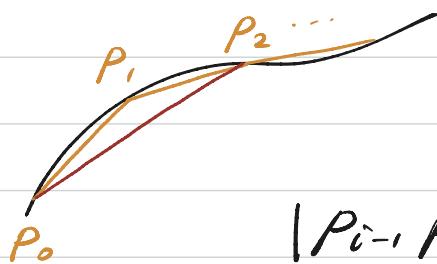


Last time: improper integral
This time: arc length

In this chapter we are going to study applications of integrals.

Lets first consider how to compute arc length



$$L = \lim_{n \rightarrow \infty} \sum_{i=1}^n |P_{i-1}, P_i|$$

$$\begin{aligned} |P_{i-1}, P_i| &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x \end{aligned}$$

taking limits gives f'

Let $y = f(x)$ be the curve

If f' is continuous, arc length for $x \in [a, b]$

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Similarly, let $x = g(y)$ be the curve, then arc length for $y \in [c, d]$

$$L = \int_c^d \sqrt{1 + (g'(y))^2} dy$$

Example 1. $y = e^x$ $0 \leq x \leq 2$

$$\Rightarrow L = \int_0^2 \sqrt{1 + (e^x)^2} dx$$

$$\text{or } L = \int_1^{e^2} \sqrt{1 + \left(\frac{1}{y}\right)^2} dy \quad \text{as } x = \ln y$$

Example 2 $x^2 + y^2 = 1$ unit circle

Using symmetry take the upper half circle

$$y = \sqrt{1 - x^2} \quad -1 \leq x \leq 1$$

$$\Rightarrow L = \int_{-1}^1 \sqrt{1 + \left(\frac{2x}{2\sqrt{1-x^2}}\right)^2} dx$$

Arc length function

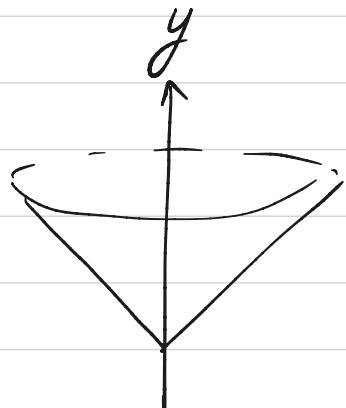
$$S(x) = \int_a^x \sqrt{1 + (f'(t))^2} dt$$

$$\text{Example 3. } f(x) = x^2 - \frac{\ln x}{8}$$

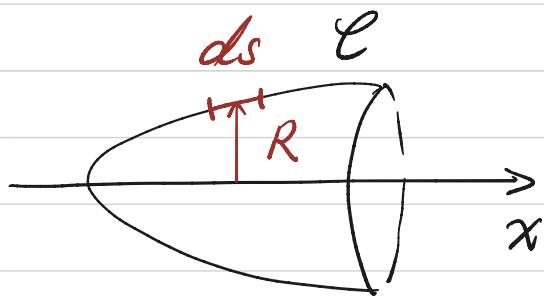
$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)^2} \, dx \\ &= \int_1^x \sqrt{1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2}} \, dx \\ &= \int_1^x 2t + \frac{1}{8t} \, dt \\ &= \left[t^2 + \frac{1}{8} \ln t \right]_1^x \\ &= x^2 + \frac{\ln x}{8} - 1 \end{aligned}$$

Surface of revolution

A surface of revolution is formed by rotating a curve about a line (usually the x or y -axis).



e.g. rotating $y = x$ about y -axis



e.g. rotating $y = \ln x$ about x -axis

To define the area, recall the surface area of a cylinder is $2\pi R l$.

If we take infinitesimal line segment ds , the small piece is approximate cylinder

$$S \approx \sum_{i=0}^n 2\pi f(x_i) \cdot ds$$

$$\Rightarrow S = \int_a^b 2\pi f(x) dx$$

Using arc length

$$ds = \sqrt{1 + (f'(x))^2} dx$$

$$\Rightarrow S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$$

Given curve $y=f(x)$, rotation about x -axis

$$S = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$$

Similarly, given curve $x=g(y)$, rotation about y -axis.

$$S = \int_c^d 2\pi x \sqrt{1 + (x')^2} dy$$