

# MATH541 Functional Analysis, Spring 2021

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April 3, 2021

Warning: I'm typing the notes slowly. Given that lecture recordings are not uploaded regularly, you can expect no updates for weeks.

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# 1 Baire's Category Theorem 20210125

Ref: A Course in Functional Analysis, John B. Conway, 1985

1. Metric space
2. Chicago suburb distance  $\mathbb{R}^b$  compact = closed and bounded no longer true
3. Cauchy sequence, completeness
4. Open, closed ball
5. Nowhere dense set, dense set, closure, interior.

$$Y \text{ is nowhere dense} \iff \bar{Y}^C \text{ is open and dense.}$$

**Theorem 1.1** (Baire's theorem). *In a complete metric space, the countable union of nowhere dense sets is again nowhere dense.*

**Lemma 1.2.** *The intersection of open dense sets is again open dense.*

Using the above lemma + induction to prove Baire's theorem.

Dense, nowhere dense, somewhere dense. [Stack Exchange](#) Theorem in notes: countable intersection of open dense is dense, then countable union does not have interior points. Need  $X$  complete metric space, so that the limit point is in  $X$ .

# 2 Baire's Category Theorem Cont. 20210127

Last time: open set, closed sets, theorem: let  $(X, d)$  be a complete metric space,  $O_n$  open dense, then  $\cap_n O_n$  is dense.

1. [intuition](#) dense set  $\cong$ , taking away a countable set of points
2. [proof idea](#) completeness  $\rightarrow$  geometric series.
3. Use Baire's theorem to show no function  $f : [0, 1] \rightarrow \mathbb{R}$  continuous exactly at  $\mathbb{Q}$

4. **proof** hard works is to find complete metric space and makes the theorem work
5. Normed space. A normed space is complete if absolute convergent sequences are convergent. Banach space.
6. isometry
7.  $\|f(x)\|_{C(K)} = \sup_{k \in K} |f(k)|.$

**Question 2.1.** Let  $C_b(\mathbb{R})$  be the set of continuous and bounded function. Is  $C_b(\mathbb{R}) = C(K)$  for some compact  $K$ ? — Yes.

Want to do: Start with Banach space, create new ones.

**Lemma 2.2.** Let  $T : X \rightarrow Y$  be a linear map between **normed spaces**. *TFRE*

1.  $T$  is continuous.
2.  $T$  is continuous at 0.
3.  $\|T\|_{op} = \sup_{\|x\| \leq 1} \|Tx\|$  is finite.
4.  $T$  is Lipschitz.

Homogeneity, duality

**Lemma 2.3.** Let  $X$  be a normed sapce and  $Y$  be Banach. Then the vector space  $L(X, Y)$  with the norm  $\|\cdot\|_{op}$  becomes a Banach space.

$L(\text{normed}, \text{Banach})$  is Banach.

**Corollary 2.4.**  $X^* = L(X, \mathbb{C})$  is Banach.

### 3 Basic Banach Space Theory 20210129

*proof of Lemma 2.3. Step 1.*  $(T_n)$  Cauchy implies  $(T_n(x_k))$  Cauchy.

*Step 2.* Let  $f(x) := \lim T_n(x)$ . Prove  $\limsup \|T_n(x) - f(x)\| = 0$ .

$$\begin{aligned}\|T_n - T\| &= \|T_n - \lim T_m\| = \lim \|T_n - T_m\| \\ &\leq \limsup_{m,n \geq N} \|T_n - T_m\| < \epsilon\end{aligned}$$

$\|T_n - T\| < \epsilon$  implies  $\|T_n(x) - T(x)\| < \epsilon$ , and so  $\limsup \|T_n(x) - f(x)\| = 0$ .

*Step 3.*  $f$  is bounded, and  $T_n \rightarrow f$ . □

**Corollary 3.1.**  $X$  Banach, then  $L(X, X) = L(X)$  is Banach algebra.

**Definition 3.2.** A **Banach algebra** is a Banach space  $(\mathcal{A}, \|\cdot\|)$  together with a product  $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ , with  $\|ab\| \leq \|a\|\|b\|$ .

1. closed subset of Banach is Banach.
2.  $K(X, Y) := \{T : X \rightarrow Y \mid \overline{T(B_X)} \text{ compact}\}$  is closed.
3. In finite dimension, linear bounded  $T$  is compact.

**Definition 3.3** (Totally bounded).

$$\forall \epsilon, \exists N \text{ s.t. } Y \subset \bigcup_{j=1}^N B(x_j, \epsilon)$$

This is equivalent to relatively compact. [Ref](#)

**Theorem 3.4.**

$$K(H, H)^{**} = B(H, H)$$

(We'll this theorem later.)

**Theorem 3.5.**

$$\exists \iota : X \rightarrow X^{**}; \iota(x)(f) = f(x), \text{ with } f : X \rightarrow \mathbb{K}$$

1.  $\iota$  is an isometry.
2.  $\overline{\iota(x)}$  is the completion of  $X$ .

Part 1 follows from Hahn-Banach.

**Definition 3.6.**  $(X, d)$  is a metric space. A **completion**  $(Y, d')$  is given by

1.  $\iota : X \rightarrow Y$  is an isometry.
2.  $\iota(X)$  is dense.
3.  $(Y, d')$  is complete.

Completion is unique.

## 4 Basic Banach Space Theory Cont. 20210201

Completion problem: see Theorem 3.5

*proof of Theorem 3.5.*

**Claim 4.1.**  $\|\iota(x)\|_{X^{**}} \leq \|x\|_X$ .

Note that

$$\begin{aligned} \|\iota(x)\|_{X^{**}} &= \sup_{\|f(x)\|_{X^*} \leq 1} |\iota(x)(f)| && \text{(by definition)} \\ &= \sup_{\|f(x)\|_{X^*} \leq 1} |f(x)| && (\iota \text{ inclusion}) \\ &\leq \sup_{\|f\|_{X^*} \leq 1} \|x\| \leq \|x\|. \end{aligned}$$

By definition  $\|f\|_{X^*} \leq 1 \iff |f(x)| \leq \|x\|$ . □

For a normed space the completion achieves in  $X^{**}$ .

Banach space

**Lemma 4.2.**  $C_b(x, x_0)$  is a Banach space.

$$C_b(x, x_0) = \{ f : X \rightarrow \mathbb{R} \mid \text{continuous and } \exists C, |f(x)| \leq Cd(x, x_0) \}.$$

Norm:  $\|f\| = \sup_x \frac{|f(x)|}{d(x, x_0)}.$

An embedding isometry  $\iota : X \rightarrow C_b(X)^*; \iota(x)(f) = f(x)$ . Hint: use evaluation map

$$\sup_{\|f\| \leq 1} |f(x) - f(x_0)| = d(x, x_0).$$

Distance attaining function is  $f(x) = d(x, x_0)$ , where  $x \neq x_0$ .

**Theorem 4.3** (Hahn-Banach Extension). *Given a vector space  $X$ , a sublinear map  $q : X \rightarrow \mathbb{R}$  s.t.*

$$q(x + y) \leq q(x) + q(y) \text{ (subadditive) and } q(sx) = sq(x), s > 0.$$

*Let  $Y \subset X$  and  $f : Y \rightarrow \mathbb{R}$  linear, with  $f \leq q$ , then  $\exists F : X \rightarrow \mathbb{R}$  linear  $F \leq q$  and  $F|_Y = f$ .*

**warning** This theorem is completely algebraic. There is no topology.

**Lemma 4.4.** *We can always add an extra dimension.*

*Proof. Step 1.*  $Y \subset X = \{ y + tx_0 \mid t \in \mathbb{R} \}$ . Candidates for  $F$  (extend 1-dim):  
 $F(y + tx_0) = F(y) + tF(x_0) = f(y) + ta_0$  for some  $a_0$ . What is  $a_0$ ? **trick**

$$\begin{aligned}
F(y + tx_0) &\leq q(y + tx_0) &\implies f(y) + ta_0 &\leq q(y + tx_0) \\
F(y - tx_0) &\leq q(y - tx_0) && f(y) - sa_0 \leq q(y - tx_0) \\
\implies a_0 &\leq \frac{q(y + tx_0) - f(y)}{t}, t > 0 &\implies a_0 &\leq \inf \frac{q(y + tx_0) - f(y)}{t}, t > 0 \\
a_0 &\geq \frac{f(y) - q(y - sx_0)}{s}, s > 0 && a_0 \geq \sup \frac{f(y) - q(y - sx_0)}{s}, s > 0
\end{aligned}$$

Check the sup is less than inf:

$$\begin{aligned}
\frac{f(y) - q(y - sx_0)}{s} &\leq \frac{q(z + tx_0) - f(z)}{t} \\
\iff f(y)t - q(y - sx_0)t &\leq q(z + tx_0)s - f(z)s \\
f(y)t + f(z)s &\leq q(z + tx_0)s + q(y - sx_0)t \\
f(yt + sz) &\leq q(yt + tsx_0 - tsx_0 + sz) \\
&\leq q(yt - stx_0) + q(tsx_0 + sz) \\
&\leq tq(y - sx_0) + sq(tx_0 + z)
\end{aligned}$$

This exactly fits the assumption, so we can pick  $a_0 = \sup \frac{f(y) - q(y - sx_0)}{s}$ .

*Step 2.* Use Zorn's lemma. Consider

$$\mathcal{L} = \{ (Z, F) \mid Y \subset Z, F \leq q \text{ on } Z, F|_Y = f \}.$$

Order on the set:  $(Z_1, F_1) \leq (Z_2, F_2)$  if  $Z_1 \subset Z_2$  and  $F_2|_{Z_1} = F_1$ . Every chain has an upper bound  $Z_\infty = \cup Z_i, F = \cup F_i$ . Hence there exists a maximal element  $(Z_{\max}, F_{\max}) \in \mathcal{L}$ .

**Claim 4.5.**  $Z_{\max} = X$ .

If not,  $\exists x_0 \notin Z_{\max}$  apply lemma to  $F_{\max}$ ,  $Z_{\max} + \mathbb{R}x_0$  admits  $F'_{\max}$ . Contradiction.  $\square$

**Remark 4.6.** Hahn-Banach is also true for  $\mathbb{C}$ .



## 5 Hahn-Banach Theorem 20210203

**Lemma 5.1.** Take  $C$  convex,  $0 \in C$ . The *Minkowski functional*

$$q_C(x) = \inf \{ \lambda \mid x \in \lambda C \}$$

is sublinear.

*Proof.*  $x, y \in V$ . Let  $\epsilon > 0$ , choose  $\lambda, \mu$  s.t.  $x \in \lambda C, y \in \mu C$ .

$$q_C(x) \leq \lambda \leq (1 + \epsilon) q_C(x)$$

$$q_C(y) \leq \mu \leq (1 + \epsilon) q_C(y).$$

Then  $z = \frac{\lambda}{\lambda + \mu} \frac{x}{\lambda} + \frac{\mu}{\lambda + \mu} \frac{y}{\mu} \in C$ . Therefore  $x + y = (\lambda + \mu) \left( \frac{x}{\lambda + \mu} + \frac{y}{\lambda + \mu} \right)$ . So

$$q_C(x + y) \leq \lambda + \mu \leq (1 + \epsilon) (q_C(x) + q_C(y)).$$

Send  $\epsilon \rightarrow 0$ . □

**Corollary 5.2.** Let  $C, D$  be nonempty convex sets  $C \cap D = \emptyset$ . There there exists  $f : V \rightarrow \mathbb{R}$  s.t.  $f(x) \leq f(y)$  for all  $x \in C, y \in D$ .

*Proof.* Take  $x_0 \in C, y_0 \in D$ . trick Shifting trick: let

$$B := C - D - (x_0 - y_0),$$

where  $C - D := \{x - y \mid x \in C, y \in D\}$ . Since  $x - y \neq 0, y_0 - x_0 \notin B$ . Let  $Y = \mathbb{R}(y_0 - x_0)$ .

**Claim 5.3.**  $q_B(x_0 - y_0) \geq 1$ .

Define  $f(t(y_0 - x_0)) = t$ , then  $f \leq q_B$ . Hahn-Banach extension gives  $F : V \rightarrow \mathbb{R}$ , with  $F \leq q$  and  $F(y_0 - x_0) = 1$ . Note that  $q_B(x - y - (x_0 - y_0)) \leq 1$  implies

$$\begin{aligned} F(x - y - (x_0 - y_0)) &\leq 1 \\ \implies F(x - y) - F(x_0 - y_0) &\leq 1 \\ F(x) &\leq F(y) + 1 - F(y_0 - x_0) = F(y) \end{aligned}$$

□

**Theorem 5.4.** For  $X$  a normed space and  $q(x) = \|x\|$ ,  $X$  subset of complex vector space,  $\forall x$  with unit norm,  $\exists$  a complex linear functional  $f \leq \|\cdot\|$  with  $|f(x)| = 1$ .

*Proof.* Consider  $X$  as a real normed space. Take  $x_0$  in  $X$  and let  $Y = \mathbb{R}x_0 + i\mathbb{R}x_0$ ,  $\|x_0\|$ . Define  $f(zx_0) = \operatorname{Re}(z)$ . Note that  $f \leq q$  as

$$f(zx_0) = \operatorname{Re}(z) \leq |z| = \|zx_0\| \leq (zx_0).$$

Then  $\exists F : X \rightarrow \mathbb{R}$  with  $F(x) \leq \|x\|$  real linear and  $F(x_0) = 1$ .

Fabrication: want to define  $G(x) = F(x) - iF(ix)$ . If  $G$  is complex linear and  $F = \operatorname{Re} G$ ,  $G(x) = \operatorname{Re} G(x) + i\operatorname{Im} G(x) = F(x) - \operatorname{Re}(iG(x))$ .

**Claim 5.5.** 1.  $G(x) = F(x) - iF(ix)$  is complex linear

2.  $|G(x)| \leq \|x\|$

□

## 6 Hahn-Banach Theorem Cont. 20210205

**Theorem 6.1** (Complex version Hahn-Banach). Let  $X$  be a complex vector space. If  $f : Y \rightarrow \mathbb{C}$  is a complex linear functional on a complex linear subspace  $Y \subset X$ , and  $q : X \rightarrow [0, \infty]$  a sublinear function and  $q(zx) = q(x)$ ,  $|z| = 1$  (semi-norm). If  $|f| \leq q$ , then there exists  $F : X \rightarrow \mathbb{C}$ , such that  $|F| \leq q$ ,  $F|_Y = f$

*Proof.* Apply the real Hahn-Banach to  $\tilde{f} = \operatorname{Re} f$ .  $\tilde{F} : X \rightarrow \mathbb{R}$ . Define a new  $F$  by

$$F(x) = \tilde{F}(x) - i\tilde{F}(ix).$$

Check  $F$  is complex linear.

□

Hahn-Banach separation.

**Lemma 6.2.** *Let  $C$  be a convex set and  $q_C$  is a Minkowski functional*

1.  $x \in C$  then  $q_C(x) \leq 1$
2.  $x \notin C$  then  $q_C(x) \geq 1$ .

$$\{x \mid q_C(x) < 1\} \subset X \subset \{x \mid q_C(x) \leq 1\}.$$

And the inclusions are strict.

*Proof.*  $q_C(y) = \inf\{\lambda \mid \frac{y}{\lambda} \in C\}$ . For part 1,  $x \in C$  so  $q_C(x) \leq \lambda = 1$ .

For part 2, assume  $q_C(x) < 1$ , then  $\exists \lambda < 1$  such that  $\frac{x}{\lambda} \in C$ . This (together with convexity) implies

$$x = (1 - \lambda) \cdot 0 + \lambda \cdot \frac{x}{\lambda} \in C,$$

contradiction. □

$C$  may or may not contain the boundary.

1. Topology
2. filter
3. continuous

**Definition 6.3.** A **filter** on a set  $X$  is a subset  $\mathcal{F} \subset 2^X$  such that

1. If  $A, B \in \mathcal{F}$  then  $A \cap B \in \mathcal{F}$
2. If  $A \subset B$  and  $A \in \mathcal{F}$ , then  $B \in \mathcal{F}$ .

It is **nontrivial** if  $\forall A \in \mathcal{F}, A \neq \emptyset$ .

**Definition 6.4.** A **neighbourhood filter** is a collection  $(\mathcal{F}_X)_{x \in X}$  of filters.

**Remark 6.5.**

1. (Topology  $\Rightarrow$  Filter)

Given topology  $\tau$ ,  $\mathcal{F}_X$  is generated by the non-empty open sets.

$$\mathcal{F}_X = \{ A \subset X \mid \exists O \text{ open, } x \in O \subset A \}.$$

Neighbourhood filter.

2. (Filter  $\Rightarrow$  Topology)

Given a filter  $\mathcal{F}_X$ , define  $O$  is open iff  $\forall x \in O, O \in \mathcal{F}_X$ . intuition A topology can equivalently be defined by open sets or neighbourhood filters.

**Lemma 6.6.**  $(\tau^{\mathcal{F}})^{\tau} = \tau$ .

**Definition 6.7.**  $f$  is **continuous** at  $x$  if  $\forall B \in \mathcal{F}_{f(x)}, f^{-1}(B) \in \mathcal{F}_X$ .

Recall: If  $f : X \rightarrow Y$  continuous and  $K \subset X$  compact, then  $f(K)$  compact

**Definition 6.8.** A space  $(X, +, \cdot, \tau)$  is a **topological vector spaces** if

1.  $(X, +, \cdot)$  is a vector space
2.  $+$  :  $X \times X \rightarrow X$  continuous  
 $\cdot$  :  $\mathbb{K} \times X \rightarrow X$  continuous

**Example 6.9.** 1.  $\mathbb{R}^2$  with the Chicago railway metric is not a topological vector space.  $+$  not continuous.

2. Let  $(\Omega, \Sigma, \mu)$  be a measure space. Define

$$L_0(\Omega, \Sigma, \mu) = \{ f : \Omega \rightarrow \mathbb{R} \mid f \text{ measurable, } \mu(|f| > \epsilon) \rightarrow 0 \text{ as } \epsilon \rightarrow \infty \}.$$

Define

$$d(f, 0) := \inf \{ \epsilon \mid \mu(|f| > \epsilon) < \epsilon \}, \quad d(f, g) = d(f - g, 0).$$

This is a translation invariant metric. Hence a translation invariant topological vector space.

## 7 Vector space 20210208

1. Topological space
2. Topological vector space  $(X, +, \cdot, \tau)$ , in particular, the translation map  $T_x : X \rightarrow X; y \mapsto T_x(y) = x + y$  is a homeomorphism
3. Application to Hahn-Banach
4. Tychonoff's theorem

Motivational lemma

**Lemma 7.1.** *Let  $X$  be a topological vector space,  $f : X \rightarrow \mathbb{R}$  be a linear nonzero continuous map, then the image of an open convex set is open.*

*Proof.* If  $f$  is linear and  $O$  is convex then  $f(O)$  is convex. Convex sets of  $\mathbb{R}$  is intervals.

Assume  $f(O) = (a, b]$  or  $[a, b]$ . That is there is a  $x \in O$ ,  $f(x) = \sup_{y \in O} f(y)$ , then  $f(x) = b$ . Since  $f(x_0) \neq 0$  with  $f(x_0) = 1$ , ( $f \neq 0$ ), we consider  $x(t) = x + tx_0$ . Then  $O$  open implies there is a  $t_0$ , for all  $|t| < t_0$ ,  $x + tx_0 \in O$  (translation is continuous). But now

$$f(x + tx_0) = f(x) + tf(x_0) = b + t \cdot 1 > b.$$

Contradiction. □

later Extension is continuous.

**Theorem 7.2** (Tychonoff). *For each  $j \in J$ , let  $X_j$  be a topological space. If each  $X_j$  is compact, then  $X = \prod_{j \in J} X_j$  is compact in the product topology.*

Clarification:  $x = (x_i)_{i \in I}$ ,  $O$  is a neighborhood of  $x$  if there are  $i_j$ ,  $O_j$  such that  $O = \{ (y_i) \mid y_{i_j} \in O_{i_j} \}$ .

**Example 7.3.** Let  $X_i$  be a metric space, the index set  $I = \mathbb{N}$ . Now the following defines a distance of the product topology

$$d((x_n), (y_n)) = \sum_{n \geq 0} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)}.$$

$$\{ (y_n) \mid d((x_n), (y_n)) < \epsilon \} \quad \supset \quad \{ (y_n) \mid \text{dist}(x_j, y_j) < \frac{\epsilon}{2}, j = 1, \dots, n \}.$$

*Proof.* Assume  $d(x_j, y_j) \leq \frac{\epsilon}{2}$  for all  $j$ . Then

$$\begin{aligned} d((x_n), (y_n)) &= \sum_{n \geq 0} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} \\ &\leq \sum_{n=1}^m 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} + \sum_{n > m} 2^{-n} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon \end{aligned}$$

(choose large  $m$  so that the second term is less than  $\frac{\epsilon}{2}$ ). *Any continuity condition only depends on finitely many terms*  $\square$

1. non-trivial filter, filter converges to  $x$ , ultra filter

**intuition** Filter is the analogue of sequence converging to something. They want to be small.

**Definition 7.4.** We say that a filter  $\mathcal{F}$  converges to  $x$  if  $\mathcal{F} \supset \mathcal{N}_x$ .

Every neighbourhood is contained in the filter.

**Definition 7.5.** A maximal non-trivial filter is called a **ultra filter**.

**Remark 7.6.** Let  $\mathcal{U}$  be an ultra filter then for every  $A \subset X$ , either  $A \in \mathcal{U}$  or  $A^C \in \mathcal{U}$ .

*Proof.* Fix  $A \subset X$ .

Case 1  $A \in \mathcal{U}$  done.

Case 2  $A \notin \mathcal{U}$  then  $A^C \in \mathcal{U}$ . (Prove by contradiction, assume  $A^C \notin \mathcal{U}$ ) Define  $\tilde{\mathcal{U}}$  to be the smallest filter which contains  $A^C$  and elements in  $\mathcal{U}$ . (Show  $\tilde{\mathcal{U}}$  is again a filter). Indeed this new filter  $\tilde{\mathcal{U}}$  is closed by superset. Need to show if  $\tilde{A}, \tilde{B} \in \tilde{\mathcal{U}}$  implies  $\tilde{A} \cap \tilde{B} \in \tilde{\mathcal{U}}$ .

- $\tilde{A}, \tilde{B} \in \mathcal{U}$  done.
- $\tilde{A}, \tilde{B} \supset A^C$  done.
- $\tilde{A} \in \mathcal{U}, \tilde{B} \supset A^C$ . We know  $\tilde{B} \supset A^C$  implies  $\tilde{B}^C \subset A$ , and we know  $\tilde{A} \neq A$ , so  $\tilde{A} \cap \tilde{B} = \emptyset$ .

Then  $\tilde{\mathcal{U}}$  is a filter, contradicting to the fact  $\mathcal{U}$  is an ultra filter.

□

**Corollary 7.7.** *Every ultra filter on an interval converges.*

**Lemma 7.8.**  *$(X, \tau)$  is compact iff every ultra filter converges.*

*Proof.* [Ref.](#)

( $\Rightarrow$ ) Let  $(X, \tau)$  be compact and  $\mathcal{U}$  be an ultra filter. Assume  $\mathcal{U}$  does not converge to any point. Then  $\forall x \in X, \mathcal{N}_x \not\subset \mathcal{U}$ . Then every point has a neighbourhood  $O_x$  which is not in  $\mathcal{U}$ .

Take the open cover  $\cup_x O_x$  of  $X$ ,  $O_x$  as above. By compactness, there is a finite subcover  $O_{x_1} \cup \dots \cup O_{x_n}$ . Since  $\mathcal{U}$  is an ultra filter,  $O_{x_i}^C \in \mathcal{U}$ , and the finite intersection

of  $O_{x_i}^C$ 's is in  $\mathcal{U}$ . But

$$\left(\bigcap_{i=1}^n O_{x_i}^C\right)^C = \bigcup_{i=1}^n O_{x_i} = X$$

implies  $\bigcap_{i=1}^n O_{x_i}^C = \emptyset \in \mathcal{U}$ , contradiction.

( $\Leftarrow$ ) Let  $X \subset \bigcup_x O_x$ ,  $O_x$  open. Assume that  $X \not\subset \bigcup_{i=1}^n O_{x_i}$  for any finite subset of indices. Then  $\bigcap_{i=1}^n O_{x_i}^C \neq \emptyset$ . Define

$$\mathcal{F} = \left\{ A \mid \exists i_1, \dots, i_n \text{ s.t. } \bigcap_{i=1}^n O_{x_i}^C \subset A \right\}.$$

This is a filter, let  $\mathcal{U}$  be the ultra filter contains  $\mathcal{F}$ . Then  $\mathcal{U}$  converges, say to some  $x_0 \in X$ , then  $\mathcal{N}_{x_0} \subset \mathcal{U}$ . Then there is a neighbourhood of  $x_0$  which is contained in  $\mathcal{U}$ , and then  $O_x^C \in \mathcal{F} \subset \mathcal{U}$ . But  $O_x \cap O_x^C = \emptyset$ , contradiction.  $\square$

*proof of Theorem 7.2. Ref.*

Let  $X = (\prod_i X_i, \tau_i)$ ,  $\mathcal{F}$  be an ultra filter. Let  $\pi_i : X \rightarrow X_i$  be the projection to the  $i$ -th term. Note that  $\pi_i(\mathcal{F})$  is also an ultra filter, so it converges to some  $x_i \in X_i$ . Then  $\mathcal{F}$  converges to  $(x_i)_{i \in I}$ .

**Claim 7.9.** *Let  $x = (x_i)_{i \in I}$ , if  $O \in \mathcal{N}_x$  then  $O \in \mathcal{U}$ .*

This means  $O \supset O_{i_1} \times \dots \times O_{i_n} \times X_{j_1} \times X_{j_1} \times \dots$ . Now  $\pi_{i_k}^{-1}(O_{i_k}) = W_k$  open and belongs to  $\mathcal{U}$ , as  $O_{i_k} \in \mathcal{U}$ . Hence, the finite intersection of  $W_k$ 's is in  $\mathcal{U}$ . Then  $O \in \mathcal{U}$ .  $\square$

## 8 Locally Convex Topological Vector Spaces 20210210

Recall

1. Topological vector spaces  $(X, +, \cdot)$
2. Tychonoff theorem
3. **intuition** An ultra filter is a generalisation of sequence converging to a point.



**Definition 8.1.** A topological vector space is called **locally convex** if  $\forall x, \forall O \in \mathcal{N}_x$ ,  $\exists W$  convex such that  $x \in W \subset O$ .

**Example 8.2.** 1. Let  $X$  is a normed space,  $\mathcal{N}_x = \{ O \mid \exists x > 0, \text{int}(B_r) + x \subset O \}$ .  
 2. Let  $X = C^\infty(\mathbb{R})$ ,  $K$  a compact subset, with semi-norm  $\|f\|_{K,n} = \sup_{x \in K} \sup_{1 \leq i \leq n} |f^{(i)}(x)|$ .  
 (This is a semi-norm because  $\text{supp } f$  can be in  $K^C$ ) The resulting topology is locally convex.

**Example 8.3** (Non-examples).

1.  $L_0 = \{ f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ measurable} \}$ , with

$$d(f, 0) = \inf \{ \epsilon \mid \mu(|f| > \epsilon) < \epsilon \}.$$

2.  $\|f\|_p = (E|f|^p)^{1/p}$  with  $0 < p < 1$ .  $B_p = \{ f \mid \|f\|_p < 1 \}$ . (Cannot put in a convex set if it is infinite dimension). The first example is when  $p \rightarrow 0$ . (E is expectation?)

**Theorem 8.4.** Let  $(X, \tau)$  be a topological space, the following are equivalent.

1.  $X$  is a Locally convex topological vector spaces (LCTVS)
2.  $\exists (q_i)_{i \in I}$  of semi-norms on  $X$
3.  $O \ni N_0$  iff  $\exists i, \exists r$  s.t.  $\{ x \mid q_i(x) < r \} \subset O$ .

**intuition** The topology is determined by many different shaped balls. Open iff contain one of the balls.

## Proof of Theorem 8.4

( $\Leftarrow$ ) Take a point  $x \in X$  and  $O$  is an open neighbourhood of  $x$ . Define a translation map  $T_{-x} : X \rightarrow X$ , by  $T_{-x}(y) = y - x$ . Note that  $T_{-x}$  is a homeomorphism, so  $T_{-x}(O) =: W$  is an open neighbourhood of 0. By (iii),  $\exists i$  s.t.  $\{ y \mid q_i(y) < 1 \} \subset W$ . Define  $V = x + \tilde{W} = \{ \tilde{y} \mid q_i(\tilde{y} - x) < 1 \} \subset O$ .

**Definition 8.5.** A set  $W \ni 0$  is called **absolutely convex** if

$$\sum_{j=1}^n |\lambda_j| \leq 1 \implies \sum_{j=1}^n \lambda_j x_j \in W.$$

**Definition 8.6.** A set  $W$  is **balanced** if  $|z| = 1$ ,  $zw = w$  for all  $w \in W$ .

**Remark 8.7.**  $W$  is absolutely convex if  $W$  is convex and balanced.

( $\Rightarrow$ ) Prove existence of seminorms. Take  $\mathbb{K} = \mathbb{R}$  let  $O$  be open and  $\exists W \subset O$  containing 0 and convex. Since  $- : X \rightarrow X$ ;  $-x \mapsto x$  is continuous, we know  $(-)^{-1}(W) \supset V$  is convex,  $V \in \mathcal{N}_x$  (Aside:  $W \cap -W$  is convex and balanced).

Define  $q_V(x) = \inf \{ \lambda \mid \frac{x}{\lambda} \in V \}$

**Lemma 8.8.**  $q_V$  is a semi-norm.

That is,  $q_V(\lambda x) = |\lambda|q_V(x)$  and subadditive  $q_V(x + y) \leq q_V(x) + q_V(y)$ .

Then

$$\frac{1}{4} \subset \{ y \mid q_V(y) < \frac{1}{2} \} (\text{ball of some semi-norm}) \subset V.$$

For every neighbourhood of 0 can choose a semi-norm

For  $\mathbb{K} = \mathbb{C}$ . Want for any set  $O$ , find a  $W$  which is convex and contained in  $\cap_{|z|=1} zO$  (in a intersection of rotations).  $(\cap_{|z|=1} zO)^C = \cup_{|z|=1} (zO)^C$ .

Question: Is  $B = \cup_{|z|=1} (zO)^C$  closed? – Yes. Let  $T = \{ z \mid |z| = 1 \}$ . The map  $T \times X \rightarrow X$ ;  $(z, x) \mapsto zx$  is continuous and  $T$  is compact.

**Lemma 8.9.**  $B$  is closed. (A compact translation of a closed set is closed.)

*Proof.* Let  $A$  be an ordered index set,  $x_\alpha \in B$ ,  $x_\alpha \rightarrow x$  meaning for a neighbourhood  $O$  of  $x$ ,  $\exists \alpha_0$ ,  $\forall \alpha > \alpha_0$ ,  $x \in O$ . □

Then  $0 \notin B$ , and  $\exists W \subset \bigcap_{|z|=1} zO)^C$  convex and  $\bigcap_{|z|=1} zw$  is balanced convex set.

## 9 Hahn-Banach Separation Theorem 20210212

**Lemma 9.1.** *Let  $X, Y$  be locally convex topological vector spaces. A linear map  $T : X \rightarrow Y$  is continuous if and only if  $T$  is continuous at 0.*

**Propersition 9.2.** *Let  $X$  be a locally convex topological space and  $f : X \rightarrow \mathbb{R}$  be a linear and continuous map. Let  $W$  be an open convex neighbourhood of 0. Then either  $f(W) = \{0\}$  or  $f(W)$  is open.*

**Theorem 9.3** (Hahn-Banach Separation Theorem). *Let  $C$  be a **convex nonempty** subset in a topological space  $X$  and  $x \notin C$ , then*

1. *there exists a linear map  $f : X \rightarrow \mathbb{R}$  such that  $f(y) \leq f(x), \forall y \in C$ ,*
2. *if in addition  $X$  is a locally convex topological vector space and  $C$  is open, then  $f$  is continuous, nontrivial and  $f(y) < f(x), \forall y \in C$ .*

*Proof.* (1) Let  $x_0 \in C$ , then  $\tilde{C} = C - \{x_0\}$  contains 0, by Lemma 5.1, the Minkowski functional  $q_{\tilde{C}} = \inf\{\lambda \mid y \in \lambda\tilde{C}\}$  is sublinear. Let  $V = \mathbb{R}(x - x_0)$  and define  $f(t(x - x_0)) = t$ , which is linear. Then  $x - x_0 \notin C - \{x_0\}$ . By Lemma 6.2,  $y \in C$  implies  $q_{\tilde{C}}(y - x_0) \leq 1$ . Therefore

$$f(y - x_0) \leq f(x - x_0) = 1 \implies f(y) \leq f(x).$$

(2) Now if  $C$  is open then  $\tilde{C} = C - \{x_0\}$  is open (here we only require a topological space, we don't actually need locally convexity). Consider  $g : X \times X \rightarrow X$ ,  $g(x, y) = x - y$ . This map is continuous,  $0 \in \tilde{C}$ .

There exists  $V_1, V_2$  neighbourhoods of 0, such that  $V_1 - V_2 \subset \tilde{C}$ . Define  $V = V_1 \cap V_2$  ( $V$  is a neighbourhood of 0). Then  $0 \in V - V \subset \tilde{C}$ . By previous part  $f|_{\tilde{C}} \leq 1$ .

Check the following later Hence

$$f(V - V) \subset f(\tilde{C}) \subset \{y \mid f(y) \leq r\}.$$

Then for all  $y = a - b \in V - V$ ,  $f(y) \leq 1$  and  $-y = b - a \in V - V$  so  $f(-y) \leq 1$ . This means  $f$  is bounded. Hence  $f$  is continuous at 0. By previous Lemma,  $f$  is continuous and  $f(\tilde{C})$  is open (image of open convex set is open). Then  $f(y - x_0) < 1$  for all  $y \in C$ .  $\square$

**Theorem 9.4.** *Let  $C, D$  be **nonempty convex** sets. If  $C \cap D = \emptyset$ , then there is a linear functional  $f$  on  $X$  such that  $f(x) < f(y)$ , for all  $x \in C, y \in D$ .*

*Proof.* **trick** Consider  $\tilde{C} = C - D = \{x - y \mid x \in C, y \in D\}$ . Note that  $\tilde{C}$  is open if either  $C$  or  $D$  is open, and  $0 \notin \tilde{C}$ . Now shift the set, i.e. let  $\tilde{D} = \tilde{C} - \{(x_0 - y_0)\}$ . Apply previous theorem  $0 \notin \tilde{C}$ , so there exists a  $f \neq 0$  and continuous,  $f(z) < f(0)$ , for all  $z \in \tilde{C} = C - D$ . Say  $z = x - y$ , for  $x \in C$  and  $y \in D$ . Then  $f(x) < f(y)$ .  $\square$

**Theorem 9.5.** *Let  $C$  be a **closed convex** set and  $D$  be a **compact convex** set in a locally convex topological vector space. Then there exists a continuous nontrivial  $f$  and  $r < s$  such that  $f(x) < r < s < f(y)$  for all  $x \in D$  and  $y \in C$ .*

*Proof.* Assume  $C$  is closed and  $D$  is compact.  $C^C$  is open,  $D \cap C = \emptyset$ . For any  $x \in D$  there is a  $W_x$  convex such that  $(x + W_x) \cap C = \emptyset$ .

Consider the open sets  $x + \frac{W_x}{2}$ , their union  $\cup(x + \frac{W_x}{2})$  gives an open cover of  $D$ . Then there is a finite subcover  $D \subset \cup_i(x_i + \frac{W_{x_i}}{2})$ . Take  $W = \cap_i \frac{W_{x_i}}{2}$ , and let  $y = d + w \in D + W$ . Then there exists an  $x_j$  such that  $d = x_j + \frac{W_{x_j}}{2}$ . Therefore,

$$y = d + w \in x_j + \frac{W_{x_j}}{2} + W \subset x_j + \frac{W_{x_j}}{2} + \frac{W_{x_j}}{2} \not\subset C.$$

(Convexity implies  $\frac{W_{x_j}}{2} + \frac{W_{x_j}}{2} \subset W_{x_j}$ .) **analogue** Triangle inequality on metric spaces.

Hence we have a strict separation between  $D+W$  and  $C$ , and we can find a nontrivial continuous  $f$  such that  $f(x) < f(d+w)$ , where  $x \in C$ ,  $d \in D$  and  $w \in W$ . Note that  $f(D)$  is compact as  $D$  is compact, so  $f(D)$  is a closed interval  $[a, b]$ . Then

$$f(D+W) = f(D) + f(W) = [a, b] + (-\alpha, \beta)$$

( $f(W)$  is a neighbourhood of 0 check). So for all  $x \in C$ ,

$$f(x) \leq a - \alpha < a \leq \inf\{f(y) \mid y \in D\}.$$

□

### Example 9.6.

1. Let  $X$  be a normed space, and  $C = \{x \mid \|x\| \leq 1\} = \bar{B}_X$ . Take  $x_0$  such that  $\|x_0\| > 1$ , then  $D = \{x_0\}$  compact. There exists  $f$  such that  $f(x) \leq 1$ ,  $\|f\| \leq 1$  and  $f(x_0) > 1$ .
2. Take a ball  $B_X$  and a triangle  $D$ .

Next, we want to make the separation line unique.

## 10 Weak Topology 20210215

**Definition 10.1.** Let  $X$  be a Banach space and  $Y \subset X^*$  a subspace. Then  $\sigma(X, Y)$ -**topology** is the coarsest topology making all the functional  $y \in Y$  continuous. This means the semi-norms defining this topology are given by

$$q_{y_1, \dots, y_n}(x) = \max_{i=1, \dots, n} |y_i(x)|.$$

Every locally convex space is given by semi-norms. Semi-norms are indexed by finite subsets of  $Y$ .

**Theorem 10.2.** *The dual space of  $(X, \sigma(X, Y))$  is  $Y$  (as a set). That is,*

$$(X, \sigma(X, Y))^* = Y.$$

Note the two spaces only equal as a set, not necessarily as a topological space. Because  $Y$  on the LHS can be taken as a algebraic dual without topological assumptions, whereas  $Y$  on the RHS is a topological vector space (may with its own norm).

**Remark 10.3.** Let  $X$  be a locally convex topological vector space and  $Y$  a Banach space or locally convex topological vector space, then  $L(X, Y)$  is also a locally convex topological vector space.

*Proof. Step 1.*  $Y \subset (X, \sigma(X, Y))^*$ .

**Claim 10.4.** *For every  $y \in Y$ ,  $f_y(x) = y(x)$  is continuous with respect to the new topology.*

It suffice to show  $f$  is continuous at 0:  $\forall \epsilon, \exists O \in \sigma(X, Y)$  containing 0, such that if  $x \in O$ , then  $|f(x)| < \epsilon$ . ( $f(0) = 0$ ). In this new topology open neighbourhood means there exists a semi-norm in system such that  $O \supset \{x \mid q(x) < \delta\}$ , i.e there exists some  $B_q(\delta) \subset O$ . This is equivalent to say  $|f(x)| \leq C \cdot q(x)$ , for some semi-norm  $q$ . compare In Banach space we don't have a choice of the norm, so we require  $|f(x)| \leq C \cdot \|x\|$ .

In our case, the semi-norm  $q_y(x) = |y(x)|$  does the job, because  $|f_y(x)| = |y(x)| = q_y(x)$ . More generally, the semi-norm is given by  $q_y(x) = \max_j |y_j(x)|$ .

*Step 2.*  $(X, \sigma(X, Y))^* \subset Y$ .

Let  $f : X \rightarrow \mathbb{K}$  be continuous. By definition there exists a  $q$  such that  $|f(x)| \leq q(x)$  and  $q(x) = \max_j |y_j(x)|$ . Fix  $y_1, \dots, y_n$  and define a map

$$\begin{aligned} \phi : X &\longrightarrow \mathbb{K}^n \\ x &\longmapsto (y_1(x), \dots, y_n(x)). \end{aligned}$$

Then  $\phi(X) \subset \mathbb{K}^n$  is a subspace. Denote  $Z = \phi(X)$ , then  $z = (y_1(x), \dots, y_n(x))$ .

Consider the map

$$\begin{aligned}\psi : Z &\longrightarrow \mathbb{K} \\ z &\longmapsto f(z).\end{aligned}$$

This map is well-defined, linear, and  $|\psi(z)| \leq \max_j |z_j| = \|z\|_\infty$ . By Hahn-Banach, there exists  $\tilde{\psi} : l_\infty^m \rightarrow \mathbb{K}$ , such that  $\tilde{\psi}|_Z(z) = \psi(z)$  and  $\|\tilde{\psi}\| = \|\psi\| \leq \|z\|_\infty$ . Note that  $\tilde{\psi}(z) \in (l_\infty^m)^* = l_1^m$ . This means there exists  $\alpha_1, \dots, \alpha_n$  such that  $\tilde{\psi}(z) = \sum_j \alpha_j z_j$ . This means

$$f(x) = \psi(\phi(x)) = \tilde{\psi}(\phi(x)) = \sum_j \alpha_j \phi_j(x) = f_y(x),$$

where  $y = \sum_j \alpha_j y_j$ . □

**Example 10.5.** Let  $X$  be a space and take  $Y = X^*$ . Then

- $\sigma(X, X^*)$  is called the **weak topology** of  $X$  and  $(X, \sigma(X, X^*)) = X^*$ ,
- $\sigma(X^*, X)$  is called the **weak\* topology** of  $X^*$  and  $(X^*, \sigma(X^*, X)) = X$ .

## 11 Weak Topology cont. 20210219

**Theorem 11.1** (Goldstine). *Let  $X$  be a Banach space, then the image of the closed unit ball  $B_X \subset X$  under the canonical embedding  $\iota$  into the closed unit ball  $B_{X^{**}}$  of the bidual space  $X^{**}$  is weak\*-dense.*

$$\overline{B_X}^{\sigma(X^{**}, X^*)} = B_{X^{**}}$$

**intuition** The unit ball with weak\*-topology is compact. In finite dimension, close + bounded = compact. Generalisations of finite dimension.

*Proof.* Recall that  $X^{**}$  is a locally convex topological vector space with respect to  $\sigma(X^{**}, X^*)$ -topology. This topology is given by the semi-norm  $q(x^{**}) = \max_j |x^{**}(x_j^*)|$ , with  $x_1^*, \dots, x_j^* \in X^*$ .

The canonical embedding  $\iota : X \rightarrow X^{**}$ , is an isometry (Hahn-Banach Theorem) and  $\iota|_{B_X} : B_X \rightarrow B_{X^{**}}$ . We want to show the closure  $\overline{\iota(B_X)}$  with respect to the  $\sigma(X^{**}, X^*)$  topology satisfies  $\overline{\iota(B_X)} = B_{X^{**}}$ . Prove by contradiction.

Assume that  $x^{**} \notin \overline{\iota(B_X)}$ , with  $\|x^{**}\|_{X^{**}} \leq 1$ . Note that  $\overline{\iota(B_X)}$  is closed, compact and convex. By Hahn-Banach separation (Theorem 9.5), there exists a nontrivial continuous map  $f : X^{**} \rightarrow \mathbb{R}$  so that  $|f(\iota(x))| \leq 1 < s < |f(x^{**})|$  for all  $x \in B_X$ . On one hand we have

$$\|f\|_{X^{**}} = \sup_{\|x\| \leq 1} |f(x^{**})| = \sup_{\|x\| \leq 1} |f(\iota(x))| \leq 1.$$

Then by definition,

$$|x^{**}(f)| \leq \|x^{**}\|_{X^{**}} \cdot \|f\|_{X^{**}} \leq 1.$$

On the other hand we have  $|x^{**}(f)| = f(x^{**}) > 1$ . Contradiction.  $\square$

**Example 11.2.** Let  $X = C_0 = \{(x_n) \mid \lim_n x_n = 0\}$ , with  $\|(x_n)\| = \sup_n \|x_n\|$ . Then  $X^* = l_1$  because

$$\begin{aligned} \|y_n\|_1 &= \sum_n y_n = \sup_k \sum_{i=1}^k |y_k| \\ &= \sup_k \langle y, \epsilon_1, \dots, \epsilon_k, 0, \dots, 0 \rangle. \end{aligned}$$

where  $\epsilon_i = \text{sgn}(y_i)$  and  $\langle y, z \rangle = \sum y_n z_n$ . And  $X^{**} = l_\infty = \{(x_n) \mid \sup_n |x_n| < \infty\}$ .

What is  $\sigma(l_\infty, l_1)$ -topology? The answer is pointwise convergence on bounded set. Consider bounded sequences  $x^\alpha$  ( $\|x^\alpha\| \leq C$ ). Then  $x^\alpha \rightarrow x \in l_\infty$  iff for all  $y \in l_1$ ,  $x^\alpha(y) \rightarrow x(y)$ .

For bounded sets  $\|x^\alpha\| \leq 1, \forall \alpha$ ,

$$x^\alpha \rightarrow x \iff x_n^\alpha \rightarrow x_n, \forall n.$$

( $\Rightarrow$ ) Take  $y_n = (0, \dots, 1, \dots, 0) \in l_1$ .



( $\Leftarrow$ ) Let  $y \in l_1$  and  $\epsilon > 0$  then there exists  $n_0$  such that  $\sum_{n>n_0} |y_n| < \frac{\epsilon}{2}$ . There exists  $\alpha_0$  such that any  $\alpha > \alpha_0$ ,  $|x_n^\alpha - x_n| < \frac{\epsilon}{2}$  for all  $n > n_0$ . We need

$$|x^\alpha(y) - x(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Let  $y^N = (y_1, \dots, y_N, 0, \dots, 0)$ ,  $y^N \rightarrow y$  in  $\sigma(l_\infty, l_1)$  because we can use pointwise convergence.

## 12 Alaoglu's Theorem 20210222

Alaoglu's Theorem says that the closed unit ball in  $X^*$  is compact in the weak\*-topology.

**Theorem 12.1** (Alaoglu). *Given a topological vector space  $X$ , and let  $B_{X^*} = \{x^* \in X^* \mid \|x^*\| \leq 1\}$  be the closed unit ball in  $X^*$ . Then  $B_{X^*}$  is compact in  $X^*$  with respect to the weak\*-topology on  $X^*$ .*

*Proof.* [Ref.](#) or see Conway p.134

Let the set  $D_x = \{z \in \mathbb{K} \mid |z| \leq 1\}$ . Consider the product  $D := \prod_{x \in B_x} D_x$ . Since  $D_x$  is compact in  $\mathbb{K}$ , Tychonov's theorem says that  $D$  compact in the product topology. Elements in  $D$  are functionals, given by  $\mu \in K$ ,  $\mu(x) = \mu_x \in D \subset \mathbb{C}$ , although they need not to be linear.

The inclusion

$$\iota : B_{X^*} \subset \prod_{x \in B_x} D =: K$$

is given by

$$\iota(x^*)(x) = x^*(x).$$

Note that  $\iota(B_{X^*}) \subset K$ . Indeed, if  $\|x\| \leq 1$  and  $\|x^*\| \leq 1$ , then  $|x^*(x)| \leq 1 \in D$ .

**Claim 12.2.**  $\iota(B_{X^*})$  is closed. Hence,  $\iota(B_{X^*})$  is a compact subspace of  $K$ .

*Proof of the claim.* Take a net  $(x_\alpha^*)$  in  $B_{X^*}$  which converges to  $f \in D$  pointwisely. So  $f(x) = \lim_{\alpha \rightarrow \infty} x_\alpha^*(x)$ . In particular  $|f(x)| \leq 1$  for all  $\|x\| \leq 1$ . (Need to show  $f$  is in the range. We can not take  $\mathbb{N}$  as index set, instead replacing  $\mathbb{N}$  by a partially ordered set. Usually the index set is given by the neighbourhood basis of  $f$ . Let  $O_i \in \mathcal{N}_f$ ,  $i = 1, 2$ , then  $O_1 \cap O_2 \in \mathcal{N}_f$  and  $O_1 \cap O_2 \geq O_i$ .)

For  $x \in X$ , define  $F(x) = \beta^{-1}f(\beta x)$  for some  $\beta$  such that  $\|\beta x\| \leq 1$  (check this is well defined). Then  $F$  agrees with  $f$  on  $B_X$ . We claim that  $F$  is linear. Take  $x_i \in X$ ,  $i = 1, 2$ . Consider  $y = \frac{x_1 + x_2}{\|x_1\| + \|x_2\|}$ . If we take  $\lambda = \frac{\|x_1\|}{\|x_1\| + \|x_2\|}$ , then by convexity  $y = \lambda \frac{x_1}{\|x_1\|} + (1 - \lambda) \frac{x_2}{\|x_2\|} \in B_X$ . Then

$$\begin{aligned} f(y) &= \lim_{\alpha} x_\alpha^*(y) = \lim_{\alpha} x_\alpha^* \left( \frac{x_1}{\|x_1\| + \|x_2\|} \right) + x_\alpha^* \left( \frac{x_2}{\|x_1\| + \|x_2\|} \right) \\ &= f \left( \frac{x_1}{\|x_1\| + \|x_2\|} \right) + f \left( \frac{x_2}{\|x_1\| + \|x_2\|} \right). \end{aligned}$$

So

$$\begin{aligned} F(x_1 + x_2) &= f(y) \cdot (\|x_1\| + \|x_2\|) \\ &= \left( f \left( \frac{x_1}{\|x_1\| + \|x_2\|} \right) + f \left( \frac{x_2}{\|x_1\| + \|x_2\|} \right) \right) \cdot (\|x_1\| + \|x_2\|) \\ &= F(x_1) + F(x_2). \end{aligned}$$

We have a linear functional  $F \in X^*$  satisfying  $|F(x)| \leq 1$  when  $\|x\| \leq 1$ . This means  $\|F\|_{X^*} \leq 1$ . So  $F \in B_{X^*}$   $\square$

**Definition 12.3.** A Banach space is **reflexive** if  $X^{**} = X$ .

Goal: to show  $X$  is reflexive iff  $X^*$  is reflexive.

**Propersition 12.4.** A closed subspace of a reflexive Banach space is reflexive.

*Proof.* The following diagram is commutative. (Check)

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^{**} \\ j \uparrow & & \uparrow j^{**} \\ Y & \xrightarrow{\iota_Y} & Y^{**} \end{array}$$

*Step 1.*  $Y^{**} = Y$ . Take an element  $y^{**} \in Y^{**}$ , note that

$$j^{**}(y^{**})(x^*) = y^{**} \circ j^*(x^*) = y^{**}(x^* \circ j) = x^*|_Y \in Y^*.$$

So we can apply  $y^{**}$  to this element, and define  $\phi(x^*) = y^{**}(x^*|_Y)$

**Lemma 12.5.** *If  $T : Y \rightarrow X$  is isometric, then  $T^{**} : Y^{**} \rightarrow X^{**}$  is also isometric.*

The above lemma says  $Y^{**}$  embeds isometrically into  $X^{**}$  (we will prove this later). If in addition,  $X^{**} = X$ , we deduce that for every  $y^{**}$  there exists an  $x \in X$  such that

$$y^{**}(x^*|_Y) = x^*(x).$$

We want to show  $x \in Y$ . We claim that  $y^{**} \in Y$ , otherwise by Hahn-Banach separation there exists  $x^*$  such that  $x^*(y^{**}) = 1$  and  $x^*|_Y = 0$ . The last equation says  $x^*(x) = y^{**}(x^*|_Y) = x^*|_Y = 0$ . A contradiction (as  $y^{**} \in Y^{**} \subset X^{**} = X$ ).  $\square$

**Lemma 12.6.** *If  $T : X \rightarrow Y$  is isometric then  $T^* : Y^* \rightarrow X^*$  sends closed unit ball to closed unit ball.*

*Proof.* Note that  $T^*(B_{Y^*}) \subset B_{X^*}$ . Indeed,

$$\begin{aligned} \|T^*\| &= \sup_{\|y^*\| \leq 1} \|T^*(y^*)\| = \sup_{\|y^*\| \leq 1} \|y^* \circ T\| \\ &= \sup_{\|y^*\| \leq 1, |x| \leq 1} |y^* \circ T(x)| = \sup_{\|y^*\| \leq 1, |x| \leq 1} |y^*(x)| \leq 1. \end{aligned}$$

So  $|T^*(y^*)| \leq \|T^*\| \|y^*\| \leq 1$ .

To show  $T^*$  is onto, take  $x^* \in B_{X^*}$ . Can define  $f(Tx) = x^*(x)$ ,  $\|f\| \leq 1$ . By Hahn-Banach there exists  $y^*$  such that  $y^*(Tx) = f(Tx) = x^*(x)$ .  $T^*(y) = x^*$ .  $\square$

**Lemma 12.7.** *If  $T : Y \rightarrow X$  is a surjection, then  $T^* : X^* \rightarrow Y^*$  is an isometry.*

*Proof of the Lemma 12.5.* The previous two lemma gives the result.  $\square$

## 13 Reflexive Spaces 20210224

**Theorem 13.1.**  *$X$  is reflexive  $\iff X^*$  is reflexive.*

*Proof.* ( $\Rightarrow$ ) Assume that  $X = X^{**}$ . Then  $B_{X^*}$  is closed in  $\sigma(X^*, X^{**}) = \sigma(X^*, X)$ . Take an element  $x^{***}$  in  $B_{X^{***}}$ , there exists a sequence  $x_\alpha^* \rightarrow x^{***}$  in  $\sigma(X^{***}, X^{**})$  topology. Since  $B_{X^*}$  is closed in  $\sigma(X^*, X)$ , there is an  $x^*$  such that  $x_\alpha^* \rightarrow x^*$ . This means  $x^{***} = x^*$ .

( $\Leftarrow$ ) If  $X^*$  is reflexive then  $X^{**}$  is reflexive, but  $X \subset X^{**}$  as a closed subspace.  $\square$

**Remark 13.2.**  $X$  is reflexive iff  $B_{X^*}$  is  $\sigma(X^*, X^{**})$  closed.

**Definition 13.3.** A Banach space is called **uniformly convex**, if  $\forall \epsilon > 0, \exists \delta > 0$  such that  $\|x\| \leq 1, \|y\| \leq 1$  and  $\|x - y\| > \epsilon$ , then  $\|\frac{x+y}{2}\| \leq 1 - \delta$ .

**Lemma 13.4.** *Take  $(x_n)$  a sequence with*

$$\limsup_n \|x_n\| \leq 1 \quad \text{and} \quad \liminf_n \inf_{m>n} \left\| \frac{x_n + x_m}{2} \right\| = 1.$$

*Then  $(x_n)$  is Cauchy.*

*Proof.* Let  $\epsilon > 0$ . Since  $\limsup_n \|x_n\| \leq 1$ , we can choose  $\epsilon_0 > 0, \exists n_0$  such that  $\|x_n\| \leq 1 + \epsilon_0$ , for all  $n > n_0$ . So  $\|\frac{x_n}{1+\epsilon_0}\| \leq 1$ , for all  $n > n_0$ . Then

$$\left\| \frac{x_n + x_m}{2(1 + \epsilon_0)} \right\| = \left\| \frac{x_n + x_m}{2} \right\| \cdot \frac{1}{1 + \epsilon_0} \geq \frac{1}{(1 + \epsilon_0)^2},$$

for all  $n > n_0$ .

Taking  $\frac{1}{(1+\epsilon_0)^2} = 1 - \delta$ . Using uniform convexity (contrapositive), we have  $\forall n, \exists m$

$$\left\| \frac{x_n - x_m}{2(1 + \epsilon_0)} \right\| < \epsilon.$$

Conclusion: Above shows  $\forall \epsilon, \exists n_0, \forall n > n_0, \exists m$ , such that  $\|x_n - x_m\| < 2\epsilon(1 + \epsilon_0)$ .

We use this for  $\epsilon = 2^{-k}$ , then there exists a converging subsequence  $x_{n_k}$  such that  $\|x_{n_k} - x_{n_{k+1}}\| \leq 2^{-k}$ .  $\square$

**Theorem 13.5** (Milman-Pettis). *Uniformly convex Banach spaces are reflexive.*

*Proof.* See: [Ref.](#)

Let  $x^{**} \in B_{X^{**}}$ ,  $\|x^{**}\| = 1$ . Then by definition of  $\|x^{**}\|$ , for all  $n$ , there exists  $x_n^* \in B_{X^*}$ , such that  $x^{**}(x_n^*) \geq 1 - \frac{1}{n}$ . Since  $B_X \subset B_{X^{**}}$  is dense in  $\sigma(X^{**}, X^*)$ . Let  $q_n(y) = |x_n^*(y)|$ . There exists  $(x_k)$  in  $B_X$  such that

$$|q_n(x^{**} - x_k)| = |x_n^*(x_k) - x^{**}(x_n^*)| \leq \frac{1}{2k}, \text{ for } n = 1, \dots, k.$$

In particular, apply the above to  $n = k$ , then

$$|x_k^*(x_k) - x^{**}(x_k^*)| \leq \frac{1}{2k} \implies -\frac{1}{2k} + x^{**}(x_k^*) \leq x_k^*(x_k).$$

Recall  $x^{**}(x_k^*) \geq 1 - \frac{1}{k}$ . So  $1 - \frac{3}{2k} \leq x_k^*(x_k) \leq 1$  (RHS because  $x_n^*$  is in the unit ball).

Then take  $m > k$ , we have

$$2 - \frac{6}{2k} \leq 1 - \frac{3}{2k} + 1 - \frac{3}{2m} \leq x_k^*(x_k) + x_m^*(x_m) \leq x_k^*(x_k + x_m) \leq \|x_k + x_m\| \leq 2. \quad (1)$$

Taking  $\liminf$  on both sides we get  $\liminf \left\| \frac{x_k + x_m}{2} \right\| = 1$ , and  $\limsup \|x_k\| \leq 1$ . By the above lemma  $(x_n)$  is Cauchy.

**Remark 13.6.** Assume there are two sequences  $x_n, \tilde{x}_n$  satisfies the property (1), then then  $\lim x_n = \lim \tilde{x}_n$ .

Now if  $(y_n^*)$  is another family using the above construction, then there exists  $(\tilde{x}_n)$  in  $B_X$  such that

$$|y_n^*(\tilde{x}_k) - x^{**}(x_n^*)| \leq \frac{1}{2k}.$$

Then  $x^*(x_k) \rightarrow x^{**}(x)$  and  $y^*(\tilde{x}_k) \rightarrow x^{**}(y)$  implies  $x^{**} = \lim x_n = \lim \tilde{x}_n$  in  $\sigma(X^{**}, X^*)$ .  $\square$

## 14 Reflexive Spaces cont. 20210226

Real analysis:  $L_p(\Omega, \Sigma, \mu) = \{[f] \mid f : \Omega \rightarrow \mathbb{K}, f \text{ measurable}, \int |f|^p d\mu < \infty\}$ , where  $\Omega$  is a set,  $\Sigma$  is a  $\sigma$ -algebra and  $\mu$  is a  $\sigma$ -additive measure. Recall

- Simple functions  $f = \sum_{j=1}^n \alpha_j 1_{E_j}$  are dense.

- $\|f\|_p = \sup_{\|g\|_{p'} \leq 1} |\int fg d\mu|$ .

Use Hölder inequality, say  $\|f\|_p = 1$ , then  $g = \text{sgn}(f) \cdot |f|^{p/p'}$ .

**Corollary 14.1.** If  $1 \leq p \leq \infty$ , then  $L_{p'}$  embeds isometrically into  $L_p^*$ ,

$$\begin{aligned} \iota_{p'} : L_{p'} &\rightarrow L_p^* \\ g &\mapsto \left( \iota_{p'}(g) : f \mapsto \iota_{p'}(g)(f) = \int fg d\mu \right) \end{aligned}$$

and  $\|f\|_p = \|\iota_{p'}(g) : L_p \rightarrow \mathbb{K}\|$ .

**Theorem 14.2.** Let  $1 < p < \infty$  and assume  $L_p$  is reflexive. Then  $L_{p'}^* = L_p$ .

(Here we check isometric isomorphism, there are two type of isomorphisms for Banach spaces, see more [here](#))

*Proof.* Let  $\varphi : L_{p'} \rightarrow \mathbb{K}$  with  $\|\varphi\|_{L_{p'}^*} = 1$ . Recall  $L_{p'} \hookrightarrow L_p^*$  is an isometry. By Hahn-Banach extension, there exists a  $\hat{\varphi} : L_p^* \rightarrow \mathbb{K}$ , with  $\hat{\varphi}|_{L_{p'}} = \varphi$ .

$$\begin{array}{ccc} L_{p'} & \xhookrightarrow{\iota_{p'}} & L_p^* \\ \varphi \downarrow & \swarrow \exists \hat{\varphi} & \\ \mathbb{K} & & \end{array}$$

To show  $\iota_{p'}$  is surjective, take  $\eta \in L_p^*$ . If we can find  $g \in L_{p'}$  such that  $\int fg \, d\mu = \eta(f)$ , then  $\iota_{p'}(g) = \eta$  and we are done. Such a  $g$  exists by commutativity and reflexivity

$$\varphi(g) = \hat{\varphi}(\iota_{p'}(g)) = \iota_{p'}(g)(f) = \int fg \, d\mu \implies \iota_{p'}(g)(f) = \eta(f).$$

□

**Example 14.3** (Discrete case). Let  $\Omega = I$ ,  $\Sigma = 2^I$ ,  $\mu$  be the counting measure. If  $I = \mathbb{N}$ , then

$$L_p(\mathbb{N}, \Sigma, \mu) = \ell_p = \{ (x_n) \mid \sum_n |x_n|^p < \infty \}.$$

What is the  $f$  defining the functional  $\varphi : \ell(\mathbb{N}) \rightarrow \mathbb{K}$ ? Well,  $f$  is given by a sequence  $(y_n) = ((0, 0, \dots, \frac{1}{n}, \dots, 0, 0))$ . One can show that the

$$\|y_n\|_{p'} = \sup_n \left( \sum_{k=1}^n |y_k|^{p'} \right)^{\frac{1}{p'}} = \sup \{ \varphi(x_n) \mid \|x_n\| \leq 1 \}.$$

Prove using Hölder.

**Theorem 14.4.**  $\ell_p^* = \ell_{p'}$  for  $1 < p < \infty$ .

**Remark 14.5.** For  $I = \mathbb{N}$ , let  $c_0 = \{ (x_n) \in \ell_\infty \mid \lim x_n = 0 \}$ . Then  $c_0^* = \ell^1$ ,  $c_0^{**} = \ell_1^* = \ell_\infty$ .

**Corollary 14.6.**  $B_{\ell_1} \subset B_{\ell_\infty^*}$  is  $\sigma(\ell_\infty^*, \ell_\infty)$ -dense.

This means for any  $\varphi \in \ell_\infty^*$ , for any  $f_i \in \ell_\infty$ , there exists  $g \in \ell_1$ , with  $\|g\|_{\ell_1} \leq \|\varphi\|$ , such that

$$|\varphi(f_i) - f_j(g)| \leq \epsilon \text{ i.e. arbitrarily closed.}$$

Or there exists a net  $(g_\alpha) \in \ell_1$  with  $\|g_\alpha\|_{\ell_1} \leq \|\varphi\|$ , such that

$$\varphi(f) = \lim_\alpha f(g_\alpha) = \lim_\alpha \sum_{n \in \mathbb{N}} f(n)g_\alpha(n).$$

**Remark 14.7.** Let  $\varphi : \ell_\infty \rightarrow \mathbb{K}$ , and assume  $\varphi(1) = 1$ . TFAE

- $\|\varphi\| = 1$
- $\forall g \geq 0, \varphi(g) \geq 0$ .

We call this **positive functionals**.

Define the **state space**  $S(\ell_\infty) = \{ \varphi \mid \varphi(1) = 1, \|\varphi\| = 1 \}$ . Then discrete probability measures are dense in the state space. Indeed if  $\varphi(1) = 1$  and  $\|\varphi\| = 1$ , then there is  $g_\alpha \in \ell_1$  with  $g_\alpha(1) = 1$ ,  $\|g_\alpha\| \leq 1$  and  $g_\alpha(f) \rightarrow \varphi(f)$ . That is  $g_\alpha \rightarrow \varphi$  in  $\sigma(\ell_\infty^*, \ell_\infty)$ .

**Lemma 14.8.**  $\|g_\alpha\|_{\ell_1} = 1$  and  $\sum_n g_\alpha(n) = 1$  implies  $g_\alpha \geq 0$ .

This means  $g_\alpha$  are discrete probability measures because  $\varphi(f) = \lim_\alpha \sum_{n \in \mathbb{N}} f(n)g_\alpha(n)$  exists.

**Theorem 14.9.** Let be  $\varphi : C(K) \rightarrow \mathbb{C}$  be such that  $\varphi(1) = 1$  and  $\|\varphi\| = 1$ . Then there exists a net  $(x_j)_{j=1}^{n(\alpha)} (\lambda_j^\alpha)_{j=1}^{n(\alpha)}$ , where  $\sum \lambda_j^\alpha = 1$  such that

$$\varphi(f) = \lim_\alpha \sum_{j=1}^{n(\alpha)} f(x_j^\alpha) \cdot \lambda_j^\alpha.$$

*Proof.* The Banach space  $C(K)$  embeds into the Banach space  $\ell_\infty(K)$  (view this as a discrete index set, no topology) isometrically via  $\iota(f)(k) = f(k)$ .



$$\begin{array}{ccc}
C(K) & \xhookrightarrow{\iota} & \ell_\infty(K) \\
\varphi \downarrow & \swarrow \exists \hat{\varphi} & \\
\mathbb{K} & & 
\end{array}$$

As previous seen,  $\hat{\varphi}$  exists by Hahn-Banach extension. Also have  $\hat{\varphi}(1) = 1$ ,  $\|\hat{\varphi}\| = 1$  and then  $\hat{\varphi} \in S(\ell_\infty(K))$ . By previous remark, and also the fact that every function in  $\ell_1$  is support on a countable number of points

$$\hat{\varphi}(F) = \lim_\alpha \sum_{(t_j)} F((t_j^\alpha)) \cdot \lambda_j^\alpha$$

where  $\sum_\alpha \lambda_j^\alpha = 1$ . Can replace LHS of this equation by  $\lim_\alpha \lim_M \sum_{j=1}^M \lambda_j^{\alpha, M} \cdot F(t_j^\alpha)$  with  $\sum_{j=1}^M \lambda_j^{\alpha, M} = 1$  (technical detail skipped). But

$$F = \iota(f) = \lim_{\alpha'} \sum_{j=1}^{M(\alpha')} \lambda_j^{\alpha'} \cdot f(t_j^{\alpha'}).$$

□

Consider  $C[0, 1]$ . It is separable (admits a countable dense subset), whereas  $\ell_\infty(\mathbb{N})$  is non-separable.

**Corollary 14.10.** *If  $\varphi : C[0, 1] \rightarrow \mathbb{C}$ , with  $\varphi(1) = 1$  and  $\|\varphi\| = 1$ . Then there exists a sequence  $(t_j^n)(\lambda_j^n)$ , where  $\sum \lambda_j^n = 1$  such that*

$$\varphi(f) = \lim_\alpha \sum_{j=1}^{M(n)} f(x_j^n) \cdot \lambda_j^n.$$

## 15 Riesz-Thorin Theorem 20210301

**Theorem 15.1** (Riesz-Thorin). *Let  $A$  be a linear operator and let  $1 \leq p_0, p_1, q_0, q_1 \leq \infty$  where  $p_0 \neq p_1$  and  $q_0 \neq q_1$ . Suppose  $A : L_{p_0} \rightarrow L_{q_0}$  is bounded and  $A : L_{p_1} \rightarrow L_{q_1}$  is bounded. Let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

where  $\theta \in (0, 1)$ . Then

$$\|A\|_{L_p \rightarrow L_q} \leq \|A\|_{L_{p_0} \rightarrow L_{q_0}}^{1-\theta} \cdot \|A\|_{L_{p_1} \rightarrow L_{q_1}}^\theta.$$

If we call  $\|A\|_{L_{p_0} \rightarrow L_{q_0}}^{1-\theta} = M_0$  and  $\|A\|_{L_{p_1} \rightarrow L_{q_1}}^\theta = M_1$ , then  $\|A\|_{L_p \rightarrow L_q} \leq M_0^{1-\theta} \cdot M_1^\theta$ .

In class  $L_p$  is replaced with  $\ell_p$ , but there is a more generalized version in literature. I leave  $L_p$  in the Theorem to remind myself this fact. For  $1 < p < q < r < \infty$ ,  $L_p \cap L_q \subset L_r \subset L_p + L_q$ . In our case (finite dimensional), the same matrix makes sense and  $A : \ell_{p_0} \cap \ell_{p_1} \rightarrow \ell_{q_0} + \ell_{q_1}$ .

We will use the following lemma to prove Riesz-Thorin Theorem.

**Lemma 15.2** (Hadamard's Three-Line Theorem). *Suppose  $f(z)$  is bounded and continuous function on  $0 \leq \operatorname{Re}(z) \leq 1$  and analytic in the interior. Denote*

$$M_\theta = \sup_{y \in \mathbb{R}} |f(\theta + iy)|.$$

Then  $M_\theta \leq M_0^{1-\theta} M_1^\theta$  for  $\theta \in (0, 1)$ .

If we control the function on boundary then we control the function in the interior.

**Example 15.3.** Map from a strip to a disk. Let  $f(z) = \sum a_n z^n$  be an analytic function,  $a_0 = f(0) = \frac{1}{2\pi i} \int \frac{f(z)}{z} dz$ . Then

$$|a_0| \leq \int |f(z)| dz = \frac{1}{2\pi i} \int f(e^{i\theta}) d\theta \leq \sup |f(e^{i\theta})|.$$

*Proof.* [Ref.](#)

Recall  $\ell_p \hookrightarrow \ell_{p'}^*$  isometrically. So

$$\|A\|_{\ell_p \rightarrow \ell_q} = \sup \left\{ \sum_{kj} y_j \cdot A_{jk} \cdot x_k \mid \sum |x_i|^p \leq 1, \sum |y_j|^{q'} \leq 1 \right\}.$$

Assume  $\sum |x_i|^p = 1$  and  $\sum |y_j|^{q'} = 1$ . Define a function

$$x_k(z) = \frac{x_k}{|x_k|} |x_k|^{p \cdot \left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)} \quad \text{and} \quad y_j(z) = \frac{y_j}{|y_j|} |y_j|^{q' \cdot \left(\frac{1-z}{q_0'} + \frac{z}{q_1'}\right)}.$$

Then  $F(z) = \sum_{jk} y_j(z) \cdot A_{jk} \cdot x_k(z)$  is also analytic. Take  $0 \leq \operatorname{Re}(z) \leq 1$  and define  $G(z) = M_0^{z-1} M_1^{-z} F(z)$ .

**Claim 15.4.**  $|G(it)| \leq 1$  and  $|G(1+it)| \leq 1$ .

Take  $z = it$ , then

$$\begin{aligned} G(it) &= \sum_{jk} \frac{x_k}{|x_k|} |x_k|^{p \cdot \left(\frac{1-it}{p_0} + \frac{it}{p_1}\right)} \cdot A_{jk} \cdot \frac{y_j}{|y_j|} |y_j|^{q' \cdot \left(\frac{1-it}{q_0'} + \frac{it}{q_1'}\right)} \\ &= \sum_{jk} \alpha_k |x_k|^{\frac{p}{p_0}} \cdot A_{jk} \cdot \beta_j |y_j|^{\frac{q'}{q_0'}} \\ &= \|A\|_{\ell_{p_0} \rightarrow \ell_{q_0}} \cdot \left\| \sum_{jk} \alpha_k |x_k|^{\frac{p}{p_0}} \right\|^{\frac{p_0}{p}} \cdot \left\| \sum_{jk} \beta_j |y_j|^{\frac{q'}{q_0'}} \right\|^{\frac{q_0'}{q'}} \leq 1, \end{aligned}$$

where  $|\alpha_k|, |\beta_j| = 1$  (???). Similarly for  $G(1+it)$ .

The Three-Line Lemma gives  $|G(\theta)| \leq 1$ . Note that

$$\begin{aligned} G(\theta) &= M_0^{\theta-1} M_1^{-\theta} \sum_{jk} \frac{x_k}{|x_k|} |x_k|^{p \cdot \left(\frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right)} \cdot A_{jk} \cdot \frac{y_j}{|y_j|} |y_j|^{q' \cdot \left(\frac{1-\theta}{q_0'} + \frac{\theta}{q_1'}\right)} \\ &= M_0^{\theta-1} M_1^{-\theta} \sum_{jk} \frac{x_k}{|x_k|} |x_k|^{p \cdot \frac{1}{p}} \cdot A_{jk} \cdot \frac{y_j}{|y_j|} |y_j|^{q' \cdot \frac{1}{q'}} \\ &= M_0^{\theta-1} M_1^{-\theta} \sum_{jk} x_k \cdot A_{jk} \cdot y_j. \end{aligned}$$

This implies  $|\sum_{jk} x_k \cdot A_{jk} \cdot y_j| \leq M_0^{1-\theta} M_1^\theta$ . □

**Corollary 15.5.** Assume  $x, y$  are complex numbers and  $r \leq s \leq r'$  then

$$(|x+y|^r + |x-y|^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (|x|^s + |y|^s)^{\frac{1}{s}}.$$

**Example 15.6.** When  $r = 2$ ,  $x, y \in \mathbb{R}$ , then we get the parallelogram law

$$(|x+y|^2 + |x-y|^2)^{\frac{1}{2}} = (x^2 + 2xy + y^2 + x^2 - 2xy + y^2)^{\frac{1}{2}} = \sqrt{2} \cdot (x^2 + y^2)^{\frac{1}{2}}.$$

*Proof.* Take the matrix  $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ . Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = (|x+y|^r + |x-y|^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (|x|^s + |y|^s)^{\frac{1}{s}}$$

For the case  $s \geq 2$ ,

$$\|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2} = \sup \left\{ \max(|x+y|, |x-y|) \mid |x| \leq 1, |y| \leq 1 \right\} \leq 2.$$

$$\|A\|_{\ell_2^2 \rightarrow \ell_2^2} \leq (|x+y|^2 + |x-y|^2)^{\frac{1}{2}} = \sqrt{2} \cdot (x^2 + y^2)^{\frac{1}{2}} \leq \sqrt{2}.$$

Using Riesz-Thorin Theorem we obtain

$$\|A\|_{\ell_s^2 \rightarrow \ell_s^2} \leq 2^{1-\theta} \cdot \sqrt{2}^\theta = 2^{1-\frac{\theta}{2}} = 2^{1-\frac{1}{s}},$$

with the last step given by  $\frac{1}{s} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$ .

For  $1 \leq s \leq 2$ , we note that  $r \leq s \leq r'$  implies  $s' \leq r$ . It suffices to consider  $r = s'$ . Again Riesz-Thorin Theorem gives

$$\|A\|_{\ell_s \rightarrow \ell_s} \leq \|A\|_{\ell_1 \rightarrow \ell_\infty}^{1-\theta} \cdot \|A\|_{\ell_2 \rightarrow \ell_2}^\theta \leq 1^{1-\theta} \cdot \sqrt{2}^\theta = 2^{\frac{\theta}{s'}} = 2^{1-\frac{1}{s}},$$

with  $\frac{1}{s} = \frac{1-\theta}{1} + \frac{\theta}{2}$  and  $\frac{1}{s'} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$ . Note that

$$\|A\|_{\ell_1 \rightarrow \ell_\infty} = \max_{jk} |A_{jk}|.$$

□

## 16 Clarkson's inequality 20210303

Clarkson's inequality  $\implies$  Uniform convexity  $\implies L_p$  is reflexive  $\implies L_p^* = L_{p'}$

We want to use the Clarkson's inequality (proof ref. Boa) to prove uniform convexity of  $L_p$ .

**Theorem 16.1** (Reformulation of Riesz-Thorin). *Let  $A$  be a matrix. Consider  $F(x, y) = \log \|A\|_{\ell_{1/x} \rightarrow \ell_{1/y}}$ . Then  $F$  is a convex function.*

Now let  $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} : \ell_p^2(\mathbb{C}) \rightarrow \ell_q^2(\mathbb{C})$ . Thus  $\|A\|_{\ell_s \rightarrow \ell_r} \leq 2^{1-\frac{1}{s}}$  for all  $s \leq r \leq s'$ .

We have seen [Ref.](#)

- $\|A\|_{\ell_2^2 \rightarrow \ell_2^2} \leq \sqrt{2}$ , and  $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$  is unitary (preserves inner product).
- $\|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2} = 2$ .
- $\|A\|_{\ell_1^2 \rightarrow \ell_\infty^2} = 1$ .

**Remark 16.2.**  $\|A\|_{\ell_p^2 \rightarrow \ell_q^2} = \|A\|_{\ell_{q'}^2 \rightarrow \ell_{p'}^2}$ .

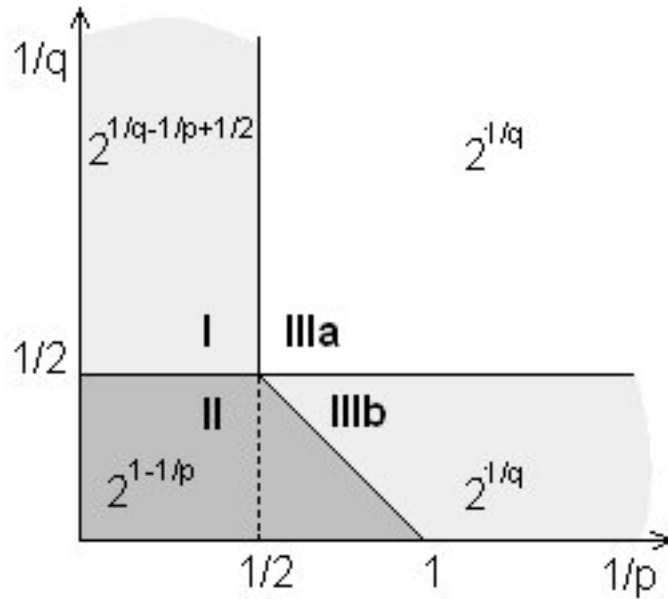


Figure 1: Picture taken from [here](#)

Explanation of the picture: by the value at a point, I mean the power of 2. (If I call the value  $\alpha$ , then  $2^\alpha$  is an upper bound for  $\|A\|_{\ell_p^2 \rightarrow \ell_q^2}$ .)

- (Region III) The point  $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, \frac{1}{2})$  corresponds to  $\|A\|_{\ell_2^2 \rightarrow \ell_2^2}$  and has value

$$\log_2(\sqrt{2}) = \frac{1}{2}.$$

- (Region IIIa) The point  $(\frac{1}{p}, \frac{1}{q}) = (1, 1)$  corresponds to  $\|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2}$  and has value  $\log_2(2) = 1$ . By the remark above  $\|A\|_{\ell_1^2 \rightarrow \ell_1^2} = \|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2} = 2$ , so the point  $(\frac{1}{p}, \frac{1}{q}) = (1, 1)$  also has value 1.
- (Region IIIa) Using convexity, for  $2 < p < \infty$ , point  $(\frac{1}{p}, \frac{1}{q})$  on the line  $y = x$  has value  $\frac{1}{q}$ .
- (Region IIIb) The point  $(\frac{1}{p}, \frac{1}{q}) = (1, \infty)$  corresponds to  $\|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2}$  and has value  $\log_2(1) = 0$ .
- (Region IIIb) Using convexity, for  $2 < p < \infty$ , point  $(\frac{1}{p}, \frac{1}{q})$  on the line  $y = 1 - x$  has value  $\frac{1}{q}$ . Vertical lines between the lines  $y = x$  and  $y = 1 - x$  has value  $\frac{1}{q}$ .
- (Region II) For  $1 \leq s \leq 2$  we have

$$\|A\|_{\ell_s \rightarrow \ell_{s'}} \leq \|A\|_{\ell_1 \rightarrow \ell_\infty}^{1-\theta} \cdot \|A\|_{\ell_2 \rightarrow \ell_2}^\theta \leq 2^{1-\frac{1}{s}} = 2^{\frac{1}{s'}}.$$

So  $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{s}, \frac{1}{s'})$  has value  $1 - \frac{1}{s}$ .

$$\|A\|_{\ell_{s'} \rightarrow \ell_{s'}} = \|A\|_{\ell_s \rightarrow \ell_s} \leq \|A\|_{\ell_1 \rightarrow \ell_\infty}^{1-\theta} \cdot \|A\|_{\ell_2 \rightarrow \ell_2}^\theta \leq 2^{1-\frac{1}{s}}.$$

For  $s \leq r \leq s'$  (???)

$$\|A\|_{\ell_s \rightarrow \ell_r} = \|A\|_{\ell_s \rightarrow \ell_s}^{1-\theta} \cdot \|A\|_{\ell_s \rightarrow \ell_{s'}}^\theta \leq (2^{1-\frac{1}{s}})^{1-\theta} \cdot (2^{1-\frac{1}{s'}})^\theta = 2^{1-\frac{1}{s'}}.$$

**Theorem 16.3** (Minkowski's inequality). *Let  $L_p(\ell_q)$  and  $\ell_q(L_p)$  be the space of functions with the norm*

$$\|f\|_{L_p(\ell_q)} = \left( \int \left( \sum_k |f_k(\omega)|^q \right)^{\frac{p}{q}} d\mu \right)^{\frac{1}{p}},$$

$$\|f\|_{\ell_q(L_p)} = \left( \sum_k \left( \int |f_k(\omega)|^p d\mu \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

*If  $p \leq q$ , then  $L_p(\ell_q) \subset \ell_q(L_p)$  and  $\ell_p(L_q) \subset L_q(\ell_p)$ .*

*Proof.* We want to show  $\|f\|_{\ell_q(L_p)} \leq \|f\|_{L_p(\ell_q)}$ , i.e.

$$\left( \sum_k \left( \int |f_k(\omega)|^p d\mu \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq \left( \int \left( \sum_k |f_k(\omega)|^p \right)^{\frac{p}{q}} d\mu \right)^{\frac{1}{p}}.$$

Let  $p \leq q$  and  $r = \frac{q}{p} \geq 1$ . The continuous version of triangle inequality says  $\|\int g d\mu\|_r \leq \int \|g\|_r d\mu$ . (Prove this first for simple function and approximation.) Define  $g(\omega) = |f_k(\omega)|^q$ , then

$$\left\| \int g(\omega) d\mu \right\|_{\ell_r} \leq \int \|g(\omega)\|_{\ell_r} d\mu$$

By definition of  $\|\cdot\|_{\ell_r}$

$$\left( \sum_k \left( \int |f_k(\omega)|^p d\mu \right)^r \right)^{\frac{1}{r}} \leq \int \left( \sum_k |f_k(\omega)|^{pr} \right)^{\frac{1}{r}} d\mu,$$

so

$$\left( \sum_k \left( \int |f_k(\omega)|^q d\mu \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \leq \int \left( \sum_k |f_k(\omega)|^q \right)^{\frac{p}{q}} d\mu.$$

Taking  $q$ -th root on both sides gives the first inclusion. The second inclusion is proved using triangle inequality in  $\ell_p$ .  $\square$

## 17 Uniform convexity of $L_p$ 20210305

Generalize the scalar valued inequality to function valued inequality.

**Theorem 17.1.** For  $f, g \in L_p$  and  $r \leq p \leq s$  then

$$(\|f + g\|_p^r + \|f - g\|_p^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (\|f\|_p^s + \|g\|_p^s)^{\frac{1}{s}}.$$

*Proof.* Recall (Minkowski inequality or generalized Fubini Theorem).

$$L_p(\ell_r) \subset \ell_r(L_p) \quad \text{if } p \leq r \quad \text{and} \quad (2)$$

$$\ell_s(L_p) \subset L_p(\ell_s) \quad \text{if } s \leq p \quad (3)$$

Let  $f, g \in L_p(\Omega, \Sigma, \mu)$  then

$$\begin{aligned}
LHS &= (\|f + g\|_p^r + \|f - g\|_p^r)^{\frac{1}{r}} \\
&\leq \left( \int |f(\omega) + g(\omega)|^r + |f(\omega) - g(\omega)|^r \frac{p}{r} d\mu \right)^{\frac{1}{p}} \quad (\text{by (2)}) \\
&\leq 2^{1-\frac{1}{s}} \cdot \left( \int |f(\omega)|^s + |g(\omega)|^s \frac{1}{s} \cdot p d\mu \right)^{\frac{1}{p}} \quad (\text{by Corollary (15.5)}) \\
&\leq 2^{1-\frac{1}{s}} \cdot \left( \int (|f(\omega)|^p)^{\frac{s}{p}} + (|g(\omega)|^p)^{\frac{s}{p}} d\mu \right)^{\frac{1}{s}} = RHS. \quad (\text{by (3)})
\end{aligned}$$

□

Now we show the above theorem implies uniform convexity.

**Theorem 17.2.** *The space  $L_p$  is uniformly convex for  $1 < p < \infty$ . In particular,  $L_p$  is reflexive.*

We need to show  $\forall \epsilon > 0, \exists \delta > 0$  with  $\|f\|_p \leq 1, \|g\|_p \leq 1$  and  $\|f - g\|_p > \epsilon$  then  $\|\frac{f+g}{2}\|_p \leq 1 - \delta$ .

**Example 17.3.** When  $p = 2, X = L_2(\Omega, \mathbb{R})$ . Fixing  $\epsilon > 0$ , if we take  $\delta = \frac{\epsilon}{8}$  then

$$\begin{aligned}
(\|f + g\|_2^2 + \|f - g\|_2^2)^{\frac{1}{2}} &\leq \sqrt{2} \cdot (\|f\|_2^2 + \|g\|_2^2)^{\frac{1}{2}} \leq \sqrt{2} \cdot \sqrt{2} \\
\text{and } \|f + g\|_2^2 + \|f - g\|_2^2 &> \|f + g\|_2^2 + \epsilon^2.
\end{aligned}$$

So  $\|f + g\|_2^2 + \epsilon^2 \leq 4$ , i.e.  $\|\frac{f+g}{2}\|_2 \leq \sqrt{1 - \frac{\epsilon^2}{4}} \leq 1 - \frac{\epsilon}{8} = 1 - \delta$ .

*Proof.* Assume  $p \geq 2, s = \min(p, p')$  and  $r = \max(p, p')$ , so that  $s \leq p \leq r$ . Fixing  $\epsilon > 0$ , and assume  $\|f\|_p \leq 1, \|g\|_p \leq 1$  and  $\|f - g\|_p > \epsilon$ . Then Theorem 17.1 gives

$$(\|f + g\|_p^r + \|f - g\|_p^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (\|f\|_p^s + \|g\|_p^s)^{\frac{1}{s}} \leq 2^{1-\frac{1}{s}} \cdot 2^{\frac{1}{s}} = 2.$$

Same as previous example

$$\left( \left\| \frac{f+g}{2} \right\|_p^r + \left( \frac{\epsilon}{2} \right)^r \right)^{\frac{1}{r}} < \left( \left\| \frac{f+g}{2} \right\|_p^r + \left\| \frac{f-g}{2} \right\|_p^r \right)^{\frac{1}{r}} \leq 1.$$



So we can choose  $\delta$  ( $\delta = O(\frac{\epsilon}{2})^r$ ). Note that when  $p \rightarrow \infty$ ,  $(\frac{\epsilon}{2})^r \rightarrow 0$ . □

**Example 17.4.** For  $1 < p, q, \infty$ , the Sobolov space

$$W_{p,q}^m = \left\{ f \in C(\mathbb{R}) \mid \|f\| = \left( \int \left( \sum_{k=1}^m |f^{(k)}(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty \right\}$$

is uniformly convex. Uniformly convex and reflexive properties pass to subspaces. (Uniform convexity is a property of two points.) We can embed  $W_{p,q}^m$  into  $L_p(\ell_q^m) = Y$  and show  $Y$  is uniformly convex.

Our goal is to find  $r, s$  so that

$$(\|F + G\|_Y^r + \|F - G\|_Y^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (\|F\|_Y^s + \|G\|_Y^s)^{\frac{1}{s}}.$$

We need the inclusions  $L_p(\ell_q(\ell_r)) \subset \ell_r(L_p(\ell_q))$  and  $L_s(\ell_p(\ell_q)) \subset L_p(\ell_q(\ell_s))$ . These require  $p, q \leq r$  and  $s \leq p, q$ . Hence  $s = \min(p, q, p', q')$  and  $r = \max(p, q, p', q')$ . Check this gives the above inequality.

## 18 Uniform Boundedness and Open Mapping 20210308

**Theorem 18.1** (Uniform boundedness principle). *Let  $X$  be a Banach space and  $Y$  a normed vector space. Suppose that  $\mathcal{F}$  is a collection of continuous linear operators from  $X$  to  $Y$ . If  $\mathcal{F}$  is pointwise bounded:*

$$\sup_{T \in \mathcal{F}} \|T(x)\|_Y < \infty, \forall x \in X$$

*then  $\mathcal{F}$  is norm-bounded:*

$$\sup_{T \in \mathcal{F}} \|T\|_{B(X,Y)} = \sup_{T \in \mathcal{F}, \|x\|=1} \|T(x)\|_Y < \infty.$$

Application: If  $\{T_n\} \subset L(X, Y)$  is a sequence such that  $\lim_n T_n(x) = y$  exists for all  $x$ , then  $\sup_n \|T_n\| < \infty$ .

*Proof.* [Ref.](#) Let  $\mathcal{F}$  be a family and start with a subset (not a subspace)

$$X_n = \{ x \mid \sup_{T \in \mathcal{F}} \|Tx\| \leq n \} \subset X.$$

**Claim 18.2.**  $X_n$  is closed.

Assume  $x_\alpha \rightarrow x$  and we have  $\|Tx_\alpha\| \leq n$  for all  $\alpha$  and  $T \in \mathcal{F}$ . Since  $T$  is continuous,  $\lim \|Tx\| = \limsup_\alpha \|Tx_\alpha\| \leq n$  (not clear what the first limit is taking with respect to), and

$$\|Tx\| = \|T \lim_\alpha x_\alpha\| = \lim_\alpha \|Tx_\alpha\| \leq \limsup_\alpha \|Tx_\alpha\| \leq n.$$

Note that  $\cup_n X_n = X$  by assumption, and  $X_1 \subset X_2 \subset \cdots \subset X_n$ .

Assume that the  $\text{int}(X_n) = \emptyset$  for all  $n$ , then  $O_n = X_n^c$  is dense for all  $n$ . Baire's Category Theorem gives  $\cap_n O_n$  is dense, in particular nonempty. But  $\cap_n O_n = (\cup_n X_n)^c = \emptyset$  gives a contradiction. So there exists  $n$  such that  $\text{int}(X_n) \neq \emptyset$ .

Take  $x_0 \in X$ ,  $\delta > 0$  and  $\|y\| < \delta$  be such that make  $B_\delta(x_0) \subset X_n$ . Then  $\|T(x_0+y)\| \leq n$  for all  $T \in \mathcal{F}$ . Therefore

$$\|T(y)\| = \left\| \frac{T(x_0+y) - T(x_0-y)}{2} \right\| \leq \frac{\|T(x_0+y)\| + \|T(x_0-y)\|}{2} \leq n.$$

and

$$\|T(y)\| = \left\| T\left(\frac{y}{\|y\|} \cdot \frac{\delta}{2}\right) \right\| \cdot \frac{2\|y\|}{\delta} \leq n \cdot \frac{2\|y\|}{\delta}$$

implies  $\|T\| \leq \frac{2n}{\delta} \implies \sup_{T \in \mathcal{F}} \|T\| \leq \frac{2n}{\delta}$  □

This argument also works for convex maps with values in another space.

A famous example is the following.

**Example 18.3.** Consider  $X = C[-\pi, \pi]$ . Define the **truncation of Fourier series**

$$P_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikt}, \text{ where } \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Note that  $P_n \in L(X, X)$ . Recall in  $L_2$ ,  $\|f\|_2 = \left( \sum_k |\hat{f}(k)|^2 \right)^{\frac{1}{2}}$  and  $P_n(f) \rightarrow f$  in  $L_2$ .

If we had that  $P_n(f) \rightarrow f$  uniformly:  $\lim_{n \rightarrow \infty} P_n(f) = f$  for all  $f \in X$ . That is, if  $\{P_n\}$  were pointwise bounded:  $\sup_n \|P_n(f)\| < \infty$ , then uniform boundedness would imply  $\sup_n \|P_n\| < \infty$ . We will prove later that  $\|P_n\| \geq C(1 + \ln n)$  (see Theorem 19.1 below).

This gives a contradiction. So there exists a continuous  $f$  such that  $\lim_{n \rightarrow \infty} \sum_{k=-n}^n \hat{f}(k) e^{ikt}$  diverges. Another fact says that the space of trigonometric polynomials  $p(t) = \sum_{k=-n}^n a_k e^{ikt}$  are dense, and  $P_n(p) \rightarrow p$  uniformly. The partial Fourier series converges almost everywhere.

Let  $X$  be a Banach space,  $D \subset X$ . What does it mean to be bounded? Two answers

1.  $\exists R$  such that  $D \subset RB_X$
2.  $D$  is weakly bounded:  $\forall x^* \in X^*$ ,  $x^*(D) \subset (-R_x, R_x)$  (or  $\{z \mid |z| \leq R_x\}$  in complex case)

With respect to the weak topology, weak bounded implies norm bounded.

**Corollary 18.4.** *Let  $X$  be a Banach space,  $D \subset X$  such that  $x^*(D)$  is bounded in  $\mathbb{K}$  for all  $x^* \in X^*$ . Then  $D$  is bounded.*

*Proof.* Let  $\mathcal{F} = \{\varphi_x \mid x \in D\} \subset L(X^*, \mathbb{K})$ , where  $\varphi_x(x^*) = x^*(x)$ . We know

$$\sup_{x \in D} \varphi_x(x^*) = \sup_{x \in D} |x^*(x)| < \infty, \forall x^*.$$

Uniform boundedness principle implies  $\sup_{x \in D} \|\varphi_x\| \leq C$ . Then  $D$  is bounded, because

$$\sup_{x \in D} \|x\| = \sup_{x \in D} \sup_{\|x^*\| \leq 1} |x^*(x)| = \sup_{x \in D} \|\varphi_x\| \leq C.$$

□

**Theorem 18.5** (Open mapping theorem). *Let  $X$  and  $Y$  be Banach spaces and  $T : X \rightarrow Y$  be linear and surjective. Then  $T$  is open.*

*Proof. Step 1.* Let  $\epsilon > 0$  and  $Y_n = \overline{T(B_X(0, n\epsilon))}$ . Then  $Y = \cup_n Y_n$ . Uniform boundedness principle implies one of the  $Y_n$ 's has nonempty interior. So there exists  $\tilde{x}$  and  $\delta > 0$  such that,  $B_Y(\tilde{x}, \delta) \subset Y_n$ . WLOG we can assume  $\tilde{x} = 0$ , so  $B_Y(0, \delta) \subset Y_n$ . Hence for some  $\delta' > 0$ , we have  $B_Y(0, \delta) \subset \overline{T(B_X(0, \epsilon))}$ . Our goal is to remove this closure.

*Step 2.* Choose  $\epsilon_k$  so that  $\sum \epsilon_k < \epsilon$ . According to the previous step, we know that there exists  $\delta_k$  such that  $B_Y(0, \delta_k) \subset \overline{T(B_X(0, \epsilon_k))}$  for all  $k$ . WLOG we can assume  $\delta_k \rightarrow 0$  because we can always take smaller value for  $\delta$ 's.

Now let  $y \in Y$  with  $\|y\| < \delta_0$ . Since  $B_Y(0, \delta_0) \subset \overline{T(B_X(0, \epsilon_0))}$  we can find  $x_0$  in  $B_X(0, \epsilon_0)$  such that  $\|y - T(x_0)\| < \delta_1$ . Call  $y_1 = y - T(x_0)$ . Then we can find we can find  $x_1$  in  $B_X(0, \epsilon_1)$  such that  $\|y - T(x_0) - T(x_1)\| = \|y_1 - T(x_1)\| < \delta_2$ . Iterate this step and we have a sequence of  $x_k$  such that

$$\|y - T(x_0) - T(x_1) - \cdots - T(x_k)\| < \delta_k. \quad (4)$$

Since  $\delta_k \rightarrow 0$ ,  $y = \sum_k T(x_k)$  by construction. Moreover  $\sum_k x_k$  converges to some point  $x \in X$  because  $\|\sum_k x_k\| \leq \sum_k \|x_k\| \leq \sum_k \epsilon_k < \epsilon < \infty$  (completeness of the Banach space). Note that  $\|x\| < \epsilon$  and passing limit of inequality (4) gives  $\|y - T(x)\| = 0$ . So  $y = T(x) \in T(B_X(0, \epsilon))$ . This proves for all  $\epsilon$ , there exists  $\delta$  such that  $B_Y(0, \delta) \subset T(B_X(0, \epsilon))$ . (**trick** Write  $x$  and  $y$  as converging sequences and use the completeness of Banach spaces).

*Step 3.* Take  $O$  an open set. □

**Example 18.6.**

- There exists a map  $T : \ell_\infty \rightarrow \ell_2$  which is linear and onto.

- There is no map  $T : \ell_\infty \rightarrow \ell_{4/3}$ .

## 19 $\|P_n\|$ is Unbounded 20210310

**Theorem 19.1.** Let  $X = C[-\pi, \pi]$  and let

$$P_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikt}, \text{ where } \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Then  $\|P_n\| \geq C(1 + \ln n)$ .

**Lemma 19.2.** If  $T : C(K) \rightarrow C(K)$ , then

$$\|T\| = \sup_{x \in K} \|T^*(\delta_x)\|_{C(K)^*},$$

where  $\delta_x \in C(K)^*$  is defined by  $\delta_x(f) = f(x)$ .

*Proof.* Certainly

$$\|T\| = \|T^*\| = \sup_{\varphi \in C(K)^*, \|\varphi\| \leq 1} \|T^*(\varphi)\| \geq \sup_{x \in K} \|T^*(\delta_x)\|.$$

It remains to show “ $\leq$ ”.

*Step 1.* Take  $\varphi = \sum_x \alpha_x \delta_x$ , we first prove  $\|\varphi\| = \sum_x |\alpha_x|$ . One one hand  $\|\varphi\| \leq \sum_x |\alpha_x|$  because

$$|\varphi(f)| = \left| \sum_x \alpha_x f(x) \right| \leq \sum_x |\alpha_x| \cdot |f(x)| \leq \sum_x |\alpha_x| \cdot \|f\|_\infty.$$

To show the other direction, we need to find  $\tilde{f}(x_j) = \epsilon_j$ , with  $|\epsilon| = 1$ , for a compact topological space  $K$ . Recall Urysohn’s lemma.

**Lemma 19.3** (Urysohn’s lemma). A topological space  $(X, \tau)$  is normal if and only if for every pair of disjoint nonempty closed subsets  $C, D \subset X$  there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  for all  $x \in C$  and  $f(x) = 1$  for all  $x \in D$ .

More generally, for  $O_i \subset X$  disjoint open subsets, and  $x_i \in O_i$ , we can find a function positive function  $f \in C(K)$  such that  $f_i(x_i) = 1$ ,  $\text{supp } f_i \subset O_i$  and  $\sum_i f_i = 1$ . Here we only need  $K = [0, 1]$ ,  $f_i(x_i) = 1$ ,  $\text{supp } f_i \subset O_i$  and  $\sum_i f_i \leq 1$ .

Define  $\tilde{f}(x) = \sum_j \epsilon_j f_j(x)$ , with  $|\epsilon_j| = 1$ . Then

$$|\varphi(\tilde{f})| = \left| \sum_j \alpha_j \delta_{x_j} \right| = \left| \sum_j \epsilon_j \alpha_j f_j(x_j) \right| = \sum_j |\epsilon_j| \cdot |\alpha_j| = \sum_j |\alpha_j|.$$

Existence of such an  $\tilde{f}$  gives  $\|\varphi\| \geq \sum_j |\alpha_j|$ . This shows that for disjoint  $x_j$ 's,  $\left\| \sum_j \alpha_j \delta_{x_j} \right\| = \sum_j |\alpha_j|$ .

*Step 2.* For an arbitrary  $\varphi \in C(K)^*$ . Recall we have the following extension

$$\begin{array}{ccc} C(K) & \xhookrightarrow{\iota} & \ell_\infty(K) \\ \varphi \downarrow & \nearrow \exists \hat{\varphi} & \\ \mathbb{K} & & \end{array}$$

Then there exists a family  $\{\varphi_\alpha\} \subset \ell_\infty(K)$  with

$$\varphi_\alpha(f) = \sum_j \lambda_j(\alpha) f(x_j) \quad \text{and} \quad \|\varphi_\alpha\|_\infty = \sum_j |\lambda_j(\alpha)| = 1.$$

Denote  $\varphi(f) = \lim_\alpha \varphi_\alpha(f)$ . Then  $\varphi_\alpha \rightarrow \varphi$  in  $\sigma(C(K)^*, C(K))$ -topology. This implies for  $T : C(K_1) \rightarrow C(K_2)$ ,

$$\begin{aligned} \|T^*(\varphi)\| &= \sup_{\|f\|_{C(K_1)} \leq 1} |T^*(\varphi)(f)| = \sup_{\|f\|_{C(K_1)} \leq 1} |\varphi(T(f))| \\ &= \sup_{\|f\|_{C(K_1)} \leq 1} \left| \lim_\alpha \varphi_\alpha(T(f)) \right| \leq \sup_{\|f\|_{C(K_1)} \leq 1} \limsup_\alpha |\varphi_\alpha(T(f))|. \end{aligned}$$

Note that

$$\begin{aligned} |\varphi_\alpha(T(f))| &= \left| \sum_j \lambda_j(\alpha) \cdot (T(f))(x_j) \right| = \left| \sum_j \lambda_j(\alpha) \cdot T^*(\delta_{x_j})(f) \right| \\ &\leq \sum_j |\lambda_j(\alpha)| \cdot \|T^*(\delta_{x_j})\| \leq \sum_j |\lambda_j(\alpha)| \cdot \|f\|_\infty \leq \sup_{x_j} \|T(\delta_{x_j})\|. \end{aligned}$$

This gives  $\|T\| = \|T^*\| \leq \|T^*(\delta_x)\|$  and thus the equality.

In short, we could use the fact that the convex hull of the  $\delta$  measures are weak\*-dense in the unit ball of  $C(K)^*$ .  $\square$

**Lemma 19.4.** *Let  $K = [0, 2\pi]$ ,  $\mu$  be a measure on  $K$  and  $F(x, y)$  be a continuous functional in two variables. Define an integral operator  $T : C(K) \rightarrow C(K)$  by*

$$T_F(h)(x) = \int_K F(x, y)h(y) \, d\mu(y).$$

*Then  $\|T_F\| = \sup_y \int_K |F(x, y)| \, d\mu(x)$ .*

*Proof.* We know  $\|T_F\| = \sup_x \|T_F^*(\delta_x)\|$  and  $T_F^*(\delta_x)(f) = \int_K F(x, y)f(y) \, d\mu(y)$ . This is given by integration against  $h(y) = F(x, y)$ . Note that

$$\|T_F^*(\delta_x)\|_{C(K)^*} = \|h\|_{L^1(\mu)} = \int_K |F(x, y)| \, d\mu(y).$$

$\square$

Often this lemma is used for groups:  $G$  is a compact group and  $\mu$  a measure on  $G$ . We can prove the existence of Haar measure, which means there exists  $\mu$  such that  $\int f(gh) \, d\mu(h) = \int f(h) \, d\mu(h)$  for all  $g$ . The integral is invariant under translation. In the case of  $K = [-\pi, \pi]$ , this is the Lebesgue measure  $\lambda$ .

**Lemma 19.5.** *Let  $f : G \rightarrow G$  be a continuous map, and define a translation invariant operator  $T : C(G) \rightarrow C(G)$  by*

$$T_{f_1}(f_2)(g) = \int_K f_1(gh^{-1})f_2(h) \, d\mu(h).$$

*Then  $\|T_{f_1}\| = \int |f_1(h)| \, d\mu(h)$ .*

The norm does not see translation by  $g$  and thus the supremum disappears. In particular, on  $K = [-\pi, \pi]$ ,

$$T_{f_1}(f_2)(s) = \int_K f_1(s - t)f_2(t) \frac{1}{2\pi} \, dt.$$

*Proof.* Let  $F(g, h) = f_1(gh^{-1})$ , and  $T_{f_1}$  as above. Then the previous lemma and right invariant suggests

$$\|T_{f_1}\| = \sup_g \int_K |f_1(gh^{-1})| d\mu(x) = \int_K |f_1(h^{-1})| d\mu(x).$$

□

*Proof of Theorem 19.1.* Consider  $P_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikt}$ , where  $\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt$ .

By substitution

$$P_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikt} = \sum_{k=-n}^n \int_{-\pi}^{\pi} f(s) e^{-iks} \frac{1}{2\pi} ds \cdot e^{ikt} = \int_{-\pi}^{\pi} \sum_{k=-n}^n e^{ik(t-s)} f(s) \frac{1}{2\pi} ds.$$

Thus we take  $f_1(s) = \sum_{k=-n}^n e^{ik(t-s)}$ . By previous Lemma,

$$\|P_n\| = \int_{-\pi}^{\pi} \left| \sum_{k=-n}^n e^{ik(t-s)} \right| ds.$$

Sum of geometric series gives for  $s \neq 0$ ,

$$\sum_{k=-n}^n e^{iks} = \frac{e^{-ins} - e^{i(n+1)s}}{1 - e^{is}}.$$

Multiplying both sides by  $e^{-is/2}$  we get

$$\left| \frac{e^{-i(n+1/2)s} - e^{i(n+1/2)s}}{e^{is/2} - e^{-is/2}} \right| = \left| \frac{\sin((n+1/2)s)}{\sin(s)} \right|.$$

Note that  $\sin(s) \sim s$  when  $s \sim 0$ , and  $|\sin(ns)| \sim 1$  when  $s \sim \frac{\pi}{2n} + 2l\pi$ ,  $l \in \mathbb{N}$ . So there are  $s_j$ 's such that on the interval  $I_j = \{s \mid |s - s_j| \leq \frac{1}{4\pi n}\}$ ,  $s_j \sim \frac{j\pi}{2n}$  and  $|\sin(ns)| \geq \frac{1}{4}$ . This implies

$$\int \left| \frac{\sin(ns)}{\sin(s)} \right| ds \geq \sum_{j=1}^n \frac{1}{4} \int_{I_j} d\lambda \frac{n}{j} = \sum_{j=1}^n \frac{1}{4} |I_j| \frac{n}{j} \sim \text{const.} \sum_{j=1}^n \frac{1}{j}.$$

So the integral is unbounded.

□



**Remark 19.6.**

1.  $\int_{-\pi}^{\pi} f_n(s) \, ds = 0$ . ( $f_n$  should be referring to the oscillating function  $\sin(ns)$  but I'm not sure).
2. The kernel  $K(t, s) = \sum_{k=-n}^n \overline{h_k(t)} h_k(s)$  appears a lot in solutions of PDEs.

## **20 Krein–Milman Theorem 20210312**

Lecture recording missing.