

Spring 2021 MATH231 Section CDQ Discussion

WF 9-10am

This document can be found [here](#) or on [my website](#). I will continue update it until the end of semester.

Contact

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- Office hour: Wed 10-11am¹

Zoom

- Please use your cameras and microphones in breakout rooms.
- Interrupt me/using the “Raise Hand” feature on Zoom to ask questions.
- You can call me into your breakout room/return to the main room to ask for help.
- It is also possible for me to join your breakout rooms randomly to check if you have any questions.

Worksheet

- Worksheet can be found on [Moodle](#) under Groupwork folder.
- Ask for hints when you get stuck on a problem.

Submission

- Submit on [Moodle](#) under Groupwork folder.
- **1 submission per group.** Once a file is uploaded, everyone in the same group will be able to see/edit the file. ²
- Group remains the same until each midterm.
- 1st worksheet of the week is due on **Thursday** at **8AM** CST. ³
- 2nd worksheet of the week is due on **Saturday** at **8AM** CST.
- Worksheet solutions available at 12:30PM CST on the due date.

Grading

Worksheets are graded with 2, 1 or 0.

2 - the worksheet uploaded is satisfactory

1 - the worksheet uploaded is unsatisfactory and needs improvement. Your TA will comment on what should be improved for next time.

0 - the worksheet was not uploaded

¹Office hour is run for all students in MATH231

²Groups are assigned randomly by Moodle

³Central Standard Time

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Worksheet 1

Recall

Theorem 1.1 (Fundamental Theorem of Calculus). [Ref p.26](#)

Part 1 If $f(x)$ is **continuous** over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_a^x f(t) \, dt, \quad x \in [a, b],$$

then $F'(x) = f(x)$ over $[a, b]$.

Part 2 If $f(x)$ is **continuous** over an interval $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$ i.e. $F'(x) = f(x)$, then

$$\int_a^b f(x) \, dx = F(a) - F(b).$$

Example 1.2. Let

$$g(x) = \int_a^{b(x)} f(t) \, dt.$$

Apply chain rule and FTC

$$g'(x) = \frac{d}{dx} \int_c^{b(x)} f(t) \, dt = b'(x) \cdot f(b(x)).$$

Worksheet 2

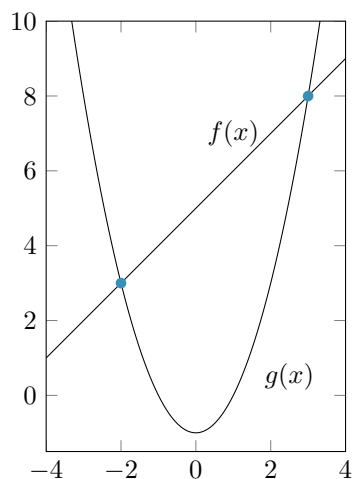
Recall

- **Substitution rule/Change of variable:** let $u = g(x)$, then

$$\int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du. \quad (\text{Q1-3})$$

- **Compute area between curves.** (Q4-7)

- Draw the graph.
- Find intersection points by solving $f(x) = g(x)$, say they are $x = a$ and $x = b$.
- Area = $\int_a^b f(x) - g(x) \, dx$.



Worksheet 3

Volume of a solid of revolution: Slices of volume are circles. [Ref](#)

$$\text{Vol} = \int_a^b \pi f(x)^2 \, dx.$$

Q3. Slices are squares/triangles.

$$\text{Vol} = \int_a^b \text{Area of slices} \, dx.$$

E.g.

$$\text{Vol} = \int_a^b f(x)^2 \, dx.$$

Worksheet 4

Volume by cylindrical shells:

$$\text{Vol} = \int_a^b 2\pi r \cdot f(x) \, dx.$$

Rotation about y -axis: $r = x$.

Rotation about the vertical line $x = a$: $r = |a - x|$.

Worksheet 5

Recall

- Since $\sin x$ is oscillating between -1 and 1, $\lim_{x \rightarrow \infty} \sin x$ does not exist.
- we can use L'Hopital's Rule to compute indeterminate forms " $\frac{0}{0}$ " and " $\frac{\infty}{\infty}$ ".

Theorem 5.1 (L'Hopital's Rule).

Assumptions:

$$\begin{aligned}f(x) &\rightarrow 0 \quad \text{as } x \rightarrow a, \\g(x) &\rightarrow 0, \\g'(x) &\neq 0.\end{aligned}$$

Conclusion:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Warning: check the assumptions before applying L'Hopital's Rule.

Worksheet 6

Integration by parts:

$$(uv)' = u'v + uv' \implies \int u \, dv = uv - \int v \, du.$$

Choose u based on which of these comes first, (search “[integration by parts what to choose as \$u\$](#) ”):

- (1) **L**ogarithmic functions: $\ln x$
- (2) **I**nverse trigonometric functions: $\arcsin x$
- (3) **A**lgebraic functions: x
- (4) **T**rigonometric functions: $\sin x$
- (5) **E**xponential functions: e^x

Worksheet 7

Worksheet 8

Recall: $1 + \tan^2 x = \sec^2 x$.

Worksheet 9

Substitution rule/Change of variable: let $u = g(x)$, then

$$\int f(g(x)) \cdot g'(x) \, dx = \int f(u) \, du.$$

Integration by parts:

$$\int u \, dv = uv - \int v \, du.$$

Worksheet 10

Partial fractions decomposition: Find A, B such that

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} - \frac{B}{x-b}.$$

The following table is from this [website](#)

Factor in denominator	Term in partial fraction decomposition
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^k$	$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}, k = 1, 2, 3, \dots$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$
$(ax^2 + bx + c)^k$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}, k = 1, 2, 3, \dots$

Typo in solution

WS10 Q3.

$$\frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2} + \frac{E}{x - 1} + \frac{F}{(x - 1)^2} + \frac{G}{(x - 1)^3}.$$

Worksheet 11

Improper integrals: There are two types of improper integrals $\int_a^b f(x) \, dx$:

(1) a or b (or both) infinite, e.g. $\int_1^\infty \frac{1}{x} \, dx$.

(2) The function $f(x)$ blows up in the interval $[a, b]$, e.g. $\int_0^1 \ln x \, dx$.

To compute improper integrals, e.g.:

$$\int_1^\infty \frac{1}{x} \, dx = \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} \, dx.$$

Worksheet 12

Simpson's rule: Let x_i 's be equally spaced points,

$$\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)).$$

Coefficient is 4 for odd i , $i \neq 0, n$; coefficient is 2 for even i , $i \neq 0, n$.

If you want to use the formula $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$, for complex a . Recall how we obtain this formula. We do trig substitution, $x = a \tan \theta$, and use the identity $1 + \tan^2 \theta = \sec^2 \theta$. This identity is still valid for complex θ . So with $x = i \tan \theta$, $\theta = \arctan(-ix)$,

$$\begin{aligned} I &= \int \frac{1}{x^2 - 1} dx = \int \frac{1}{x^2 + i^2} dx \\ &= \int \frac{1}{i^2 \tan^2 \theta + i^2} i \sec^2 \theta d\theta \end{aligned}$$

(Here we need the derivative of $\tan \theta$, but you can check this is $\sec^2 \theta$ in the complex case)

$$= -i \int \frac{1}{\sec^2 \theta} \sec^2 \theta d\theta = -i\theta + C = -i \arctan(-ix) + C.$$

The above checked our formula is valid for complex a . Hence, we can plug in the value of \arctan

$$\arctan t = \frac{1}{i} \ln \sqrt{\frac{1+it}{1-it}} = \frac{i}{2} [\ln(1-it) - \ln(1+it)].$$

So

$$\arctan(-ix) = \frac{1}{i} \ln \sqrt{\frac{1+x}{1-x}} = \frac{i}{2} [\ln(1-x) - \ln(1+x)].$$

$$I = -i \cdot \frac{i}{2} [\ln(1-x) - \ln(1+x)] + C = \frac{1}{2} [\ln(1-x) - \ln(1+x)] + D.$$

Note that D should be a real constant as I is real.

Worksheet 13

Let ds be the arclength differential.

Arc length:

$$L = \int ds = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Surface area:

$$A = \int 2\pi x ds = \int 2\pi x \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Worksheet 14

$$\text{Force} = \int \rho g \cdot \text{depth}(y) \cdot \text{width}(y) \, dy.$$

Worksheet 15

Integral test: If $f(x)$ is **continuous, positive and decreasing** on $[N, \infty)$. Then

$$\begin{aligned}\int_N^\infty f(x) \, dx \text{ converges} &\implies \sum_{n=N}^\infty f(n) \text{ converges;} \\ \int_N^\infty f(x) \, dx \text{ diverges} &\implies \sum_{n=N}^\infty f(n) \text{ diverges.}\end{aligned}$$

Warning: check the assumptions before applying integral test.

Worksheet 16

Geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 \dots = \begin{cases} \frac{1}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } |r| \geq 1 \end{cases}$$

Integral test: If $f(x)$ is continuous, positive and decreasing on $[N, \infty)$. Then

$$\begin{aligned} \int_N^{\infty} f(x) \, dx \text{ converges} &\implies \sum_{n=N}^{\infty} f(n) \text{ converges;} \\ \int_N^{\infty} f(x) \, dx \text{ diverges} &\implies \sum_{n=N}^{\infty} f(n) \text{ diverges.} \end{aligned}$$

Error estimate: Assume $\sum_{n=1}^{\infty} a_n$ converges

$$S = \overbrace{a_1 + a_2 + \dots + a_n}^{\text{partial sum } S_n} + \underbrace{a_{n+1} + a_{n+2} + \dots}_{\text{reminders } R_n}.$$

Note that the first term in R_n is a_{n+1} ,

$$\int_{n+1}^{\infty} f(x) \, dx < R_n < \int_n^{\infty} f(x) \, dx.$$

Worksheet 17

Alternating series test: Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n . If

- $\lim_{n \rightarrow \infty} b_n = 0$;
- $\{b_n\}$ is a decreasing sequence the series,

then $\sum_n a_n$ is convergent.

Ratio test: Let $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

- $L < 1$, $\sum_n a_n$ convergent;
- $L > 1$, $\sum_n a_n$ divergent;
- $L = 1$ no conclusion.

Worksheet 18

Comparison test: If $a_n, b_n > 0$ and $a_n \leq b_n$ for all large n then

- $\sum b_n$ converges, then a_n also converges;
- $\sum a_n$ diverges, then b_n also diverges.

Limit comparison test: Given $\sum a_n, \sum b_n$, with $a_n, b_n > 0$ If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = C$ for some $C \neq 0, C \neq \infty$. Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Worksheet 19

Ratio test: Let $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

Root test: Let $L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

Worksheet 20

Radius of convergence: For a power series of the form

$$\sum_{n=0}^{\infty} c_n (x - a)^n,$$

where a and c_n are numbers. The **radius of convergence** is a number $R \in [0, \infty]$ s.t.

- the series converges if $|x - a| < R$;
- the series diverges if $|x - a| > R$.

To find R , compute $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$. Apply ratio test and check the boundary case.

Recall

Ratio test: Let $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$

- $L < 1$, $\sum_n a_n$ converges absolutely;
- $L > 1$, $\sum_n a_n$ diverges;
- $L = 1$ no conclusion.

Worksheet 21

Power series expansion: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, for $|x| < R$ then we can differentiate and integrate $f(x)$:

- $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1};$
- $\int f(t) dt = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C$