The Yamabe Problem

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Outline

Motivation

In 2D case

Uniformazation Theorem

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The theorem is consequence of the fact that every Riemann surface has a conformal metric with constant Gaussian curvature.

Definition

Two Riemannian metrics g and h are **conformal** if there exists positive function $f \in C^\infty(M)$ such that $h = e^{2f} g$.

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- Riemannian curvature tensor, n^4 components
- Ricci curvature, n^2 components
- scalar curvature, 1 component

Question: Which curvature to choose?

The Yamabe Problem

Given a compact Riemannian manifold (M, g) with $n = \dim M \geq 3$, find a metric conformal to g with constant scalar curvature.

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Given two metrics g and $\tilde{g}$, the transformation law between the scalar curvatures S and $\tilde{S}$,

$$\tilde{S} = \varphi^{1-p}(a\Delta\varphi + S\varphi).$$

Here φ satisfies $\tilde{g} = \varphi^{p-2}g$ and $a = \dfrac{4(n-1)}{n-2}$, $p = \dfrac{2n}{n-2}$ are constants.

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Define $\Box = a\Delta + S$ and call it the **conformal Laplacian**. Let $\tilde{S} = \lambda = \text{const}$. Then

$$\Box\varphi = \lambda\varphi^{p-1}. \qquad (\star)$$

Equation ($\star$) is the Euler-Lagrange equation for the **Yamabe functional**

$$Q_g(\varphi) = \frac{\int_M a|\nabla\varphi|^2 + S\varphi^2 \,\mathrm{d}V_g}{\left(\int_M |\varphi|^p \,\mathrm{d}V_g\right)^{2/p}} = \frac{E(\varphi)}{\|\varphi\|_p^2}.$$

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By Hölder's inequality $Q_g(\varphi)$ is bounded below so we can take the infimum

Definition

The **Yamabe invariant** is the constant

$$\begin{aligned}\lambda(M) &= \inf\{Q_g(\varphi) \mid \varphi \in C^\infty(M) \text{ and positive}\} \\ &= \inf\{Q_g(\varphi) \mid \varphi \in L_1^2(M)\}.\end{aligned}$$

$\lambda(M)$ is an invariant of the conformal class of (M, g).

Main Results

Theorem A (Yamabe, Trudinger, Aubin)

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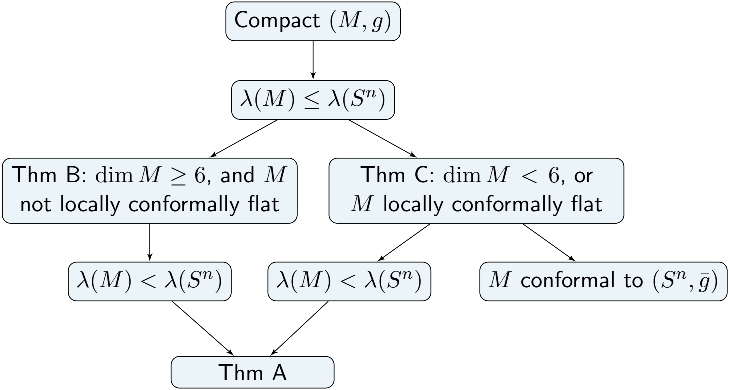
Theorem B (Aubin)

If M has dimension $n \geq 6$ and M is not locally conformally flat, then $\lambda(M) < \lambda(S^n)$.

Theorem C (Schoen)

If M has dimension $n = 3, 4, 5$ or M is locally conformally flat, then either $\lambda(M) < \lambda(S^n)$ or M is conformal to the n-sphere.


Compact (M, g)
$\lambda(M) \leq \lambda(S^n)$
Thm B: $\dim M \geq 6$, and M not locally conformally flat
Thm C: $\dim M < 6$, or M locally conformally flat
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M conformal to $(S^n, \bar{g})$
Thm A

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Example

- The stereographic map σ is a conformal diffeomorphism.
- Rotations, $\sigma^{-1}\tau_v\sigma$ and $\sigma^{-1}\delta_\alpha\sigma$ are conformal diffeomorphisms.

The Yamabe Problem on the Sphere

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Moreover, one can prove the following.

Theorem

The Yamabe functional $Q_g(\varphi)$ on $(S^n, \bar{g})$ is minimized by

- *constant multiples of $\bar{g}$;*
- *the images of $\bar{g}$ under conformal diffeomorphisms.*

These are the only metrics conformal to $\bar{g}$ with constant scalar curvature.

An Upper Bound for $\lambda(M)$

Lemma (Aubin)

For any compact Riemannian manifold (M, g) of dimension $n \geq 3$, $\lambda(M) \leq \lambda(S^n) = \Lambda$.

- Goal: to find a function φ makes $Q_g(\varphi) \leq \Lambda$.

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- Goal: to find a function φ makes $Q_g(\varphi) \leq \Lambda$.
- Consider $\varphi = \eta \cdot u_\alpha(x)$ where

  $$\eta \text{ cut off function and } u_\alpha(x) = \left(\frac{|x|^2+\alpha^2}{\alpha}\right)^{(n-2)/2}.$$

- $$Q_g(\varphi) = \frac{\int_M a|\nabla\varphi|^2 + S\varphi^2 \, \mathrm{d}V_g}{\|\varphi\|_p^2} \leq (1 + C\epsilon)(\Lambda + C\alpha).$$

Proof of Theorem A

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- Direct approach: construct a minimizing sequence (u_i), with $\|u_i\|_p = 1$ such that $Q_g(u_i) \to \lambda(M)$. This does not work: Although $\varphi = \lim u_i \in L_1^2(M)$, there is no guarantee for $\|\varphi\|_p \neq 0$, because the inclusion $L_1^2 \subset L^p$ is not compact.
- Instead we seek for a subcritical solution. The following equation is call **subcritical equation**

$$\Box\varphi = \lambda_s \, \varphi^{s-1}. \qquad (\star')$$

$Q^s(\varphi) = \frac{E(\varphi)}{\|\varphi\|_s^2}$, $\lambda_s = \inf\{Q^s(\varphi) : \varphi \in C^\infty(M)\}$.

Proof of Thm A.

Step 1. Subcritical solution φ_s exists,

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Step 2. Properties of λ_s. If $\int_M \mathrm{d}V_g = 1$, then for $2 \le s \le p$,

- $|\lambda_s|$ is non-increasing;
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- $|\lambda_s|$ is non-increasing;
- If $\lambda(M) \ge 0$, then $\lambda_s \ge 0$;
- λ_s is continuous from the left:

  Definiton of λ_s $\exists u$ s.t. $Q^s(u) < \lambda_s + \epsilon$;

  Continuity of $\|u\|_s$ as a function of s:

$$\lambda_{s'} \le Q^{s'}(u) < \lambda_s + 2\epsilon, \text{ as } s' \to s^-.$$

Step 3. Suppose $\lambda(M) < \Lambda$, then the subcritical solution $\varphi_s \in L^r$ for $s < s_0 < p < r$. As $s \to p$, $\exists(\varphi_{s_j})$ a subsequence that converges uniformly and $\varphi = \lim \varphi_{s_j}$ is the solution.

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An intermediate step to show $\varphi_s \in L^r$:

$$\|w\|_p^2 \leq (1+\epsilon)\frac{(1+\delta)^2}{1+2\delta} \cdot \frac{\lambda_s}{\Lambda} \cdot \|w\|_p^2 + C'_\epsilon \cdot \|w\|_2^2.$$

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- Uniform boundedness in $L^r \implies C^{2,\alpha} \overset{\text{subsq}}{\implies} C^2$;
- Arzela-Ascoli Thm gives a converging subsequence in C^2;
- φ solves the Yamabe equation (needs Step 2), and $\varphi \in C^\infty(M)$ (ellptic regularity).

Remark

The above proof requires $\lambda(M) \geq 0$ (Step 2).
The fact that $\Lambda = \lambda(S^n) > 0$ completes the proof.

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Remarks on Theorem B and C

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Estimation of $E(\varphi)$:

$$E(\varphi) \leq \begin{cases} \Lambda \|\varphi\|_p^2 - C|W(P)|^2\alpha^4 + o(\alpha^4) & n > 6 \\ \Lambda \|\varphi\|_p^2 - C|W(P)|^2\alpha^4 \ln(1/\alpha) + O(\alpha^4) & n = 6 \end{cases}$$

M locally conformally flat $\iff$ the conformal part: $W \equiv 0$.

Theorem C (Schoen)

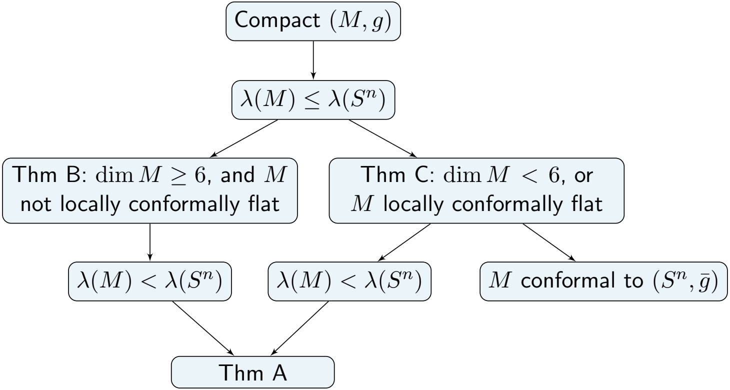
If M has dimension $n = 3, 4, 5$ or M is locally conformally flat, then either $\lambda(M) < \lambda(S^n)$ or M is conformal to the n-sphere.

Estimation of $E(\varphi)$:

$$E(\varphi) \leq \Lambda \|\varphi\|_p^2 - C\mu\alpha^{-k} + O(\alpha^{-k-1}).$$

Identify μ with "mass". The positive mass theorem gives $\mu > 0$.

Summary



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