Spring 2021 MATH231 Section CDQ Discussion

WF 9-10am

This document can be found here or on my website. I will continue update it until the end of semester.

Contact

- TA for section CDQ: Xinran Yu
- Email: xinran4@illinois.edu. Please included MATH231 in your email subject.
- Office hour: Wed 10-11am[1]

Zoom

- Please use your cameras and microphones in breakout rooms.
- Interrupt me/using the "Raise Hand" feature on Zoom to ask questions.
- You can call me into your breakout room/return to the main room to ask for help.
- It is also possible for me to join your breakout rooms randomly to check if you have any questions.

Worksheet

- Worksheet can be found on Moodle under Groupwork folder.
- Ask for hints when you get stuck on a problem.

Submission

- Submit on Moodle under Groupwork folder.
- **1 submission per group**. Once a file is uploaded, everyone in the same group will be able to see/edit the file. [2]
- Group remains the same until each midterm.
- 1st worksheet of the week is due on **Thursday** at **8AM** CST. [3]
- 2nd worksheet of the week is due on **Saturday** at **8AM** CST.
- Worksheet solutions available at 12:30PM CST on the due date.

Grading

Worksheets are graded with 2, 1 or 0.

2 - the worksheet uploaded is satisfactory

1 - the worksheet uploaded is unsatisfactory and needs improvement. Your TA will comment on what should be improved for next time.

0 - the worksheet was not uploaded

[1]Office hour is run for all students in MATH231
[2]Groups are assigned randomly by Moodle
[3]Central Standard Time

Contents

Worksheet 1

Recall

Theorem 1.1 (Fundamental Theorem of Calculus)**.** *Ref p.26*

Part 1 *If $f(x)$ is* ***continuous*** *over an interval $[a,b]$, and the function $F(x)$ is defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \quad x \in [a,b],$$

then $F'(x) = f(x)$ over $[a,b]$.

Part 2 *If $f(x)$ is* ***continuous*** *over an interval $[a,b]$, and $F(x)$ is any antiderivative of $f(x)$ i.e. $F'(x) = f(x)$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(a) - F(b).$$

Example 1.2. Let

$$g(x) = \int_a^{b(x)} f(t)\,\mathrm{d}t.$$

Apply chain rule and FTC

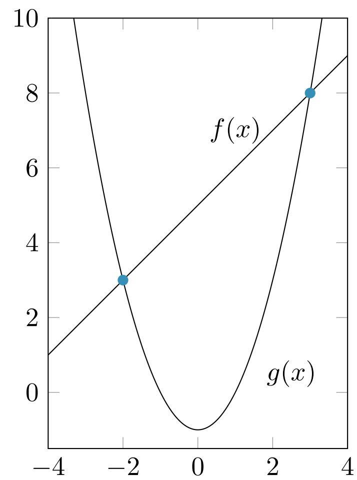
$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_c^{b(x)} f(t)\,\mathrm{d}t = b'(x) \cdot f(b(x)).$$

Worksheet 2

Recall

- **Substitution rule/Change of variable**: let $u = g(x)$, then
$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u. \tag{Q1-3}$$

- **Compute area between curves.** (Q4-7)
  - Draw the graph.
  - Find intersection points by solving $f(x) = g(x)$, say they are $x = a$ and $x = b$.
  - Area $= \int_a^b f(x) - g(x)\,\mathrm{d}x$.



Worksheet 3

Volume of a solid of revolution: Slices of volume are circles. Ref

$$\text{Vol } = \int_a^b \pi f(x)^2 \, \mathrm{d}x.$$

Q3. Slices are squares/triangles.

$$\text{Vol } = \int_a^b \text{ Area of slices } \mathrm{d}x.$$

E.g.

$$\text{Vol } = \int_a^b f(x)^2 \, \mathrm{d}x.$$

Worksheet 4

Volume by cylindrical shells:

$$\text{Vol } = \int_a^b 2\pi r \cdot f(x)\, \mathrm{d}x.$$

Rotation about y-axis: $r = x$.
Rotation about the vertical line $x = a$: $r = |a - x|$.

Worksheet 5

Recall

- Since $\sin x$ is oscillating between -1 and 1, $\lim\limits_{x\to\infty} \sin x$ does not exists.
- we can use L'Hopital's Rule to compute indeterminate forms "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$".

Theorem 5.1 (L'Hopital's Rule)**.**
Assumptions:

$$\begin{aligned}
&\boldsymbol{f(x) \to 0} \qquad \textit{as } x \to a,\\
&\boldsymbol{g(x) \to 0},\\
&\boldsymbol{g'(x) \neq 0}.
\end{aligned}$$

Conclusion:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Warning: check the assumptions before applying L'Hopital's Rule.

Worksheet 6

Integration by parts:

$$(uv)' = u'v + uv' \implies \int \boldsymbol{u}\,\mathbf{d}\boldsymbol{v} = \boldsymbol{uv} - \int \boldsymbol{v}\,\mathbf{d}\boldsymbol{u}.$$

Choose u based on which of these comes first, (search "integration by parts what to choose as u"):

(1) **L**ogarithmic functions: $\ln x$

(2) **I**nverse trigonometric functions: $\arcsin x$

(3) **A**lgebraic functions: x

(4) **T**rigonometric functions: $\sin x$

(5) **E**xponential functions: e^x

Worksheet 7

Worksheet 8

Recall: $1 + \tan^2 x = \sec^2 x$.

Worksheet 9

Substitution rule/Change of variable: let $u = g(x)$, then

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u.$$

Integration by parts:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Worksheet 10

Partial fractions decomposition: Find A, B such that

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} - \frac{B}{x-b}.$$

The following table is from this website

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k},\ k = 1, 2, 3, \ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k},\ k = 1, 2, 3, \ldots$

Typo in solution

WS10 Q3.

$$\frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{E}{x-1} + \frac{F}{(x-1)^{\mathbf{2}}} + \frac{G}{(x-1)^{\mathbf{3}}}.$$

Worksheet 11

Improper integrals: There are two types of improper integrals $\int_a^b f(x)\,\mathrm{d}x$:

(1) a or b (or both) infinite, e.g $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$.

(2) The function $f(x)$ blows up in the interval $[a, b]$, e.g $\int_0^1 \ln x\,\mathrm{d}x$.

To compute improper integrals, e.g.:

$$\int_1^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{b\to\infty} \int_1^b \frac{1}{x}\,\mathrm{d}x.$$

Worksheet 12

Simpson's rule: Let x_i's be equally spaced points,

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)).$$

Coefficient is 4 for odd i, $i \neq 0, n$; coefficient is 2 for even i, $i \neq 0, n$.

If you want to use the formula $\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\arctan(\frac{x}{a}) + C$, for complex a. Recall how we obtain this formula. We do trig substitution, $x = a\tan\theta$, and use the identity $1+\tan^2\theta = \sec^2\theta$. This identity is still valid for complex θ. So with $x = i\tan\theta$, $\theta = \arctan(-ix)$,

$$\begin{aligned} I = \int \frac{1}{x^2-1}\,\mathrm{d}x &= \int \frac{1}{x^2+i^2}\,\mathrm{d}x \\ &= \int \frac{1}{i^2\tan^2\theta + i^2} i\sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

(Here we need the derivative of $\tan\theta$, but you can check this is $\sec^2\theta$ in the complex case)

$$= -i\int \frac{1}{\sec^2\theta}\sec^2\theta\,\mathrm{d}\theta = -i\theta + C = -i\arctan(-ix) + C.$$

The above checked our formula is valid for complex a. Hence, we can plug in the value of arctan

$$\arctan t = \frac{1}{i}\ln\sqrt{\frac{1+it}{1-it}} = \frac{i}{2}\left[\ln(1-it) - \ln(1+it)\right].$$

So

$$\arctan(-ix) = \frac{1}{i}\ln\sqrt{\frac{1+x}{1-x}} = \frac{i}{2}\left[\ln(1-x) - \ln(1+x)\right].$$

$$I = -i\cdot\frac{i}{2}\left[\ln(1-x) - \ln(1+x)\right] + C = \frac{1}{2}\left[\ln(1-x) - \ln(1+x)\right] + D.$$

Note that D should be a real constant as I is real.

Worksheet 13

Let $\mathrm{d}s$ be the arclength differential.

Arc length:

$$L = \int \mathrm{d}s = \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Surface area:

$$A = \int 2\pi x\,\mathrm{d}s = \int 2\pi x \cdot \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

Worksheet 14

$$\text{Force} = \int \rho g \cdot \text{depth}(y) \cdot \text{width}(y)\, \mathrm{d}y.$$

Worksheet 15

Integral test: If $f(x)$ is **continuous, positive and decreasing** on $[N, \infty)$. Then

$$\int_N^\infty f(x)\,\mathrm{d}x \text{ converges} \implies \sum_{n=N}^\infty f(n) \text{ converges;}$$
$$\int_N^\infty f(x)\,\mathrm{d}x \text{ diverges} \implies \sum_{n=N}^\infty f(n) \text{ diverges.}$$

Warning: check the assumptions before applying integral test.

Worksheet 16

Geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 \cdots = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } |r| \geq 1 \end{cases}$$

Integral test: If $f(x)$ is continuous, positive and decreasing on $[N, \infty)$. Then

$$\int_N^{\infty} f(x)\,\mathrm{d}x \text{ converges} \implies \sum_{n=N}^{\infty} f(n) \text{ converges;}$$

$$\int_N^{\infty} f(x)\,\mathrm{d}x \text{ diverges} \implies \sum_{n=N}^{\infty} f(n) \text{ diverges.}$$

Error estimate: Assume $\sum_{n=1}^{\infty} a_n$ converges

$$S = \overbrace{a_1 + a_2 + \cdots + a_n}^{\text{partial sum } S_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{\text{reminders } R_n}.$$

Note that the first term in R_n is a_{n+1},

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x < R_n < \int_n^{\infty} f(x)\,\mathrm{d}x.$$

Worksheet 17

Alternating series test: Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. If

- $\lim\limits_{n\to\infty} b_n = 0$;
- $\{b_n\}$ is a decreasing sequence the series,

then $\sum_n a_n$ is convergent.

Ratio test: Let $L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- $L < 1$, $\sum_n a_n$ convergent;
- $L > 1$, $\sum_n a_n$ divergent;
- $L = 1$ no conclusion.

Worksheet 18

Comparison test: If a_n, $b_n > 0$ and $a_n \leq b_n$ for all large n then

- $\sum b_n$ converges, then a_n also converges;
- $\sum a_n$ diverges, then b_n also diverges.

Limit comparison test: Given $\sum a_n$, $\sum b_n$, with $a_n,\ b_n > 0$ If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = C$ for some $C \neq 0$, $C \neq \infty$. Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Worksheet 19

Ratio test: Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

Root test: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

Worksheet 20

Radius of convergence: For a power series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

where a and c_n are numbers. The **radius of convergence** is a number $R \in [0, \infty]$ s.t.

- the series converges if $|x-a| < R$;
- the series diverges if $|x-a| > R$.

To find R, compute $L = \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$. Apply ratio test and check the boundary case.

Recall

Ratio test: Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

Worksheet 21

Power series expansion: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $|x| < R$ then we can differentiate and integrate $f(x)$:

- $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$;
- $\int f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C$.

Worksheet 22

Taylor series:

$$\begin{aligned} f\left(x\right) &= \sum_{n=0}^{\infty} \frac{f^{(n)}\left(a\right)}{n!}\left(x-a\right)^n \\ &= f\left(a\right) + f'\left(a\right)\left(x-a\right) + \frac{f''\left(a\right)}{2!}\left(x-a\right)^2 + \frac{f'''\left(a\right)}{3!}\left(x-a\right)^3 + \cdots \end{aligned}$$

Maclaurin series: taking Taylor Series about $x = 0$

$$\begin{aligned} f\left(x\right) &= \sum_{n=0}^{\infty} \frac{f^{(n)}\left(0\right)}{n!}x^n \\ &= f\left(0\right) + f'\left(0\right)x + \frac{f''\left(0\right)}{2!}x^2 + \frac{f'''\left(0\right)}{3!}x^3 + \cdots \end{aligned}$$

Examples to remember:

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1 \\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \cos x &= \sum_{n=0}^{\infty} \frac{\left(-1\right)^n x^{2n}}{(2n)!} \\ \sin x &= \sum_{n=0}^{\infty} \frac{\left(-1\right)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$