Introduction to Einstein-Maxwell Equations

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Abstract

Maxwell's equations reveal the fundamental relations between electricity and magnetism. They can be formulated in various equivalent ways under the idea of unification. In this talk, I will formulate Maxwell's equations using differential forms and tensor calculus. We will see there are at least two benefits of tensorizing spacetime, electric-magnetic field, and energy-momentum: this provides a covariant formulation of Maxwell's equation; and, we obtain a crucial term that appears in Einstein field equation of general relativity.

Contents

1 Equivalence formulation of Maxwell's equations

Our plan is to rewrite Maxwell's equations from different point of view and introducing tensor calculus and Riemannian metric. More precisely we will do the following

$$\begin{tikzcd}
\text{differential forms on } \mathbb{R}^3 \text{ (vector calculus)} \arrow[d, "\text{unify spacetime}"] \\
\text{tensor} \arrow[d, "\text{identify } E, B \text{ with forms and using Hodge star}"] \\
\text{Hodge-Maxwell theorem}
\end{tikzcd}$$

1.1 A brief history

1784-1786 Coulomb did experiments on electric repulsion and attraction (Coulomb force)

1837 Faraday studied Coulomb's law and discovered electro-magnetic induction (electric field lines)

1855, 1861, 1864 Maxwell formulated Faraday's experiments using mathematical language

1880 Heaviside introduced permittivity ϵ_0

We will start with the differential geometric formulation of Maxwell's equations.

1.2 In differential form

$$\text{Gauss' law: } \nabla \cdot E = \frac{\rho}{\epsilon_0} \tag{E}$$

$$\nabla \cdot B = 0 \tag{M}$$

$$\text{Faraday's law of induction: } \nabla \wedge E = -\frac{\partial B}{\partial t} \tag{MI}$$

$$\text{Ampère's circuital law: } \nabla \wedge B = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t} \tag{EI}$$

Here

- ρ = charge density
- $\epsilon_0 = \frac{1}{4\pi} \cdot \frac{1}{9 \cdot 10^9}$ Fm^{-1}, the permittivity of free space
- J = current density
- $\mu_0 = \frac{4\pi}{10^7}$ NA^{-2}, the permeability of free space
- $\mu_0 \epsilon_0 = \frac{1}{c^2}$

1.2.1 Meaning of the equations

- Equation (E): electric charges causes electric fields.
- Equation (M): magnetic monopoles does not exists.
- Equation (MI): changing magnetic fields induces electric fields.
- Equation (EI): electric current causes magnetic fields and changing electric fields induces magnetic fields. The second term (Maxwell's addition) is a correction in order to get a compatible equation to the continuity equation. See below.

1.2.2 Maxwell's addition

Continuity equation (electric charge conservation) says

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot J.$$

Hence for any volume V, the total charge $Q = \int_V \rho dV$ satisfies $\frac{\partial Q}{\partial t} = \int_{\partial V} -\nabla \cdot J dA$.

Note that Equation (M) and Equation (MI) are consistent by taking divergence; whereas Equation (E) and Equation (EI) without Maxwell's addition only holds if $J = 0$. Maxwell's addition is designed to cancel $\nabla \cdot J$.

Note that we can move time derivatives to the left hand side and arrange the equations into two groups. This will becomes clear when we convert Maxwell's equation to tensor form.

homogeneous	inhomogeneous
$\nabla \cdot B = 0$	$\nabla \cdot E = \dfrac{\rho}{\epsilon_0}$
$\nabla \wedge E + \dfrac{\partial B}{\partial t} = 0$	$\nabla \wedge B - \dfrac{1}{c^2}\dfrac{\partial E}{\partial t} = \mu_0 J$

2 Maxwell's equations in tensor calculus

2.1 Levi-Civita symbol

Consider the Levi-Civita symbol ϵ_{ijk}. It has properties

$$(v \times w)_k = \epsilon_{ijk} v^j w^k.$$
$$\epsilon_{ija}\epsilon_{akl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \implies \epsilon_{ijk}\epsilon_{ijk} = 6.$$

2.2 4-vector setting

Take the *spacetime coordinate* $x^\mu = (ct, x)$[1] and define $\partial_\mu = (\frac{1}{c}\partial_t, \nabla)$. (We need to multiply t by speed of light c to match the unit of space). Rotations in $\mathbb{R}^3$ are unified to Lorentz transformations in spacetime. Let

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{c^2}\partial_t^2 + \Delta$. ($\Delta$ with plus sign).

In this section, we are only considering flat spacetime (also known as *Minkowski spacetime*) so it seems introducing $\eta^{\mu\nu}$ is not necessary. The benefit of this will become clear later when we modify $\eta^{\mu\nu}$ by a symmetric 2-tensor.

Note that -1 in $\eta_{\mu\nu}$ reflects the crucial difference between ct and x. A coordinate change purely in space corresponds to the usual rotational matrix;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

[1] Here c is needed to match the physical dimensions.

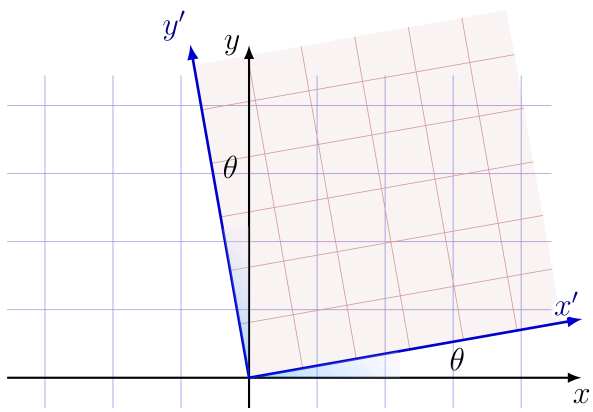

Figure 1: Lorentz transformation on (x, y)-plane

Whereas coordinate change in ct and x corresponds to

$$\eta_{\mu\nu} = \begin{bmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

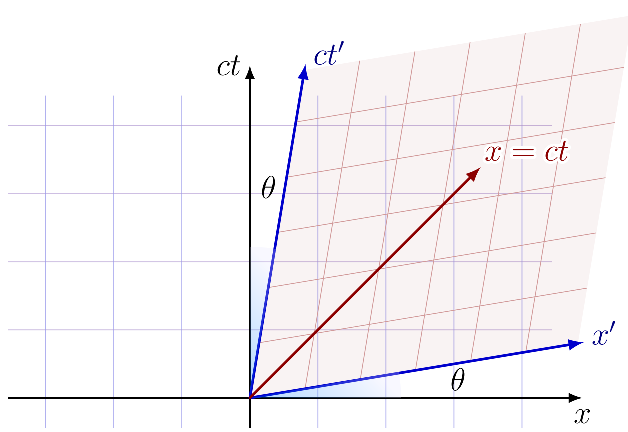

Figure 2: Lorentz transformation on (ct, x)-plane

Remark: rotation in 3-dim is translated to symmetries in 4-dim.

Definition 2.1. We define the 4-vector current J^μ and field strength tensor $F_{\mu\nu}$[2] to

[2]antisymmetric $(0, 2)$-tensor

be

$$
\begin{aligned}
J^\mu &= (\rho c, J), \\
F^{\mu\nu} &= \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
F_{0i} &= -\frac{E_i}{c}, \\
F_{ij} &= \epsilon_{ijk} B_k.
\end{aligned}
$$

2.3 Maxwell's equations in tensor form

Using tensor setting the Maxwell's equations are

$$F_{[\mu\nu,\gamma]} = 0 \text{ or } \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta,\nu} = 0. \tag{M*}$$
$$F^{\mu\nu}{}_{,\nu} = \mu_0 J^\nu, \tag{E*}$$

If we introduce $\frac{1}{2} G^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$, then we can simplify Equation (M*) further.

homogeneous	inhomogeneous
$G^{\mu\nu}{}_{,\nu} = 0$	$F^{\mu\nu}{}_{,\nu} = \mu_0 J^\nu$

Homogeneity suggests that Equation (M) and (MI) comes from the one on left and Equation (E) and (EI) come from the one on right.

Second equation with $\mu, \nu, \gamma = i, j, k$ gives Equation (M)

Proof. Multiplying the equation by ϵ_{ijk} and using $-2B_i = \epsilon_{ijk} F^{jk}$ we obtain

$$0 = \epsilon_{ijk} F_{[ij,k]} = 3\epsilon_{ijk} F_{ij,k} = 3 \cdot (2\nabla \cdot B).$$

□

Second equation with $\mu, \nu, \gamma = 0, j, k$ gives Equation (MI)

Proof. Start with

$$F_{0j,k} + F_{k0,j} + F_{jk,0} = 0.$$

Substitute the values of F_{jk} gives

$$\frac{1}{c}\Big(- E_{j,k} + E_{k,j} + \epsilon_{kjl}\frac{\partial B_l}{\partial t}\Big) = 0.$$

Multiplying the equation by ϵ_{ijk} we obtain

$$0 = \frac{(2\nabla \wedge E)_i}{c} + \frac{2}{c}\frac{\partial B_i}{\partial t}.$$

$\square$

2.4 4-potentials

Definition 2.2. We define the *4-potential* A^μ to be

$$A^\mu = (\frac{\phi}{c}, A).$$

Here

- $\phi =$ electrostatic potential, defined by $E = -\nabla\phi$.
- $A =$ magnetic vector potential, defined in magnetostatics by $B = \nabla \wedge A$.
- In electrostatics, there is no time dependence, so $\nabla \wedge E = 0$. This means we can write E as gradient of some function which is the potential. Similar for B.

Then

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and using Maxwell's equation we obtain

$$B = \nabla \wedge A,$$
$$E = -\nabla\phi - \frac{\partial A}{\partial t}.$$

Note that 4-potential is transformed as

$$\tilde{A}^\mu = A^\mu + \partial^\mu\lambda.$$

We can check

$$\begin{aligned}\tilde{F}^{\mu\nu} &= \partial^\mu\tilde{A}^\nu - \partial^\nu\tilde{A}^\mu = \partial^\mu(A^\nu + \partial^\nu\lambda) - \partial^\nu(A^\mu + \partial^\mu\lambda)\\ &= \partial^\mu A^\nu + \partial^\mu\partial^\nu\lambda - \partial^\nu A^\mu - \partial^\nu\partial^\mu\lambda = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}\end{aligned}$$

2.5 Energy-momentum conservation

$$\text{Energy density: } \rho_{en} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2,$$
$$\text{energy flux: } J_{en} = \frac{1}{\mu_0}E \wedge B.$$

This implies $\dfrac{\partial \rho_{en}}{\partial t} = -\nabla J_{en}$.

We now introduce the energy-momentum tensor, which plays a key role in the Einstein field equation.

Definition 2.3. We define *energy momentum tensor* $T^{\mu\nu}$ to be

$$T^{\mu\nu} = \frac{1}{\mu_0}F^{\alpha\mu}F_{\alpha}{}^{\nu} - \frac{1}{4\mu_0}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}.$$

T^{ij} is known as the *Maxwell stress tensor*.

Remark 2.4.

1. Note that $T^{00} = \rho_{en}$, $T^{0i} = J^i_{en}$ and $T^{\mu\nu}{}_{,\mu} = 0$.
2. The Newtonian energy is given by integrating T^{00}: $\int_V T^{00}dx$, whose value is proportional to the first components of the *4-potential* $p^\mu = (\frac{E}{c}, p)$. So entries of $T_{\mu\nu}$ has physical dimensions of energy density.
3. Computing the energy of a stationary time-like object (e.g. a massive particle) gives $E = mc^2$.

Proof.

$$\begin{aligned}
T^{\mu\nu}{}_{,\mu} &= \frac{1}{\mu_0}F^{\alpha\mu}F_{\alpha}{}^{\nu}{}_{,\mu} - \frac{1}{2\mu_0}F^{\alpha\beta,\nu}F_{\alpha\beta} \\
&= \frac{1}{2\mu_0}F^{\alpha\mu}(F_{\alpha}{}^{\nu}{}_{,\mu} - F_{\mu}{}^{\nu}{}_{,\alpha}) - \frac{1}{2\mu_0}F^{\alpha\mu,\nu}F_{\alpha\mu} \\
&= \frac{1}{2\mu_0}F^{\alpha\mu}(F_{\alpha}{}^{\nu}{}_{,\mu} + F^{\nu}{}_{\mu,\alpha}) + \frac{1}{2\mu_0}F^{\alpha\mu,\nu}F_{\mu\alpha} \\
&= \frac{1}{2\mu_0}F^{\alpha\mu}(F_{\alpha}{}^{\nu}{}_{,\mu} + F^{\nu}{}_{\mu,\alpha} + F_{\mu\alpha,}{}^{\nu}) \\
&= 0. && \text{(by Maxwell's second equation)}
\end{aligned}$$

$\square$

So far Maxwell's equation get simplify by unification

- Maxwell's electric-magnetic field
- Lorentz's spacetime
- unknown: vacuum energy, Higgs boson, quantum gravity.

3 Hodge-Maxwell Theorem

Now we leave out the flatness. We identify E and B as two forms via the following (here I use Einstein summation)

$$\begin{aligned} E &= E_i dx^i, \\ B &= B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \\ J &= J_\mu dx^\mu, \end{aligned}$$

Let $F = E \wedge dx^0 + B$ be the Faraday 2-form. One can check $B = *(dx^0 \wedge E)$ and $dx^0 \wedge E = - * B$. [3] We have,

$$G^{\mu\nu}{}_{,\nu} = 0 \iff dF = 0. \tag{M**}$$

$$F^{\mu\nu}{}_{,\nu} = \mu_0 J^\nu \iff d * F = *\mu_0 J, \tag{E**}$$

and the energy-momentum tensor is given by the same formula in Definition 2.3.

4 In curved spacetime

The benefit of having tensor calculus and hodge star operator is we can generalize Maxwell's equations to a curved spacetime. We do this by perturbating the metric (inner product) $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$. Then $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (in Minkowski space) is replaced by $F^{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ [4, Section3]. Then Maxwell's equations become

homogeneous	inhomogeneous
$\nabla_{[\alpha} F_{\beta\gamma]} = 0$	$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu$

Aside, the process of replacing the partial derivatives with covariant derivatives is similar to how we generalize the Laplace type operator. It might be more useful to look at the later case.

[3] I am not going to specify what Hodge star means. One can find detail in [1, Section 3] or [2]

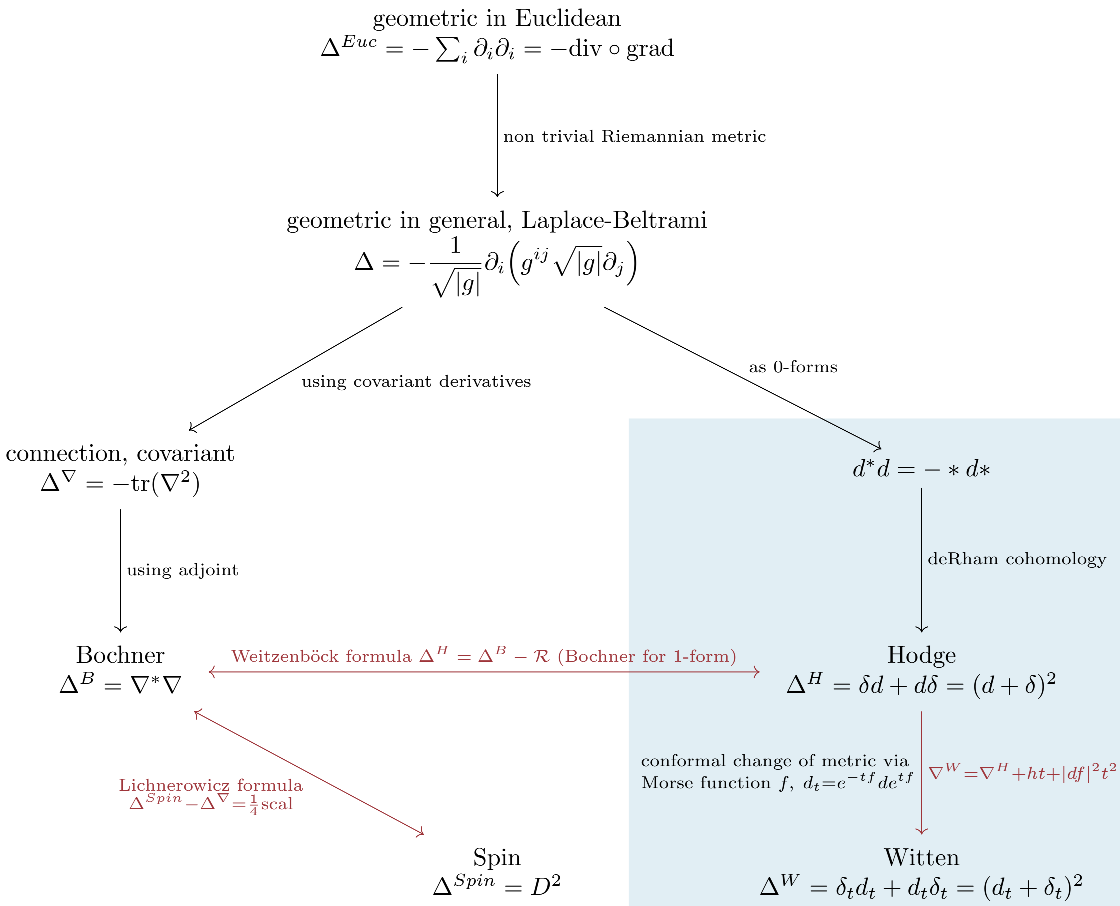

Figure 3: Generalized laplacian

5 Einstein field equation

We now have the terminology needed for the Einstein field equation. Due to time limitation of the talk, I will write down the equation without a proof.

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2} \mathrm{scal}\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

Here

- G is the gravitational constant
- $T_{\mu\nu}$ energy-momentum tensor which is extremely large
- $\dfrac{8\pi G}{c^4} = 2.068 \times 10^{-43} \, \dfrac{\mathrm{sec}^2}{\mathrm{kg} \cdot \mathrm{m}}$

Remark 5.1.

1. The Einstein-Maxwell equations of gravitation and electromagnetism consist of the Einstein field equation with $\nabla_\mu T^{\mu\nu} = 0$.

2. One may add the effect of cosmological constant and obtain generalized version of the above equation

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}\mathrm{scal}\, g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

Note that we have Einstein condition when the right hand side vanishes. This says Minkowski metric is a trivial solution of the Einstein field equation. There are also non trivial solutions: the Schwarzschild and Kerr metrics [3, Section 6].

Remark 5.2.

1. Non trivial solutions contains singularities at a finite distance from the source. These are called event horizons.

2. At infinity, singularities are normalized, so spacetime looks flat far away and Maxwell field $T_{\mu\nu}$ dominates the behavior.

References

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[4] Wytler Cordeiro dos Santos, *Introduction to Einstein-Maxwell equations and the Rainich conditions*, 2016, avaliable online at https://arxiv.org/pdf/1606.08527.pdf.