

# Fall 2021 MATH241 Discussion

This document can be found on [my website](#), named as “Discussion notes”. If needed, I will update further information in the same document.

## Time & location

- Section BDH: TR 3-4pm, 441 Altgeld.
- Section BDI: TR 4-5pm, 441 Altgeld.

## Contact

- Email: [xinran4@illinois.edu](mailto:xinran4@illinois.edu). Please included MATH241 and your section number in your email subject. If you don't get a reply in two days, feel free to send me a reminder.
- **Any math question should be post on [Campuswire](#)** because it can be answered much quicker than email and we TAs can type math symbols on Campuswire.
- Office hour: Wed 4-6pm on Zoom.<sup>1</sup>

## Covid related

- **Face mask is required** all the time during discussion.
- Face shield is in general not acceptable unless one holds a [DRES accommodation letter](#).<sup>2</sup>
- **According to university policy, student who does not wear a proper face covering will be asked to put on one or to leave the class room.**
- **If the student refuse to leave, I'll have to dismiss the class and report this to the undergraduate office.**
- In the case one tested positive, status can be checked by your instructor. I'll have to verify it before giving any excuse of being absent due to covid.<sup>3</sup>

## Worksheet

- I will print copies and bring it to the classroom.
- You'll find group number written at the up right corner.
- If you prefer to work on a electronic version, you can find the worksheets on [Moodle](#) under "Worksheets" folder.
- The worksheets are long and you might not be able to finish them in calss, so you don't have to write down every single step.
- **Solutions will be avaiable on Moodle at 5pm** after discussion sections are done.
- Ask for hints when you get stuck on a problem.

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<sup>1</sup>Office hour is run for all students in MATH241, regardless of section.

<sup>2</sup>See accommodation below for more information.

<sup>3</sup>Also see the grading section below.

## Grading

- **Attendance is required** in order to get full grade for discussion.
- You shouldn't come to the class if you are sick.
- **The lowest 4 scores will be dropped** in order to remediate unforeseen illness, change of location or any possible reason for missing a class.
- If you were ill for more than 4 classes and want to see if you could be excused from that, you'll need to provide documents such as DRES letter to your instructor.
- Worksheets will be graded in a scale of 0-5. They are **not** graded for correctness.
- Most likely you'll get a full mark. In case you are interested, here is a sample grading scale:

- 5 Most likely you'll get a full mark
- 4 Being late or leave early for 15 min
- 3 Being late or leave early for 25 min
- 2 Being late or leave early for 35 min
- 1 Not doing anything at all during the class
- 0 Not showing up for any reason.

## Accommodation

- Please contact the Disability Resources & Educational Services ([DRES](#)), if you need any sort of accommodation.
- You'll need to email **both your instructor and me** once you get the accommodation letter.

## Contents

|   |  |    |
|---|--|----|
| 1 | Worksheet 1                                  | 4  |
| 2 | Open discussion: Sep 2nd                     | 5  |
| 3 | Open discussion: Sep 16th & Exam 1 checklist | 6  |
| 4 | Open discussion: Sep 21st                    | 8  |
| 5 | Open discussion: Sep 23rd                    | 10 |
| 6 | Open discussion: Oct 5th                     | 12 |

## Worksheet 1

### Chain rule

If  $h(x) = g \circ f(x)$ , then

$$h'(x) = g'(f(x)) \cdot f'(x).$$

### Arc length of parameterized curve

Given a parameterized curve  $(x(t), y(t))$ , then the arc length between  $(x(a), y(a))$  and  $(x(b), y(b))$  is

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

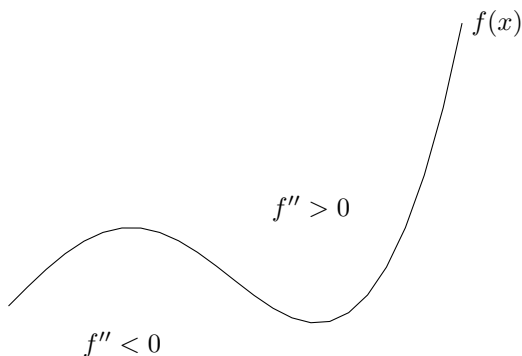
### First and second derivative tests

We use the first and second derivative tests to determine local minimum and maximum.

*First derivative tests.* Compute  $f'(x) = 0$  to find critical points.

*Second derivative tests.*

- If  $f''(x) > 0$  for all  $x$  in the interval, then  $f$  is concave upward  $\implies$  local minimum.
- If  $f''(x) < 0$  for all  $x$  in the interval, then  $f$  is concave downward  $\implies$  local maximum.



### Taylor series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots \end{aligned}$$

### Substitution rule/Change of variable

Let  $u = g(x)$ , then

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du.$$

## Open discussion: Sep 2nd

1. Find the angle between the planes  $2x + y + 2z = 3$  and  $4y + 3z = 1$ .

Hint: Use normal vectors.

2. Determine if the following vectors lie in the same plane or not:

$$\mathbf{a} = \langle 1, 4, -7 \rangle, \mathbf{b} = \langle 2, -1, 4 \rangle \text{ and } \mathbf{c} = \langle 0, 9, -18 \rangle.$$

Hint: Consider  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ , draw a picture represent  $\mathbf{b} \times \mathbf{c}$ .

3. Find a counter example for the following:

$$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} \implies \mathbf{b} = \mathbf{c}.$$

Hint: Consider a triangle  $ABC$ . Let  $\overrightarrow{AB} = \mathbf{a}$ ,  $\overrightarrow{CA} = \mathbf{b}$  and  $\overrightarrow{CB} = \mathbf{c}$ . What is  $\mathbf{a} \times (\mathbf{b} - \mathbf{c})$ ?

4. Derive the following formula:

- Area of a parallelogram determined by the vectors  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\text{Area} = |\mathbf{a} \times \mathbf{b}|.$$

Hint: Notes that  $\mathbf{a} \times \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \sin \theta$ .

- Volume of a parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ :

$$\text{Volume} = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

Hint: Try to find the height of this parallelepiped.

## Open discussion: Sep 16th & Exam 1 checklist

1. Find the following limit or explain why it does not exist.

- $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$

DNE. Hint: Approach  $(0,0)$  along  $x = 0$  and  $y = 0$  respectively.

- $\lim_{(x,y) \rightarrow (1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2}$

$$\lim_{(x,y) \rightarrow (1,1)} \frac{2x^2 - xy - y^2}{x^2 - y^2} = \lim_{(x,y) \rightarrow (1,1)} \frac{(2x+y)(x-y)}{(x-y)(x+y)} = \lim_{(x,y) \rightarrow (1,1)} \frac{2x+y}{x+y} = \frac{3}{2}.$$

- $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + \sin y}{3x^2 + y}$

DNE. Hint: Approach  $(0,0)$  along  $x = 0$  we have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + \sin y}{3x^2 + y} = \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1.$$

Approach  $(0,0)$  along  $y = 0$  we have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + \sin y}{3x^2 + y} = \lim_{x \rightarrow 0} \frac{x^2 + 1}{3x^2} = \frac{1}{3}.$$

2. Determine whether the following functions are continuous at  $(0,0)$  or not.

- $f(x,y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

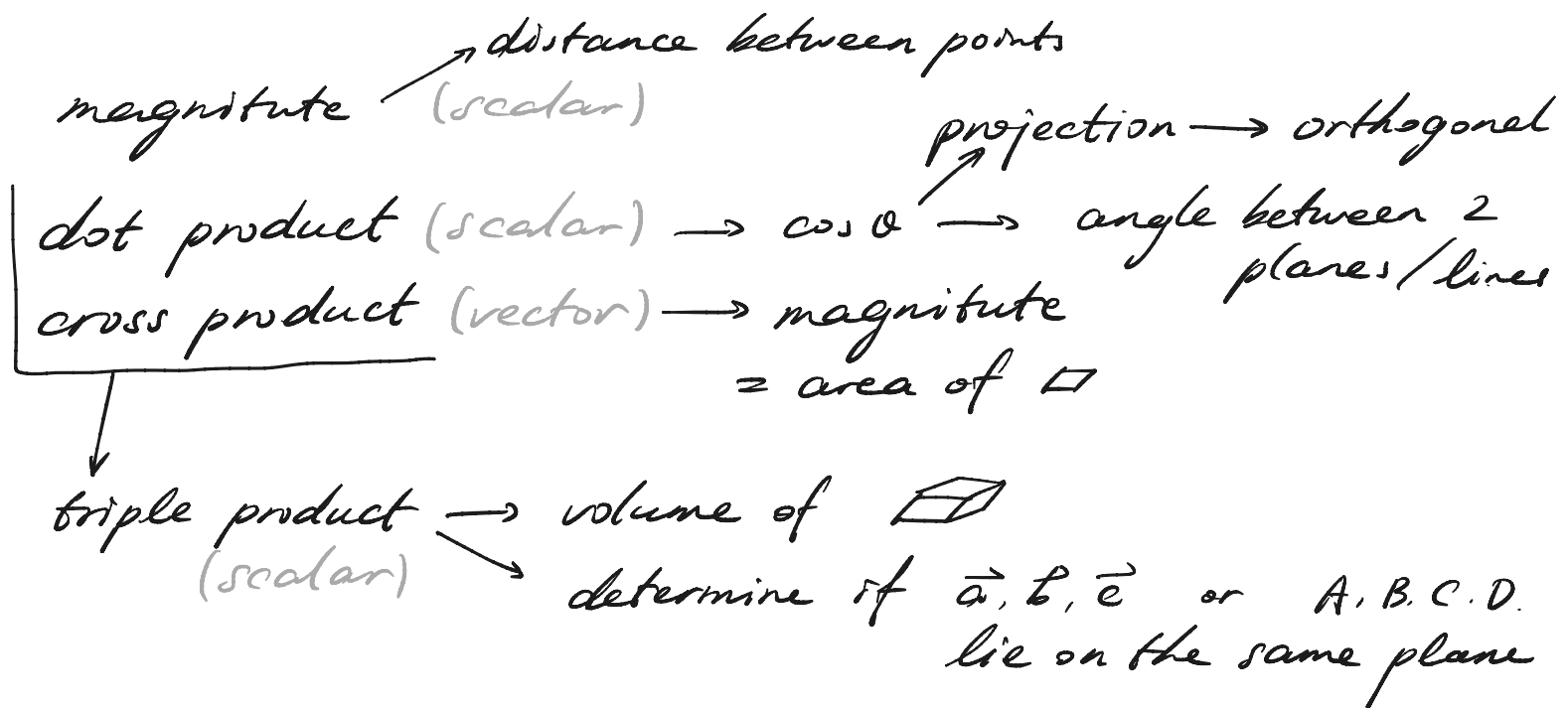
The function is not continuous at  $(0,0)$ . Hint: use  $y = ax$ .

- $f(x,y) = \begin{cases} x \sin \frac{1}{y} + y \sin \frac{1}{x} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$

The function is continuous at  $(0,0)$ :

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| \leq |x| + |y| \xrightarrow{(x,y) \rightarrow (0,0)} 0.$$

## Exam 1 Check list.



line  $\leftrightarrow$  directional vector  $\vec{v}$   
+  
a point on the line  $P$

plane  $\leftrightarrow$  normal vector  $\vec{n}$   
+  
a point on the plane  $P$

$\hookrightarrow$  e.g. given a line  $l$   
find the plane  $\parallel$  to  $l$  and contain a certain pt

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match level sets & figure

- check certain pts.
  - symmetry (odd/even function, switch  $x$  and  $y$ )
  - $f(x, y) \rightarrow \infty$  at some pt /  $x, y \rightarrow \infty$ .
- 

know the equations of quadric surfaces  
computing limits.

## Open discussion: Sep 21st

1. Let  $f(x, y) = e^{xy}$  and  $g(x, y) = f(\sin(x^2 + y), x^3 + 2y + 1)$ . Compute  $g_x$ .

Hint: This question really tests your understanding about composing functions. If you write

$$g(x, y) = e^{\sin(x^2+y)(x^3+2y+1)}$$

everything should be familiar.

2. Let  $f(x, y) = x^2y$ . Find the derivative of  $f$  in the direction of  $\mathbf{v} = \langle 3, 4 \rangle$  at the point  $P = (2, \frac{1}{4})$ .

$$\nabla f(x, y) = \langle 2xy, x^2 \rangle, D_{\mathbf{v}}f(P) = \langle 1, 4 \rangle \cdot \frac{\langle 3, 4 \rangle}{|\langle 3, 4 \rangle|} = \frac{19}{5}.$$

3. Find the direction for which the directional derivative of  $f(x, y) = 3x^2 - 4xy + 2y^2$  at  $P = (\frac{1}{2}, 1)$ . What is the maximum value?

$$\nabla f(x, y) = \langle 6x - 4y, -4x + y \rangle,$$

$$\nabla f(P) = \langle -1, 2 \rangle.$$

The maximum value of the directional derivative occurs when  $\nabla f$  and the unit vector point in the same direction. Note that

$$D_{\mathbf{v}}f = |\nabla f| \cos \theta, \theta \in [-1, 1].$$

We can take  $\cos \theta = 1$ , i.e.  $\theta = 0$  to maximize the directional derivative. The maximum is the magnitude  $|\nabla f(P)| = \sqrt{5}$ .

4. Given a differentiable function  $f(u, v)$ , and let  $g(x, y) = f(x \cos y, \sin(x) + x^2y)$ . Using the following information to compute  $\nabla g$  at  $P = (\pi, 2\pi)$ .

|                  | $f$ | $g$ | $f_u$ | $f_v$ |
|------------------|-----|-----|-------|-------|
| $(\pi, 2\pi)$    | 1   | 0   | 4     | 0     |
| $(-\pi, 4\pi^3)$ | 3   | -2  | 1     | 2     |

Chain rule:

$$\begin{aligned} \nabla g(x, y) &= \langle g_x(x, y), g_y(x, y) \rangle \\ &= \langle f_u(u, v) \cdot u_x(x, y) + f_v(u, v) \cdot v_x(x, y), f_u(u, v) \cdot u_y(x, y) + f_v(u, v) \cdot v_y(x, y) \rangle. \end{aligned}$$

Note that you need to use the value of  $f_u, f_v$  at the point  $(u(x, y), v(x, y)) = (-\pi, 4\pi^3)$  not at the point  $(\pi, 2\pi)$ .

$$\begin{aligned} \nabla g(P) &= \langle 1 \cdot \cos 2\pi + 2 \cdot (\cos \pi + 2 \cdot \pi \cdot 2\pi), 1 \cdot \pi \cdot (-\sin 2\pi) + 2 \cdot \pi^2 \rangle \\ &= \langle 4\pi^2 - 1, 2\pi^2 \rangle. \end{aligned}$$



5. Open-ended question:

- What is an intuitive explanation of the 1-dimensional chain rule?  
Could it be generalized to the chain rule in higher dimension?

We only care the value of  $f$  locally when finding its derivative. In a sufficiently small neighborhood  $U$  around  $x$ ,  $f$  is approximately a linear function. And notice that the chain rule works for linear functions!

For higher dimension, chain rule is essentially matrix multiplication. It is beyond the this course but if you're interested search Jacobian matrix.

- What is a geometric intuition for directional derivatives? Say  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

If the domain is 1-dimensional, say  $y = f(x)$  then we know  $f'(x)$  gives the slope of the *tangent line*. The 2-dimensional generalization of a tangent line would be a *tangent plane*  $T$ .

Say we are considering the directional derivative  $\nabla_{\mathbf{v}}f(P)$  The extra vector  $\mathbf{v}$  appear in the directional derivative gives a *slicing plane*  $A$  (the plane  $A$  is passing through the point  $P$  and parallel to the  $z$ -axis and  $\mathbf{v}$ ). The directional derivative is given by the intersecting line of the tangent plane  $T$  and the slicing plane  $A$ .

To see a figure, visit [this clickable link](#).

## Open discussion: Sep 23rd

1. Let  $P$  be a point in  $\mathbb{R}^2$ . Given that  $D_{\langle 1,0 \rangle} f(P) = \frac{1}{3}$  and  $D_{\langle 0,-4 \rangle} f(P) = 4$ . Compute  $D_{\langle 3,-2 \rangle} f(P)$ .

Using definition of directional derivatives, we have

$$\begin{aligned}\frac{1}{3} &= D_{\langle 1,0 \rangle} f(P) = \nabla f(P) \cdot \langle 1,0 \rangle, \\ 4 &= D_{\langle 0,-4 \rangle} f(P) = \nabla f(P) \cdot \frac{\langle 0,-4 \rangle}{|\langle 0,-4 \rangle|} = \nabla f(P) \cdot \langle 0,-1 \rangle.\end{aligned}$$

Decomposing  $\langle 3,-2 \rangle$ , we have

$$\begin{aligned}D_{\langle 3,-2 \rangle} f(P) &= \nabla f \cdot \frac{\langle 3,-2 \rangle}{|\langle 3,-2 \rangle|} = \nabla f \cdot \frac{\langle 3,-2 \rangle}{\sqrt{13}} \\ &= \nabla f \cdot \left( \frac{3}{\sqrt{13}} \langle 1,0 \rangle + \frac{2}{\sqrt{13}} \langle 0,-1 \rangle \right) \\ &= \frac{3}{\sqrt{13}} \cdot \frac{1}{3} + \frac{2}{\sqrt{13}} \cdot 4 = \frac{9}{\sqrt{13}}.\end{aligned}$$

2. Suppose  $f$  is a differentiable function. Let  $\mathbf{v}$  be some unit vector. At some point  $P$ , suppose  $|\nabla f(P)| = 10$  and  $D_{\mathbf{v}} f(P) = 5\sqrt{3}$ . What is the angle between  $\nabla f(P)$  and  $\mathbf{v}$ .

Let's call the angle  $\theta$ , then

$$\cos \theta = \frac{\nabla f(P) \cdot \mathbf{v}}{|\nabla f(P)| \cdot |\mathbf{v}|} = \frac{D_{\mathbf{v}} f(P)}{|\nabla f(P)|} = \frac{\sqrt{3}}{2}.$$

3. Prove that magnitude of directional derivative  $D_{\mathbf{v}} f$  is the same as component of its projection  $\text{proj}_{\mathbf{v}} \nabla$  on  $\mathbf{v}$ . That is  $\text{comp}_{\mathbf{v}} \nabla f = D_{\mathbf{v}} f$ .

$$\text{comp}_{\mathbf{v}} \nabla f = |\nabla f| \cdot \cos \theta = |\nabla f| \cdot \frac{\nabla f \cdot \mathbf{v}}{|\nabla f| \cdot |\mathbf{v}|} = \nabla f \cdot \mathbf{v} = D_{\mathbf{v}} f.$$

4. Let  $z = f(x,y)$ . Prove the gradient is always perpendicular to the level curves.

Take the level curve given by  $c = f(x,y)$ , where  $c$  is a constant. Suppose we parametrize the level curve using the variable  $t$ , then

$$c = f(x(t), y(t)).$$

Take derivative with respect to  $t$  on both sides, we get

$$\begin{aligned}0 &= f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t) \\ &= \nabla f(x(t), y(t)) \cdot \langle x'(t), y'(t) \rangle.\end{aligned}$$

Note that  $\langle x'(t), y'(t) \rangle$  gives the tangent direction of the level curve.

5. Use gradients and level surfaces to find the tangent plane of the graph of  $z = f(x, y)$  at  $P = (x_0, y_0, z_0)$ .

Consider the function

$$g(x, y, z) = f(x, y) - z.$$

The graph of  $z = f(x, y)$  is exactly the level set  $g^{-1}(0) = \{(x, y, z) \mid g(x, y, z) = 0\}$ . The previous question also generalize to 3-dimension (you can prove this using the same method). We know the normal vector  $\mathbf{n}$  to the surface is given by

$$\nabla g = \langle g_x, g_y, g_z \rangle = \langle f_x, f_y, -1 \rangle.$$

So the equation of the tangent plane is

$$f_x(P) \cdot (x - x_0) + f_y(P) \cdot (y - y_0) - (z - z_0) = 0.$$

## Open discussion: Oct 5th

I took the problems from an MIT open course. [Here](#) is the detailed solution. I won't type the solution again. The plan is to go through these two problems in detail on the blackboard.

For each problem ask yourself the following question:

- What is the function  $f$  we are trying to maximize/minimize?
- What is the constraint function  $g$ ? (We require  $g = \text{const.}$ )

Then

- Set  $L = f - \lambda g$  and write down the equation  $\nabla L = \nabla f - \lambda \nabla g = 0$ .
- Find critical points of  $L$  and determine if it gives the maximum/minimum.

1. Find the maximum and minimum values of  $f(x, y) = x^2 + x + 2y^2$  on the unit circle.

2. Find the minimum and maximum values of  $f(x, y) = x^2 - xy + y^2$  on the quarter circle  $x^2 + y^2 = 1, x, y \geq 0$ .