

## Ch 11. Infinite sequence and series

Def. An sequence is an infinite list of numbers written in a definite order.

Notation:  $\{a_1, a_2, \dots, a_n, \dots\}$ ,  $\{a_n\}$  or  $\{a_n\}_{n=1}^{\infty}$ .

Examples  $\{1, 2, 3, 4, \dots\}$   
 $\{7, 1, 8, 2, 8, \dots\}$

Some sequences can be defined by giving a formula for the  $n$ -th term  $a_n$

Examples 1.  $a_n = \left(\frac{1}{2}\right)^n$   $\{a_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \dots\right\}$

2.  $a_n = (-1)^n$   $\{a_n\} = \{-1, 1, -1, 1, \dots\}$

3.  $a_n = \frac{n}{n+1}$   $\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$

Some sequences may not have a simple / explicit defining equation

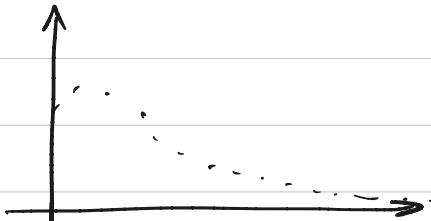
Examples 1.  $a_n$  = the digit in the  $n$ -th decimal place of  $\pi$

2. The Fibonacci sequence

$$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$$
$$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

A sequence "is" a function  $f$  that only takes values on natural numbers. So we will study properties such as graph and convergency.

Example



$$\lim_{n \rightarrow \infty} a_n = \infty$$

$$L < \infty$$

Def. A sequence has limit  $L$  if for any  $\varepsilon$  there is an  $N$  s.t. if  $n > N$  then  $|a_n - L| < \varepsilon$

We say  $\{a_n\}$  converges to  $L$ .

Intuition



Def.  $\lim_{n \rightarrow \infty} a_n = \infty$  means that for every positive number  $M$  there is an integer  $N$  s.t. if  $n > N$  then  $a_n > M$ .

## Examples

$$1. \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

$$2. \lim_{n \rightarrow \infty} \frac{1}{n^r} = \begin{cases} 0 & \text{if } r > 0 \\ \infty & \text{if } r < 0 \end{cases}$$

$$3. \lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DNE} & \text{if } r < 1 \end{cases}$$

## Limit law for sequences

if  $\{a_n\}, \{b_n\}$  are convergent sequences then

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n \quad c \text{ const.}$$

$$\lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$$

$$\lim_{n \rightarrow \infty} a_n^p = \left( \lim_{n \rightarrow \infty} a_n \right)^p \quad p > 0 \quad a_n > 0$$

## Squeeze Theorem

$$b_n \leq a_n \leq c_n \Rightarrow \lim_{n \rightarrow \infty} a_n = L$$

$\downarrow$        $\downarrow$

$L$        $L$

Thm If  $\lim_{n \rightarrow \infty} |a_n| = 0$  then  $\lim_{n \rightarrow \infty} a_n = 0$

If  $f$  is continuous

$$\lim_{n \rightarrow \infty} a_n = L \Rightarrow \lim_{n \rightarrow \infty} f(a_n) = f(L)$$

Example 1.

$$\begin{aligned} \lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{n}\right) &= \sin\left(\lim_{n \rightarrow \infty} \frac{\pi}{n}\right) \\ &= \sin 0 = 0 \end{aligned}$$

Example 2.  $\lim_{n \rightarrow \infty} \frac{\ln(n+2)}{\ln(1+4n)}$

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\ln(x+2)}{\ln(1+4x)} &\stackrel{\text{L'Hopital}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x+2}}{\frac{1}{1+4x} \cdot 4} \\ &= \lim_{x \rightarrow \infty} \frac{4x+1}{4(x+2)} = 1 \end{aligned}$$

Example 3.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} e^{\ln \left(1 + \frac{1}{n}\right)^n}$$

$$= e^{\lim_{n \rightarrow \infty} n \ln \left(1 + \frac{1}{n}\right)}$$

$$= e$$

$$\lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1 + \frac{1}{n}} \cdot -\frac{1}{n^2}}{-\frac{1}{n^2}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

Last time: sequence

This time: series

Def. We call  $\sum_{n=1}^{\infty} a_n$  or  $\sum a_n$  a series and

$s_N = \sum_{n=1}^N a_n = a_1 + a_2 + \dots + a_N$  the partial sums.

Note that  $\{s_N\}$  is itself a sequence. So it makes sense to talk about if  $\{s_N\}$  converges or not.

Def. The series  $\sum a_n$  is called convergent if its partial sum is convergent. Otherwise  $\sum a_n$  is called divergent.

Example 1. (geometric series)

Consider  $a_n = r^n$   $r$ : common ratio.

$$a_0 = r^0 = 1 \quad s_0 = a_0 = 1$$

$$a_1 = r^1 = r \quad s_1 = a_0 + a_1 = 1 + r$$

$$a_2 = r^2 = r^2 \quad s_2 = a_0 + a_1 + a_2 = 1 + r + r^2$$

⋮

⋮

$$s_N = \underbrace{1 + r + r^2 + \dots + r^N}_{}$$

we are interested in this sum.

$$\text{Let } R_N = \sum_{n=0}^N r^n = 1 + r + r^2 + \cdots + r^N$$

↓      ↓      ↓

$$rR_N = r + r^2 + \cdots + r^N + r^{N+1}$$

$$\Rightarrow R_N - rR_N = 1 - r^{N+1}$$

$$\Rightarrow R_N = \frac{1 - r^{N+1}}{1 - r}$$

$$\sum_{n=0}^{\infty} r^n = \lim_{N \rightarrow \infty} R_N$$

$\forall n \geq 0$

$$= \begin{cases} \frac{1}{1-r} & \text{if } -1 < r < 1 \quad \text{conv.} \\ \infty & \text{if } r \geq 1 \quad \leftarrow \text{div.} \\ \text{DNE} & \text{if } r \leq -1 \quad \leftarrow \end{cases}$$

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} \quad \text{because} \quad \sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r^n$$

Example 2. Compute  $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$  using the above

$$\sum_{n=1}^{\infty} 2^{2n} 6^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 6 \cdot 6^{-n}$$

note that  $n$  start from 1

$$= 6 \cdot \sum_{n=1}^{\infty} \underbrace{\left(\frac{4}{6}\right)^n}_{\frac{2}{3}^n} = 6 \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}}$$

$$= 6 \cdot 2 = 12$$

Example 3. (harmonic series)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$S_2 = 1 + \frac{1}{2}$$

$$\begin{aligned} S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \\ &= 1 + \frac{2}{2} \end{aligned}$$

$$\begin{aligned} S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}} \\ &> 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}} \\ &= 1 + \frac{3}{2} \quad \frac{1}{2} \quad \frac{1}{2} \end{aligned}$$

⋮  
⋮  
⋮

$$S_{2^n} = 1 + \frac{n}{2} \xrightarrow{n \rightarrow \infty} \infty$$

Hence  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges

Example 4.  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$

Note that  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} \rightarrow 1$$

Theorem  $\sum_{n=1}^{\infty} a_n$ ,  $\sum_{n=1}^{\infty} b_n$  converges,  $c$  const.

$$\sum_{n=1}^{\infty} a_n \pm b_n = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$$

Example 5  $\sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \frac{1}{2^n}$

$a_n$        $b_n$

We have  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} - 1$

$$= 2 - 1 = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So the original series converges to  
 $3 \cdot 1 + 1 = 4$

Then  $\sum_{n=1}^{\infty} a_n$  converges  $\Rightarrow \lim_{n \rightarrow \infty} a_n = 0$

pf. By definition, we know  $\lim_{n \rightarrow \infty} s_n = L$  for some real number  $L$ .

$$\Rightarrow \lim_{n \rightarrow \infty} s_{n-1} = \lim_{n \rightarrow \infty} s_n = L$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (s_n - s_{n-1})$$

$$= \lim_{n \rightarrow \infty} s_{n-1} - \lim_{n \rightarrow \infty} s_n$$

$$= L - L = 0$$

Corollary (The divergence test)

If  $\lim_{n \rightarrow \infty} a_n \neq 0$  then  $\sum_{n=1}^{\infty} a_n$  diverges

Examples 1.  $\sum_{n=1}^{\infty} (-1)^n$

2.  $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$  all diverges

3.  $\sum_{n=1}^{\infty} \frac{n}{n+1}$

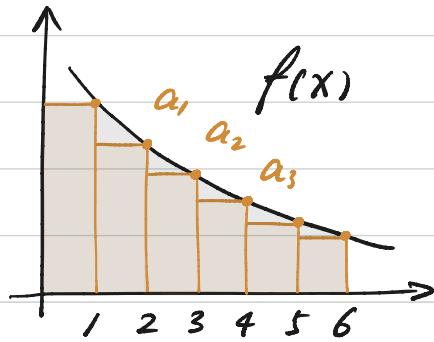
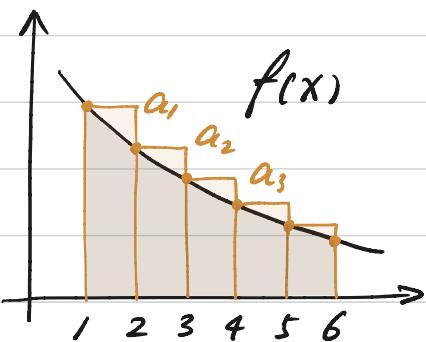
Last time computing series.  
 This time integral test & estimates.

We have been computing exact value of a series so far for some special cases. However, in general it is quite difficult. In those cases, we are interested in finding an estimate.

Thm (the integral test)

Suppose  $f(x) > 0$  is a continuous decreasing function for  $x \geq 1$  such that  $a_n = f(n)$ . Then

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \Leftrightarrow \int_1^{\infty} f(x) dx \text{ conv.}$$



Moreover,

$$\int_1^{\infty} f(x) dx \leq \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x) dx \quad (\star)$$

$$\text{Error } R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n = \sum_{n=N+1}^{\infty} a_n$$

$$\int_{N+1}^{\infty} f(x) dx \leq R_N \leq \int_N^{\infty} f(x) dx$$

Example 1.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  conv.

$f(x) = \frac{1}{x^2} > 0$  for  $x \geq 1$  cont. and decreasing

$$\int_1^{\infty} \frac{1}{x^2} dx \text{ conv.}$$

$$f'(x) = -2x^{-3} < 0 \quad \text{for } x \geq 1$$

Example 2.  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  is conv. if  $p > 1$   
div if  $p \leq 1$

Example 3.  $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$  conv.

$f(x) = \frac{1}{x^2+1} > 0$  for  $x \geq 1$ , cont. and decreasing

$$f'(x) = -(x^2+1)^{-2} \cdot 2x < 0 \quad \text{for } x \geq 1$$

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{t \rightarrow \infty} [\arctan x]_1^t \\ &= \lim_{t \rightarrow \infty} \arctan t - \frac{\pi}{4} \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty \end{aligned}$$

Last time: integral test.

This time: the comparison test

The idea of the comparison test for sequences is similar to that for integrals.

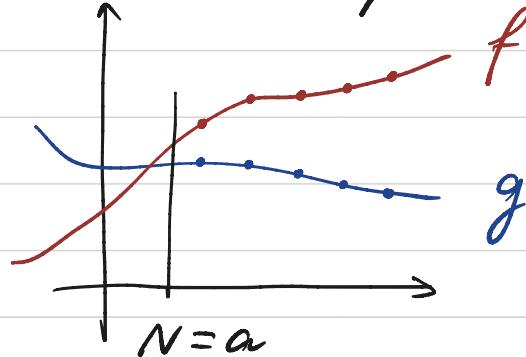
Thm (the comparison test)

Suppose  $\sum a_n$  and  $\sum b_n$  are series with positive terms and  $a_n \geq b_n$  for  $n \geq N$ .

$$\begin{aligned}\sum a_n \text{ conv} &\Rightarrow \sum b_n \text{ conv.} \\ \sum b_n \text{ div} &\Rightarrow \sum a_n \text{ div}\end{aligned}$$

Compare the above with the comparison test in Ch 7.

$$\begin{aligned}a_n &\leftrightarrow f \\ b_n &\leftrightarrow g \\ \sum &\leftrightarrow \int \\ N &\leftrightarrow a\end{aligned}$$



Example 1.  $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$

$$2n^2 + 4n + 3 \geq 2n^2 \quad \text{for } n \geq 1$$

$$\Rightarrow \underbrace{\frac{5}{2n^2 + 4n + 3}}_{a_n} \leq \underbrace{\frac{5}{2n^2}}_{b_n} \quad N=1 \text{ here}$$

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \Rightarrow \sum_{n=1}^{\infty} a_n \text{ conv.}$$

Example 2.  $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Note that  $\ln n > 1$  for  $n > e$

$$\Rightarrow \underbrace{\frac{\ln n}{n}}_{a_n} > \underbrace{\frac{1}{n}}_{b_n} \quad \text{for } n > e$$

$\sum_{n=3}^{\infty} \frac{1}{n}$  div  $\Rightarrow \sum_{n=3}^{\infty} a_n$  div  $\Rightarrow \sum_{n=1}^{\infty} a_n$  div.

eg. we can take  $N = 3$

Thm (the limit comparison test)

Suppose  $\sum a_n$ ,  $\sum b_n$  are series with positive terms and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \in (0, \infty)$

Then  $\sum a_n$  conv  $\Leftrightarrow \sum b_n$  conv.

Example 3.  $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$  Take  $b_n = \frac{1}{2^n}$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n \rightarrow \infty} \frac{2^n}{2^n - 1} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{1}{2^n}} = 1 \in (0, \infty) \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ conv} \Rightarrow \sum a_n \text{ conv.}$$

dominant part is  $2n^2$

Example 4.  $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5+n^5}} \quad \sqrt{n^5} = n^{5/2}$

Take  $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{\sqrt{n}}$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \frac{2n^2 + 3n}{\sqrt{5+n^2}} \cdot \frac{\sqrt{n}}{2} \\
 &= \lim_{n \rightarrow \infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5+n^2}} \\
 &= \lim_{n \rightarrow \infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{2}{2} = 1
 \end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}} \text{div} \Rightarrow \sum a_n \text{div}$$

Last time: comparison tests  
 This time: alternating series.

So far we've studied series with positive terms. In this section we will study series whose terms are alternating (e.g.  $a_{2n} > 0$ ,  $a_{2n+1} < 0$ ).

Examples 1.  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

2.  $\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - \dots$

The following theorem tells us how to determine if an alternating series converges or not.

Thm (Alternating series test)

Given  $\sum_{n=0}^{\infty} (-1)^n a_n$ ,  $\underline{a_n > 0}$  if  
 ②  $\underline{a_{n+1} \leq a_n}$  for all  $n$  and  $\lim_{n \rightarrow \infty} a_n = 0$  ③  
 then  $\sum_{n=0}^{\infty} (-1)^n a_n$  converges.

pf. Consider the even partial sums (which have positive terms because  $a_{n+1} \leq a_n$ )

$$s_{2n} = s_{2n-2} + \underbrace{(a_{2n-1} - a_{2n})}_{\geq 0} \geq s_{2n-2} \quad (n \geq 1)$$

$\{s_{2n}\}$  is a positive nonincreasing sequence hence converges, say

$$\lim_{n \rightarrow \infty} s_{2n} = S$$

Then the partial sum converges by limit law

$$\begin{aligned}\lim_{n \rightarrow \infty} s_{2n+1} &= \lim_{n \rightarrow \infty} s_{2n} + a_{2n+1} \\ &= S + 0 = S\end{aligned}$$

Moreover, from the above proof, we see that if  $\lim_{n \rightarrow \infty} a_n$  div, the series div. So

divergence test still holds

Example 1. (alternating harmonic series)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \text{ conv.}$$

Check:  $a_n = \frac{1}{n} > 0$

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

So the alternating series test tells the series conv.

Example 2.  $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^3 + 1}$  conv.

Check •  $a_n = \frac{n^2}{n^3 + 1} \geq 0$  for all  $n$ .

- $a_{n+1} \leq a_n$  for  $n \geq 2$  because the function  $f(x) = \frac{x^2}{x^3 + 1}$  is decreasing (not obvious)

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0 \text{ when } x > \sqrt[3]{2}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{n^3 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0$$

Apply the alternating series test for  $n \geq 2$ .

$$\Rightarrow \sum_{n=2}^{\infty} (-1)^n a_n \text{ conv.}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n a_n = \underbrace{a_0 - a_1}_{\text{finite number}} + \underbrace{\sum_{n=2}^{\infty} (-1)^n a_n}_{< \infty \text{ as conv.}}$$

$$< \infty$$

$$\sqrt[3]{2} \leq 2$$

## Estimating alternating series.

Thm (Alternating series estimation thm)

Given  $\sum_{n=0}^{\infty} (-1)^n a_n$ ,  $a_n > 0$  satisfying

$a_{n+1} \leq a_n$  and  $\lim_{n \rightarrow \infty} a_n = 0$

then  $|R_n| = |s - s_n| \leq a_{n+1}$

pf. Recall that  $s_n$  is positive and nonincreasing

Let  $s = \lim_{n \rightarrow \infty} s_n$ . Then  $s \leq s_n$  for all  $n$ .

Similarly,  $s \geq s_{2n+1}$  (odd partial sums)

$$\Rightarrow |s - s_m| = \begin{cases} s - s_m \leq s_{m+1} - s_m & m \text{ odd} \\ -(s - s_m) \leq -(s_{m+1} - s_m) & m \text{ even} \end{cases}$$

$m+1$  even       $\uparrow$        $-s \leq -s_{m+1}$   
              odd

$$\Rightarrow |s - s_m| \leq |s_{m+1} - s_m| = a_{m+1}$$

Last time: alternating series test

This time: absolute convergence and more tests

Def A series  $\sum a_n$  is called absolutely convergent if the series of absolute values  $\sum |a_n|$  is convergent.

Def A series  $\sum a_n$  is called conditionally conv. if it is convergent but not abs. conv.

Note that absolutely conv is stronger than convergent. That is,

$$\text{abs. conv.} \Rightarrow \text{conv.}$$

pf. Observe that  $-a_n \leq |a_n| \leq a_n$

$$\Rightarrow 0 \leq a_n + |a_n| \leq 2|a_n|$$

Apply the comparison test  $\sum A_n \underset{\sum B_n}{\sim} \sum B_n$ .

$$\sum B_n \text{ conv.} \Rightarrow \sum A_n \text{ conv.}$$

then  $\sum a_n = \underbrace{\sum A_n}_{<\infty} - \underbrace{\sum |a_n|}_{<\infty} < \infty$ .

by one of the properties of series

Example 1.  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$  conv.  $\sum_{n=1}^{\infty} \frac{1}{n}$  not conv.

Hence we say  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$  is conditionally conv.  
but not absolutely conv.

Example 2.  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$  and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  both conv.

Hence we say  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$  is conditionally conv.  
and also absolutely conv.

### Thm (the ratio test)

Given a series  $\sum a_n$ , let

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$\text{if } L \begin{cases} < 1 \\ > 1 \\ = 1 \end{cases} \text{ then } \begin{cases} \sum a_n \text{ abs. conv.} \\ \sum a_n \text{ div} \\ \text{no conclusion} \end{cases}$$

### Thm (the root test)

Given a series  $\sum a_n$ , let

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

$$\text{if } L \begin{cases} < 1 \\ > 1 \\ = 1 \end{cases} \text{ then } \begin{cases} \sum a_n \text{ abs. conv.} \\ \sum a_n \text{ div} \\ \text{no conclusion} \end{cases}$$

Remark

1. Note that we have absolute conv.
2.  $L=1$  case examples

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div} \quad & \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \text{ conv}$$

but in both cases  $L$  (for the ratio test) is given by

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

3. prototype for both tests: geometric series

$$\left. \begin{aligned} L_1 &= \lim_{n \rightarrow \infty} \left| \frac{r^{n+1}}{r^n} \right| \\ L_2 &= \lim_{n \rightarrow \infty} \sqrt[n]{|r|^n} \end{aligned} \right\} = \lim_{n \rightarrow \infty} |r| = |r|$$

we know

$|r| < 1$  conv.

$|r| > 1$  div.

Example 1.  $\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$  ↪ a sign for ratio test

$$\lambda = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)^2}{(2(n+1)-1)!}}{\frac{n^2}{(2n-1)!}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+1) \cdot (2n) \cdot n^2} = 0 < 1$$

Hence the series abs. conv. by ratio test.

Example 2.  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

$$\lambda = \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)^2+1}}{\frac{(-1)^n}{n^2+1}} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{n^2+1}{n^2+2n+2} = 1$$

The ratio test is not useful.

Instead one can use the alternating series test to conclude this series conv. and use comparison test for abs. conv.

$$\hookrightarrow A_n = \frac{1}{n^2+1} \leq B_n = \frac{1}{n^2}$$

Example 3.  $\sum_{n=0}^{\infty} \left( \frac{3n+1}{4-2n} \right)^{2n}$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( \frac{3n+1}{4-2n} \right)^{2n} \right|} \\ &= \lim_{n \rightarrow \infty} \left( \frac{3n+1}{2n-4} \right)^2 \\ &= \lim_{n \rightarrow \infty} \frac{9n^2 + 6n + 1}{4n^2 - 16n + 16} \\ &= \frac{9}{4} > 1 \end{aligned}$$

The series converges absolutely by root test.

Example 4.  $\sum_{n=4}^{\infty} \left( 1 + \frac{1}{n} \right)^{-n^2}$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left( 1 + \frac{1}{n} \right)^{-n^2} \right|} \\ &= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^{-n} \\ &= \frac{1}{e} < 1 \end{aligned}$$

The series converges absolutely by root test.

For strategy of choosing conv. tests

See "Supplementary Resources" on course webpage