Homework Solutions

MATH231

Spring 2022

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Homework 0

1. Calculating Limits

   - $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$

     $$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2}(x+3) = 5.$$

   - $\lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$

     Consider x as a "constant"

     $$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2-2xh-h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh-h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

   - $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2+x}\right)$

     $$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right) = \lim_{x\to 0} \frac{x^2+x-x}{x(x^2+x)} = \lim_{x\to 0} \frac{x^2}{x^2(x+1)} = \lim_{x\to 0} \frac{1}{x+1} = 1.$$

2. The Chain Rule

   - $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln(x+\sin x)$

     $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln(x+\sin x) = (1+\cos x)\dfrac{1}{x+\sin x}.$

   - $\dfrac{\mathrm{d}}{\mathrm{d}x} \cos(x^2e^x)$

     $\dfrac{\mathrm{d}}{\mathrm{d}x} \cos(x^2e^x) = -(2xe^x + x^2e^x)\sin(x^2e^x).$

3. Implicit Differentiation: Solve for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following implicit function.

   - $x^2+y^2=r^2$, where r is a constant

     Differentiate on both sides w.r.t. x gives $2x+2yy'=0$. Hence $y' = -\dfrac{x}{y}$.

   - $\dfrac{x+y}{x-y} = x$

     The above is equivalent to $x+y = x^2-xy$. Differentiate on both sides w.r.t. x gives $1+y' = 2x-y-xy' \iff (1+x)y' = 2x-y-1$. Hence $y' = \dfrac{2x-y-1}{x+1}$.

4. Linear Approximations and Differentials: Find the Taylor polynomials of degree two approximating the given function centered at the given point.

   - $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

     $f' = 2\cos(2x)$, $f'' = -4\sin(2x)$. So $f \sim -2\Big(x - \dfrac{\pi}{2}\Big)$.

   - $f(x) = e^x$ at $a = 1$

     $f' = f'' = e^x$. So $f \sim e + e(x-1) + \dfrac{e}{2}(x-1)^2$.

5. Mean Value Theorem: Determine if the Mean Value Theorem can be applied to the following function on the the given closed interval.

   Both intervals are closed. It suffices to check that these functions are continuous on the given interval. One can do this by computing the derivative exists.

   - $f(x) = 3 + \sqrt{x}, x \in [0, 4]$

     Here $f' = \frac{1}{2\sqrt{x}}$.

   - $f(x) = \dfrac{x}{1+x}, x \in [1, 3]$

     Here $f' = \frac{1}{(1+x)^2}$.

6. L'Hospital's Rule

   - $\displaystyle\lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

     Check that as $x \to 2$, $x^3 - 7x^2 + 10x \to 0$ and $x^2 + x - 6 \to 0$ so L'Hospital's rule applies. Then

     $$\lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x\to 2} \frac{3x^2 - 14x + 10}{2x + 1} = -\frac{6}{5}.$$

     The last step uses division property of limits.

   - $\displaystyle\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}$

     As exp and ln are continuous functions

     $$\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}} = \exp\left[\ln\left(\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}\right)\right] = \exp\left(\lim_{x\to\infty} \frac{\ln(e^x + x)}{x}\right).$$

     Check that as $x \to \infty$, $e^x + x \to \infty$ so L'Hospital's rule applies.

     $$\begin{aligned} RHS &= \exp\left(\lim_{x\to\infty} \frac{e^x + 1}{e^x + x}\right) \\ &= \exp\left(\lim_{x\to\infty} 1 + \frac{1 - x}{e^x + x}\right) \qquad \text{(Check that L'Hospital's rule applies)} \\ &= \exp\left(1 + \lim_{x\to\infty} \frac{-1}{e^x + 1}\right) = e. \end{aligned}$$

- $\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right)$

$$\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}.$$

Check that as $x\to\infty$, $\ln\left(1+\frac{3}{x}\right)$, $\frac{1}{x}\to 0$ so L'Hospital's rule applies.

$$RHS = \lim_{x\to\infty}\frac{\frac{-3/x^2}{1+3/x}}{-1/x^2} = \lim_{x\to\infty}\frac{3}{1+\frac{3}{x}} = 3.$$

7. The Fundamental Theorem of Calculus: Find the derivative of the following

- $\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t$

Apply FTC

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t = \frac{1}{x^3+1}.$$

- $\int_1^{\sqrt{x}} \sin t\,\mathrm{d}t$

Let $u(x)=\sqrt{x}$. Apply chain rule and FTC

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t = \sin u(x)\cdot\frac{\mathrm{d}u}{\mathrm{d}x} = \sin u(x)\cdot\frac{1}{2\sqrt{x}} = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

- $\int_x^{2x} t^3\,\mathrm{d}t$

Using subtraction property of integral,

$$\int_x^{2x} t^3\,\mathrm{d}t = \int_0^{2x} t^3\,\mathrm{d}t - \int_0^x t^3\,\mathrm{d}t.$$

Apply FTC to each term

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{2x} t^3\,\mathrm{d}t = 16x^3 - x^3 = 15x^3.$$

8. Substitution Rule

- $\displaystyle\int_{\frac{1}{2}}^{0} \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$

  Take $u = 1-4x^2$, then $\mathrm{d}u = -8x\,\mathrm{d}x$ and $\mathrm{d}x = -\dfrac{1}{8}\,\mathrm{d}u$.

$$\int_{\frac{1}{2}}^{0} \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x = \int_{0}^{1} -\frac{1}{8\sqrt{u}}\,\mathrm{d}u = -\frac{1}{4}\sqrt{u}\Big|_{0}^{1} = -\frac{1}{4}.$$

- $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)}\,\mathrm{d}x$

  Take $u = \sin(\pi x)$, then $\mathrm{d}u = \pi\cos(\pi x)\,\mathrm{d}x$ and $\mathrm{d}x = -\dfrac{1}{8}\,\mathrm{d}u$.

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)}\,\mathrm{d}x = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{1}{\pi u^2}\,\mathrm{d}u = -\frac{1}{u}\Big|_{\frac{\sqrt{2}}{2}}^{1} = \frac{\sqrt{2}-1}{\pi}.$$

- $\displaystyle\int_{0}^{1} xe^{4x^2+3}\,\mathrm{d}x$

  Take $u = 4x^2+3$, then $\mathrm{d}u = 8x\,\mathrm{d}x$

$$\int_{0}^{1} xe^{4x^2+3}\,\mathrm{d}x = \frac{1}{8}\int_{3}^{7} e^{u}\,\mathrm{d}u = \frac{1}{8}\sqrt{u}\Big|_{3}^{7} = \frac{e^7-e^3}{8}.$$

Homework 1

The following solutions provide a possible way to solve the problems. Any other reasonable solution is accepted.

1. Integration by parts (Note that the following integrals are indefinite. You need to add constants to your final answer.) You may also need to use substitution rule.

   - $\displaystyle\int \frac{\ln x}{x^2}\,\mathrm{d}x$ [3pt]

     Take $u = \ln x, v = -\dfrac{1}{x}$. Then
     $$\int \frac{\ln x}{x^2}\,\mathrm{d}x = \int \ln x\,\mathrm{d}\Big(-\frac{1}{x}\Big) = -\frac{1}{x}\ln x - \int -\frac{1}{x}\,\mathrm{d}(\ln x) = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

   - $\displaystyle\int x^2 \sin x\,\mathrm{d}x$ [4pt]

     Take $u = x^2, v = -\cos x$. Then
     $$\int x^2 \sin x\,\mathrm{d}x = -x^2\cos x - \int -\cos x\,\mathrm{d}(x^2) = -x^2\cos x + 2\int x\cos x\,\mathrm{d}x.$$
     To evaluate $\displaystyle\int x\cos x\,\mathrm{d}x$ we use integration by parts again with $u = x, v = \sin x$.
     $$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + \tilde{C}.$$
     Final answer: $\quad -x^2\cos x + 2x\sin x + 2\cos x + C.$

   - $\displaystyle\int (\ln x)^2\,\mathrm{d}x$ [4pt]

     Take $u = (\ln x)^2, v = x$. Then
     $$\int (\ln x)^2\,\mathrm{d}x = x(\ln x)^2 - \int x\,\mathrm{d}(\ln x)^2 = x(\ln x)^2 - \int \ln x\,\mathrm{d}x.$$
     To evaluate $\displaystyle\int \ln x\,\mathrm{d}x$ we use integration by parts again, with $u = \ln x, v = x$.
     $$\int \ln x\,\mathrm{d}x = x\ln x - \int x\,\mathrm{d}(\ln x) = x\ln x - \int 1\,\mathrm{d}x = x\ln x - x + C.$$
     Final answer: $\quad x(\ln x)^2 - 2x\ln x + 2x + C.$

   - $\displaystyle\int \arccos x\,\mathrm{d}x$ [4pt]

     Take $u = \arccos x, v = x$. Then
     $$\int \arccos x\,\mathrm{d}x = x\arccos x - \int x\,\mathrm{d}(\arccos x) = x\arccos x - \int -\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

To evaluate $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$ we use substitution rule with $u = 1 - x^2,\ \mathrm{d}u = -2x\,\mathrm{d}x$.

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = -\int \frac{1}{2\sqrt{u}}\,\mathrm{d}u = -\sqrt{u} + C.$$

Final answer: $\quad x \arccos x - \sqrt{1-x^2} + C.$

- $\int e^{\sqrt{x}}\,\mathrm{d}x$ [4pt]

  Using substitution rule with $t = \sqrt{x}$ we obtain

  $$\int e^{\sqrt{x}}\,\mathrm{d}x = 2\int te^t\,\mathrm{d}t$$

  Integration by parts: take $u = t, v = e^t$. Then

  $$RHS = 2\int t\,\mathrm{d}(e^t) = 2\left(te^t - \int e^t\,\mathrm{d}t\right) = 2(te^t - e^t + \tilde{C}) = 2(\sqrt{x}-1)e^{\sqrt{x}} + C.$$

2. Trigonometric integration: Evaluate the following integral of the form $\int \sin^n x \cos^m x\,\mathrm{d}x$.

> You need specify the values for θ, so that you can get rid of absolute values.

- $\int \sin^2 x \cos^3 x\,\mathrm{d}x$ [3pt]

  Note that

  $$\int \sin^2 x \cos^3 x\,\mathrm{d}x = \int \sin^2 x \cos^2 x \cdot \cos x\,\mathrm{d}x = \int \sin^2 x(1-\sin^2 x)\cdot \cos x\,\mathrm{d}x.$$

  Apply substitution rule with $u = \sin x,\ \mathrm{d}u = \cos x\,\mathrm{d}x$. So

  $$RHS = \int u^2(1-u^2)\,\mathrm{d}u = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

- $\int \cos^4 x\,\mathrm{d}x$ [4pt]

  Note that

  $$\begin{aligned}\cos^4 x = \cos^2 x \cos^2 x &= \frac{1}{4}(1+\cos(2x))^2 = \frac{1}{4}(1 + 2\cos(2x) + \cos^2(2x))\\ &= \frac{1}{4}\left(1 + 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\right) = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x).\end{aligned}$$

  Hence

  $$\begin{aligned}\int \cos^4 x\,\mathrm{d}x &= \int \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\,\mathrm{d}x\\ &= \frac{3}{8}\int 1\,\mathrm{d}x + \frac{1}{2}\int \cos(2x)\,\mathrm{d}x + \frac{1}{8}\int \cos(4x)\,\mathrm{d}x\\ &= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.\end{aligned}$$

3. Trigonometric substitution

- $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x$ [6pt]

  Let $x = 3\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x &= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\cdot 3\cos\theta\,\mathrm{d}\theta = \int \frac{27\sin^2\theta\cos\theta}{3\cos\theta}\,\mathrm{d}\theta \\
&= \int 9\sin^2\theta\,\mathrm{d}\theta = \int \frac{9(1-\cos 2\theta)}{2}\,\mathrm{d}\theta = \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C \\
&= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C = \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{x}{3}\sqrt{1-\left(\frac{x}{3}\right)^2}\right) + C \\
&= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C.
\end{aligned}$$

- $\displaystyle\int \frac{1}{\sqrt{25+x^2}}\,\mathrm{d}x$ [6pt]

  Let $x = 5\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\mathrm{d}x = 5\sec^2\theta\,\mathrm{d}\theta$

$$\begin{aligned}
\int \frac{1}{\sqrt{25+x^2}}\,\mathrm{d}x &= \int \frac{5\sec^2\theta}{\sqrt{25+25\tan^2\theta}}\,\mathrm{d}\theta = \int \frac{5\sec^2\theta}{\sqrt{25\sec^2\theta}}\,\mathrm{d}\theta \\
&= \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C.
\end{aligned}$$

- $\displaystyle\int \frac{1}{\sqrt{x^2+2x}}\,\mathrm{d}x$ [6pt]

  Note that
$$\int \frac{1}{\sqrt{x^2+2x}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x+1)^2-1}}\,\mathrm{d}x.$$
  Let $x+1 = \sec\theta, 0 \le \theta < \frac{\pi}{2}$, then $\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$

$$\begin{aligned}
RHS &= \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}\,\mathrm{d}\theta = \int \frac{\sec\theta\tan\theta}{\tan\theta}\,\mathrm{d}x \\
&= \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|x+1+\sqrt{(x+1)^2-1}\right| + C \\
&= \ln\left|x+1+\sqrt{x^2+2x}\right| + C.
\end{aligned}$$

- $\displaystyle\int (x-2)^3\sqrt{5+4x-x^2}\,\mathrm{d}x$ [6pt]

  Note that

$$\int (x-2)^3\sqrt{5+4x-x^2}\,\mathrm{d}x = \int (x-2)^3\sqrt{9-(x-2)^2}\,\mathrm{d}x.$$

  Let $x-2 = 3\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$

$$RHS = \int (3\sin\theta)^3 \cdot \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,\mathrm{d}\theta = 3^5\int \sin^3\theta\cos^2\theta\,\mathrm{d}\theta.$$

  To solve $\displaystyle\int \sin^3\theta\cos^2\theta\,\mathrm{d}\theta$, apply substitution rule with $u = \cos\theta,\ \mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$.
  So

$$\begin{aligned}\int \sin^3\theta\cos^2\theta\,\mathrm{d}\theta &= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\,\mathrm{d}\theta = -\int (1-u^2)u^2\,\mathrm{d}u \\ &= \frac{u^5}{5} - \frac{u^3}{3} + \tilde{C} = \frac{\cos^5\theta}{5} - \frac{\cos^3\theta}{3} + \tilde{C}\end{aligned}$$

  Final answer:

$$\begin{aligned}&3^5\left[\frac{1}{5}\left(\frac{\sqrt{9-(x-2)^2}}{3}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{9-(x-2)^2}}{3}\right)^3\right] + C \\ =&3^5\left[\frac{1}{5}\left(\frac{\sqrt{5+4x-x^2}}{3}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{5+4x-x^2}}{3}\right)^3\right] + C.\end{aligned}$$

  There's another way to do this problem. We will discuss that in problem session.