MATH231 CALCULUS II

XINRAN YU

Contents

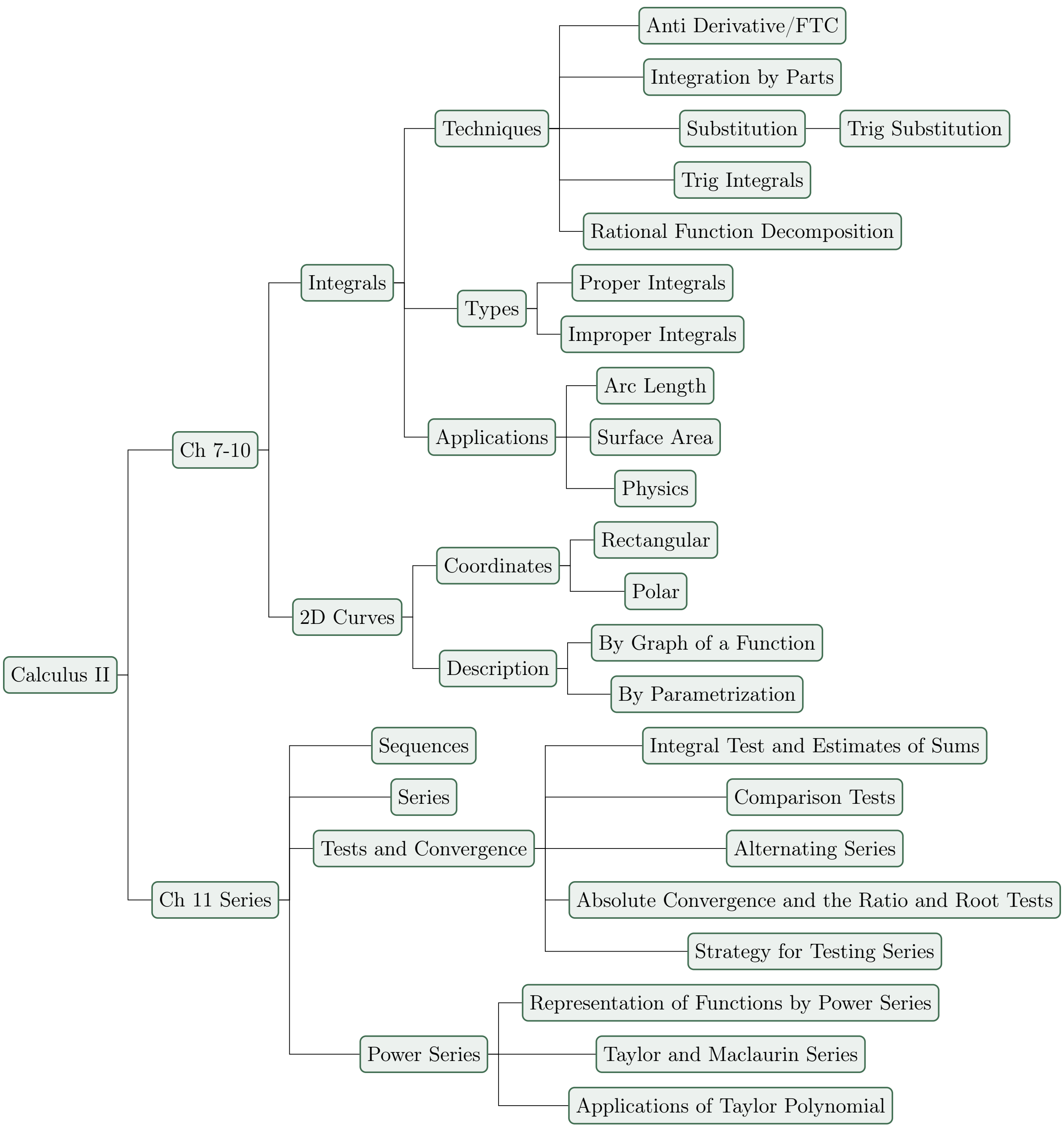
University of Illinois at Urbana-Champaign
Date: Spring 2025.

NB:

- §[number] in the margin refers to the section covered in the book

  James Stewart, *Calculus: Early Transcendentals*, 8th edition, 2016.
- "Typo22" means there is a typo in the handwritten notes from Spring 2022.

An overview of Calculus II



I. Calculus I Review

- Examples can be found in the assignment HW0.
- Make sure you understand how to take *derivatives* and compute *limits* and the basic *integrals*, as these concepts are essential for Calculus II.

1. Limits, Derivatives, and Integrals

1. **Limits Laws.**

(i) $\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
(ii) $\lim_{x\to c}(f(x)\cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
(iii) $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ if $\lim_{x\to c} g(x) \neq 0$

2. **L'Hôpital's Rule.** When $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, we may apply L'Hôpital's Rule

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Example 1.1 ("Counterexample" to L'Hôpital's Rule)**.**

$$\lim_{x\to 0} \frac{x+\sin x}{x} = DNE.$$

L'Hôpital's Rule does not apply because $\lim_{x\to 0} x + \sin x = 1 \neq 0$. •

3. **Derivatives.**

Definition 1.2.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Derivative Rules

- Product Rule: $\dfrac{d}{dx}[f(x)g(x))] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$,
- Chain Rule: $\dfrac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$.

Derivatives of elementary functions

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x,$$
$$\frac{d}{dx}\ln x = \frac{1}{x}, \quad \frac{d}{dx}\tan x = \sec^2 x.$$

2. Integration

1. **Definition.** Reversing the process of differentiation:

Definition 2.1.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(a + \Delta x \cdot i) \cdot \Delta x.$$

2. **Integration Laws.**

- $\int (f(x) \pm g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\,\mathrm{d}x,$
- $\int c \cdot f(x)\,\mathrm{d}x = c \cdot \int f(x)\,\mathrm{d}x.$

3. **Fundamental Theorem of Calculus.**

Theorem 2.2 (Fundamental Theorem of Calculus (FTC))**.** *(i) If f is continuous on $[a,b]$, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$ and differentiable on (a,b), and $F'(x) = f(x)$ for all $x \in (a,b)$.

(ii) If f is continuous on $[a,b]$ and F is an antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

4. **Substitution Rule.** Let $u = g(x)$, then:

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,du.$$

II. Chapter 7

1. Integration by Parts

[1] §7.1

[2] Week 1

- **New method for integrals**: Integration by parts provides a technique to evaluate integrals of *products* of functions.
- **Inverse of the product rule**: If $\frac{d}{\mathrm{d}x}(uv) = u\frac{dv}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$, then

$$\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u \tag{IBP}$$

1. **Motivation.** Using the *Fundamental Theorem of Calculus*, we know the antiderivative of basic functions such as x and e^x:

$$\int x\ \mathrm{d}x = \frac{x^2}{2} + C,$$
$$\int e^x\ \mathrm{d}x = e^x + C.$$

However, for functions like $\ln x$ or $\tan x$, a new tool is required: **Integration by Parts**.

2. **Formula Derivation.** **Key idea: product rule for differentiation**

Proof. Let u, v be functions of x. Recall that the product rule for differentiation formula says

$$(uv)' = u'v + uv'.$$

Integrating both sides with respect to x gives

$$\int uv'\ \mathrm{d}x = uv - \int u'v\ \mathrm{d}x.$$

Rearranging gives the formula for integration by parts:

$$\int u\ \mathrm{d}v = uv - \int v\ \mathrm{d}u.$$

$\square$

3. **Example.**

[3] There were typo in the statement of this question before.

Example 1.1 (Evaluate $\int \ln x\ \mathrm{d}x$)**.** Take $u = \ln x$ and $dv =\ \mathrm{d}x$. Then:

$$\begin{aligned}\int \ln x\ \mathrm{d}x &= x\ln x - \int x \cdot \frac{1}{x}\ \mathrm{d}x\\ &= x\ln x - \int 1\ \mathrm{d}x\\ &= x\ln x - x + C.\end{aligned}$$

•

Example 1.2 (Evaluate $\int x\ln x\ \mathrm{d}x$)**.** Take $u = \ln x$ and $dv = x\ \mathrm{d}x$. Then $\mathrm{d}u = \frac{1}{x}\ \mathrm{d}x$ and

[4] Here's how to solve $\int x\ln x\ \mathrm{d}x$

$v = \dfrac{1}{2}x^2$:

$$\begin{aligned}\int \ln x \,\mathrm{d}x &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,\mathrm{d}x\\ &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\,\mathrm{d}x\\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C.\end{aligned}$$

•

4. **Steps to Apply IBP.**

(i) Identify u and $\mathrm{d}v$(the LIATE rule can be used, see below).
(ii) Compute $\mathrm{d}u$ and v.
(iii) Substitute the above expressions into the IBP formula and evaluate.

Choosing u and v (LIATE Rule): When applying integration by parts, the choice of u and $\mathrm{d}v$ can be guided by the LIATE rule. Take u to be the function that appears earlier in the list.

- **L**ogarithmic functions $\ln x, \log_a x$
- **I**nverse trigonometric functions $\arcsin x, \arctan x$ etc.
- **A**lgebraic functions x^a
- **T**rigonometric functions $\sin x, \cos x$ etc.
- **E**xponential functions e^x, a^x

However, in general, there is no easy way to immediately determine which function to choose as u. In practice, you won't need to remember this rule, as the computation becomes second nature.

5. **More Examples.**

Example 1.3 (Evaluate $\int \arctan x \,\mathrm{d}x$)**.** Let $u = \arctan x$ and $dv = \mathrm{d}x$. Then:

$$\mathrm{d}u = \frac{1}{1+x^2}\,\mathrm{d}x, \quad v = x$$

Substitute into the integration by parts formula:

$$\begin{aligned}\int \arctan x \,\mathrm{d}x &= x \arctan x - \int x \cdot \frac{1}{1+x^2}\,\mathrm{d}x\\ &= x \arctan x - \frac{1}{2}\ln\left(1+x^2\right) + C.\end{aligned}$$

The last step is done by substituting $w = 1 + x^2$:

$$\int x \cdot \frac{1}{1+x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}w}{w} = \ln w + C.$$

•

As in the example above, there are situations where both *integration by parts* and *substitution* are needed.

Here's another example.

Example 1.4 (Evaluate $\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x$)**.** Take $u = x^2$ and $dv = \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$, so:

$$\mathrm{d}u = 2x\,\mathrm{d}x, \quad v = \sqrt{1+x^2}.$$

Substitute into the integration by parts formula:

$$\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x = \int x^2 \cdot \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x = x^2\sqrt{1+x^2} - \int 2x\sqrt{1+x^2}\,\mathrm{d}x.$$

To compute $\int 2x\sqrt{1+x^2}\,\mathrm{d}x$: Using substitution, let $w = 1+x^2$, then $\mathrm{d}w = 2x\,\mathrm{d}x$. We have

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x = \int \sqrt{w}\,\mathrm{d}w = \frac{2}{3}w^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C.$$

So the original integral is:

$$\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x = x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2} + C.$$

•

You may notices that there is no need to apply the IBP at all. Here's another way to solve the same problem.

Example 1.5 (The same problem with substitution)**.** Using substitution rule, we let $u = 1+x^2$ so that $\mathrm{d}u = 2x\,\mathrm{d}x$. Then

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+x^2}}\,\mathrm{d}x &= \int \frac{x^2 \cdot x}{\sqrt{u}}\,\mathrm{d}x = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,\mathrm{d}u \\ &= \frac{1}{2}\int u^{1/2}\,\mathrm{d}u - \frac{1}{2}\int u^{-1/2}\,\mathrm{d}u \\ &= \frac{1}{3}u^{3/2} - u^{1/2} + C \\ &= \frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C.\end{aligned}$$

•

2. Trigonometric Integrals

[5] §7.2

- **Particular type of integral**:
$$\int \sin^n x \cos^m x \, \mathrm{d}x.$$
- **Tools to use**:
  - Trig formulae and identities (to reduce the powers of $\sin x$ or $\cos x$)
  - Substitution rule
  - Integration by parts
- **To memorize**:
$$\sin^2 x + \cos^2 x = 1,$$
$$\sin(2x) = 2\sin x \cos x, \qquad \cos(2x) = \cos^2 x - \sin^2 x$$
The other formulae can be derived from the above.

In this section, we are interested in solving integrals of the form:

$$\int \sin^n x \cos^m x \, \mathrm{d}x$$

where n, m are integers. (You will see in the homework that n and m could be noninteger).

We first recall the trigonometric formulae and identities.

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\sin x = \pm\sqrt{\frac{1-\cos(2x)}{2}} \qquad \cos x = \pm\sqrt{\frac{1+\cos(2x)}{2}}$$

It suffices to remember the equations in red; the rest can be derived from them. (Try it).

1. **First Example of Trigonometric Integral.** Before discussing the general approach, let's first look at an example to motivate the method and the overall strategy.

Example 2.1 (Evaluate $\int \sin^3 x \cos^2 x \; \mathrm{d}x$)**.** Apply the substitution rule. Let $u = \cos x$, then $\mathrm{d}u = -\sin x \, \mathrm{d}x$. We get

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, \mathrm{d}x &= \int \sin^2 x \cos^2 x \cdot \sin x \, \mathrm{d}x && \text{(substitution } u = \cos x\text{)} \\
&= \int (1-u^2)u^2(-\,\mathrm{d}u) \\
&= -\int (u^4 - u^2) \, \mathrm{d}u \\
&= -\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C && \text{(Constant } C \text{ indefinite integral)} \\
&= -\left(\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}\right) + C.
\end{aligned}$$

•

2. **Steps to Evaluate Trig Integrals.**

(i) Identify the type of the integrand
- If n or m is odd, substitution rule is needed. E.g. n is odd, rewrite

$$\int \sin^n x \cos^m x \, \mathrm{d}x = \int \sin^{n-1} x \cos^m x \, \sin x \, \mathrm{d}x.$$

Then take $u = \cos x$ so that $\mathrm{d}u = -\sin x \, \mathrm{d}x$.
- If both of the powers are even, use trig formulae to reduce the powers of $\sin x$ and $\cos x$'s.

(ii) Be careful with the sign!

3. **More Examples.** In the next example, you will see that the half-angle formulae are particularly useful when dealing with even powers of sine and cosine.

Example 2.2 (Evaluate $\int \cos^4 \; \mathrm{d}x$)**.** Use $\cos^2 x = \dfrac{1+\cos(2x)}{2}$ to lower the order of $\cos x$'s, we get:

$$\begin{aligned}
\int \cos^4 x \, \mathrm{d}x &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 \mathrm{d}x \\
&= \int \frac{1 + 2\cos(2x) + (\cos^2(2x))^2}{4} \, \mathrm{d}x && \text{(pull the constant 1/4 out)} \\
&= \frac{1}{4} \int 1 + 2\cos(2x) + \frac{1+\cos(4x)}{2} \, \mathrm{d}x \\
&= \frac{3}{8}x + \frac{\sin(2x)}{4} + \frac{1}{32}\sin(4x) + C.
\end{aligned}$$

•

[6] Week 2

Similarly we can compute

$$\int \tan^n x \sec^m x \, \mathrm{d}x.$$

Recall that $(\tan x)' = \sec^2 x$ and $(\sec x)' = \sec x \tan x$.

Example 2.3 (Evaluate $\int \tan x \sec^4 x \, \mathrm{d}x$).

$$\int \tan x \sec^4 x \, \mathrm{d}x = \int \tan x \sec^2 x \sec^2 x \, \mathrm{d}x = \int \tan x (1 + \tan^2 x) \cdot \sec^2 x \, \mathrm{d}x$$

(substitution $u = \tan x$)

$$= \int u(1 + u^2) \, \mathrm{d}u$$

$$= \frac{u^2}{2} + \frac{u^4}{4} + C \qquad \text{(Constant } C \text{ due to indefinite integral)}$$

$$= \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C.$$

•

Example 2.4 (Another way to compute $\int \tan x \sec^4 x \, \mathrm{d}x$). Take $u = \sec x$ then $\mathrm{d}u = 2\sec^2 x \tan x \, \mathrm{d}x$. We have

$$\int \tan x \sec^4 x \, \mathrm{d}x = \int \tan x \sec^2 x \sec^2 x \, \mathrm{d}x = \int \frac{1}{2} \, \mathrm{d}u = \frac{u^2}{4} + C' = \frac{\sec^4 x}{4} + C'.$$

•

Note that the two method gives the SAME answer. Here's why

$$\frac{\sec^4 x}{4} + C' = \frac{(1 + \tan x)^2}{4} + C' = \frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + \frac{1}{4} + C'.$$

The constant C and C' satisfies the relation: $C = \frac{1}{4} + C'$.

4. **Beyond Calculus II.** Why do we study $\int \sin^n x \cos^m x \, \mathrm{d}x$?

Integrals of the this form are studied for their broad applications in mathematics, physics, and engineering. These integrals appear in Fourier analysis, wave mechanics, and signal processing, where sine and cosine functions serve as fundamental building blocks.

3. Trigonometric substitution

[7] §7.3

- **Particular type of integral**: integral involving square root of quadric polynomials.
- **Tools to use**: Trig substitutions (the idea comes from the trig identities)

	x	Range of θ	$\mathrm{d}x$	$\sqrt{\cdots}$ becomes
$\sqrt{a^2 - x^2}$	$a\sin(\theta)$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$a\cos(\theta) \, \mathrm{d}\theta$	$a\cos(\theta)$
$\sqrt{x^2 + a^2}$	$a\tan(\theta)$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a\sec^2(\theta) \, \mathrm{d}\theta$	$a\sec(\theta)$
$\sqrt{x^2 - a^2}$	$a\sec(\theta)$	$0 \le \theta \le \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$	$a\sec(\theta)\tan(\theta) \, \mathrm{d}\theta$	$a\tan(\theta)$

1. **Trig Substitution Rule.** In this section, we consider integrals containing square roots of the form

$$\sqrt{a^2 - x^2} \qquad \sqrt{x^2 + a^2} \qquad \sqrt{x^2 - a^2}.$$

We use trigonometric substitutions:

	x	Range of θ	$\mathrm{d}x$	$\sqrt{\cdots}$ becomes
$\sqrt{a^2 - x^2}$	$a\sin(\theta)$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$a\cos(\theta)\ \mathrm{d}\theta$	$a\cos(\theta)$
$\sqrt{x^2 + a^2}$	$a\tan(\theta)$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$a\sec^2(\theta)\ \mathrm{d}\theta$	$a\sec(\theta)$
$\sqrt{x^2 - a^2}$	$a\sec(\theta)$	$0 \le \theta \le \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$	$a\sec(\theta)\tan(\theta)\ \mathrm{d}\theta$	$a\tan(\theta)$

Note that we use trigonometric identities to simplify the square root expressions.

For example:

$$\begin{aligned} x = a\sin\theta \implies \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2\sin^2\theta} \\ &= \sqrt{a^2\cos^2\theta} = |a\cos\theta|. \end{aligned}$$

Warning: We have to specify the range of θ so that we can get rid of $|\ |$.

2. **Steps to Apply Trig Substitutions.**

(i) **Identify the integrand type**: there are three types

$$\sqrt{a^2 - x^2} \qquad \sqrt{x^2 + a^2} \qquad \sqrt{x^2 - a^2}.$$

(ii) **Choose an appropriate substitution**: Use the table or trigonometric identities to eliminate the square root by substituting x with a trigonometric function.
(iii) Always **specify the range of θ** to ensure x is the positive root $+\sqrt{\cdots}$.
(iv) **Back-substitution**: Express the trig functions $\sin\theta, \tan\theta \cdots$ in terms of x (trig identities and Calculus I knowledge are needed, and rewrite θ using inverse trig functions.

3. **Examples.**

Example 3.1 (Evaluate $\int \sqrt{9 - x^2}\ \mathrm{d}x$). Take $x = 3\sin\theta$, with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then

$$\int \sqrt{9 - x^2}\ \mathrm{d}x = \int \sqrt{9 - (3\sin\theta)^2} \cdot 3\cos\theta\ \mathrm{d}\theta = \int |3\cos\theta| \cdot 3\cos\theta\ \mathrm{d}\theta$$

$$\left(\text{Need } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ so that } \cos\theta \ge 0\right)$$

$$\begin{aligned} &= \int 9\cos^2\theta\ \mathrm{d}\theta = \int 9\frac{1 + \cos\theta}{2}\ \mathrm{d}\theta = \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \\ &= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C = \frac{9}{2}\arcsin\frac{x}{3} + \frac{9}{2}\frac{x}{3}\sqrt{1 - \frac{x^2}{3^2}} + C \qquad \text{(Back-substitution)} \\ &= \frac{9}{2}\arcsin\frac{x}{3} + \frac{x\sqrt{9 - x^2}}{2} + C. \end{aligned}$$

For the back-substitution step: note that $\sin\theta = \dfrac{x}{3}, \cos\theta = \sqrt{1-\sin^2\theta}$ (by trig identity) $= \sqrt{1-\dfrac{x^3}{9}} = \dfrac{\sqrt{9-x^2}}{3}$, and $\theta = \arcsin\dfrac{x}{3}$. •

Example 3.2 (Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x$)**.** Take $x = 2\tan\theta$, with $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, then

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+4}}\,\mathrm{d}x &= \int \frac{1}{(2\tan\theta)^2\sqrt{4\tan^2\theta+4}}(2\sec^2\theta)\,\mathrm{d}\theta \\
&= \int \frac{2\sec^2\theta}{4\tan^2\theta\cdot|2\sec\theta|}\,\mathrm{d}\theta = \int \frac{\sec\theta}{4\tan^2\theta}\,\mathrm{d}\theta \\
&\qquad\qquad \left(\text{Need } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \text{ so that } \sec\theta > 0\right) \\
&= \int \frac{\cos\theta}{4\sin^2\theta}\,\mathrm{d}\theta \qquad (\text{Using substitution: } u = \sin\theta,\ \mathrm{d}u = \cos\theta\,\mathrm{d}\theta) \\
&= \int \frac{1}{4u^2}\,\mathrm{d}u = -\frac{1}{4u} + C \qquad (\text{Back-substitution}) \\
&= -\frac{1}{4\sin\theta} + C = -\frac{\sqrt{4+x^2}}{x} + C.
\end{aligned}$$

For the back-substitution step: Note that $\tan\theta = \dfrac{x}{2}$, so $\theta = \arctan\dfrac{x}{2}$. To rewrite $\sin\theta$, observe that $\tan\theta = \dfrac{Y}{X}$ and $\sin\theta = \dfrac{Y}{R}$. Solving for R, we get $R = \sqrt{X^2+Y^2} = \sqrt{4+x^2}$, which implies $\sin\theta = \dfrac{x}{\sqrt{4+x^2}}$. •

Example 3.3 (Evaluate $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x$)**.** Complete the square: $3-2x-x^2 = -(x^2+2x+1-1)+3 = -(x+1)^2+4$. Take $u = x+1$

$$\begin{aligned}
\int \frac{x}{\sqrt{3-2x-x^2}}\,\mathrm{d}x = \int \frac{u-1}{\sqrt{4-u^2}}\,\mathrm{d}u &= \int \frac{2\sin\theta-1}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,\mathrm{d}\theta \\
&= \int \frac{2\sin\theta-1}{|2\cos\theta|}2\cos\theta\,\mathrm{d}\theta \\
&\qquad \left(\text{Need } -\frac{\pi}{2} < \theta < \frac{\pi}{2}. \text{ Note that } \cos\theta \text{ is strictly positive.}\right) \\
&= \int 2\sin\theta - 1\,\mathrm{d}\theta = -2\cos\theta - \theta + C. \\
&= -\sqrt{4-u^2} - \arcsin\frac{u}{2} + C \qquad (\text{Back-substitution}) \\
&= -\sqrt{3-2x-x^2} - \arcsin\frac{x+1}{2} + C.
\end{aligned}$$

•

4. **Motivation.** Why do we study these types of integrals?

Because they frequently arise in problems related to *arc length* and *surface area* calculations. These integrals help us model and solve real-world geometric problems, such as determining the length of a curve or the area of a surface of revolution. Their importance will become evident as we explore further in Chapter 8.

[8] §7.4

- **Particular type of integral**: integral involving rational functions.
- **Tools to use**: partial fraction decomposition
  - A *proper* rational function $R(x) = \dfrac{P(x)}{Q(x)}$ satisfies $\deg P(x) < \deg Q(x)$.
  - An *improper* rational function satisfies $\deg P(x) \geq \deg Q(x)$.
  - Improper rational functions are converted into proper ones via polynomial long division:
$$\frac{P(x)}{Q(x)} = F(x) + \frac{\tilde{P}(x)}{Q(x)}.$$

Factor in denominator	Terms in the decomposition of a *proper rational function*
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

1. **Rational Functions.**

Definition 4.1. A ***rational function*** is a function of the form $R(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

If $\deg P < \deg Q$, it is ***proper***; otherwise, it is ***improper***.

Example 4.2.

$$R_1(x) = \frac{1}{x+1}, \qquad R_2(x) = \frac{2x+1}{(x+1)^2}, \qquad R_3(x) = \frac{x^3-3}{(x-7)(x+5)}.$$

The first two are proper; whereas the last one is improper. •

In this section we solve the integral of the type $\displaystyle\int R(x)\,\mathrm{d}x$. The strategy is to rewrite $R(x)$ as a sum of simpler rational functions (using long division and partial fraction decomposition). Then use the substitution rule to solve the integral.

2. **Partial Fraction Decomposition.** We start with proper rational functions. The following table lists the terms that appear in the decomposition of a proper rational function.

[9] Note that "proper" is necessary, otherwise, there will be a polynomial term appear in the decomposition. See Example 4.4.

Factor in denominator	Terms in the decomposition of a *proper rational function*
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Improper rational functions are converted into proper ones via polynomial long division:

$$R(x)=\frac{P(x)}{Q(x)}=F(x)+\frac{\tilde{P}(x)}{Q(x)},$$

where $F(x)$ is a polynomial. Then we can apply partial fraction decomposition to $\frac{\tilde{P}(x)}{Q(x)}$.

Example 4.3. R_2 is proper, so we set

$$R_2(x)=\frac{2x+1}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2}.$$

Compare: R_3 is improper

$$R_3(x)=\frac{x^3-3}{(x-7)(x+5)}=\frac{x(x-7)(x+5)+2x^2+35x-3}{(x-7)(x+5)}=x+\frac{2x^2+35x-3}{(x-7)(x+5)}.$$

Then decompose to the second term $\dfrac{2x^2+35x-3}{(x-7)(x+5)}$ using the table, we set

$$\frac{2x^2+35x-3}{(x-7)(x+5)}=\frac{A}{x-7}+\frac{B}{x+5}.$$

•

3. **Examples of integrals of rational functions.**

Example 4.4 (Evaluate $\displaystyle\int\frac{x}{x+4}\,\mathrm{d}x$)**.** The integrand is improper, so we first apply long division:

$$\frac{x}{x+4}=\frac{x+4-4}{x+4}=1-\frac{4}{x+4}.$$

So

$$\int\frac{x}{x+4}\,\mathrm{d}x=\int 1-\frac{4}{x+4}\,\mathrm{d}x=(x+4)-4\ln|x+4|+C.$$

•

If Q is a product of distinct linear factors,

$$Q = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n),$$

we take

$$R = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}.$$

Example 4.5 (Evaluate $\displaystyle\int \frac{1}{x^2 - 4}\,\mathrm{d}x$)**.** The integrand is proper and the denominator factors as $x^2 - 4 = (x-2)(x+2)$. Using partial fraction decomposition:

$$\frac{1}{x^2 - 4} = \frac{A}{x-2} + \frac{B}{x+2}, \quad \text{where } A, B \text{ are constants.}$$

The numerator gives

$$(A+B)x + 2(A-B) = 1 \implies A = -B = \frac{1}{4}.$$

Plug this back into the integral, we have

$$\begin{aligned}
\int \frac{1}{x^2-4}\,\mathrm{d}x &= \int \left(\frac{1}{4(x-2)} - \frac{1}{4(x+2)}\right)\mathrm{d}x \\
&= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C \\
&= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C.
\end{aligned}$$

•

If Q contains distinct irreducible quadratic factors, take the corresponding quadratic form

$$R = (\text{fraction with linear terms}) + \cdots + \frac{Ax+B}{ax^2+bx+c}.$$

Example 4.6 (Evaluate $\displaystyle\int \frac{5x^2+2}{x(x^2+2x+2)}\,\mathrm{d}x$)**.** The integrand is proper. To decomposition the fraction, we set

$$\frac{5x^2+2}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}, \quad \text{where } A, B, C \text{ are constants.}$$

The numerator gives

$$Ax^2 + 2ax + 2a + Bx^2 + Cx = 5x^2 + 1 \implies \begin{cases} A + B = 5 \\ 2A + C = 0 \\ 2A = 2 \end{cases} \implies \begin{cases} A = 1 \\ B = 4 \\ C = -2 \end{cases}.$$

Plug this back into the integral, we have

$$
\begin{aligned}
\int \frac{5x^2+2}{x(x^2+2x+2)}\,\mathrm{d}x &= \int \frac{1}{x}\,\mathrm{d}x + \int \frac{4x-2}{x^2+2x+2}\,\mathrm{d}x \\
&= \ln|x| + \int \frac{4x-2}{x^2+2x+2}\,\mathrm{d}x = \ln|x| + \int \frac{4x-2}{(x+1)^2+1}\,\mathrm{d}x \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Substitution: } u = x+1) \\
&= \ln|x| + \int \frac{4u-6}{u^2+1}\,\mathrm{d}u = \ln|x| + \int \frac{4u}{u^2+1}\,\mathrm{d}u - \int \frac{6}{u^2+1}\,\mathrm{d}u + C \\
&= \ln|x| + 2\int \frac{1}{w+1}\,\mathrm{d}w - 6\arctan u + C \\
&= \ln|x| + 2\ln|w| - 6\arctan u + C \\
&= \ln|x| + 2\ln|u^2+1| - 6\arctan(x+1) + C \\
&= \ln|x| + 2\ln|x^2+2x+2| - 6\arctan(x+1) + C.
\end{aligned}
$$

•

[10] Typo22

If Q contains a repeated linear factor, say $(ax+b)^r$, include terms of the form:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}.$$

Similarly, for a repeated quadratic factor, say $(ax^2+bx+c)^r$, include terms of the form:

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Example 4.7 (Evaluate $\int \frac{4x}{x^3-x^2-x+1}\,\mathrm{d}x$)**.** The integrand is proper and the denominator factors as :

$$x^3-x^2-x+1 = (x-1)^2(x+1).$$

Using partial fraction decomposition, we set

$$\frac{4x}{x^3-x^2-x+1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}, \quad \text{where } A, B, C \text{ are constants.}$$

The numerator gives

$$A(x+1)(x-1) + B(x-1) + C(x-1)^2 = (A+C)x^2 + (B-2C)x + (-A+B+C) = 4x,$$

so

$$A = 1, B = 2, C = -1.$$

Thus, the integral becomes:

$$
\begin{aligned}
\int \frac{4x}{x^3-x^2-x+1}\,\mathrm{d}x &= \int \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\,\mathrm{d}x \\
&= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C.
\end{aligned}
$$

•

4. **Steps to Evaluate Integrals of Rational Function.**

(i) Apply **long division** to improper rational functions.
(ii) **Factorize** the denominator of the proper rational functions.
(iii) Set up the terms that appear (see the table above) in the **partial fraction decomposition** and solve for the constants.
(iv) Apply the **integration** techniques learned in the preceding sections.

5. Approximate Integration

[11] §7.7

- **Approximating integral**
- **Tools to use**: Midpoint Rule, Trapezoidal Rule, and Simpson's Rule.
- No need to memorize the statements. Know how to use them.

1. **Motivation.** In general, it is difficult to compute the antiderivative of a function and apply the Fundamental Theorem of Calculus, even with techniques we have learned so far. Therefore, we seek an *approximate* value of the integral.

Recall from Calculus I, the integral is defined as the limit of Riemann sums:

Definition 5.1.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x.$$

Since we are interested in an approximate value of the integral, instead of taking $n \to \infty$, we sum over a finite number of intervals:

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} f(\xi_i)\Delta x.$$

For the finite sum above:

- If $\xi_i = a + \Delta x \cdot (i-1)$, it is a *left* endpoint approximation.
- If $\xi_i = a + \Delta x \cdot i$, it is a *right* endpoint approximation.
- If $\xi_i == a + \Delta x \cdot \dfrac{2i-1}{2}$ is the *midpoint*, it is a midpoint approximation.

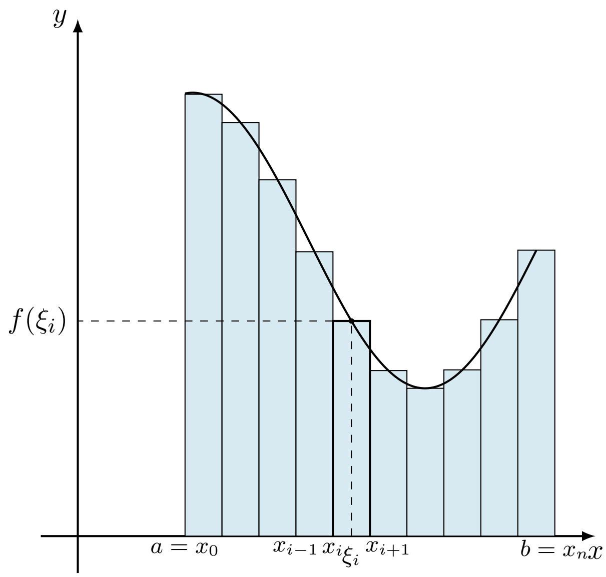


FIGURE 1. Riemann sum

2. **The Midpoint, Trapezoidal and Simpson's Rules.** We usually use the midpoint approximation. The formula is explicitly written as:

Theorem 5.2 (Midpoint rule)**.**

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x \approx M_n &= \left(f(\overline{x_1}) + f(\overline{x_2}) + \cdots + f(\overline{x_n})\right)\Delta x \\ &= \sum_{i=1}^{n} f\Big(a + \Delta x \cdot \frac{2i-1}{2}\Big) \cdot \Delta x,\end{aligned}$$

where $\overline{x_i}$ are the midpoints and Δx is the width of each subinterval.

Another way to approximate the integral is the trapezoidal rule:

Theorem 5.3 (Trapezoidal rule)**.**

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x \approx T_n &= \frac{\Delta x}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right] \\ &= \frac{\Delta x}{2}\left(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\right).\end{aligned}$$

Note that

$$T_n = \left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \cdots + \frac{f(x_{n-1})+f(x_n)}{2}\right)\Delta x.$$

Each term $\dfrac{f(x_{i-1})+f(x_i)}{2}\Delta x$ is the area of one trapezoid.

Similar to trapezoidal rule, another rule to approxiamte the integral is

Theorem 5.4 (Simpson's Rule)**.**

$$\int_a^b f(x)\, \mathrm{d}x \approx S_n = \frac{\Delta x}{3}\left[f(a) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(b)\right].$$

3. **Error of Approximation.**

$$E_M = \int_a^b f(x)\, \mathrm{d}x - M_n, \qquad E_T = \int_a^b f(x)\, \mathrm{d}x - T_n \qquad E_S = \int_a^b f(x)\, \mathrm{d}x - S_n.$$

Error bounds: For $a \le x \le b$, suppose $|f''(x)| \le K$ for the trapezoidal rule and suppose $|f^{(4)}(x)| \le K$ for Simpson's rule, then:

$$|E_M| \le \frac{K(b-a)^3}{24n^2}, \qquad |E_T| \le \frac{K(b-a)^3}{12n^2} \qquad |E_S| \le \frac{K(b-a)^5}{180n^4}.$$

4. **Example.**

Example 5.5. Let $f(x) = x^2$ on the interval $[1, 4]$. Determine the number of subintervals n required such that the error E_M in the Midpoint Rule approximation satisfies

$$|E_M| < 0.1.$$

Solution. The error bound for the Midpoint Rule is given by:

$$|E_M| \le \frac{K(b-a)^3}{24n^2}$$

where $a = 1, b = 4$ and $K = \max_{c\in[1,4]} |f''(c)| = 2$.

Substitute into the error formula we have:

$$|E_M| \le \frac{2 \cdot (4-1)^3}{24n^2} = \frac{9}{4n^2} < 0.1 \implies n \ge \sqrt{\frac{9}{0.4}} \approx 4.74.$$

Since n must be an integer to ensure $|E_M| < 0.1$, the smallest number n is 5. •

6. Improper Integrals

[12] §7.8

[13] Week 4

- **Improper integrals**: deal with unbounded intervals or functions.
- **Tools to use**: taking limit of a proper integral. E.g.
  - Type I: $\displaystyle\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x$
  - Type II: $\displaystyle\int_a^c f(x)\,\mathrm{d}x = \lim_{t\to c}\int_a^t f(x)\,\mathrm{d}x$
- **Comparison test**: (A) continuous, (B) nonnegative, (C) $f(x) \le g(x)$ then

$$\int_a^\infty f(x)\,\mathrm{d}x \text{ converges} \quad\Longrightarrow\quad \int_a^\infty g(x)\,\mathrm{d}x \text{ converges}$$
$$\int_a^\infty f(x)\,\mathrm{d}x \text{ diverges} \quad\Longleftarrow\quad \int_a^\infty g(x)\,\mathrm{d}x \text{ diverges}$$

- **To memorize**:

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x \qquad \begin{cases} \text{converges} & \text{p} > 1 \\ \text{diverges} & \text{p} \le 1 \end{cases}$$

In Chapter 5 (Calculus I), we studied definite integrals of the form $\displaystyle\int_a^b f(x)\,\mathrm{d}x$, where:

- $f(x)$ is piecewise continuous, and
- a, b are real numbers.

Such integrals are known as **proper integrals** (note that this has nothing to do with proper fractional functions $R(x) = \dfrac{P(x)}{Q(x)}$).

1. **Definition of Improper Integrals.** In this section, we extend our discussion to **improper integrals**, which arises in two main cases: when the limits of integration are infinite or when the function being integrated has discontinuities. We define two types of improper integrals.

Definition 6.1 (Type I improper integral)**.**

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x,$$
$$\int_{-\infty}^b f(x)\,\mathrm{d}x := \lim_{t\to-\infty}\int_t^b f(x)\,\mathrm{d}x,$$
$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x := \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to-\infty}\int_t^a f(x)\,\mathrm{d}x + \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x.$$

Definition 6.2. An improper integral is ***convergent*** if the above limit exists; otherwise, it is ***divergent***.

Definition 6.3 (Type II improper integral)**.** If $f(x)$ has a discontinuity at some point $c \in [a, b]$, we define

$$\begin{aligned}
\int_a^c f(x)\,\mathrm{d}x &:= \lim_{t\to c}\int_a^t f(x)\,\mathrm{d}x,\\
\int_c^b f(x)\,\mathrm{d}x &:= \lim_{t\to c}\int_t^b f(x)\,\mathrm{d}x,\\
\int_a^b f(x)\,\mathrm{d}x &:= \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \lim_{t\to c}\int_a^t f(x)\,\mathrm{d}x + \lim_{t\to c}\int_t^b f(x)\,\mathrm{d}x.
\end{aligned}$$

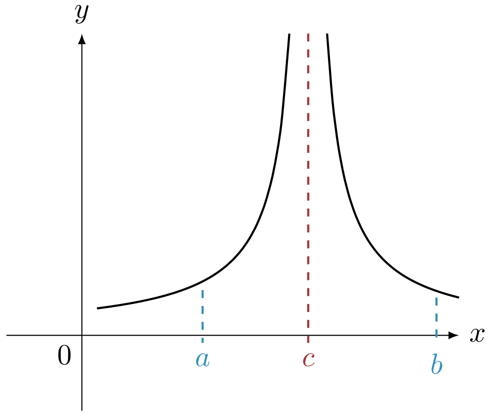


FIGURE 2. Type II indefinite integral

It is important to review the techniques for taking limits from Calculus I. For reference, see Chapter 2: Limits and Derivatives of the textbook.

2. **Steps to Evaluate Improper Integrals.**

(i) **Identify all points** where the integral is improper, including points at infinity and discontinuities.
(ii) **Decompose the integral** into subintervals such that each integral is proper.
(iii) **Express the improper integral** as a limit of proper integrals.
(iv) **Evaluate** the proper integrals and take the limit.

3. **Examples.**

Example 6.4 (Evaluate $\int_0^\infty e^{-x}\,\mathrm{d}x$)**.**

$$\int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-e^{-t}+e^0\right) = 1.$$

•

Example 6.5 (Evaluate $\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x, p<1$)**.**

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,\mathrm{d}x = \lim_{t\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \lim_{t\to\infty}\frac{1}{p-1}\left(\frac{1}{t^{p-1}}-1\right) = \frac{1}{p-1}.$$

Note that this integral diverges if $p > 1$. •

Example 6.6 (Evaluate $\int_0^1 \frac{1}{x-1}\,\mathrm{d}x$)**.**

$$\int_0^1 \frac{1}{x-1}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,\mathrm{d}x = \lim_{t\to 1^-}\big[\ln|x-1|\big]_0^t = \lim_{t\to 1^-}\ln|t-1| = -\infty.$$

•

[14] Week 5

4. **Comparison Test for Improper Integrals.**

Comparison tests can establish the convergence or divergence of improper integrals. Suppose $f(x)$ and $g(x)$ are **continuous, nonnegative** functions and $0 \le f(x) \le g(x)$ for $x \ge a$. Then

$$\int_a^\infty g(x)\,\mathrm{d}x \text{ converges} \implies \int_a^\infty f(x)\,\mathrm{d}x \text{ converges}$$

$$\int_a^\infty g(x)\,\mathrm{d}x \text{ diverges} \impliedby \int_a^\infty f(x)\,\mathrm{d}x \text{ diverges}$$

Example 6.7 (Example to remember)**.**

$$\int_a^\infty \frac{1}{x^p}\,\mathrm{d}x \qquad \begin{cases} \text{converges} & \mathrm{p} > 1 \\ \text{diverges} & \mathrm{p} \le 1 \end{cases}$$

•

5. **Steps for Applying the Comparison Test.**

(i) **Determine the dominant term** of the integrand as x approaches infinity or a discontinuity, typically a power function.
(ii) **Identify the exponent** p in the power function and use Example 6.7 to make an initial guess.
(iii) **Find suitable functions** $f(x)$ and $g(x)$ for the comparison test.
(iv) **Justify the inequality** $0 \le f(x) \le g(x)$.
(v) **Apply the comparison test** to confirm the guess.

6. **Examples.**

Example 6.8 (Show that $I = \int_1^\infty \frac{1+e^{-x}}{x}\,\mathrm{d}x$ diverges)**.** *Step 1.* Note that the integrand is dominated by $\frac{1}{x}$ as $x \to \infty$.

Step 2. This corresponds to the case $p = 1$, and we aim to justify divergence.

Step 3. We set $f(x) = \frac{1+e^{-x}}{x}$ and $g(x) = \frac{1}{x}$. *Step 4.* Note that $0 \le f(x) \le g(x)$ for all $x \ge 1$. *Step 5.* Moreover, $p = 1$ so the integral

$$\begin{aligned}\int_1^\infty g(x)\,\mathrm{d}x &= \int_1^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,\mathrm{d}x \\ &= \lim_{t\to\infty}[\ln x]_1^t = \lim_{t\to\infty}(\ln t - \ln 1) = \infty. \qquad \text{(The same computation as Example 6.5)}\end{aligned}$$

By the comparison test, $I = \int_1^\infty \frac{1+e^{-x}}{x}\,\mathrm{d}x$ also diverges.

•

Example 6.9 (Apply the comparison test to $I = \displaystyle\int_1^\infty \frac{1}{\sqrt{x^6+1}}\,\mathrm{d}x$)**.** *Step 1.* Note that the integrand is dominated by $\dfrac{1}{\sqrt{x^6}} = \dfrac{1}{x^3}$ as $x \to \infty$.

Step 2. This corresponds to the case $p = 3$, and we aim to justify convergence.

Step 3. We set $g(x) = \dfrac{1}{\sqrt{x^6+1}}$ and compare it to $f(x) = \dfrac{1}{\sqrt{x^6}}$.

Step 4. Note that for $x \geq 1$,

$$0 \leq \sqrt{x^6} \leq \sqrt{x^6+1} \quad \Longrightarrow \quad 0 \leq \frac{1}{\sqrt{x^6+1}} \leq \frac{1}{\sqrt{x^6}}.$$

Step 5. Moreover, $p = 3 > 1$ so the integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^6}}\,\mathrm{d}x = \int_1^\infty x^{-3}\,\mathrm{d}x$ converges.

By the comparison test, the integral I also converges. •

Example 6.10 (Apply the comparison test to $I = \displaystyle\int_2^\infty \frac{\cos^2 x}{x^2}\,\mathrm{d}x$)**.**

[15] Read the remaining examples we haven't discussed in class.

Step 1. Note that the integrand is dominated by $\dfrac{1}{x^2}$ as $x \to \infty$.

Step 2. This corresponds to the case $p = 2$, and we aim to justify convergence.

Step 3. We set $g(x) = \dfrac{\cos^2 x}{x^2}$ and compare it to $f(x) = \dfrac{1}{x^2}$.

Step 4. Note that for $0 \leq \cos^2 x \leq 1$ for all x, so

$$0 \leq \frac{\cos^2 x}{x^2} \leq \frac{1}{x^2}.$$

Step 5. Moreover, $p = 2 > 1$ so the integral $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^2}}$ converges. By the comparison test, the integral I also converges. •

Example 6.11 (Apply the comparison test to $I = \displaystyle\int_3^\infty \frac{1}{x - e^{-x}}\,\mathrm{d}x$)**.** *Step 1.* Note that the integrand is dominated by $\dfrac{1}{x}$ as $x \to \infty$.

Step 2. This corresponds to the case $p = 1$, and we aim to justify divergence.

Step 3. We set $f(x) = \dfrac{1}{x - e^{-x}}$ and compare it to $g(x) = \dfrac{1}{x}$.

Step 4. Note that for $0 < e^{-x} < x$ with $x > 3$, so

$$0 < x - e^{-x} \leq x < \infty \quad \Longrightarrow \quad 0 < \frac{1}{x} < \frac{1}{x - e^{-x}}.$$

Step 5. Moreover, $p = 1$ so the integral $\displaystyle\int_3^\infty \frac{1}{x}$ diverges. By the comparison test, the integral I also diverges. •

III. Chapter 8

In this chapter, we will explore the applications of the techniques we have learned so far. We will apply integration methods to problems involving arc length, surface area, and other geometric quantities.

1. Arc Length

[16] §8.1

- **Infinitesimal line element**:

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \sqrt{1 + (g'(y))^2}\,\mathrm{d}y.$$

- **Arc length**:

$$L = \int_A^B \mathrm{d}s = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \int_c^d \sqrt{1 + (g'(y))^2}\,\mathrm{d}y.$$

Let's first consider how to compute arc length

1. **Derivation of the Arc Length Formula.**

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|, \text{ where } |P_{i-1}P_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i.$$

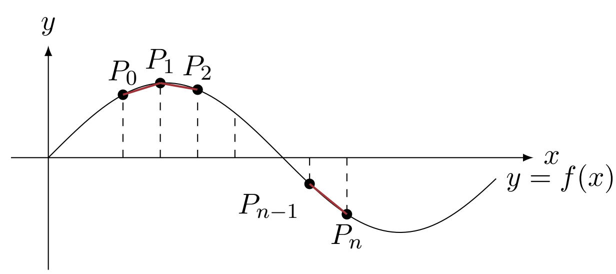


Figure 3. Arc length

Suppose the curve is given by the graph of some differentiable function $y = f(x)$. Then, when taking the limit $\Delta x \to 0$, the expression $\frac{\Delta y}{\Delta x} \to f'$. This suggests

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

Definition 1.1. We define the ***infinitesimal line element***

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \sqrt{1 + (g'(y))^2}\,\mathrm{d}y.$$

Then the ***arc length*** L of a curve is given by

$$L = \int_A^B \mathrm{d}s.$$

In particular, if the curve is given by $y = f(x)$ for $x \in [a, b]$, where f is continuous and differentiable, then

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Similarly, if the curve is given by $x = g(y)$ for $y \in [c, d]$, where g is continuous and differentiable, then:

$$L = \int_c^d \sqrt{1 + (g'(y))^2}\, dy.$$

2. **Examples.**

Example 1.2. Let $y = e^x$ for $x \in [0, 2]$:

$$L = \int_0^2 \sqrt{1 + (e^x)^2}\, \mathrm{d}x.$$

Alternatively, using $x = \ln y$, we rewrite the integral:

$$L = \int_1^{e^2} \sqrt{1 + \frac{1}{y^2}}\, \mathrm{d}y.$$

•

Example 1.3. Let $y^2 + x^2 = 1$ (Unit Circle).

We first compute the arc length of the upper half circle and then use symmetry to get the arc length of the full circle. The upper half of the circle $(y \geq 0)$ is given by

$$y = \sqrt{1 - x^2}, \quad -1 \leq x \leq 1.$$

Therefore,

$$L = \int_{-1}^{1} \sqrt{1 + \left(-\frac{x}{\sqrt{1 - x^2}}\right)^2}\, \mathrm{d}x = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\, \mathrm{d}x = \cdots = \pi.$$

The arc length of the full circle is given by $2L = 2\pi$. •

3. **Arc Length Function.** Given a curve $y = f(x)$, the arc length function $s(x)$ from $x = a$ to $x = b$ is:

$$s(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, \mathrm{d}t.$$

Example 1.4. Let $f(x) = x^2 - \dfrac{\ln x}{8}$ for $x \in [1, \infty)$. Then

$$\begin{aligned} s(x) &= \int_1^x \sqrt{1 + \left(2t - \frac{1}{8t}\right)^2}\, \mathrm{d}x = \int_1^x \sqrt{1 + 4t^2 - \frac{1}{2} + \frac{1}{64t^2}}\, \mathrm{d}t \\ &= \int_1^x 2t + \frac{1}{8t}\, \mathrm{d}t = t^2 + \frac{\ln t}{8}\Big|_1^x = x^2 + \frac{\ln x}{8} - 1. \end{aligned}$$

•

[17] Typo22

4. **Steps to Compute Arc Length.**

(i) Check if the function is **differentiable** and determine which variable to use.
(ii) Write down the corresponding **infinitesimal line element** $\mathrm{d}s$.
(iii) Set up the arc length integral. Be careful with the limits of integration.

2. Area of a Surface of Revolution

[18] 8.2

1. **Derivation of the Surface Area of Revolution formula.**

- **Infinitesimal area element**:
$$\mathrm{d}A = 2\pi R\,\mathrm{d}s.$$
- **Surface area**:
$$A = \int \mathrm{d}A.$$

A surface of revolution is formed by rotating a curve about a line (e.g. the x- or y-axis).

To derive the area, recall the surface area of a cylinder is $2\pi Rl$.

If we take infinitesimal line segments $\mathrm{d}s$, the small piece is approximately a cylinder.

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x)\,\mathrm{d}s = \int_a^b 2\pi f(x)\,\mathrm{d}s.$$

This expression needs to be rewritten in terms of x to make it computable.

Recall from last section $s(x) = \int_a^x \sqrt{1+(f'(t))^2}\,\mathrm{d}t$. This tells us

$$\mathrm{d}s = \sqrt{1+(f'(t))^2}\,\mathrm{d}x.$$

We can rewrite the surface areas as follows.

For a curve $y = f(x)$ rotated about the x-axis, the surface area $A = \int \mathrm{d}A$ is:

$$\begin{aligned} A &= \int_{x_1}^{x_2} 2\pi f(x)\sqrt{1+(f'(x))^2}\,\mathrm{d}x \\ &= \int_{y_1}^{y_2} 2\pi y\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y. && \left(\frac{\mathrm{d}x}{\mathrm{d}y}\text{ is given by implicit differentiation}\right)\end{aligned}$$

If the curve $x = g(y)$ is rotated about the y-axis, then:

$$\begin{aligned} A &= \int_{y_1}^{y_2} 2\pi g(y)\sqrt{1+(g'(y))^2}\,\mathrm{d}y \\ &= \int_{x_1}^{x_2} 2\pi x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x. && \left(\frac{\mathrm{d}y}{\mathrm{d}x}\text{ is given by implicit differentiation}\right)\end{aligned}$$

2. **Examples.**

Example 2.1. Let $y = \sqrt{9-x^2}$ for $x \in [-2, 2]$. Rotating about the x-axis, compute the surface area:

$$\begin{aligned}
\mathrm{d}s &= \sqrt{1+(y')^2}\,\mathrm{d}x = \sqrt{1+\left(-\frac{x}{\sqrt{9-x^2}}\right)^2}\,\mathrm{d}x = \sqrt{1+\frac{x^2}{9-x^2}}\,\mathrm{d}x \\
&= \sqrt{\frac{9-x^2+x^2}{9-x^2}}\,\mathrm{d}x = \frac{3}{\sqrt{9-x^2}}\,\mathrm{d}x.
\end{aligned}$$

$$\begin{aligned}
A &= \int_{-2}^{2} 2\pi f(x)\,\mathrm{d}s = \int_{-2}^{2} 2\pi\sqrt{9-x^2}\cdot\frac{3}{\sqrt{9-x^2}}\,\mathrm{d}x \\
&= \int_{-2}^{2} 6\pi\,\mathrm{d}x = 6\pi(2-(-2)) = 24\pi.
\end{aligned}$$

•

Note that if the axis is shifted by 1, (that is, rotated about the $y=-1$ axis), then $R = f(x)+1$.

Example 2.2. Let $y = e^x$ for $x \in [0, 2]$. Rotating about the y-axis, set up the surface area of revolution. The radius is given by $R(y) = \ln(y)$. So

$$\mathrm{d}s = \sqrt{1+\frac{1}{y^2}}\,\mathrm{d}y,$$

and the surface area is given by

[19] Typo25

$$A = \int \mathrm{d}A = \int_1^{e^2} 2\pi\ln(y)\sqrt{1+\frac{1}{y^2}}\,\mathrm{d}y.$$

•

3. **Steps to Compute Surface Area.**

(i) Determine whether R is a function of x or y; this will determine the variable used in the integration.
(ii) Once you select the variable (either x or y), write down the corresponding **infinitesimal line element** $\mathrm{d}s$.
(iii) Set up the **infinitesimal area element** $\mathrm{d}A$ and the surface area integral. Be careful with the limits of integration.

3. Applications to physics and engineering

[20] 8.3

We likely won't have time to cover this in class, but you're welcome to read it if you're interested in the applications of the integral techniques we've learned. I'm happy to discuss any questions during discussion, office hours, or whenever we meet.

1. **Hydrostatic Pressure and Force.** The force F exerted by a fluid on a submerged plate is given by

$$F = mg = \rho g A d$$

where ρ is the fluid density, g is gravitational acceleration, A is the surface area and d is the depth/width.

Example 3.1. Compute the force on one end of a submerged cylinder with radius 3 and depth 10. Here we have $\rho = \rho(y)$ and $d = 7 - y$ is a constant. Since the infinitesimal area ΔA is given by

$$\Delta A = 2\sqrt{9 - y_i^2}\Delta y,$$

taking limits as $\Delta y \to 0$, we have d

$$\mathrm{d}A = 2\sqrt{9 - y^2}\ \mathrm{d}y.$$

Substitute into the force, we have

$$F = \int_{-3}^{3} \rho g(7 - y)\ \mathrm{d}A = \int_{-3}^{3} (7 - y)\rho g\sqrt{9 - y^2}\ \mathrm{d}y.$$

•

2. **Moments and Center of Mass.** For a lamina with density ρ, the total mass of the lamina is:

$$M = \int \rho(x, y)\ \mathrm{d}A.$$

The moment about the x-axis is:

$$M_x = \rho \int_a^b f(x) \cdot \frac{f(x)}{2}\ \mathrm{d}x.$$

The moment about the y-axis is:

$$M_y = \rho \int_a^b xf(x)\ \mathrm{d}x.$$

The center of mass $(\overline{x}, \overline{y}) = (\frac{M_y}{M}, \frac{M_x}{M})$. (Notice the swap in x and y).

Example 3.2. Find the center of mass of a semicircular plate, suppose ρ is a constant:

$$\begin{aligned}\overline{y} &= \frac{1}{\rho A} \cdot \rho \int_{-r}^{r} \frac{1}{2} f(x)^2\ \mathrm{d}x = \frac{1}{\frac{1}{2}\pi r^2} \int_{-r}^{r} \frac{1}{2}(r^2 - x^2)\ \mathrm{d}x && \text{(Use symmetry)} \\ &= \frac{2}{\pi r^2} \int_0^r r^2 - x^2\ \mathrm{d}x = \frac{2}{\pi r^2} \left[r^2 x - \frac{x^3}{3} \right]_0^r = \frac{4r}{3\pi}.\end{aligned}$$

•

IV. Chapter 10

In this chapter, we introduce an alternative method for representing curves on the 2D plane. Within this framework, we will also explore the reformulated expressions for arc length and the surface area of a revolution.

1. Curves Defined by Parametric Equations

[21] §10.1

[22] Week 6

- **New Concept**: Parametrization

1. **Parametrization.** Consider a particle moving along a curve as follows: The curve cannot be

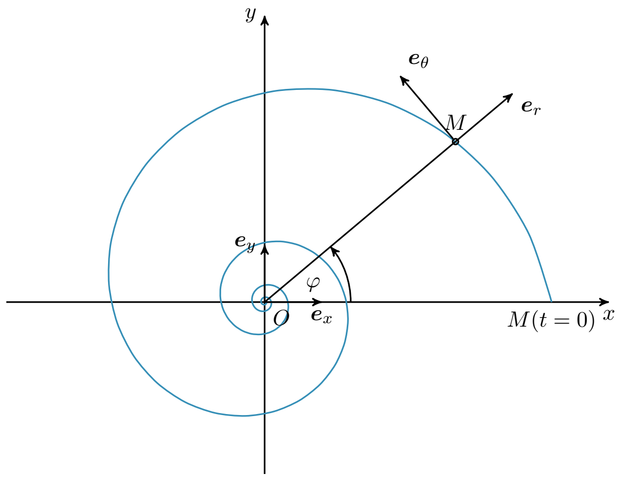


Figure 4. A plot of the Golden Spiral.

expressed as $y = f(x)$ because it fails the vertical line test. However, by introducing a parameter t, representing the angle, the golden spiral can be represented as a parametric curve:

$$x(t) = ae^{bt}\cos(t), \quad t \geq 0,$$
$$y(t) = ae^{bt}\sin(t).$$

(It can also be represented in polar coordinates (see §10.3) as $r(\theta) = ae^{b\theta}$.)

Definition 1.1. The system of equations

$$x = f(t),$$
$$y = g(t).$$

is called a ***parametric equation/parametrization***, and the resulting curve $\mathscr{C}$ is called a ***parametric curve***. We call t a ***parameter***.

Example 1.2 (Unit Circle)**.**

$$x = \cos t, 0 \leq t < 2\pi$$
$$y = \sin t.$$

Note that $\cos^2 t + \sin^2 t = 1$, which implies $x^2 + y^2 = 1$. Thus, these equations parametrize a unit circle. •

Note that parameterizations are **not unique**.

Example 1.3 (Circle with Opposite Orientation)**.**

$$x = \sin 2t, , 0 \le t < \pi$$
$$y = \cos 2t.$$

These parametrize the same circle as in Example 1.1, but with opposite orientation. •

Example 1.4 (Circle of Radius r Centered at (a, b))**.** The equation of the circle is

$$(x-a)^2 + (y-b)^2 = r^2.$$

The corresponding parametrization is given by:

$$x = a + r\cos t, 0 \le t < 2\pi$$
$$y = b + r\sin t.$$

Here r represents the scaling and a, b represents the translation. •

Example 1.5 (Parabola)**.** The curve $x = 6 - 4y^2$ can be parametrized directly as:

$$x = 6 - 4t^2, t \in \mathbb{R}$$
$$y = t.$$

•

We can compute the x- and y-intercepts, critical points, and tangent lines (recall this is $y - y_0 = y'(x_0)(x - x_0)$) just as we do in rectangular coordinates. There will be examples in the discussion worksheet.

2. Calculus with Parametric Curves

[23] §10.2

- **Calculus of Parametric Curves**:

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$
$$L = \int_{t_1}^{t_2} \mathrm{d}s,$$
$$S = \int_{t_1}^{t_2} \mathrm{d}A = \int_{t_1}^{t_2} 2\pi R\,\mathrm{d}s.$$

Calculus techniques can be applied to analyze parametrized curves. For a curve parametrized as $x = f(t)$ and $y = g(t)$, we can compute the following.

- **Tangent slope:** when $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$, the chain rule says $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}.$$

Remark 2.1. *(i)* $\frac{\mathrm{d}x}{\mathrm{d}t} \neq 0$ *is required to take the quotient.*

(ii) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0, \dfrac{\mathrm{d}y}{\mathrm{d}t} \neq 0$ *corresponds to the vertical line* $y = ct$.
(iii) $\dfrac{\mathrm{d}x}{\mathrm{d}t} \neq 0, \dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$ *corresponds to the horizontal line* $x = ct$.

We now introduce the substitution $x = x(t),\ \mathrm{d}x = \dfrac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$:

- **Infinitesimal line element**

$$\mathrm{d}s = \sqrt{(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{(1 + \left(\frac{\mathrm{d}y/\,\mathrm{d}t}{\mathrm{d}x/\,\mathrm{d}t}\right)^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

- **Arc length:**

$$L = \int_a^b \sqrt{(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

- **Surface area (for revolution):**

$$S = \int_a^b 2\pi R\,\mathrm{d}s, \quad \text{where } \mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

1. **Example.**

Example 2.2. Consider the parametrization

$$\begin{aligned} x &= \cos^2 t, 0 \leq t \leq \frac{\pi}{4}. \\ y &= \sin^2 t. \end{aligned}$$

The *infinitesimal line element* is given by

$$\begin{aligned} \mathrm{d}s &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,\mathrm{d}t = \sqrt{(-2\cos t\sin t)^2 + (2\sin t\cos t)^2}\,\mathrm{d}t \\ &= \sqrt{4\cos^2 t\sin^2 t + 4\sin^2 t\cos^2 t}\,\mathrm{d}t = \sqrt{8\sin^2 t\cos^2 t}\,\mathrm{d}t \\ &= \sqrt{2}\sin(2t)\,\mathrm{d}t. \end{aligned}$$

Hence the *arc length* is given by

$$L = \int\,\mathrm{d}s = \int_0^{\frac{\pi}{4}} \sqrt{2}\sin(2t)\,\mathrm{d}t = -\frac{\sqrt{2}}{2}\cos(2t)\Big|_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}.$$

Rotated about the x-axis, the *surface area* is given by

$$
\begin{aligned}
A &= \int_0^{\frac{\pi}{4}} 2\pi \cdot \sin^2(t) \cdot \sqrt{2}\sin(2t)\ \mathrm{d}t = 2\sqrt{2}\pi \int_0^{\frac{\pi}{4}} \sin^2(t) \cdot \sin(2t)\ \mathrm{d}t \\
&= 2\sqrt{2}\pi \int_0^{\frac{\pi}{4}} \frac{1-\cos(2t)}{2} \cdot \sin(2t)\ \mathrm{d}t \\
&= \sqrt{2}\pi \int_0^{\frac{\pi}{4}} \sin(2t)\ \mathrm{d}t + \sqrt{2}\pi \int_0^{\pi/4} \frac{1}{2}\sin(4t)\ \mathrm{d}t \\
&= \sqrt{2}\pi \left[-\frac{\cos(2t)}{2}\right]_0^{\pi/4} - \sqrt{2}\pi \left[-\frac{\cos(4t)}{8}\right]_0^{\frac{\pi}{4}} \\
&= \sqrt{2}\pi \left(\frac{1}{2} - \left(\frac{1}{8} + \frac{1}{8}\right)\right) = \sqrt{2}\pi \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi\sqrt{2}}{4}.
\end{aligned}
$$

•

So far, our examples involve parametric curves easily expressed as the graph of a differentiable function. The next example shows that parametric curves are more general than those defined by a differentiable function, as seen in Chapter 8.

Example 2.3. Consider the parametrization

$$
\begin{aligned}
x &= 3\cos(\pi t), 0 \le t \le \frac{1}{2}. \\
y &= 5t + 2.
\end{aligned}
$$

Note that as t increases, the x-coordinate oscillates, while the y-coordinate increases. You can use an online plotter, such as GeoGebra, to view the graph in the 2D plane.

The *infinitesimal line element* is given by

$$
\mathrm{d}s = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ \mathrm{d}t = \sqrt{\left(-3\pi\sin(\pi t)\right)^2 + 5^2}\ \mathrm{d}t = \sqrt{9\pi^2\sin^2(\pi t) + 25}\ \mathrm{d}t.
$$

Hence the *arc length* is given by

$$
L = \int\ \mathrm{d}s = \int_0^{1/2} \sqrt{9\pi^2\sin^2(\pi t) + 25}\ \mathrm{d}t.
$$

Rotated about the y-axis the *surface area* is given by

$$\begin{aligned}
A &= \int_0^{1/2} 2\pi \cdot 3\cos(\pi t) \cdot \sqrt{9\pi^2 \sin^2(\pi t) + 25}\,\mathrm{d}t && (\text{Substitution } u = \sin(\pi t),\ \mathrm{d}u = \pi\cos(\pi t)) \\
&= \int_0^{1/2} 6 \cdot \sqrt{9\pi^2 u^2 + 25}\,\mathrm{d}u && \left(\text{Trig integral } u = \frac{5}{3\pi}\tan\theta,\ \mathrm{d}u = \frac{5}{3\pi}\sec^2\theta\,\mathrm{d}\theta\right) \\
&= \int_0^{\arctan(3\pi/5)} 6 \cdot \sqrt{25\tan^2\theta + 25}\frac{5}{3\pi} \cdot \sec^2\theta\,\mathrm{d}\theta = \int_0^{\arctan(3\pi/5)} 6 \cdot 5\sec\theta \cdot \frac{5}{3\pi}\sec^2\theta\,\mathrm{d}\theta \\
&= \frac{25}{\pi}\int_0^{\arctan(3\pi/5)} \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{25}{\pi}\Big(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big)\Big|_0^{\arctan(3\pi/5)} \approx 43.0705.
\end{aligned}$$

To evaluate the last quantity, use $\sec\theta = \sqrt{1+\tan^2\theta} = \dfrac{\sqrt{25+9\pi^2}}{5}$. •

3. Polar Coordinates

[24] §10.3

- **New Concept**: Polar Coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases} \iff \begin{cases} r = \sqrt{x^2+y^2}, \\ \theta = \tan^{-1}\left(\dfrac{y}{x}\right). \end{cases}$$

- **Examples**:
  - Circles $r = R$ or $r = a\cos\theta + b\sin\theta$.
  - Cardioids $r = a \pm a\cos\theta$ or $r = a \pm a\sin\theta$.
  - Limaçons $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$, $a \neq b$.

In this section, we will explore an alternative method for representing points on the Euclidean plane.

1. **Curves in Polar Coordinates.** In polar coordinates, a point (r, θ) is represented as:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

Conversely:

$$\begin{aligned} r &= \sqrt{x^2+y^2}, \text{ if } r > 0 \\ \theta &= \arctan\left(\frac{y}{x}\right). \end{aligned}$$

When $r < 0$, the angle θ get added by π. See Example 3.2.

Example 3.1. Converting $(x, y) = (1, 1)$ to polar coordinates Given $(x, y) = (1, 1)$:

$$\begin{aligned} r &= \sqrt{1^2+1^2} = \sqrt{2}, \\ \theta &= \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}. \end{aligned}$$

Thus, the polar coordinates are $(\sqrt{2}, \pi/4)$.

Note that the polar coordinate for this point is **not unique**. We can also pick $(\sqrt{2}, 2\pi + \frac{\pi}{4})$. •

Example 3.2. Convert $(r, \theta) = (-\sqrt{2}, \frac{\pi}{4})$ to rectangular coordinates: We know

$$x = r\cos\theta = -\sqrt{2} \cdot \frac{1}{\sqrt{2}} = -1,$$
$$y = r\sin\theta = -\sqrt{2} \cdot \frac{1}{\sqrt{2}} = -1.$$

The point is symmetric to $(1, 1)$ with respect to the origin, as shown in the previous example. You can check that $(r, \theta) = (-\sqrt{2}, \pi + \frac{\pi}{4})$ corresponds to $(x, y) = (1, 1)$. •

Some curves, such as circles or spirals, can be expressed as simple functions in terms of polar coordinates

$$F(r, \theta) = 0.$$

We will explore how to compute arc length and surface area using polar coordinates.

2. **Examples.**

Example 3.3 (circle centered at the origin)**.** In rectangular coordinates, a circle of radius R centered at the origin is given by $x^2 + y^2 = R^2$. In polar coordinates, this is given by $r = R, \theta \in [0, 2\pi]$.

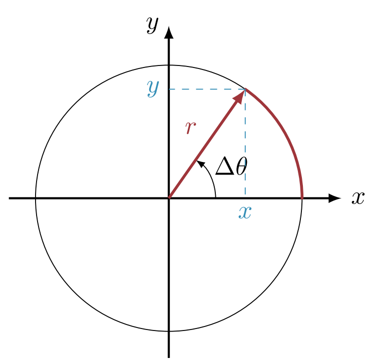


Figure 5. Circle of radius r

•

Example 3.4. Consider a circle centered at $(0, \frac{1}{2})$ with radius $\frac{1}{2}$, then $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$. We convert this into polar coordinates by plug in $x = r\cos\theta, y = r\sin\theta$:

$$\begin{aligned} x^2 + (y - \frac{1}{2})^2 = \frac{1}{4} &\iff r^2\cos^2\theta + (r\sin\theta - \frac{1}{2})^2 = \frac{1}{4} \\ &\iff r^2\cos^2\theta + r^2\sin^2\theta - r\sin\theta + \frac{1}{4} = \frac{1}{4} \\ &\iff r^2 - r\sin\theta = 0. \end{aligned}$$

Since $r > 0$, this equation is equivalent to $r = \sin\theta$.

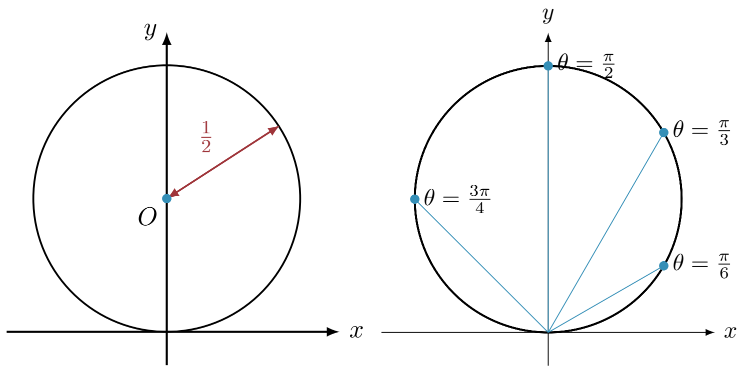


FIGURE 6. Circle of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$

Note that with the points winding around the *full* circle once when $\theta \in [0, \pi]$. •

In general, in polar coordinates, the equations $r = R$ or $r = a\cos\theta + b\sin\theta$ represent circles.

Example 3.5 ($r = a + b\sin\theta$)**.** The polar curve $r = a + b\sin\theta$ gives a cardioid.

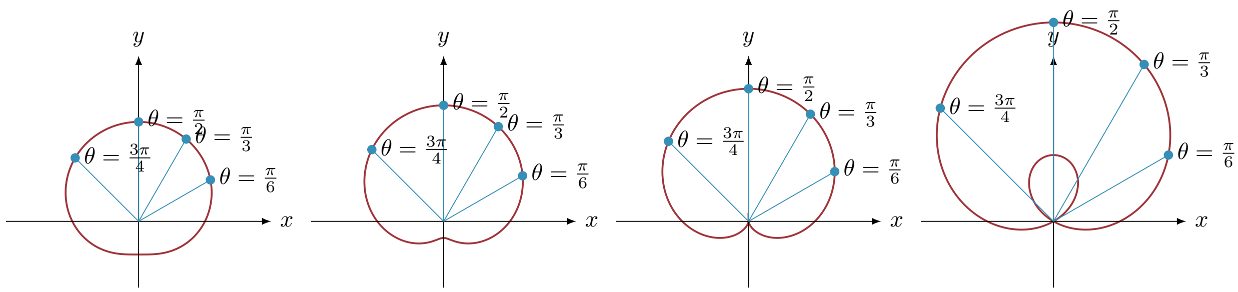


FIGURE 7. Convex limaçon $r = 1 + \frac{1}{2}\sin\theta$, dimpled limaçon $r = 1 + \frac{3}{4}\sin\theta$, cardioid $r = 1 + \sin\theta$ and limaçon with inner loop $r = 1 + 2\sin\theta$.

•

Note that as $b \to 0$, the polar curve converges to a circle centered at the origin.

In general, in polar coordinates, the equations $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$ represent

- Cardioids: if $a = b$.
- Limaçons with an inner loop: if $a < b$.
- Limaçons without an inner loop: if $a > b$.

4. Areas and Lengths in Polar Coordinates

[25] §10.4

[26] Week 7

- **Calculus with Polar Coordinates**:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, \mathrm{d}\theta \qquad \text{(area enclosed by } r = f(\theta)\text{)}$$

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta.$$

$$L = \int_{\theta_1}^{\theta_2} \mathrm{d}s,$$

$$S = \int_{\theta_1}^{\theta_2} \mathrm{d}A = \int_{\theta_1}^{\theta_2} 2\pi R \, \mathrm{d}s. \qquad \text{(surface area of revolution of a polar curve)}$$

1. **Tangent.** Now consider a polar curve of the form $r = f(\theta)$. Then,

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta.$$

The derivative of the parametrization with respect to θ is given by

$$\frac{dx}{d\theta} = f'\cos\theta - f\sin\theta, \quad \frac{dy}{d\theta} = f'\sin\theta + f\cos\theta.$$

We can compute its tangent by the chain rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'\cos\theta + f\sin\theta}{f'\sin\theta - f\cos\theta}$$

Example 4.1. Let $r = 1 + \sin\theta$. Compute $\dfrac{dy}{dx}$.

$$x = (1+\sin\theta)\cos\theta, \quad y = (1+\sin\theta)\sin\theta.$$

Differentiating,

$$\frac{dx}{d\theta} = \cos\theta \cdot \sin\theta - (1+\sin\theta)\cos\theta, \quad \frac{dy}{d\theta} = \cos\theta\cdot\cos\theta - (1+\sin\theta)\sin\theta.$$

Thus,

$$\frac{dy}{dx} = \frac{\cos\theta + 2\cos\theta\sin\theta}{\cos^2\theta - \sin^2\theta - \sin\theta} = \frac{\cos\theta + \sin(2\theta)}{\cos(2\theta) - \sin\theta}.$$

Note that:

$$\lim_{\theta\to\frac{3\pi}{2}^-} \frac{dy}{dx} = \lim_{\theta\to\frac{3\pi}{2}^-} \frac{\cos\theta + \sin(2\theta)}{\cos(2\theta) - \sin\theta} = \lim_{\theta\to\frac{3\pi}{2}^-} \frac{-\cos\theta + 2\cos(2\theta)}{-2\sin(2\theta) - \cos\theta} = -\infty. \qquad \text{(L'H)}$$

This means the tangent blows up at $\dfrac{3\pi}{2}$. •

2. **Area Enclosed by Polar Curves.** For $r = f(\theta)$, the area of a sector is approximately

$$\Delta A \approx \frac{1}{2} r^2 \Delta\theta.$$

Using a Riemann sum,

$$A \approx \sum_{i=1}^{n} \frac{1}{2} \left[f(\xi_i)\right]^2 \Delta\theta \qquad \Longrightarrow \qquad A = \int_{\theta_1}^{\theta_2} \frac{1}{2} \left[f(\xi)\right]^2 \, \mathrm{d}\theta = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, \mathrm{d}\theta.$$

Example 4.2. Find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.

$$\begin{aligned} A &= \frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 \, \mathrm{d}\theta = \frac{1}{2} \int_{0}^{\pi/4} \cos^2(2\theta) \, \mathrm{d}\theta && \text{(Integrand is an even function)} \\ &= \int_{0}^{\pi/4} \frac{1 + \cos(4\theta)}{2} \, \mathrm{d}\theta = \frac{1}{2} \left[\theta + \frac{1}{4} \sin(4\theta)\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{4}. \end{aligned}$$

[27]
Typo22

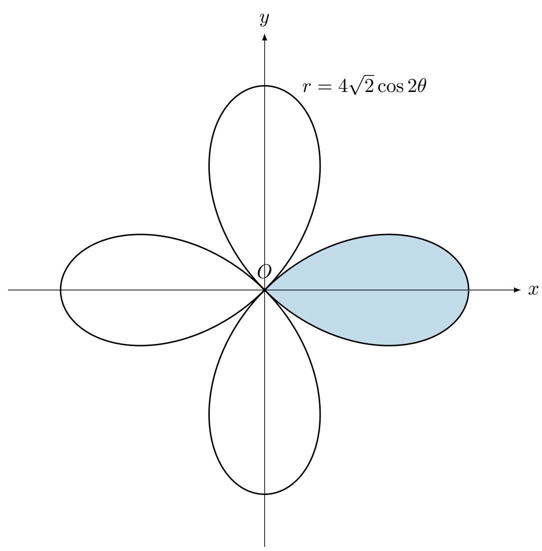


FIGURE 8. Four-leaf $r = \cos(2\theta)$

3. **Arc Length.** We compute the infinitesimal line element $\mathrm{d}s$ as follows:

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2} \, \mathrm{d}\theta.$$

Note that

$$\begin{aligned} \left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 &= (r')^2 \cos^2\theta - 2rr' \cos\theta \sin\theta + r^2 \sin^2\theta \\ &\quad + (r')^2 \sin^2\theta + 2rr' \sin\theta \cos\theta + r^2 \cos^2\theta \\ &= (r')^2 + r^2, \qquad \text{where } r' = f'(\theta). \end{aligned}$$

So

$$\mathrm{d}s = \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = \sqrt{\Big(f(\theta)\Big)^2 + \Big(f'(\theta)\Big)^2}\,\mathrm{d}\theta.$$

The arc length of a polar curve is given by:

$$L = \int \mathrm{d}s = \int_{\theta_1}^{\theta_2} \sqrt{\Big(f(\theta)\Big)^2 + \Big(f'(\theta)\Big)^2}\,\mathrm{d}\theta.$$

Example 4.3. Find the arc length of $r = \theta$, $0 \le \theta \le 1$.

$$L = \int_0^1 \sqrt{\theta^2 + 1}\,\mathrm{d}\theta.$$

We have seen this integral in §7. Using the substitution $\theta = \tan x$, $d\theta = \sec^2 x\,\mathrm{d}x$, we have

$$\begin{aligned} L &= \int_0^{\pi/4} \sec x \sec^2 x\,\mathrm{d}x && \text{(IBP with } u = \sec x,\ v = \tan x) \\ &= \sec x \tan x - \int_0^{\pi/4} \tan x \cdot \tan x \sec x\,\mathrm{d}x = \sec x \tan x - \int_0^{\pi/4} \tan^2 x \sec x\,\mathrm{d}x \\ &= \sec x \tan x - \int_0^{\pi/4} (\sec^2 x - 1) \sec x\,\mathrm{d}x = \sec x \tan x - L + \int_0^{\pi/4} \sec x\,\mathrm{d}x \end{aligned}$$

This implies

$$\begin{aligned} 2L &= \sec x \tan x + \int_0^{\pi/4} \sec x\,\mathrm{d}x \\ &= \sec x \tan x + \ln|\sec x + \tan x|\Big|_0^{\pi/4} = \frac{1}{2}\Big(\sqrt{2} + \ln\Big(1 + \sqrt{2}\Big)\Big). \end{aligned}$$

•

5. Summary of Arc Length and Area Integrals

Here is a summary of the integrals we learned in Chapters 8 and 10. In practice, you only need to remember the formulae in bold text; the others can be derived from them using the chain rule (for differentiation) and the substitution rule (for integration).

	$\mathrm{d}s$	surface area of revolution $\mathrm{d}A$	surface area enclosed by curve
(x, y)	$\boldsymbol{\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}}$ $= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ $= \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}\,\mathrm{d}y$	$\boldsymbol{2\pi R\,\mathrm{d}s}$, R is a function of x or y	$\int_{x_1}^{x_2} f(x)\,\mathrm{d}x$ or $\int_{y_1}^{y_2} g(y)\,\mathrm{d}y$
$\big(x(t), y(t)\big)$ $x' = \frac{\mathrm{d}x}{\mathrm{d}t}$ $y' = \frac{\mathrm{d}y}{\mathrm{d}t}$	$\sqrt{(x')^2 + (y')^2}\,\mathrm{d}t$	$2\pi R\,\mathrm{d}s$, R is a function of t	$\int_{t_1}^{t_2} f(x(t))\,x'\,\mathrm{d}t$ or $\int_{t_1}^{t_2} g(y(t))\,y'\,\mathrm{d}y$
(r, θ), $r = r(\theta)$, $r' = \frac{\mathrm{d}r}{\mathrm{d}\theta}$	$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$ $= \boldsymbol{\sqrt{r^2 + (r')^2}\,\mathrm{d}\theta}$	$2\pi R\,\mathrm{d}s$, R is a function of θ	$\boldsymbol{\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2\,\mathrm{d}\theta}$

Note that the last row can also be derived from the second row, but it is convenient to remember them.

V. Chapter 11: Infinite Sequences and Series

[28] Week 8

In this chapter, we introduce sequences and series. We will focus on how to test for convergence using tools like the integral test, comparison tests, and the ratio and root tests. We will also discuss alternating and absolutely convergent series, along with strategies for analyzing them. Finally, we will explore power series and Taylor and Maclaurin series, and how to use them to approximate functions.

1. Sequences

[29] §11.1

- **New Concept**: Sequence, limit of sequence, sequence converges/diverges
- **Example to memorize**:
  - $$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} 0 & \text{if } p > 0 \\ 1 & \text{if } p = 0 \\ \infty & \text{if } p < 0 \end{cases}$$
  - $$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DNE} & \text{if } r < -1 \end{cases}$$

Definition 1.1. A ***sequence*** is an infinite list of members written with an order. We denote the sequence as $\{a_1, a_2, \ldots, a_n, \ldots\}$, $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

Example 1.2 (sequences)**.**

(i) $\{1, 2, 3, 9, \ldots\}$.
(ii) $\{7, 1, 8, 2, 8, \ldots\}$.

•

Some sequences can be defined by giving a formula for the n-th term a_n.

Example 1.3 (sequences given by formulae)**.**

(i) $a_n = \dfrac{1}{n}$, $\{a_n\} = \{1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots\}$.
(ii) $a_n = (-1)^{n-1}$, $\{a_n\} = \{-1, 1, -1, 1, \ldots\}$.
(iii) $a_n = \dfrac{1}{3^n}$, $\{a_n\} = \{\dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27}, \ldots\}$.

•

Some sequences may not have a simple/explicit defining equation.

Example 1.4 (sequences without explicit formulae)**.**

(i) $a_n =$ the digit in the n-th decimal place of π.

(ii) The Fibonacci sequence: $a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$
$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$.

•

Remark 1.5. *A sequence can be thought of as a function f defined only on the natural numbers. Therefore, we can examine properties such as the graph and convergence. For example,*

$$\lim_{n\to\infty} a_n = 0.$$

Definition 1.6. A sequence has ***limit*** L if for any $\epsilon > 0$, there is an N such that if $n > N$, then $|a_n - L| < \epsilon$. (We write this as

$$\forall\epsilon > 0, \exists N \text{ s.t. if } n > N, \text{ then } |a_n - L| < \epsilon.$$

We say a_n ***converges*** to L and denote it as $\lim_{n\to\infty} a_n = L$.

Remark 1.7 (Intuition)**.** $\lim_{n\to\infty} a_n = \infty$ *means that for every positive number M, there is an integer N such that if $n > N$, then $a_n > M$.*

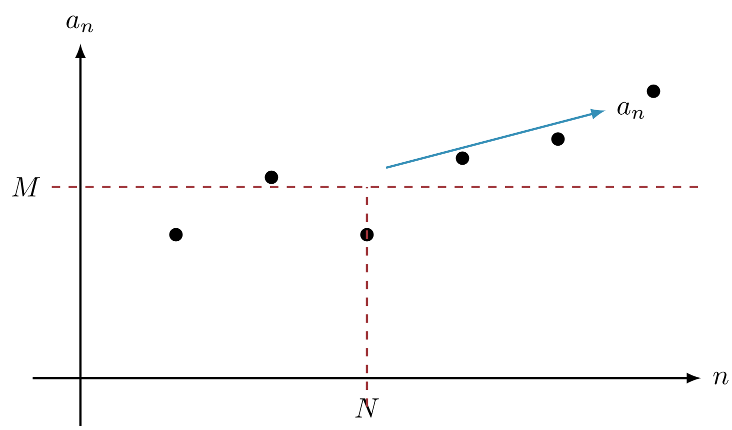


FIGURE 9. Illustration of the definition of $\lim_{n\to\infty} a_n = \infty$. Beyond some N, all a_n exceed M.

Example 1.8 (limit of a sequence)**.**

(i) $\displaystyle\lim_{n\to\infty} \frac{n}{n+1} = 0 = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$

$$\lim_{n\to\infty} \frac{1}{n^p} = \begin{cases} 0 & \text{if } p > 0 \\ 1 & \text{if } p = 1 \\ \infty & \text{if } p < 0 \end{cases}$$

(ii)

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ \text{DNE} & \text{if } r < -1 \end{cases}$$

•

1. **Limit Laws for Sequences.**

- **Tools for evaluate limits**: limit law, squeeze theorem,
- **Continuous function commutes with limit**:

$$f \text{ continuous} \implies \lim_{n\to\infty} f(a_n) = f\Big(\lim_{n\to\infty} a_n\Big).$$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then

(i) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$.
(ii) $\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty} a_n$, c constant.
(iii) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.
(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$.
(v) $\lim_{n\to\infty}(a_n)^p = \Big(\lim_{n\to\infty} a_n\Big)^p$.

Note that if the convergent condition fails, the equality could also fail. For example, if $a_n = (-1)^n$ and $b_n = \frac{1}{n}$. $\lim_{n\to\infty} a_n = \text{DNE}$, but $\lim_{n\to\infty} a_n b_n = 0$.

Theorem 1.9 (Squeeze Theorem)**.** *If $b_n \le a_n \le c_n$ holds for every $n \ge N$ (N is some natural number) and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} a_n = L$.*

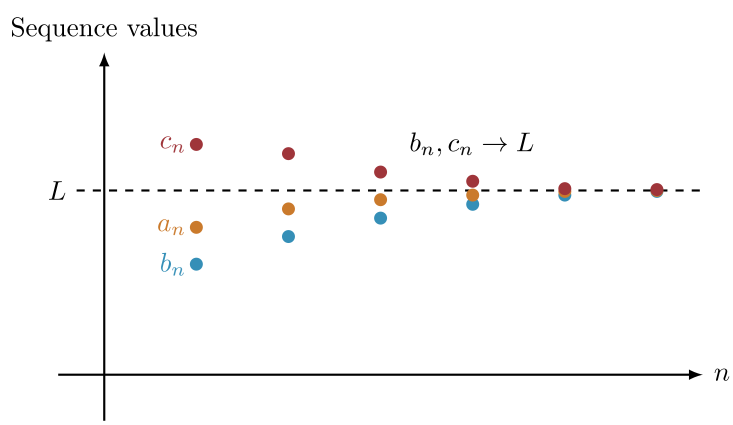


FIGURE 10. Visualization of the Squeeze Theorem: if $b_n \le a_n \le c_n$ and both b_n and c_n converge to L, then a_n also converges to L.

Example 1.10. If $a_n = (-1)^n \frac{1}{n}$, $b_n = -\frac{1}{n}$ and $c_n = \frac{1}{n}$. Then $b_n \le a_n \le c_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$ implies $\lim_{n\to\infty} a_n = 0$ by the Squeeze Theorem. •

Theorem 1.11.

(i) If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.
(ii) Continuous function commutes with limit. If f is ***continuous****, then $\lim_{n\to\infty} a_n = L$ implies*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L).$$

Example 1.12. (i) $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty}\frac{1}{n}\right) = \sin(0) = 0.$

(ii) $\lim_{n\to\infty}\frac{\ln(n+2)}{\ln(1+4n)}$. Note that this is same as

$$\lim_{x\to\infty}\frac{\ln(x+2)}{\ln(1+4x)} = \lim_{x\to\infty}\frac{\frac{1}{x+2}}{\frac{4}{1+4x}} = \lim_{x\to\infty}\frac{1+4x}{4(x+2)} = 1. \qquad \text{(L'Hôpital's rule)}$$

(iii)

$$\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{x\to\infty} e^{\left(1+\frac{1}{n}\right)^n} = e^{\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n} = e.$$

(Can apply L'Hôpital's rule to compute the limit)

•

2. Series

[30] §11.2

- **New concept**: series, partial sum, converges/diverges
- **Examples to memorize**:
  - Geometric series
$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r} & \text{if } |r|<1, \\ \infty & \text{if } r \geq 1 \\ \text{DNE} & \text{if } r \leq -1. \end{cases}$$
  - Harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.
- **Tools to study series**: Series laws, $\sum a_n$ converges $\implies a_n \to 0$.

Definition 2.1. We call $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ a ***series***, and

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \ldots + a_N$$

the ***partial sum***.

Remark 2.2. *Note that S_N is itself a sequence. So it makes sense to talk about whether S_N converges or not.*

Definition 2.3. The series $\sum a_n$ is called ***convergent*** if its partial sum is convergent. Otherwise, $\sum a_n$ is called ***divergent***.

Example 2.4 (Geometric Series)**.** Consider $a_n = r^n$, where r is the common ratio.

$$\begin{aligned} a_0 &= 1, & S_0 &= a_0 = 1, \\ a_1 &= r, & S_1 &= a_0 + a_1 = 1 + r, \\ a_2 &= r^2, & S_2 &= a_0 + a_1 + a_2 = 1 + r + r^2. \\ & & &\vdots \\ & & S_N &= 1 + r + r^2 + \ldots + r^N. \end{aligned}$$

Let $R_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + \ldots + r^N$, then

$$\begin{aligned} rR_N &= r + r^2 + \ldots + r^{N+1}, \\ R_N - rR_N &= 1 - r^{N+1}, \\ R_N &= \frac{1 - r^{N+1}}{1 - r}, \quad \text{for } r \neq 1. \end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1, \\ \infty & \text{if } r \geq 1 \\ \text{DNE} & \text{if } r \leq -1. \end{cases}$$

Also, note that $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ because $\sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r^n$. Thus, the starting point matters. •

[31]
Typo22

Example 2.5. Compute $\sum_{n=1}^{\infty} 2^{2n} \cdot 6^{1-n}$ using the formula from the previous example.

$$\begin{aligned} \sum_{n=1}^{\infty} 2^2 \cdot 6^{1-n} &= \sum_{n=1}^{\infty} 4^n \cdot 6 \cdot \left(\frac{1}{6}\right)^n = 6 \cdot \sum_{n=1}^{\infty} \left(\frac{4}{6}\right)^n = 6 \cdot \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n \qquad (\text{Here } r = \frac{2}{3}) \\ &= 6 \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 6 \cdot 2 = 12. \end{aligned}$$

•

Example 2.6 (Harmonic Series)**.** $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

Partial sums:

$$\begin{aligned}
S_2 &= 1 + \frac{1}{2}, \\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}, \\
S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \\
&> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{3}{2}, \\
S_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n} > 1 + \frac{n}{2} \xrightarrow{n\to\infty} \infty.
\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. •

Example 2.7 (Telescope series)**.** Check that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

We note that:

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

$$\begin{aligned}
S_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1} \xrightarrow{n\to\infty} 1.
\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1. •

Theorem 2.8. *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converge, and* c *is a constant, then*

(i) $\sum_{n=1}^{\infty} a_n \pm b_n = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$.

(ii) $\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$.

Example 2.9. Evaluate $\sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \frac{1}{2^n}$.

We have

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} - 1 = 2 - 1 = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

So the original series converges to $3 \cdot 1 + 1 = 4$. •

Theorem 2.10. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. By definition, we know if the series converges to some real number L, we have

$$\begin{aligned} &\lim_{n\to\infty} S_{N-1} = \lim_{n\to\infty} S_N = L \\ \implies &\lim_{N\to\infty} a_N = \lim_{N\to\infty} (S_N - S_{N-1}) = \lim_{N\to\infty} S_N - \lim_{n\to\infty} S_{N-1} = L - L = 0. \end{aligned}$$

□

Corollary 2.11. *If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Remark 2.12. *Note that when* $\lim_{n\to\infty} a_n = 0$, *there is no conclusion. For example,*

$$\lim_{n\to\infty} \frac{1}{n(n+1)} = 0, \text{ we know } \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

$$\lim_{n\to\infty} \frac{1}{n} = 0, \text{ but } \qquad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

Example 2.13 (diverges series)**.**

(i) $\sum_{n=1}^{\infty} (-1)^n$.

(ii) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$.

(iii) $\sum_{n=1}^{\infty} \frac{n}{n+1}$.

•

3. The Integral Test and Estimates of Sums

[32] §11.3

- **New tool for testing convergency**: **The Integral Test** f positive, continuous, decreasing for $x \geq 1$, and let $a_n = f(n)$. Then:

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\, dx \text{ converges.}$$

We have been computing the exact value of a series so far for some special cases. However, in general, this is quite difficult. In those cases, we are interested in finding an estimate.

1. **The Integral Test.**

Theorem 3.1. *Suppose* $f(x) > 0$ *is a continuous and decreasing function for* $x \geq 1$, *and let* $a_n = f(n)$. *Then:*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\, dx \text{ converges.}$$

Moreover:

$$\sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^{\infty} f(x)\, \mathrm{d}x.$$

The error of this estimate is given by

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n.$$

We have

$$\int_{N+1}^{\infty} f(x)\, \mathrm{d}x \leq R_N \leq \int_{N}^{\infty} f(x)\, \mathrm{d}x.$$

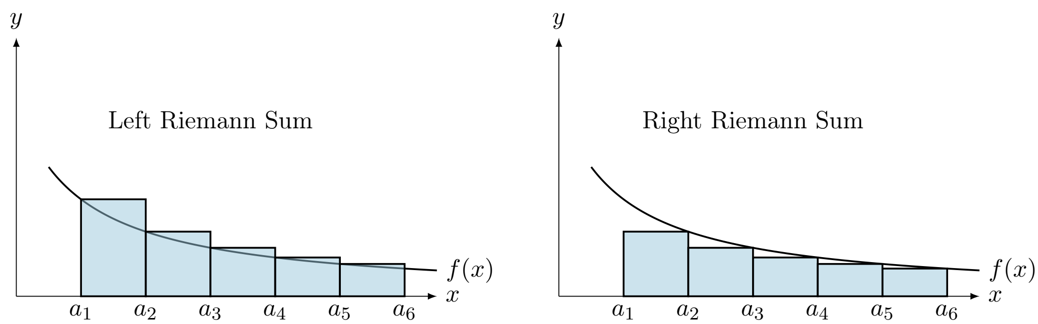


FIGURE 11. Upper and lower bounds for the integral test

Example 3.2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Let $f(x) = \dfrac{1}{x^2}$. For $x \geq 1$, $f(x)$ is continuous, positive, and decreasing. Then $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, \mathrm{d}x$ converges implies $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. •

Example 3.3.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1. \end{cases}$$

Recall $p > 1$, $\int_1^{\infty} \dfrac{1}{x^p}\, \mathrm{d}x$ converges. For $p \leq 1$, it diverges. Apply the integral test. •

Example 3.4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges •

Let $f(x) = \dfrac{1}{x^2+1} > 0$. For $x \leq 1$, we check f is continuous and decreasing:

$$f'(x) = -(x^2+1)^{-2} \cdot 2x < 0, x \geq 1.$$

Apply the integral test as follows:

$$\int_0^{\infty} \frac{1}{x^2+1}\, \mathrm{d}x = \lim_{t \to \infty} \left(\arctan x|_1^t \right) = \lim_{t \to \infty} \left(\arctan t - \frac{\pi}{4} \right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \leq \infty.$$

So the series converges.

Example 3.5. There is an example in the discussion worksheet regarding the error estimate. See Question 2 in Worksheet w8-2. •

4. The Comparison Tests

[33] §11.4

[34] Week 10

- **New tool for testing convergency**:
  - **The (Direct) Comparison Test for Series**: $0 \le a_n \le b_n$ for all $n \ge N$, then

$$\sum b_n \text{ converges} \implies \sum a_n \text{ converges,}$$
$$\sum a_n \text{ diverges} \implies \sum b_n \text{ diverges.}$$

  - **The Limit Comparison Test**: $0 \le a_n \le b_n$ for all $n \ge N$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ where } 0 < c < \infty.$$

Then

$$\sum b_n \text{ converges} \iff \sum a_n \text{ converges.}$$

The idea of the comparison test for sequences is similar to that for integrals.

1. **The (Direct) Comparison Test for Series.**

Theorem 4.1. *Suppose* $\sum a_n$ *and* $\sum b_n$ *are series with positive terms, and* $a_n \le b_n$ *for all* $n \ge N$. *Then:*

- *If* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
- *If* $\sum a_n$ *diverges, then* $\sum b_n$ *diverges.*

Here we can compare the comparison test for series with the comparison test for integrals::

- a_n and b_n play the roles of f and g.
- Integrals are replaced by summations.
- The lower bound $x \ge a$ is replaced by $n \ge N$.

.

Example 4.2. Show that $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges.

Note that $2n^2 + 4n + 3 \le 2n^2$ for $n \le 1$. This implies

$$a_n := \frac{5}{2n^2+4n+3} \le \frac{5}{2n^2} =: b_n.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \implies \sum_{n=1}^{\infty} a_n \text{ converges.}$$

•

Example 4.3. Show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

Note that $\ln n > 1$ for $n < e$. This implies

$$a_n := \frac{\ln n}{n} \leq \frac{1}{n} =: b_n, n \geq 3.$$

Therefore,

$$\sum_{n=3}^{\infty} \frac{1}{n} \text{ diverges.} \quad \Longrightarrow \quad \sum_{n=3}^{\infty} a_n \text{ converges.} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.}$$

•

2. **The Limit Comparison Test.**

Theorem 4.4. *Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms, and:*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \text{ where } 0 < c < \infty.$$

Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Example 4.5. Show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges.

Take $b_n = \frac{1}{2^n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1.$$

Apply the limit comparison test, we conclude that $\sum b_n$ converges implies $\sum \frac{1}{2^n - 1}$ converges. •

Example 4.6. Show that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ diverges.

Take $b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{\sqrt{n}}$ (this is the dominant part). Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{5+n^5}}}{\frac{2}{\sqrt{n}}} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{1/2}}{2\sqrt{5 + n^5}} = \lim_{n\to\infty} \frac{2 + \frac{3}{n}}{2\sqrt{\frac{5}{n^5} + 1}} = \frac{2}{2} = 1.$$

Apply the limit comparison test, we conclude that $\sum b_n$ diverges implies $\sum a_n$ diverges. •

5. Alternating Series

[35] §11.5

- **New Concept**: alternating series
- **The Alternating Series Test**:

$$a_n \text{ positive, decreasing, limit goes to zero, then} \quad \Longrightarrow \quad \sum_{n=0}^{\infty} (-1)^n a_n \text{ converges.}$$

- **Example to memorize**: The alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converges.

So far, we have studied series with positive terms. In this section, we will study series whose terms are alternating series, such as

- $\sum \frac{(-1)^{n+1}}{n}$ (alternating harmonic series).
- $\sum (-1)^n a_n$, where $a_n > 0$ and terms alternate in sign.

1. **The Alternating Series Test.** The following theorem tells us how to determine if an alternating series converges or diverges.

Theorem 5.1. *Given an alternating series* $\sum_{n=0}^{\infty} (-1)^n a_n$, *if*

(i) $a_n > 0$,
(ii) (decreasing a_n*)* $a_{n+1} \leq a_n$ *for all* n,
(iii) $\lim_{n\to\infty} a_n = 0$,

then $\sum_{n=0}^{\infty} (-1)^n a_n$ *converges.*

[36] to complete

Proof. □

Moreover, from the above proof, we see that if $\lim_{n\to\infty} a_n$ diverges, the series also diverges. So the **diverges test still holds**.

Example 5.2 (Alternating Harmonic Series)**.** Show that $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. We first check that the alternating series test applies:

- $a_n = \frac{1}{n} > 0$.
- $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ for all n.
- $\lim_{n\to\infty} a_n = 0$.

Thus, the alternating series test tells us that the series converges. •

Example 5.3. Show that $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n^3+1}$ converges. We first check that the alternating series test applies:

- $a_n = \frac{n^2}{n^3+1} > 0$.

- $a_{n+1} < a_n$ for $n \leq 2$ because the function $f(x) = \dfrac{x^2}{x^3+1}$ is decreasing (not obvious, we compute the derivative):

$$f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0, \text{ where } x > \sqrt[3]{2}.$$

- $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^3}} = 0.$

Apply the alternating series test to $\displaystyle\sum_{n=2}^{\infty}(-1)^n a_n$ (because we need $n \leq 2$), we conclude that $\displaystyle\sum_{n=2}^{\infty}(-1)^n a_n$ converges. So $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n = a_0 - a_1 + \sum_{n=2}^{\infty}(-1)^n a_n$ also converges. •

2. **Estimating alternating series.**

Theorem 5.4 (Alternating series estimation)**.** *Given* $\displaystyle\sum_{n=0}^{\infty}(-1)^n a_n$, $a_n > 0$ *satisfying*

- $a_n > 0$.
- $a_{n+1} \leq \frac{1}{n} = a_n$ *for all* n.
- $\lim_{n\to\infty} a_n = 0$.

Then $|R_n| = |S - S_n| \leq a_{n+1}$.

Proof. □

[37] revise

Last time: alternating series test. This time: absolute convergence and more tests.

6. Absolute Convergence and the Ratio and Root Tests

[38] §11.6

- **New Concept**: absolute convergence and conditional convergence.
- **New tool for testing convergency**: **The Ratio/Root Test**:

$$L_{ratio} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, \qquad L_{root} = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

Then:
- If $L < 1$, the series converges absolutely.
- If $L > 1$, the series diverges.
- If $L = 1$, the test is inconclusive.

Definition 6.1. A series $\sum a_n$ is called ***absolutely convergent*** if the series of absolute values $\sum |a_n|$ converges.

Definition 6.2. A series $\sum a_n$ is called ***conditionally convergent*** if it converges but is not absolutely convergent.

Note that absolute convergence is stronger than convergence

$$\text{If } \sum |a_n| \text{ converges, then } \sum a_n \text{ converges.}$$

Proof. Observe that:

$$-a_n \leq |a_n| \leq a_n \quad \Longrightarrow \quad 0 \leq a_n + |a_n| \leq 2|a_n|.$$

We call $A_n = a_n + |a_n|$, $B_n = 2|a_n|$. By the comparison test, $\sum B_n$ converges implies $\sum |A_n|$ converges. Then

$$\sum a_n = \sum A_n - \sum |a_n| < \infty.$$

□

1. **Examples.**

Example 6.3. The series $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent because:

- $\sum \frac{1}{n}$ diverges (harmonic series).
- $\sum \frac{(-1)^{n+1}}{n}$ converges by the alternating series test.

•

Example 6.4. The series $\sum \frac{(-1)^{n+1}}{n^2}$ is conditionally convergent because:

- $\sum \frac{1}{n^2}$ converges by the p-series test with $p = 2 > 1$.
- $\sum \frac{(-1)^{n+1}}{n^2}$ converges by the alternating series test.

•

2. **Ratio Test and Root Test.**

Theorem 6.5 (Ratio Test)**.** *Given a series $\sum a_n$, let:*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then:

- *If $L < 1$, the series converges absolutely.*
- *If $L > 1$, the series diverges.*
- *If $L = 1$, the test is inconclusive.*

Theorem 6.6 (Root Test)**.** *Given a series $\sum a_n$, let:*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then:

- *If $L < 1$, the series converges absolutely.*
- *If $L > 1$, the series diverges.*
- *If $L = 1$, the test is inconclusive.*

Remark 6.7. *(i) Note that we have absolute convergence.*
(ii) $L = 1$ case examples

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges and } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges.}$$

But in both cases L (for the ratio test) is given by

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

(iii) Prototype for both tests are the geometric series:

- $L_{ratio} = \lim_{n\to\infty} \left| \frac{r^{n+1}}{r^n} \right| = \lim_{n\to\infty} |r| = |r|$.
- $L_{root} = \lim_{n\to\infty} \sqrt[n]{|r|^n} = \lim_{n\to\infty} |r| = |r|$.

Recall that $|r| < 1$ corresponds to convergent series; and that $|r| > 1$ corresponds to divergent series.

3. **Examples.**

Example 6.8. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{(2(n+1)-1)!}}{\frac{n^2}{(2n-1)!}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n)n^2} = 0 < 1.$$

Hence the series converges absolutely by ratio test. •

Example 6.9. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}$

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)^2+1}}{\frac{(-1)^n}{n^2+1}} \right| = \lim_{n\to\infty} \frac{n^2+1}{2n^2+2n+2} = 1.$$

The ratio test has no conclusion. •

Instead, one can use the alternating series test to conclude that this series converges and the comparison test (with $A_n = \frac{1}{n^2+1} \leq B_n = \frac{1}{n^2}$) for absolute convergence.

Example 6.10. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$

$$L = \lim_{n\to\infty} \left| \sqrt[n]{\left(\frac{3n+1}{4-2n}\right)^{2n}} \right| = \lim_{n\to\infty} \left(\frac{3n+1}{4-2n}\right)^2 = \lim_{n\to\infty} \frac{9n^2+6n+1}{4n^2-16n+16} = \frac{9}{4} > 1.$$

Hence the series diverges absolutely by root test. •

[39] Typo22

Example 6.11. $\displaystyle\sum_{n=4}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$

$$L=\lim_{n\to\infty}\left|\frac{\dfrac{(n+1)^2}{(2(n+1)-1)!}}{\dfrac{n^2}{(2n-1)!}}\right|=\lim_{n\to\infty}\frac{(n+1)^2}{(2n+1)(2n)n^2}=0<1.$$

Hence the series converges absolutely by ratio test. Hence the series converges absolutely by root test. •

For strategy of choosing converges tests, see "Supplementary Resources" on course webpage.

7. Power Series

[40] §11.8

Definition 7.1. A ***power series centered at*** a is a series of the form

$$\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+\dots$$

Here, x is a variable, and c_n are coefficients.

Example 7.2. Take $a=0$, then

$$\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+\cdots+c_nx^n+\cdots.$$

This is a polynomial with infinitely many terms. Moreover if $c_n=1$ for all n, then

$$f(x)=1+x+x^2+\cdots=\sum_{n=0}^{\infty}x^n.$$

This is a geometric series, we know it converges when $|x|<1$. •

The above example shows that a power series may converge for some values of x and diverge for others. We use convergence tests to determine this.

Example 7.3. When does $\displaystyle\sum_{n=0}^{\infty}\frac{(x-3)^n}{n}$ converges?

Using ratio test:

$$L=\lim_{n\to\infty}\left|\frac{\dfrac{(x-3)^{n+1}}{n+1}}{\dfrac{(x-3)^n}{n}}\right|=\lim_{n\to\infty}\frac{|x-3|}{1+\dfrac{1}{n}}=|x-3|.$$

Hence the series converges absolutely when $|x-3|<1$ (i.e. $2<x<4$) and $|x-3|>1$ (i.e. $x<2$ or $x>4$) diverges by ratio test.

Now we analysis the boundary cases:

- When $x=2$, $\sum a_n=\dfrac{(-1)^n}{n}$ converges.
- When $x=4$, $\sum a_n=\dfrac{1}{n}$ diverges.

Conclusion: the series converges when $x \in [2, 4)$. •

Theorem 7.4. *For a power series $\sum c_n(x-a)^n$, there are three possibilities:*

(i) The series converges only at $x = a$.
(ii) The series converges for all x.
(iii) There exists $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Definition 7.5. The number R is called the ***radius of convergence***. The ***interval of convergence*** is the interval that consists of all values of x for which the power series converges.

Example 7.6. For the series $\sum \frac{(x-3)^n}{n}$, the radius of convergence is $R = 2$, and the interval of convergence is $[2, 4)]$. •

Example 7.7. Compute the radius of converges and integral of converges for $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^n}$.

Using ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+1}}}{\frac{n(x+2)^n}{3^n}} \right| = \lim_{n\to\infty} \frac{|x+3|}{3\left(1+\frac{1}{n}\right)} = \frac{|x+2|}{3}.$$

The series converges when $\frac{|x+2|}{3} < 1$, so the radius of converges is $R = 3$.

Now we analysis the boundary cases:

- When $x = -5$, $\sum a_n = \frac{(-1)^n n}{3}$ diverges.
- When $x = 1$, $\sum a_n = \frac{n}{3}$ diverges.

So the interval of convergence is $x \in (-5, 1)$. •

8. Representation of Functions by Power Series

[41] §11.9

In this section, we will learn how to represent some functions as power series. An application of this technique is the approximation of certain integrals that do not have elementary antiderivatives.

We start by discussing how to find the power series representation through substitution, integration, and differentiation.

Recall we have seen that

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n, \quad \text{for } |u| < 1.$$

Example 8.1. Find the power series for $\frac{1}{1+x^2}$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^2 = \sum_{n=0}^{\infty}(-1)^n x^{2n}, \quad \text{for } |x| < 1.$$

Take $u = (-x^2)$, then $|u| = |-x^2| = x^2 < 1$. So we have $|x| < 1$. •

Example 8.2. Find the power series for $\dfrac{1}{2+x}$.

$$\frac{1}{2+x} = \frac{1}{2}\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\frac{1}{1-\left(-\frac{x}{2}\right)}$$

(If $|x| < 2$, then $|u| = \left|\frac{x}{2}\right| < 1$, we may use the Equation of $\frac{1}{1-u}$.)

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x)^n.$$

•

1. **Term-by-Term Differentiation and Integration.**

Theorem 8.3. *If* $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$*, then* $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *is differentiable within* $(a-R, a+R)$*.*

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$\int f(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} nc_n\frac{(x-a)^{n+1}}{n+1}.$$

Proof. One can prove this by computing the differentiation:

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \quad \text{for } |x-a| < R.$$

and the integration:

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n\,\mathrm{d}x = C + \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1}, \quad \text{for } |x-a| < R.$$

□

2. **Examples.**

Example 8.4.

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}x}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = \frac{\mathrm{d}x}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} nx^{n-1} \text{ when } |x| < 1.$$

•

Example 8.5. Recall by the Fundamental Theorem of Calculus,

$$\ln(1+x) - \ln(1+0) = \int_0^x \frac{1}{1+t}\,\mathrm{d}t.$$

This implies (note that $\ln(1+0)=0$) for $|x|<1$,

$$\begin{aligned}\ln(1+x) &= \int_0^x \frac{1}{1-(-t)}\,\mathrm{d}t = \int_0^x \sum_{n=0}^{\infty}(-t)^n\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty}\int_0^x (-1)^n t^n\,\mathrm{d}t = \sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{n+1}}{n+1}\right]_{t=0}^{x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}.\end{aligned}$$

Thus:

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}, \quad \text{for } |x|<1.$$

•

Example 8.6. Another solution for solving $\ln(1+x)$.

$$\begin{aligned}\ln(1+x) &= \int \sum_{n=0}^{\infty}(-1)^n x^n\,\mathrm{d}x \qquad (\text{Take } u=-t, \text{ need } |u|=|-t|<1, \text{ i.e. } |t|<1) \\ &= \int \sum_{n=0}^{\infty}(-1)^n x^n\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \int x^n\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} + C, \quad \text{when } |x|<1.\end{aligned}$$

To determine C, take $x=0$, we have

$$\ln(1+0) = 0 = C.$$

•

Example 8.7 ($\arctan(x)$)**.** the Fundamental Theorem of Calculus,

$$\arctan(x) - \arctan(0) = \int_0^x \frac{1}{1+t^2}\,\mathrm{d}t.$$

This implies (note that $\arctan(0)=0$) for $|x|<1$,

$$\begin{aligned}\arctan(x) &= \int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \int_0^x \sum_{n=0}^{\infty}(-t^2)^n\,\mathrm{d}t \\ &= \sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n}\,\mathrm{d}t = \sum_{n=0}^{\infty}(-1)^n\left[\frac{t^{2n+1}}{2n+1}\right]_{t=0}^{x} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}, \quad \text{when } |x|<1.\end{aligned}$$

Thus:

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}, \quad \text{for } |x|<1.$$

•

Example 8.8. Another solution for solving $\arctan(x)$.

$$\begin{aligned}\arctan(x) &= \int \frac{1}{1+x^2} = \int \sum_{n=0}^{\infty}(-x^2)^n\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \int x^{2n}\,\mathrm{d}x \\ &= \int \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}\,\mathrm{d}x + C, \quad \text{when } |x|<1.\end{aligned}$$

To determine C, take $x=0$, we have

$$\arctan(0) = 0 = C.$$

9. Taylor and Maclaurin Series

[42]
§11.10

Theorem 9.1. *Suppose the function $f(x)$ has a power series representation at a given by:*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R.$$

Then $c_n = \frac{f^{(n)}(a)}{n!}$.

Proof. We compute derivatives:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} c_n n(x-a)^{n-1}, \\ f''(x) &= \sum_{n=2}^{\infty} c_n n(n-1)(x-a)^{n-2}. \end{aligned}$$

Taking $x = a$ yields

$$\begin{aligned} f'(a) &= c_1, && (C_1 \text{ is the only non-vanishing term}) \\ f''(a) &= 2!c_2, \\ &\vdots \\ f^{(n)}(a) &= n!c_n. \end{aligned}$$

□

Definition 9.2. We define the ***Taylor series*** of f centered at a as:

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad |x-a| < R.$$

When $a = 0$, this is called the ***Maclaurin series***:

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n.$$

Example 9.3 ($f(x) = e^x$ at $a = 0$)**.** The derivatives of $f(x)$ are given by

$$f^{(n)}(x) = e^x \quad \text{for all } n.$$

So $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. We compute the radius of convergence:

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = 0 < 1.$$

The radius of convergence is ∞.

Example 9.4 ($f(x) = \sin(x)$ at $a = 0$). The derivatives of $f(x)$ are given by

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f'''(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x).$$

Higher order derivatives repeat. So

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

(Note that sin is an odd function). The radius of convergence $R = \infty$, as

$$L = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0 < 1.$$

•

Example 9.5 ($f(x) = \cos(x)$ at $a = 0$). Check that

$$f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f'''(x) = \sin(x), \quad f^{(4)}(x) = \cos(x).$$

So

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad \text{even function.}$$

The radius of convergence is again $R = \infty$. •

10. Applications of Taylor Polynomials

[43] §11.11

1. **Estimating Integrals.** Let's consider a particular integral:

Example 10.1. Compute $\int_0^\infty e^{-x^2}\,\mathrm{d}x$.

Step 1. Get the Maclaurin series of $\int_0^\infty e^{-x^2}\,\mathrm{d}x$*.*. Recall the Maclaurin series expansion for e^{-x^2}:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

Integrating term by term:

$$\begin{aligned} \int e^{-x^2}\,\mathrm{d}x &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} + C. \end{aligned}$$

Evaluate at $x = 0$ and $x = 1$ gives

$$\int_0^1 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1^{2n+1}}{2n+1} + C - \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{0^{2n+1}}{2n+1} + C \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}.$$

To estimate the integral, we use the first five terms:

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475.$$

Using the alternating series estimation theorem, the error is bounded by:

$$|R| < |a_6| = \frac{1}{(2 \cdot 6 + 1)!} < 0.001.$$

•

2. **Approximating Functions.** We denote the N-th degree Taylor polynomial of f at a as:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

For example, $N = 1$

$$T_1(x) = f(a) + f'(a)(x-a).$$

This is the tangent line of the function f.

The error in this approximation is given by:

$$R_N(x) = f(x) - T_N(x).$$

Taylor's inequality states:

$$|R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!},$$

where M is an upper bound on $|f^{(N+1)}(x)|$ for x in the interval of interest.

Example 10.2. Let $f(x) = \sqrt[3]{x}$ with $N = 2$ at $a = 8$.

We compute:

$$f(x) = \sqrt[3]{x}, \quad f'(x) = \frac{1}{3}x^{-2/3}, \quad f''(x) = -\frac{2}{9}x^{-5/3}.$$

$$f(8) = \sqrt[3]{8} = 2, \quad f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}, \quad f''(8) = -\frac{2}{9} \cdot 8^{-5/3} = -\frac{2}{9} \cdot \frac{1}{32} = -\frac{1}{144}.$$

Then the second-degree Taylor polynomial is:

$$T_2(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2.$$

To estimate the error, we use Taylor's inequality:

$$|R_2(x)| \leq \frac{M|x-8|^3}{3!},$$

where M is an upper bound on $|f^{(3)}(x)| = \left|\frac{10}{27}x^{-8/3}\right|$ for x in the interval of interest.

For x near $a = 8$, the maximum value of $|f^{(3)}(x)|$ occurs at $x = 7$, so

$$|f^{(3)}(x)| \leq |f^{(3)}(7)| \leq \frac{10}{27} \cdot 7^{-8/3} < 0.0021.$$

Finally, the error is bounded by:

$$|R_2(x)| < \frac{M}{3!} < 0.0004.$$

•

11. List of Common Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty.$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad |x| < \infty.$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad |x| < \infty.$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1.$$

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$