x) = g(x)$ , say they are  $x = a$  and  $x = b$ .
- Area =  $\int_a^b f(x) - g(x) \, dx$ .



## Worksheet 3

**Volume of a Solid of Revolution:** Slices of volume are circles. [Ref](#)

$$\text{Vol} = \int_a^b \pi f(x)^2 \, dx.$$

Q3. Slices are squares/triangles.

$$\text{Vol} = \int_a^b \text{Area of slices} \, dx.$$

E.g.

$$\text{Vol} = \int_a^b f(x)^2 \, dx.$$

## Worksheet 4

Volume by Cylindrical Shells:

$$\text{Vol} = \int_a^b 2\pi r \cdot f(x) \, dx.$$

Rotation about  $y$ -axis:  $r = x$ .

Rotation about the vertical line  $x = a$ :  $r = |a - x|$ .

## Worksheet 5

Recall

- Since  $\sin x$  is oscillating between -1 and 1,  $\lim_{x \rightarrow \infty} \sin x$  does not exist.
- we can use L'Hopital's Rule to compute indeterminate forms " $\frac{0}{0}$ " and " $\frac{\infty}{\infty}$ ".

**Theorem 5.1** (L'Hopital's Rule).

*Assumptions:*

$$\begin{aligned}f(x) &\rightarrow 0 \quad \text{as } x \rightarrow a, \\g(x) &\rightarrow 0, \\g'(x) &\neq 0.\end{aligned}$$

*Conclusion:*

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

**Warning:** check the assumptions before applying L'Hopital's Rule.

## Worksheet 6

Integration by Parts:

$$(uv)' = u'v + uv' \implies \int u \, dv = uv - \int v \, du.$$

Choose  $u$  based on which of these comes first, (search “[integration by parts what to choose as  \$u\$](#) ”):

- (1) **L**ogarithmic functions:  $\ln x$
- (2) **I**nverse trigonometric functions:  $\arcsin x$
- (3) **A**lgebraic functions:  $x$
- (4) **T**rigonometric functions:  $\sin x$
- (5) **E**xponential functions:  $e^x$



## Worksheet 7

## Worksheet 8

Recall:  $1 + \tan^2 x = \sec^2 x$ .

## Worksheet 9

**Substitution rule/Change of variable:** let  $u = g(x)$ , then

$$\int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du.$$

**Integration by Parts:**

$$\int u \, dv = uv - \int v \, du.$$

## Worksheet 10

**Partial Fractions decomposition:** Find  $A, B$  such that

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} - \frac{B}{x-b}.$$

The following table is from this [website](#)

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^k$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}, k = 1, 2, 3, \dots$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$
$(ax^2 + bx + c)^k$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}, k = 1, 2, 3, \dots$

Typo in solution

WS10 Q3.

$$\frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2} + \frac{E}{x - 1} + \frac{F}{(x - 1)^2} + \frac{G}{(x - 1)^3}.$$

## Worksheet 11

**Improper integrals:** There are two types of improper integrals  $\int_a^b f(x) \, dx$ :

(1)  $a$  or  $b$  (or both) infinite, e.g.  $\int_1^\infty \frac{1}{x} \, dx$ .

(2) The function  $f(x)$  blows up in the interval  $[a, b]$ , e.g.  $\int_0^1 \ln x \, dx$ .

To compute improper integrals, e.g.:

$$\int_1^\infty \frac{1}{x} \, dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} \, dx$$

## Worksheet 12

**Simpson's rule:** Let  $x_i$ 's be equally spaced points,

$$\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)).$$

Coefficient is 4 for odd  $i$ ,  $i \neq 0, n$ ; coefficient is 2 for even  $i$ ,  $i \neq 0, n$ .

If you want to use the formula  $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ , for complex  $a$ . Recall how we obtain this formula. We do trig substitution,  $x = a \tan \theta$ , and use the identity  $1 + \tan^2 \theta = \sec^2 \theta$ . This identity is still valid for complex  $\theta$ . So with  $x = i \tan \theta$ ,  $\theta = \arctan(-ix)$

$$\begin{aligned} I &= \int \frac{1}{x^2 - 1} dx = \int \frac{1}{x^2 + i^2} dx \\ &= \int \frac{1}{i^2 \tan^2 \theta + i^2} i \sec^2 \theta d\theta \end{aligned}$$

(Here we need the derivative of  $\tan \theta$ , but you can check this is  $\sec^2 \theta$  in the complex case)

$$= -i \int \frac{1}{\sec^2 \theta} \sec^2 \theta d\theta = -i\theta + C = -i \arctan(-ix) + C$$

The above checked our formula is valid for complex  $a$ . Hence, we can plug in the value of  $\arctan$

$$\arctan t = \frac{1}{i} \ln \sqrt{\frac{1+it}{1-it}} = \frac{i}{2} [\ln(1-it) - \ln(1+it)].$$

So

$$\arctan(-ix) = \frac{1}{i} \ln \sqrt{\frac{1+x}{1-x}} = \frac{i}{2} [\ln(1-x) - \ln(1+x)].$$

$$I = -i \cdot \frac{i}{2} [\ln(1-x) - \ln(1+x)] + C = \frac{1}{2} [\ln(1-x) - \ln(1+x)] + D.$$

Note that  $D$  should be a real constant as  $I$  is real.

## Worksheet 13

Let  $ds$  be the arclength differential.

**Arc length:**

$$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

**Surface area:**

$$A = \int 2\pi x ds = \int 2\pi x \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

## Worksheet 14

$$\text{Force} = \int \rho g \cdot \text{depth}(y) \cdot \text{width}(y) \, dy$$



## Worksheet 15

**Integral test:** If  $f(x)$  is **continuous, positive and decreasing** on  $[N, \infty)$ . Then

$$\begin{aligned}\int_N^\infty f(x) \, dx \text{ converges} &\implies \sum_{n=N}^\infty f(n) \text{ converges} \\ \int_N^\infty f(x) \, dx \text{ diverges} &\implies \sum_{n=N}^\infty f(n) \text{ diverges}\end{aligned}$$

**Warning:** check the assumptions before applying integral test.

## Worksheet 16

Geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 \dots = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } |r| \geq 1 \end{cases}$$

Integral test: If  $f(x)$  is continuous, positive and decreasing on  $[N, \infty)$ . Then

$$\begin{aligned} \int_N^{\infty} f(x) \, dx \text{ converges} &\implies \sum_{n=N}^{\infty} f(n) \text{ converges} \\ \int_N^{\infty} f(x) \, dx \text{ diverges} &\implies \sum_{n=N}^{\infty} f(n) \text{ diverges} \end{aligned}$$

**Error estimate:** Assume  $\sum_{n=1}^{\infty} a_n$  converges

$$S = \overbrace{a_1 + a_2 + \dots + a_n}^{\text{partial sum } S_n} + \underbrace{a_{n+1} + a_{n+2} + \dots}_{\text{reminders } R_n}$$

Note that the first term in  $R_n$  is  $a_{n+1}$ .

$$\int_{n+1}^{\infty} f(x) \, dx < R_n < \int_n^{\infty} f(x) \, dx.$$

## Worksheet 17

## Worksheet 18

**Alternating Series Test:** Suppose that we have a series  $\sum a_n$  and either  $a_n = (-1)^n b_n$  or  $a_n = (-1)^{n+1} b_n$  where  $b_n \geq 0$  for all  $n$ . If

- $\lim_{n \rightarrow \infty} b_n = 0$
- $\{b_n\}$  is a decreasing sequence the series,

then  $\sum_n a_n$  is convergent.

**Ratio test:** Let  $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

- $L < 1$ ,  $\sum_n a_n$  convergent,
- $L > 1$ ,  $\sum_n a_n$  divergent,
- $L = 1$  no conclusion.