Dec 9 Expanding Ricci solitons asymptotic to cones.

I. Background.

$n = 2, 3$ no conical singularity of compact RF.

$n \geq 4$ conical singularity $(\mathbb{R}_+ \times N^{n-1}, dr^2 + r^2 h_N)$

ex. FIK shrinking Ricci soliton

$$\underbrace{(M^4, g, \nabla f)}_{Bl_p(\mathbb{C}^2)} \quad \text{satisfies} \quad -2Ric = -g + \mathcal{L}_{\nabla f} g$$

RF on $t \in (-\infty, 0)$ $\quad g(t) = -t \varphi_t^* g$ where

$$\frac{d}{dt} \varphi_t = -\frac{1}{t} \nabla f, \quad \varphi_{-1} = id.$$

It is $U(2)$ invariant. asy. conic to $(\mathbb{C}^2, \gamma)$. cone with positive curvature operator.

Since $t < 0$, max principle $\Rightarrow \big(\partial_t R = \Delta R + |Ric|^2$
$\Rightarrow R > 0\big)$

There also exists FIK expander resolving the cone $(M^4, \tilde{g}, \nabla \tilde{f})$: $-2Ric_{\tilde{g}} = \tilde{g} + \mathcal{L}_{\nabla \tilde{f}} \tilde{g}$.

For $t > 0$ $\quad \tilde{g}(t) = t \varphi_t^* \tilde{g}$.

<u>Asymptotically conical expanders</u>

$\exists\, K \subset\subset M$, $\Phi : \cancel{\mathbb{R}^n \setminus B_R(0)} \longrightarrow M \setminus K$ diffeo.
$[r_0, \infty) \times N^{n-1}$
w/ cone metric γ.

s.t. $|\nabla^k (\Phi^* \tilde{g} - \gamma)|_\gamma = O(r^{-2-k})$

$$\Phi^* \nabla \tilde{f} = -\frac{1}{2} r \partial_r.$$

Rmk • $n=4$, FIK is the only known AC shrinker and indeed the only Kähler one.

• $(M^4, \lambda_i^2 g, p) \xrightarrow[\lambda_i \to 0]{GH}$ to AC, smooth away from tip

Rmk • real $n=4$, Conlon-Deruelle-Sun. criterion for whether Kähler cone admits smooth expander.

Question: resolving cones that may appear as singularities? ($R_\gamma > 0$ for cone).

• existence / uniqueness.
• resolving conical singularity on compact space.

Existence / uniqueness

• Bryant ($\mathbb{R}_+ \times S^{n-1}$, $dr^2 + r^2\lambda^2$ ground).
any λ admit an AC expanders.

• Chodosh - for symmetric cones, $Rm_\gamma > 0$
$\Rightarrow$ unique expander in this class.

• Schulze - Simon, any $Rm_\gamma > 0$ cone on $(\mathbb{R}_+ \times S^{n-1}, \gamma)$ admits a unique expander in this class.

• When $n=4$, $R_\gamma > 0 \Rightarrow R_h > 6$
$\Rightarrow N^3 = (S^3/P_1) \# \cdots \# (S^3/P_k) \# \, l \, (S^2 \times S^1)$.

$n \geq 5$

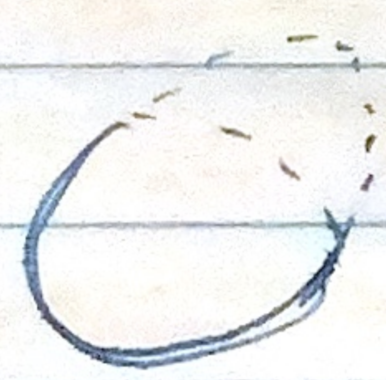
• Angenent - Knopf '17, expanders AC to $(\mathbb{R}_+ \times S^p \times S^2, \gamma)$. In some cases, on $B^{p+1} \times S^2$ or $S^p \times B^{q+1}$ $\longrightarrow$ $dr^2 + r^2\lambda_1^2 g_{S^p} + r^2\lambda_2^2 g_{S^2}$
round metric.

Close to Ricci flat cone, many distinct expanders)
ODE methods.

- $n=4$. Niehaus - Wink $(\mathbb{R}_+ \times S^2 \times S^1,\ dr^2 + r^2\lambda_1^2 g_{S^2} + r^2\lambda_2^2 g_{S^1})$
  $B^3 \times S^1$ case, any $\lambda_1, \lambda_2 \Rightarrow$ expander.
  $B^2 \times S^2$ case, some $\lambda_1, \lambda_2 \Rightarrow$ expander.
  nonexistence of symmetric expanders. ~~Rm_g~~

Resolving compact cone singularities.

- Gianniotis - Schulze '16
  isolated conical singularities, modelled on
  $Rm_g > 0$ cones on compact manifold.



resolved by gluing in expander
(using stability for $Rm_g > 0$ expander).

- Lavoyer '24 : edge-cone singularity on compact manifold $Rm_g > 0$ cones.

II. Chen - Bamler

Q. expander AC to cones with $R_\gamma > 0$.

Thm A Any $(\mathbb{R}_+ \times S^3/\Gamma, \gamma)$ with $R_\gamma > 0$ admits an (orbifold) expander.

Thm B Given M^4 smooth, with isolated orbifold singularity with $\partial M = N^3$.
The following map

$$\mathcal{M}(M^4) = \{\text{AC expanding Ricci solitons with } R>0\}/\sim$$

$$\downarrow \pi$$

$$\mathrm{Cone}(N^3) = \{\text{cone metrics on } \mathbb{R}_+ \times N^3 \text{ with } R>0\}.$$

is proper, with well-defined $\mathbb{Z}$-degree $\deg_{exp}(M^4)$. $\deg \neq 0 \Rightarrow \pi$ surjective.

Cor (Thm B $\Rightarrow$ Thm A)

$(\mathbb{R}^4, \delta_{ij}, -\frac{1}{2} r\partial_r)$ Gaussian soliton.

$\nu[\delta_{ij}] = 0$ in general, if $(\mathbb{R}^4, g)$ AC to $(\mathbb{R}^4, \delta_{ij})$

$\Rightarrow \underbrace{\nu[g]}_{0 \geq} \geq \underbrace{\nu[\delta_{ij}]}_{=0}$

So $\pi^{-1}(\text{flat metric}) = \{\text{Gaussian soliton}\} \Rightarrow \deg = 1$.

ν = entropy

if γ is regular value of π.

$$\deg_{exp}(M^4) = \sum_{p \in \pi^{-1}(X)} (-1)^{\mathrm{index}(L_g)}.$$

Question:

Q1. $\deg_{exp}(B^3 \times S^1)$ or $\deg_{exp}(B^2 \times S^2)$?

Q2. $\deg_{exp}(M_1 \#_0 M_2)$ related to $\deg_{exp}(M_i)$?

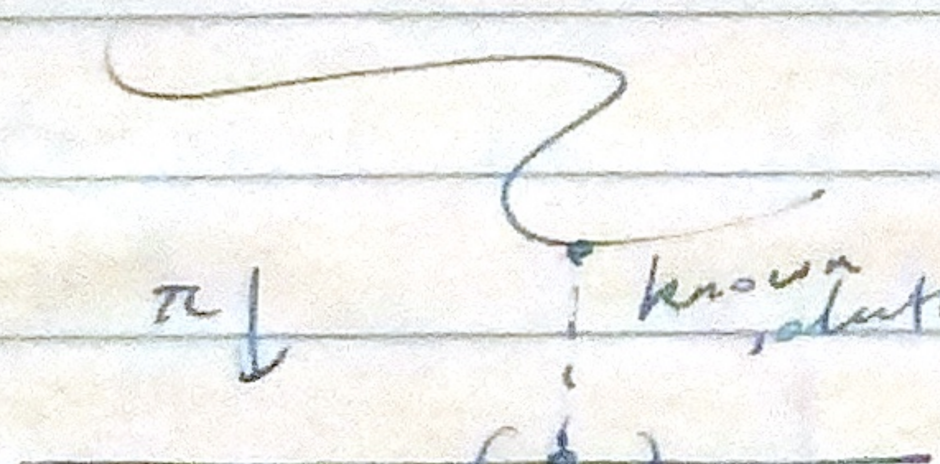
<u>Set up.</u> an elliptic BVP.

$-2Ric = g + \mathcal{L}_X g$ weakly elliptic.

Goal: on fixed M^4, prescribe asy. to cone $(\mathbb{R}_+ \times N, \gamma)$ solve for g.

Analogues: elliptic equation.	boundary value.
– MCF expander	prescribed AC.
– CCE filling	conformal infinity
– Plateau problem $f: M \to \mathbb{R}^n$ stationary submanifold	$f|_{\partial M}: \partial M \to \mathbb{R}^n$.

General outline



moduli space of solution to elliptic equation.

BV.

Step 1. local deformation / implicit func. thm.

2. construct space $\mathcal{M}$. est properness.

orientation on $\mathcal{M}$. est degree.

Expanding Ricci soliton.

① Given $(M, g, \nabla f)$ expander, search for (g', X') solving expander soliton equation near by.

$$Q_g[\tilde g] = -2\mathrm{Ric}_{\tilde g} + \tilde g + \mathcal{L}_{\underbrace{\boxed{\nabla f - \mathrm{div}_g \tilde g + \frac{1}{2}\nabla_g \mathrm{tr}_g \tilde g}}_{\tilde X}} \tilde g$$

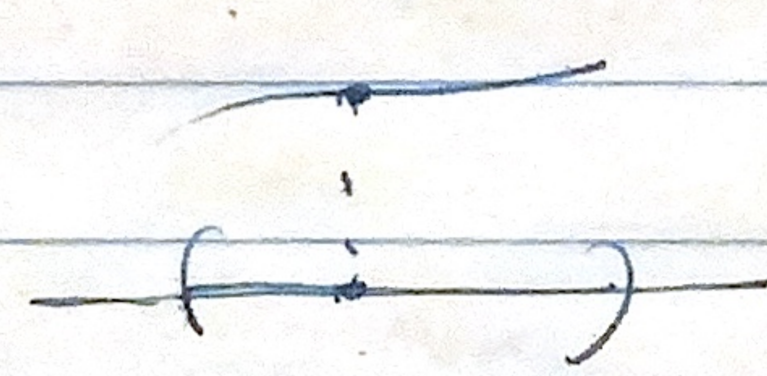
similar to DeTurck's trick. strictly elliptic.

Weighted space $\|u\|_{L^2_f} = \int u^2 e^{-f}$.

Linearization $(DQ_g)_g[h] = -\Delta h + \nabla_{\nabla f} h - 2\mathrm{Rm}\cdot h$.

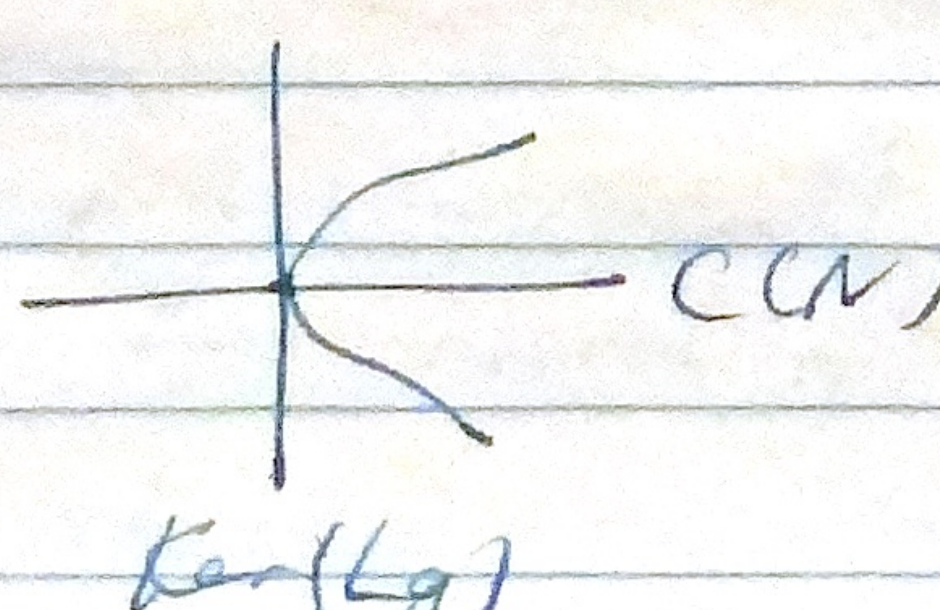
$=: L_g h$.

For $\mathrm{Rm}_g > 0$, $L_g > 0$



In general, $R_g > 0$ $k = \mathrm{Ker}(L_g h)$ ~~may~~ may not be empty, $\dim \mathrm{Ker} < \infty$. local structure of $\{\text{solutions } \tilde g\}$ Banach inf dim structure.

codim k submanifold of $C(M) \times \mathrm{Ker}(L_g)$, tangent to $\{0\} \times \mathrm{Ker}(L_g)$.

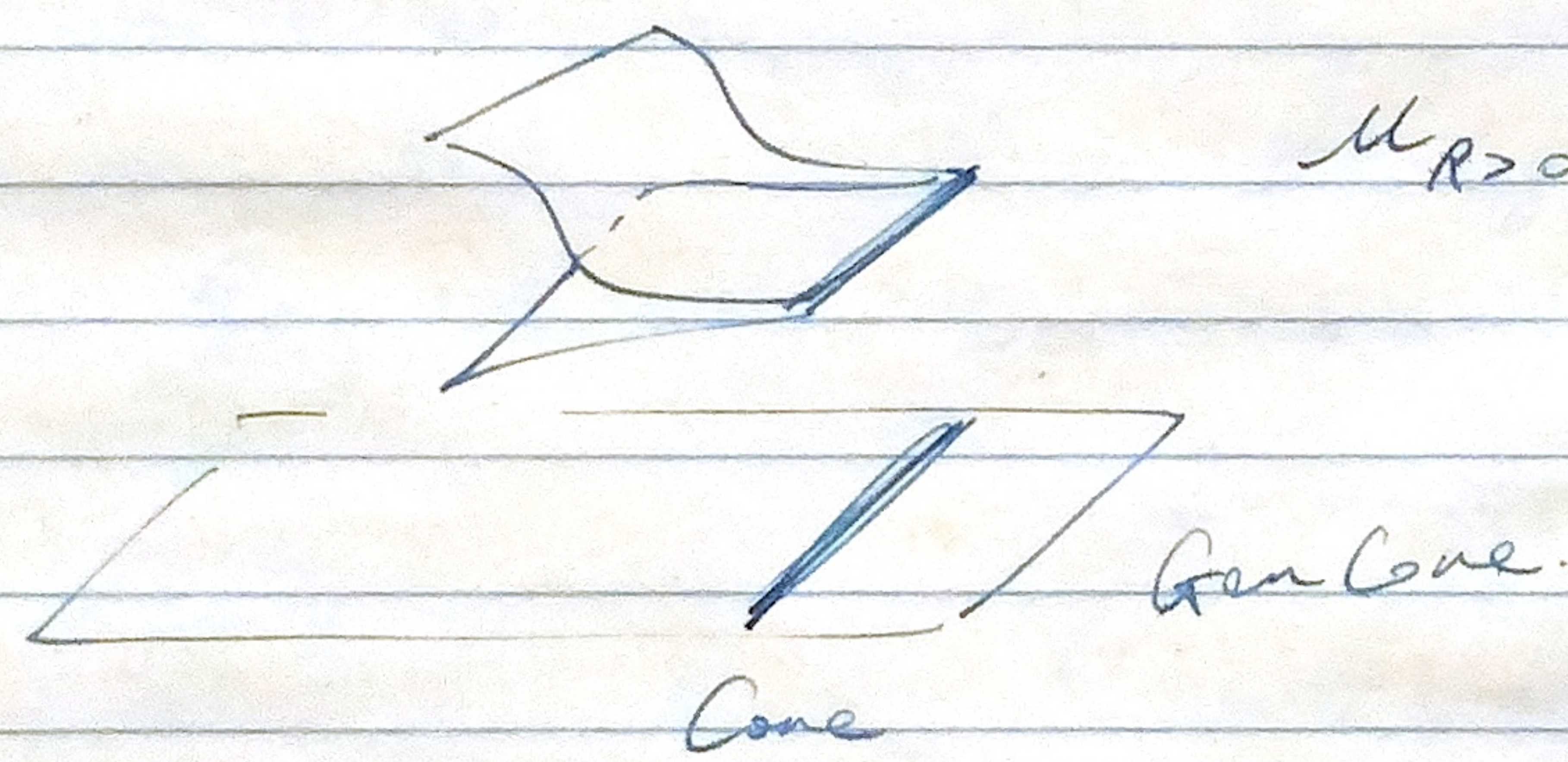


Global def.

$\mathcal{M}_{R>0}(M^4) = \{(g, X, \gamma)$ expanders, (g, X) AC to cone γ on $\mathbb{R}_+ \times N$, s.t. $\Psi^* X = -\frac{1}{2} r\partial_r\} / \sim$

$(g, X, \gamma) \sim (\tilde{g}, \tilde{X}, \tilde{\gamma})$ if $\gamma = \tilde{\gamma}$, and $\exists\ \Psi : M \to M$ compactly support, s.t. $\Psi^* \tilde{g} = g$, $\Psi^* \tilde{X} = X$.

Actually need
Gen Cone $(N^3) = \{dr^2 + r^?(dr \otimes \beta + \beta \otimes dr) + r^2 h_{N^3}\}$



Some additional ingredients.

- properness: sequence (g_i, X_i, γ_i) s.t. $\gamma_i \to \gamma$
Does $(g_i, X_i) \to (g, X)$ subconvergence?
need uniform (Rm_{g_i}) bound.

blow up arguement, if $|Rm| \to \infty$.

$\begin{cases} \text{interior} \longrightarrow \text{rescaling} \longrightarrow \text{Ricci flat ALE.} \\ \text{bdy. OK by expander} \end{cases}$

ruled out by topological assumptions of H_2, H_1