

MATH541 Functional Analysis, Spring 2021

Lectures delivered by Marius Junge

Notes by Xinran Yu

April 4, 2021

Warning: I'm typing the notes slowly. Given that lecture recordings are not uploaded regularly, you can expect no updates for weeks.

The first several lectures contains a review on the materials from Real Analysis, which I will omit in this notes.

Contents

1	Baire's Category Theorem 20210125	3
2	Baire's Category Theorem Cont. 20210127	3
3	Basic Banach Space Theory 20210129	4
4	Basic Banach Space Theory Cont. 20210201	6
5	Hahn-Banach Theorem 20210203	9
6	Hahn-Banach Theorem Cont. 20210205	10
7	Vector Space 20210208	13

8	Locally Convex Topological Vector Spaces 20210210	16
9	Hahn-Banach Separation Theorem 20210212	18
10	Weak Topology 20210215	21
11	Weak Topology Cont. 20210219	23
12	Alaoglu's Theorem 20210222	25
13	Reflexive Spaces 20210224	27
14	Reflexive Spaces Cont. 20210226	30
15	Riesz-Thorin Theorem 20210301	33
16	Clarkson's Inequality 20210303	36
17	Uniform convexity of L_p 20210305	39
18	Uniform Boundedness and Open Mapping 20210308	41
19	$\ P_n\ $ is Unbounded 20210310	44
20	Krein–Milman Theorem 20210312	49
21	Krein–Milman Theorem Cont. 20210315	49

1 Baire's Category Theorem 20210125

Ref: A Course in Functional Analysis, John B. Conway, 1985

1. Metric space
2. Chicago suburb distance \mathbb{R}^b compact = closed and bounded no longer true
3. Cauchy sequence, completeness
4. Open, closed ball
5. Nowhere dense set, dense set, closure, interior.

$$Y \text{ is nowhere dense} \iff \bar{Y}^C \text{ is open and dense.}$$

Theorem 1.1 (Baire's theorem). *In a complete metric space, the countable union of nowhere dense sets is again nowhere dense.*

Lemma 1.2. *The intersection of open dense sets is again open dense.*

Using the above lemma + induction to prove Baire's theorem.

Dense, nowhere dense, somewhere dense. [Stack Exchange](#) Theorem in notes: countable intersection of open dense is dense, then countable union does not have interior points. Need X complete metric space, so that the limit point is in X .

2 Baire's Category Theorem Cont. 20210127

Last time: open set, closed sets, theorem: let (X, d) be a complete metric space, O_n open dense, then $\cap_n O_n$ is dense.

1. **intuition** dense set \cong , taking away a countable set of points
2. **proof idea** completeness \rightarrow geometric series.
3. Use Baire's theorem to show no function $f : [0, 1] \rightarrow \mathbb{R}$ continuous exactly at \mathbb{Q}
4. **proof** hard works is to find complete metric space and makes the theorem work
5. Normed space. A normed space is complete if absolute convergent sequences are convergent. Banach space.

6. isometry

7. $\|f(x)\|_{C(K)} = \sup_{k \in K} |f(k)|.$

Question 2.1. Let $C_b(\mathbb{R})$ be the set of continuous and bounded function. Is $C_b(\mathbb{R}) = C(K)$ for some compact K ? — Yes.

Want to do: Start with Banach space, create new ones.

Lemma 2.2. Let $T : X \rightarrow Y$ be a linear map between *normed spaces*. *TFRE*

1. T is continuous.

2. T is continuous at 0.

3. $\|T\|_{op} = \sup_{\|x\| \leq 1} \|Tx\|$ is finite.

4. T is Lipschitz.

Homogeneity, duality

Lemma 2.3. Let X be a normed sapce and Y be Banach. Then the vector space $L(X, Y)$ with the norm $\|\cdot\|_{op}$ becomes a Banach space.

$L(\text{normed}, \text{Banach})$ is Banach.

Corollary 2.4. $X^* = L(X, \mathbb{C})$ is Banach.

3 Basic Banach Space Theory 20210129

proof of Lemma 2.3. Step 1. (T_n) Cauchy implies $(T_n(x_k))$ Cauchy.

Step 2. Let $f(x) := \lim T_n(x)$. Prove $\limsup \|T_n(x) - f(x)\| = 0$.

$$\begin{aligned}\|T_n - T\| &= \|T_n - \lim T_m\| = \lim \|T_n - T_m\| \\ &\leq \limsup_{m,n \geq N} \|T_n - T_m\| < \epsilon\end{aligned}$$

$\|T_n - T\| < \epsilon$ implies $\|T_n(x) - T(x)\| < \epsilon$, and so $\limsup \|T_n(x) - f(x)\| = 0$.

Step 3. f is bounded, and $T_n \rightarrow f$. □

Corollary 3.1. X Banach, then $L(X, X) = L(X)$ is Banach algebra.

Definition 3.2. A **Banach algebra** is a Banach space $(\mathcal{A}, \|\cdot\|)$ together with a product $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, with $\|ab\| \leq \|a\|\|b\|$.

1. closed subset of Banach is Banach.
2. $K(X, Y) := \{T : X \rightarrow Y \mid \overline{T(B_X)} \text{ compact}\}$ is closed.
3. In finite dimension, linear bounded T is compact.

Definition 3.3 (Totally bounded).

$$\forall \epsilon, \exists N \text{ s.t. } Y \subset \bigcup_{j=1}^N B(x_j, \epsilon)$$

This is equivalent to relatively compact. [Ref](#)

Theorem 3.4.

$$K(H, H)^{**} = B(H, H)$$

(We'll this theorem later.)

Theorem 3.5.

$$\exists \iota : X \rightarrow X^{**}; \iota(x)(f) = f(x), \text{ with } f : X \rightarrow \mathbb{K}$$

1. ι is an isometry.
2. $\overline{\iota(x)}$ is the completion of X .

Part 1 follows from Hahn-Banach.

Definition 3.6. (X, d) is a metric space. A **completion** (Y, d') is given by

1. $\iota : X \rightarrow Y$ is an isometry.
2. $\iota(X)$ is dense.
3. (Y, d') is complete.

Completion is unique.

4 Basic Banach Space Theory Cont. 20210201

Completion problem: see Theorem 3.5

proof of Theorem 3.5.

Claim 4.1. $\|\iota(x)\|_{X^{**}} \leq \|x\|_X$.

Note that

$$\begin{aligned} \|\iota(x)\|_{X^{**}} &= \sup_{\|f(x)\|_{X^*} \leq 1} |\iota(x)(f)| && \text{(by definition)} \\ &= \sup_{\|f(x)\|_{X^*} \leq 1} |f(x)| && (\iota \text{ inclusion}) \\ &\leq \sup_{\|f\|_{X^*} \leq 1} \|x\| \leq \|x\|. \end{aligned}$$

By definition $\|f\|_{X^*} \leq 1 \iff |f(x)| \leq \|x\|$. □

For a normed space the completion achieves in X^{**} .

Banach space

Lemma 4.2. $C_b(x, x_0)$ is a Banach space.

$$C_b(x, x_0) = \{ f : X \rightarrow \mathbb{R} \mid \text{continuous and } \exists C, |f(x)| \leq Cd(x, x_0) \}.$$

Norm: $\|f\| = \sup_x \frac{|f(x)|}{d(x, x_0)}.$

An embedding isometry $\iota : X \rightarrow C_b(X)^*; \iota(x)(f) = f(x)$. Hint: use evaluation map

$$\sup_{\|f\| \leq 1} |f(x) - f(x_0)| = d(x, x_0).$$

Distance attaining function is $f(x) = d(x, x_0)$, where $x \neq x_0$.

Theorem 4.3 (Hahn-Banach Extension). *Given a vector space X , a sublinear map $q : X \rightarrow \mathbb{R}$ s.t.*

$$q(x + y) \leq q(x) + q(y) \text{ (subadditive) and } q(sx) = sq(x), s > 0.$$

Let $Y \subset X$ and $f : Y \rightarrow \mathbb{R}$ linear, with $f \leq q$, then $\exists F : X \rightarrow \mathbb{R}$ linear $F \leq q$ and $F|_Y = f$.

warning This theorem is completely algebraic. There is no topology.

Lemma 4.4. *We can always add an extra dimension.*

Proof. Step 1. $Y \subset X = \{ y + tx_0 \mid t \in \mathbb{R} \}$. Candidates for F (extend 1-dim):
 $F(y + tx_0) = F(y) + tF(x_0) = f(y) + ta_0$ for some a_0 . What is a_0 ? **trick**

$$\begin{array}{ll} F(y + tx_0) \leq q(y + tx_0) & \implies f(y) + ta_0 \leq q(y + tx_0) \\ F(y - tx_0) \leq q(y - tx_0) & f(y) - sa_0 \leq q(y - sx_0) \end{array}$$

$$\begin{aligned} \Rightarrow \quad a_0 &\leq \frac{q(y+tx_0) - f(y)}{t}, t > 0 & \Rightarrow \quad a_0 &\leq \inf \frac{q(y+tx_0) - f(y)}{t}, t > 0 \\ a_0 &\geq \frac{f(y) - q(y-sx_0)}{s}, s > 0 & a_0 &\geq \sup \frac{f(y) - q(y-sx_0)}{s}, s > 0 \end{aligned}$$

Check the sup is less than inf:

$$\begin{aligned} \frac{f(y) - q(y-sx_0)}{s} &\leq \frac{q(z+tx_0) - f(z)}{t} \\ \Leftrightarrow \quad f(y)t - q(y-sx_0)t &\leq q(z+tx_0)s - f(z)s \\ f(y)t + f(z)s &\leq q(z+tx_0)s + q(y-sx_0)t \\ f(yt + sz) &\leq q(yt + tsx_0 - tsx_0 + sz) \\ &\leq q(yt - tsx_0) + q(tsx_0 + sz) \\ &\leq tq(y - sx_0) + sq(tx_0 + z) \end{aligned}$$

This exactly fits the assumption, so we can pick $a_0 = \sup \frac{f(y) - q(y-sx_0)}{s}$.

Step 2. Use Zorn's lemma. Consider

$$\mathcal{L} = \{ (Z, F) \mid Y \subset Z, F \leq q \text{ on } Z, F|_Y = f \}.$$

Order on the set: $(Z_1, F_1) \leq (Z_2, F_2)$ if $Z_1 \subset Z_2$ and $F_2|_{Z_1} = F_1$. Every chain has an upper bound $Z_\infty = \cup Z_i, F = \cup F_i$. Hence there exists a maximal element $(Z_{\max}, F_{\max}) \in \mathcal{L}$.

Claim 4.5. $Z_{\max} = X$.

If not, $\exists x_0 \notin Z_{\max}$ apply lemma to F_{\max} , $Z_{\max} + \mathbb{R}x_0$ admits F'_{\max} . Contradiction. \square

Remark 4.6. Hahn-Banach is also true for \mathbb{C} .

5 Hahn-Banach Theorem 20210203

Lemma 5.1. *Take C convex, $0 \in C$. The **Minkowski functional***

$$q_C(x) = \inf \{ \lambda \mid x \in \lambda C \}$$

is sublinear.

Proof. $x, y \in V$. Let $\epsilon > 0$, choose λ, μ s.t. $x \in \lambda C, y \in \mu C$.

$$q_C(x) \leq \lambda \leq (1 + \epsilon) q_C(x)$$

$$q_C(y) \leq \mu \leq (1 + \epsilon) q_C(y).$$

Then $z = \frac{\lambda}{\lambda + \mu} \frac{x}{\lambda} + \frac{\mu}{\lambda + \mu} \frac{y}{\mu} \in C$. Therefore $x + y = (\lambda + \mu) \left(\frac{x}{\lambda + \mu} + \frac{y}{\lambda + \mu} \right)$. So

$$q_C(x + y) \leq \lambda + \mu \leq (1 + \epsilon) (q_C(x) + q_C(y)).$$

Send $\epsilon \rightarrow 0$. □

Corollary 5.2. *Let C, D be nonempty convex sets $C \cap D = \emptyset$. There there exists $f : V \rightarrow \mathbb{R}$ s.t. $f(x) \leq f(y)$ for all $x \in C, y \in D$.*

Proof. Take $x_0 \in C, y_0 \in D$. trick Shifting trick: let

$$B := C - D - (x_0 - y_0),$$

where $C - D := \{x - y \mid x \in C, y \in D\}$. Since $x - y \neq 0, y_0 - x_0 \notin B$. Let $Y = \mathbb{R}(y_0 - x_0)$.

Claim 5.3. $q_B(x_0 - y_0) \geq 1$.

Define $f(t(y_0 - x_0)) = t$, then $f \leq q_B$. Hahn-Banach extension gives $F : V \rightarrow \mathbb{R}$, with $F \leq q$ and $F(y_0 - x_0) = 1$. Note that $q_B(x - y - (x_0 - y_0)) \leq 1$ implies

$$\begin{aligned} F(x - y - (x_0 - y_0)) &\leq 1 \\ \implies F(x - y) - F(x_0 - y_0) &\leq 1 \\ F(x) &\leq F(y) + 1 - F(y_0 - x_0) = F(y) \end{aligned}$$

□

Theorem 5.4. For X a normed space and $q(x) = \|x\|$, X subset of complex vector space, $\forall x$ with unit norm, \exists a complex linear functional $f \leq \|\cdot\|$ with $|f(x)| = 1$.

Proof. Consider X as a real normed space. Take x_0 in X and let $Y = \mathbb{R}x_0 + i\mathbb{R}x_0$, $\|x_0\|$. Define $f(zx_0) = \operatorname{Re}(z)$. Note that $f \leq q$ as

$$f(zx_0) = \operatorname{Re}(z) \leq |z| = \|zx_0\| \leq (zx_0).$$

Then $\exists F : X \rightarrow \mathbb{R}$ with $F(x) \leq \|x\|$ real linear and $F(x_0) = 1$.

Fabrication: want to define $G(x) = F(x) - iF(ix)$. If G is complex linear and $F = \operatorname{Re} G$, $G(x) = \operatorname{Re} G(x) + i\operatorname{Im} G(x) = F(x) - \operatorname{Re}(iG(x))$.

Claim 5.5.

1. $G(x) = F(x) - iF(ix)$ is complex linear
2. $|G(x)| \leq \|x\|$

□

6 Hahn-Banach Theorem Cont. 20210205

Theorem 6.1 (Complex version Hahn-Banach). Let X be a complex vector space. If $f : Y \rightarrow \mathbb{C}$ is a complex linear functional on a complex linear subspace $Y \subset X$, and $q : X \rightarrow [0, \infty]$ a sublinear function and $q(zx) = q(x)$, $|z| = 1$ (semi-norm). If $|f| \leq q$, then there exists $F : X \rightarrow \mathbb{C}$, such that $|F| \leq q$, $F|_Y = f$

Proof. Apply the real Hahn-Banach to $\tilde{f} = \operatorname{Re} f$. $\tilde{F} : X \rightarrow \mathbb{R}$. Define a new F by

$$F(x) = \tilde{F}(x) - i\tilde{F}(ix).$$

Check F is complex linear. □

Hahn-Banach separation.

Lemma 6.2. *Let C be a convex set and q_C is a Minkowski functional*

1. $x \in C$ then $q_C(x) \leq 1$
2. $x \notin C$ then $q_C(x) \geq 1$.

$$\{x \mid q_C(x) < 1\} \subset X \subset \{x \mid q_C(x) \leq 1\}.$$

And the inclusions are strict.

Proof. $q_C(y) = \inf\{\lambda \mid \frac{y}{\lambda} \in C\}$. For part 1, $x \in C$ so $q_C(x) \leq \lambda = 1$.

For part 2, assume $q_C(x) < 1$, then $\exists \lambda < 1$ such that $\frac{x}{\lambda} \in C$. This (together with convexity) implies

$$x = (1 - \lambda) \cdot 0 + \lambda \cdot \frac{x}{\lambda} \in C,$$

contradiction. □

C may or may not contain the boundary.

1. Topology
2. filter
3. continuous

Definition 6.3. A **filter** on a set X is a subset $\mathcal{F} \subset 2^X$ such that

1. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
2. If $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

It is **nontrivial** if $\forall A \in \mathcal{F}, A \neq \emptyset$.

Definition 6.4. A **neighbourhood filter** is a collection $(\mathcal{F}_x)_{x \in X}$ of filters.

Remark 6.5.

1. (Topology \Rightarrow Filter)

Given topology τ , \mathcal{F}_X is generated by the non-empty open sets.

$$\mathcal{F}_X = \{ A \subset X \mid \exists O \text{ open, } x \in O \subset A \}.$$

Neighbourhood filter.

2. (Filter \Rightarrow Topology)

Given a filter \mathcal{F}_X , define O is open iff $\forall x \in O, O \in \mathcal{F}_X$. **intuition** A topology can equivalently be defined by open sets or neighbourhood filters.

Lemma 6.6. $(\tau^{\mathcal{F}})^{\tau} = \tau$.

Definition 6.7. f is **continuous** at x if $\forall B \in \mathcal{F}_{f(x)}, f^{-1}(B) \in \mathcal{F}_X$.

Recall: If $f : X \rightarrow Y$ continuous and $K \subset X$ compact, then $f(K)$ compact

Definition 6.8. A space $(X, +, \cdot, \tau)$ is a **topological vector spaces** if

1. $(X, +, \cdot)$ is a vector space
2. $+: X \times X \rightarrow X$ continuous
 $\cdot: \mathbb{K} \times X \rightarrow X$ continuous

Example 6.9. 1. \mathbb{R}^2 with the Chicago railway metric is not a topological vector space. $+$ not continuous.

2. Let (Ω, Σ, μ) be a measure space. Define

$$L_0(\Omega, \Sigma, \mu) = \{ f : \Omega \rightarrow \mathbb{R} \mid f \text{ measurable, } \mu(|f| > \epsilon) \rightarrow 0 \text{ as } \epsilon \rightarrow \infty \}.$$

Define

$$d(f, 0) := \inf \{ \epsilon \mid \mu(|f| > \epsilon) < \epsilon \}, \quad d(f, g) = d(f - g, 0).$$

This is a translation invariant metric. Hence a translation invariant topological vector space.

7 Vector Space 20210208

1. Topological space
2. Topological vector space $(X, +, \cdot, \tau)$, in particular, the translation map $T_x : X \rightarrow X; y \mapsto T_x(y) = x + y$ is a homeomorphism
3. Application to Hahn-Banach
4. Tychonoff's theorem

Motivational lemma

Lemma 7.1. *Let X be a topological vector space, $f : X \rightarrow \mathbb{R}$ be a linear nonzero continuous map, then the image of an open convex set is open.*

Proof. If f is linear and O is convex then $f(O)$ is convex. Convex sets of \mathbb{R} are intervals.

Assume $f(O) = (a, b]$ or $[a, b]$. That is there is a $x \in O$, $f(x) = \sup_{y \in O} f(y)$, then $f(x) = b$. Since $f(x_0) \neq 0$ with $f(x_0) = 1$, ($f \neq 0$), we consider $x(t) = x + tx_0$. Then O open implies there is a t_0 , for all $|t| < t_0$, $x + tx_0 \in O$ (translation is continuous). But now

$$f(x + tx_0) = f(x) + tf(x_0) = b + t \cdot 1 > b.$$

Contradiction. □

later Extension is continuous.

Theorem 7.2 (Tychonoff). *For each $j \in J$, let X_j be a topological space. If each X_j is compact, then $X = \prod_{j \in J} X_j$ is compact in the product topology.*

Clarification: $x = (x_i)_{i \in I}$, O is a neighborhood of x if there are i_j , O_j such that $O = \{ (y_i) \mid y_{i_j} \in O_{i_j} \}$.

Example 7.3. Let X_i be a metric space, the index set $I = \mathbb{N}$. Now the following

defines a distance of the product topology

$$d((x_n), (y_n)) = \sum_{n \geq 0} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)}.$$

$$\{ (y_n) \mid d((x_n), (y_n)) < \epsilon \} \quad \supset \quad \{ (y_n) \mid \text{dist}(x_j, y_j) < \frac{\epsilon}{2}, j = 1, \dots, n \}.$$

Proof. Assume $d(x_j, y_j) \leq \frac{\epsilon}{2}$ for all j . Then

$$\begin{aligned} d((x_n), (y_n)) &= \sum_{n \geq 0} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} \\ &\leq \sum_{n=1}^m 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} + \sum_{n > m} 2^{-n} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon \end{aligned}$$

(choose large m so that the second term is less than $\frac{\epsilon}{2}$). *Any continuity condition only depends on finitely many terms* \square

1. non-trivial filter, filter converges to x , ultra filter

intuition Filter is the analogue of sequence converging to something. They want to being small.

Definition 7.4. We say that a **filter** \mathcal{F} **converges to** x if $\mathcal{F} \supset \mathcal{N}_x$.

Every neighbourhood is contained in the filter.

Definition 7.5. A maximal non-trivial filter is called a **ultra filter**.

Remark 7.6. Let \mathcal{U} be an ultra filter then for every $A \subset X$, either $A \in \mathcal{U}$ or $A^C \in \mathcal{U}$.

Proof. Fix $A \subset X$.

Case 1. $A \in \mathcal{U}$ done.

Case 2. $A \notin \mathcal{U}$ then $A^C \in \mathcal{U}$. (Prove by contradiction, assume $A^C \notin \mathcal{U}$) Define $\tilde{\mathcal{U}}$ to be the smallest filter which contains A^C and elements in \mathcal{U} . (Show $\tilde{\mathcal{U}}$ is again a filter). Indeed this new filter $\tilde{\mathcal{U}}$ is closed by superset. Need to show if $\tilde{A}, \tilde{B} \in \tilde{\mathcal{U}}$ implies $\tilde{A} \cap \tilde{B} \in \tilde{\mathcal{U}}$.

- $\tilde{A}, \tilde{B} \in \mathcal{U}$ done.
- $\tilde{A}, \tilde{B} \supset A^C$ done.
- $\tilde{A} \in \mathcal{U}, \tilde{B} \supset A^C$. We know $\tilde{B} \supset A^C$ implies $\tilde{B}^C \subset A$, and we know $\tilde{A} \neq A$, so $\tilde{A} \cap \tilde{B} = \emptyset$.

Then $\tilde{\mathcal{U}}$ is a filter, contradicting to the fact \mathcal{U} is an ultra filter. \square

Corollary 7.7. *Every ultra filter on an interval converges.*

Lemma 7.8. *(X, τ) is compact iff every ultra filter converges.*

Proof. [Ref.](#)

(\Rightarrow) Let (X, τ) be compact and \mathcal{U} be an ultra filter. Assume \mathcal{U} does not converge to any point. Then $\forall x \in X, \mathcal{N}_x \not\subset \mathcal{U}$. Then every point has a neighbourhood O_x which is not in \mathcal{U} .

Take the open cover $\cup_x O_x$ of X , O_x as above. By compactness, there is a finite subcover $O_{x_1} \cup \dots \cup O_{x_n}$. Since \mathcal{U} is an ultra filter, $O_{x_i}^C \in \mathcal{U}$, and the finite intersection of $O_{x_i}^C$'s is in \mathcal{U} . But

$$\left(\bigcap_{i=1}^n O_{x_i}^C \right)^C = \bigcup_{i=1}^n O_{x_i} = X$$

implies $\cap_{i=1}^n O_{x_i}^C = \emptyset \in \mathcal{U}$, contradiction.

(\Leftarrow) Let $X \subset \cup_x O_x$, O_x open. Assume that $X \not\subset \cup_{i=1}^n O_{x_i}$ for any finite subset of

indices. Then $\cap_{i=1}^n O_{x_i}^C \neq \emptyset$. Define

$$\mathcal{F} = \left\{ A \mid \exists i_1, \dots, i_n \text{ s.t. } \bigcap_{i=1}^n O_{x_i}^C \subset A \right\}.$$

This is a filter, let \mathcal{U} be the ultra filter contains \mathcal{F} . Then \mathcal{U} converges, say to some $x_0 \in X$, then $\mathcal{N}_{x_0} \subset \mathcal{U}$. Then there is a neighbourhood of x_0 which is contained in \mathcal{U} , and then $O_x^C \in \mathcal{F} \subset \mathcal{U}$. But $O_x \cap O_x^C = \emptyset$, contradiction. \square

proof of Theorem 7.2. Ref.

Let $X = (\prod_i X_i, \tau_i)$, \mathcal{F} be an ultra filter. Let $\pi_i : X \rightarrow X_i$ be the projection to the i -th term. Note that $\pi_i(\mathcal{F})$ is also an ultra filter, so it converges to some $x_i \in X_i$. Then \mathcal{F} converges to $(x_i)_{i \in I}$.

Claim 7.9. *Let $x = (x_i)_{i \in I}$, if $O \in \mathcal{N}_x$ then $O \in \mathcal{U}$.*

This means $O \supset O_{i_1} \times \dots \times O_{i_n} \times X_{j_1} \times X_{j_1} \times \dots$. Now $\pi_{i_k}^{-1}(O_{i_k}) = W_k$ open and belongs to \mathcal{U} , as $O_{i_k} \in \mathcal{U}$. Hence, the finite intersection of W_k 's is in \mathcal{U} . Then $O \in \mathcal{U}$. \square

8 Locally Convex Topological Vector Spaces 20210210

Recall

1. Topological vector spaces $(X, +, \cdot)$
2. Tychonoff theorem
3. **intuition** An ultra filter is a generalisation of sequence converging to a point.

Definition 8.1. A topological vector space is called **locally convex** if $\forall x, \forall O \in \mathcal{N}_x$, $\exists W$ convex such that $x \in W \subset O$.

Example 8.2.

1. Let X is a normed space, $\mathcal{N}_x = \{ O \mid \exists x > 0, \text{int}(B_r) + x \subset O \}$.
2. Let $X = C^\infty(\mathbb{R})$, K a compact subset, with semi-norm $\|f\|_{K,n} = \sup_{x \in K} \sup_{1 \leq i \leq n} |f^{(i)}(x)|$.

(This is a semi-norm because $\text{supp } f$ can be in K^C) The resulting topology is locally convex.

Example 8.3 (Non-examples).

1. $L_0 = \{ f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ measurable} \}$, with

$$d(f, 0) = \inf \{ \epsilon \mid \mu(|f| > \epsilon) < \epsilon \}.$$

2. $\|f\|_p = (E|f|^p)^{1/p}$ with $0 < p < 1$. $B_p = \{ f \mid \|f\|_p < 1 \}$. (Cannot put in a convex set if it is infinite dimension). The first example is when $p \rightarrow 0$. (E is expectation?)

Theorem 8.4. *Let (X, τ) be a topological space, the following are equivalent.*

1. X is a Locally convex topological vector spaces (LCTVS)
2. $\exists (q_i)_{i \in I}$ of semi-norms on X
3. $O \ni N_0$ iff $\exists i, \exists r$ s.t. $\{ x \mid q_i(x) < r \} \subset O$.

intuition The topology is determined by many different shaped balls. Open iff contain one of the balls.

Proof of Theorem 8.4

(\Leftarrow) Take a point $x \in X$ and O is an open neighbourhood of x . Define a translation map $T_{-x} : X \rightarrow X$, by $T_{-x}(y) = y - x$. Note that T_{-x} is a homeomorphism, so $T_{-x}(O) =: W$ is an open neighbourhood of 0. By (iii), $\exists i$ s.t. $\{ y \mid q_i(y) < 1 \} \subset W$. Define $V = x + \tilde{W} = \{ \tilde{y} \mid q_i(\tilde{y} - x) < 1 \} \subset O$.

Definition 8.5. A set $W \ni 0$ is called **absolutely convex** if

$$\sum_{j=1}^n |\lambda_j| \leq 1 \implies \sum_{j=1}^n \lambda_j x_j \in W.$$

Definition 8.6. A set W is **balanced** if $|z| = 1, zw = w$ for all $w \in W$.

Remark 8.7. W is absolutely convex if W is convex and balanced.

(\Rightarrow) Prove existence of seminorms. Take $\mathbb{K} = \mathbb{R}$ let O be open and $\exists W \subset O$ containing 0 and convex. Since $- : X \rightarrow X; -x \mapsto x$ is continuous, we know $(-)^{-1}(W) \supset V$ is convex, $V \in \mathcal{N}_x$ (Aside: $W \cap -W$ is convex and balanced).

Define $q_V(x) = \inf \{ \lambda \mid \frac{x}{\lambda} \in V \}$

Lemma 8.8. q_V is a semi-norm.

That is, $q_V(\lambda x) = |\lambda|q_V(x)$ and subadditive $q_V(x + y) \leq q_V(x) + q_V(y)$.

Then

$$\frac{1}{4} \subset \{ y \mid q_V(y) < \frac{1}{2} \} \text{ (ball of some semi-norm)} \subset V.$$

For every neighbourhood of 0 can choose a semi-norm

For $\mathbb{K} = \mathbb{C}$. Want for any set O , find a W which is convex and contained in $\cap_{|z|=1} zO$ (in a intersection of rotations). $(\cap_{|z|=1} zO)^C = \cup_{|z|=1} (zO)^C$.

Question: Is $B = \cup_{|z|=1} (zO)^C$ closed? – Yes. Let $T = \{ z \mid |z| = 1 \}$. The map $T \times X \rightarrow X; (z, x) \mapsto zx$ is continuous and T is compact.

Lemma 8.9. B is closed. (A compact translation of a closed set is closed.)

Proof. Let A be an ordered index set, $x_\alpha \in B$, $x_\alpha \rightarrow x$ meaning for a neighbourhood O of x , $\exists \alpha_0, \forall \alpha > \alpha_0, x \in O$. □

Then $0 \notin B$, and $\exists W \subset \cap_{|z|=1} zO)^C$ convex and $\cap_{|z|=1} zw$ is balanced convex set.

9 Hahn-Banach Separation Theorem 20210212

Lemma 9.1. Let X, Y be locally convex topological vector spaces. A linear map $T : X \rightarrow Y$ is continuous if and only if T is continuous at 0.

Propersition 9.2. *Let X be a locally convex topological space and $f : X \rightarrow \mathbb{R}$ be a linear and continuous map. Let W be an open convex neighbourhood of 0. Then either $f(W) = \{0\}$ or $f(W)$ is open.*

Theorem 9.3 (Hahn-Banach Separation Theorem). *Let C be a **convex nonempty** subset in a topological space X and $x \notin C$, then*

1. *there exists a linear map $f : X \rightarrow \mathbb{R}$ such that $f(y) \leq f(x), \forall y \in C$,*
2. *if in addition X is a locally convex topological vector space and C is open, then f is continuous, nontrivial and $f(y) < f(x), \forall y \in C$.*

Proof. (1) Let $x_0 \in C$, then $\tilde{C} = C - \{x_0\}$ contains 0, by Lemma 5.1, the Minkowski functional $q_{\tilde{C}} = \inf\{\lambda \mid y \in \lambda\tilde{C}\}$ is sublinear. Let $V = \mathbb{R}(x - x_0)$ and define $f(t(x - x_0)) = t$, which is linear. Then $x - x_0 \notin C - \{x_0\}$. By Lemma 6.2, $y \in C$ implies $q_{\tilde{C}}(y - x_0) \leq 1$. Therefore

$$f(y - x_0) \leq f(x - x_0) = 1 \implies f(y) \leq f(x).$$

(2) Now if C is open then $\tilde{C} = C - \{x_0\}$ is open (here we only require a topological space, we don't actually need locally convexity). Consider $g : X \times X \rightarrow X$, $g(x, y) = x - y$. This map is continuous, $0 \in \tilde{C}$.

There exists V_1, V_2 neighbourhoods of 0, such that $V_1 - V_2 \subset \tilde{C}$. Define $V = V_1 \cap V_2$ (V is a neighbourhood of 0). Then $0 \in V - V \subset \tilde{C}$. By previous part $f|_{\tilde{C}} \leq 1$.

check Hence

$$f(V - V) \subset f(\tilde{C}) \subset \{y \mid f(y) \leq 1\}.$$

Then for all $y = a - b \in V - V$, $f(y) \leq 1$ and $-y = b - a \in V - V$ so $f(-y) \leq 1$. This means f is bounded. Hence f is continuous at 0. By previous Lemma, f is continuous and $f(\tilde{C})$ is open (image of open convex set is open). Then $f(y - x_0) < 1$ for all $y \in C$. \square

Theorem 9.4. *Let C, D be **nonempty convex** sets. If $C \cap D = \emptyset$, then there is a linear functional f on X such that $f(x) < f(y)$, for all $x \in C, y \in D$.*

Proof. **trick** Consider $\tilde{C} = C - D = \{x - y \mid x \in C, y \in D\}$. Note that \tilde{C} is open if either C or D is open, and $0 \notin \tilde{C}$. Now shift the set, i.e. let $\tilde{D} = \tilde{C} - \{(x_0 - y_0)\}$. Apply previous theorem $0 \notin \tilde{C}$, so there exists a $f \neq 0$ and continuous, $f(z) < f(0)$, for all $z \in \tilde{C} = C - D$. Say $z = x - y$, for $x \in C$ and $y \in D$. Then $f(x) < f(y)$. \square

Theorem 9.5. *Let C be a **closed convex** set and D be a **compact convex** set in a locally convex topological vector space. Then there exists a continuous nontrivial f and $r < s$ such that $f(x) < r < s < f(y)$ for all $x \in D$ and $y \in C$.*

Proof. Assume C is closed and D is compact. C^C is open, $D \cap C = \emptyset$. For any $x \in D$ there is a W_x convex such that $(x + W_x) \cap C = \emptyset$.

Consider the open sets $x + \frac{W_x}{2}$, their union $\cup(x + \frac{W_x}{2})$ gives an open cover of D . Then there is a finite subcover $D \subset \cup_i(x_i + \frac{W_{x_i}}{2})$. Take $W = \cap_i \frac{W_{x_i}}{2}$, and let $y = d + w \in D + W$. Then there exists an x_j such that $d = x_j + \frac{W_{x_j}}{2}$. Therefore,

$$y = d + w \in x_j + \frac{W_{x_j}}{2} + W \subset x_j + \frac{W_{x_j}}{2} + \frac{W_{x_j}}{2} \not\subset C.$$

(Convexity implies $\frac{W_{x_j}}{2} + \frac{W_{x_j}}{2} \subset W_{x_j}$.) **analogue** Triangle inequality on metric spaces.

Hence we have a strict separation between $D + W$ and C , and we can find a nontrivial continuous f such that $f(x) < f(d + w)$, where $x \in C, d \in D$ and $w \in W$. Note that $f(D)$ is compact as D is compact, so $f(D)$ is a closed interval $[a, b]$. Then

$$f(D + W) = f(D) + f(W) = [a, b] + (-\alpha, \beta)$$

($f(W)$ is a neighbourhood of 0 **check**). So for all $x \in C$,

$$f(x) \leq a - \alpha < a \leq \inf\{f(y) \mid y \in D\}.$$

□

Example 9.6.

1. Let X be a normed space, and $C = \{x \mid \|x\| \leq 1\} = \bar{B}_X$. Take x_0 such that $\|x_0\| > 1$, then $D = \{x_0\}$ compact. There exists f such that $f(x) \leq 1$, $\|f\| \leq 1$ and $f(x_0) > 1$.
2. Take a ball B_X and a triangle D .

Next, we want to make the separation line unique.

10 Weak Topology 20210215

Definition 10.1. Let X be a Banach space and $Y \subset X^*$ a subspace. Then $\sigma(X, Y)$ -topology is the coarsest topology making all the functional $y \in Y$ continuous. This means the semi-norms defining this topology are given by

$$q_{y_1, \dots, y_n}(x) = \max_{i=1, \dots, n} |y_i(x)|.$$

Every locally convex space is given by semi-norms. Semi-norms are indexed by finite subsets of Y .

Theorem 10.2. *The dual space of $(X, \sigma(X, Y))$ is Y (as a set). That is,*

$$(X, \sigma(X, Y))^* = Y.$$

Note the two spaces only equal as a set, not necessarily as a topological space. Because Y on the LHS can be taken as a algebraic dual without topological assumptions, whereas Y on the RHS is a topological vector space (may with its own norm).

Remark 10.3. Let X be a locally convex topological vector space and Y a Banach

space or locally convex topological vector space, then $L(X, Y)$ is also a locally convex topological vector space.

Proof. Step 1. $Y \subset (X, \sigma(X, Y))^*$.

Claim 10.4. *For every $y \in Y$, $f_y(x) = y(x)$ is continuous with respect to the new topology.*

It suffice to show f is continuous at 0: $\forall \epsilon, \exists O \in \sigma(X, Y)$ containing 0, such that if $x \in O$, then $|f(x)| < \epsilon$. ($f(0) = 0$). In this new topology open neighbourhood means there exists a semi-norm in system such that $O \supset \{x \mid q(x) < \delta\}$, i.e there exists some $B_q(\delta) \subset O$. This is equivalent to say $|f(x)| \leq C \cdot q(x)$, for some semi-norm q . compare In Banach space we don't have a choice of the norm, so we require $|f(x)| \leq C \cdot \|x\|$.

In our case, the semi-norm $q_y(x) = |y(x)|$ does the job, because $|f_y(x)| = |y(x)| = q_y(x)$. More generally, the semi-norm is given by $q_y(x) = \max_j |y_j(x)|$.

Step 2. $(X, \sigma(X, Y))^* \subset Y$.

Let $f : X \rightarrow \mathbb{K}$ be continuous. By definition there exists a q such that $|f(x)| \leq q(x)$ and $q(x) = \max_j |y_j(x)|$. Fix y_1, \dots, y_n and define a map

$$\begin{aligned} \phi : X &\longrightarrow \mathbb{K}^n \\ x &\longmapsto (y_1(x), \dots, y_n(x)). \end{aligned}$$

Then $\phi(X) \subset \mathbb{K}^n$ is a subspace. Denote $Z = \phi(X)$, then $z = (y_1(x), \dots, y_n(x))$. Consider the map

$$\begin{aligned} \psi : Z &\longrightarrow \mathbb{K} \\ z &\longmapsto f(x). \end{aligned}$$

This map is well-defined, linear, and $|\psi(z)| \leq \max_j |z_j| = \|z\|_\infty$. By Hahn-Banach, there exists $\tilde{\psi} : l_\infty^m \rightarrow \mathbb{K}$, such that $\tilde{\psi}|_Z(z) = \psi(z)$ and $\|\tilde{\psi}\| = \|\psi\| \leq \|z\|_\infty$. Note that $\tilde{\psi}(z) \in (l_\infty^m)^* = l_1^m$. This means there exists $\alpha_1, \dots, \alpha_n$ such that $\tilde{\psi}(z) = \sum_j \alpha_j z_j$.

This means

$$f(x) = \psi(\phi(x)) = \tilde{\psi}(\phi(x)) = \sum_j \alpha_j \phi_j(x) = f_y(x),$$

where $y = \sum_j \alpha_j y_j$. □

Example 10.5. Let X be a space and take $Y = X^*$. Then

- $\sigma(X, X^*)$ is called the **weak topology** of X and $(X, \sigma(X, X^*)) = X^*$,
- $\sigma(X^*, X)$ is called the **weak* topology** of X^* and $(X^*, \sigma(X^*, X)) = X$.

11 Weak Topology Cont. 20210219

Theorem 11.1 (Goldstine). *Let X be a Banach space, then the image of the closed unit ball $B_X \subset X$ under the canonical embedding ι into the closed unit ball $B_{X^{**}}$ of the bidual space X^{**} is weak*-dense.*

$$\overline{B_X}^{\sigma(X^{**}, X^*)} = B_{X^{**}}$$

intuition The unit ball with weak*-topology is compact. In finite dimension, close + bounded = compact. Generalisations of finite dimension.

Proof. Recall that X^{**} is a locally convex topological vector space with respect to $\sigma(X^{**}, X^*)$ -topology. This topology is given by the semi-norm $q(x^{**}) = \max_j |x^{**}(x_j^*)|$, with $x_1^*, \dots, x_j^* \in X^*$.

The canonical embedding $\iota : X \rightarrow X^{**}$, is an isometry (Hahn-Banach Theorem) and $\iota|_{B_X} : B_X \rightarrow B_{X^{**}}$. We want to show the closure $\overline{\iota(B_X)}$ with respect to the $\sigma(X^{**}, X^*)$ topology satisfies $\overline{\iota(B_X)} = B_{X^{**}}$. Prove by contradiction.

Assume that $x^{**} \notin \overline{\iota(B_X)}$, with $\|x^{**}\|_{X^{**}} \leq 1$. Note that $\overline{\iota(B_X)}$ is closed, compact and convex. By Hahn-Banach separation (Theorem 9.5), there exists a nontrivial continuous map $f : X^{**} \rightarrow \mathbb{R}$ so that $|f(\iota(x))| \leq 1 < s < |f(x^{**})|$ for all $x \in B_X$. On

one hand we have

$$\|f\|_{X^{**}} = \sup_{\|x\| \leq 1} |f(x^{**})| = \sup_{\|x\| \leq 1} |f(\iota(x))| \leq 1.$$

Then by definition,

$$|x^{**}(f)| \leq \|x^{**}\|_{X^{**}} \cdot \|f\|_{X^{**}} \leq 1.$$

On the other hand we have $|x^{**}(f)| = f(x^{**}) > 1$. Contradiction. \square

Example 11.2. Let $X = C_0 = \{ (x_n) \mid \lim_n x_n = 0 \}$, with $\|(x_n)\| = \sup_n \|x_n\|$. Then $X^* = l_1$ because

$$\begin{aligned} \|y_n\|_1 &= \sum_n y_n = \sup_k \sum_{i=1}^k |y_k| \\ &= \sup_k \langle y, \epsilon_1, \dots, \epsilon_k, 0, \dots, 0 \rangle. \end{aligned}$$

where $\epsilon_i = \text{sgn}(y_i)$ and $\langle y, z \rangle = \sum y_n z_n$. And $X^{**} = l_\infty = \{ (x_n) \mid \sup_n |x_n| < \infty \}$.

What is $\sigma(l_\infty, l_1)$ -topology? The answer is pointwise convergence on bounded set. Consider bounded sequences x^α ($\|x^\alpha\| \leq C$). Then $x^\alpha \rightarrow x \in l_\infty$ iff for all $y \in l_1$, $x^\alpha(y) \rightarrow x(y)$.

For bounded sets $\|x^\alpha\| \leq 1, \forall \alpha$,

$$x^\alpha \rightarrow x \iff x_n^\alpha \rightarrow x_n, \forall n.$$

(\Rightarrow) Take $y_n = (0, \dots, 1, \dots, 0) \in l_1$.

(\Leftarrow) Let $y \in l_1$ and $\epsilon > 0$ then there exists n_0 such that $\sum_{n > n_0} |y_n| < \frac{\epsilon}{2}$. There exists α_0 such that any $\alpha > \alpha_0$, $|x_n^\alpha - x_n| < \frac{\epsilon}{2}$ for all $n > n_0$. We need

$$|x^\alpha(y) - x(y)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Let $y^N = (y_1, \dots, y_N, 0, \dots, 0)$, $y^N \rightarrow y$ in $\sigma(l_\infty, l_1)$ because we can use pointwise convergence.

12 Alaoglu's Theorem 20210222

Alaoglu's Theorem says that the closed unit ball in X^* is compact in the weak*-topology.

Theorem 12.1 (Alaoglu). *Given a topological vector space X , and let $B_{X^*} = \{x^* \in X^* \mid \|x^*\| \leq 1\}$ be the closed unit ball in X^* . Then B_{X^*} is compact in X^* with respect to the weak*-topology on X^* .*

Proof. [Ref.](#) or see Conway p.134

Let the set $D_x = \{z \in \mathbb{K} \mid |z| \leq 1\}$. Consider the product $D := \prod_{x \in B_x} D_x$. Since D_x is compact in \mathbb{K} , Tychonov's theorem says that D compact in the product topology. Elements in D are functionals, given by $\mu \in K$, $\mu(x) = \mu_x \in D \subset \mathbb{C}$, although they need not to be linear.

The inclusion

$$\iota : B_{X^*} \subset \prod_{x \in B_x} D =: K$$

is given by

$$\iota(x^*)(x) = x^*(x).$$

Note that $\iota(B_{X^*}) \subset K$. Indeed, if $\|x\| \leq 1$ and $\|x^*\| \leq 1$, then $|x^*(x)| \leq 1 \in D$.

Claim 12.2. $\iota(B_{X^*})$ is closed. Hence, $\iota(B_{X^*})$ is a compact subspace of K .

Proof of the claim. Take a net (x_α^*) in B_{X^*} which converges to $f \in D$ pointwisely. So $f(x) = \lim_{\alpha \rightarrow \infty} x_\alpha^*(x)$. In particular $|f(x)| \leq 1$ for all $\|x\| \leq 1$. (Need to show f is in the range. We can not take \mathbb{N} as index set, instead replacing \mathbb{N} by a partially ordered set. Usually the index set is given by the neighbourhood basis of f . Let $O_i \in \mathcal{N}_f$, $i = 1, 2$, then $O_1 \cap O_2 \in \mathcal{N}_f$ and $O_1 \cap O_2 \geq O_i$.)

For $x \in X$, define $F(x) = \beta^{-1}f(\beta x)$ for some β such that $\|\beta x\| \leq 1$ (check this is well defined). Then F agrees with f on B_X . We claim that F is linear. Take

$x_i \in X$, $i = 1, 2$. Consider $y = \frac{x_1 + x_2}{\|x_1\| + \|x_2\|}$. If we take $\lambda = \frac{\|x_1\|}{\|x_1\| + \|x_2\|}$, then by convexity $y = \lambda \frac{x_1}{\|x_1\|} + (1 - \lambda) \frac{x_2}{\|x_2\|} \in B_X$. Then

$$\begin{aligned} f(y) &= \lim_{\alpha} x_{\alpha}^*(y) = \lim_{\alpha} x_{\alpha}^* \left(\frac{x_1}{\|x_1\| + \|x_2\|} \right) + x_{\alpha}^* \left(\frac{x_2}{\|x_1\| + \|x_2\|} \right) \\ &= f \left(\frac{x_1}{\|x_1\| + \|x_2\|} \right) + f \left(\frac{x_2}{\|x_1\| + \|x_2\|} \right). \end{aligned}$$

So

$$\begin{aligned} F(x_1 + x_2) &= f(y) \cdot (\|x_1\| + \|x_2\|) \\ &= \left(f \left(\frac{x_1}{\|x_1\| + \|x_2\|} \right) + f \left(\frac{x_2}{\|x_1\| + \|x_2\|} \right) \right) \cdot (\|x_1\| + \|x_2\|) \\ &= F(x_1) + F(x_2). \end{aligned}$$

We have a linear functional $F \in X^*$ satisfying $|F(x)| \leq 1$ when $\|x\| \leq 1$. This means $\|F\|_{X^*} \leq 1$. So $F \in B_{X^*}$. \square

Definition 12.3. A Banach space is **reflexive** if $X^{**} = X$.

Goal: to show X is reflexive iff X^* is reflexive.

Propersition 12.4. *A closed subspace of a reflexive Banach space is reflexive.*

Proof. The following diagram is commutative. (Check)

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^{**} \\ j \uparrow & & \uparrow j^{**} \\ Y & \xrightarrow{\iota_Y} & Y^{**} \end{array}$$

Step 1. $Y^{**} = Y$. Take an element $y^{**} \in Y^{**}$, note that

$$j^{**}(y^{**})(x^*) = y^{**} \circ j^*(x^*) = y^{**}(x^* \circ j) = x^*|_Y \in Y^*.$$

So we can apply y^{**} to this element, and define $\phi(x^*) = y^{**}(x^*|_Y)$

Lemma 12.5. *If $T : Y \rightarrow X$ is isometric, then $T^{**} : Y^{**} \rightarrow X^{**}$ is also isometric.*

The above lemma says Y^{**} embeds isometrically into X^{**} (we will prove this later). If in addition, $X^{**} = X$, we deduce that for every y^{**} there exists an $x \in X$ such that

$$y^{**}(x^*|_Y) = x^*(x).$$

We want to show $x \in Y$. We claim that $y^{**} \in Y$, otherwise by Hahn-Banach separation there exists x^* such that $x^*(y^{**}) = 1$ and $x^*|_Y = 0$. The last equation says $x^*(x) = y^{**}(x^*|_Y) = x^*|_Y = 0$. A contradiction (as $y^{**} \in Y^{**} \subset X^{**} = X$). \square

Lemma 12.6. *If $T : X \rightarrow Y$ is isometric then $T^* : Y^* \rightarrow X^*$ sends closed unit ball to closed unit ball.*

Proof. Note that $T^*(B_{Y^*}) \subset B_{X^*}$. Indeed,

$$\begin{aligned} \|T^*\| &= \sup_{\|y^*\| \leq 1} \|T^*(y^*)\| = \sup_{\|y^*\| \leq 1} \|y^* \circ T\| \\ &= \sup_{\|y^*\| \leq 1, |x| \leq 1} |y^* \circ T(x)| = \sup_{\|y^*\| \leq 1, |x| \leq 1} |y^*(x)| \leq 1. \end{aligned}$$

So $|T^*(y^*)| \leq \|T^*\| \|y^*\| \leq 1$.

To show T^* is onto, take $x^* \in B_{X^*}$. Can define $f(Tx) = x^*(x)$, $\|f\| \leq 1$. By Hahn-Banach there exists y^* such that $y^*(Tx) = f(Tx) = x^*(x)$. $T^*(y) = x^*$. \square

Lemma 12.7. *If $T : Y \rightarrow X$ is a surjection, then $T^* : X^* \rightarrow Y^*$ is an isometry.*

Proof of the Lemma 12.5. The previous two lemma gives the result. \square

13 Reflexive Spaces 20210224

Theorem 13.1. X is reflexive $\iff X^*$ is reflexive.

Proof. (\Rightarrow) Assume that $X = X^{**}$. Then B_{X^*} is closed in $\sigma(X^*, X^{**}) = \sigma(X^*, X)$. Take an element x^{***} in $B_{X^{**}}$, there exists a sequence $x_\alpha^* \rightarrow x^{***}$ in $\sigma(X^{**}, X^{**})$ topology. Since B_{X^*} is closed in $\sigma(X^*, X)$, there is an x^* such that $x_\alpha^* \rightarrow x^*$. This means $x^{***} = x^*$.

(\Leftarrow) If X^* is reflexive then X^{**} is reflexive, but $X \subset X^{**}$ as a closed subspace. \square

Remark 13.2. X is reflexive iff B_{X^*} is $\sigma(X^*, X^{**})$ closed.

Definition 13.3. A Banach space is called **uniformly convex**, if $\forall \epsilon > 0, \exists \delta > 0$ such that $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| > \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Lemma 13.4. Take (x_n) a sequence with

$$\limsup_n \|x_n\| \leq 1 \quad \text{and} \quad \liminf_n \inf_{m>n} \left\| \frac{x_n + x_m}{2} \right\| = 1.$$

Then (x_n) is Cauchy.

Proof. Let $\epsilon > 0$. Since $\limsup_n \|x_n\| \leq 1$, we can choose $\epsilon_0 > 0, \exists n_0$ such that $\|x_n\| \leq 1 + \epsilon_0$, for all $n > n_0$. So $\|\frac{x_n}{1+\epsilon_0}\| \leq 1$, for all $n > n_0$. Then

$$\left\| \frac{x_n + x_m}{2(1 + \epsilon_0)} \right\| = \left\| \frac{x_n + x_m}{2} \right\| \cdot \frac{1}{1 + \epsilon_0} \geq \frac{1}{(1 + \epsilon_0)^2},$$

for all $n > n_0$.

Taking $\frac{1}{(1+\epsilon_0)^2} = 1 - \delta$. Using uniform convexity (contrapositive), we have $\forall n, \exists m$

$$\left\| \frac{x_n - x_m}{2(1 + \epsilon_0)} \right\| < \epsilon.$$

Conclusion: Above shows $\forall \epsilon, \exists n_0, \forall n > n_0, \exists m$, such that $\|x_n - x_m\| < 2\epsilon(1 + \epsilon_0)$.

We use this for $\epsilon = 2^{-k}$, then there exists a converging subsequence x_{n_k} such that $\|x_{n_k} - x_{n_{k+1}}\| \leq 2^{-k}$. \square

Theorem 13.5 (Milman-Pettis). *Uniformly convex Banach spaces are reflexive.*

Proof. [Ref.](#)

Let $x^{**} \in B_{X^{**}}$, $\|x^{**}\| = 1$. Then by definition of $\|x^{**}\|$, for all n , there exists $x_n^* \in B_{X^*}$, such that $x^{**}(x_n^*) \geq 1 - \frac{1}{n}$. Since $B_X \subset B_{X^{**}}$ is dense in $\sigma(X^{**}, X^*)$. Let $q_n(y) = |x_n^*(y)|$. There exists (x_k) in B_X such that

$$|q_n(x^{**} - x_k)| = |x_n^*(x_k) - x^{**}(x_n^*)| \leq \frac{1}{2k}, \text{ for } n = 1, \dots, k.$$

In particular, apply the above to $n = k$, then

$$|x_k^*(x_k) - x^{**}(x_k^*)| \leq \frac{1}{2k} \implies -\frac{1}{2k} + x^{**}(x_k^*) \leq x_k^*(x_k).$$

Recall $x^{**}(x_k^*) \geq 1 - \frac{1}{k}$. So $1 - \frac{3}{2k} \leq x_n^*(x_k) \leq 1$ (RHS because x_n^* is in the unit ball).

Then take $m > k$, we have

$$2 - \frac{6}{2k} \leq 1 - \frac{3}{2k} + 1 - \frac{3}{2m} \leq x_k^*(x_k) + x_m^*(x_m) \leq x_k^*(x_k + x_m) \leq \|x_k + x_m\| \leq 2. \quad (1)$$

Taking \liminf on both sides we get $\liminf \| \frac{x_k + x_m}{2} \| = 1$, and $\limsup \|x_k\| \leq 1$. By the above lemma (x_n) is Cauchy.

Remark 13.6. Assume there are two sequences x_n, \tilde{x}_n satisfies the property (1), then then $\lim x_n = \lim \tilde{x}_n$.

Now if (y_n^*) is another family using the above construction, then there exists (\tilde{x}_n) in B_X such that

$$|y_n^*(\tilde{x}_k) - x^{**}(x_n^*)| \leq \frac{1}{2k}.$$

Then $x^*(x_k) \rightarrow x^{**}(x)$ and $y^*(\tilde{x}_k) \rightarrow x^{**}(y)$ implies $x^{**} = \lim x_n = \lim \tilde{x}_n$ in $\sigma(X^{**}, X^*)$. \square

14 Reflexive Spaces Cont. 20210226

Real analysis: $L_p(\Omega, \Sigma, \mu) = \{ [f] \mid f : \Omega \rightarrow \mathbb{K}, f \text{ measurable}, \int |f|^p d\mu < \infty \}$, where Ω is a set, Σ is a σ -algebra and μ is a σ -additive measure. Recall

- Simple functions $f = \sum_{j=1}^n \alpha_j 1_{E_j}$ are dense.

- $\|f\|_p = \sup_{\|g\|_{p'} \leq 1} \left| \int fg d\mu \right|.$

Use Hölder inequality, say $\|f\|_p = 1$, then $g = \text{sgn}(f) \cdot |f|^{p/p'}$.

Corollary 14.1. *If $1 \leq p \leq \infty$, then $L_{p'}$ embeds isometrically into L_p^* ,*

$$\begin{aligned} \iota_{p'} : L_{p'} &\rightarrow L_p^* \\ g &\mapsto \left(\iota_{p'}(g) : f \mapsto \iota_{p'}(g)(f) = \int fg d\mu \right) \end{aligned}$$

and $\|f\|_p = \|\iota_{p'}(g) : L_p \rightarrow \mathbb{K}\|$.

Theorem 14.2. *Let $1 < p < \infty$ and assume L_p is reflexive. Then $L_{p'}^* = L_p$.*

(Here we check isometric isomorphism, there are two type of isomorphisms for Banach spaces, see more [here](#))

Proof. Let $\varphi : L_{p'} \rightarrow \mathbb{K}$ with $\|\varphi\|_{L_{p'}^*} = 1$. Recall $L_{p'} \hookrightarrow L_p^*$ is an isometry. By Hahn-Banach extension, there exists a $\hat{\varphi} : L_p^* \rightarrow \mathbb{K}$, with $\hat{\varphi}|_{L_{p'}} = \varphi$.

$$\begin{array}{ccc} L_{p'} & \xhookrightarrow{\iota_{p'}} & L_p^* \\ \varphi \downarrow & \swarrow \exists \hat{\varphi} & \\ \mathbb{K} & & \end{array}$$

To show $\iota_{p'}$ is surjective, take $\eta \in L_p^*$. If we can find $g \in L_{p'}$ such that $\int fg d\mu = \eta(f)$,

then $\iota_{p'}(g) = \eta$ and we are done. Such a g exists by commutativity and reflexivity

$$\varphi(g) = \hat{\varphi}(\iota_{p'}(g)) = \iota_{p'}(g)(f) = \int fg \, d\mu \implies \iota_{p'}(g)(f) = \eta(f).$$

□

Example 14.3 (Discrete case). Let $\Omega = I$, $\Sigma = 2^I$, μ be the counting measure. If $I = \mathbb{N}$, then

$$L_p(\mathbb{N}, \Sigma, \mu) = \ell_p = \{ (x_n) \mid \sum_n |x_n|^p < \infty \}.$$

What is the f defining the functional $\varphi : \ell(\mathbb{N}) \rightarrow \mathbb{K}$? Well, f is given by a sequence $(y_n) = ((0, 0, \dots, \frac{1}{n}, \dots, 0, 0))$. One can show that the

$$\|y_n\|_{p'} = \sup_n \left(\sum_{k=1}^n |y_k|^{p'} \right)^{\frac{1}{p'}} = \sup \{ \varphi(x_n) \mid \|x_n\| \leq 1 \}.$$

Prove using Hölder.

Theorem 14.4. $\ell_p^* = \ell_{p'}$ for $1 < p < \infty$.

Remark 14.5. For $I = \mathbb{N}$, let $c_0 = \{ (x_n) \in \ell_\infty \mid \lim x_n = 0 \}$. Then $c_0^* = \ell^1$, $c_0^{**} = \ell_1^* = \ell_\infty$.

Corollary 14.6. $B_{\ell_1} \subset B_{\ell_\infty^*}$ is $\sigma(\ell_\infty^*, \ell_\infty)$ -dense.

This means for any $\varphi \in \ell_\infty^*$, for any $f_i \in \ell_\infty$, there exists $g \in \ell_1$, with $\|g\|_{\ell_1} \leq \|\varphi\|$, such that

$$|\varphi(f_i) - f_j(g)| \leq \epsilon \text{ i.e. arbitrarily closed.}$$

Or there exists a net $(g_\alpha) \in \ell_1$ with $\|g_\alpha\|_{\ell_1} \leq \|\varphi\|$, such that

$$\varphi(f) = \lim_\alpha f(g_\alpha) = \lim_\alpha \sum_{n \in \mathbb{N}} f(n) g_\alpha(n).$$

Remark 14.7. Let $\varphi : \ell_\infty \rightarrow \mathbb{K}$, and assume $\varphi(1) = 1$. TFAE

- $\|\varphi\| = 1$
- $\forall g \geq 0, \varphi(g) \geq 0$.

We call this **positive functionals**.

Define the **state space** $S(\ell_\infty) = \{ \varphi \mid \varphi(1) = 1, \|\varphi\| = 1 \}$. Then discrete probability measures are dense in the state space. Indeed if $\varphi(1) = 1$ and $\|\varphi\| = 1$, then there is $g_\alpha \in \ell_1$ with $g_\alpha(1) = 1$, $\|g_\alpha\| \leq 1$ and $g_\alpha(f) \rightarrow \varphi(f)$. That is $g_\alpha \rightarrow \varphi$ in $\sigma(\ell_\infty^*, \ell_\infty)$.

Lemma 14.8. $\|g_\alpha\|_{\ell_1} = 1$ and $\sum_n g_\alpha(n) = 1$ implies $g_\alpha \geq 0$.

This means g_α are discrete probability measures because $\varphi(f) = \lim_\alpha \sum_{n \in \mathbb{N}} f(n) g_\alpha(n)$ exists.

Theorem 14.9. Let be $\varphi : C(K) \rightarrow \mathbb{C}$ be such that $\varphi(1) = 1$ and $\|\varphi\| = 1$. Then there exists a net $(x_j)_{j=1}^{n(\alpha)} (\lambda_j^\alpha)_{j=1}^{n(\alpha)}$, where $\sum \lambda_j^\alpha = 1$ such that

$$\varphi(f) = \lim_\alpha \sum_{j=1}^{n(\alpha)} f(x_j^\alpha) \cdot \lambda_j^\alpha.$$

Proof. The Banach space $C(K)$ embeds into the Banach space $\ell_\infty(K)$ (view this as a discrete index set, no topology) isometrically via $\iota(f)(k) = f(k)$.

$$\begin{array}{ccc} C(K) & \xhookrightarrow{\iota} & \ell_\infty(K) \\ \varphi \downarrow & \swarrow \text{---} \exists \hat{\varphi} & \\ \mathbb{K} & & \end{array}$$

As previous seen, $\hat{\varphi}$ exists by Hahn-Banach extension. Also have $\hat{\varphi}(1) = 1$, $\|\hat{\varphi}\| = 1$ and then $\hat{\varphi} \in S(\ell_\infty(K))$. By previous remark, and also the fact that every function in ℓ_1 is support on a countable number of points

$$\hat{\varphi}(F) = \lim_\alpha \sum_{(t_j)} F((t_j^\alpha)) \cdot \lambda_j^\alpha$$

where $\sum_{\alpha} \lambda_j^{\alpha} = 1$. Can replace LHS of this equation by $\lim_{\alpha} \lim_M \sum_{j=1}^M \lambda_j^{\alpha, M} \cdot F(t_j^{\alpha})$ with $\sum_{j=1}^M \lambda_j^{\alpha, M} = 1$ (technical detail skipped). But

$$F = \iota(f) = \lim_{\alpha'} \sum_{j=1}^{M(\alpha')} \lambda_j^{\alpha'} \cdot f(t_j^{\alpha'}).$$

□

Consider $C[0, 1]$. It is separable (admits a countable dense subset), whereas $\ell_{\infty}(\mathbb{N})$ is non-separable.

Corollary 14.10. *If $\varphi : C[0, 1] \rightarrow \mathbb{C}$, with $\varphi(1) = 1$ and $\|\varphi\| = 1$. Then there exists a sequence $(t_j^n)(\lambda_j^n)$, where $\sum \lambda_j^n = 1$ such that*

$$\varphi(f) = \lim_{\alpha} \sum_{j=1}^{M(n)} f(x_j^n) \cdot \lambda_j^n.$$

15 Riesz-Thorin Theorem 20210301

Theorem 15.1 (Riesz-Thorin). *Let A be a linear operator and let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ where $p_0 \neq p_1$ and $q_0 \neq q_1$. Suppose $A : L_{p_0} \rightarrow L_{q_0}$ is bounded and $A : L_{p_1} \rightarrow L_{q_1}$ is bounded. Let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

where $\theta \in (0, 1)$. Then

$$\|A\|_{L_p \rightarrow L_q} \leq \|A\|_{L_{p_0} \rightarrow L_{q_0}}^{1-\theta} \cdot \|A\|_{L_{p_1} \rightarrow L_{q_1}}^{\theta}.$$

If we call $\|A\|_{L_{p_0} \rightarrow L_{q_0}}^{1-\theta} = M_0$ and $\|A\|_{L_{p_1} \rightarrow L_{q_1}}^{\theta} = M_1$, then $\|A\|_{L_p \rightarrow L_q} \leq M_0^{1-\theta} \cdot M_1^{\theta}$.

In class L_p is replaced with ℓ_p , but there is a more generalized version in literature. I leave L_p in the Theorem to remind myself this fact. For $1 < p < q < r < \infty$, $L_p \cap L_q \subset L_r \subset L_p + L_q$. In our case (finite dimensional), the same matrix makes sense and $A : \ell_{p_0} \cap \ell_{p_1} \rightarrow \ell_{q_0} + \ell_{q_1}$.

We will use the following lemma to prove Riesz-Thorin Theorem.

Lemma 15.2 (Hadamard's Three-Line Theorem). *Suppose $f(z)$ is bounded and continuous function on $0 \leq \operatorname{Re}(z) \leq 1$ and analytic in the interior. Denote*

$$M_\theta = \sup_{y \in \mathbb{R}} |f(\theta + iy)|.$$

Then $M_\theta \leq M_0^{1-\theta} M_1^\theta$ for $\theta \in (0, 1)$.

If we control the function on boundary then we control the function in the interior.

Example 15.3. Map from a strip to a disk. Let $f(z) = \sum a_n z^n$ be an analytic function, $a_0 = f(0) = \frac{1}{2\pi i} \int \frac{f(z)}{z} dz$. Then

$$|a_0| \leq \int |f(z)| dz = \frac{1}{2\pi i} \int f(e^{i\theta}) d\theta \leq \sup |f(e^{i\theta})|.$$

Proof. [Ref.](#)

Recall $\ell_p \hookrightarrow \ell_{p'}^*$ isometrically. So

$$\|A\|_{\ell_p \rightarrow \ell_q} = \sup \left\{ \sum_{kj} y_j \cdot A_{jk} \cdot x_k \mid \sum |x_i|^p \leq 1, \sum |y_j|^{q'} \leq 1 \right\}.$$

Assume $\sum |x_i|^p = 1$ and $\sum |y_j|^{q'} = 1$. Define a function

$$x_k(z) = \frac{x_k}{|x_k|} |x_k|^{p \cdot \left(\frac{1-z}{p_0} + \frac{z}{p_1}\right)} \quad \text{and} \quad y_j(z) = \frac{y_j}{|y_j|} |y_j|^{q' \cdot \left(\frac{1-z}{q_0'} + \frac{z}{q_1'}\right)}.$$

Then $F(z) = \sum_{jk} y_j(z) \cdot A_{jk} \cdot x_k(z)$ is also analytic. Take $0 \leq \operatorname{Re}(z) \leq 1$ and define $G(z) = M_0^{z-1} M_1^{-z} F(z)$.

Claim 15.4. $|G(it)| \leq 1$ and $|G(1+it)| \leq 1$.

Take $z = it$, then

$$\begin{aligned}
G(it) &= \sum_{jk} \frac{x_k}{|x_k|} |x_k|^{p \cdot \left(\frac{1-it}{p_0} + \frac{it}{p_1}\right)} \cdot A_{jk} \cdot \frac{y_j}{|y_j|} |y_j|^{q' \cdot \left(\frac{1-it}{q_0'} + \frac{it}{q_1'}\right)} \\
&= \sum_{jk} \alpha_k |x_k|^{\frac{p}{p_0}} \cdot A_{jk} \cdot \beta_j |y_j|^{\frac{q'}{q_0'}} \\
&= \|A\|_{\ell_{p_0} \rightarrow \ell_{q_0'}} \cdot \left\| \sum_{jk} \alpha_k |x_k|^{\frac{p}{p_0}} \right\|^{\frac{p_0}{p}} \cdot \left\| \sum_{jk} \beta_j |y_j|^{\frac{q'}{q_0'}} \right\|^{\frac{q_0'}{q'}} \leq 1,
\end{aligned}$$

where $|\alpha_k|, |\beta_k| = 1$ (???). Similarly for $G(1 + it)$.

The Three-Line Lemma gives $|G(\theta)| \leq 1$. Note that

$$\begin{aligned}
G(\theta) &= M_0^{\theta-1} M_1^{-\theta} \sum_{jk} \frac{x_k}{|x_k|} |x_k|^{p \cdot \left(\frac{1-\theta}{p_0} + \frac{\theta}{p_1}\right)} \cdot A_{jk} \cdot \frac{y_j}{|y_j|} |y_j|^{q' \cdot \left(\frac{1-\theta}{q_0'} + \frac{\theta}{q_1'}\right)} \\
&= M_0^{\theta-1} M_1^{-\theta} \sum_{jk} \frac{x_k}{|x_k|} |x_k|^{p \cdot \frac{1}{p}} \cdot A_{jk} \cdot \frac{y_j}{|y_j|} |y_j|^{q' \cdot \frac{1}{q'}} \\
&= M_0^{\theta-1} M_1^{-\theta} \sum_{jk} x_k \cdot A_{jk} \cdot y_j.
\end{aligned}$$

This implies $|\sum_{jk} x_k \cdot A_{jk} \cdot y_j| \leq M_0^{1-\theta} M_1^\theta$. □

Corollary 15.5. Assume x, y are complex numbers and $r \leq s \leq r'$ then

$$(|x + y|^r + |x - y|^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (|x|^s + |y|^s)^{\frac{1}{s}}.$$

Example 15.6. When $r = 2$, $x, y \in \mathbb{R}$, then we get the parallelogram law

$$(|x + y|^2 + |x - y|^2)^{\frac{1}{2}} = (x^2 + 2xy + y^2 + x^2 - 2xy + y^2)^{\frac{1}{2}} = \sqrt{2} \cdot (x^2 + y^2)^{\frac{1}{2}}.$$

Proof. Take the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = (|x + y|^r + |x - y|^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (|x|^s + |y|^s)^{\frac{1}{s}}$$

For the case $s \geq 2$,

$$\|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2} = \sup \left\{ \max(|x+y|, |x-y|) \mid |x| \leq 1, |y| \leq 1 \right\} \leq 2.$$

$$\|A\|_{\ell_2^2 \rightarrow \ell_2^2} \leq (|x+y|^2 + |x-y|^2)^{\frac{1}{2}} = \sqrt{2} \cdot (x^2 + y^2)^{\frac{1}{2}} \leq \sqrt{2}.$$

Using Riesz-Thorin Theorem we obtain

$$\|A\|_{\ell_s^2 \rightarrow \ell_s^2} \leq 2^{1-\theta} \cdot \sqrt{2}^\theta = 2^{1-\frac{\theta}{2}} = 2^{1-\frac{1}{s}},$$

with the last step given by $\frac{1}{s} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$.

For $1 \leq s \leq 2$, we note that $r \leq s \leq r'$ implies $s' \leq r$. It suffices to consider $r = s'$. Again Riesz-Thorin Theorem gives

$$\|A\|_{\ell_s \rightarrow \ell_s} \leq \|A\|_{\ell_1 \rightarrow \ell_\infty}^{1-\theta} \cdot \|A\|_{\ell_2 \rightarrow \ell_2}^\theta \leq 1^{1-\theta} \cdot \sqrt{2}^\theta = 2^{\frac{\theta}{s'}} = 2^{1-\frac{1}{s}},$$

with $\frac{1}{s} = \frac{1-\theta}{1} + \frac{\theta}{2}$ and $\frac{1}{s'} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$. Note that

$$\|A\|_{\ell_1 \rightarrow \ell_\infty} = \max_{jk} |A_{jk}|.$$

□

16 Clarkson's Inequality 20210303

Clarkson's inequality \implies Uniform convexity $\implies L_p$ is reflexive $\implies L_p^* = L_{p'}$

We want to use the Clarkson's inequality (proof ref. Boa) to prove uniform convexity of L_p .

Theorem 16.1 (Reformulation of Riesz-Thorin). *Let A be a matrix. Consider $F(x, y) = \log \|A\|_{\ell_{1/x} \rightarrow \ell_{1/y}}$. Then F is a convex function.*

Now let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} : \ell_p^2(\mathbb{C}) \rightarrow \ell_q^2(\mathbb{C})$. Thus $\|A\|_{\ell_s \rightarrow \ell_r} \leq 2^{1-\frac{1}{s}}$ for all $s \leq r \leq s'$.

We have seen

- $\|A\|_{\ell_2^2 \rightarrow \ell_2^2} \leq \sqrt{2}$, and $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ is unitary (preserves inner product).
- $\|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2} = 2$.
- $\|A\|_{\ell_1^2 \rightarrow \ell_\infty^2} = 1$.

Remark 16.2. $\|A\|_{\ell_p^2 \rightarrow \ell_q^2} = \|A\|_{\ell_{q'}^2 \rightarrow \ell_{p'}^2}$.

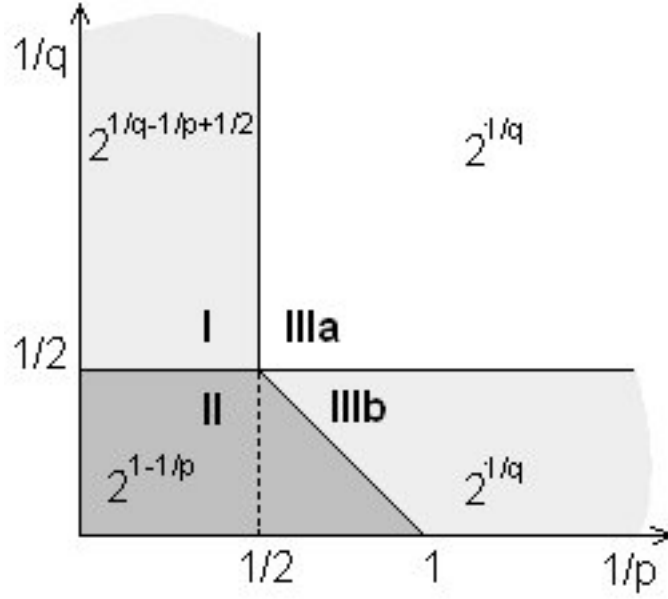


Figure 1: Picture taken from [here](#), p.1366

Explanation of the picture: by the value at a point, I mean the power of 2. (If I call the value α , then 2^α is an upper bound for $\|A\|_{\ell_p^2 \rightarrow \ell_q^2}$.)

- (Region III) The point $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, \frac{1}{2})$ corresponds to $\|A\|_{\ell_2^2 \rightarrow \ell_2^2}$ and has value $\log_2(\sqrt{2}) = \frac{1}{2}$.
- (Region IIIa) The point $(\frac{1}{p}, \frac{1}{q}) = (1, 1)$ corresponds to $\|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2}$ and has value $\log_2(2) = 1$. By the remark above $\|A\|_{\ell_1^2 \rightarrow \ell_1^2} = \|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2} = 2$, so the point $(\frac{1}{p}, \frac{1}{q}) = (1, 1)$ also has value 1.
- (Region IIIa) Using convexity, for $2 < p < \infty$, point $(\frac{1}{p}, \frac{1}{q})$ on the line $y = x$ has value $\frac{1}{q}$.

- (Region IIIb) The point $(\frac{1}{p}, \frac{1}{q}) = (1, \infty)$ corresponds to $\|A\|_{\ell_\infty^2 \rightarrow \ell_\infty^2}$ and has value $\log_2(1) = 0$.
- (Region IIIb) Using convexity, for $2 < p < \infty$, point $(\frac{1}{p}, \frac{1}{q})$ on the line $y = 1 - x$ has value $\frac{1}{q}$. Vertical lines between the lines $y = x$ and $y = 1 - x$ has value $\frac{1}{q}$.
- (Region II) For $1 \leq s \leq 2$ we have

$$\|A\|_{\ell_s \rightarrow \ell_{s'}} \leq \|A\|_{\ell_1 \rightarrow \ell_\infty}^{1-\theta} \cdot \|A\|_{\ell_2 \rightarrow \ell_2}^\theta \leq 2^{1-\frac{1}{s}} = 2^{\frac{1}{s'}}.$$

So $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{s}, \frac{1}{s'})$ has value $1 - \frac{1}{s}$.

$$\|A\|_{\ell_{s'} \rightarrow \ell_{s'}} = \|A\|_{\ell_s \rightarrow \ell_s} \leq \|A\|_{\ell_1 \rightarrow \ell_\infty}^{1-\theta} \cdot \|A\|_{\ell_2 \rightarrow \ell_2}^\theta \leq 2^{1-\frac{1}{s}}.$$

For $s \leq r \leq s'$ (???)

$$\|A\|_{\ell_s \rightarrow \ell_r} = \|A\|_{\ell_s \rightarrow \ell_s}^{1-\theta} \cdot \|A\|_{\ell_s \rightarrow \ell_{s'}}^\theta \leq (2^{1-\frac{1}{s}})^{1-\theta} \cdot (2^{1-\frac{1}{s'}})^\theta = 2^{1-\frac{1}{s'}}.$$

Theorem 16.3 (Minkowski's inequality). *Let $L_p(\ell_q)$ and $\ell_q(L_p)$ be the space of functions with the norm*

$$\begin{aligned} \|f\|_{L_p(\ell_q)} &= \left(\int \left(\sum_k |f_k(\omega)|^q \right)^{\frac{p}{q}} d\mu \right)^{\frac{1}{p}}, \\ \|f\|_{\ell_q(L_p)} &= \left(\sum_k \left(\int |f_k(\omega)|^p d\mu \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}. \end{aligned}$$

If $p \leq q$, then $L_p(\ell_q) \subset \ell_q(L_p)$ and $\ell_p(L_q) \subset L_q(\ell_p)$.

Proof. We want to show $\|f\|_{\ell_q(L_p)} \leq \|f\|_{L_p(\ell_q)}$, i.e.

$$\left(\sum_k \left(\int |f_k(\omega)|^p d\mu \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq \left(\int \left(\sum_k |f_k(\omega)|^p \right)^{\frac{q}{p}} d\mu \right)^{\frac{1}{p}}.$$

Let $p \leq q$ and $r = \frac{q}{p} \geq 1$. The continuous version of triangle inequality says $\|\int g d\mu\|_r \leq \int \|g\|_r d\mu$. (Prove this first for simple function and approximation.) Define $g(\omega) = |f_k(\omega)|^q$, then

$$\left\| \int g(\omega) d\mu \right\|_{\ell_r} \leq \int \|g(\omega)\|_{\ell_r} d\mu$$

By definition of $\|\cdot\|_{\ell_r}$

$$\left(\sum_k \left(\int |f_k(\omega)|^p d\mu \right)^r \right)^{\frac{1}{r}} \leq \int \left(\sum_k |f_k(\omega)|^{pr} \right)^{\frac{1}{r}} d\mu,$$

so

$$\left(\sum_k \left(\int |f_k(\omega)|^q d\mu \right)^{\frac{q}{p}} \right)^{\frac{p}{q}} \leq \int \left(\sum_k |f_k(\omega)|^q \right)^{\frac{p}{q}} d\mu.$$

Taking q -th root on both sides gives the first inclusion. The second inclusion is proved using triangle inequality in ℓ_p . \square

17 Uniform convexity of L_p 20210305

Generalize the scalar valued inequality to function valued inequality.

Theorem 17.1. *For $f, g \in L_p$ and $r \leq p \leq s$ then*

$$(\|f + g\|_p^r + \|f - g\|_p^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (\|f\|_p^s + \|g\|_p^s)^{\frac{1}{s}}.$$

Proof. Recall (Minkowski inequality or generalized Fubini Theorem).

$$L_p(\ell_r) \subset \ell_r(L_p) \quad \text{if } p \leq r \quad \text{and} \quad (2)$$

$$\ell_s(L_p) \subset L_p(\ell_s) \quad \text{if } s \leq p \quad (3)$$

Let $f, g \in L_p(\Omega, \Sigma, \mu)$ then

$$\begin{aligned} LHS &= (\|f + g\|_p^r + \|f - g\|_p^r)^{\frac{1}{r}} \\ &\leq \left(\int |f(\omega) + g(\omega)|^r + |f(\omega) - g(\omega)|^r d\mu \right)^{\frac{1}{r}} \quad (\text{by (2)}) \\ &\leq 2^{1-\frac{1}{s}} \cdot \left(\int |f(\omega)|^s + |g(\omega)|^s d\mu \right)^{\frac{1}{s}} \quad (\text{by Corollary (15.5)}) \\ &\leq 2^{1-\frac{1}{s}} \cdot \left(\int (|f(\omega)|^p)^{\frac{s}{p}} + (|g(\omega)|^p)^{\frac{s}{p}} d\mu \right)^{\frac{1}{s}} = RHS. \quad (\text{by (3)}) \end{aligned}$$

\square

Now we show the above theorem implies uniform convexity.

Theorem 17.2. *The space L_p is uniformly convex for $1 < p < \infty$. In particular, L_p is reflexive.*

We need to show $\forall \epsilon > 0, \exists \delta > 0$ with $\|f\|_p \leq 1, \|g\|_p \leq 1$ and $\|f - g\|_p > \epsilon$ then $\|\frac{f+g}{2}\|_p \leq 1 - \delta$.

Example 17.3. When $p = 2, X = L_2(\Omega, \mathbb{R})$. Fixing $\epsilon > 0$, if we take $\delta = \frac{\epsilon}{8}$ then

$$\begin{aligned} (\|f + g\|_2^2 + \|f - g\|_2^2)^{\frac{1}{2}} &\leq \sqrt{2} \cdot (\|f\|_2^2 + \|g\|_2^2)^{\frac{1}{2}} \leq \sqrt{2} \cdot \sqrt{2} \\ \text{and } \|f + g\|_2^2 + \|f - g\|_2^2 &> \|f + g\|_2^2 + \epsilon^2. \end{aligned}$$

So $\|f + g\|_2^2 + \epsilon^2 \leq 4$, i.e. $\|\frac{f+g}{2}\|_2 \leq \sqrt{1 - \frac{\epsilon^2}{4}} \leq 1 - \frac{\epsilon}{8} = 1 - \delta$.

Proof. Assume $p \geq 2, s = \min(p, p')$ and $r = \max(p, p')$, so that $s \leq p \leq r$. Fixing $\epsilon > 0$, and assume $\|f\|_p \leq 1, \|g\|_p \leq 1$ and $\|f - g\|_p > \epsilon$. Then Theorem 17.1 gives

$$(\|f + g\|_p^r + \|f - g\|_p^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (\|f\|_p^s + \|g\|_p^s)^{\frac{1}{s}} \leq 2^{1-\frac{1}{s}} \cdot 2^{\frac{1}{s}} = 2.$$

Same as previous example

$$\left(\left\| \frac{f+g}{2} \right\|_p^r + \left(\frac{\epsilon}{2} \right)^r \right)^{\frac{1}{r}} < \left(\left\| \frac{f+g}{2} \right\|_p^r + \left\| \frac{f-g}{2} \right\|_p^r \right)^{\frac{1}{r}} \leq 1.$$

So we can choose δ ($\delta = O(\frac{\epsilon}{2})^r$). Note that when $p \rightarrow \infty, (\frac{\epsilon}{2})^r \rightarrow 0$. □

Example 17.4. For $1 < p, q, \infty$, the Sobolov space

$$W_{p,q}^m = \left\{ f \in C(\mathbb{R}) \mid \|f\| = \left(\int \left(\sum_{k=1}^m |f^{(k)}(x)|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty \right\}$$

is uniformly convex. Uniformly convex and reflexive properties pass to subspaces. (Uniform convexity is a property of two points.) We can embed $W_{p,q}^m$ into $L_p(\ell_q^m) = Y$ and show Y is uniformly convex.

Our goal is to find r, s so that

$$(\|F + G\|_Y^r + \|F - G\|_Y^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (\|F\|_Y^s + \|G\|_Y^s)^{\frac{1}{s}}.$$

We need the inclusions $L_p(\ell_q(\ell_r)) \subset \ell_r(L_p(\ell_q))$ and $L_s(\ell_p(\ell_q)) \subset L_p(\ell_q(\ell_s))$. These require $p, q \leq r$ and $s \leq p, q$. Hence $s = \min(p, q, p', q')$ and $r = \max(p, q, p', q')$. Check this gives the above inequality.

18 Uniform Boundedness and Open Mapping 20210308

Theorem 18.1 (Uniform boundedness principle). *Let X be a Banach space and Y a normed vector space. Suppose that \mathcal{F} is a collection of continuous linear operators from X to Y . If \mathcal{F} is pointwise bounded:*

$$\sup_{T \in \mathcal{F}} \|T(x)\|_Y < \infty, \forall x \in X$$

then \mathcal{F} is norm-bounded:

$$\sup_{T \in \mathcal{F}} \|T\|_{B(X,Y)} = \sup_{T \in \mathcal{F}, \|x\|=1} \|T(x)\|_Y < \infty.$$

Application: If $\{T_n\} \subset L(X, Y)$ is a sequence such that $\lim_n T_n(x) = y$ exists for all x , then $\sup_n \|T_n\| < \infty$.

Proof. [Ref.](#) Let \mathcal{F} be a family and start with a subset (not a subspace)

$$X_n = \{x \mid \sup_{T \in \mathcal{F}} \|Tx\| \leq n\} \subset X.$$

Claim 18.2. X_n is closed.

Assume $x_\alpha \rightarrow x$ and we have $\|Tx_\alpha\| \leq n$ for all α and $T \in \mathcal{F}$. Since T is continuous, $\lim \|Tx\| = \lim \sup_\alpha \|Tx_\alpha\| \leq n$ (not clear what the first limit is taking with respect to), and

$$\|Tx\| = \|T \lim_\alpha x_\alpha\| = \lim_\alpha \|Tx_\alpha\| \leq \limsup_\alpha \|Tx_\alpha\| \leq n.$$

Note that $\cup_n X_n = X$ by assumption, and $X_1 \subset X_2 \subset \cdots \subset X_n$.

Assume that the $\text{int}(X_n) = \emptyset$ for all n , then $O_n = X_n^c$ is dense for all n . Baire's Category Theorem gives $\cap_n O_n$ is dense, in particular nonempty. But $\cap_n O_n = (\cup_n X_n)^c = \emptyset$ gives a contradiction. So there exists n such that $\text{int}(X_n) \neq \emptyset$.

Take $x_0 \in X$, $\delta > 0$ and $\|y\| < \delta$ be such that make $B_\delta(x_0) \subset X_n$. Then $\|T(x_0 + y)\| \leq n$ for all $T \in \mathcal{F}$. Therefore

$$\|T(y)\| = \left\| \frac{T(x_0 + y) - T(x_0 - y)}{2} \right\| \leq \frac{\|T(x_0 + y)\| + \|T(x_0 - y)\|}{2} \leq n.$$

and

$$\|T(y)\| = \left\| T\left(\frac{y}{\|y\|} \cdot \frac{\delta}{2}\right) \right\| \cdot \frac{2\|y\|}{\delta} \leq n \cdot \frac{2\|y\|}{\delta}$$

implies $\|T\| \leq \frac{2n}{\delta} \implies \sup_{T \in \mathcal{F}} \|T\| \leq \frac{2n}{\delta}$ □

This argument also works for convex maps with values in another space.

A famous example is the following.

Example 18.3. Consider $X = C[-\pi, \pi]$. Define the **truncation of Fourier series**

$$P_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikt}, \text{ where } \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Note that $P_n \in L(X, X)$. Recall in L_2 , $\|f\|_2 = (\sum_k |\hat{f}(k)|^2)^{\frac{1}{2}}$ and $P_n(f) \rightarrow f$ in L_2 .

If we had that $P_n(f) \rightarrow f$ uniformly: $\lim_{n \rightarrow \infty} P_n(f) = f$ for all $f \in X$. That is, if $\{P_n\}$ were pointwise bounded: $\sup_n \|P_n(f)\| < \infty$, then uniform boundedness would imply $\sup_n \|P_n\| < \infty$. We will prove later that $\|P_n\| \geq C(1 + \ln n)$ (see Theorem 19.1 below).

This gives a contradiction. So there exists a continuous f such that $\lim_{n \rightarrow \infty} \sum_{k=-n}^n \hat{f}(k) e^{ikt}$

diverges. Another fact says that the space of trigonometric polynomials $p(t) = \sum_{k=-n}^n a_k e^{ikt}$ are dense, and $P_n(p) \rightarrow p$ uniformly. The partial Fourier series converges almost everywhere.

Let X be a Banach space, $D \subset X$. What does it mean to be bounded? Two answers

1. $\exists r$ such that $D \subset RB_X$
2. D is weakly bounded: $\forall x^* \in X^*$, $x^*(D) \subset (-R_x, R_x)$ (or $\{z \mid |z| \leq R_x\}$ in complex case)

With respect to the weak topology, weak bounded implies norm bounded.

Corollary 18.4. *Let X be a Banach space, $D \subset X$ such that $x^*(D)$ is bounded in \mathbb{K} for all $x^* \in X^*$. Then D is bounded.*

Proof. Let $\mathcal{F} = \{\varphi_x \mid x \in D\} \subset L(X^*, \mathbb{K})$, where $\varphi_x(x^*) = x^*(x)$. We know

$$\sup_{x \in D} \varphi_x(x^*) = \sup_{x \in D} |x^*(x)| < \infty, \forall x^*.$$

Uniform boundedness principle implies $\sup_{x \in D} \|\varphi_x\| \leq C$. Then D is bounded, because

$$\sup_{x \in D} \|x\| = \sup_{x \in D} \sup_{\|x^*\| \leq 1} |x^*(x)| = \sup_{x \in D} \|\varphi_x\| \leq C.$$

□

Theorem 18.5 (Open mapping theorem). *Let X and Y be Banach spaces and $T : X \rightarrow Y$ be linear and surjective. Then T is open.*

Proof. Step 1. Let $\epsilon > 0$ and $Y_n = \overline{T(B_X(0, n\epsilon))}$. Then $Y = \cup_n Y_n$. Uniform boundedness principle implies one of the Y_n 's has nonempty interior. So there exists \tilde{x} and $\delta > 0$ such that, $B_Y(\tilde{x}, \delta) \subset Y_n$. WLOG we can assume $\tilde{x} = 0$, so $B_Y(0, \delta) \subset Y_n$. Hence for some $\delta' > 0$, we have $B_Y(0, \delta) \subset \overline{T(B_X(0, \epsilon))}$. Our goal is to remove this closure.

Step 2. Choose ϵ_k so that $\sum \epsilon_k < \epsilon$. According to the previous step, we know that there exists δ_k such that $B_Y(0, \delta_k) \subset \overline{T(B_X(0, \epsilon_k))}$ for all k . WLOG we can assume $\delta_k \rightarrow 0$ because we can always take smaller value for δ 's.

Now let $y \in Y$ with $\|y\| < \delta_0$. Since $B_Y(0, \delta_0) \subset \overline{T(B_X(0, \epsilon_0))}$ we can find x_0 in $B_X(0, \epsilon_0)$ such that $\|y - T(x_0)\| < \delta_1$. Call $y_1 = y - T(x_0)$. Then we can find x_1 in $B_X(0, \epsilon_1)$ such that $\|y - T(x_0) - T(x_1)\| = \|y_1 - T(x_1)\| < \delta_2$. Iterate this step and we have a sequence of x_k such that

$$\|y - T(x_0) - T(x_1) - \dots - T(x_k)\| < \delta_k. \quad (4)$$

Since $\delta_k \rightarrow 0$, $y = \sum_k T(x_k)$ by construction. Moreover $\sum_k x_k$ converges to some point $x \in X$ because $\|\sum_k x_k\| \leq \sum_k \|x_k\| \leq \sum_k \epsilon_k < \epsilon < \infty$ (completeness of the Banach space). Note that $\|x\| < \epsilon$ and passing limit of inequality (4) gives $\|y - T(x)\| = 0$. So $y = T(x) \in T(B_X(0, \epsilon))$. This proves for all ϵ , there exists δ such that $B_Y(0, \delta) \subset T(B_X(0, \epsilon))$. (**trick** Write x and y as converging sequences and use the completeness of Banach spaces).

Step 3. Take O an open set. □

Example 18.6.

- There exists a map $T : \ell_\infty \rightarrow \ell_2$ which is linear and onto.
- There is no map $T : \ell_\infty \rightarrow \ell_{4/3}$.

19 $\|P_n\|$ is Unbounded 20210310

Theorem 19.1. Let $X = C[-\pi, \pi]$ and let

$$P_n(f) = \sum_{k=-n}^n \hat{f}(k) e^{ikt}, \text{ where } \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

Then $\|P_n\| \geq C(1 + \ln n)$.

Lemma 19.2. *If $T : C(K) \rightarrow C(K)$, then*

$$\|T\| = \sup_{x \in K} \|T^*(\delta_x)\|_{C(K)^*},$$

where $\delta_x \in C(K)^*$ is defined by $\delta_x(f) = f(x)$.

Proof. Certainly

$$\|T\| = \|T^*\| = \sup_{\varphi \in C(K)^*, \|\varphi\| \leq 1} \|T^*(\varphi)\| \geq \sup_{x \in K} \|T^*(\delta_x)\|.$$

It remains to show “ \leq ”.

Step 1. Take $\varphi = \sum_x \alpha_x \delta_x$, we first prove $\|\varphi\| = \sum_x |\alpha_x|$. One one hand $\|\varphi\| \leq \sum_x |\alpha_x|$ because

$$|\varphi(f)| = \left| \sum_x \alpha_x f(x) \right| \leq \sum_x |\alpha_x| \cdot |f(x)| \leq \sum_x |\alpha_x| \cdot \|f\|_\infty.$$

To show the other direction, we need to find $\tilde{f}(x_j) = \epsilon_j$, with $|\epsilon_j| = 1$, for a compact topological space K . Recall Urysohn’s lemma.

Lemma 19.3 (Urysohn’s lemma). *A topological space (X, τ) is normal if and only if for every pair of disjoint nonempty closed subsets $C, D \subset X$ there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $x \in D$.*

More generally, for $O_i \subset X$ disjoint open subsets, and $x_i \in O_i$, we can find a function positive function $f \in C(K)$ such that $f_i(x_i) = 1$, $\text{supp } f_i \subset O_i$ and $\sum_i f_i = 1$. Here we only need $K = [0, 1]$, $f_i(x_i) = 1$, $\text{supp } f_i \subset O_i$ and $\sum_i f_i \leq 1$.

Define $\tilde{f}(x) = \sum_j \epsilon_j f_j(x)$, with $|\epsilon_j| = 1$. Then

$$|\varphi(\tilde{f})| = \left| \sum_j \alpha_j \delta_{x_j} \right| = \left| \sum_j \epsilon_j \alpha_j f_j(x_j) \right| = \sum_j |\epsilon_j| \cdot |\alpha_j| = \sum_j |\alpha_j|.$$

Existence of such an \tilde{f} gives $\|\varphi\| \geq \sum_j |\alpha_j|$. This shows that for disjoint x_j 's, $\left\| \sum_j \alpha_j \delta_{x_j} \right\| = \sum_j |\alpha_j|$.

Step 2. For an arbitrary $\varphi \in C(K)^*$. Recall we have the following extension

$$\begin{array}{ccc} C(K) & \xhookrightarrow{\iota} & \ell_\infty(K) \\ \varphi \downarrow & \swarrow \exists \hat{\varphi} & \\ \mathbb{K} & & \end{array}$$

Then there exists a family $\{\varphi_\alpha\} \subset \ell_\infty(K)$ with

$$\varphi_\alpha(f) = \sum_j \lambda_j(\alpha) f(x_j) \quad \text{and} \quad \|\varphi_\alpha\|_\infty = \sum_j |\lambda_j(\alpha)| = 1.$$

Denote $\varphi(f) = \lim_\alpha \varphi_\alpha(f)$. Then $\varphi_\alpha \rightarrow \varphi$ in $\sigma(C(K)^*, C(K))$ -topology. This implies for $T : C(K_1) \rightarrow C(K_2)$,

$$\begin{aligned} \|T^*(\varphi)\| &= \sup_{\|f\|_{C(K_1)} \leq 1} |T^*(\varphi)(f)| = \sup_{\|f\|_{C(K_1)} \leq 1} |\varphi(T(f))| \\ &= \sup_{\|f\|_{C(K_1)} \leq 1} \left| \lim_\alpha \varphi_\alpha(T(f)) \right| \leq \sup_{\|f\|_{C(K_1)} \leq 1} \limsup_\alpha |\varphi_\alpha(T(f))|. \end{aligned}$$

Note that

$$\begin{aligned} |\varphi_\alpha(T(f))| &= \left| \sum_j \lambda_j(\alpha) \cdot (T(f))(x_j) \right| = \left| \sum_j \lambda_j(\alpha) \cdot T^*(\delta_{x_j})(f) \right| \\ &\leq \sum_j |\lambda_j(\alpha)| \cdot \|T^*(\delta_{x_j})\| \leq \sum_j |\lambda_j(\alpha)| \cdot \|f\|_\infty \leq \sup_{x_j} \|T(\delta_{x_j})\|. \end{aligned}$$

This gives $\|T\| = \|T^*\| \leq \|T^*(\delta_x)\|$ and thus the equality.

In short, we could use the fact that the convex hull of the δ measures are weak*-dense in the unit ball of $C(K)^*$. \square

Lemma 19.4. *Let $K = [0, 2\pi]$, μ be a measure on K and $F(x, y)$ be a continuous functional in two variables. Define an integral operator $T : C(K) \rightarrow C(K)$ by*

$$T_F(h)(x) = \int_K F(x, y) h(y) d\mu(y).$$

Then $\|T_F\| = \sup_y \int_K |F(x, y)| \, d\mu(x)$.

Proof. We know $\|T_F\| = \sup_x \|T_F^*(\delta_x)\|$ and $T_F^*(\delta_x)(f) = \int_K F(x, y)f(y) \, d\mu(y)$. This is given by integration against $h(y) = F(x, y)$. Note that

$$\|T_F^*(\delta_x)\|_{C(K)^*} = \|h\|_{L^1(\mu)} = \int_K |F(x, y)| \, d\mu(y).$$

□

Often this lemma is used for groups: G is a compact group and μ a measure on G . We can prove the existence of Haar measure, which means there exists μ such that $\int f(gh) \, d\mu(h) = \int f(h) \, d\mu(h)$ for all g . The integral is invariant under translation. In the case of $K = [-\pi, \pi]$, this is the Lebesgue measure λ .

Lemma 19.5. *Let $f : G \rightarrow G$ be a continuous map, and define a translation invariant operator $T : C(G) \rightarrow C(G)$ by*

$$T_{f_1}(f_2)(g) = \int_K f_1(gh^{-1})f_2(h) \, d\mu(h).$$

Then $\|T_{f_1}\| = \int |f_1(h)| \, d\mu(h)$.

The norm does not see translation by g and thus the supremum disappears. In particular, on $K = [-\pi, \pi]$,

$$T_{f_1}(f_2)(s) = \int_K f_1(s-t)f_2(t) \frac{1}{2\pi} \, dt.$$

Proof. Let $F(g, h) = f_1(gh^{-1})$, and T_{f_1} as above. Then the previous lemma and right invariant suggests

$$\|T_{f_1}\| = \sup_g \int_K |f_1(gh^{-1})| \, d\mu(x) = \int_K |f_1(h^{-1})| \, d\mu(x).$$

□

Proof of Theorem 19.1. Consider $P_n(f) = \sum_{k=-n}^n \hat{f}(k)e^{ikt}$, where $\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt} dt$.

By substitution

$$P_n(f) = \sum_{k=-n}^n \hat{f}(k)e^{ikt} = \sum_{k=-n}^n \int_{-\pi}^{\pi} f(s)e^{-iks} \frac{1}{2\pi} ds \cdot e^{ikt} = \int_{-\pi}^{\pi} \sum_{k=-n}^n e^{ik(t-s)} f(s) \frac{1}{2\pi} ds.$$

Thus we take $f_1(s) = \sum_{k=-n}^n e^{ik(t-s)}$. By previous Lemma,

$$\|P_n\| = \int_{-\pi}^{\pi} \left| \sum_{k=-n}^n e^{ik(t-s)} \right| ds.$$

Sum of geometric series gives for $s \neq 0$,

$$\sum_{k=-n}^n e^{iks} = \frac{e^{-ins} - e^{i(n+1)s}}{1 - e^{is}}.$$

Multiplying both sides by $e^{-is/2}$ we get

$$\left| \frac{e^{-i(n+1/2)s} - e^{i(n+1/2)s}}{e^{is/2} - e^{-is/2}} \right| = \left| \frac{\sin((n+1/2)s)}{\sin(s)} \right|.$$

Note that $\sin(s) \sim s$ when $s \sim 0$, and $|\sin(ns)| \sim 1$ when $s \sim \frac{\pi}{2n} + 2l\pi$, $l \in \mathbb{N}$. So there are s_j 's such that on the interval $I_j = \{s \mid |s - s_j| \leq \frac{1}{4\pi n}\}$, $s_j \sim \frac{j\pi}{2n}$ and $|\sin(ns)| \geq \frac{1}{4}$. This implies

$$\int \left| \frac{\sin(ns)}{\sin(s)} \right| ds \geq \sum_{j=1}^n \frac{1}{4} \int_{I_j} d\lambda \frac{n}{j} = \sum_{j=1}^n \frac{1}{4} |I_j| \frac{n}{j} \sim \text{const.} \sum_{j=1}^n \frac{1}{j}.$$

So the integral is unbounded. □

Remark 19.6.

1. $\int_{-\pi}^{\pi} f_n(s) ds = 0$. (f_n should be referring to the oscillating function $\sin(ns)$ but I'm not sure).
2. The kernel $K(t, s) = \sum_{k=-n}^n \overline{h_k(t)} h_k(s)$ appears a lot in solutions of PDEs.

20 Krein–Milman Theorem 20210312

Lecture recording missing. I'll try to type the lecture notes later.

Let X be a locally convex topological space. Recall that a set $C \subset X$ is convex if and only if for all $x, y \in C$, $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda)y \in C$.

Definition 20.1. Let C be convex set. A point $x \in C$ is called **extreme** if $x = \lambda y + (1 - \lambda)z$ with $y, z \in C$ implies $\lambda \in \{0, 1\}$ or $y = z = x$. We denote the set of extreme points of C as $\text{Ext}(C)$.

Warning: the set of extreme points need not to be closed.

Remark 20.2. Let $x_1, x_2, \dots, x_m \in \mathbb{R}^n$. Then

$$\text{Ext}(\{\text{conv}(x_i) \mid 1 \leq i \leq m\}) \subset \{x_1, x_2, \dots, x_m\}.$$

Theorem 20.3 (Krein–Milman Theorem). *Let X be a locally convex topological vector space and let C be a nonempty, convex, compact subset of X . Then C is equal to the closure of the convex hull of the extreme points of C , i.e. $C = \overline{\text{conv}(\text{Ext}(C))}$. In particular, C contains the extreme points.*

21 Krein–Milman Theorem Cont. 20210315

Applications of Krein–Milman Theorem

Example 21.1. Consider $K = \{T \in L(\ell_1^n, \ell_1^n) \mid \|T\| \leq 1\}$. Krein–Milman Theorem gives $K = \overline{\text{conv}(\text{Ext}(K))}$. The map T has an associated matrix $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Compute

$$\|T\| = \sup_{\sum_j |\alpha_j| \leq 1} \sum_i \left| \sum_j a_{ij} \alpha_j \right| \leq \sum_i \sum_j |a_{ij}| \cdot |\alpha_j| \leq \max_j \sum_i |a_{ij}|.$$

This implies $K = B(L(\ell_1^n, \ell_1^n)) = B(\ell_\infty^n(\ell_1^n)) = \prod B(\ell_1^n)$ (the RHS is the product of n copies of $B(\ell_1^n)$). Recall

$$\text{Ext}(K_1, K_2, \dots, K_n) = \{ (x_1, x_2, \dots, x_n) \mid x_j \in \text{Ext}(K_j) \}.$$

So $\text{Ext}(B(\ell_1^n)) = \{ \pm e_k \mid 1 \leq k \leq n \}$ implies

$$\text{Ext}(K) = \{ (\epsilon_1 e_{k_1}, \epsilon_2 e_{k_2}, \dots, \epsilon_n e_{k_n}) \mid k_j \in \{1, \dots, n\} \text{ and } |\epsilon| = 1 \}.$$

Write this as a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

where $a_{ij} \in \{0, \epsilon_j\}$. The extreme points are matrices with exactly one entry of absolute value one in each column (reputation in rows is allowed).

Example 21.2. Let K to be the set of bistochastic matrices. That is,

$$K = \{ A = (a_{ij})_{1 \leq i, j \leq n} \mid a_{ij} \geq 0, \sum_i a_{ij} = 1, \forall i, \text{ and } \sum_j a_{ij} = 1, \forall j \},$$

$$A = \begin{matrix} & \begin{matrix} 1 & 1 & 1 & 1 \end{matrix} \\ \begin{matrix} 1 \\ 1 \\ 1 \\ 1 \end{matrix} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \end{matrix}.$$

These matrices are contained in the set

$$S = \{ A = (a_{ij}) \mid \|T\| \leq 1 \text{ and } \|T^t\| \leq 1 \}.$$

Clearly the identity matrix and more generally all permutation matrices are extreme points of S . By Birkhoff's Theorem (which we will not prove), the extreme points of S are exactly the permutation matrices.

For $n = 2$, there is a nice decomposition for the permutation matrices, namely

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The general case is proved by the Hall's Marriage Theorem.

Example 21.3. Let K be the set of non-increasing convex functions. That is,

$$K = \{ f \mid f' \leq 0, f'' \geq 0 \text{ and } f(0) = 1 \}.$$

The extreme points are in $\text{Ext}(K) = \{ e^{-\lambda x} \mid \lambda \geq 0 \}$.

The prove is not so simple. One needs first to show these exponential functions are extreme points and then use the fact that every function can be written as a convex combination $f(x) = \int g(x, y) e^{-xy} d\mu(y)$ with $\int g(y) d\mu(y) = 1$.

Lemma 21.4 (Contraction lemma). *Let C be a subset of a locally convex and Hausdorff topological space X . Let $x \in \text{int}(C)$, $y \in \partial C$ and $\lambda \in [0, 1)$, then $z = (1 - \lambda)x + \lambda y \in \text{int}(C)$.*

Proof. Assume X is a normed space. (The proof for a general case is similar using semi-norms.) Fix $\lambda \in [0, 1)$ WLOG assume $x = 0$. For any $\tilde{y} \in \text{int}(C)$, there exists $\delta > 0$ so that the set

$$B = (1 - \lambda)0 + \lambda(\tilde{y} + B_\delta) = \lambda\tilde{y} + \lambda B_\delta = \{ \gamma = \lambda(\tilde{y} + a) \mid \|a\| \leq \delta \} \subset C.$$

Now choose a sequence $\tilde{y}_n \rightarrow y$.

Claim 21.5. *There exists $n \in \mathbb{N}$ so that $\lambda y + \lambda \frac{\delta_n}{2} B \subset C$.*

Let $z_n = (1 - \lambda)0 + \lambda\tilde{y}_n$, then

$$\|z - z_n\| = \|\lambda(y - \tilde{y}_n)\| = \lambda\|y - \tilde{y}_n\| \rightarrow 0.$$

So for some n , $\|z - z_n\| < \lambda \frac{\delta_n}{2}$, $z \in z_n + \lambda \frac{\delta_n}{2} B_{\delta_n} \subset C$ and $z + \lambda \frac{\delta_n}{2} B_{\delta_n} \subset z_n + \lambda \delta_n B_{\delta_n} \subset C$.
So $z \in \text{int}(C)$. □

The following proof was done in previous lecture.

Proof of Theorem 20.3

We had

$$\mathcal{F} = \{ L \mid L \text{ nonempty proper convex open subset of } C \}.$$

Step 1. $L_{\max} = L$ in the family \mathcal{F} .

Step 2. Define the affine map $T_\lambda^x(y) = (1 - \lambda)x + \lambda y$ (preserves convex combination). $T_\lambda^x(C) \subset L$ for all $x \in C$ and $\lambda \in [0, 1)$.

- Observe that if L is maximal, then $\bar{L} = C$. Using contradiction: $T_\lambda^x(L) \subset L \implies L \subset (T_\lambda^x)^{-1}(L)$ which is open and convex.
- $T_\lambda^x(\bar{L}) \subset \text{int}(L)$ by contraction lemma.

Step 3. If O is open, then $O \cup L = C$ or $O \subset L$. If $O \subset L$ done. Otherwise $O \cup L$ is open convex. Take $a, b \in O$ then $a\lambda + b(1 - \lambda) \in O \subset O \cup L$, similarly for $a, b \in L$. If $a \in O, b \in L$ and $0 < \lambda < 1$, then $a\lambda + b(1 - \lambda) = T_\lambda^b(a) \in L$ because $a \in C = \bar{L}$. Maximally says $O \cup L$ is not a subset so $O \cup L = C$.

Step 4. $C \setminus L$ has at most one point. Take two disjoint neighborhoods V_1, V_2 and apply the alternative to $L \cup V_1$. If $L \cup V_1 = C$, then $V_2 \subset L \cup V_1$, which gives a contradiction.

Step 5.

Lemma 21.6. *Let $l : X \rightarrow \mathbb{R}$ be a continuous real linear functional. Then*

$$\sup_{x \in C} l(x) = \sup_{x \in \text{Ext}(C)} l(x).$$

Proof. By compactness the above supreme is attained at some $x_0 \in C$. Consider the set $F = \{ x \in C \mid l(x) = l(x_0) \}$. This F is a face, because $C \setminus F = \{ x \in C \mid l(x) < l(x_0) \}$ is open. Also $C \setminus F$ is convex: $x_1, x_2 \in C \setminus F$, then $l(\lambda x_1 + (1 - \lambda)x_2) = \lambda l(x_1) + (1 - \lambda)l(x_2) < l(x_0)$. So $C \setminus F \in \mathcal{F}$. By Zorn's lemma, there exists \tilde{L}_{\max} such that $C \setminus F \subset \tilde{L}_{\max}$ and we know $\tilde{L} = C \setminus \{x_1\}$ for some $x_1 \in \text{Ext}(C)$. This means $x_1 \notin C \setminus F$. So $l(x_1) = l(x_0)$. \square

Now we can finish the proof of Krein-Milman Theorem: $C = \overline{\text{conv}(\text{Ext}(C))}$.

Suppose this does not hold, that is $K := \overline{\text{conv}(\text{Ext}(C))}$ is contained properly in C . Then there exists $x_0 \in C \setminus K$. By Hahn-Banach, there exists a linear functional l so that $l(x_0) > \sup_{x \in K} l(x) \geq \sup_{y \in \text{Ext}(C)} l(y)$. But $l(x_0) = l(x_1)$ for some $x_1 \in \text{Ext}(C)$. This gives a contradiction.

Corollary 21.7. *For a convex continuous function f ,*

$$\sup_{x \in C} f(x) = \sup_{x \in \text{Ext}(C)} f(x).$$

Proof. This is because we can write $x = \lim_{\alpha} \sum_i \lambda_i^{\alpha} x_i$, and estimate

$$f(x) \leq \lim_{\alpha} \sum_i \lambda_i^{\alpha} f(x_i) \leq \sup_{x \in \text{Ext}(C)} f(x).$$

□