MATH541 Functional Analysis, Spring 2021

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Warning: I'm typing the notes slowly. Given that lecture recordings are not uploaded regularly, you can expect no updates for weeks.

Contents

1 Baire's Categroy Theorem 20210125

Ref: A Course in Functional Analysis, John B. Conway, 1985

1. Metric space
2. Chicago suburb distance $\mathbb{R}^b$ compact = closed and bounded no longer true
3. Cauchy sequence, completeness
4. Open, closed ball
5. Nowhere dense set, dense set, closure, interior.

$$Y \text{ is nowhere dense} \iff \bar{Y}^C \text{ is open and dense.}$$

Theorem 1.1 (Baire's theorem)**.** *In a complete metric space, the countable union of nowhere dense sets is again nowhere dense.*

Lemma 1.2. *The intersection of open dense sets is again open dense.*

Using the above lemma + induction to prove Baire's theorem.

Dense, nowhere dense, somewhere dense. Stack Exchange Theorem in notes: countable intersection of open dense is dense, then countable union does not have interior points. Need X complete metric space, so that the limit point is in X.

2 Baire's Categroy Theorem Cont. 20210127

Last time: open set, closed sets, theorem: let (X, d) be a complete metric space, O_n open dense, then $\cap_n O_n$ is dense.

1. intuition dense set $\cong$, taking away a countable set of points
2. proof idea completeness $\to$ geometric series.
3. Use Baire's theorem to show no function $f : [0, 1] \to \mathbb{R}$ continuous exactly at $\mathbb{Q}$

4. proof hard works is to find complete metric space and makes the theorem work

5. Normed space. A normed space is complete if absolute convergent sequences are convergent. Banach space.

6. isometry

7. $\|f(x)\|_{C(K)} = \sup_{k \in K} |f(k)|$.

Question 2.1. Let $C_b(\mathbb{R})$ be the set of continuous and bounded function. Is $C_b(\mathbb{R}) = C(K)$ for some compact K? — Yes.

Want to do: Start with Banach space, create new ones.

Lemma 2.2. *Let $T : X \to Y$ be a linear map between* ***normed spaces****. TFRE*

1. *T is continuous.*
2. *T is continuous at 0.*
3. *$\|T\|_{op} = \sup_{\|x\| \leq 1} \|Tx\|$ is finite.*
4. *T is Lipschitz.*

Homogeneity, duality

Lemma 2.3. *Let X be a normed sapce and Y be Banach. Then the vector space $L(X, Y)$ with the norm $\| \|_{op}$ becomes a Banach space.*

$L(\text{normed}, \text{Banach})$ is Banach.

Corollary 2.4. *$X^* = L(X, \mathbb{C})$ is Banach.*

3 Basic Banach Space Theory 20210129

proof of Lemma 2.3. Step 1. (T_n) Cauchy implies $(T_n(x_k))$ Cauchy.

Step 2. Let $f(x) := \lim T_n(x)$. Prove $\limsup \|T_n(x) - f(x)\| = 0$.

$$\|T_n - T\| = \|T_n - \lim T_m\| = \lim \|T_n - T_m\|$$
$$\leq \limsup_{m,n \geq N} \|T_n - T_m\| < \epsilon$$

$\|T_n - T\| < \epsilon$ implies $\|T_n(x) - T(x)\| < \epsilon$, and so $\limsup \|T_n(x) - f(x)\| = 0$.

Step 3. f is bounded, and $T_n \to f$. □

Corollary 3.1. *X Banach, then $L(X, X) = L(X)$ is Banach algebra.*

Definition 3.2. A **Banach algebra** is a Banach space $(\mathcal{A}), \| \ \|$ together with a product $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, with $\|ab\| \leq \|a\|\|b\|$.

1. closed subset of Banach is Banach.
2. $K(X, Y) := \{ T : X \to Y \mid \overline{T(B_X)} \text{ compact} \}$ is closed .
3. In finite dimension, linear bounded T is compact.

Definition 3.3 (Totally bounded)**.**

$$\forall \epsilon, \exists N \text{ s.t. } Y \subset \bigcup_{j=1}^{N} B(x_j, \epsilon)$$

This is equivalent to relatively compact. Ref

Theorem 3.4.

$$K(H, H)^{**} = B(H, H)$$

(We'll this theorem later.)

Theorem 3.5.

$$\exists \iota : X \to X^{**};\ \iota(x)(f) = f(x),\ \textit{with } f : X \to \mathbb{K}$$

1. *ι is an isometry.*
2. *$\overline{\iota(x)}$ is the completion of X.*

Part 1 follows from Hahn-Banach.

Definition 3.6. (X, d) is a metric space. A **completion** (Y, d') is given by

1. $\iota : X \to Y$ is an isometry.
2. $\iota(X)$ is dense.
3. (Y, d') is complete.

Completion is unique.

4 Basic Banach Space Theory Cont. 20210201

Completion problem: see Theorem 3.5

proof of Theorem 3.5.

Claim 4.1. $\|\iota(x)\|_{X^{**}} \le \|x\|_X$.

Note that

$$\begin{aligned} \|\iota(x)\|_{X^{**}} &= \sup_{\|f(x)\|_{X^*} \le 1} |\iota(x)(f)| && \text{(by definition)} \\ &= \sup_{\|f(x)\|_{X^*} \le 1} |f(x)| && (\iota \text{ inclusion}) \\ &\le \sup_{\|f\|_{X^*} \le 1} \|x\| \le \|x\|. \end{aligned}$$

By definition $\|f\|_{X^*} \le 1 \iff |f(x)| \le \|x\|$. $\square$

For a normed space the completion achieves in X^{**}.

Banach space

Lemma 4.2. *$C_b(x, x_0)$ is a Banach space.*

$$C_b(x, x_0) = \{\, f : X \to \mathbb{R} \mid \text{ continuous and } \exists\, C, |f(x)| \le C d(x, x_0) \,\}.$$

Norm: $\|f\| = \sup_x \frac{|f(x)|}{d(x,x_0)}$.

An embedding isometry $\iota : X \to C_b(X)^*; i(x)(f) = f(x)$. Hint: use evaluation map

$$\sup_{\|f\| \le 1} |f(x) - f(x_0)| = d(x, x_0).$$

Distance attaining function is $f(x) = d(x, x_0)$, where $x \ne x_0$.

Theorem 4.3 (Hahn-Banach Extension)**.** *Given a vector space X, a sublinear map $q : X \to \mathbb{R}$ s.t.*

$$q(x + y) \le q(x) + q(y) \text{ (subadditive) and } q(sx) = sq(x),\ s > 0.$$

Let $Y \subset X$ and $f : Y \to \mathbb{R}$ linear, with $f \le q$, then $\exists F : X \to \mathbb{R}$ linear $F \le q$ and $F|_Y = f$.

warning This theorem is completely algebraic. There is no topology.

Lemma 4.4. *We can always add an extra dimension.*

Proof. Step 1. $Y \subset X = \{\, y + tx_0 \mid t \in \mathbb{R} \,\}$. Candidates for F (extend 1-dim): $F(y + tx_0) = F(y) + tF(x_0) = f(y) + ta_0$ for some a_0. What is a_0? trick

$$
\begin{aligned}
F(y+tx_0) &\le q(y+tx_0) \qquad \Longrightarrow \qquad & f(y)+ta_0 &\le q(y+tx_0)\\
F(y-tx_0) &\le q(y-tx_0) & f(y)-sa_0 &\le q(y-sx_0)
\end{aligned}
$$

$$
\Longrightarrow \quad
\begin{aligned}
a_0 &\le \frac{q(y+tx_0)-f(y)}{t}, t>0\\
a_0 &\ge \frac{f(y)-q(y-sx_0)}{s}, s>0
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
a_0 &\le \inf \frac{q(y+tx_0)-f(y)}{t}, t>0\\
a_0 &\ge \sup \frac{f(y)-q(y-sx_0)}{s}, s>0
\end{aligned}
$$

Check the sup is less than inf:

$$
\begin{aligned}
\frac{f(y)-q(y-sx_0)}{s} &\le \frac{q(z+tx_0)-f(z)}{t}\\
\iff \quad f(y)t-q(y-sx_0)t &\le q(z+tx_0)s-f(z)s\\
f(y)t+f(z)s &\le q(z+tx_0)s+q(y-sx_0)t\\
f(yt+sz) &\le q(yt+tsx_0-tsx_0+sz)\\
&\le q(yt-stx_0)+q(tsx_0+sz)\\
&\le tq(y-sx_0)+sq(tx_0+z)
\end{aligned}
$$

This exactly fits the assumption, so we can pick $a_0 = \sup \frac{f(y)-q(y-sx_0)}{s}$.

Step 2. Use Zorn's lemma. Consider

$$
\mathcal{L} = \{\, (Z,F) \mid Y \subset Z, F \le q \text{ on } Z, F|_Y = f \,\}.
$$

Order on the set: $(Z_1,F_1) \le (Z_2,F_2)$ if $Z_1 \subset Z_2$ and $F_2|Z_1 = F_1$. Every chain has an upper bound $Z_\infty = \cup Z_i, F = \cup F_i$. Hence there exists a maximal element $(Z_{\max}, F_{\max}) \in \mathcal{L}$.

Claim 4.5. *$Z_{max} = X$.*

If not, $\exists x_0 \notin Z_{\max}$ apply lemma to $F_{\max}$, $Z_{\max}+\mathbb{R}x_0$ admits $F'_{\max}$. Contradiction. □

Remark 4.6. Hahn-Banach is also true for $\mathbb{C}$.

5 Hahn-Banach Theorem 20210203

Lemma 5.1. *Take C convex, $0 \in C$. The **Minkowski functional***

$$q_C(x) = \inf\{\, \lambda \mid x \in \lambda C \,\}$$

is sublinear.

Proof. $x, y \in V$. Let $\epsilon > 0$, choose λ, μ s.t. $x \in \lambda C, y \in \mu C$.

$$q_C(x) \le \lambda \le (1+\epsilon)\, q_C(x)$$
$$q_C(y) \le \mu \le (1+\epsilon)\, q_C(y).$$

Then $z = \frac{\lambda}{\lambda+\mu}\frac{x}{\lambda} + \frac{\mu}{\lambda+\mu}\frac{y}{\mu} \in C$. Therefore $x + y = (\lambda+\mu)\left(\frac{x}{\lambda+\mu} + \frac{y}{\lambda+\mu}\right)$. So

$$q_C(x+y) \;\le\; \lambda+\mu \;\le\; (1+\epsilon)\,(q_C(x)+q_C(y)).$$

Send $\epsilon \to 0$. $\square$

Corollary 5.2. *Let C, D be nonempty convex sets $C \cap D = \varnothing$. There there exists $f : V \to \mathbb{R}$ s.t. $f(x) \le f(y)$ for all $x \in C, y \in D$.*

Proof. Take $x_0 \in C, y_0 \in D$. trick Shifting trick: let

$$B := C - D - (x_0 - y_0),$$

where $C - D := \{\, x - y \mid x \in C, y \in D \,\}$. Since $x - y \neq 0$, $y_0 - x_0 \notin B$. Let $Y = \mathbb{R}(y_0 - x_0)$.

Claim 5.3. $q_B(x_0 - y_0) \ge 1$.

Define $f(t(y_0 - x_0)) = t$, then $f \le q_B$. Hahn-Banach extension gives $F : V \to \mathbb{R}$, with $F \le q$ and $F(y_0 - x_0) = 1$. Note that $q_B(x - y - (x_0 - y_0)) \le 1$ implies

$$\begin{aligned} & F(x - y - (x_0 - y_0)) \le 1 \\ \implies\ & F(x-y) - F(x_0 - y_0) \le 1 \\ & \qquad F(x) \le F(y) + 1 - F(y_0 - x_0) = F(y) \end{aligned}$$

□

Theorem 5.4. *For X a normed space and $q(x) = \|x\|$, X subset of complex vector space, $\forall x$ with unit norm, $\exists$ a complex linear functional $f \leq \|\cdot\|$ with $|f(x)| = 1$.*

Proof. Consider X as a real normed space. Take x_0 in X and let $Y = \mathbb{R}x_0 + i\mathbb{R}x_0$, $\|x_0\|$. Define $f(zx_0) = \operatorname{Re}(z)$. Note that $f \leq q$ as

$$f(zx_0) = \operatorname{Re}(z) \leq |z| = \|zx_0\| \leq (zx_0).$$

Then $\exists F : X \to \mathbb{R}$ with $F(x) \leq \|x\|$ real linear and $F(x_0) = 1$.

Fabrication: want to define $G(x) = F(x) - iF(ix)$. If G is complex linear and $F = \operatorname{Re} G$, $G(x) = \operatorname{Re} G(x) + \operatorname{Im} G(x) = F(x) - \operatorname{Re}(iG(x))$.

Claim 5.5. *1. $G(x) = F(x) - iF(ix)$ is complex linear*

2. $|G(x)| \leq \|x\|$

□

6 Hahn-Banach Theorem Cont. 20210205

Theorem 6.1 (Complex version Hahn-Banach)**.** *Let X be a complex vector space. If $f : Y \to \mathbb{C}$ is a complex linear functional on a complex linear subspace $Y \subset X$, and $q : X \to [0, \infty]$ a sublinear function and $q(zx) = q(x)$, $|z| = 1$ (semi-norm). If $|f| \leq q$, then there exists $F : X \to \mathbb{C}$, such that $|F| \leq q, F|_Y = f$*

Proof. Apply the real Hahn-Banach to $\tilde{f} = \operatorname{Re} f$. $\tilde{F} : X \to \mathbb{R}$. Define a new F by

$$\boxed{F(x) = \tilde{F}(x) - i\tilde{F}(ix).}$$

Check F is complex linear. □

Hahn-Banach separation.

Lemma 6.2. *Let C be a convex set and q_C is a Minkowski functional*

1. $x \in C$ *then* $q_C(x) \leq 1$
2. $x \notin C$ *then* $q_C(x) \geq 1$.

$$\{ x \mid q_C(x) < 1 \} \subset X \subset \{ x \mid q_C(x) \leq 1 \}.$$

And the inclusions are strict.

Proof. $q_C(y) = \inf\{ \lambda \mid \frac{y}{\lambda} \in C \}$. For part 1, $x \in C$ so $q_C(x) \leq \lambda = 1$.

For part 2, assume $q_C(x) < 1$, then $\exists \lambda < 1$ such that $\frac{x}{\lambda} \in C$. This (together with convexity) implies

$$x = (1 - \lambda) \cdot 0 + \lambda \cdot \frac{x}{\lambda} \in C,$$

contradiction. □

C may or may not contain the boundary.

1. Topology
2. filter
3. continuous

Definition 6.3. A **filter** on a set X is a subset $\mathcal{F} \subset 2^X$ such that

1. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
2. If $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

It is **nontrivial** if $\forall A \in \mathcal{F}$, $A \neq \emptyset$.

Definition 6.4. A **neighbourhood filter** is a collection $(\mathcal{F}_X)_{x\in X}$ of filters.

Remark 6.5.

1. (Topology $\Rightarrow$ Filter)
   Given topology τ, $\mathcal{F}_X$ is generated by the non-empty open sets.

   $$\mathcal{F}_X = \{ A \subset X \mid \exists O \text{ open }, x \in O \subset A \}.$$

   Neighbourhood filter.

2. (Filter $\Rightarrow$ Topology)
   Given a filter $\mathcal{F}_X$, define O is open iff $\forall x \in O$, $O \in \mathcal{F}_X$. intuition A topology can equivalently be defined by open sets or neighbourhood filters.

Lemma 6.6. $(\tau^{\mathcal{F}})^{\tau} = \tau$.

Definition 6.7. f is **continuous** at x if $\forall B \in \mathcal{F}_{f(x)}, f^{-1}(B) \in \mathcal{F}_X$.

Recall: If $f : X \to Y$ continuous and $K \subset X$ compact, then $f(K)$ compact

Definition 6.8. A space $(X, +, \cdot, \tau)$ is a **topological vector spaces** if

1. $(X, +, \cdot)$ is a vector space
2. $+ : X \times X \to X$ continuous
   $\cdot : \mathbb{K} \times X \to X$ continuous

Example 6.9.

1. $\mathbb{R}^2$ with the Chicago railway metric is not a topological vector space. $+$ not continuous.
2. Let (Ω, Σ, μ) be a measure space. Define

   $$L_0(\Omega, \Sigma, \mu) = \{ f : \Omega \to \mathbb{R} \mid f \text{ measurable}, \mu(|f| > \epsilon) \to 0 \text{ as } \epsilon \to \infty \}.$$

   Define
   $$d(f, 0) := \inf\{ \epsilon \mid \mu(|f| > \epsilon) < \epsilon \},\ d(f, g) = d(f - g, 0).$$

This is a translation invariant metric. Hence a translation invariant topological vector space.

7 Vector space 20210208

1. Topological space
2. Topological vector space $(X, +, \cdot, \tau)$, in particular, the translation map $T_x : X \to X; y \mapsto T_x(y) = x + y$ is a homeomorphism
3. Application to Hahn-Banach
4. Tychonoff's theorem

Motivational lemma

Lemma 7.1. *Let X be a topological vector space, $f : X \to \mathbb{R}$ be a linear nonzero continuous map, then the image of an open convex set is open.*

Proof. If f is linear and O is convex then $f(O)$ is convex. Convex sets of $\mathbb{R}$ is intervals.

Assume $f(O) = (a, b]$ or $[a, b]$. That is there is a $x \in O$, $f(x) = \sup_{y \in O} f(y)$, then $f(x) = b$. Since $f(x_0) \neq 0$ with $f(x_0) = 1$, $(f \neq 0)$, we consider $x(t) = x + tx_0$. Then O open implies there is a t_0, for all $|t| < t_0$, $x + tx_0 \in O$ (translation is continuous). But now

$$f(x + tx_0) = f(x) + tf(x_0) = b + t \cdot 1 > b.$$

Contradiction. $\square$

later Extension is continuous.

Theorem 7.2 (Tychonoff). *For each $j \in J$, let X_j be a topological space. If each X_j is compact, then $X = \prod_{j \in J} X_j$ is compact in the product topology.*

Clarification: $x = (x_i)_{i \in I}$, O is a neighborhood of x if there are i_j, O_j such that $O = \{ (y_i) \mid y_{i_j} \in O_{i_j} \}$.

Example 7.3. Let X_i be a metric space, the index set $I = \mathbb{N}$. Now the following defines a distance of the product topology

$$d((x_n),(y_n)) = \sum_{n \geq 0} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)}.$$

$$\{ (y_n) \mid d((x_n),(y_n)) < \epsilon \} \quad \supset \quad \{ (y_n) \mid \mathrm{dist}(x_j, y_j) < \frac{\epsilon}{2}, j = 1, \cdots n \}.$$

Proof. Assume $d(x_j, y_j) \leq \frac{\epsilon}{2}$ for all j. Then

$$\begin{aligned} d((x_n),(y_n)) &= \sum_{n \geq 0} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} \\ &\leq \sum_{n=1}^{m} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} + \sum_{n > m} 2^{-m} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq \epsilon \end{aligned}$$

(choose large m so that the second term is less than $\frac{\epsilon}{2}$). *Any continuity condition only depends on finitely many terms* $\square$

1. non-trivial filter, filter converges to x, ultra filter

intuition Filter is the analogue of sequence converging to something. They want to being small.

Definition 7.4. We say that **a filter $\mathcal{F}$ converges to** x if $\mathcal{F} \supset \mathcal{N}_x$.

Every neighbourhood is contained in the filter.

Definition 7.5. A maximal non-trivial filter is called a **ultra filter**.

Remark 7.6. Let $\mathcal{U}$ be an ultra filter then for every $A \subset X$, either $A \in \mathcal{U}$ or $A^C \in \mathcal{U}$.

Proof. Fix $A \subset X$.

Case 1 $A \in \mathcal{U}$ done.

Case 2 $A \notin \mathcal{U}$ then $A^C \in \mathcal{U}$. (Prove by contradiction, assume $A^C \notin \mathcal{U}$) Define $\tilde{\mathcal{U}}$ to be the smallest filter which contains A^C and elements in $\mathcal{U}$. (Show $\tilde{\mathcal{U}}$ is again a filter). Indeed this new filter $\tilde{\mathcal{U}}$ is closed by superset. Need to show if $\tilde{A}, \tilde{B} \in \tilde{\mathcal{U}}$ implies $\tilde{A} \cap \tilde{B} \in \tilde{\mathcal{U}}$.

- $\tilde{A}, \tilde{B} \in \mathcal{U}$ done.
- $\tilde{A}, \tilde{B} \supset A^C$ done.
- $\tilde{A} \in \mathcal{U}, \tilde{B} \supset A^C$. We know $\tilde{B} \supset A^C$ implies $\tilde{B}^C \subset A$, and we know $\tilde{A} \neq A$, so $\tilde{A} \cap \tilde{B} = \emptyset$.

Then $\tilde{\mathcal{U}}$ is a filter, contradicting to the fact $\mathcal{U}$ is an ultra filter.

□

Corollary 7.7. *Every ultra filter on an interval converges.*

Lemma 7.8. (X, τ) *is compact iff every ultra filter converges.*

Proof. Ref.

($\Rightarrow$) Let (X, τ) be compact and $\mathcal{U}$ be an ultra filter. Assume $\mathcal{U}$ does not converge to any point. Then $\forall x \in X$, $\mathcal{N}_x \not\subset \mathcal{U}$. Then every point has a neighbourhood O_x which is not in $\mathcal{U}$.

Take the open cover $\cup_x O_x$ of X, O_x as above. By compactness, there is a finite subcover $O_{x_1} \cup \cdots \cup O_{x_n}$. Since $\mathcal{U}$ is an ultra filter, $O_{x_i}^C \in \mathcal{U}$, and the finite intersection of $O_{x_i}^C$'s is in $\mathcal{U}$. But

$$\left(\bigcap_{i=1}^n O_{x_i}^C\right)^C = \bigcup_{i=1}^n O_{x_i} = X$$

implies $\cap_{i=1}^n O_{x_i}^C = \emptyset \in \mathcal{U}$, contradiction.

($\Leftarrow$) Let $X \subset \cup_x O_x$, O_x open. Assume that $X \not\subset \cup_{i=1}^n O_{x_i}$ for any finite subset of indices. Then $\cap_{i=1}^n O_{x_i}^C \neq \emptyset$. Define

$$\mathcal{F} = \left\{ A \,\middle|\, \exists i_1, \cdots, i_n \text{ s.t. } \bigcap_{i=1}^n O_{x_i}^C \subset A \right\}.$$

This is a filter, let $\mathcal{U}$ be the ultra filter contains $\mathcal{F}$. Then $\mathcal{U}$ converges, say to some $x_0 \in X$, then $\mathcal{N}_{x_0} \subset \mathcal{U}$. Then there is a neighbourhood of x_0 which is contained in $\mathcal{U}$, and then $O_x^C \in \mathcal{F} \subset \mathcal{U}$. But $O_x \cap O_x^C = \emptyset$, contradiction. □

proof of Theorem 7.2. Ref.

Let $X = (\prod_i X_i, \tau_i)$, $\mathcal{F}$ be an ultra filter. Let $\pi_i : X \to X_i$ be the projection to the i-th term. Note that $\pi_i(\mathcal{F})$ is also an ultra filter, so it converges to some $x_i \in X_i$. Then $\mathcal{F}$ converges to $(x_i)_{i\in I}$.

Claim 7.9. *Let $x = (x_i)_{i\in I}$, if $O \in \mathcal{N}_x$ then $O \in \mathcal{U}$.*

This means $O \supset O_{i_1} \times \cdots \times O_{i_n} \times X_{j_1} \times X_{j_1} \times \cdots$. Now $\pi_{i_k}^{-1}(O_{i_k}) = W_k$ open and belongs to $\mathcal{U}$, as $O_{i_k} \in \mathcal{U}$. Hence, the finite intersection of W_k's is in $\mathcal{U}$. Then $O \in \mathcal{U}$. □

8 Locally Convex Topological Vector Spaces (LCTVS) 20210210

Recall

1. Topological vector spaces $(X, +, \cdot)$
2. Tychonoff theorem

3. intuition An ultra filter is a generalisation of sequence converging to a point.

Definition 8.1. A topological vector space is called **locally convex** if $\forall x$, $\forall O \in \mathcal{N}_x$, $\exists W$ convex such that $x \in W \subset O$.

Example 8.2.

1. Let X is a normed space, $\mathcal{N}_x = \{ O \mid \exists x > 0, \operatorname{int}(B_r) + x \subset O \}$.
2. Let $X = C^\infty(\mathbb{R})$, K a compact subset, with semi-norm $\|f\|_{K,n} = \sup_{x\in K} \sup_{1\le i\le n} |f^{(i)}(x)|$. (This is a semi-norm because $\operatorname{supp} f$ can be in K^C) The resulting topology is locally convex.

Example 8.3 (Non-examples)**.**

1. $L_0 = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ measurable} \}$, with
$$d(f, 0) = \inf\{ \epsilon \mid \mu(|f| > \epsilon) < \epsilon \}.$$
2. $\|f\|_p = (E|f|^p)^{l/p}$ with $0 < p < 1$. $B_p = \{ f \mid \|f\|_p < 1 \}$. (Cannot put in a convex set if it is infinite dimension). The first example is when $p \to 0$. (E is expectation?)

Theorem 8.4. *Let (X, τ) be a topological space, the following are equivalent.*

1. *X is a Locally convex topological vector spaces (LCTVS)*
2. *$\exists (q_i)_{i\in I}$ of semi-norms on X*
3. *$O \ni N_0$ iff $\exists i$, $\exists r$ s.t. $\{ x \mid q_i(x) < r \} \subset O$.*

intuition The topology is determined by many different shaped balls. Open iff contain one of the balls.

Proof of Theorem 8.4

($\Leftarrow$) Take a point $x \in X$ and O is an open neighbourhood of x. Define a translation map $T_{-x} : X \to X$, by $T_{-x}(y) = y - x$. Note that T_{-x} is a homeomorphism, so

$T_{-x}(O) =: W$ is an open neighbourhood of 0. By (iii), $\exists i$ s.t. $\{ y \mid q_i(y) < 1 \} \subset W$. Define $V = x + \tilde{W} = \{ \tilde{y} \mid q_i(\tilde{y} - x < 1) \} \subset O$.

Definition 8.5. A set $W \ni 0$ is called **absolutely convex** if

$$\sum_{j=1}^{n} |\lambda_j| \leq 1 \implies \sum_{j=1}^{n} \lambda_j x_j \in W.$$

Definition 8.6. A set W is **balanced** if $|z| = 1$, $zw = w$ for all $w \in W$.

Remark 8.7. *W is absolutely convex if W is convex and balanced.*

($\Rightarrow$) Prove existance of seminorms. Take $\mathbb{K} = \mathbb{R}$ let O be open and $\exists W \subset O$ containing 0 and convex. Since $- : X \to X$; $-x \mapsto x$ is continuous, we know $(-)^{-1}(W) \supset V$ is convex, $V \in \mathcal{N}_x$ (Aside: $W \cap -W$ is convex and balanced).

Define $q_V(x) = \inf\{ \lambda \mid \frac{x}{\lambda} \in V \}$

Lemma 8.8. *q_V is a semi-norm.*
That is, $q_V(\lambda x) = |\lambda| q_V(x)$ and subadditive $q_V(x + y) \leq q_V(x) + q_V(y)$.

Then

$$\frac{1}{4} \subset \{ y \mid q_V(y) < \frac{1}{2} \}(\text{ball of some semi-norm}) \subset V.$$

For every neighbourhood of 0 can choose a semi-norm

For $\mathbb{K} = \mathbb{C}$. Want for any set O, find a W which is convex and contained in $\cap_{|z|=1} zO$ (in a intersection of rotations). $(\cap_{|z|=1} zO)^C = \cup_{|z|=1} (zO)^C$.

Question: Is $B = \cup_{|z|=1}(zO)^C$ closed? – Yes. Let $T = \{ z \mid |z| = 1 \}$. The map $T \times X \to X$; $(z, x) \mapsto zx$ is continuous and T is compact.

Lemma 8.9. *B is closed. (A compact translation of a closed set is closed.)*

Proof. Let A be an ordered index set, $x_\alpha \in B$, $x_\alpha \to x$ meaning for a neighbourhood O of x, $\exists \alpha_0$, $\forall \alpha > \alpha_0$, $x \in O$. $\square$

Then $0 \notin B$, and $\exists W \subset \cap_{|z|=1} zO)^C$ convex and $\cap_{|z|=1} zw$ is balanced convex set.

9 Hahn-Banach Separation Theorem 20210212

Lemma 9.1. *Let X, Y be locally convex topological vector spaces. A linear map $T : X \to Y$ is continuous if and only if T is continuous at 0.*

Propersition 9.2. *Let X be a locally convex topological space and $f : X \to \mathbb{R}$ be a linear and continuous map. Let W be an open convex neighbourhood of 0. Then either $f(W) = \{0\}$ or $f(W)$ is open.*

Theorem 9.3 (Hahn-Banach Separation Theorem)**.** *Let C be a **convex nonempty** subset in a topological space X and $\boldsymbol{x \notin C}$, then*

1. *there exists a linear map $f : X \to \mathbb{R}$ such that $f(y) \leq f(x)$, $\forall y \in C$,*
2. *if in addition X is a locally convex topological vector space and C is open, then f is continuous, nontrivial and $f(y) < f(x)$, $\forall y \in C$.*

Proof. (1) Let $x_0 \in C$, then $\tilde{C} = C - \{x_0\}$ contains 0, by Lemma 5.1, the Minkowski functional $q_{\tilde{C}} = \inf\{\, \lambda \mid y \in \lambda\tilde{C}\,\}$ is sublinear. Let $V = \mathbb{R}(x - x_0)$ and define $f(t(x - x_0)) = t$, which is linear. Then $x - x_0 \notin C - \{x_0\}$. By Lemma 6.2, $y \in C$ implies $q_{\tilde{C}}(y - x_0) \leq 1$. Therefore

$$f(y - x_0) \leq f(x - x_0) = 1 \implies f(y) \leq f(x).$$

(2) Now if C is open then $\tilde{C} = C - \{x_0\}$ is open (here we only require a topological space, we don't actually need locally convexity). Consider $g : X \times X \to X$, $g(x, y) = x - y$. This map is continuous, $0 \in \tilde{C}$.

There exists V_1, V_2 neighbourhoods of 0, such that $V_1 - V_2 \subset \tilde{C}$. Define $V = V_1 \cap V_2$ (V is a neighbourhood of 0). Then $0 \in V - V \subset \tilde{C}$. By previous part $f|_{\tilde{C}} \leq 1$. Check the following later Hence

$$f(V - V) \subset f(\tilde{C}) \subset \{\, y \mid f(y) \leq r \,\}.$$

Then for all $y = a - b \in V - V$, $f(y) \leq 1$ and $-y = b - a \in V - V$ so $f(-y) \leq 1$. This means f is bounded. Hence f is continuous at 0. By previous Lemma, f is continuous and $f(\tilde{C})$ is open (image of open convex set is open). Then $f(y - x_0) < 1$ for all $y \in C$. $\square$

Theorem 9.4. *Let C, D be* ***nonempty convex*** *sets. If $C \cap D = \emptyset$, then there is a linear functional f on X such that $f(x) < f(y)$, for all $x \in C, y \in D$.*

Proof. trick Consider $\tilde{C} = C - D = \{\, x - y \mid x \in C, y \in D \,\}$. Note that $\tilde{C}$ is open if either C or D is open, and $0 \notin \tilde{C}$. Now shift the set, i.e. let $\tilde{D} = \tilde{C} - \{(x_0 - y_0)\}$. Apply previous theorem $0 \notin \tilde{C}$, so there exists a $f \neq 0$ and continuous, $f(z) < f(0)$, for all $z \in \tilde{C} = C - D$. Say $z = x - y$, for $x \in C$ and $y \in D$. Then $f(x) < f(y)$. $\square$

Theorem 9.5. *Let C be a* ***closed convex*** *set and D be a* ***compact convex*** *set in a locally convex topological vector space. Then there exists a continuous nontrivial f and $r < s$ such that $f(x) < r < s < f(y)$ for all $x \in D$ and $y \in C$.*

Proof. Assume C is closed and D is compact. C^C is open, $D \cap C = \emptyset$. For any $x \in D$ there is a W_x convex such that $(x + W_x) \cap C = \emptyset$.

Consider the open sets $x + \frac{W_x}{2}$, their union $\cup(x + \frac{W_x}{2})$ gives an open cover of D. Then there is a finite subcover $D \subset \cup_i(x_i + \frac{W_{x_i}}{2})$. Take $W = \cap_i \frac{W_{x_i}}{2}$, and let $y = d + w \in D + W$. Then there exists an x_j such that $d = x_j + \frac{W_{x_j}}{2}$. Therefore,

$$y = d + w \in x_j + \frac{W_{x_j}}{2} + W \subset x_j + \frac{W_{x_j}}{2} + \frac{W_{x_j}}{2} \not\subset C.$$

(Convexity implies $\frac{W_{x_j}}{2}+\frac{W_{x_j}}{2} \subset W_{x_j}$.) analogue Triangle inequality on metric spaces.

Hence we have a strict separation between $D+W$ and C, and we can find a nontrivial continuous f such that $f(x) < f(d+w)$, where $x \in C$, $d \in D$ and $w \in W$. Note that $f(D)$ is compact as D is compact, so $f(D)$ is a closed interval $[a, b]$. Then

$$f(D+W) = f(D) + f(W) = [a, b] + (-\alpha, \beta)$$

($f(W)$ is a neighbourhood of 0 check). So for all $x \in C$,

$$f(x) \leq a - \alpha < a \leq \inf\{\, f(y) \mid y \in D \,\}.$$

$\square$

Example 9.6.

1. Let X be a normed space, and $C = \{\, x \mid \|x\| \leq 1 \,\} = \bar{B_X}$. Take x_0 such that $\|x_0\| > 1$, then $D = \{x_0\}$ compact. There exists f such that $f(x) \leq 1$, $\|f\| \leq 1$ and $f(x_0) > 1$.
2. Take a ball B_X and a triangle D.

Next, we want to make the separation line unique.

10 Weak Topology 20210215

Definition 10.1. Let X be a Banach space and $Y \subset X^*$ a subspace. Then $\boldsymbol{\sigma(X, Y)}$**-topology** is the coarsest topology making all the functional $y \in Y$ continuous. This means the semi-norms defining this topology are given by

$$q_{y_1,\cdots,y_n}(x) = \max_{i=1,\cdots,n} |y_i(x)|.$$

Every locally convex space is given by semi-norms. Semi-norms are indexed by finite subsets of Y.

Theorem 10.2. *The dual space of* $(X, \sigma(X,Y))$ *is* Y *(as a set). That is,*

$$\boxed{(X, \sigma(X,Y))^* = Y.}$$

Note the two spaces only equal as a set, not necessarily as a topological space. Because Y on the LHS can be taken as a algebraic dual without topological assumptions, whereas Y on the RHS is a topological vector space (may with its own norm).

Remark 10.3. Let X be a locally convex topological vector space and Y a Banach space or locally convex topological vector space, then $L(X,Y)$ is also a locally convex topological vector space.

Proof. Step 1. $Y \subset (X, \sigma(X,Y))^*$.

Claim 10.4. *For every* $y \in Y$, $f_y(x) = y(x)$ *is continuous with respect to the new topology.*

It suffice to show f is continuous at 0: $\forall \epsilon$, $\exists O \in \sigma(X,Y)$ containing 0, such that if $x \in O$, then $|f(x)| < \epsilon$. ($f(0) = 0$). In this new topology open neighbourhood means there exists a semi-norm in system such that $O \supset \{x \mid q(x) < \delta\}$, i.e there exists some $B_q(\delta) \subset O$. This is equivalent to say $|f(x)| \leq C \cdot q(x)$, for some semi-norm q. compare In Banach space we don't have a choice of the norm, so we require $|f(x)| \leq C \cdot \|x\|$.

In our case, the semi-norm $q_y(x) = |y(x)|$ does the job, because $|f_y(x)| = |y(x)| = q_y(x)$. More generally, the semi-norm is given by $q_y(x) = \max_j |y_j(x)|$.

Step 2. $(X, \sigma(X,Y))^* \subset Y$.

Let $f : X \to \mathbb{K}$ be continuous. By definition there exists a q such that $|f(x)| \leq q(x)$

and $q(x) = \max_j |y_j(x)|$. Fix $y_1, \cdots, y_n$ and define a map

$$\begin{aligned} \phi : X &\longrightarrow \mathbb{K}^n \\ x &\longmapsto (y_1(x), \cdots, y_n(x)). \end{aligned}$$

Then $\phi(X) \subset \mathbb{K}^n$ is a subspace. Denote $Z = \phi(X)$, then $z = (y_1(x), \cdots, y_n(x))$. Consider the map

$$\begin{aligned} \psi : Z &\longrightarrow \mathbb{K} \\ z &\longmapsto f(x). \end{aligned}$$

This map is well-defined, linear, and $|\psi(z)| \leq \max_j |z_j| = \|z\|_\infty$. By Hahn-Banach, there exists $\tilde{\psi} : l_\infty^m \to \mathbb{K}$, such that $\tilde{\psi}|_Z(z) = \psi(z)$ and $\|\tilde{\psi}\| = \|\psi\| \leq \|z\|_\infty$. Note that $\tilde{\psi}(z) \in (l_\infty^m)^* = l_1^m$. This means there exists $\alpha_1, \cdots, \alpha_n$ such that $\tilde{\psi}(z) = \sum_j \alpha_j z_j$. This means

$$f(x) = \psi(\phi(x)) = \tilde{\psi}(\phi(x)) = \sum_j \alpha_j \phi_j(x) = f_y(x),$$

where $y = \sum_j \alpha_j y_j$. □

Example 10.5. Let X be a space and take $Y = X^*$. Then

- $\sigma(X, X^*)$ is called the **weak topology** of X and $(X, \sigma(X, X^*)) = X^*$,
- $\sigma(X^*, X)$ is called the **weak* topology** of X^* and $(X^*, \sigma(X^*, X)) = X$.

11 Weak Topology cont. 20210219

Theorem 11.1 (Goldstine)**.** *Let X be a Banach space, then the image of the closed unit ball $B_X \subset X$ under the canonical embedding ι into the closed unit ball $B_{X^{**}}$ of the bidual space X^{**} is weak*-dense.*

$$\overline{B_X}^{\sigma(X^{**}, X^*)} = B_{X^{**}}$$

intuition The unit ball with weak*-topology is compact. In finite dimension, close + bounded = compact. Generalisations of finite dimension.

Proof. Recall that X^{**} is a locally convex topological vector space with respect to $\sigma(X^{**}, X^*)$-topology. This topology is given by the semi-norm $q(x^{**}) = \max_j |x^{**}(x_j^*)|$, with $x_1^*, \cdots, x_j^* \in X^*$.

The canonical embedding $\iota : X \to X^{**}$, is an isometry (Hahn-Banach Theorem) and $\iota|_{B_X} : B_X \to B_{X^{**}}$. We want to show the closure $\overline{\iota(B_X)}$ with respect to the $\sigma(X^{**}, X^*)$ topology satisfies $\overline{\iota(B_X)} = B_X^{**}$. Prove by contradiction.

Assume that $x^{**} \notin \overline{\iota(B_X)}$, with $\|x^{**}\|_{X^{**}} \leq 1$. Note that $\overline{\iota(B_X)}$ is closed, compact and convex. By Hahn-Banach separation (Theorem 9.5), there exists a nontrivial continuous map $f : X^{**} \to \mathbb{R}$ so that $|f(\iota(x))| \leq 1 < s < |f(x^{**})|$ for all $x \in B_X$. On one hand we have

$$\|f\|_{X^{**}} = \sup_{\|x\| \leq 1} |f(x^{**})| = \sup_{\|x\| \leq 1} |f(\iota(x))| \leq 1.$$

Then by definition,

$$|x^{**}(f)| \leq \|x^{**}\|_{X^{**}} \cdot \|f\|_{X^{**}} \leq 1.$$

On the other hand we have $|x^{**}(f)| = f(x^{**}) > 1$. Contradiction. □

Example 11.2. Let $X = C_0 = \{ (x_n) \mid \lim_n x_n = 0 \}$, with $\|(x_n)\| = \sup_n \|x_n\|$. Then $X^* = l_1$ because

$$\begin{aligned} \|y_n\|_1 &= \sum_n y_n = \sup_k \sum_{i=1}^k |y_k| \\ &= \sup_k < y, \epsilon_1, \cdots, \epsilon_k, 0, \cdots, 0 > . \end{aligned}$$

where $\epsilon_i = \operatorname{sgn}(y_i)$ and $< y, z >= \sum y_n z_n$. And $X^{**} = l_\infty = \{ (x_n) \mid \sup_n |x_n| < \infty \}$.

What is $\sigma(l_\infty, l_1)$-topology? The answer is pointwise convergence on bounded set. Consider bounded sequences x^α ($\|x^\alpha\| \leq C$). Then $x^\alpha \to x \in l_\infty$ iff for all $y \in l_1$, $x^\alpha(y) \to x(y)$.

For bounded sets $\|x^\alpha\| \leq 1$, $\forall \alpha$,

$$x^\alpha \to x \quad \Longleftrightarrow \quad x_n^\alpha \to x_n, \forall n.$$

($\Rightarrow$) Take $y_n = (0, \cdots, 1, \cdots, 0) \in l_1$.

($\Leftarrow$) Let $y \in l_1$ and $\epsilon > 0$ then there exists n_0 such that $\sum\limits_{n>n_0} |y_n| < \frac{\epsilon}{2}$. There exists α_0 such that any $\alpha > \alpha_0$, $|x_n^\alpha - x_n| < \frac{\epsilon}{2}$ for all $n > n_0$. We need

$$|x^\alpha(y) - x(y)| \leq < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Let $y^N = (y_1, \cdots, y_N, 0, \cdots, 0)$, $y^N \to y$ in $\sigma(l_\infty, l_1)$ because we can use pointwise convergence.

12 Alaoglu's Theorem 20210222

Alaoglu's Theorem says that the closed unit ball in X^* is compact in the weak*-topology.

Theorem 12.1 (Alaoglu). *Given a topological vector space X, and let $B_{X^*} = \{ x^* \in X^* \mid \|x^*\| \leq 1 \}$ be the closed unit ball in X^*. Then B_{X^*} is compact in X^* with respect to the weak*- topology on X^*.*

Proof. Ref. or see Conway p.134

Let the set $D_x = \{ z \in \mathbb{K} \mid |z| \leq 1 \}$. Consider the product $D := \prod_{x \in B_x} D_x$. Since D_x is compact in $\mathbb{K}$, Tychonov's theorem says that D compact in the product topology. Elements in D are functionals, given by $\mu \in K$, $\mu(x) = \mu_x \in D \subset \mathbb{C}$, although they need not to be linear.

The inclusion

$$\iota : B_{X^*} \subset \prod_{x \in B_x} D =: K$$

is given by

$$\iota(x^*)(x) = x^*(x).$$

Note that $\iota(B_{X^*}) \subset K$. Indeed, if $\|x\| \leq 1$ and $\|x^*\| \leq 1$, then $|x^*(x)| \leq 1 \in D$.

Claim 12.2. *$\iota(B_{X^*})$ is closed. Hence, $\iota(B_{X^*})$ is a compact subspace of K.*

Proof of the claim. Take a net (x^*_α) in B_{X^*} which converges to $f \in D$ pointwisely. So $f(x) = \lim_{\alpha\to\infty} x^*_\alpha(x)$. In particular $|f(x)| \leq 1$ for all $\|x\| \leq 1$. (Need to show f is in the range. We can not take $\mathbb{N}$ as index set, instead replacing $\mathbb{N}$ by a partially ordered set. Usually the index set is given by the neighbourhood basis of f. Let $O_i \in \mathcal{N}_f$, $i = 1, 2$, then $O_1 \cap O_2 \in \mathcal{N}_f$ and $O_1 \cap O_2 \geq O_i$.)

For $x \in X$, define $F(x) = \beta^{-1} f(\beta x)$ for some β such that $\|\beta x\| \leq 1$ (check this is well defined). Then F agrees with f on B_X. We claim that F is linear. Take $x_i \in X$, $i = 1, 2$. Consider $y = \frac{x_1+x_2}{\|x_1\|+\|x_2\|}$. If we take $\lambda = \frac{\|x_1\|}{\|x_1\|+\|x_2\|}$, then by convexity $y = \lambda \frac{x_1}{\|x_1\|} + (1-\lambda)\frac{x_2}{\|x_2\|} \in B_X$. Then

$$\begin{aligned} f(y) = \lim_\alpha x^*_\alpha(y) &= \lim_\alpha x^*_\alpha\left(\frac{x_1}{\|x_1\|+\|x_2\|}\right) + x^*_\alpha\left(\frac{x_2}{\|x_1\|+\|x_2\|}\right) \\ &= f\left(\frac{x_1}{\|x_1\|+\|x_2\|}\right) + f\left(\frac{x_2}{\|x_1\|+\|x_2\|}\right). \end{aligned}$$

So

$$\begin{aligned} F(x_1+x_2) &= f(y)\cdot(\|x_1\|+\|x_2\|) \\ &= \left(f\left(\frac{x_1}{\|x_1\|+\|x_2\|}\right) + f\left(\frac{x_2}{\|x_1\|+\|x_2\|}\right)\right)\cdot(\|x_1\|+\|x_2\|) \\ &= F(x_1) + F(x_2). \end{aligned}$$

We have a linear functional $F \in X^*$ satisfying $|F(x)| \leq 1$ when $\|x\| \leq 1$. This means $\|F\|_{X^*} \leq 1$. So $F \in B_{X^*}$ $\square$

Definition 12.3. A Banach space is **reflexive** if $X^{**} = X$.

Goal: to show X is reflexive iff X^* is reflexive.

Propersition 12.4. *A closed subspace of a reflexive Banach space is reflexive.*

Proof. The following diagram is commutative. (Check)

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X^{**} \\ j\big\uparrow & & \big\uparrow j^{**} \\ Y & \xrightarrow{\iota_Y} & Y^{**} \end{array}$$

Step 1. $Y^{**} = Y$. Take an element $y^{**} \in Y^{**}$, note that

$$j^{**}(y^{**})(x^*) = y^{**} \circ j^*(x^*) = y^{**}(x^* \circ j) = x^*|_Y \in Y^*.$$

So we can apply y^{**} to this element, and define $\phi(x^*) = y^{**}(x^*|_Y)$

Lemma 12.5. *If $T : Y \to X$ is isometric, then $T^{**} : Y^{**} \to X^{**}$ is also isometric.*

The above lemma says Y^{**} embeds isometrically into X^{**} (we will prove this later). If in addition, $X^{**} = X$, we deduce that for every y^{**} there exists an $x \in X$ such that

$$y^{**}(x^*|_Y) = x^*(x).$$

We want to show $x \in Y$. We claim that $y^{**} \in Y$, otherwise by Hahn-Banach separation there exists x^* such that $x^*(y^{**}) = 1$ and $x^*|_Y = 0$. The last equation says $x^*(x) = y^{**}(x^*|_Y) = x^*|_Y = 0$. A contradiction (as $y^{**} \in Y^{**} \subset X^{**} = X$). □

Lemma 12.6. *If $T : X \to Y$ is isometric then $T^* : Y^* \to X^*$ sends closed unit ball to closed unit ball.*

Proof. Note that $T^*(B_{Y^*}) \subset B_{X^*}$. Indeed,

$$\begin{aligned} \|T^*\| &= \sup_{\|y^*\|\le 1} \|T^*(y^*)\| = \sup_{\|y^*\|\le 1} \|y^* \circ T\| \\ &= \sup_{\|y^*\|\le 1, |x|\le 1} |y^* \circ T(x)| = \sup_{\|y^*\|\le 1, |x|\le 1} |y^*(x)| \le 1. \end{aligned}$$

So $|T^*(y^*)| \le \|T^*\|\|y^*\| \le 1$.

To show T^* is onto, take $x^* \in B_{X^*}$. Can define $f(Tx) = x^*(x)$, $\|f\| \le 1$. By Hahn-Banach there exists y^* such that $y^*(Tx) = f(Tx) = x^*(x)$. $T^*(y) = x^*$. □

Lemma 12.7. *If $T : Y \to X$ is a surjection, then $T^* : X^* \to Y^*$ is an isometry.*

Proof of the Lemma 12.5. The previous two lemma gives the result. □

13 Reflexive Spaces 20210224

Theorem 13.1. *X is reflexive $\iff$ X^* is reflexive.*

Proof. ($\Rightarrow$) Assume that $X = X^{**}$. Then B_{X^*} is closed in $\sigma(X^*, X^{**}) = \sigma(X^*, X)$. Take an element x^{***} in $B_{X^{***}}$, there exists a sequence $x^*_\alpha \to x^{***}$ in $\sigma(X^{***}, X^{**})$ topology. Since B_{x^*} is closed in $\sigma(X^*, X)$, there is an x^* such that $x^*_\alpha \to x^*$. This means $x^{***} = x^*$.

($\Leftarrow$) If X^* is reflexive then X^{**} is reflexive, but $X \subset X^{**}$ as a closed subspace. □

Remark 13.2. X is reflexive iff B_{X^*} is $\sigma(X^*, X^{**})$ closed.

Definition 13.3. A Banach space is called **uniformly convex**, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| > \epsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Lemma 13.4. *Take (x_n) a sequence with*

$$\limsup_n \|x_n\| \leq 1 \quad \text{and} \quad \lim_n \inf_{m>n} \left\| \frac{x_n + x_m}{2} \right\| = 1.$$

Then (x_n) is Cauchy.

Proof. Let $\epsilon > 0$. Since $\limsup_n \|x_n\| \leq 1$, we can choose $\epsilon_0 > 0$, $\exists n_0$ such that $\|x_n\| \leq 1 + \epsilon_0$, for all $n > n_0$. So $\|\frac{x_n}{1+\epsilon_0}\| \leq 1$, for all $n > n_0$. Then

$$\left\| \frac{x_n + x_m}{2(1+\epsilon_0)} \right\| = \left\| \frac{x_n + x_m}{2} \right\| \cdot \frac{1}{1+\epsilon_0} \geq \frac{1}{(1+\epsilon_0)^2},$$

for all $n > n_0$.

Taking $\frac{1}{(1+\epsilon_0)^2} = 1 - \delta$. Using uniform convexity (contrapositive), we have $\forall n$, $\exists m$

$$\left\| \frac{x_n - x_m}{2(1+\epsilon_0)} \right\| < \epsilon.$$

Conclusion: Above shows $\forall \epsilon$, $\exists n_0$, $\forall n > n_0$, $\exists m$, such that $\|x_n - x_m\| < 2\epsilon(1+\epsilon_0)$.

We use this for $\epsilon = 2^{-k}$, then there exists a converging subsequence x_{n_k} such that $\|x_{n_k} - x_{n_{k+1}}\| \leq 2^{-k}$. $\square$

Theorem 13.5 (Milman-Pettis)**.** *Uniformly convex Banach spaces are reflexive.*

Proof. See: Ref.

Let $x^{**} \in B_{X^{**}}$, $\|x^{**}\| = 1$. Then by definition of $\|x^{**}\|$, for all n, there exists $x_n^* \in B_{X^*}$, such that $x^{**}(x_n^*) \geq 1 - \frac{1}{n}$. Since $B_X \subset B_{X^{**}}$ is dense in $\sigma(X^{**}, X^*)$. Let $q_n(y) = |x_n^*(y)|$. There exists (x_k) in B_X such that

$$|q_n(x^{**} - x_k)| = |x_n^*(x_k) - x^{**}(x_n^*)| \leq \frac{1}{2k}, \text{ for } n = 1, \cdots, k.$$

In particular, apply the above to $n = k$, then

$$|x_k^*(x_k) - x^{**}(x_k^*)| \leq \frac{1}{2k} \implies -\frac{1}{2k} + x^{**}(x_k^*) \leq x_k^*(x_k).$$

Recall $x^{**}(x_k^*) \geq 1 - \frac{1}{k}$. So $1 - \frac{3}{2k} \leq x_n^*(x_k) \leq 1$ (RHS because x_n^* is in the unit ball).

Then take $m > k$, we have

$$2 - \frac{6}{2k} \leq 1 - \frac{3}{2k} + 1 - \frac{3}{2m} \leq x_k^*(x_k) + x_m^*(x_m) \leq x_k^*(x_k + x_m) \leq \|x_k + x_m\| \leq 2. \quad (1)$$

Taking $\liminf$ on both sides we get $\liminf \|\frac{x_k + x_m}{2}\| = 1$, and $\limsup \|x_k\| \leq 1$. By the above lemma (x_n) is Cauchy.

Remark 13.6. Assume there are two sequences x_n, $\tilde{x}_n$ satisfies the property (1), then then $\lim x_n = \lim \tilde{x}_n$.

Now if (y_n^*) is another family using the above construction, then there exists $(\tilde{x}_n)$ in B_X such that

$$|y_n^*(\tilde{x}_k) - x^{**}(x_n^*)| \leq \frac{1}{2k}.$$

Then $x^*(x_k) \to x^{**}(x)$ and $y^*(\tilde{x}_k) \to x^{**}(y)$ implies $x^{**} = \lim x_n = \lim \tilde{x}_n$ in $\sigma(X^{**}, X^*)$. $\square$

14 Reflexive Spaces cont. 20210226

Real analysis: $L_p(\Omega, \Sigma, \mu) = \{ [f] \mid f : \Omega \to \mathbb{K}, f \text{ measurable}, \int |f|^p \,\mathrm{d}\mu < \infty \}$, where Ω is a set, Σ is a σ-algebra and μ is a σ-additive measure. Recall

- Simple functions $f = \sum_{j=1}^n \alpha_j 1_{E_j}$ are dense.

- $\|f\|_p = \sup\limits_{\|g\|_{p'} \leq 1} |\int fg \,\mathrm{d}\mu|$.

  Use Hölder inequality, say $\|f\|_p = 1$, then $g = \mathrm{sgn}(f) \cdot |f|^{p/p'}$.

Corollary 14.1. *If $1 \leq p \leq \infty$, then $L_{p'}$ embeds isometrically into L_p^*,*

$$\begin{aligned} \iota_{p'} : L_{p'} &\to L_p^* \\ g &\mapsto \left(\iota_{p'}(g) : f \mapsto \iota_{p'}(g)(f) = \int fg \,\mathrm{d}\mu \right) \end{aligned}$$

and $\|f\|_p = \|\iota_{p'}(g) : L_p \to \mathbb{K}\|$.

Theorem 14.2. *Let $1 < p < \infty$ and assume L_p is reflexive. Then $L_{p'}^* = L_p$.*

(Here we check isometric isomorphism, there are two type of isomorphisms for Banach spaces, see more here)

Proof. Let $\varphi : L_{p'} \to \mathbb{K}$ with $\|\varphi\|_{L^*_{p'}} = 1$. Recall $L_{p'} \hookrightarrow L^*_p$ is an isometry. By Hahn-Banach extension, there exists a $\hat{\varphi} : L^*_p \to \mathbb{K}$, with $\hat{\varphi}|_{L_{p'}} = \varphi$.

$$\begin{tikzcd} L_{p'} \arrow[r, hook, "\iota_{p'}"] \arrow[d, "\varphi"'] & L^*_p \arrow[dl, dashed, "\exists\hat{\varphi}"] \\ \mathbb{K} & \end{tikzcd}$$

To show $\iota_{p'}$ is surjective, take $\eta \in L^*_p$. If we can find $g \in L_{p'}$ such that $\int fg \,\mathrm{d}\mu = \eta(f)$, then $\iota_{p'}(g) = \eta$ and we are done. Such a g exists by commutativity and reflexivity

$$\varphi(g) = \hat{\varphi}(\iota_{p'}(g)) = \iota_{p'}(g)(f) = \int fg \,\mathrm{d}\mu \quad\implies\quad \iota_{p'}(g)(f) = \eta(f).$$

$\square$

Example 14.3 (Discrete case)**.** Let $\Omega = I$, $\Sigma = 2^I$, μ be the counting measure. If $I = \mathbb{N}$, then

$$L_p(\mathbb{N}, \Sigma, \mu) = \ell_p = \{\, (x_n) \mid \sum_n |x_n|^p < \infty \,\}.$$

What is the f defining the functional $\varphi : \ell(\mathbb{N}) \to \mathbb{K}$? Well, f is given by a sequence $(y_n) = ((0, 0, \cdots, \frac{1}{n}, \cdots, 0, 0))$. One can show that the

$$\|y_n\|_{p'} = \sup_n \left(\sum_{k=1}^{n} |y_k|^{p'} \right)^{\frac{1}{p'}} = \sup\{\, \varphi(x_n) \mid \|x_n\| \leq 1 \,\}.$$

Prove using Hölder.

Theorem 14.4. *$\ell^*_p = \ell_{p'}$ for $1 < p < \infty$.*

Remark 14.5. For $I = \mathbb{N}$, let $c_0 = \{\, (x_n) \in \ell_\infty \mid \lim x_n = 0 \,\}$. Then $c_0^* = \ell^1$, $c_0^{**} = \ell^*_1 = \ell_\infty$.

Corollary 14.6. *$B_{\ell_1} \subset B_{\ell^*_\infty}$ is $\sigma(\ell^*_\infty, \ell_\infty)$-dense.*

This means for any $\varphi \in \ell_\infty^*$, for any $f_i \in \ell_\infty$, there exists $g \in \ell_1$, with $\|g\|_{\ell_1} \leq \|\varphi\|$, such that

$$|\varphi(f_i) - f_j(g)| \leq \epsilon \text{ i.e. arbitrarily closed.}$$

Or there exists a net $(g_\alpha) \in \ell_1$ with $\|g_\alpha\|_{\ell_1} \leq \|\varphi\|$, such that

$$\varphi(f) = \lim_\alpha f(g_\alpha) = \lim_\alpha \sum_{n \in \mathbb{N}} f(n) g_\alpha(n).$$

Remark 14.7. Let $\varphi : \ell_\infty \to \mathbb{K}$, and assume $\varphi(1) = 1$. TFAE

- $\|\varphi\| = 1$

- $\forall g \geq 0$, $\varphi(g) \geq 0$.

We call this **positive functionals**.

Define the **state space** $S(\ell_\infty) = \{ \varphi \mid \varphi(1) = 1, \|\varphi\| = 1 \}$. Then discrete probability measures are dense in the state space. Indeed if $\varphi(1) = 1$ and $\|\varphi\| = 1$, then there is $g_\alpha \in \ell_1$ with $g_\alpha(1) = 1$, $\|g_\alpha\| \leq 1$ and $g_\alpha(f) \to \varphi(f)$. That is $g_\alpha \to \varphi$ in $\sigma(\ell_\infty^*, \ell_\infty)$.

Lemma 14.8. *$\|g_\alpha\|_{\ell_1} = 1$ and $\sum_n g_\alpha(n) = 1$ implies $g_\alpha \geq 0$.*

This means g_α are discrete probability measures because $\varphi(f) = \lim_\alpha \sum_{n \in \mathbb{N}} f(n) g_\alpha(n)$ exists.

Theorem 14.9. *Let be $\varphi : C(K) \to \mathbb{C}$ be such that $\varphi(1) = 1$ and $\|\varphi\| = 1$. Then there exists a net $(x_j)_{j=1}^{n(\alpha)} (\lambda_j^\alpha)_{j=1}^{n(\alpha)}$, where $\sum \lambda_j^\alpha = 1$ such that*

$$\varphi(f) = \lim_\alpha \sum_{j=1}^{n(\alpha)} f(x_j^\alpha) \cdot \lambda_j^\alpha.$$

Proof. The Banach space $C(K)$ embeds into the Banach space $\ell_\infty(K)$ (view this as a discrete index set, no topology) isometrically via $\iota(f)(k) = f(k)$.

$$\begin{tikzcd} C(K) \arrow[r, hook, "\iota"] \arrow[d, "\varphi"'] & \ell_\infty(K) \arrow[dl, dashed, "\exists\hat{\varphi}"] \\ \mathbb{K} & \end{tikzcd}$$

As previous seen, $\hat{\varphi}$ exists by Hahn-Banach extension. Also have $\hat{\varphi}(1) = 1$, $\|\hat{\varphi}\| = 1$ and then $\hat{\varphi} \in S(\ell_\infty(K))$. By previous remark, and also the fact that every function in ℓ_1 is support on a countable number of points

$$\hat{\varphi}(F) = \lim_\alpha \sum_{(t_j)} F((t_j^\alpha)) \cdot \lambda_j^\alpha$$

where $\sum_\alpha \lambda_j^\alpha = 1$. Can replace LHS of this equation by $\lim_\alpha \lim_M \sum_{j=1}^M \lambda_j^{\alpha,M} \cdot F(t_j^\alpha)$ with $\sum_{j=1}^M \lambda_j^{\alpha,M} = 1$ (technical detail skipped). But

$$F = \iota(f) = \lim_{\alpha'} \sum_{j=1}^{M(\alpha')} \lambda_j^{\alpha'} \cdot f(t_j^{\alpha'}).$$

$\square$

Consider $C[0,1]$. It is separable (admits a countable dense subset), whereas $\ell_\infty(\mathbb{N})$ is non-separable.

Corollary 14.10. *If $\varphi : C[0,1] \to \mathbb{C}$, with $\varphi(1) = 1$ and $\|\varphi\| = 1$. Then there exists a sequence $(t_j^n)(\lambda_j^n)$, where $\sum \lambda_j^n = 1$ such that*

$$\varphi(f) = \lim_\alpha \sum_{j=1}^{M(n)} f(x_j^n) \cdot \lambda_j^n.$$

15 Riesz-Thorin Theorem 20210301

Theorem 15.1 (Riesz-Thorin)**.** *Let A be a linear operator and let $1 \le p_0, p_1, q_0, q_1 \le \infty$ where $p_0 \neq p_1$ and $q_0 \neq q_1$. Suppose $A : L_{p_0} \to L_{q_0}$ is bounded and $A : L_{p_1} \to L_{q_1}$ is bounded. Let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

where $\theta \in (0,1)$. *Then*

$$\|A\|_{L_p \to L_q} \leq \|A\|^{1-\theta}_{L_{p_0} \to L_{q_0}} \cdot \|A\|^{\theta}_{L_{p_1} \to L_{q_1}}.$$

If we call $\|A\|^{1-\theta}_{L_{p_0} \to L_{q_0}} = M_0$ *and* $\|A\|^{\theta}_{L_{p_1} \to L_{q_1}} = M_1$, *then* $\|A\|_{L_p \to L_q} \leq M_0^{1-\theta} \cdot M_1^{\theta}$.

In class L_p is replaced with ℓ_p, but there is a more generalized version in literature. I leave L_p in the Theorem to reminds myself this fact. For $1 < p < q < r < \infty$, $L_p \cap L_q \subset L_r \subset L_p + L_q$. In our case (finite dimensional), the same matrix makes sense and $A : \ell_{p_0} \cap \ell_{p_1} \to \ell_{q_0} + \ell_{q_1}$.

We will use the following lemma to prove Riesz-Thorin Theorem.

Lemma 15.2 (Hadamard's Three-Line Theorem)**.** *Suppose* $f(z)$ *is bounded and continuous function on* $0 \leq \mathrm{Re}(z) \leq 1$ *and analytic in the interior. Denote*

$$M_\theta = \sup_{y \in \mathbb{R}} |f(\theta + iy)|.$$

Then $M_\theta \leq M_0^{1-\theta} M_1^{\theta}$ *for* $\theta \in (0,1)$.

If we control the function on boundary then we control the function in the interior.

Example 15.3. Map from a strip to a disk. Let $f(z) = \sum a_n z^n$ be an analytic function, $a_0 = f(0) = \frac{1}{2\pi i} \int \frac{f(z)}{z} \, \mathrm{d}z$. Then

$$|a_0| \leq \int |f(z)| \, \mathrm{d}z = \frac{1}{2\pi i} \int f(e^{i\theta}) \, \mathrm{d}\theta \leq \sup |f(e^{i\theta})|.$$

Proof. Ref.

Recall $\ell_p \hookrightarrow \ell^*_{p'}$ isometrically. So

$$\boxed{\|A\|_{\ell_p \to \ell_q} = \sup \Big\{ \sum_{kj} y_j \cdot A_{jk} \cdot x_k \ \Big| \ \sum |x_i|^p \leq 1, \sum |y_j|^{q'} \leq 1 \Big\}.}$$

Assume $\sum |x_i|^p = 1$ and $\sum |y_j|^{q'} = 1$. Define a function

$$x_k(z) = \frac{x_k}{|x_k|}|x_k|^{p\cdot\left(\frac{1-z}{p_0}+\frac{z}{p_1}\right)} \quad \text{and} \quad y_j(z) = \frac{y_j}{|y_j|}|y_j|^{q'\cdot\left(\frac{1-z}{{q_0}'}+\frac{z}{{q_1}'}\right)}.$$

Then $F(z) = \sum_{jk} y_j(z) \cdot A_{jk} \cdot x_k(z)$ is also analytic. Take $0 \le \mathrm{Re}(z) \le 1$ and define $G(z) = M_0^{z-1}M_1^{-z}F(z)$.

Claim 15.4. *$|G(it)| \le 1$ and $|G(1+it)| \le 1$.*

Take $z = it$, then

$$\begin{aligned} G(it) &= \sum_{jk} \frac{x_k}{|x_k|}|x_k|^{p\cdot\left(\frac{1-it}{p_0}+\frac{it}{p_1}\right)} \cdot A_{jk} \cdot \frac{y_j}{|y_j|}|y_j|^{q'\cdot\left(\frac{1-it}{{q_0}'}+\frac{it}{{q_1}'}\right)} \\ &= \sum_{jk} \alpha_k |x_k|^{\frac{p}{p_0}} \cdot A_{jk} \cdot \beta_j |y_j|^{\frac{q'}{{q_0}'}} \\ &= \|A\|_{\ell_{p_0}\to\ell_{q_0}} \cdot \left\| \sum_{jk} \alpha_k |x_k|^{\frac{p}{p_0}} \right\|^{\frac{p_0}{p_0}} \cdot \left\| \sum_{jk} \beta_j |y_j|^{\frac{q'}{{q_0}'}} \right\|^{\frac{{q_0}'}{{q_0}'}} \le 1, \end{aligned}$$

where $|\alpha_k|, |\beta_k| = 1$ (???). Similarly for $G(1+it)$.

The Three-Line Lemma gives $|G(\theta)| \le 1$. Note that

$$\begin{aligned} G(\theta) &= M_0^{\theta-1}M_1^{-\theta} \sum_{jk} \frac{x_k}{|x_k|}|x_k|^{p\cdot\left(\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\right)} \cdot A_{jk} \cdot \frac{y_j}{|y_j|}|y_j|^{q'\cdot\left(\frac{1-\theta}{{q_0}'}+\frac{\theta}{{q_1}'}\right)} \\ &= M_0^{\theta-1}M_1^{-\theta} \sum_{jk} \frac{x_k}{|x_k|}|x_k|^{p\cdot\frac{1}{p}} \cdot A_{jk} \cdot \frac{y_j}{|y_j|}|y_j|^{q'\cdot\frac{1}{q'}} \\ &= M_0^{\theta-1}M_1^{-\theta} \sum_{jk} x_k \cdot A_{jk} \cdot y_j. \end{aligned}$$

This implies $|\sum_{jk} x_k \cdot A_{jk} \cdot y_j| \le M_0^{1-\theta}M_1^{\theta}$. □

Corollary 15.5. *Assume x, y are complex numbers and $r \le s \le r'$ then*

$$(|x+y|^r + |x-y|^r)^{\frac{1}{r}} \le 2^{1-\frac{1}{s}} \cdot (|x|^s + |y|^s)^{\frac{1}{s}}.$$

Example 15.6. When $r = 2$, $x, y \in \mathbb{R}$, then we get the parallelogram law

$$(|x+y|^2 + |x-y|^2)^{\frac{1}{2}} = (x^2 + 2xy + y^2 + x^2 - 2xy + y^2)^{\frac{1}{2}} = \sqrt{2} \cdot (x^2+y^2)^{\frac{1}{2}}.$$

Proof. Take the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then

$$A \begin{pmatrix} x \\ y \end{pmatrix} = (|x+y|^r + |x-y|^r)^{\frac{1}{r}} \leq 2^{1-\frac{1}{s}} \cdot (|x|^s + |y|^s)^{\frac{1}{s}}$$

For the case $s \geq 2$,

$$\|A\|_{\ell^2_\infty \to \ell^2_\infty} = \sup \left\{ \max(|x+y|, |x-y|) \;\middle|\; |x| \leq 1,\, |y| \leq 1 \right\} \leq 2.$$

$$\|A\|_{\ell^2_2 \to \ell^2_2} \leq (|x+y|^2 + |x-y|^2)^{\frac{1}{2}} = \sqrt{2} \cdot (x^2+y^2)^{\frac{1}{2}} \leq \sqrt{2}.$$

Using Riesz-Thorin Theorem we obtain

$$\|A\|_{\ell^2_s \to \ell^2_s} \leq 2^{1-\theta} \cdot \sqrt{2}^{\,\theta} = 2^{1-\frac{\theta}{2}} = 2^{1-\frac{1}{s}},$$

with the last step given by $\frac{1}{s} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$.

For $1 \leq s \leq 2$, we note that $r \leq s \leq r'$ implies $s' \leq r$. It suffices to consider $r = s'$. Again Riesz-Thorin Theorem gives

$$\|A\|_{\ell_s \to \ell_s} \leq \|A\|^{1-\theta}_{\ell_1 \to \ell_\infty} \cdot \|A\|^{\theta}_{\ell_2 \to \ell_2} \leq 1^{1-\theta} \cdot \sqrt{2}^{\,\theta} = 2^{\frac{1}{s'}} = 2^{1-\frac{1}{s}},$$

with $\frac{1}{s} = \frac{1-\theta}{1} + \frac{\theta}{2}$ and $\frac{1}{s'} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$. Note that

$$\|A\|_{\ell_1 \to \ell_\infty} = \max_{jk} |A_{jk}|.$$

□

16 Clarkson's inequality 20210303

Clarkson's inequality $\Longrightarrow$ Uniform convexity $\Longrightarrow$ L_p is reflexive $\Longrightarrow$ $L_p^* = L_{p'}$

We want to use the Clarkson's inequality (proof ref. Boa) to prove uniform convexity of L_p.

Theorem 16.1 (Reformulation of Riesz-Thorin)**.** *Let A be a matrix. Consider $F(x,y) = \log \|A\|_{\ell_{1/x} \to \ell_{1/y}}$. Then F is a convex function.*

Now let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} : \ell_p^2(\mathbb{C}) \to \ell_q^2(\mathbb{C})$. Thus $\|A\|_{\ell_s \to \ell_r} \leq 2^{1-\frac{1}{s}}$ for all $s \leq r \leq s'$.

We have seen Ref.

- $\|A\|_{\ell_2^2 \to \ell_2^2} \leq \sqrt{2}$, and $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ is unitary (preserves inner product).
- $\|A\|_{\ell_\infty^2 \to \ell_\infty^2} = 2$.
- $\|A\|_{\ell_1^2 \to \ell_\infty^2} = 1$.

Remark 16.2. $\|A\|_{\ell_p^2 \to \ell_q^2} = \|A\|_{\ell_{q'}^2 \to \ell_{p'}^2}$.

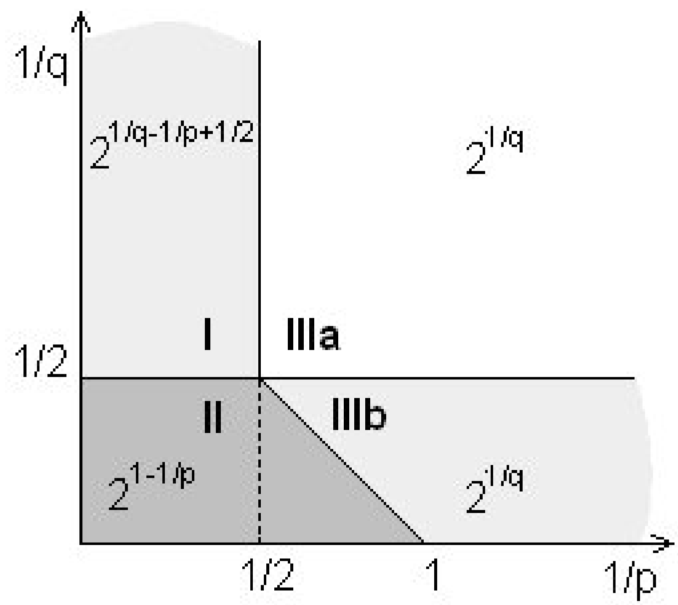

Figure 1: Picture taken from here

Explanation of the picture: by the value at a point, I mean the power of 2. (If I call the value α, then 2^α is an upper bound for $\|A\|_{\ell_p^2 \to \ell_q^2}$.)

- (Region III) The point $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, \frac{1}{2})$ corresponds to $\|A\|_{\ell_2^2 \to \ell_2^2}$ and has value

$\log_2(\sqrt{2}) = \frac{1}{2}$.

- (Region IIIa) The point $(\frac{1}{p}, \frac{1}{q}) = (1,1)$ corresponds to $\|A\|_{\ell^2_\infty \to \ell^2_\infty}$ and has value $\log_2(2) = 1$. By the remark above $\|A\|_{\ell^2_1 \to \ell^2_1} = \|A\|_{\ell^2_\infty \to \ell^2_\infty} = 2$, so the point $(\frac{1}{p}, \frac{1}{q}) = (1,1)$ also has value 1.
- (Region IIIa) Using convexity, for $2 < p < \infty$, point $(\frac{1}{p}, \frac{1}{q})$ on the line $y = x$ has value $\frac{1}{q}$.
- (Region IIIb) The point $(\frac{1}{p}, \frac{1}{q}) = (1, \infty)$ corresponds to $\|A\|_{\ell^2_\infty \to \ell^2_\infty}$ and has value $\log_2(1) = 0$.
- (Region IIIb) Using convexity, for $2 < p < \infty$, point $(\frac{1}{p}, \frac{1}{q})$ on the line $y = 1 - x$ has value $\frac{1}{q}$. Vertical lines between the lines $y = x$ and $y = 1 - x$ has value $\frac{1}{q}$.
- (Region II) For $1 \le s \le 2$ we have

$$\|A\|_{\ell_s \to \ell_{s'}} \le \|A\|^{1-\theta}_{\ell_1 \to \ell_\infty} \cdot \|A\|^{\theta}_{\ell_2 \to \ell_2} \le 2^{1-\frac{1}{s}} = 2^{\frac{1}{s'}}.$$

So $(\frac{1}{p}, \frac{1}{q}) = (\frac{1}{s}, \frac{1}{s'})$ has value $1 - \frac{1}{s}$.

$$\|A\|_{\ell_{s'} \to \ell_{s'}} = \|A\|_{\ell_s \to \ell_s} \le \|A\|^{1-\theta}_{\ell_1 \to \ell_\infty} \cdot \|A\|^{\theta}_{\ell_2 \to \ell_2} \le 2^{1-\frac{1}{s}}.$$

For $s \le r \le s'$ (???)

$$\|A\|_{\ell_s \to \ell_r} = \|A\|^{1-\theta}_{\ell_s \to \ell_s} \cdot \|A\|^{\theta}_{\ell_s \to \ell_{s'}} \le (2^{1-\frac{1}{s'}})^{1-\theta} \cdot (2^{1-\frac{1}{s'}})^{\theta} = 2^{1-\frac{1}{s'}}.$$

Theorem 16.3 (Minkowski's inequality)**.** *Let $L_p(\ell_q)$ and $\ell_q(L_p)$ be the space of functions with the norm*

$$\|f\|_{L_p(\ell_q)} = \left(\int \left(\sum_k |f_k(\omega)|^q \right)^{\frac{p}{q}} \mathrm{d}\mu \right)^{\frac{1}{p}},$$

$$\|f\|_{\ell_q(L_p)} = \left(\sum_k \left(\int |f_k(\omega)|^p \, \mathrm{d}\mu \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}.$$

If $p \le q$, then $L_p(\ell_q) \subset \ell_q(L_p)$ and $\ell_p(L_q) \subset L_q(\ell_p)$.

Proof. We want to show $\|f\|_{\ell_q(L_p)} \le \|f\|_{L_p(\ell_q)}$, i.e.

$$\left(\sum_k \left(\int |f_k(\omega)|^p \,\mathrm{d}\mu\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \le \left(\int \left(\sum_k |f_k(\omega)|^p\right)^{\frac{p}{q}} \mathrm{d}\mu\right)^{\frac{1}{p}}.$$

Let $p \le q$ and $r = \frac{q}{p} \ge 1$. The continuous version of triangle inequality says $\|\int g\,\mathrm{d}\mu\|_r \le \int \|g\|_r\,\mathrm{d}\mu$. (Prove this first for simple function and approximation.) Define $g(\omega) = |f_k(\omega)|^q$, then

$$\left\|\int g(\omega)\,\mathrm{d}\mu\right\|_{\ell_r} \le \int \|g(\omega)\|_{\ell_r}\,\mathrm{d}\mu$$

By definition of $\|\cdot\|_{\ell_r}$

$$\left(\sum_k \left(\int |f_k(\omega)|^p \,\mathrm{d}\mu\right)^{r}\right)^{\frac{1}{r}} \le \int \left(\sum_k |f_k(\omega)|^{pr}\right)^{\frac{1}{r}} \mathrm{d}\mu,$$

so

$$\left(\sum_k \left(\int |f_k(\omega)|^q \,\mathrm{d}\mu\right)^{\frac{q}{p}}\right)^{\frac{p}{q}} \le \int \left(\sum_k |f_k(\omega)|^{q}\right)^{\frac{p}{q}} \mathrm{d}\mu.$$

Taking q-th root on both sides gives the first inclusion. The second inclusion is proved using triangle inequality in ℓ_p. $\square$

17 Uniform convexity of L_p 20210305

Generalize the scalar valued inequality to function valued inequality.

Theorem 17.1. *For f, $g \in L_p$ and $r \le p \le s$ then*

$$(\|f+g\|_p^r + \|f-g\|_p^r)^{\frac{1}{r}} \le 2^{1-\frac{1}{s}} \cdot (\|f\|_p^s + \|g\|_p^s)^{\frac{1}{s}}.$$

Proof. Recall (Minkowski inequality or generalized Fubini Theorem).

$$L_p(\ell_r) \subset \ell_r(L_p) \quad \text{if } p \le r \quad \text{and} \tag{2}$$
$$\ell_s(L_p) \subset L_p(\ell_s) \quad \text{if } s \le p \tag{3}$$

Let $f,\, g \in L_p(\Omega, \Sigma, \mu)$ then

$$\begin{aligned}
LHS &= (\|f+g\|_p^r + \|f-g\|_p^r)^{\frac{1}{r}} \\
&\le \left(\int |f(\omega)+g(\omega)|^r + |f(\omega)-g(\omega)|^r)^{\frac{p}{r}} \,\mathrm{d}\mu\right)^{\frac{1}{p}} && \text{(by (2))} \\
&\le 2^{1-\frac{1}{s}} \cdot \left(\int |f(\omega)|^s + |g(\omega)|^s)^{\frac{1}{s}\cdot p} \,\mathrm{d}\mu\right)^{\frac{1}{p}} && \text{(by Corollary (15.5))} \\
&\le 2^{1-\frac{1}{s}} \cdot \left(\int (|f(\omega)|^p)^{\frac{s}{p}} + (|g(\omega)|^p)^{\frac{s}{p}} \,\mathrm{d}\mu\right)^{\frac{1}{s}} = RHS. && \text{(by (3))}
\end{aligned}$$

□

Now we show the above theorem implies uniform convexity.

Theorem 17.2. *The space L_p is uniformly convex for $1 < p < \infty$. In particular, L_p is reflexive.*

We need to show $\forall \epsilon > 0$, $\exists \delta > 0$ with $\|f\|_p \le 1$, $\|g\|_p \le 1$ and $\|f-g\|_p > \epsilon$ then $\|\frac{f+g}{2}\|_p \le 1-\delta$.

Example 17.3. When $p = 2$, $X = L_2(\Omega, \mathbb{R})$. Fixing $\epsilon > 0$, if we take $\delta = \frac{\epsilon}{8}$ then

$$\begin{aligned}
&(\|f+g\|_2^2 + \|f-g\|_2^2)^{\frac{1}{2}} \le \sqrt{2} \cdot (\|f\|_2^2 + \|g\|_2^2)^{\frac{1}{2}} \le \sqrt{2} \cdot \sqrt{2} \\
\text{and} \quad &\|f+g\|_2^2 + \|f-g\|_2^2 > \|f+g\|_2^2 + \epsilon^2.
\end{aligned}$$

So $\|f+g\|_2^2 + \epsilon^2 \le 4$, i.e. $\|\frac{f+g}{2}\|_2 \le \sqrt{1 - \frac{\epsilon^2}{4}} \le 1 - \frac{\epsilon}{8} = 1 - \delta$.

Proof. Assume $p \ge 2$, $s = \min(p, p')$ and $r = \max(p, p')$, so that $s \le p \le r$. Fixing $\epsilon > 0$, and assume $\|f\|_p \le 1$, $\|g\|_p \le 1$ and $\|f-g\|_p > \epsilon$. Then Theorem 17.1 gives

$$(\|f+g\|_p^r + \|f-g\|_p^r)^{\frac{1}{r}} \le 2^{1-\frac{1}{s}} \cdot (\|f\|_p^s + \|g\|_p^s)^{\frac{1}{s}} \le 2^{1-\frac{1}{s}} \cdot 2^{\frac{1}{s}} = 2.$$

Same as previous example

$$\left(\left\|\frac{f+g}{2}\right\|_p^r + \left(\frac{\epsilon}{2}\right)^r\right)^{\frac{1}{r}} < \left(\left\|\frac{f+g}{2}\right\|_p^r + \left\|\frac{f-g}{2}\right\|_p^r\right)^{\frac{1}{r}} \le 1.$$

So we can choose δ ($\delta = O(\frac{\epsilon}{2})^r$). Note that when $p \to \infty$, $(\frac{\epsilon}{2})^r \to 0$. $\square$

Example 17.4. For $1 < p, q, \infty$, the Sobolev space

$$W^m_{p,q} = \left\{ f \in C(\mathbb{R}) \;\middle|\; \|f\| = \left(\int \left(\sum_{k=1}^{m} \left| f^{(k)}(x) \right|^q \right)^{\frac{p}{q}} \right)^{\frac{l}{q}} < \infty \right\}$$

is uniformly convex. Uniformly convex and reflexive properties pass to subspaces. (Uniform convexity is a property of two points.) We can embeds $W^m_{p,q}$ into $L_p(\ell^m_q) = Y$ and show Y is uniformly convex.

Our goal is to find r, s so that

$$\left(\|F+G\|^r_Y + \|F-G\|^r_Y \right)^{\frac{l}{r}} \le 2^{1-\frac{1}{s}} \cdot \left(\|F\|^s_Y + \|G\|^s_Y \right)^{\frac{1}{s}}.$$

We need the inclusions $L_p(\ell_q(\ell_r)) \subset \ell_r(L_p(\ell_q))$ and $L_s(\ell_p(\ell_q)) \subset L_p(\ell_q(\ell_s))$. These require $p, q \le r$ and $s \le p, q$. Hence $s = \min(p, q, p', q')$ and $r = \max(p, q, p', q')$. Check this gives the above inequality.

18 Uniform Boundedness and Open Mapping 20210308

Theorem 18.1 (Uniform boundedness principle)**.** *Let X be a Banach space and Y a normed vector space. Suppose that $\mathcal{F}$ is a collection of continuous linear operators from X to Y. If $\mathcal{F}$ is pointwise bounded:*

$$\sup_{T \in \mathcal{F}} \|T(x)\|_Y < \infty, \forall x \in X$$

then $\mathcal{F}$ is norm-bounded:

$$\sup_{T \in \mathcal{F}} \|T\|_{B(X,Y)} = \sup_{T \in \mathcal{F}, \|x\|=1} \|T(x)\|_Y < \infty.$$

Application: If $\{T_n\} \subset L(X, Y)$ is a sequence such that $\lim_n T_n(x) = y$ exists for all x, then $\sup_n \|T_n\| < \infty$.

Proof. Ref. Let $\mathcal{F}$ be a family and start with a subset (not a subspace)

$$X_n = \{\, x \mid \sup_{T\in\mathcal{F}} \|Tx\| \le n \,\} \subset X.$$

Claim 18.2. *X_n is closed.*

Assume $x_\alpha \to x$ and we have $\|Tx_\alpha\| \le n$ for all α and $T \in \mathcal{F}$. Since T is continuous, $\lim \|Tx\| = \limsup_\alpha \|Tx_\alpha\| \le n$ (not clear what the first limit is taking with respect to), and

$$\|Tx\| = \|T \lim_\alpha x_\alpha\| = \lim_\alpha \|Tx_\alpha\| \le \limsup_\alpha \|Tx_\alpha\| \le n.$$

Note that $\cup_n X_n = X$ by assumption, and $X_1 \subset X_2 \subset \cdots \subset X_n$.

Assume that the $\operatorname{int}(X_n) = \varnothing$ for all n, then $O_n = X_n^c$ is dense for all n. Baire's Category Theorem gives $\cap_n O_n$ is dense, in particular nonempty. But $\cap_n O_n = (\cup_n X_n)^c = \varnothing$ gives a contradiction. So there exists n such that $\operatorname{int}(X_n) \ne \varnothing$.

Take $x_0 \in X$, $\delta > 0$ and $\|y\| < \delta$ be such that make $B_\delta(x_0) \subset X_n$. Then $\|T(x_0+y)\| \le n$ for all $T \in \mathcal{F}$. Therefore

$$\|T(y)\| = \left\| \frac{T(x_0+y) - T(x_0-y)}{2} \right\| \le \frac{\|T(x_0+y)\| + \|T(x_0-y)\|}{2} \le n.$$

and

$$\|T(y)\| = \left\| T\left(\frac{y}{\|y\|} \cdot \frac{\delta}{2} \right) \right\| \cdot \frac{2\|y\|}{\delta} \le n \cdot \frac{2\|y\|}{\delta}$$

implies $\|T\| \le \frac{2n}{\delta} \implies \sup_{T\in\mathcal{F}} \|T\| \le \frac{2n}{\delta}$ $\square$

This argument also works for convex maps with values in another space.

A famous example is the following.

Example 18.3. Consider $X = C[-\pi,\pi]$. Define the **truncation of Fourier series**

$$P_n(f) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}, \text{ where } \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, \mathrm{d}t.$$

Note that $P_n \in L(X, X)$. Recall in L_2, $\|f\|_2 = \left(\sum_k |\hat{f}(k)|^2\right)^{\frac{1}{2}}$ and $P_n(f) \to f$ in L_2.

If we had that $P_n(f) \to f$ uniformly: $\lim_{n\to\infty} P_n(f) = f$ for all $f \in X$. That is, if $\{P_n\}$ were pointwise bounded: $\sup_n \|P_n(f)\| < \infty$, then uniform boundedness would imply $\sup_n \|P_n\| < \infty$. We will prove later that $\|P_n\| \geq C(1 + \ln n)$ (see Theorem 19.1 below).

This gives a contradiction. So there exists a continuous f such that $\lim_{n\to\infty} \sum_{k=-n}^{n} \hat{f}(k)e^{ikt}$ diverges. Another fact says that the space of trigonometric polynomials $p(t) = \sum_{k=-n}^{n} a_k e^{ikt}$ are dense, and $P_n(p) \to p$ uniformly. The partial Fourier series converges almost everywhere.

Let X be a Banach space, $D \subset X$. What does it mean to be bounded? Two answers

1. $\exists R$ such that $D \subset RB_X$
2. D is weakly bounded: $\forall x^* \in X^*$, $x^*(D) \subset (-R_x, R_x)$ (or $\{\, z \mid |z| \leq R_x \,\}$ in complex case)

With respect to the weak topology, weak bounded implies norm bounded.

Corollary 18.4. *Let X be a Banach space, $D \subset X$ such that $x^*(D)$ is bounded in $\mathbb{K}$ for all $x^* \in X^*$. Then D is bounded.*

Proof. Let $\mathcal{F} = \{\, \varphi_x \mid x \in D \,\} \subset L(X^*, \mathbb{K})$, where $\varphi_x(x^*) = x^*(x)$. We know

$$\sup_{x\in D} \varphi_x(x^*) = \sup_{x\in D} |x^*(x)| < \infty, \forall x.$$

Uniform boundedness principle implies $\sup_{x\in D} \|\varphi_x\| \leq C$. Then D is bounded, because

$$\sup_{x\in D} \|x\| = \sup_{x\in D} \sup_{\|x^*\|\leq 1} |x^*(x)| = \sup_{x\in D} \|\varphi_x\| \leq C.$$

$\square$

Theorem 18.5 (Open mapping theorem)**.** *Let X and Y be Banach spaces and $T : X \to Y$ be linear and surjective. Then T is open.*

Proof. Step 1. Let $\epsilon > 0$ and $Y_n = \overline{T(B_X(0, n\epsilon))}$. Then $Y = \cup_n Y_n$. Uniform boundedness principle implies one of the Y_n's has nonempty interior. So there exists $\tilde{x}$ and $\delta > 0$ such that, $B_Y(\tilde{x}, \delta) \subset Y_n$. WLOG we can assume $\tilde{x} = 0$, so $B_Y(0, \delta) \subset Y_n$. Hence for some $\delta' > 0$, we have $B_Y(0, \delta) \subset \overline{T(B_X(0, \epsilon))}$. Our goal is to remove this closure.

Step 2. Choose ϵ_k so that $\sum \epsilon_k < \epsilon$. According to the previous step, we know that there exists δ_k such that $B_Y(0, \delta_k) \subset \overline{T(B_X(0, \epsilon_k))}$ for all k. WLOG we can assume $\delta_k \to 0$ because we can always take smaller value for δ's.

Now let $y \in Y$ with $\|y\| < \delta_0$. Since $B_Y(0, \delta_0) \subset \overline{T(B_X(0, \epsilon_0))}$ we can find x_0 in $B_X(0, \epsilon_0)$ such that $\|y - T(x_0)\| < \delta_1$. Call $y_1 = y - T(x_0)$. Then we can find we can find x_1 in $B_X(0, \epsilon_1)$ such that $\|y - T(x_0) - T(x_1)\| = \|y_1 - T(x_1)\| < \delta_2$. Iterate this step and we have a sequence of x_k such that

$$\|y - T(x_0) - T(x_1) - \cdots - T(x_k)\| < \delta_k. \tag{4}$$

Since $\delta_k \to 0$, $y = \sum_k T(x_k)$ by construction. Moreover $\sum_k x_k$ converges to some point $x \in X$ because $\|\sum_k x_k\| \le \sum_k \|x_k\| \le \sum_k \epsilon_k < \epsilon < \infty$ (completeness of the Banach space). Note that $\|x\| < \epsilon$ and passing limit of inequality (4) gives $\|y - T(x)\| = 0$. So $y = T(x) \in T(B_X(0, \epsilon))$. This proves for all ϵ, there exists δ such that $B_Y(0, \delta) \subset T(B_X(0, \epsilon))$. (trick Write x and y as converging sequences and use the completeness of Banach spaces).

Step 3. Take O an open set. □

Example 18.6.

- There exists a map $T : \ell_\infty \to \ell_2$ which is linear and onto.

- There is no map $T : \ell_\infty \to \ell_{4/3}$.

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Theorem 19.1. *Let $X = C[-\pi, \pi]$ and let*

$$P_n(f) = \sum_{k=-n}^{n} \hat{f}(k) e^{ikt}, \text{ where } \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} \, \mathrm{d}t.$$

Then $\|P_n\| \geq C(1 + \ln n)$.

Lemma 19.2. *If $T : C(K) \to C(K)$, then*

$$\|T\| = \sup_{x \in K} \|T^*(\delta_x)\|_{C(K)^*},$$

where $\delta_x \in C(K)^*$ is defined by $\delta_x(f) = f(x)$.

Proof. Certainly

$$\|T\| = \|T^*\| = \sup_{\varphi \in C(K)^*, \|\varphi\| \leq 1} \|T^*(\varphi)\| \geq \sup_{x \in K} \|T^*(\delta_x)\|.$$

It remains to show "$\leq$".

Step 1. Take $\varphi = \sum_x \alpha_x \delta_x$, we first prove $\|\varphi\| = \sum_x |\alpha_x|$. One one hand $\|\varphi\| \leq \sum_x |\alpha_x|$ because

$$|\varphi(f)| = \Big| \sum_x \alpha_x f(x) \Big| \leq \sum_x |\alpha_x| \cdot |f(x)| \leq \sum_x |\alpha_x| \cdot \|f\|_\infty.$$

To show the other direction, we need to find $\tilde{f}(x_j) = \epsilon_j$, with $|\epsilon| = 1$, for a compact topological space K. Recall Urysohn's lemma.

Lemma 19.3 (Urysohn's lemma)**.** *A topological space (X, τ) is normal if and only if for every pair of disjoint nonempty closed subsets $C, D \subset X$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $x \in D$.*

More generally, for $O_i \subset X$ disjoint open subsets, and $x_i \in O_i$, we can find a function positive function $f \in C(K)$ such that $f_i(x_i) = 1$, $\operatorname{supp} f_i \subset O_i$ and $\sum_i f_i = 1$. Here we only need $K = [0,1]$, $f_i(x_i) = 1$, $\operatorname{supp} f_i \subset O_i$ and $\sum_i f_i \leq 1$.

Define $\tilde{f}(x) = \sum_j \epsilon_j f_j(x)$, with $|\epsilon_j| = 1$. Then

$$|\varphi(\tilde{f})| = \Big| \sum_j \alpha_j \delta_{x_j} \Big| = \Big| \sum_j \epsilon_j \alpha_j f_j(x_j) \Big| = \sum_j |\epsilon_j| \cdot |\alpha_j| = \sum_j |\alpha_j|.$$

Existence of such an $\tilde{f}$ gives $\|\varphi\| \geq \sum_j |\alpha_j|$. This shows that for disjoint x_j's, $\Big\| \sum_j \alpha_j \delta_{x_j} \Big\| = \sum_j |\alpha_j|$.

Step 2. For an arbitrary $\varphi \in C(K)^*$. Recall we have the following extension

$$\begin{tikzcd} C(K) \arrow[r, hook, "\iota"] \arrow[d, "\varphi"'] & \ell_\infty(K) \arrow[dl, dashed, "\exists \hat{\varphi}"] \\ \mathbb{K} & \end{tikzcd}$$

Then there exists a family $\{\varphi_\alpha\} \subset \ell_\infty(K)$ with

$$\varphi_\alpha(f) = \sum_j \lambda_j(\alpha) f(x_j) \quad \text{and} \quad \|\varphi_\alpha\|_\infty = \sum_j |\lambda_j(\alpha)| = 1.$$

Denote $\varphi(f) = \lim_\alpha \varphi_\alpha(f)$. Then $\varphi_\alpha \to \varphi$ in $\sigma(C(K)^*, C(K))$-topology. This implies for $T : C(K_1) \to C(K_2)$,

$$\begin{aligned} \|T^*(\varphi)\| &= \sup_{\|f\|_{C(K_1)} \leq 1} |T^*(\varphi)(f)| = \sup_{\|f\|_{C(K_1)} \leq 1} |\varphi(T(f))| \\ &= \sup_{\|f\|_{C(K_1)} \leq 1} |\lim_\alpha \varphi_\alpha(T(f))| \leq \sup_{\|f\|_{C(K_1)} \leq 1} \limsup_\alpha |\varphi_\alpha(T(f))|. \end{aligned}$$

Note that

$$\begin{aligned} |\varphi_\alpha(T(f))| &= \Big| \sum_j \lambda_j(\alpha) \cdot (T(f))(x_j) \Big| = \Big| \sum_j \lambda_j(\alpha) \cdot T^*(\delta_{x_j})(f) \Big| \\ &\leq \sum_j |\lambda_j(\alpha)| \cdot \|T^*(\delta_{x_j})\| \leq \sum_j |\lambda_j(\alpha)| \cdot \|f\|_\infty \leq \sup_{x_j} \|T(\delta_{x_j})\|. \end{aligned}$$

This gives $\|T\| = \|T^*\| \leq \|T^*(\delta_x)\|$ and thus the equality.

In short, we could use the fact that the convex hull of the δ measures are weak*-dense in the unit ball of $C(K)^*$. $\square$