

## Ch 11. Infinite sequence and series

Def. An sequence is an infinite list of numbers written in a definite order.

Notation:  $\{a_1, a_2, \dots, a_n, \dots\}$ ,  $\{a_n\}$  or  $\{a_n\}_{n=1}^{\infty}$ .

Examples  $\{1, 2, 3, 4, \dots\}$   
 $\{7, 1, 8, 2, 8, \dots\}$

Some sequences can be defined by giving a formula for the  $n$ -th term  $a_n$

Examples 1.  $a_n = \left(\frac{1}{2}\right)^n$   $\{a_n\} = \left\{\frac{1}{2}, \frac{1}{4}, \dots\right\}$

2.  $a_n = (-1)^n$   $\{a_n\} = \{-1, 1, -1, 1, \dots\}$

3.  $a_n = \frac{n}{n+1}$   $\{a_n\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$

Some sequences may not have a simple / explicit defining equation

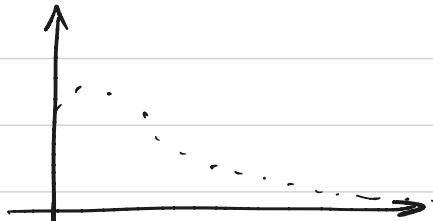
Examples 1.  $a_n$  = the digit in the  $n$ -th decimal place of  $\pi$

2. The Fibonacci sequence

$$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$$
$$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

A sequence "is" a function  $f$  that only takes values on natural numbers. So we will study properties such as graph and convergency.

Example



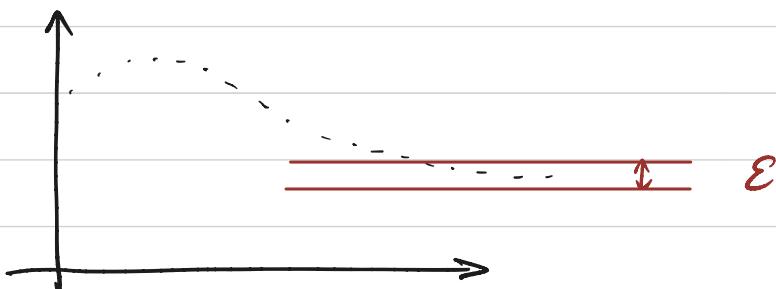
$$\lim_{n \rightarrow \infty} a_n = 0$$

$$L < \infty$$

Def. A sequence has limit  $L$  if for any  $\varepsilon$  there is an  $N$  s.t. if  $n > N$  then  $|a_n - L| < \varepsilon$

We say  $\{a_n\}$  converges to  $L$ .

Intuition



Def.  $\lim_{n \rightarrow \infty} a_n = \infty$  means that for every positive number  $M$  there is an integer  $N$  s.t. if  $n > N$  then  $a_n > M$ .

## Examples

$$1. \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

$$2. \lim_{n \rightarrow \infty} \frac{1}{n^r} = \begin{cases} 0 & \text{if } r > 0 \\ \infty & \text{if } r < 0 \end{cases}$$

$$3. \lim_{n \rightarrow \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

## Limit law for sequences

if  $\{a_n\}, \{b_n\}$  are convergent sequences then

$$\lim_{n \rightarrow \infty} (a_n \pm b_n) = \lim_{n \rightarrow \infty} a_n \pm \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} c a_n = c \lim_{n \rightarrow \infty} a_n \quad c \text{ const.}$$

$$\lim_{n \rightarrow \infty} a_n b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$$

$$\lim_{n \rightarrow \infty} a_n^p = (\lim_{n \rightarrow \infty} a_n)^p \quad p > 0 \quad a_n > 0$$

## Squeeze Theorem

$$b_n \leq a_n \leq c_n \Rightarrow \lim_{n \rightarrow \infty} a_n = L$$

$\downarrow$        $\downarrow$

$L$        $L$

Thm If  $\lim_{n \rightarrow \infty} |a_n| = 0$  then  $\lim_{n \rightarrow \infty} a_n = 0$

If  $f$  is continuous

$$\lim_{n \rightarrow \infty} a_n = L \Rightarrow \lim_{n \rightarrow \infty} f(a_n) = f(L)$$

Example 1.

$$\lim_{n \rightarrow \infty} \sin\left(\frac{\pi}{n}\right) = \sin\left(\lim_{n \rightarrow \infty} \frac{\pi}{n}\right)$$

$$= \sin 0 = 0$$

Example 2.  $\lim_{n \rightarrow \infty} \frac{\ln(n+2)}{\ln(1+4n)}$

$$\lim_{x \rightarrow \infty} \frac{\ln(x+2)}{\ln(1+4x)} \stackrel{\text{L'Hopital}}{=} \lim_{x \rightarrow \infty} \frac{\frac{1}{x+2}}{\frac{1}{1+4x} \cdot 4}$$

$$= \lim_{x \rightarrow \infty} \frac{4x+1}{4(x+2)} = 1$$