Preliminary to Ch7 of <u>Hamilton's Ricci flow</u>

- Ric flow $\qquad \partial_t g = -2\,Ric \qquad\qquad$ (RF)

- normalized $\qquad \partial_t g = -2\,Ric + \frac{2}{n} r g \qquad\qquad$ (NRF)

- Kulkani – Nomizu product

$$(\omega \otimes \eta)_{ijkl} = \omega_{ik}\eta_{jl} + \omega_{jl}\eta_{ik} - \omega_{il}\eta_{jk} - \omega_{jk}\eta_{il}$$

- decomposition of Rm

$$Rm = \frac{R}{2n(n-1)} g^2 + \frac{1}{n-2} \mathring{Ric}\, g + W$$

$$|g^2| = 8n(n-1)$$

Schur's lemma $\quad n \neq 2$

$$Rm = \frac{R}{2n(n-1)} g^2 \implies R \text{ const.}$$

pf. $\quad Ric = \frac{1}{n} R g$

2nd Bianchi id $\quad d\,scal = 2\, div(Ric)$

$LHS = dR$

$RHS = 2 g^{ij} \nabla_i (Ric_{jk}) dx^k$

$= 2 g^{ij} g_{jk} \nabla_i \left(\frac{R}{n}\right) dx^k = \frac{2}{n} dR$

if $n > 2 \longrightarrow dR = 0 \implies R$ const.

$\hookrightarrow \quad R(g_\infty) \equiv \text{const.} \qquad \nabla Rm(g_\infty) = 0$

but $\left|\nabla Rm(g_\infty)\right| (x_\infty, 0) > \theta R(g_\infty)^{3/2} (x_\infty, 0) > 0$

Ch 5 Perelman's no collapsing

Einstein - Hilbert

$$E(g) = \int R_g \, d\mu$$

$$\frac{dE}{ds} = \int_M \langle \partial_s g, \tfrac{1}{2} R g - Ric \rangle \, d\mu$$

gradient flow of E. not parabolic

$$\partial_t g = 2 \nabla E(g) = \underbrace{Rg}_{} - 2Ric$$

want to get rid of this. (comes from $\partial_s d\mu$)
then RHS becomes RHS of Ric flow

To get rid of $\partial_s d\mu$.

- define $\mathcal{E}(g) = \int_M R e^{-f} d\mu$

$$\frac{d\mathcal{E}}{ds} = -\int_M \langle \partial_s g, Ric \rangle e^{-f} d\mu$$

$$+ \int_M (-\Delta \operatorname{tr}_g(\partial_s g) + \nabla_i \nabla_j \partial_s g_{ij}) e^{-f} d\mu$$

get rid of part of this by considering
$\int_M |\nabla f|^2 e^{-f} d\mu$ w/ $\partial_s (e^{-f} d\mu) = 0$

- Take $\mathcal{F}(g, f) = \int_M (R_g + |\nabla f|^2) e^{-f} d\mu$

$$= \int_M (R_g + 2\Delta f - |\nabla f|^2) e^{-f} d\mu$$

then $\frac{d}{ds}\mathcal{F} = -\int_M \langle \partial_s g, Ric + \nabla\nabla f \rangle e^{-f} d\mu.$

$\longrightarrow$ gradient flow of $\mathcal{F}$ is

$$\begin{cases} \partial_t g = -2(Ric + \nabla\nabla f) \\ \partial_t f = -R - \Delta f \end{cases} \qquad (\star)$$

- monotonicity formula

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 2\int_M |Ric + \nabla\nabla f|^2 e^{-f} d\mu \geq 0$$

- pullback via diffeo $(\star)$ becomes

$$\begin{cases} \partial_t \tilde{g} = -2Ric_{\tilde{g}} \\ \partial_t \tilde{f} = -R_{\tilde{g}} - \Delta \tilde{f} + |\nabla \tilde{f}|^2 \end{cases}$$

$$\tilde{g} = \psi^* g \qquad \tilde{f} = \psi^* f.$$

- same monotonicity formula

$$\frac{d}{dt}\mathcal{F}(g(t), f(t)) = 0 \iff \partial_t g = \mathcal{L}_{\nabla f}\, g$$

Def Perelman's Entropy $\quad$ $\mathcal{F}$ + scaling

$\mathcal{W}(g, f, \tau)$

$$= \int_M \left(\tau (R + |\nabla f|^2) + (f - n) \right) (4\pi\tau)^{-n/2} e^{-f} \, d\mu$$

scaling factor $\tau > 0$

Consider
$$\begin{cases} \partial_t g = -2Ric \\ \partial_t f = -\Delta f - R + |\nabla f|^2 + \frac{n}{2\tau} \\ \partial_t \tau = -1 \end{cases}$$

- entropy monotonicity

$$\frac{d}{dt} \mathcal{W} = 2\tau \int_M \left| Ric + \nabla\nabla f - \frac{1}{2\tau} g \right|^2 \underbrace{u}_{(4\pi\tau)^{-n/2} e^{-f}} \, d\mu \geq 0$$

Def μ-invariant $\quad$ monotonicity w.r.t time

$$\mu(g, \tau) = \inf \left\{ \mathcal{W} : f \in C^\infty, \int_M (4\pi\tau)^{-n/2} e^{-f} \, d\mu = 1 \right\} > \infty.$$

§ 5.4.3

logarithmic Sobolev ineq. $\Rightarrow$ $\mathcal{W}$ bdd below

Def κ-noncollapsed below the scale ρ

- $\rho \in (0,\infty]$ $\quad \kappa > 0$ $\quad$ if $\forall\, B(x,r)$, $r<\rho$
- $|Rm(y)| \leq r^{-2}$ $\quad \forall\, y \in B(x,r)$

$\triangleright$ $$\frac{\mathrm{Vol}\, B(x,r)}{r^n} \geq \kappa$$

Thm 5.35 (Perelman: no local collapsing)

- $g(t)$, $\quad t \in [0,T)$ $\quad$ RF sol. on closed M^n
- $T < \infty$

$\triangleright$ $\forall\, \rho \in (0,\infty)$, $\exists\, \kappa = \kappa(g(0), T, \rho) > 0$ s.t. $g(t)$ is κ-noncollapsed below the scale ρ for all $t \in [0,T)$.

Remark 5.36. *Perelman's entropy monotonicity formula rules out local collapse for finite time solutions of the Ricci flow on closed manifolds. The idea of the proof is that if a metric g is κ-collapsed at a point p on a distance scale r for κ small and r bounded, then $\mathcal{W}\left(g,f,r^2\right)$ is large and negative, on the order of $\log \kappa$ for f concentrated in a ball of radius r centered at p. This contradicts the monotonicity formula.* of μ

Ch 6. compactness thm $\qquad$ Thm 6.35

- $\{(M_i^n, g_i(t), O_i)\}_{i\in\mathbb{N}}$

base pt $t=0$

$t \in (\alpha, \omega)$

complete, pointed sol. of RF s.t.

- $|Rm(g_i(t))|_{g_i(t)} \leq C$ on $M_i^n \times (\alpha, \omega)$ for some $C < \infty$.

- $\mathrm{inj}_{g_i(0)}(O_i) \geq \delta > 0$

$\Rightarrow$ $\exists$ subseq. $\xrightarrow{C^k} (M_\infty^n, g_\infty(t), O_\infty)$

a completed, pointed sol. of RF

$|Rm(g_\infty)|_{g_\infty} \leq C$ on $M_i^n \times (\alpha, \omega)$

pf use Arzela - Ascoli

§ 7.1 Thm (7.2)

- $\left|\frac{1}{n-2}\mathring{Ric}\, g\right|^2 + |W|^2 < \frac{2\varepsilon_n R^n}{n(n-1)}$

$\varepsilon_n = \frac{1}{5}, \frac{1}{10}, \frac{2}{(n-2)(n+1)}$ for $n = 4, 5, \geq 6$

- unique solution to IVP $\begin{cases} (NRF) & \text{for } t \in [0, \infty) \\ g(0) = g_0 \end{cases}$

- $t \to \infty$ $\quad g(t) \to g_\infty$
  - converges exp fast in C^k norm
  - $scal(g_\infty) \equiv$ const.

- $M \cong$ spherical space form

pinching estimate

const sec. curv. Rm

- $\mathring{Rm} = Rm - \frac{2R}{n(n-1)} Id_{\Lambda^2}$

  estimate how far is g from h:

  $sec_h \equiv$ const > 0

Prop 7.4

- Pinching $\longrightarrow$ $|\nabla Rm|$ estimate

  - $(M^n, g(t))$ $\quad n \geq 3.$

    closed $\quad$ $scal_{g(t)} > 0$ $\quad$ Ric flow sol.

  - $|\mathring{R}m| \leq K R^{1-\varepsilon}$ $\quad K < \infty,\ \varepsilon > 0$

  ▷ $\forall \eta > 0,\ \theta > 0,\ \exists\, C = C(g_0, \eta, \theta) < \infty$ s.t.

  if $R(\bar{x}, \bar{t}) \geq C$, $\quad R(\bar{x}, \bar{t}) \geq \eta \cdot \max_{M^3 \times [0, \bar{t}]} R$

  $\Rightarrow \quad |\nabla Rm|(\bar{x}, \bar{t}) \leq \theta R^{3/2}(\bar{x}, \bar{t})$

  Ric scales as g^{-1}, $\quad |\nabla Rm| \sim g^{-3/2}$

pf by contradiction

$|\mathring{Rm}|$ bdd + $R > 0$ $\Rightarrow$ $|Rm| < CR$

if Prop false, then $\exists\, \eta > 0,\ \theta > 0$ s.t. $\forall\, C_i \to \infty$
$\exists\, (x_i, t_i)$ s.t.

$$R(x_i, t_i) \geq \max\left\{ C_i,\ \eta \cdot \max_{M \times [0, t_i]} R \right\}$$

and $|\nabla Rm|(x_i, t_i) \geq \theta\, R^{3/2}(x_i, t_i)$

Perelman no local collapsing (5.41)
+ compactness thm (6.35)

$\longrightarrow$ $\exists\, g_i(t)$ dilated sol.

complete ancient sol.

$$g_i(t) = R(x_i, t_i)\, g(t_i + R(x_i, t_i)^{-1} t) \longrightarrow (M^n_\infty, g_\infty)$$

Rm_∞ bdd $\quad t \in (-\infty, \omega) \quad \omega > 0$
$R_\infty > 0$

$|\mathring{Rm}_\infty| = 0$ on $M^n_\infty \times (-\infty, \omega)$

$$\Rightarrow \quad Rm(g_\infty) = \frac{2 R_\infty}{n(n-1)}\, Id_{\Lambda^2}.$$

Ex 7.2

$$F_\delta = \frac{|\mathring{Rm}|^2}{R^{2-\delta}}$$

compute $\frac{\partial F_\delta}{\partial t}$ and get a curvature term X

$$X = -2R\left(\mathring{\mathrm{Rm}}\right)_{ijk\ell}\left(B_{ijk\ell} + B_{ikj\ell}\right) + \left(|\mathrm{Rc}|^2 - 2\frac{R^2}{n(n-1)}\right)\left|\mathring{\mathrm{Rm}}\right|^2 + 4\frac{R^2}{n(n-1)}\left(\mathring{\mathrm{Rm}}\right)_{pijq}\left(\mathring{\mathrm{Rm}}\right)_{qijp}.$$

X vanishes for const. sec. curv metric.

$$\frac{\partial F_\delta}{\partial t} = \Delta F_\delta + \frac{2(1-\delta)}{R}\langle \nabla R, \nabla F_\delta\rangle - \frac{2}{R^{4-\delta}}\left|R\nabla_i R_{jk\ell m} - \nabla_i R \cdot R_{jk\ell m}\right|^2$$

$$- \frac{\delta(1-\delta)}{R^{4-\delta}}\left|\mathring{\mathrm{Rm}}\right|^2 |\nabla R|^2 + \frac{2}{R^{3-\delta}}\left(\delta\,|\mathrm{Rc}|^2\left|\mathring{\mathrm{Rm}}\right|^2 - 2X\right), \tag{7.7}$$

7.11. $\delta > 0$ small + (7.9)

$$|\mathring{Rm}|^2 = \left|\frac{1}{n-2}\mathring{Ric} \owedge g\right|^2 + |W|^2 \le (1-\delta)^2 \frac{2\varepsilon_n R^2}{n(n-1)} \quad (\star)$$

$\triangleright$ $X \ge \frac{\delta}{n} R^2 |\mathring{Rm}|^2$ $\quad(\star^{\ddagger})$ (7.11)

if $(\star)$ holds at $t=0$,

$\triangleright$ then $(\star^{\ddagger})$ holds for all $t \ge 0$

$\triangleright$ $\exists\, K < \infty$, $\varepsilon > 0$ s.t. $|\mathring{Rm}| \le K R^{1-\varepsilon}$ $\quad$ (7.12) pf.

lower bound of X $\Rightarrow$ upper bound of $\frac{\partial F_\delta}{\partial t}$

if ($\star^{\neq}$) holds, get (7.12) by maximum principle

use contradiction to check this

$\exists$ $t_0 \in (0, T)$ s.t. ($\star^{=}$) holds. RHS > 0

but upper bound of $\frac{\partial F}{\partial t}$ + ($\star^{=}$)

$$\Rightarrow \quad 0 \leq \frac{\partial F}{\partial t} \leq -\frac{4\delta}{nR} |\mathring{Rm}|^2 < 0$$