

## Ch 11. Infinite sequence and series

Def. An sequence is an infinite list of numbers written in a definite order.

Notation:  $\{a_1, a_2, \dots, a_n, \dots\}$ ,  $\{a_n\}$  or  $\{a_n\}_{n=1}^{\infty}$ .

Examples  $\{1, 2, 3, 4, \dots\}$   
 $\{7, 1, 8, 2, 8, \dots\}$

Some sequences can be defined by giving a formula for the  $n$ -th term  $a_n$

Examples 1.  $a_n = \left(\frac{1}{2}\right)^n$   $\{a_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \dots\right\}$

2.  $a_n = (-1)^n$   $\{a_n\} = \{-1, 1, -1, 1, \dots\}$

3.  $a_n = \frac{n}{n+1}$   $\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$

Some sequences may not have a simple / explicit defining equation

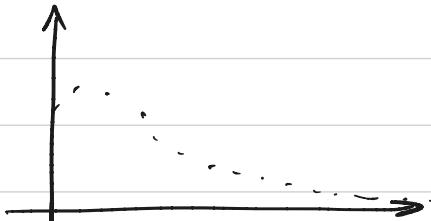
Examples 1.  $a_n$  = the digit in the  $n$ -th decimal place of  $\pi$

2. The Fibonacci sequence

$$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$$
$$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

A sequence "is" a function  $f$  that only takes values on natural numbers. So we will study properties such as graph and convergency.

Example



$$\lim_{n \rightarrow \infty} a_n = 0$$

$$L < \infty$$

Def. A sequence has limit  $L$  if for any  $\varepsilon$  there is an  $N$  s.t. if  $n > N$  then  $|a_n - L| < \varepsilon$

We say  $\{a_n\}$  converges to  $L$ .

Intuition



Def.  $\lim_{n \rightarrow \infty} a_n = \infty$  means that for every positive number  $M$  there is an integer  $N$  s.t. if  $n > N$  then  $a_n > M$ .

## Examples

$$1. \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

$$2. \lim_{n \rightarrow \infty} \frac{1}{n^r} = \begin{cases} 0 & \text{if } r > 0 \\ \infty & \text{if } r < 0 \end{cases}$$

$$3. \lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DNE} & \text{if } r < 1 \end{cases}$$

## Limit law for sequences

if  $\{a_n\}, \{b_n\}$  are convergent sequences then

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n \quad c \text{ const.}$$

$$\lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$$

$$\lim_{n \rightarrow \infty} a_n^p = (\lim_{n \rightarrow \infty} a_n)^p \quad p > 0 \quad a_n > 0$$

## Squeeze Theorem

$$b_n \leq a_n \leq c_n \Rightarrow \lim_{n \rightarrow \infty} a_n = L$$

$\downarrow$        $\downarrow$

$L$        $L$

Thm If  $\lim_{n \rightarrow \infty} |a_n| = 0$  then  $\lim_{n \rightarrow \infty} a_n = 0$

If  $f$  is continuous

$$\lim_{n \rightarrow \infty} a_n = L \Rightarrow \lim_{n \rightarrow \infty} f(a_n) = f(L)$$

Example 1.

$$\begin{aligned} \lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{n}\right) &= \sin\left(\lim_{n \rightarrow \infty} \frac{\pi}{n}\right) \\ &= \sin 0 = 0 \end{aligned}$$

Example 2.  $\lim_{n \rightarrow \infty} \frac{\ln(n+2)}{\ln(1+4n)}$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln(x+2)}{\ln(1+4x)} &\stackrel{\text{L'Hopital}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x+2}}{\frac{1}{1+4x} \cdot 4} \\ &= \lim_{x \rightarrow \infty} \frac{4x+1}{4(x+2)} = 1 \end{aligned}$$

Example 3.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} e^{\ln \left(1 + \frac{1}{n}\right)^n}$$
$$= e^{\lim_{n \rightarrow \infty} n \ln \left(1 + \frac{1}{n}\right)}$$

$$= e$$



$$\lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{n}} \cdot -\frac{1}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

Last time: sequence

This time: series

Def. We call  $\sum_{n=1}^{\infty} a_n$  or  $\sum a_n$  a series and

$s_N = \sum_{n=1}^N a_n = a_1 + a_2 + \dots + a_N$  the partial sums.

Note that  $\{s_N\}$  is itself a sequence. So it makes sense to talk about if  $\{s_N\}$  converges or not.

Def. The series  $\sum a_n$  is called convergent if its partial sum is convergent. Otherwise  $\sum a_n$  is called divergent.

Example 1. (geometric series)

Consider  $a_n = r^n$   $r$ : common ratio.

$$a_0 = r^0 = 1 \quad s_0 = a_0 = 1$$

$$a_1 = r^1 = r \quad s_1 = a_0 + a_1 = 1 + r$$

$$a_2 = r^2 = r^2 \quad s_2 = a_0 + a_1 + a_2 = 1 + r + r^2$$

⋮

⋮

⋮

$$s_N = \underbrace{1 + r + r^2 + \dots + r^N}_{}$$

we are interested in this sum.

$$\text{Let } R_N = \sum_{n=0}^N r^n = 1 + r + r^2 + \cdots + r^N$$

$r + r^2 + \cdots + r^N + r^{N+1}$

$$\Rightarrow R_N - rR_N = 1 - r^{N+1}$$

$$\Rightarrow R_N = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{n=0}^{\infty} r^n = \lim_{N \rightarrow \infty} R_N$$

$\forall n \geq 0$

$$= \begin{cases} \frac{1}{1-r} & \text{if } -1 < r < 1 \quad \text{conv.} \\ \infty & \text{if } r \geq 1 \quad \leftarrow \text{div.} \\ \text{DNE} & \text{if } r \leq -1 \quad \leftarrow \end{cases}$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \quad \text{because} \quad \sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r^n$$

Example 2. Compute  $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$  using the above

$$\sum_{n=1}^{\infty} 2^{2n} 6^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 6 \cdot 6^{-n}$$

note that  $n$  start from 1

$$= 6 \cdot \sum_{n=1}^{\infty} \underbrace{\left(\frac{4}{6}\right)^n}_{\frac{2}{3}^n} = 6 \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}}$$

$$= 6 \cdot 2 = 12$$

Example 3. (harmonic series)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$S_2 = 1 + \frac{1}{2}$$

$$\begin{aligned} S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \\ &= 1 + \frac{2}{2} \end{aligned}$$

$$\begin{aligned} S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} \\ &> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} \\ &= 1 + \frac{3}{2} \quad \frac{1}{2} \quad \frac{1}{2} \end{aligned}$$

⋮  
⋮  
⋮

$$S_{2^n} = 1 + \frac{n}{2} \xrightarrow{n \rightarrow \infty} \infty$$

Hence  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges

Example 4.  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

Note that  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \rightarrow 1$$

Theorem  $\sum_{n=1}^{\infty} a_n$ ,  $\sum_{n=1}^{\infty} b_n$  converges,  $c$  const.

$$\sum_{n=1}^{\infty} a_n \pm b_n = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

Example 5  $\sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \frac{1}{2^n}$

$a_n$        $b_n$

We have  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - 1$

$$= 2 - 1 = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the original series converges to  
 $3 \cdot 1 + 1 = 4$

Then  $\sum_{n=1}^{\infty} a_n$  converges  $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$

pf. By definition, we know  $\lim_{n \rightarrow \infty} s_n = L$  for some real number  $L$ .

$$\Rightarrow \lim_{n \rightarrow \infty} s_{n-1} = \lim_{n \rightarrow \infty} s_n = L$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1})$$

$$= \lim_{n \rightarrow \infty} s_{n-1} - \lim_{n \rightarrow \infty} s_n$$

$$= L - L = 0$$

Corollary (The divergence test)

If  $\lim_{n \rightarrow \infty} a_n \neq 0$  then  $\sum_{n=1}^{\infty} a_n$  diverges

Examples 1.  $\sum_{n=1}^{\infty} (-1)^n$

2.  $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$  all diverges

3.  $\sum_{n=1}^{\infty} \frac{n}{n+1}$

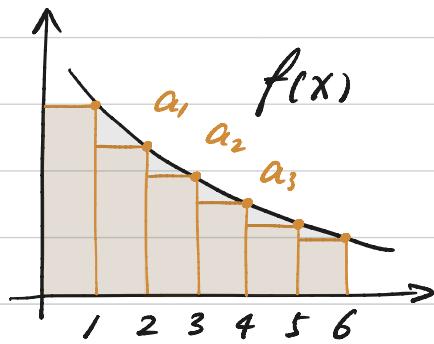
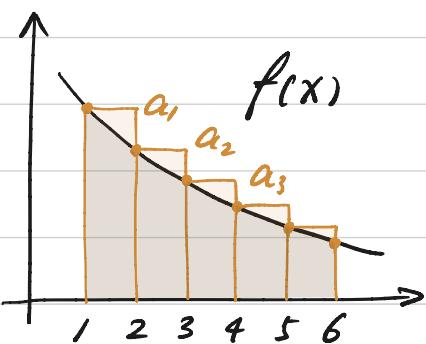
Last time computing series.  
 This time integral test & estimates.

We have been computing exact value of a series so far for some special cases. However, in general it is quite difficult. In those cases, we are interested in finding an estimate.

Thm (the integral test)

Suppose  $f(x) > 0$  is a continuous decreasing function for  $x \geq 1$  such that  $a_n = f(n)$ . Then

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \Leftrightarrow \int_1^{\infty} f(x) dx \text{ conv.}$$



Moreover,

$$\int_1^{\infty} f(x) dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x) dx \quad (\star)$$

$$\text{Error } R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n = \sum_{n=N+1}^{\infty} a_n$$

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx$$

Example 1.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  conv.

$f(x) = \frac{1}{x^2} > 0$  for  $x \geq 1$  cont. and decreasing

$$\int_1^{\infty} \frac{1}{x^2} dx \text{ conv.}$$

$$f'(x) = -2x^{-3} < 0 \quad \text{for } x \geq 1$$

Example 2.  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is conv. if  $p > 1$   
div if  $p \leq 1$

Example 3.  $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$  conv.

$f(x) = \frac{1}{x^2+1} > 0$  for  $x \geq 1$ , cont. and decreasing

$$f'(x) = -(x^2+1)^{-2} \cdot 2x < 0 \quad \text{for } x \geq 1$$

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{t \rightarrow \infty} [\arctan x]_1^t \\ &= \lim_{t \rightarrow \infty} \arctan t - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty \end{aligned}$$

Last time: integral test.

This time: the comparison test

The idea of the comparison test for sequences is similar to that for integrals.

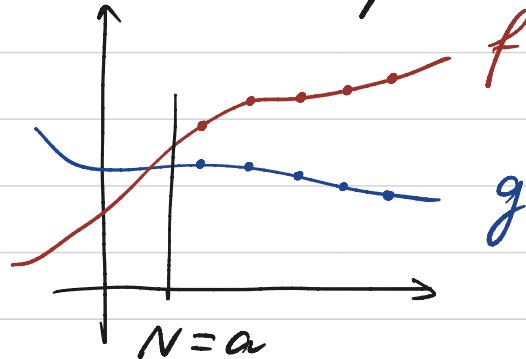
Thm (the comparison test)

Suppose  $\sum a_n$  and  $\sum b_n$  are series with positive terms and  $a_n \geq b_n$  for  $n \geq N$ .

$$\begin{aligned}\sum a_n \text{ conv} &\Rightarrow \sum b_n \text{ conv.} \\ \sum b_n \text{ div} &\Rightarrow \sum a_n \text{ div}\end{aligned}$$

Compare the above with the comparison test in Ch 7.

$$\begin{aligned}a_n &\leftrightarrow f \\ b_n &\leftrightarrow g \\ \sum &\leftrightarrow \int \\ N &\leftrightarrow a\end{aligned}$$



Example 1.  $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$

$$2n^2 + 4n + 3 \geq 2n^2 \quad \text{for } n \geq 1$$

$$\Rightarrow \underbrace{\frac{5}{2n^2 + 4n + 3}}_{a_n} \leq \underbrace{\frac{5}{2n^2}}_{b_n} \quad \text{N=1 here}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv.}$$

Example 2.  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Note that  $\ln n > 1$  for  $n > e$

$$\Rightarrow \frac{\ln n}{n} > \frac{1}{n} \quad \text{for } n > e$$

$$\sum_{n=3}^{\infty} \frac{1}{n} \text{ div} \Rightarrow \sum_{n=3}^{\infty} a_n \text{ div} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ div}$$

eg. we can take  $N = 3$

Thm (the limit comparison test)

Suppose  $\sum a_n$ ,  $\sum b_n$  are series with positive terms and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \in (0, \infty)$

Then  $\sum a_n$  conv  $\Leftrightarrow \sum b_n$  conv.

Example 3.  $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$  Take  $b_n = \frac{1}{2^n}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{2^n}} = 1 \in (0, \infty) \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ conv} \Rightarrow \sum a_n \text{ conv.}$$

dominant part is  $2n^2$

Example 4.  $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5+n^5}} \quad \sqrt{n^5} = n^{5/2}$

Take  $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{\sqrt{n}}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n}{\sqrt{5+n^2}} \cdot \frac{\sqrt{n}}{2} \\ &= \lim_{n \rightarrow \infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5+n^2}} \\ &= \lim_{n \rightarrow \infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{2}{2} = 1 \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \text{div} \Rightarrow \sum a_n \text{div}$$

Last time: comparison tests  
 This time: alternating series.

So far we've studied series with positive terms. In this section we will study series whose terms are alternating (e.g.  $a_{2n} > 0$ ,  $a_{2n+1} < 0$ ).

Examples 1.  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

2.  $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - \dots$

The following theorem tells us how to determine if an alternating series converges or not.

Thm (Alternating series test)

Given  $\sum_{n=0}^{\infty} (-1)^n a_n$ ,  $\underline{a_n > 0}$  if  
 ②  $\underline{a_{n+1} \leq a_n}$  for all  $n$  and  $\lim_{n \rightarrow \infty} a_n = 0$  ③  
 then  $\sum_{n=0}^{\infty} (-1)^n a_n$  converges.

pf. Consider the even partial sums (which have positive terms because  $a_{n+1} \leq a_n$ )

$$s_{2n} = s_{2n-2} + \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0} \geq s_{2n-2} \quad (n \geq 1)$$

$\{s_{2n}\}$  is a positive nonincreasing sequence hence converges, say

$$\lim_{n \rightarrow \infty} s_{2n} = S$$

Then the partial sum converges by limit law

$$\begin{aligned}\lim_{n \rightarrow \infty} s_{2n+1} &= \lim_{n \rightarrow \infty} s_{2n} + a_{2n+1} \\ &= S + 0 = S\end{aligned}$$

Moreover, from the above proof, we see that if  $\lim_{n \rightarrow \infty} a_n$  div, the series div. So

divergence test still holds

Example 1. (alternating harmonic series)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ conv.}$$

Check:  $a_n = \frac{1}{n} > 0$

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

So the alternating series test tells the series conv.

Example 2.  $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3 + 1}$  conv.

Check •  $a_n = \frac{n^2}{n^3 + 1} \geq 0$  for all  $n$ .

- $a_{n+1} \leq a_n$  for  $n \geq 2$  because the function  $f(x) = \frac{x^2}{x^3 + 1}$  is decreasing (not obvious)

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0 \text{ when } x > \sqrt[3]{2}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0$$

Apply the alternating series test for  $n \geq 2$ .

$$\Rightarrow \sum_{n=2}^{\infty} (-1)^n a_n \text{ conv.}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n = \underbrace{a_0 - a_1}_{\text{finite number}} + \underbrace{\sum_{n=2}^{\infty} (-1)^n a_n}_{< \infty \text{ as conv.}}$$

$$< \infty$$

$$\sqrt[3]{2} \leq 2$$