Midterm Solutions

MATH231

Spring 2022

Midterm 1

Points indicated in each step are the max points at that step. For example, in Q1: if you reached $x \arctan x - \frac{1}{2}\ln|u| + C$, but forgot to substitute back you x, I'll remove no more than 2 points.

Q1–Q4. Evaluate the following integrals. You may use any method other than the hint.

1. $\displaystyle\int \arctan x \, \mathrm{d}x$

$$\begin{aligned}
\int \arctan x \, \mathrm{d}x &= x \arctan x - \int x \, \mathrm{d}(\arctan x) \\
&= x \arctan(x) - \int \frac{x}{1+x^2} \, \mathrm{d}x && \text{(IBP, 4pt)} \\
&= x \arctan x - \frac{1}{2}\int \frac{1}{u} \, \mathrm{d}u && \text{(Substitution } u = 1+x^2\text{)} \\
&= x \arctan x - \frac{1}{2}\ln|u| + C && \text{(2pt)} \\
&= x \arctan x - \frac{1}{2}\ln(1+x^2) + C. && \text{(2pt)}
\end{aligned}$$

2. $\displaystyle\int \frac{1}{\sqrt{x^2-2x}} \, \mathrm{d}x$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2-2x}} \, \mathrm{d}x &= \int \frac{1}{\sqrt{u^2-1}} \, \mathrm{d}u \\
& \qquad \text{(Completing the square and substitution } u = x-1\text{, 2pt)} \\
&= \int \frac{1}{\sqrt{\sec^2\theta - 1}} \, \mathrm{d}(\sec\theta) \\
& \text{(Substitution } u = \sec\theta\text{, either } 0 < \theta < \frac{\pi}{2} \text{ or } \pi < \theta < \frac{3\pi}{2} \text{ see comment i below)} \\
&= \int \frac{\sec\theta\tan\theta}{|\tan\theta|} \, \mathrm{d}\theta && \text{(Absolute value removed since } \tan\theta > 0\text{, 3pt)} \\
&= \int \sec\theta \, \mathrm{d}\theta && \text{(1pt)} \\
&= \ln|\sec\theta + \tan\theta| + C = \ln|u + \sqrt{u^2-1}| + C \\
&= \ln|x - 1 + \sqrt{x^2-2x}| + C. && \text{(2pt, see comment ii below)}
\end{aligned}$$

Comment:

i. Strictly speaking, since that square root is the the denominator, we are not allowed to have $\tan\theta = 0$. So $\theta = 0$ (respectively $\theta = \pi$) is excluded from the range. However, I will not remove points if you write $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

ii. You need Calc I knowledge to compute $\int \sec\theta \, \mathrm{d}\theta$. I didn't intend to test you on this, though we go through how to integrate this in a problem session. Hence, removing 2 points is reasonable if you reach that integral but didn't compute it.

3. $\displaystyle\int \frac{5x}{(x-2)(x+3)}\,\mathrm{d}x$

To decompose, set $\dfrac{5x}{(x-2)(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{x+3} = \dfrac{(A+B)x + (3A-2B)}{(x-2)(x+3)}$. (3pt)

Solve for A, B get $A = 2$, $B = 3$. (2pt)

$$\begin{aligned}\int \frac{5x}{x^2+x-6}\,\mathrm{d}x &= \int \frac{2}{x-2} + \frac{3}{x+3}\,\mathrm{d}x \\ &= 2\int \frac{1}{x-2}\,\mathrm{d}(x-2) + 3\int \frac{1}{x+3}\,\mathrm{d}(x+3) \\ &= 2\cdot \ln|x-2| + 3\cdot \ln|x+3| + C. && \text{(3pt)}\end{aligned}$$

Absolute value is needed.

4. $\displaystyle\int 16\sin^2 x\cos^4 x\,\mathrm{d}x$

Solution 1.

$$\begin{aligned}\int 16\sin^2 x\cos^4 x\,\mathrm{d}x &= 16\int (\sin x\cos x)^2\cdot\cos^2 x\,\mathrm{d}x \\ &= 16\int \frac{\sin^2(2x)}{4}\cdot\frac{1+\cos(2x)}{2}\,\mathrm{d}x && \text{(3pt)} \\ &= 2\left(\int \sin^2(2x)\,\mathrm{d}x + \int \sin^2(2x)\cdot\cos(2x)\,\mathrm{d}x\right) \\ &= 2\left(\int \frac{1-\cos(4x)}{2}\,\mathrm{d}x + \int \frac{1}{2}\sin^2(2x)\,\mathrm{d}(\sin(2x))\right) \\ &= \int 1-\cos(4x)\,\mathrm{d}x + \int \sin^2(2x)\,\mathrm{d}(\sin(2x)) \\ &= x - \frac{\sin(4x)}{4} + \frac{\sin^3(2x)}{3} + C && \text{(7pt)}\end{aligned}$$

Solution 2. I actually didn't expect you to solve it this way. You'll see why in a second.

$$\begin{aligned}\int 16\sin^2 x\cos^4 x\,\mathrm{d}x &= \int 16\sin^2 x\cos^2 x\cos^2 x\,\mathrm{d}x \\ &= 16\int \frac{1-\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\cdot\frac{1+\cos(2x)}{2}\,\mathrm{d}x && \text{(3pt)} \\ &= 2\int \Big(1-\cos^2(2x)\Big)\cdot\Big(1+\cos(2x)\Big)\,\mathrm{d}x\end{aligned}$$

(there are different ways to expand this expression, e.g. combine the first two terms)

$$\begin{aligned} &= 2\int \left(1-\frac{1+\cos(4x)}{2}\right)\cdot\Big(1+\cos(2x)\Big)\,\mathrm{d}x \\ &= \int \Big(1-\cos(4x)\Big)\cdot\Big(1+\cos(2x)\Big)\,\mathrm{d}x \\ &= \int 1+\cos(2x)-\cos(4x)-\cos(4x)\cos(2x)\,\mathrm{d}x && \text{(5pt)}\end{aligned}$$

Now you'll have to use some knowledge from a pre-calculus course. (I'm not assuming you memorize this by heart, so I only leave 2 points for the rest of the computation.) One of the product formula for trig functions says

$$\cos(u)\cos(v) = \frac{\cos(u+v)+\cos(u-v)}{2}.$$

Therefore

$$\begin{aligned}\int \cos(4x)\cos(2x)\,\mathrm{d}x &= \frac{1}{2}\int \cos(6x)+\cos(2x)\,\mathrm{d}x \\ &= \frac{1}{12}\sin(6x)+\frac{1}{4}\sin(2x)+\tilde{C}.\end{aligned}$$

Final answer:

$$\begin{aligned}I &= x+\frac{1}{2}\sin(2x)-\frac{1}{4}\sin(4x)-\frac{1}{12}\sin(6x)-\frac{1}{4}\sin(2x)+C \\ &= x-\frac{1}{4}\sin(4x)+\frac{1}{4}\sin(2x)-\frac{1}{12}\sin(6x)+C\end{aligned}$$

You can check that $\sin^3\theta = \dfrac{3\sin\theta-\sin(3\theta)}{4}$. So these two approaches give you the same answer.

Q5–Q6. Improper integrals.

> This problem aims to test your specific knowledge of improper integrals. If you didn't use the definition in Q5 or the comparison test in Q6 at all, the maximum number of points you can get is half of the total points assigned to that part.

5. **Use the definition** to show that $\displaystyle\int_e^\infty \frac{1}{x\sqrt{\ln x}}\,\mathrm{d}x$ diverges.

$$\begin{aligned}\int_e^\infty \frac{1}{x\sqrt{\ln x}} &= \lim_{t\to\infty}\int_e^t \frac{1}{x\sqrt{\ln x}}\,\mathrm{d}x && (3\text{pt}) \\ &= \lim_{t\to\infty}\int_e^t \frac{1}{\sqrt{\ln x}}\,\mathrm{d}(\ln x) && (3\text{pt}) \\ &= \lim_{t\to\infty} 2\sqrt{\ln x}\,\Big|_e^t \\ s &= \lim_{t\to\infty} 2\sqrt{\ln t}-2. && (\text{diverges, 2pt})\end{aligned}$$

6. **Use the comparison test** to show $\displaystyle\int_\pi^\infty \frac{x\sin^2 x+1}{x^4}\,\mathrm{d}x$ converges.

Use integral law

$$\int_\pi^\infty \frac{x\sin^2 x+1}{x^4} = \int_\pi^\infty \frac{\sin^2 x}{x^3}\,\mathrm{d}x+\int_\pi^\infty \frac{1}{x^4}\,\mathrm{d}x =: I+II. \qquad (2\text{pt})$$

Part II converges because $p=4>1$. (2pt)
Comparison test for I: Note that $0\le \sin^2 x\le 1$. (1pt)

$$\frac{\sin^2 x}{x^3}\le \frac{1}{x^3} \implies I \text{ converges.} \qquad (\text{converges by } p\text{-test, 3pt})$$

Practice midterm 2

2. Compute the surface area generated by rotating the curve $y = \sin\sqrt{x}$ about the x-axis, for $0 \le x \le \pi^2$.

   The infinitesimal line element is given by $\mathrm{d}s = \sqrt{1+(y')^2}\ \mathrm{d}x = \sqrt{1+\frac{\sin^2\sqrt{x}}{4x}}\ \mathrm{d}x$. Hence, the area integral is

$$A = \int_0^{\pi^2} 2\pi R\ \mathrm{d}s = \int_0^{\pi^2} 2\pi \sin\sqrt{x} \cdot \sqrt{1+\frac{\sin^2\sqrt{x}}{4x}}\ \mathrm{d}x \qquad (R = y)$$

   This integral doesn't seem to be solvable. Even if it is solvable, that wouldn't be something you will be asked to solve during an 45-minute exam.

6. Use the integral test to determine if $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or not.

   Consider $f(x) = \dfrac{\ln x}{x}$, $x \ge 1$. The function is continuous and positive. It is decreasing for $x > e$, since $\ln x > 1$ implies

$$f'(x) = \frac{1-\ln x}{x^2} < 0.$$

   So we can apply the integral test for $n \ge 3$. Consider

$$\int_3^{\infty} \frac{\ln x}{x}\ \mathrm{d}x = \lim_{t\to\infty} \int_{\ln 3}^{t} u\ \mathrm{d}u = \lim_{t\to\infty} \frac{1}{2}u^2\Big|_{\ln 3}^{t} = \infty.$$

   By integral test $\sum_{n=3}^{\infty} \frac{\ln n}{n}$ diverges, and so $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ also diverges.

Midterm 2

Similar to midterm 1, points indicated in each step are the max points at that step.

Q1–Q2. Compute the arc length or the surface area given by revolution for the following.

For Q1 and Q2, you don't have to write down the formula for $\mathrm{d}s$ in a separate line. 4 points is given if you correctly set up the integral. the

1. Arc length of the curve $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$ for $1 \le x \le e$.

$$\begin{aligned}\mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\,\mathrm{d}x \\ &= \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2}\,\mathrm{d}x = \left(\frac{x}{2} + \frac{1}{2x}\right)\mathrm{d}x. && \text{(3pt)}\end{aligned}$$

$$\begin{aligned}L &= \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right)\mathrm{d}x && \text{(2pt)} \\ &= \frac{1}{2}\int_1^e x + \frac{1}{x}\,\mathrm{d}x = \frac{1}{2}\left(\frac{x^2}{2} + \ln x\right)\bigg|_1^e \\ &= \frac{1}{4}\cdot(e^2 - 1) + \frac{1}{2}\cdot(\ln e - \ln 1) = \frac{e^2+1}{4}. && \text{(5pt)}\end{aligned}$$

2. Surface area generated by rotating the curve $y = \sqrt{r^2 - x^2}$ about the x-axis, where $r > 0$ and $-r \le x \le r$.

$$\begin{aligned}\mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(\frac{-2x}{2\sqrt{r^2 - x^2}}\right)^2}\,\mathrm{d}x \\ &= \sqrt{1 + \frac{x^2}{r^2 - x^2}}\,\mathrm{d}x = \sqrt{\frac{r^2}{r^2 - x^2}}\,\mathrm{d}x. && \text{(3pt)}\end{aligned}$$

$$\begin{aligned}S &= \int 2\pi \cdot y\,\mathrm{d}s = \int_{-r}^{r} 2\pi \cdot \sqrt{r^2 - x^2} \cdot \sqrt{\frac{r^2}{r^2 - x^2}}\,\mathrm{d}x && \text{(3pt)} \\ &= \int_{-r}^{r} 2\pi \cdot r\,\mathrm{d}x \\ &= 2\pi r \cdot x|_{-r}^{r} = 4\pi r^2. && \text{(4pt)}\end{aligned}$$

Q3–Q6. Sequences and divergence tests.

3. Compute $\sum_{n=0}^{\infty} \sqrt{5}^{2-2n} \cdot 2^{n+2}$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \sqrt{5}^{2-2n} \cdot 2^{n+2} &= \sum_{n=0}^{\infty} \sqrt{5}^{2} \left(\sqrt{5}^{2}\right)^{-n} \cdot 2^{n} \cdot 2^{2} \\
&= \frac{4}{5} \cdot \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^{n} && \text{(3pt)} \\
&= \frac{4}{5} \cdot \frac{1}{1-\frac{2}{5}} = \frac{4}{5} \cdot \frac{5}{3} = \frac{4}{3}. && \text{(3pt)}
\end{aligned}$$

4. Use the divergence test to determine if $\sum_{n=1}^{\infty} \left(1+\frac{1}{2n}\right)^{n}$ converges or not.

> Q4: 2pt assigned for applying the divergent test; 6pt for computing the limit of the sequence.

The series diverges because $\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{n} = \sqrt{e} \neq 0.$ (2pt)

To compute this limit:

Method 1: Using the fact that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$. Let $m = 2n$ then (2pt)

$$\begin{aligned}
\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{n} &= \lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{m/2} \\
&= \left(\lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{m}\right)^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}. && \text{(4pt)}
\end{aligned}$$

Method 2: Compute the limit directly

$$\begin{aligned}
\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{n} &= \lim_{n\to\infty} \exp\left(\ln\left(1+\frac{1}{2n}\right)^{n}\right) && \text{(exp and ln functions are inverses.)} \\
&= \exp\left(\lim_{n\to\infty} n \ln\left(1+\frac{1}{2n}\right)\right) = \exp\left(\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{2n}\right)}{\frac{1}{n}}\right).
\end{aligned}$$

To compute the limit inside exponential, apply L'Hopital's rule (x is used because we need the function to be differentiable, but n is discrete)

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{1}{1+\frac{1}{2x}} \cdot \frac{1}{2}\frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{2}}{1+\frac{1}{2x}} = \frac{1}{2}. \qquad \text{(4pt)}$$

So the final answer for the limit is $\sqrt{e} \neq 0$. (2pt)

5. Use the integral test to determine if $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or not.

Let $f(x) = xe^{-x^2}$. Note that f is a positive and continuous function defined on $[1, \infty)$. Moreover f is decreasing because (2pt)

$$f'(x) = e^{-x^2}(1 - 2x^2) \quad \Longrightarrow \quad f'(x) < 0 \text{ when } x \geq 1. \qquad \text{(2pt)}$$

The above argument guarantees that we can apply the integral test. Now compute the improper integral:

$$\begin{aligned}\int_1^{\infty} xe^{-x^2}\,\mathrm{d}x = \int_1^{\infty} \frac{1}{2}e^{-x^2}\,\mathrm{d}x^2 &= -\frac{1}{2}\lim_{t\to\infty} e^{-x^2}\Big|_1^t \\ &= -\frac{1}{2}\lim_{t\to\infty} e^{-t^2} - e^{-1} = \frac{1}{2e} \leq \infty. \qquad \text{(4pt)}\end{aligned}$$

The series converges by integral test. (2pt)

6. Use the comparison test to determine if $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^6 + n}}$ converges or not.

Note that $\frac{n}{\sqrt{n^6}}$ dominates the sequence. Note that for $n \geq 1$,

$$\begin{aligned} n^3 = \sqrt{n^6} < \sqrt{n^6 + n} &\Longrightarrow \frac{1}{\sqrt{n^6+n}} < \frac{1}{n^3} \\ &\Longrightarrow \frac{n}{\sqrt{n^6+n}} < \frac{1}{n^2}. \qquad \text{(4pt)}\end{aligned}$$

Apply comparison test with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ (we know this converges), we conclude the series converges. (2pt)