Sep 23 <u>Thm</u> (Donaldson)

M_Q C^∞-manifold $\Leftrightarrow$ Q diagonizable $/\mathbb{Z}$.

Example: $\#_k \mathbb{CP}^2 \#_l \overline{\mathbb{CP}^2}$

Indefinite case

<u>Thm</u> (Serre)

$Q: H \times H \longrightarrow \mathbb{Z}$ quadratic, indefinite. $\det Q = \pm 1$

It is completely characterized up to isomorphism by its $\operatorname{rank}(Q) = \dim_{\mathbb{Q}} H \times \mathbb{Q}$

σ_Q = signature of Q.

parity

Examples (1) $S^2 \times S^2$ $\quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_Q = 0$, rank $= 2$

(2) K3-surface

<u>Def</u> A simply connected manifold M that is complex. $K_M = \Omega^{2,0}_M \cong \mathcal{O}_X$.

<u>Thm</u> (Kodaira) Any two K3 surfaces are deformation equivalent. In particular, they are diffeomorphic.

$$\begin{array}{ll} \mathcal{X} & \mathcal{X}|_{\partial\Delta} \cong S^1 \times K3. \\ \downarrow & X_0 \cong K3_2. \\ \Delta & \end{array}$$

(X_i, J_i) $i = 1, 2$ $\qquad \varphi_i : M \xrightarrow{\text{diffeo}} X_i$

$\hookrightarrow$ complex structure

$\varphi_i^* J_i = J_{t \geq i}$

$X_4 \subseteq \mathbb{CP}^3 \qquad X_4 = \{F=0\}. \quad \deg(F) = 0.$

Lefschtz hyperplane theorem + adjunction.
$\Rightarrow X_4$ is $K3$.

Take $\mathbb{T} = \mathbb{C}^2 / \mathbb{Z}\langle\sigma\rangle \qquad \sigma : T \to T$ has 16 fixed pts.
$\qquad\qquad\qquad\qquad\qquad\qquad x \mapsto -x$

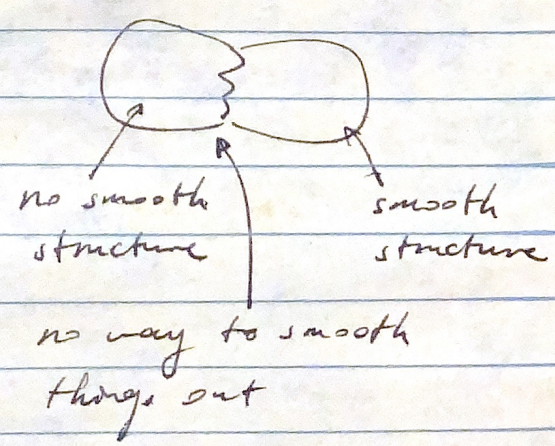
$L \to \mathbb{CP}^1$. tautoulogical bundle
$L \to \mathbb{C}^2$ a neighbourhood of zero section
$x \mapsto x^2$ contains a (-2)-curve.

$Q_{K3} \cong 2E_8 + 3H$.

$$\left.\begin{array}{ll} \sigma_{E_8} = 8 & \text{rank}(E_8) = 8 \\ \sigma_H = 0 & \text{rank}(H) = 2 \end{array}\right\} \Rightarrow \begin{array}{l} \sigma_{Q_{K3}} = 16 \\ \text{rank } Q_{K3} = 22. \end{array}$$

<u>Thm</u> (Hasse-Minkowski)
indefinite, symmetric, even unimodular forms
$$Q \cong mE_8 + nH$$
if not even $\quad \pm 1 \oplus \pm 1, \quad Q \oplus \pm 1$
$$k\,\mathbb{CP}^2 \# h\,\overline{\mathbb{CP}^2}$$

$K3 \underset{\text{homeo}}{\cong} M_{E_8 \oplus E_8} \# 3(S^2 \times S^2)$



M has a spin-structure $\iff$ $w_2(M) = 0$

<u>Thm</u> (Wu)

$\forall S \subseteq M$ oriented embedded surface

$$w_2(T_M) \cdot S = S \cdot S \pmod 2.$$
$$= Q([S], [S])$$

<u>Corollary</u> (1). $w_2(H) = 0 \Rightarrow Q_M$ is even

(2) If $H_1(M, \mathbb{Z}) = 0$ Then Q_M even $\Rightarrow w_2(T_M) = 0$

<u>Corollary</u> Q even, $\pi_1 = \langle 1 \rangle \Rightarrow M$ spin

<u>Def</u> Q is <u>spin-realizable</u> if there is a spin manifold M s.t. $Q_M = Q$.

<u>Conj.</u> (11/8) $2pE_8 + qH$ is spin realizable $\iff q \geq 3p$. $p, q \in \mathbb{Z}$.

Conj. (11/8) M^4 spin smooth, closed

$$b_2(M) \geq \frac{11}{8} |\sigma_M|.$$

Equivalent : $\sigma(2pE_8 + qH) = \sigma(2pE_8) = 16p$.

$$\frac{11}{8}\sigma_M = 22p \leq b_2 = 16p + 2q$$

rank

$\iff 6p \leq 2q$.

($\Leftarrow$) of the conjecture.

if $q \geq 3p$, then $M = \#_p K_3 \; \#_{q-3p} (S^2 \times S^2)$ gives

$$Q_M = 2p E_8 + 3p H + (q-3p) H = 2p E_8 + qH.$$

Looking at Seiberg-Witten equations + Coulomb gauge. H_1, H_2 Hilbert spaces.

$$\widetilde{SW} : H_1 \longrightarrow H_2$$

1) decomposition $\widetilde{SW} = L + C$

L: Fredholm, C: compact

2) $\exists R_0, \varepsilon > 0$ s.t.

$$B_0^{H_1}(R_0) \supseteq \widetilde{SW}^{-1}(B_0^{H_2}(\varepsilon)) \ni 0.$$

3). $Pin(2) = S^1 \cup S^1 j$ $\qquad \mathbb{H} = \mathbb{R} \oplus \mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k$

acts on H_1, H_2. and $\widetilde{SW}$ is $Pin(2)$-equivariant.

Choose $V_2 \subseteq H_2$ finite dim, ~~orthogonal~~ transverse to $\mathrm{Im}(L)^{\perp}$, $Pin(2)$ invariant

$$\left(\widetilde{SW}_{approx.} = L + Pr_{V_2} \circ C\right) : V_1 \longrightarrow V_2$$

where $V_1 = L^{-1}(V_2)$.

$\forall \varepsilon > 0$, consider $\widetilde{SW}^{-1}_{approx}\left(B^{V_2}(0,\varepsilon)\right)$

Choose V_2 large enough

$$\widetilde{SW}_{apr}\left(\partial B^{V_1}(R_0+1)\right) \subseteq \overline{V_2 \setminus B^{V_2}(\varepsilon)}$$

Let $S^{V_1} = B^{V_1}(R_0+1)/\partial B$.

$S^{V_2} = B^{V_2}(\varepsilon)/\partial B$

$\widetilde{SW}_{apr} : S^{V_1} \longrightarrow S^{V_2}$

apply stable homotopy Σ^∞. $\longrightarrow$ No finite dim approximation scheme gives any better.

Q: Can one prove monotonicity of Furuta's ratio by some monotonicity formula? $\frac{10}{8}$ "