Ch 11. Infinite sequence and series

Def. An <u>sequence</u> is an infinite list of numbers written in a definite order.

Notation: $\{a_1, a_2, \cdots, a_n, \cdots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples $\{1, 2, 3, 4, \cdots\}$
$\{7, 1, 8, 2, 8, \cdots\}$

Some sequences can be defined by giving a formula for the n-th term a_n

Examples 1. $a_n = \left(\frac{1}{2}\right)^n$ $\quad \{a_n\} = \{\frac{1}{2}, \frac{1}{4}, \cdots\}$

2. $a_n = (-1)^n$ $\quad \{a_n\} = \{-1, 1, -1, 1, \cdots\}$

3. $a_n = \frac{n}{n+1}$ $\quad \{a_n\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$

Some sequences may not have a simple / explicit defining equation

Examples 1. a_n = the digit in the n-th decimal place of π

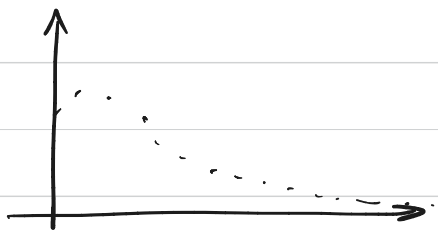
2. The Fibonacci sequence

$a_1 = 1, \ a_2 = 1, \ a_n = a_{n-1} + a_{n-2}$

$\{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}$

A sequence "is" a function f that only takes values on natural numbers. So we will study properties such as graph and convergency.

Example



$$\lim_{n\to\infty} a_n = 0$$

$L < \infty$

Def. A sequence <u>has limit L</u> if for any ε there is an N s.t. if $n > N$ then

$$|a_n - L| < \varepsilon$$

We say $\{a_n\}$ <u>converges</u> to L.

Intuition

Def. $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N s.t. if $n > N$ then $a_n > M$.

Examples

1. $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$

2. $\lim_{n\to\infty} \frac{1}{n^r} = \begin{cases} 0 & \text{if } r>0 \\ \infty & \text{if } r<0 \end{cases}$

3. $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1 \\ \infty & \text{if } r>1 \end{cases}$

Limit law for sequences

if $\{a_n\}$, $\{b_n\}$ are convergent sequences then

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n \qquad c \text{ const.}$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

$$\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p \qquad p>0 \quad a_n>0$$

Squeeze Theorem

$$b_n \le a_n \le c_n \quad \Rightarrow \quad \lim_{n\to\infty} a_n = L$$
$$\begin{array}{ccc} b_n & & c_n \\ \downarrow & & \downarrow \\ L & & L \end{array}$$

<u>Thm</u> If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

If f is continuous

$$\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} f(a_n) = f(L)$$

Example 1.

$$\lim_{n\to\infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{\pi}{n}\right)$$
$$= \sin 0 = 0$$

Example 2. $\lim_{n\to\infty} \dfrac{\ln(n+2)}{\ln(1+4n)}$

$$\lim_{x\to\infty} \frac{\ln(x+2)}{\ln(1+4x)} \overset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{\frac{1}{x+2}}{\frac{1}{1+4x}\cdot 4}$$
$$= \lim_{x\to\infty} \frac{4x+1}{4(x+2)} = 1$$