MATH541 Functional Analysis

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Spring 2021

Contents

Feb 2022: I recently revisited these notes and made some changes. First, sections are now divided based on topics rather than lectures. Second, some materials are reorganized.

The first two sections contain reviews on the materials from Real Analysis, which are excluded from these notes. In Section 1, definitions and theorems are listed without explicitly stated; in Section 2, precise definitions and theorems are stated without providing examples and proofs.

1 Baire's Categroy Theorem

Textbook: *A Course in Functional Analysis*, John B. Conway, 1985.

review From real analysis:

- metric space; Chicago suburb distance[1]; Cauchy sequence, completeness
- open, closed; closure, interior; dense [2], nowhere dense, somewhere dense
- Countable intersection of open dense is dense, then countable union does not have interior points.
- normed space, Banach space; isometry; $\|f(x)\|_{C(K)} = \sup\limits_{k \in K} |f(k)|$.
- A normed space is complete if absolute convergent sequences are convergent

Theorem 1.1 (Baire Category Theorem for Complete Metric Spaces)**.** *Let X be a complete metric space. Then countable union of nowhere dense sets is nowhere dense.*

Lemma 1.2. *The intersection of open dense sets is again open dense.* [3]

Remark 1.3.

1. Using the above lemma and induction, one can prove Theorem 1.1. (Proof idea: completeness $\to$ geometric series.)

[1] "compact = closed and bounded" no longer true

[2] Intuitively dense set $\approx$ taking away a countable set of points.

[3] There are several versions of the Baire's category theorem, for example one may refer to this notes.

2. Using Baire's theorem to show no function $f : [0,1] \to \mathbb{R}$ continuous exactly at $\mathbb{Q}$.
3. Hard work is to find complete metric space and makes the theorem to work.
4. Let $C_b(\mathbb{R})$ be the set of continuous and bounded function. Is $C_b(\mathbb{R}) = C(K)$ for some compact K? – The answer is yes.

2 Banach Space Basics

Lemma 2.1. *Let $T : X \to Y$ be a linear map between normed spaces. TFRE*

1. *T is continuous.*
2. *T is continuous at 0.*
3. *$\|T\|_{op} = \sup_{\|x\| \leq 1} \|Tx\|$ is finite.*
4. *T is Lipschitz.*

Using homogeneity and duality, we can construct new Banach space from the old ones. Informally speaking, "L(normed, Banach) is a Banach space". This is the following lemma.

Lemma 2.2. *Let X be a normed sapce and Y be Banach. Then the vector space $L(X,Y)$ with the norm $\|\cdot\|_{op}$ becomes a Banach space.*

Corollary 2.3. *$X^* = L(X, \mathbb{C})$ is Banach.*

Proof (sketch) of Lemma 2.2. Step 1. Show that if (T_n) is a Cauchy sequence, then $(T_n(x_k))$ is a Cauchy sequence.

Step 2. Let $T(x) := \lim T_n(x)$. Show that $\limsup \|T_n(x) - T(x)\| = 0$. Indeed

$$\begin{aligned}\|T_n - T\| = \|T_n - \lim T_m\| &= \lim \|T_n - T_m\| \\ &\leq \limsup_{m,n \geq N} \|T_n - T_m\| < \epsilon,\end{aligned}$$

and $\|T_n - T\| < \epsilon$ implies $\|T_n(x) - T(x)\| < \epsilon$. Hence $\limsup \|T_n(x) - T(x)\| = 0$.

Step 3. Show that T is bounded. $\square$

Definition 2.4. A **Banach algebra** is a Banach space $(\mathcal{A}, \|\cdot\|)$ together with a product $\cdot : A \times A \to A$ satisfying $\|ab\| \leq \|a\| \, \|b\|$.

Remark 2.5.

1. closed subset of Banach is Banach.
2. $K(X, Y) := \{T : X \to Y \mid \overline{T(B_X)} \text{ compact}\}$ is closed.
3. In finite dimension, linear bounded T is compact.

Corollary 2.6. *X Banach, then $L(X, X) = L(X)$ is Banach algebra.*

Definition 2.7 (Totally bounded)**.**

$$\forall \epsilon, \exists N \text{ s.t. } Y \subset \bigcup_{j=1}^{N} B(x_j, \epsilon)$$

Totally bounded is equivalent to relatively compact.

We now state a theorem which will be proved later.

Theorem 2.8.

$$K(H, H)^{**} = B(H, H).$$

Theorem 2.9. *There exists inclusion*

$$\iota : X \to X^{**}; \iota(x)(f) = f(x), \text{ with } f : X \to \mathbb{K}$$

such that ι is an isometry and $\overline{\iota(x)}$ is the completion of X.

Note that the isometry part follows form Hahn-Banach Theorem (will be proved later).

Definition 2.10. The **completion** (Y, d') of a metric space (X, d) is a metrics space satisfying

1. $\iota : X \to Y$ is an isometry.
2. ι is dense.

3. (Y, d') is complete.

Remark 2.11. Completion is unique.

Proof of Theorem 2.9. We claim that $\|\iota(x)\|_{X^{**}} \leq \|x\|_X$. Indeed

$$\begin{aligned}
\|\iota(x)\|_{X^{**}} &= \sup_{\|f(x)\|_{X^*}\leq 1} |\iota(x)(f)| && \text{(by definition)} \\
&= \sup_{\|f(x)\|_{X^*}\leq 1} |f(x)| && \text{(ι an inclusion)} \\
&\leq \sup_{\|f(x)\|_{X^*}\leq 1} \|x\| \leq \|x\| .
\end{aligned}$$

By definition $\|f\|_{X^*} \leq 1 \iff |f(x)| \leq \|x\|$.

For a normed space the completion is achieved by taking double duals X^{**}. $\square$

Lemma 2.12. *Consider a Banach space*

$$C_b(x, x_0) = \{f : X \to \mathbb{R} \mid \ \textit{continuous and } \exists C, |f(x)| \leq C \cdot d(x, x_0)\},$$

where the norm is given by $\|f\| = \sup_x \dfrac{|f(x)|}{d(x, x_0)}$. Let $\iota : X \to C_b(X)^*$; $i(x)(f) = f(x)$ be an embedding isometry. Hint: use evaluation map

$$\sup_{\|f\|\leq 1} |f(x) - f(x_0)| = d(x, x_0).$$

Distance attaining function is $f(x) = d(x, x_0)$, where $x \neq x_0$.

3 Hahn-Banach Theorem

Definition 3.1. A **sublinear** function $q : X \to \mathbb{R}$ is a function satisfying

$$q(x + y) \leq q(x) + q(y) \text{ (subadditive) } \text{ and } \ q(sx) = sq(x), s > 0.$$

Example 3.2. Norms and seminorms are examples for sublinear functions.

Theorem 3.3 (Hahn–Banach dominated extension theorem for real linear functionals)**.** *Given a real vector space X, a sublinear function $q : X \to \mathbb{R}$ and a subset $Y \subset X$, any linear functional defined on Y which is dominated above by q has at least one linear extension to all of X that is also dominated above by q. In notation this means: If $f : Y \to \mathbb{R}$, $f \leq q$, then $\exists F : X \to \mathbb{R}$ linear, $F \leq q$ and $F|_Y = f$.*
Moreover, if q is a seminorm, then $|F| \leq q$.

Remark 3.4.

1. Given a function f as above, Theorem 3.3 allows us to extend the dimension of the domain by 1.
2. This theorem is completely algebraic. There is no topology involved.
3. The theorem remains true if the requirements on q are relaxed to require only that q be a convex function.

Proof. Step 1. Denote $X = \{y + tx_0 \mid y \in Y, x_0 \in X, t \in R\}$. Then a candidate for F would be

$$F(y + tx_0) = F(y) + tF(x_0) =: f(y) + ta_0.$$

It remains to find a suitable a_0, which is given by the following computation:

$$\begin{aligned} F(y + tx_0) &\leq q(y + tx_0) & \implies && f(y) + ta_0 &\leq q(y + tx_0) \\ F(y - tx_0) &\leq q(y - tx_0) & && f(y) - sa_0 &\leq q(y - sx_0) \end{aligned}$$

$$\begin{aligned} \implies \qquad a_0 &\leq \frac{q(y + tx_0) - f(y)}{t}, t > 0 \implies & a_0 &\leq \inf_{t>0} \frac{q(y + tx_0) - f(y)}{t} \\ a_0 &\geq \frac{f(y) - q(y - sx_0)}{s}, s > 0 & a_0 &\geq \sup_{s>0} \frac{f(y) - q(y - sx_0)}{s} \end{aligned}$$

To make sure such an a_0 exists, we need to show the supreme is less than the infinimum.

Indeed

$$
\begin{aligned}
\frac{f(y)-q(y-sx_0)}{s} &\leq \frac{q(z+tx_0)-f(z)}{t}. \\
\iff f(y)t-q(y-sx_0)t &\leq q(z+tx_0)s-f(z)s \\
f(y)t+f(z)s &\leq q(z+tx_0)s+q(y-sx_0)t \\
f(yt+sz) &\leq q(yt+tsx_0-tsx_0+sz) \\
&\leq q(yt-stx_0)+q(tsx_0+sz) \\
&\leq tq(y-sx_0)+sq(tx_0+z).
\end{aligned}
$$

Hence we can choose $a_0 = \sup_{s>0} \frac{f(y)-q(y-sx_0)}{s}$.

Step 2. Consider the set

$$\mathcal{L} = \{(Z,F) \mid Y \subset Z, F \leq q \text{ on } Z, F|_Y = f\},$$

with order given by

$$(Z_1,F_1) \leq (Z_2,F_2) \iff Z_1 \subset Z_2 \text{ and } F_2|_{Z_1} = F_1.$$

Note that every chain in this set has an upper bound $Z_\infty = \bigcup Z_i$, $F = \bigcup F_i$. By Zorn's lemma, there exists a maximal element $(Z_{max}, F_{max}) \in \mathcal{L}$.

Step 3. It remains to show that $Z_{max} = X$. Assume the opposite, then there exists some element $x \in X - Z_{max}$. Now apply step 1 again to the function F_{max}, we obtain some F'_{max} defined on the set $Z_{max} + \mathbb{R}x_0$. Contradiction. $\square$

Remark 3.5. Theorem 3.3 also holds if we replace $\mathbb{R}$ by $\mathbb{C}$.