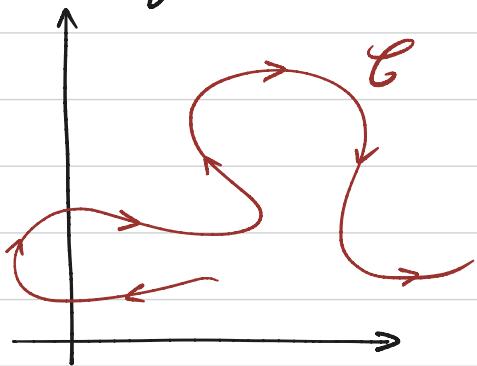


This time: curves defined by parametric eqns.
 Consider a particle moves along a curve like the following. It is not possible to write C as



$$y = f(x)$$

because C fails the vertical line test. However, if we introduce a new variable t then we can write x and y -coordinates as functions which depend on time.

Def We call t parameter

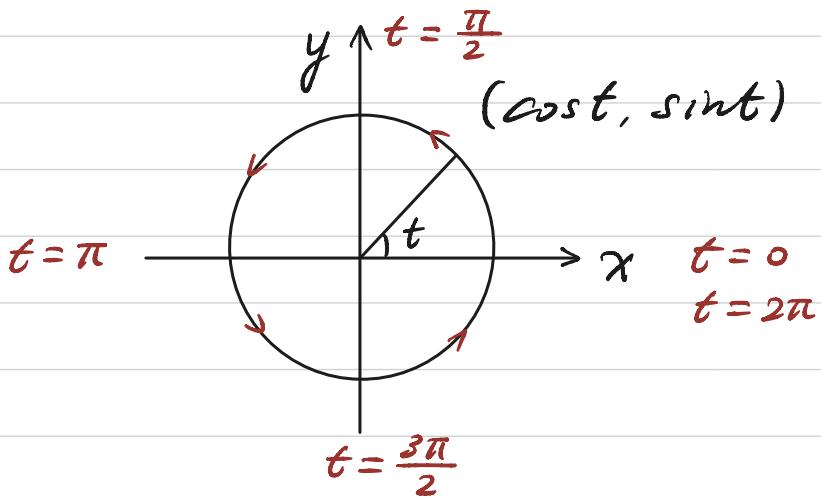
$$\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned} \quad \left. \begin{array}{l} \text{parametric equations} \\ \text{parametrization} \end{array} \right\}$$

C a parametric curve.

Example 1. Consider $x = \cos t$ $0 \leq t \leq 2\pi$

$$y = \sin t$$

Note that $\cos^2 t + \sin^2 t = 1$ implies $x^2 + y^2 = 1$
 So these gives us a unit circle.



Note that parameterization is not unique

Example 2. $x = \sin 2t \quad 0 \leq t \leq \pi$
 $y = \cos 2t$

These give the same circle but with opposite orientation.

Example 3. Find parametrization for the circle of radius r , centered at (a, b) .

$$(x-a)^2 + (y-b)^2 = r^2$$

$$\left. \begin{array}{c} \\ \end{array} \right\}$$

$$r \cos t \quad r \sin t$$

scaling

$$\Rightarrow x = a + r \cos t \quad 0 \leq t \leq 2\pi$$

$$y = b + r \sin t$$

translation

Example 4. $x = y^4 - 3y^2$

Take $y = t$ then $x = t^4 - 3t^2, t \in \mathbb{R}$

calculus with parametric curves

We can apply calculus techniques to parametrized curves.

Tangent : $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ if $\frac{dx}{dt} \neq 0$

The above is derived from chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Remark :

1. $\frac{dx}{dt} \neq 0$ so that we can take quotient.

2. $\begin{cases} \frac{dx}{dt} = 0 \\ \frac{dy}{dt} \neq 0 \end{cases}$ corresponds to vertical line $y=ct$.

3. $\begin{cases} \frac{dx}{dt} \neq 0 \\ \frac{dy}{dt} = 0 \end{cases}$ corresponds to horizontal line $x=ct$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$$

Arc length

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

$$= \int_{t_1}^{t_2} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2} \frac{dx}{dt} dt$$

substitution

$$= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

"ds

Surface area $S = \int_{t_1}^{t_2} 2\pi R ds$

Example $\begin{cases} x = \cos^2 t & 0 \leq t \leq \frac{\pi}{4} \\ y = \sin^2 t \end{cases}$

$$ds = \left((2\cos t(-\sin t))^2 + (2\sin t \cos t)^2 \right)^{\frac{1}{2}} dt$$

$$= (4\cos^2 t \sin^2 t + 4\cos^2 t \sin^2 t)^{\frac{1}{2}} dt$$

$$= 2\sqrt{2} \cos t \sin t dt$$

$$= \sqrt{2} \sin(2t) dt$$

$$\angle = \int ds = \int_0^{\frac{\pi}{4}} \sqrt{2} \sin(2t) dt$$

$$= -\frac{\sqrt{2}}{2} \cos(2t) \Big|_{t=0}^{\frac{\pi}{4}}$$

$$= -0 - \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

$$S = \int_0^{\frac{\pi}{4}} 2\pi \sin^2 t \sqrt{2} \sin(2t) dt$$

$$= \sqrt{2}\pi \int_0^{\frac{\pi}{4}} (1 - \cos(2t)) \sin(2t) dt$$

$$= \sqrt{2}\pi \left(\int_0^{\frac{\pi}{4}} \sin(2t) dt - \int_0^{\frac{\pi}{4}} \frac{1}{2} \sin(4t) dt \right)$$

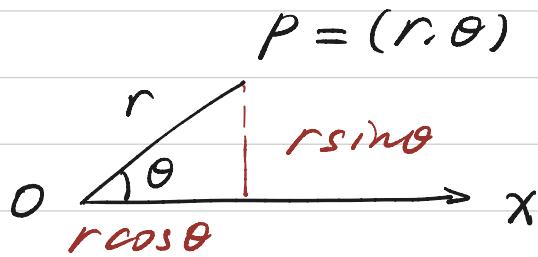
$$= \sqrt{2}\pi \left(-\frac{\cos(2t)}{2} \Big|_0^{\frac{\pi}{4}} - \frac{\cos(4t)}{8} \Big|_0^{\frac{\pi}{4}} \right)$$

$$= \sqrt{2}\pi \left(\frac{1}{2} - \left(\frac{1}{8} + \frac{1}{8} \right) \right)$$

$$= \sqrt{2}\pi \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\sqrt{2}\pi}{4}$$

Polar coordinates.

In this section we will consider a different way of writing points on the Euclidean plane.



$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan \frac{y}{x} \end{cases}$$

Polar \Rightarrow Rectangular Rectangular \Rightarrow Polar

(1.1) (1.2)

Example 1. Convert (1.1) and (1.2) into (r, θ) for $r > 0$.

$$(\sqrt{2}, \frac{\pi}{4}) \text{ and } (\sqrt{5}, \arctan 2)$$

Polar curves

Some curves like circles or rays can be written as a simple function in terms of polar coords.

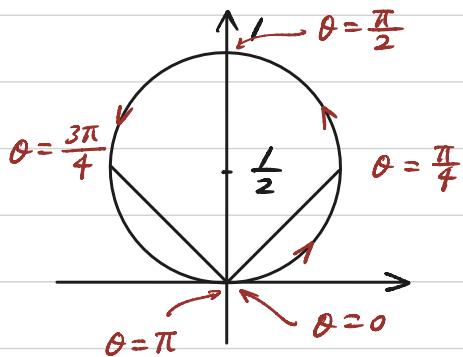
$$F(r, \theta) = 0$$

We will see how to compute arc length and surface area using polar coords.

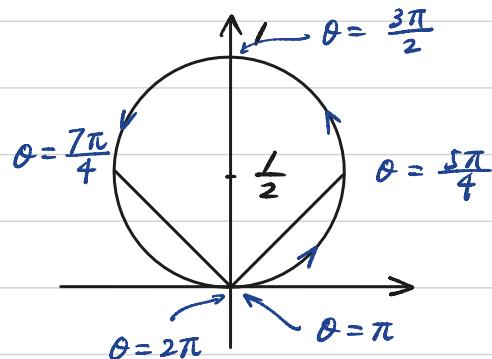
Example 2 circles

- $x^2 + y^2 = R^2 \Leftrightarrow r = R$
- $r = \sin \theta \Leftrightarrow r^2 = r \sin \theta$
- $\Leftrightarrow x^2 + y^2 = y \Leftrightarrow x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$

Note that when $\theta \in (0, \pi)$ $r > 0$
 $\theta \in (\pi, 2\pi)$ $r < 0$

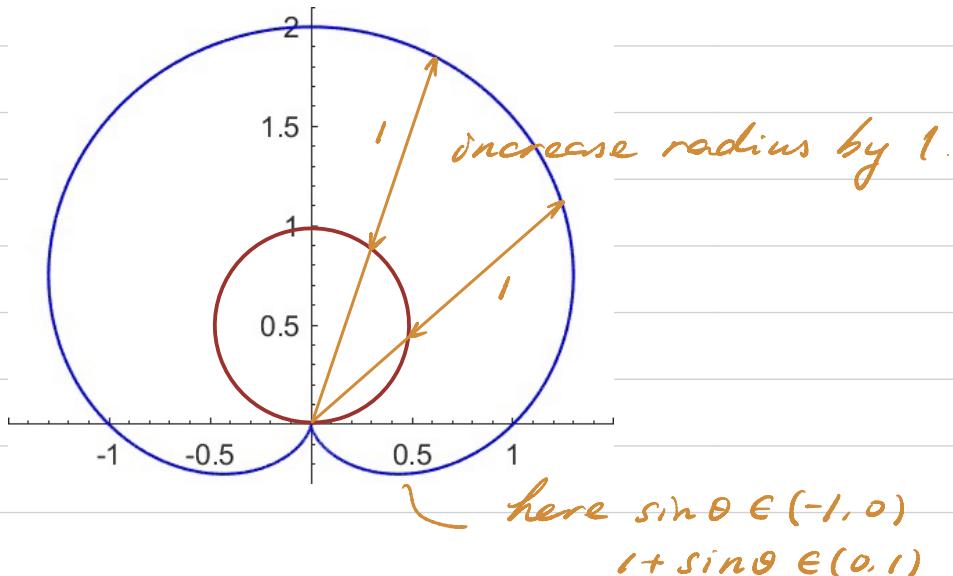


winding around
the circle once



winding around
the circle once again

Example 3. $r = 1 + \sin \theta$ cardoid



Tangent

Now consider polar curve of the form $r = f(\theta)$
then

$$x = f(\theta) \cos \theta$$

$$y = f(\theta) \sin \theta$$

we can compute its tangent by chain rule

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta) \sin \theta + f(\theta) \cos \theta}{f'(\theta) \cos \theta - f(\theta) \sin \theta}$$

Example 4. Let $r = 1 + \sin \theta$, compute $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}{\cos \theta \cos \theta - (1 + \sin \theta) \sin \theta}$$

$$= \frac{\cos \theta + 2 \cos \theta \sin \theta}{\cos^2 \theta - \sin^2 \theta - \sin \theta}$$

$$= \frac{\cos \theta + \sin 2\theta}{\cos 2\theta - \sin \theta} \quad \frac{-1}{0} \quad \text{L'H.}$$

Note that $\lim_{\theta \rightarrow (\frac{3\pi}{2})^-} \frac{dy}{dx} = \lim_{\theta \rightarrow (\frac{3\pi}{2})^-} \frac{-\sin \theta + 2 \cos 2\theta}{-2 \sin 2\theta - \cos \theta}$

$$\sin 3\pi = 0 \quad \cos 3\pi = -1 \quad = -\infty$$

$$\sin \frac{3\pi}{2} = -1 \quad \cos \frac{3\pi}{2} = 0$$