Spring 2021 MATH231 Section CDQ Discussion

WF 9-10am

This document can be found here or on my website. I will continue update it until the end of semester.

Contact

- TA for section CDQ: Xinran Yu
- Email: xinran4@illinois.edu. Please included MATH231 in your email subject.
- Office hour: Wed 10-11am[1]

Zoom

- Please use your cameras and microphones in breakout rooms.
- Interrupt me/using the "Raise Hand" feature on Zoom to ask questions.
- You can call me into your breakout room/return to the main room to ask for help.
- It is also possible for me to join your breakout rooms randomly to check if you have any questions.

Worksheet

- Worksheet can be found on Moodle under Groupwork folder.
- Ask for hints when you get stuck on a problem.

Submission

- Submit on Moodle under Groupwork folder.
- **1 submission per group**. Once a file is uploaded, everyone in the same group will be able to see/edit the file. [2]
- Group remains the same until each midterm.
- 1st worksheet of the week is due on **Thursday** at **8AM** CST. [3]
- 2nd worksheet of the week is due on **Saturday** at **8AM** CST.
- Worksheet solutions available at 12:30PM CST on the due date.

Grading

Worksheets are graded with 2, 1 or 0.

2 - the worksheet uploaded is satisfactory

1 - the worksheet uploaded is unsatisfactory and needs improvement. Your TA will comment on what should be improved for next time.

0 - the worksheet was not uploaded

[1]Office hour is run for all students in MATH231
[2]Groups are assigned randomly by Moodle
[3]Central Standard Time

Contents

1 Worksheet 1 3

2 Worksheet 2 4

3 Worksheet 3 5

4 Worksheet 4 6

5 Worksheet 5 7

6 Worksheet 6 8

7 Worksheet 7 9

8 Worksheet 8 9

9 Worksheet 9 10

10 Worksheet 10 11

11 Worksheet 11 12

12 Worksheet 12 13

13 Worksheet 13 15

14 Worksheet 14 16

15 Worksheet 15 17

16 Worksheet 16 18

17 Worksheet 17 19

18 Worksheet 18 20

19 Worksheet 19 21

Worksheet 1

Recall

Theorem 1.1 (Fundamental Theorem of Calculus)**.** *Ref p.26*

Part 1 *If $f(x)$ is* ***continuous*** *over an interval $[a, b]$, and the function $F(x)$ is defined by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \quad x \in [a, b],$$

then $F'(x) = f(x)$ over $[a, b]$.

Part 2 *If $f(x)$ is* ***continuous*** *over an interval $[a, b]$, and $F(x)$ is any antiderivative of $f(x)$ i.e. $F'(x) = f(x)$, then*

$$\int_a^b f(x)\,\mathrm{d}x = F(a) - F(b).$$

Example 1.2. Let

$$g(x) = \int_a^{b(x)} f(t)\,\mathrm{d}t.$$

Apply chain rule and FTC

$$g'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_c^{b(x)} f(t)\,\mathrm{d}t = b'(x) \cdot f(b(x)).$$

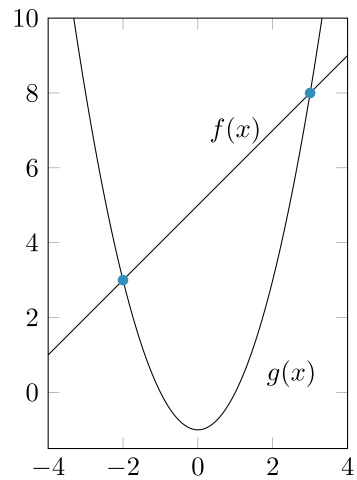
Worksheet 2

Recall

- **Substitution rule/Change of variable**: let $u = g(x)$, then

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u. \tag{Q1-3}$$

- **Compute area between curves.** (Q4-7)
  - Draw the graph.
  - Find intersection points by solving $f(x) = g(x)$, say they are $x = a$ and $x = b$.
  - Area $= \int_a^b f(x) - g(x)\,\mathrm{d}x$.



Worksheet 3

Volume of a Solid of Revolution: Slices of volume are circles. Ref

$$\text{Vol } = \int_a^b \pi f(x)^2 \, \mathrm{d}x.$$

Q3. Slices are squares/triangles.

$$\text{Vol } = \int_a^b \text{ Area of slices } \mathrm{d}x.$$

E.g.

$$\text{Vol } = \int_a^b f(x)^2 \, \mathrm{d}x.$$

Worksheet 4

Volume by Cylindrical Shells:

$$\text{Vol} = \int_a^b 2\pi r \cdot f(x)\,\mathrm{d}x.$$

Rotation about y-axis: $r = x$.
Rotation about the vertical line $x = a$: $r = |a - x|$.

Worksheet 5

Recall

- Since $\sin x$ is oscillating between -1 and 1, $\lim_{x\to\infty} \sin x$ does not exists.
- we can use L'Hopital's Rule to compute indeterminate forms "$\frac{0}{0}$" and "$\frac{\infty}{\infty}$".

Theorem 5.1 (L'Hopital's Rule)**.**
Assumptions:

$$\begin{aligned} &\boldsymbol{f(x) \to 0} \qquad \textit{as } x \to a, \\ &\boldsymbol{g(x) \to 0}, \\ &\boldsymbol{g'(x) \neq 0}. \end{aligned}$$

Conclusion:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Warning: check the assumptions before applying L'Hopital's Rule.

Worksheet 6

Integration by Parts:

$$(uv)' = u'v + uv' \implies \int \boldsymbol{u}\,\mathbf{d}\boldsymbol{v} = \boldsymbol{uv} - \int \boldsymbol{v}\,\mathbf{d}\boldsymbol{u}.$$

Choose u based on which of these comes first, (search "integration by parts what to choose as u"):

(1) **L**ogarithmic functions: $\ln x$

(2) **I**nverse trigonometric functions: $\arcsin x$

(3) **A**lgebraic functions: x

(4) **T**rigonometric functions: $\sin x$

(5) **E**xponential functions: e^x

Worksheet 7

Worksheet 8

Recall: $1 + \tan^2 x = \sec^2 x$.

Worksheet 9

Substitution rule/Change of variable: let $u = g(x)$, then

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u.$$

Integration by Parts:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u.$$

Worksheet 10

Partial Fractions decomposition: Find A, B such that

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} - \frac{B}{x-b}.$$

The following table is from this website

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}, k = 1, 2, 3, \ldots$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}, k = 1, 2, 3, \ldots$

Typo in solution

WS10 Q3.

$$\frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{E}{x-1} + \frac{F}{(x-1)^{\mathbf{2}}} + \frac{G}{(x-1)^{\mathbf{3}}}.$$

Worksheet 11

Improper integrals: There are two types of improper integrals $\int_a^b f(x)\,\mathrm{d}x$:

(1) a or b (or both) infinite, e.g $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$.

(2) The function $f(x)$ blows up in the interval $[a, b]$, e.g $\int_0^1 \ln x\,\mathrm{d}x$.

To compute improper integrals, e.g.:

$$\int_1^\infty \frac{1}{x}\ \mathrm{d}x = \lim_{b\to\infty} \int_1^b \frac{1}{x}\ \mathrm{d}x.$$

Worksheet 12

Simpson's rule: Let x_i's be equally spaced points,

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)).$$

Coefficient is 4 for odd i, $i \neq 0, n$; coefficient is 2 for even i, $i \neq 0, n$.

If you want to use the formula $\int \frac{1}{x^2+a^2}\,\mathrm{d}x = \frac{1}{a}\arctan(\frac{x}{a}) + C$, for complex a. Recall how we obtain this formula. We do trig substitution, $x = a\tan\theta$, and use the identity $1+\tan^2\theta = \sec^2\theta$. This identity is still valid for complex θ. So with $x = i\tan\theta$, $\theta = \arctan(-ix)$,

$$\begin{aligned} I = \int \frac{1}{x^2-1}\,\mathrm{d}x &= \int \frac{1}{x^2+i^2}\,\mathrm{d}x \\ &= \int \frac{1}{i^2\tan^2\theta + i^2} i\sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

(Here we need the derivative of $\tan\theta$, but you can check this is $\sec^2\theta$ in the complex case)

$$= -i\int \frac{1}{\sec^2\theta}\sec^2\theta\,\mathrm{d}\theta = -i\theta + C = -i\arctan(-ix) + C.$$

The above checked our formula is valid for complex a. Hence, we can plug in the value of arctan

$$\arctan t = \frac{1}{i}\ln\sqrt{\frac{1+it}{1-it}} = \frac{i}{2}\left[\ln(1-it) - \ln(1+it)\right].$$

So

$$\arctan(-ix) = \frac{1}{i}\ln\sqrt{\frac{1+x}{1-x}} = \frac{i}{2}\left[\ln(1-x) - \ln(1+x)\right].$$

$$I = -i\cdot\frac{i}{2}\left[\ln(1-x) - \ln(1+x)\right] + C = \frac{1}{2}\left[\ln(1-x) - \ln(1+x)\right] + D.$$

Note that D should be a real constant as I is real.

Worksheet 13

Let $\mathrm{d}s$ be the arclength differential.

Arc length:

$$L = \int \mathrm{d}s = \int \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x.$$

Surface area:

$$A = \int 2\pi x \,\mathrm{d}s = \int 2\pi x \cdot \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x.$$

Worksheet 14

$$\text{Force} = \int \rho g \cdot \text{depth}(y) \cdot \text{width}(y)\, \mathrm{d}y.$$

Worksheet 15

Integral test: If $f(x)$ is **continuous, positive and decreasing** on $[N, \infty)$. Then

$$\int_N^\infty f(x)\,\mathrm{d}x \text{ converges} \implies \sum_{n=N}^\infty f(n) \text{ converges;}$$

$$\int_N^\infty f(x)\,\mathrm{d}x \text{ diverges} \implies \sum_{n=N}^\infty f(n) \text{ diverges.}$$

Warning: check the assumptions before applying integral test.

Worksheet 16

Geometric series:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 \cdots = \begin{cases} \dfrac{1}{1-r} & \text{if } |r| < 1 \\ \infty & \text{if } |r| \geq 1 \end{cases}$$

Integral test: If $f(x)$ is continuous, positive and decreasing on $[N, \infty)$. Then

$$\int_N^{\infty} f(x)\,\mathrm{d}x \text{ converges} \implies \sum_{n=N}^{\infty} f(n) \text{ converges;}$$
$$\int_N^{\infty} f(x)\,\mathrm{d}x \text{ diverges} \implies \sum_{n=N}^{\infty} f(n) \text{ diverges.}$$

Error estimate: Assume $\sum_{n=1}^{\infty} a_n$ converges

$$S = \overbrace{a_1 + a_2 + \cdots + a_n}^{\text{partial sum } S_n} + \underbrace{a_{n+1} + a_{n+2} + \cdots}_{\text{reminders } R_n}.$$

Note that the first term in R_n is a_{n+1},

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x < R_n < \int_n^{\infty} f(x)\,\mathrm{d}x.$$

Worksheet 17

Alternating Series Test: Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geq 0$ for all n. If

- $\lim\limits_{n\to\infty} b_n = 0$;
- $\{b_n\}$ is a decreasing sequence the series,

then $\sum_n a_n$ is convergent.

Ratio test: Let $L = \lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- $L < 1$, $\sum_n a_n$ convergent;
- $L > 1$, $\sum_n a_n$ divergent;
- $L = 1$ no conclusion.

Worksheet 18

Comparison Test: If a_n, $b_n > 0$ and $a_n \leq b_n$ for all large n then

- $\sum b_n$ converges, then a_n also converges;
- $\sum a_n$ diverges, then b_n also diverges.

Limit Comparison Test: Given $\sum a_n$, $\sum b_n$, with a_n, $b_n > 0$ If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = C$ for some $C \neq 0$, $C \neq \infty$. Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Worksheet 19

Ratio test: Let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

- $L < 1$, $\sum_n a_n$ convergent;
- $L > 1$, $\sum_n a_n$ divergent;
- $L = 1$ no conclusion.

Ratio test: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- $L < 1$, $\sum_n a_n$ convergent;
- $L > 1$, $\sum_n a_n$ divergent;
- $L = 1$ no conclusion.