Ch 11. Infinite sequence and series

Def. An <u>sequence</u> is an infinite list of numbers written in a definite order.

Notation: $\{a_1, a_2, \cdots, a_n, \cdots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples $\{1, 2, 3, 4, \cdots\}$
$\{7, 1, 8, 2, 8, \cdots\}$

Some sequences can be defined by giving a formula for the n-th term a_n

Examples 1. $a_n = \left(\frac{1}{2}\right)^n$ $\quad \{a_n\} = \{\frac{1}{2}, \frac{1}{4}, \cdots\}$

2. $a_n = (-1)^n$ $\quad \{a_n\} = \{-1, 1, -1, 1, \cdots\}$

3. $a_n = \frac{n}{n+1}$ $\quad \{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$

Some sequences may not have a simple / explicit defining equation

Examples 1. a_n = the digit in the n-th decimal place of π

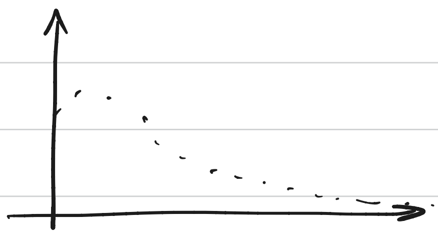
2. The Fibonacci sequence

$$a_1 = 1, \quad a_2 = 1, \quad a_n = a_{n-1} + a_{n-2}$$

$$\{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}$$

A sequence "is" a function f that only takes values on natural numbers. So we will study properties such as graph and convergency.

Example



$$\lim_{n\to\infty} a_n = 0$$

$L < \infty$

Def. A sequence <u>has limit L</u> if for any ε there is an N s.t. if $n > N$ then

$$|a_n - L| < \varepsilon$$

We say $\{a_n\}$ <u>converges</u> to L.

Intuition

Def. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N s.t. if $n > N$ then $a_n > M$.

Examples

1. $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

2. $\lim_{n\to\infty} \frac{1}{n^r} = \begin{cases} 0 & \text{if } r>0 \\ \infty & \text{if } r<0 \end{cases}$

3. $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \\ \text{DNE} & \text{if } r<1 \end{cases}$

Limit law for sequences

if $\{a_n\}$. $\{b_n\}$ are convergent sequences then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c\,a_n = c \lim_{n\to\infty} a_n \qquad c \text{ const.}$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

$$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \qquad p>0 \quad a_n>0$$

Squeeze Theorem

$$\begin{array}{ccccc} b_n & \le & a_n & \le & c_n \\ \downarrow & & & & \downarrow \\ L & & & & L \end{array} \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$

Thm If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

If f is continuous

$$\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} f(a_n) = f(L)$$

Example 1.

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$$

$$= \sin 0 = 0$$

Example 2. $\lim_{n\to\infty} \dfrac{\ln(n+2)}{\ln(1+4n)}$

L'Hopital

$$\lim_{x\to\infty} \frac{\ln(x+2)}{\ln(1+4x)} = \lim_{x\to\infty} \frac{\frac{1}{x+2}}{\frac{1}{1+4x}\cdot 4}$$

$$= \lim_{x\to\infty} \frac{4x+1}{4(x+2)} = 1$$

Example 3.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n}$$

$$= e^{\lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right)}$$

$$= e$$

$$\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{1}{1+\frac{1}{n}}\cdot -\frac{1}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

Last time: sequence
This time: series.

Def. We call $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ a <u>series</u> and

$$s_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N \text{ the } \underline{\text{partial sums}}.$$

Note that $\{s_N\}$ is itself a sequence. So it make sense to talk about if $\{s_N\}$ converges or not.

Def. The series $\sum a_n$ is called <u>convergent</u> if its partial sum is convergent. Otherwise $\sum a_n$ is called <u>divergent</u>.

Example 1. (geometric series)
Consider $a_n = r^n$ r: common ratio.

$$a_0 = r^0 = 1 \qquad s_0 = a_0 = 1$$
$$a_1 = r^1 = r \qquad s_1 = a_0 + a_1 = 1 + r$$
$$a_2 = r^2 = r^2 \qquad s_2 = a_0 + a_1 + a_2 = 1 + r + r^2$$
$$\vdots \qquad\qquad \vdots$$

$$s_N = \underbrace{1 + r + r^2 + \cdots + r^N}_{\text{we are interested in this sum.}}$$

Let $R_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \cdots + r^N$

$$rR_N = \quad r + r^2 + \cdots + r^N + r^{N+1}$$

$$\Rightarrow \quad R_N - rR_N = 1 - r^{N+1}$$

$$\Rightarrow \quad R_N = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{n=0}^{\infty} r^n = \lim_{N \to \infty} R_N$$

! $n=0$

$$= \begin{cases} \dfrac{1}{1-r} & \text{if } -1 < r < 1 \quad \text{conv.} \\ \infty & \text{if } r \geq 1 \\ \text{DNE} & \text{if } r \leq -1 \end{cases}$$

div.

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \quad \text{because} \quad \sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r_n$$

Example 2. Compute $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ using the above

$$\sum_{n=1}^{\infty} 2^{2n} 6^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 6 \cdot 6^{-n}$$

note that n start from 1

$$= 6 \cdot \sum_{\boxed{n=1}}^{\infty} \underbrace{\left(\frac{4}{6}\right)}_{= \frac{2}{3}}^{n} = 6 \cdot \frac{\boxed{\frac{2}{3}}}{1 - \frac{2}{3}}$$

$$= 6 \cdot 2 = 12$$

Example 3. (harmonic series)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$S_2 = 1 + \frac{1}{2}$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= 1 + \frac{2}{2}$$

$$S_8 = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$$

$$> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{\frac{1}{2}}$$

$$= 1 + \frac{3}{2}$$

$$\vdots$$

$$S_{2^n} = 1 + \frac{n}{2} \xrightarrow{n \to \infty} \infty$$

Hence $\boxed{\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}}$

Example 4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

Note that $\displaystyle\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \longrightarrow 1$$

Theorem $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converges, c const.

$$\sum_{n=1}^{\infty} a_n \pm b_n = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$$

Example 5 $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{3}{n(n+1)}}_{a_n} + \underbrace{\frac{1}{2^n}}_{b_n}$

We have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - 1$

$$= 2 - 1 = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the original series converges to

$$3 \cdot 1 + 1 = 4$$

Thm $\sum_{n=1}^{\infty} a_n$ convergent $\Rightarrow$ $\lim_{n\to\infty} a_n = 0$

pf. By definition, we know $\lim_{n\to\infty} S_n = L$ for some real number L.

$$\Rightarrow \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = L$$

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (S_n - S_{n-1}) \\ &= \lim_{n\to\infty} S_{n-1} - \lim_{n\to\infty} S_n \\ &= L - L = 0\end{aligned}$$

Coroallary (The divergence test)

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges

Examples

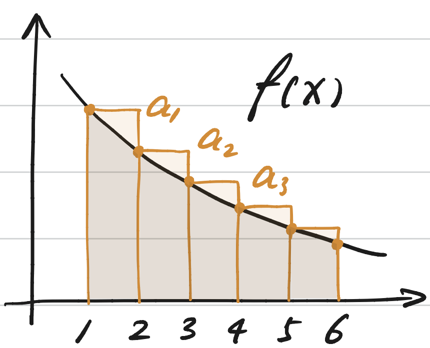
1. $\sum_{n=1}^{\infty} (-1)^n$
2. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ all diverges
3. $\sum_{n=1}^{\infty} \frac{n}{n+1}$

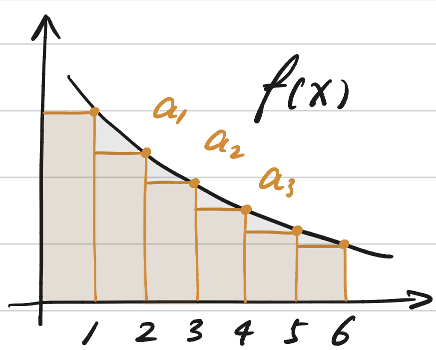
Last time computing series.
This time integral test & estimates.

We have been computing exact value of a series so far for some special cases. However, in general it is quite difficult. In those cases, we are interested in finding an estimate.

<u>Thm</u> (the integral test)
Suppose $f(x) > 0$ is a continuous decreasing function for $x \geq 1$ such that $a_n = f(n)$. Then

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \iff \int_1^{\infty} f(x)\,dx \text{ conv.}$$





Moreover,

$$\int_1^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\,dx \quad (\star)$$

Error $\quad R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n$

$$\int_{N+1}^{\infty} f(x)\,dx \leq R_N \leq \int_N^{\infty} f(x)\,dx$$

Example 1. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv.

$f(x) = \frac{1}{x^2} > 0$ for $x \geq 1$ cont. and decreasing

$\int_1^{\infty} \frac{1}{x^2}\,dx$ conv.

$f'(x) = -2x^{-3} < 0$ for $x \geq 1$

Example 2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is conv. if $p > 1$

div if $p \leq 1$

Example 3. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ conv.

$f(x) = \frac{1}{x^2+1} > 0$ for $x \geq 1$, cont. and decreasing

$f'(x) = -(x^2+1)^{-2} \cdot 2x < 0$ for $x \geq 1$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \arctan x \Big]_1^t$$

$$= \lim_{t\to\infty} \arctan t - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty$$

Last time: integral test.
This time: the comparison test

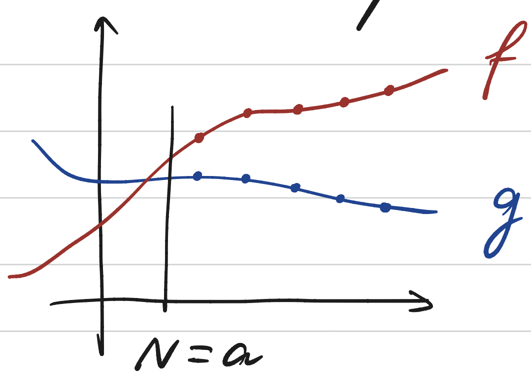
The idea of the comparison test for sequences is similar to that for integrals.

<u>Thm</u> (the comparison test)
Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $a_n \geq b_n$ for $n \geq N$.

$$\sum a_n \text{ conv} \Rightarrow \sum b_n \text{ conv.}$$
$$\sum b_n \text{ div} \Rightarrow \sum a_n \text{ div}$$

Compare the above with the comparison test in Ch 7.

$$a_n \longleftrightarrow f$$
$$b_n \longleftrightarrow g$$
$$\sum \longleftrightarrow \int$$
$$N \longleftrightarrow a$$



Example 1. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

$$2n^2+4n+3 \geq 2n^2 \quad \text{for } n \geq 1$$

$N = 1$ here

$$\Rightarrow \underbrace{\frac{5}{2n^2+4n+3}}_{a_n} \leq \underbrace{\frac{5}{2n^2}}_{b}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv.}$$

Example 2. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Note that $\ln n > 1$ for $n > e$

$$\Rightarrow \quad \underbrace{\frac{\ln n}{n}}_{a_n} > \underbrace{\frac{1}{n}}_{b_n} \quad \text{for } n > e$$

e.g. we can take $N = 3$

$$\sum_{n=3}^{\infty} \frac{1}{n} \text{ div} \Rightarrow \sum_{n=3}^{\infty} a_n \text{ div} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ div.}$$

<u>Thm</u> (the limit comparison test)

Suppose $\sum a_n$, $\sum b_n$ are series with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n} = c \in (0, \infty)$

Then $\sum a_n$ conv $\Leftrightarrow \sum b_n$ conv.

Example 3. $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ Take $b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n-1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1}$$

$$= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 \in (0, \infty)$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ conv} \Rightarrow \sum a_n \text{ conv.}$$

dominant part is $2n^2$

Example 4. $$\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \quad \leftarrow \sqrt{n^5} = n^{5/2}$$

Take $b_n = \dfrac{2n^2}{n^{5/2}} = \dfrac{2}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^2}} \cdot \frac{\sqrt{n}}{2}$$

$$= \lim_{n\to\infty} \frac{2n^{5/2}+3n^{3/2}}{2\sqrt{5+n^2}}$$

$$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{2\sqrt{\frac{5}{n^5}+1}} = \frac{2}{2} = 1$$

$\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$ div $\Rightarrow$ $\sum a_n$ div

Last time: comparison tests
This time: alternating series.

So far we've studied series with positive terms. In this section we will study series whose terms are alternating (e.g. $a_{2n} > 0$, $a_{2n+1} < 0$).

Examples 1. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

2. $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - \cdots$

The following theorem tells us how to determine if an alternating series converges or not.

Thm (Alternating series test)

Given $\sum_{n=0}^{\infty} (-1)^n a_n$, ① $a_n > 0$ if ② $a_{n+1} \leq a_n$ for all n and ③ $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges.

pf. Consider the even partial sums (which have positive terms because $a_{n+1} \leq a_n$)

$$S_{2n} = S_{2n-2} + \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0} \geq S_{2n-2} \qquad (n \geq 1)$$

$\{S_{2n}\}$ is a positive nonincreasing sequence hence converges, say

$$\lim_{n\to\infty} S_{2n} = S$$

Then the partial sum converges by limit law

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + a_{2n+1}$$

$$= S + 0 = S$$

Moreover, from the above proof, we see that if $\lim_{n\to\infty} a_n$ div, the series div. So

$\boxed{\text{divergence test still holds}}$

Example 1. (alternating harmonic series)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ conv.}$$

Check: $a_n = \frac{1}{n} > 0$

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

$$\lim_{n\to\infty} a_n = 0$$

So the alternating series test tells the series conv.

Example 2. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3+1}$ conv.

Check • $a_n = \frac{n^2}{n^3+1} \geq 0$ for all n.

• $a_{n+1} \leq a_n$ for $n \geq 2$ because the function $f(x) = \frac{x^2}{x^3+1}$ is decreasing (not obvious)

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0 \quad \text{when } x > \sqrt[3]{2}$$

$\sqrt[3]{2} \leq 2$

• $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0$

Apply the alternating series test for $n \geq 2$.

$\Rightarrow \sum_{n=2}^{\infty} (-1)^n a_n$ conv.

$\Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n = \underbrace{a_0 - a_1}_{\text{finite number}} + \underbrace{\sum_{n=2}^{\infty} (-1)^n a_n}_{<\infty \text{ as conv.}}$

$< \infty$

Estimating alternating series.

<u>Thm</u> (Alternating series estimation thm)

Given $\sum_{n=0}^{\infty} (-1)^n a_n$, $a_n > 0$ satisfying

$$a_{n+1} \leq a_n \text{ and } \lim_{n\to\infty} a_n = 0$$

then $|R_n| = |s - s_n| \leq a_{n+1}$

pf. Recall that s_{2n} is positive and nonincreasing.
Let $s = \lim_{n\to\infty} s_{2n}$. Then $s \overset{\leq}{\cancel{\geq}} s_{2n}$ for all n.
Similarly, $s \overset{\geq}{\cancel{\leq}} s_{2n+1}$ (odd partial sums)

$$\Rightarrow |s - s_m| = \begin{cases} s - s_m \leq s_{m+1} - s_m & m \text{ odd} \\ -(s - s_m) \leq -(s_{m+1} - s_m) & m \text{ even} \end{cases}$$

$m+1$ ~~even~~ odd $\quad -s \leq -s_{m+1}$

$$\Rightarrow |s - s_m| \leq |s_{m+1} - s_m| = a_{m+1}$$

Last time: alternating series test
This time: absolute convergence and more tests

Def A series $\sum a_n$ is called <u>absolutely convergent</u> if the series of absolute values $\sum |a_n|$ is convergent.

Def A series $\sum a_n$ is called <u>conditionally conv.</u> if it is convergent but not abs. conv.

Note that absolutely conv is stronger than convergent. That is,

abs. conv. $\Rightarrow$ conv.

pf. Observe that $-a_n \leq |a_n| \leq a_n$

$$\Rightarrow 0 \leq \underbrace{a_n + |a_n|}_{A_n} \leq \underbrace{2|a_n|}_{B_n}$$

Apply the comparison test

$$\sum B_n \text{ conv} \Rightarrow \sum A_n \text{ conv.}$$

then $\sum a_n = \underbrace{\sum A_n}_{<\infty} - \underbrace{\sum |a_n|}_{<\infty} < \infty.$

by one of the properties of series

Example 1. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ conv. $\sum_{n=1}^{\infty} \frac{1}{n}$ not conv.

Hence we say $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is conditionally conv. but not absolutely conv.

Example 2. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ both conv.

Hence we say $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ is conditionally conv. and also absolutely conv.

<u>Thm</u> (the ratio test)

Given a series $\sum a_n$, let

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\text{if } L \begin{cases} <1 \\ >1 \\ =1 \end{cases} \text{ then } \begin{cases} \sum a_n \text{ abs. conv.} \\ \sum a_n \text{ div} \\ \text{no conclusion} \end{cases}$$

<u>Thm</u> (the root test)

Given a series $\sum a_n$, let

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

$$\text{if } L \begin{cases} <1 \\ >1 \\ =1 \end{cases} \text{ then } \begin{cases} \sum a_n \text{ abs. conv.} \\ \sum a_n \text{ div} \\ \text{no conclusion} \end{cases}$$

Remark

1. Note that we have absolute conv.
2. $L=1$ case examples

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div} \quad \& \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \text{ conv}$$

but in both cases L (for the ratio test) is given by

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

3. prototype for both tests: geometric series

$$\left.\begin{aligned} L_1 &= \lim_{n\to\infty} \left| \frac{r^{n+1}}{r^n} \right| \\ L_2 &= \lim_{n\to\infty} \sqrt[n]{|r|^n} \end{aligned}\right\} = \lim_{n\to\infty} |r| = |r|$$

we know

$|r| < 1$ conv.

$|r| > 1$ div.

Example 1. $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$ ← a sign for ratio test

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{(2(n+1)-1)!}}{\frac{n^2}{(2n-1)!}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)\cdot(2n)\cdot n^2} = 0 < 1$$

Hence the series abs. conv. by ratio test.

Example 2. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)^2+1}}{\frac{(-1)^n}{n^2+1}} \right|$$

$$= \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} = 1$$

The ratio test is not useful.

Instead one can use the alternating series test to conclude this series conv. and use comparison test for abs. conv.

$$A_n = \frac{1}{n^2+1} \leq B_n = \frac{1}{n^2}$$

Example 3. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3n+1}{4-2n}\right)^{2n}\right|}$$

$$= \lim_{n\to\infty} \left(\frac{3n+1}{2n-4}\right)^{2}$$

$$= \lim_{n\to\infty} \frac{9n^2+6n+1}{4n^2-16n+16}$$

$$= \frac{9}{4} > 1$$

The series converges absolutely by root test.

Example 4. $\displaystyle\sum_{n=4}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\left(1+\frac{1}{n}\right)^{-n^2}\right|}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \frac{1}{e} < 1$$

The series converges absolutely by root test.

For strategy of choosing conv. tests

See "Supplementary Resources" on course webpage

This time: power series.

<u>Def</u> A <u>power series centered at a</u> is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Here x is a variable, c_n's are coefficients.

Example 1. Take $a=0$, then $\sum c_n x^n$ is

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

a polynomial with infinitely many terms.

Moreover if $c_n = 1$ for all n, then

$$f(x) = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

is a geometric series, we know it conv when $|x| < 1$.

The above example shows that a power series may converge for some values of x and diverges for other values of x. We can use convergency tests to determine that.

Example 2. $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$

Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right|$$

$$= \lim_{n\to\infty} \frac{|x-3|}{1+\frac{1}{n}} = |x-3|$$

conv. when $|x-3| < 1 \iff 2 < x < 4$

div $\qquad > 1 \qquad x < 2$ or $x > 4$

Boundary cases:

$x = 2 \quad \sum a_n = \sum \frac{(-1)^n}{n}$ conv.

$x = 4 \quad \sum a_n = \sum \frac{1}{n}$ div

Thus the power series conv when $2 \leq x < 4$.

<u>Thm</u> For a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities

(1) series conv only when $x = a$

(2) series conv for all x

(3) there is $R > 0$ s.t.

series conv for $|x-a| < R$

div for $|x-a| > R$

Def. The number R is called the <u>radius of convergence.</u>

Def The <u>interval of converges</u> is the interval that consists of all values of x for which the power series conv.

Example 2' $\quad R = 2 \quad I = [2, 4)$

Example 3. $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^n}$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+1}}}{\frac{n(x+2)^n}{3^n}} \right|$$

$$= \lim_{n\to\infty} \frac{|x+2|}{3\left(1+\frac{1}{n}\right)} = \underbrace{\frac{1}{3}|x+2|}_{\substack{<1 \\ \text{when } -5<x<1}}$$

$\Rightarrow R = 3$

Boundary cases

$x = -5 \qquad \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{3}$ div

$x = 1 \qquad \displaystyle\sum_{n=0}^{\infty} \frac{n}{3}$ div

$\Rightarrow I = (-5, 1)$