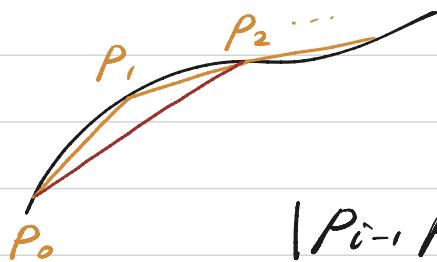


Last time: improper integral  
This time: arc length

In this chapter we are going to study applications of integrals.

Lets first consider how to compute arc length



$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n |P_{i-1}, P_i|$$

$$\begin{aligned} |P_{i-1}, P_i| &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x \end{aligned}$$

taking limits gives  $f'$

Let  $y = f(x)$  be the curve

If  $f'$  is continuous, arc length for  $x \in [a, b]$

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Similarly, let  $x = g(y)$  be the curve, then arc length for  $y \in [c, d]$

$$L = \int_c^d \sqrt{1 + (g'(y))^2} dy$$

Example 1.  $y = e^x$   $0 \leq x \leq 2$

$$\Rightarrow L = \int_0^2 \sqrt{1 + (e^x)^2} dx$$

$$\text{or } L = \int_1^{e^2} \sqrt{1 + \left(\frac{1}{y}\right)^2} dy \quad \text{as } x = \ln y$$

Example 2  $x^2 + y^2 = 1$  unit circle

Using symmetry take the upper half circle

$$y = \sqrt{1 - x^2} \quad -1 \leq x \leq 1$$

$$\Rightarrow L = \int_{-1}^1 \sqrt{1 + \left(\frac{2x}{2\sqrt{1-x^2}}\right)^2} dx$$

Arc length function

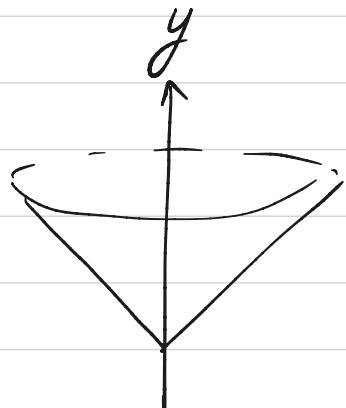
$$S(x) = \int_a^x \sqrt{1 + (f'(t))^2} dt$$

$$\text{Example 3. } f(x) = x^2 - \frac{\ln x}{8}$$

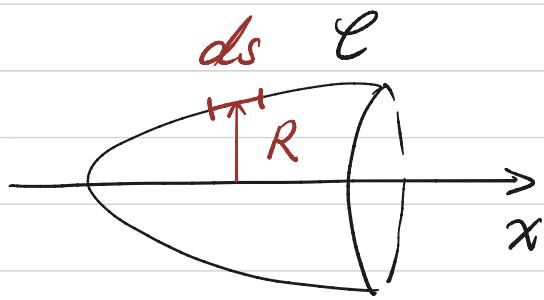
$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)^2} \, dx \\ &= \int_1^x \sqrt{1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2}} \, dx \\ &= \int_1^x 2t + \frac{1}{8t} \, dt \\ &= \left[ t^2 + \frac{1}{8} \ln t \right]_1^x \\ &= x^2 + \frac{\ln x}{8} - 1 \end{aligned}$$

## Surface of revolution

A surface of revolution is formed by rotating a curve about a line (usually the  $x$  or  $y$ -axis).



e.g. rotating  $y = x$  about  $y$ -axis



e.g. rotating  $y = \ln x$  about  $x$ -axis

To define the area, recall the surface area of a cylinder is  $2\pi R l$ .

If we take infinitesimal line segment  $ds$ , the small piece is approximate cylinder

$$S \approx \sum_{i=0}^n 2\pi f(x_i) \cdot ds$$

$$\Rightarrow S = \int_a^b 2\pi f(x) dx$$

Using arc length

$$ds = \sqrt{1 + (f'(x))^2} dx$$

$$\Rightarrow S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

Given curve  $y = f(x)$ , rotation about  $x$ -axis

$$S = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$$

Similarly, given curve  $x = g(y)$ , rotation about  $y$ -axis.

$$S = \int_c^d 2\pi x \sqrt{1 + (x')^2} dy$$

Example 1.  $y = \sqrt{9 - x^2}$   $-2 \leq x \leq 2$

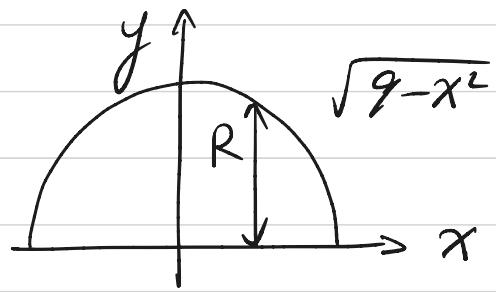
rotate about the  $x$ -axis

$$ds = \sqrt{1 + (y')^2} dx = \sqrt{1 + \left( \frac{2x}{2\sqrt{9-x^2}} \right)^2} dx$$

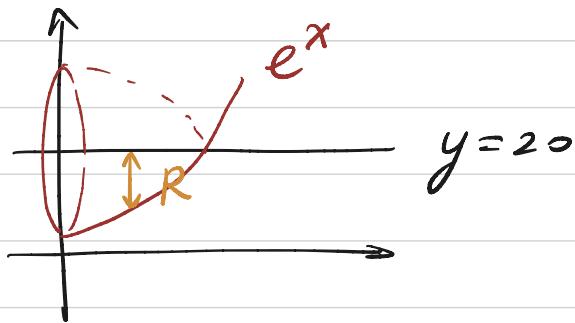
$$= \sqrt{1 + \frac{x^2}{9-x^2}} dx = \sqrt{\frac{9-x^2+x^2}{9-x^2}} dx$$

$$= \frac{3}{\sqrt{9-x^2}} dx$$

$$\begin{aligned}
 S &= \int_{-2}^2 2\pi y \, ds \\
 &= \int_{-2}^2 2\pi \sqrt{9-x^2} \frac{3}{\sqrt{9-x^2}} dx \\
 &= \int_{-2}^2 6\pi \, dx \\
 &= 24\pi
 \end{aligned}$$



Example 2.  $y = e^x$   $0 \leq x \leq 2$   
 rotate about  $y = 20$



$$R(x) = 20 - e^x$$

$$\begin{aligned}
 S &= \int_0^2 2\pi R(x) \, ds \\
 &= \int_0^2 2\pi (20 - e^x) \sqrt{1 - e^{2x}} \, dx
 \end{aligned}$$

## Application

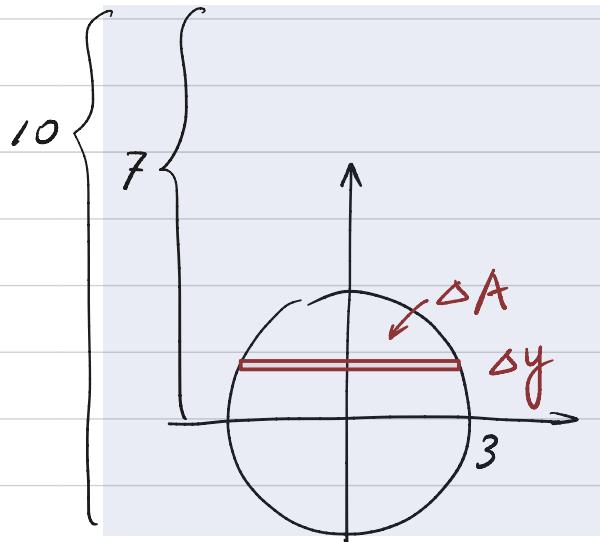
Example 1. hydrostatic pressure and force

$$F = mg = \rho g \overbrace{Ad}^{\leftarrow \text{area \& width}} \text{ of a plate}$$

density                                  acceleration due to gravity

$$\overbrace{P}^{\text{pressure}} = F/A = \rho gd$$

Compute the force on one end of a cylinder with radius 3 if it is submerged in water of depth 10



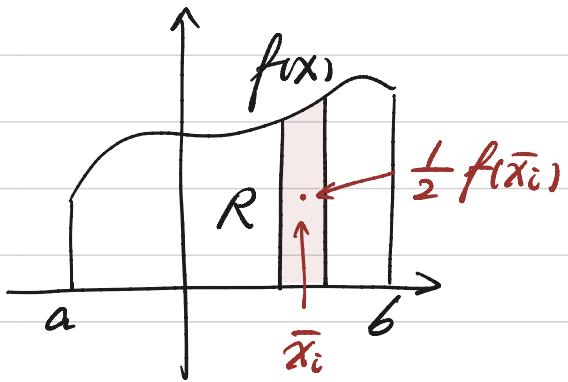
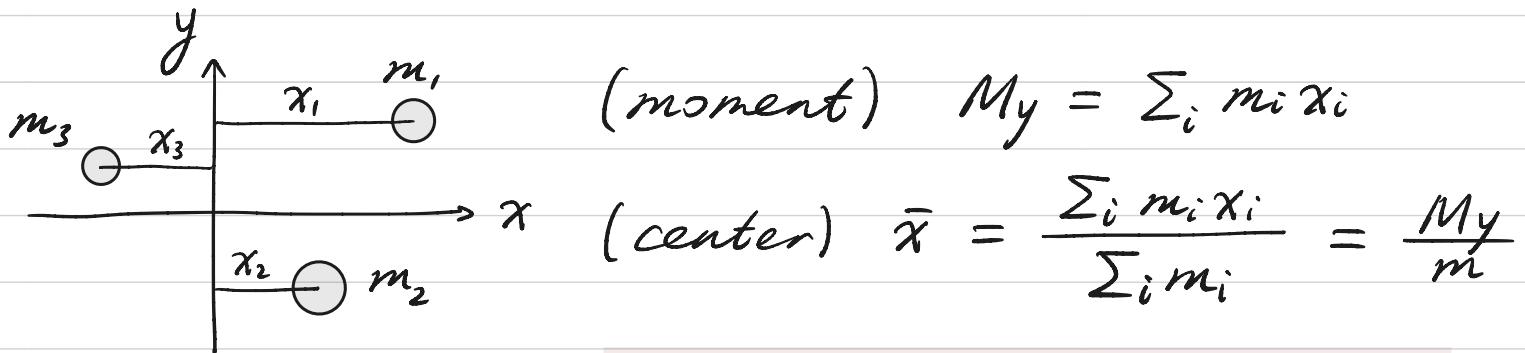
$$F = \rho g Ad \quad d = 7 - y$$

$$\Delta A \approx 2\sqrt{9-y^2} \Delta y$$

$$dA = 2\sqrt{9-y^2} dy \quad \Rightarrow \quad \Delta y \rightarrow 0$$

$$\Rightarrow F = \int_{-3}^3 (7-y) \rho g \sqrt{9-y^2} dy$$

## Example 2 moments and center of mass.



$$m = \rho A = \rho \int_a^b f(x) dx$$

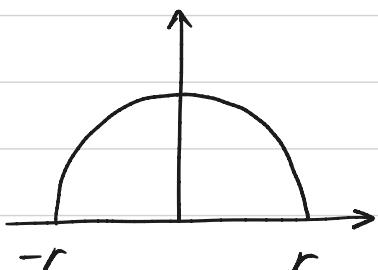
$$M_y = \rho \int_a^b x f(x) dx$$

$$M_x = \rho \int_a^b \frac{1}{2} (f(x))^2 dx$$

$$\bar{x} = \frac{M_y}{m} \quad \bar{y} = \frac{M_x}{m}$$

here  $y_i = \frac{1}{2} f(x)$  since we are taking the center of rectangle

Find the center of mass of a semicircular plate



$$\begin{aligned}\bar{y} &= \frac{1}{A} \int_{-r}^r \frac{1}{2} (f(x))^2 dx \\ &= \frac{1}{\frac{1}{2}\pi r^2} \frac{1}{2} \int_{-r}^r (r^2 - x^2) dx \\ &= \frac{2}{\pi r^2} \int_0^r r^2 - x^2 dx \\ &= \frac{2}{\pi r^2} \left[ r^2 x - \frac{x^3}{3} \right]_0^r = \frac{4r}{3\pi}\end{aligned}$$

# Summary

Infinitesimal line element

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \end{aligned}$$

Arc length  $L = \int ds$

Infinitesimal area element

$$dA = 2\pi R ds.$$

$\overbrace{\quad\quad\quad}$   
distance between point  
on curve and rotating axis

Surface area (by revolution)

$$S = \int dA.$$

Two particular cases

$$\text{about } x\text{-axis} \quad S = \int 2\pi y \, ds$$

$$\text{about } y\text{-axis} \quad S = \int 2\pi x \, ds$$