Ch 11. Infinite sequence and series

Def. An <u>sequence</u> is an infinite list of numbers written in a definite order.

Notation: $\{a_1, a_2, \cdots, a_n, \cdots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples $\{1, 2, 3, 4, \cdots\}$
$\{7, 1, 8, 2, 8, \cdots\}$

Some sequences can be defined by giving a formula for the n-th term a_n

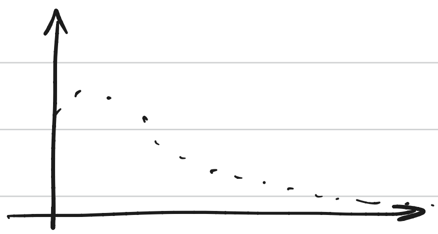
Examples
1. $a_n = \left(\frac{1}{2}\right)^n$ $\{a_n\} = \{\frac{1}{2}, \frac{1}{4}, \cdots\}$
2. $a_n = (-1)^n$ $\{a_n\} = \{-1, 1, -1, 1, \cdots\}$
3. $a_n = \frac{n}{n+1}$ $\{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$

Some sequences may not have a simple / explicit defining equation

Examples
1. a_n = the digit in the n-th decimal place of π
2. The Fibonacci sequence
$a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$
$\{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}$

A sequence "is" a function f that only takes values on natural numbers. So we will study properties such as graph and convergency.

Example



$$\lim_{n\to\infty} a_n = 0$$

$L < \infty$

Def. A sequence has limit L if for any ε there is an N s.t. if $n > N$ then

$$|a_n - L| < \varepsilon$$

We say $\{a_n\}$ converges to L.

Intuition

Def. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N s.t. if $n > N$ then $a_n > M$.

Examples

1. $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

2. $\lim_{n\to\infty} \frac{1}{n^r} = \begin{cases} 0 & \text{if } r>0 \\ \infty & \text{if } r<0 \end{cases}$

3. $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \\ \text{DNE} & \text{if } r<1 \end{cases}$

Limit law for sequences

if $\{a_n\}$. $\{b_n\}$ are convergent sequences then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c\, a_n = c \lim_{n\to\infty} a_n \qquad c \text{ const.}$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

$$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \qquad p>0 \quad a_n>0$$

Squeeze Theorem

$$b_n \le a_n \le c_n \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$
$$\begin{array}{ccc} b_n & & c_n \\ \downarrow & & \downarrow \\ L & & L \end{array}$$

<u>Thm</u> If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

If f is continuous

$$\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} f(a_n) = f(L)$$

Example 1.

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$$
$$= \sin 0 = 0$$

Example 2. $\lim_{n\to\infty} \frac{\ln(n+2)}{\ln(1+4n)}$

$$\lim_{x\to\infty} \frac{\ln(x+2)}{\ln(1+4x)} \overset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{\frac{1}{x+2}}{\frac{1}{1+4x}\cdot 4}$$
$$= \lim_{x\to\infty} \frac{4x+1}{4(x+2)} = 1$$

Example 3.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n}$$

$$= e^{\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)}$$

$$= e$$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\cdot -\frac{1}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

Last time: sequence
This time: series.

Def. We call $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ a <u>series</u> and

$$s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N$$ the <u>partial sums.</u>

Note that $\{s_N\}$ is itself a sequence. So it make sense to talk about if $\{s_N\}$ converges or not.

Def. The series $\sum a_n$ is called <u>convergent</u> if its partial sum is convergent. Otherwise $\sum a_n$ is called <u>divergent.</u>

Example 1. (geometric series)
Consider $a_n = r^n$ r: common ratio.

$$\begin{aligned}
a_0 &= r^0 = 1 & s_0 &= a_0 = 1 \\
a_1 &= r^1 = r & s_1 &= a_0 + a_1 = 1 + r \\
a_2 &= r^2 = r^2 & s_2 &= a_0 + a_1 + a_2 = 1 + r + r^2 \\
&\vdots & &\vdots
\end{aligned}$$

$$s_N = \underbrace{1 + r + r^2 + \cdots + r^N}_{\text{we are interested in this sum.}}$$

Let $R_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \cdots + r^N$

$$rR_N = r + r^2 + \cdots + r^N + r^{N+1}$$

$$\Rightarrow \quad R_N - rR_N = 1 - r^{N+1}$$

$$\Rightarrow \quad R_N = \frac{1 - r^{N+1}}{1-r}$$

$$\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} R_N$$

! $n=0$

$$= \begin{cases} \dfrac{1}{1-r} & \text{if } -1<r<1 \quad \text{conv.} \\ \infty & \text{if } r \geq 1 \\ \text{DNE} & \text{if } r \leq -1 \end{cases}$$

div.

$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ because $\sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r_n$

Example 2. Compute $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ using the above

$$\sum_{n=1}^{\infty} 2^{2n} 6^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 6 \cdot 6^{-n}$$

note that n start from 1

$$= 6 \cdot \sum_{\boxed{n=1}}^{\infty} \Big(\underbrace{\frac{4}{6}}_{=\frac{2}{3}}\Big)^n = 6 \cdot \frac{\boxed{\frac{2}{3}}}{1-\frac{2}{3}}$$

$$= 6 \cdot 2 = 12$$

Example 3. (harmonic series)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$$

$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}}$$

$$= 1 + \frac{3}{2}$$

$$\vdots$$

$$S_{2^n} = 1 + \frac{n}{2} \xrightarrow{n \to \infty} \infty$$

Hence $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}}$

Example 4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

Note that $\displaystyle\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \longrightarrow 1$$

Theorem $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converges, c const.

$$\sum_{n=1}^{\infty} a_n \pm b_n = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$$

Example 5 $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{3}{n(n+1)}}_{a_n} + \underbrace{\frac{1}{2^n}}_{b_n}$

We have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - 1$

$$= 2 - 1 = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the original series converges to

$3 \cdot 1 + 1 = 4$

Thm $\sum_{n=1}^{\infty} a_n$ convergent $\Rightarrow \lim_{n\to\infty} a_n = 0$

pf. By definition, we know $\lim_{n\to\infty} S_n = L$ for some real number L.

$$\Rightarrow \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = L$$

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (S_n - S_{n-1}) \\ &= \lim_{n\to\infty} S_{n-1} - \lim_{n\to\infty} S_n \\ &= L - L = 0 \end{aligned}$$

Coroallary (The divergence test)

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges

Examples 1. $\sum_{n=1}^{\infty} (-1)^n$

2. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ all diverges

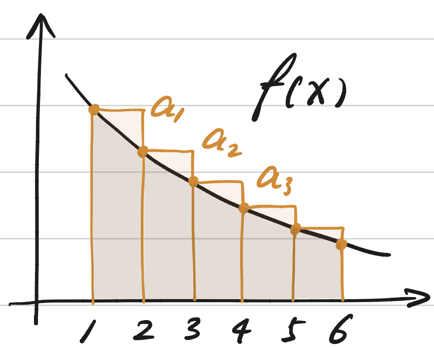
3. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

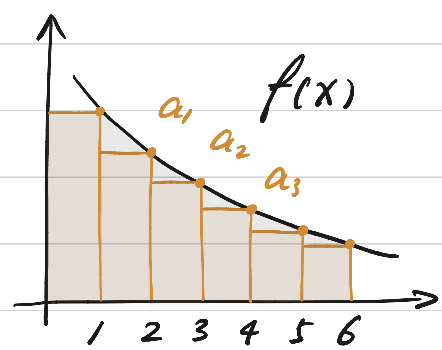
Last time computing series.
This time integral test & estimates.

We have been computing exact value of a series so far for some special cases. However, in general it is quite difficult. In those cases, we are interested in finding an estimate.

<u>Thm</u> (the integral test)
Suppose $f(x) > 0$ is a continuous decreasing function for $x \geq 1$ such that $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \iff \int_1^{\infty} f(x)\,dx \text{ conv.}$$





Moreover,

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx \quad (\star)$$

Error $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n$

$$\int_{N+1}^{\infty} f(x)\,dx \leq R_N \leq \int_N^{\infty} f(x)\,dx$$

Example 1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv.

$f(x) = \frac{1}{x^2} > 0$ for $x \geq 1$ cont. and decreasing

$\int_1^{\infty} \frac{1}{x^2}\,dx$ conv.

$f'(x) = -2x^{-3} < 0$ for $x \geq 1$

Example 2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is conv. if $p > 1$
div if $p \leq 1$

Example 3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ conv.

$f(x) = \frac{1}{x^2+1} > 0$ for $x \geq 1$, cont. and decreasing

$f'(x) = -(x^2+1)^{-2} \cdot 2x < 0$ for $x \geq 1$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \arctan x \Big]_1^t$$

$$= \lim_{t\to\infty} \arctan t - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty$$

Last time: integral test.
This time: the comparison test

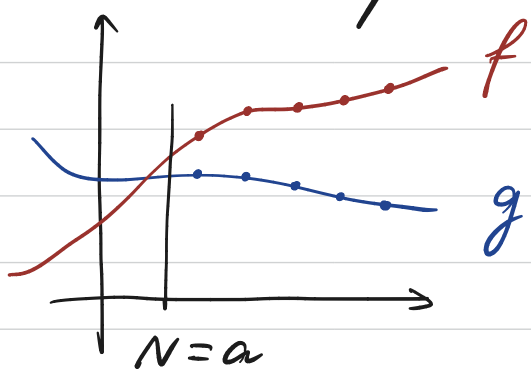
The idea of the comparison test for sequences is similar to that for integrals.

<u>Thm</u> (the comparison test)
Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $a_n \geq b_n$ for $n \geq N$.

$$\sum a_n \text{ conv} \Rightarrow \sum b_n \text{ conv.}$$
$$\sum b_n \text{ div} \Rightarrow \sum a_n \text{ div}$$

Compare the above with the comparison test in Ch 7.

$a_n \longleftrightarrow f$
$b_n \longleftrightarrow g$
$\sum \longleftrightarrow \int$
$N \longleftrightarrow a$



Example 1. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

$2n^2+4n+3 \geq 2n^2$ for $n \geq 1$

N = 1 here

$$\Rightarrow \underbrace{\frac{5}{2n^2+4n+3}}_{a_n} \leq \underbrace{\frac{5}{2n^2}}_{b_n}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} < \infty \Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv.}$$

Example 2. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Note that $\ln n > 1$ for $n > e$

$$\Rightarrow \quad \underbrace{\frac{\ln n}{n}}_{a_n} > \underbrace{\frac{1}{n}}_{b_n} \quad \text{for } n > e$$

e.g. we can take $N = 3$

$$\sum_{n=3}^{\infty} \frac{1}{n} \text{ div} \Rightarrow \sum_{n=3}^{\infty} a_n \text{ div} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ div.}$$

<u>Thm</u> (the limit comparison test)

Suppose $\sum a_n$, $\sum b_n$ are series with positive terms and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = c \in (0, \infty)$

Then $\sum a_n$ conv $\Leftrightarrow$ $\sum b_n$ conv.

Example 3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ Take $b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

$$= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 \in (0, \infty)$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ conv} \Rightarrow \sum a_n \text{ conv.}$$

dominant part is $2n^2$

Example 4. $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ $\leftarrow$ $\sqrt{n^5} = n^{5/2}$

Take $b_n = \dfrac{2n^2}{n^{5/2}} = \dfrac{2}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^2}} \cdot \frac{\sqrt{n}}{2}$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^2}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}} = \frac{2}{2} = 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ div $\Rightarrow$ $\sum a_n$ div

Last time: comparison tests
This time: alternating series.

So far we've studied series with positive terms. In this section we will study series whose terms are alternating (e.g. $a_{2n} > 0$, $a_{2n+1} < 0$).

Examples 1. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

2. $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - \cdots$

The following theorem tells us how to determine if an alternating series converges or not.

<u>Thm</u> (Alternating series test)

Given $\sum_{n=0}^{\infty} (-1)^n a_n$, ① $a_n > 0$ if

② $a_{n+1} \leq a_n$ for all n and ③ $\lim_{n\to\infty} a_n = 0$

then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

pf. Consider the even partial sums (which have positive terms because $a_{n+1} \leq a_n$)

$$S_{2n} = S_{2n-2} + \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0} \geq S_{2n-2} \qquad (n \geq 1)$$

$\{S_{2n}\}$ is a positive nonincreasing sequence hence converges, say

$$\lim_{n\to\infty} S_{2n} = S$$

Then the partial sum converges by limit law

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + a_{2n+1}$$

$$= S + 0 = S$$

Moreover, from the above proof, we see that if $\lim_{n\to\infty} a_n$ div, the series div. So

divergence test still holds

Example 1. (alternating harmonic series)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ conv.}$$

Check: $a_n = \frac{1}{n} > 0$

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

$$\lim_{n\to\infty} a_n = 0$$

So the alternating series test tells the series conv.

Example 2. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$ conv.

Check • $a_n = \frac{n^2}{n^3+1} \geq 0$ for all n.

• $a_{n+1} \leq a_n$ for $n \geq 2$ because the function $f(x) = \frac{x^2}{x^3+1}$ is decreasing (not obvious)

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0 \quad \text{when } x > \sqrt[3]{2}$$

$\sqrt[3]{2} \leq 2$

• $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$

Apply the alternating series test for $n \geq 2$.

$$\Rightarrow \sum_{n=2}^{\infty} (-1)^n a_n \text{ conv.}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n = \underbrace{a_0 - a_1}_{\text{finite number}} + \underbrace{\sum_{n=2}^{\infty} (-1)^n a_n}_{< \infty \text{ as conv.}}$$

$< \infty$

Estimating alternating series.

Thm (Alternating series estimation thm)

Given $\sum_{n=0}^{\infty} (-1)^n a_n$, $a_n > 0$ satisfying

$$a_{n+1} \leq a_n \text{ and } \lim_{n\to\infty} a_n = 0$$

then $|R_n| = |s - s_n| \leq a_{n+1}$

pf. Recall that s_{2n} is positive and nonincreasing. Let $s = \lim_{n\to\infty} s_{2n}$. Then $s \overset{\leq}{\cancel{\geq}} s_{2n}$ for all n. Similarly, $s \overset{\geq}{\cancel{\leq}} s_{2n+1}$ (odd partial sums)

$$\Rightarrow |s - s_m| = \begin{cases} s - s_m \leq s_{m+1} - s_m & m \text{ odd} \\ -(s - s_m) \leq -(s_{m+1} - s_m) & m \text{ even} \end{cases}$$

$m+1$ ~~even~~ odd $\quad -s \leq -s_{m+1}$

$$\Rightarrow |s - s_m| \leq |s_{m+1} - s_m| = a_{m+1}$$

Last time: alternating series test
This time: absolute convergence and more tests

Def A series $\sum a_n$ is called <u>absolutely convergent</u> if the series of absolute values $\sum |a_n|$ is convergent.

Def A series $\sum a_n$ is called <u>conditionally conv.</u> if it is convergent but not abs. conv.

Note that absolutely conv is stronger than convergent. That is,

$$\text{abs. conv.} \Rightarrow \text{conv.}$$

pf. Observe that $-a_n \le |a_n| \le a_n$

$$\Rightarrow 0 \le \underbrace{a_n + |a_n|}_{A_n} \le \underbrace{2|a_n|}_{B_n}.$$

Apply the comparison test

$$\sum B_n \text{ conv} \Rightarrow \sum A_n \text{ conv.}$$

then $\sum a_n = \underbrace{\sum A_n}_{<\infty} - \underbrace{\sum |a_n|}_{<\infty} < \infty.$

by one of the properties of series

Example 1. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ conv. $\sum_{n=1}^{\infty}\frac{1}{n}$ not conv.

Hence we say $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is conditionally conv. but not absolutely conv.

Example 2. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ both conv.

Hence we say $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ is conditionally conv. and also absolutely conv.

Thm (the ratio test)

Given a series $\sum a_n$, let

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$\text{if } L \begin{cases} <1 \\ >1 \\ =1 \end{cases} \text{ then } \begin{cases} \sum a_n \text{ abs. conv.} \\ \sum a_n \text{ div} \\ \text{no conclusion} \end{cases}$$

Thm (the root test)

Given a series $\sum a_n$, let

$$L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$$

$$\text{if } L \begin{cases} <1 \\ >1 \\ =1 \end{cases} \text{ then } \begin{cases} \sum a_n \text{ abs. conv.} \\ \sum a_n \text{ div} \\ \text{no conclusion} \end{cases}$$

Remark

1. Note that we have absolute conv.
2. $L=1$ case examples

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div} \quad \& \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \text{ conv}$$

but in both cases L (for the ratio test) is given by

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

3. prototype for both tests: geometric series

$$\left.\begin{aligned} L_1 &= \lim_{n\to\infty} \left| \frac{r^{n+1}}{r^n} \right| \\ L_2 &= \lim_{n\to\infty} \sqrt[n]{|r|^n} \end{aligned}\right\} = \lim_{n\to\infty} |r| = |r|$$

we know

$|r| < 1$ conv.

$|r| > 1$ div.

Example 1. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$ ← a sign for ratio test

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^2}{(2(n+1)-1)!}}{\dfrac{n^2}{(2n-1)!}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)\cdot(2n)\cdot n^2} = 0 < 1$$

Hence the series abs. conv. by ratio test.

Example 2. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

$$L = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}}{(n+1)^2+1}}{\dfrac{(-1)^n}{n^2+1}} \right|$$

$$= \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} = 1$$

The ratio test is not useful.

Instead one can use the alternating series test to conclude this series conv. and use comparison test for abs. conv.

$$A_n = \frac{1}{n^2+1} \leq B_n = \frac{1}{n^2}$$

Example 3. $\sum_{n=0}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3n+1}{4-2n}\right)^{2n}\right|}$$

$$= \lim_{n\to\infty} \left(\frac{3n+1}{2n-4}\right)^2$$

$$= \lim_{n\to\infty} \frac{9n^2+6n+1}{4n^2-16n+16}$$

$$= \frac{9}{4} > 1$$

The series converges absolutely by root test.

Example 4. $\sum_{n=4}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^{-n^2}\right|}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \frac{1}{e} < 1$$

The series converges absolutely by root test.

For strategy of choosing conv. tests

See "Supplementary Resources" on course webpage

This time: power series.

Def A <u>power series centered at a</u> is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Here x is a variable, c_n's are coefficients.

Example 1. Take $a=0$, then $\sum c_n x^n$ is

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

a polynomial with infinitely many terms.

Moreover if $c_n = 1$ for all n, then

$$f(x) = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

is a geometric series, we know it conv when $|x| < 1$.

The above example shows that a power series may converge for some values of x and diverges for other values of x. We can use convergency tests to determine that.

Example 2. $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$

Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right|$$

$$= \lim_{n\to\infty} \frac{|x-3|}{1+\frac{1}{n}} = |x-3|$$

conv. when $|x-3| < 1 \iff 2 < x < 4$

div $\qquad > 1 \qquad x < 2$ or $x > 4$

Boundary cases:

$x = 2 \quad \sum a_n = \sum \frac{(-1)^n}{n}$ conv.

$x = 4 \quad \sum a_n = \sum \frac{1}{n}$ div

Thus the power series conv when $2 \leq x < 4$.

<u>Thm</u> For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities

(1) series conv only when $x = a$

(2) series conv for all x

(3) there is $R > 0$ s.t.

series conv for $|x-a| < R$

div for $|x-a| > R$

Def. The number R is called the radius of convergence.

Def. The interval of convergence is the interval that consists of all values of x for which the power series conv.

Example 2' $\quad R = 2 \quad I = [2, 4)$

Example 3. $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^n}$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+1}}}{\frac{n(x+2)^n}{3^n}} \right|$$

$$= \lim_{n\to\infty} \frac{|x+2|}{3\left(1+\frac{1}{n}\right)} = \underbrace{\frac{1}{3}|x+2|}_{\substack{<1 \\ \text{when } -5<x<1}}$$

$\Rightarrow R = 3$

Boundary cases

$x = -5 \qquad \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{3}$ div

$x = 1 \qquad \displaystyle\sum_{n=0}^{\infty} \frac{n}{3}$ div

$\Rightarrow I = (-5, 1)$

Last time power series
This time functions as power series

In this section, we will learn how to represent some function as a power series. Application for this techique is that we may approximate certain integrals which does not have an elementary antiderivative.

We start by discussing how to find the power series representation by substitution, intergation and differentiation.

Recall we have seen $\boxed{\dfrac{1}{1-u} = \sum_{n=0}^{\infty} u^n \quad |u| < 1}$

Example 1.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^2$$

Take $u = -x^2$

$|u| = |-x^2| = x^2 < 1$

$\Longleftrightarrow \ |x| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{when } |x| < 1$$

Example 2.

$|u| = \left|-\frac{x}{2}\right| < 1 \ \Longleftrightarrow \ |x| < 2$

$$\frac{1}{2+x} = \frac{1}{2}\,\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\,\frac{1}{1-\left(-\frac{x}{2}\right)}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$

when $|x| < 2$

Term-by-term differentiation and integration

<u>Thm</u>

If $\sum c_n(x-a)^n$ has radius of convergence $R>0$ then $f(x) = \sum c_n(x-a)^n$ is differentiable on $(a-R, a+R)$ and

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)\,dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

One can prove this by computing

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(x-a)^n$$

Example 3.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$= \sum_{n=1}^{\infty} n x^{n-1} \qquad \text{when } |x|<1$$

not $n=0$.

Example 4

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt \qquad \text{FTC}$$

$$= \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \,dt$$

Take $u=-t$
$|u| = |-t| < 1$
$\Rightarrow |t| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n \,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{n+1}}{n+1}\right]_{t=0}^{x} + C$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C \quad \text{when } |x|<1$$

To determine C: take $x=0$
$\ln(1+0) = 0 = C$

$$\Rightarrow \boxed{\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad |x|<1}$$

Example 5. (did in problem session)

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \int_0^x \sum_{n=0}^{\infty} (-t^2)^n\,dt$$

Take $u = -t^2$
$|u| = |-t^2| < 1$
$\Rightarrow |t| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{t^{2n+1}}{2n+1}\right]_{t=0}^{x} + C$$

$C = 0$ as
$\arctan 0 = 0$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{when } |x| < 1}$$

This time: Taylor and Maclaurin series

<u>Thm</u> If f has a power series representation at a

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then $c_n = \dfrac{f^{(n)}(a)}{n!}$

pf. Compute $f'(x) = \sum_{n=1}^{\infty} c_n n (x-a)^{n-1}$

$$f''(x) = \sum_{n=2}^{\infty} c_n n(n-1)(x-a)^{n-2}$$

$$\vdots$$

Taking $x=a$ yields the only nonvanishing term is the 0-th order term

$$f'(a) = 1 \cdot c_1$$

$$f''(a) = 1 \cdot 2 \cdot c_2 = 2! c_2$$

$$\vdots$$

$$f^{(n)}(a) = 1 \cdot 2 \cdot 3 \cdots n \, c_n = n! \, c_n$$

We define <u>Taylor series</u> of f centered at a to be

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \qquad |x-a| < R}$$

When $a=0$ we call $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$ <u>Maclaurin series</u>.

Example 1. $f(x) = e^x$ at 0

$f^{(n)}(x) = e^x$ for all n.

$$\Rightarrow \boxed{e^x = \sum_{n\geq 0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x}$$

Radius of conv.

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = 0 < 1$$

Check this

$\Rightarrow R = \infty$ as the series always conv.

Example 2. $f(x) = \sin x$ at 0

$f'(x) = \cos x \qquad f''(x) = -\sin x$

$f^{(3)}(x) = -\cos x \qquad f^{(4)}(x) = \sin x$

only odd powers survive

$$\Rightarrow \boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all } x.}$$

odd function

again $R = \infty$ as

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0 < 1$$

Example 2' Check that

only even powers survive

$$\boxed{\cos x = \sum_{n\geq 0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x}$$

even function

Application: estimate integral.

Let's look at a particular integral

Example 3. $\int_0^1 e^{-x^2}\,dx$

- First the Maclaurin series of $\int e^{-x^2}\,dx$ is

$$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} + C$$

- Evaluate at $x=0$ and $x=1$ gives

$$\int_0^1 e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!} + C - C$$

- Say we take the first five term, the value is

$$1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475.$$

Recall the alternating series estimation, the error here is bounded by

$$|R| < |a_6| = \frac{1}{11 \cdot 5!} < 0.001.$$