

# Homework Solutions

MATH231

Spring 2022

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## Homework 0

### 1. Calculating Limits

- $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2}$

$$\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 3)}{x - 2} = \lim_{x \rightarrow 2} (x + 3) = 5.$$

- $\lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h}$

Consider  $x$  as a “constant”

$$\lim_{h \rightarrow 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 - 2xh - h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh - h^2}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x.$$

- $\lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{x^2 + x} \right)$

$$\lim_{x \rightarrow 0} \left( \frac{1}{x} - \frac{1}{x^2 + x} \right) = \lim_{x \rightarrow 0} \frac{x^2 + x - x}{x(x^2 + x)} = \lim_{x \rightarrow 0} \frac{x^2}{x^2(x + 1)} = \lim_{x \rightarrow 0} \frac{1}{x + 1} = 1.$$

### 2. The Chain Rule

- $\frac{d}{dx} \ln(x + \sin x)$

$$\frac{d}{dx} \ln(x + \sin x) = (1 + \cos x) \frac{1}{x + \sin x}.$$

- $\frac{d}{dx} \cos(x^2 e^x)$

$$\frac{d}{dx} \cos(x^2 e^x) = -(2xe^x + x^2 e^x) \sin(x^2 e^x).$$

### 3. Implicit Differentiation: Solve for $\frac{dy}{dx}$ for the following implicit function.

- $x^2 + y^2 = r^2$ , where  $r$  is a constant

Differentiate on both sides w.r.t.  $x$  gives  $2x + 2yy' = 0$ . Hence  $y' = -\frac{x}{y}$ .

- $\frac{x + y}{x - y} = x$

The above is equivalent to  $x + y = x^2 - xy$ . Differentiate on both sides w.r.t.  $x$  gives  $1 + y' = 2x - y - xy' \iff (1 + x)y' = 2x - y - 1$ . Hence  $y' = \frac{2x - y - 1}{x + 1}$ .

4. Linear Approximations and Differentials: Find the Taylor polynomials of degree two approximating the given function centered at the given point.

- $f(x) = \sin(2x)$  at  $a = \frac{\pi}{2}$

$$f' = 2 \cos(2x), f'' = -4 \sin(2x). \text{ So } f \sim -2 \left( x - \frac{\pi}{2} \right).$$

- $f(x) = e^x$  at  $a = 1$

$$f' = f'' = e^x. \text{ So } f \sim e + e(x - 1) + \frac{e}{2}(x - 1)^2.$$

5. Mean Value Theorem: Determine if the Mean Value Theorem can be applied to the following function on the the given closed interval.

Both intervals are closed. It suffices to check that these functions are continuous on the given interval. One can do this by computing the derivative exists.

- $f(x) = 3 + \sqrt{x}, x \in [0, 4]$

$$\text{Here } f' = \frac{1}{2\sqrt{x}}.$$

- $f(x) = \frac{x}{1+x}, x \in [1, 3]$

$$\text{Here } f' = \frac{1}{(1+x)^2}.$$

6. L'Hospital's Rule

- $\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

Check that as  $x \rightarrow 2$ ,  $x^3 - 7x^2 + 10x \rightarrow 0$  and  $x^2 + x - 6 \rightarrow 0$  so L'Hospital's rule applies. Then

$$\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x \rightarrow 2} \frac{3x^2 - 14x + 10}{2x + 1} = -\frac{6}{5}.$$

The last step uses division property of limits.

- $\lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{x}}$

As exp and ln are continuous functions

$$\lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{x}} = \exp \left[ \ln \left( \lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{x}} \right) \right] = \exp \left( \lim_{x \rightarrow \infty} \frac{\ln(e^x + x)}{x} \right).$$

Check that as  $x \rightarrow \infty$ ,  $e^x + x \rightarrow \infty$  so L'Hospital's rule applies.

$$\begin{aligned} RHS &= \exp \left( \lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x} \right) \\ &= \exp \left( \lim_{x \rightarrow \infty} 1 + \frac{1 - x}{e^x + x} \right) \quad (\text{Check that L'Hospital's rule applies}) \\ &= \exp \left( 1 + \lim_{x \rightarrow \infty} \frac{-1}{e^x + 1} \right) = e. \end{aligned}$$

- $\lim_{x \rightarrow \infty} x \ln \left( 1 + \frac{3}{x} \right)$

$$\lim_{x \rightarrow \infty} x \ln \left( 1 + \frac{3}{x} \right) = \lim_{x \rightarrow \infty} \frac{\ln \left( 1 + \frac{3}{x} \right)}{\frac{1}{x}}.$$

Check that as  $x \rightarrow \infty$ ,  $\ln \left( 1 + \frac{3}{x} \right), \frac{1}{x} \rightarrow 0$  so L'Hospital's rule applies.

$$RHS = \lim_{x \rightarrow \infty} \frac{\frac{-3/x^2}{1+3/x}}{-1/x^2} = \lim_{x \rightarrow \infty} \frac{3}{1 + \frac{3}{x}} = 3.$$

7. The Fundamental Theorem of Calculus: Find the derivative of the following

- $\int_1^x \frac{1}{t^3 + 1} dt$

Apply FTC

$$\frac{d}{dx} \int_1^x \frac{1}{t^3 + 1} dt = \frac{1}{x^3 + 1}.$$

- $\int_1^{\sqrt{x}} \sin t dt$

Let  $u(x) = \sqrt{x}$ . Apply chain rule and FTC

$$\frac{d}{dx} \int_1^x \frac{1}{t^3 + 1} dt = \sin u(x) \cdot \frac{du}{dx} = \sin u(x) \cdot \frac{1}{2\sqrt{x}} = \frac{\sin \sqrt{x}}{2\sqrt{x}}.$$

- $\int_x^{2x} t^3 dt$

Using subtraction property of integral,

$$\int_x^{2x} t^3 dt = \int_0^{2x} t^3 dt - \int_0^x t^3 dt.$$

Apply FTC to each term

$$\frac{d}{dx} \int_x^{2x} t^3 dt = 16x^3 - x^3 = 15x^3.$$

## 8. Substitution Rule

- $\int_{\frac{1}{2}}^0 \frac{x}{\sqrt{1-4x^2}} \, dx$

Take  $u = 1 - 4x^2$ , then  $du = -8x \, dx$  and  $dx = -\frac{1}{8} \, du$ .

$$\int_{\frac{1}{2}}^0 \frac{x}{\sqrt{1-4x^2}} \, dx = \int_0^1 -\frac{1}{8\sqrt{u}} \, du = -\frac{1}{4} \sqrt{u} \Big|_0^1 = -\frac{1}{4}.$$

- $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)} \, dx$

Take  $u = \sin(\pi x)$ , then  $du = \pi \cos(\pi x) \, dx$  and  $dx = \frac{1}{\pi} \, du$ .

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)} \, dx = \int_{\frac{\sqrt{2}}{2}}^1 \frac{1}{\pi u^2} \, du = -\frac{1}{u} \Big|_{\frac{\sqrt{2}}{2}}^1 = \frac{\sqrt{2}-1}{\pi}.$$

- $\int_0^1 x e^{4x^2+3} \, dx$

Take  $u = 4x^2 + 3$ , then  $du = 8x \, dx$

$$\int_0^1 x e^{4x^2+3} \, dx = \frac{1}{8} \int_3^7 e^u \, du = \frac{1}{8} \sqrt{u} \Big|_3^7 = \frac{e^7 - e^3}{8}.$$

## Homework 1

The following solutions provide a possible way to solve the problems. Any other reasonable solution is accepted.

1. Integration by parts (Note that the following integrals are indefinite. You need to add constants to your final answer.) You may also need to use substitution rule.

•  $\int \frac{\ln x}{x^2} dx$  [3pt]

Take  $u = \ln x, v = -\frac{1}{x}$ . Then

$$\int \frac{\ln x}{x^2} dx = \int \ln x d\left(-\frac{1}{x}\right) = -\frac{1}{x} \ln x - \int -\frac{1}{x} d(\ln x) = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

•  $\int x^2 \sin x dx$  [4pt]

Take  $u = x^2, v = -\cos x$ . Then

$$\int x^2 \sin x dx = -x^2 \cos x - \int -\cos x d(x^2) = -x^2 \cos x + 2 \int x \cos x dx.$$

To evaluate  $\int x \cos x dx$  we use integration by parts again with  $u = x, v = \sin x$ .

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + \tilde{C}.$$

Final answer:  $-x^2 \cos x + 2x \sin x + 2 \cos x + C$ .

•  $\int (\ln x)^2 dx$  [4pt]

Take  $u = (\ln x)^2, v = x$ . Then

$$\int (\ln x)^2 dx = x(\ln x)^2 - \int x d(\ln x)^2 = x(\ln x)^2 - \int \ln x dx.$$

To evaluate  $\int \ln x dx$  we use integration by parts again, with  $u = \ln x, v = x$ .

$$\int \ln x dx = x \ln x - \int x d(\ln x) = x \ln x - \int 1 dx = x \ln x - x + C.$$

Final answer:  $x(\ln x)^2 - 2x \ln x + 2x + C$ .

•  $\int \arccos x dx$  [4pt]

Take  $u = \arccos x, v = x$ . Then

$$\int \arccos x dx = x \arccos x - \int x d(\arccos x) = x \arccos x - \int -\frac{x}{\sqrt{1-x^2}} dx.$$

To evaluate  $\int \frac{x}{\sqrt{1-x^2}} dx$  we use substitution rule with  $u = 1 - x^2$ ,  $du = -2x dx$ .

$$\int \frac{x}{\sqrt{1-x^2}} dx = - \int \frac{1}{2\sqrt{u}} du = -\sqrt{u} + C.$$

Final answer:  $x \arccos x - \sqrt{1-x^2} + C$ .

•  $\int e^{\sqrt{x}} dx$  [4pt]

Using substitution rule with  $t = \sqrt{x}$  we obtain

$$\int e^{\sqrt{x}} dx = 2 \int t e^t dt$$

Integration by parts: take  $u = t, v = e^t$ . Then

$$RHS = 2 \int t d(e^t) = 2 \left( t e^t - \int e^t dt \right) = 2(t e^t - e^t + \tilde{C}) = 2(\sqrt{x} - 1)e^{\sqrt{x}} + C.$$

2. Trigonometric integration: Evaluate the following integral of the form  $\int \sin^n x \cos^m x dx$ .

You need specify the values for  $\theta$ , so that you can get rid of absolute values.

•  $\int \sin^2 x \cos^3 x dx$  [3pt]

Note that

$$\int \sin^2 x \cos^3 x dx = \int \sin^2 x \cos^2 x \cdot \cos x dx = \int \sin^2 x (1 - \sin^2 x) \cdot \cos x dx.$$

Apply substitution rule with  $u = \sin x$ ,  $du = \cos x dx$ . So

$$RHS = \int u^2(1-u^2) du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

•  $\int \cos^4 x dx$  [4pt]

Note that

$$\begin{aligned} \cos^4 x &= \cos^2 x \cos^2 x = \frac{1}{4}(1 + \cos(2x))^2 = \frac{1}{4}(1 + 2\cos(2x) + \cos^2(2x)) \\ &= \frac{1}{4} \left( 1 + 2\cos(2x) + \frac{1}{2}(1 + \cos(4x)) \right) = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x). \end{aligned}$$

Hence

$$\begin{aligned} \int \cos^4 x dx &= \int \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x) dx \\ &= \frac{3}{8} \int 1 dx + \frac{1}{2} \int \cos(2x) dx + \frac{1}{8} \int \cos(4x) dx \\ &= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C. \end{aligned}$$

### 3. Trigonometric substitution

- $\int \frac{x^2}{\sqrt{9-x^2}} dx$  [6pt]

Let  $x = 3 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ , then  $dx = 3 \cos \theta d\theta$

$$\begin{aligned} \int \frac{x^2}{\sqrt{9-x^2}} dx &= \int \frac{9 \sin^2 \theta}{\sqrt{9-9 \sin^2 \theta}} \cdot 3 \cos \theta d\theta = \int \frac{27 \sin^2 \theta \cos \theta}{3 \cos \theta} d\theta \\ &= \int 9 \sin^2 \theta d\theta = \int \frac{9(1-\cos 2\theta)}{2} d\theta = \frac{9}{2} \theta - \frac{9}{4} \sin 2\theta + C \\ &= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C = \frac{9}{2} \left( \arcsin \left( \frac{x}{3} \right) - \frac{x}{3} \sqrt{1 - \left( \frac{x}{3} \right)^2} \right) + C \\ &= \frac{9}{2} \arcsin \left( \frac{x}{3} \right) - \frac{x \sqrt{9-x^2}}{2} + C. \end{aligned}$$

- $\int \frac{1}{\sqrt{25+x^2}} dx$  [6pt]

Let  $x = 5 \tan \theta$ ,  $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ , then  $dx = 5 \sec^2 \theta d\theta$

$$\begin{aligned} \int \frac{1}{\sqrt{25+x^2}} dx &= \int \frac{5 \sec^2 \theta}{\sqrt{25+25 \tan^2 \theta}} d\theta = \int \frac{5 \sec^2 \theta}{\sqrt{25 \sec^2 \theta}} d\theta \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C \\ &= \ln \left| \frac{\sqrt{25+x^2}}{5} + \frac{x}{5} \right| + C. \end{aligned}$$

- $\int \frac{1}{\sqrt{x^2+2x}} dx$  [6pt]

Note that

$$\int \frac{1}{\sqrt{x^2+2x}} dx = \int \frac{1}{\sqrt{(x+1)^2-1}} dx.$$

Let  $x+1 = \sec \theta$ ,  $0 \leq \theta < \frac{\pi}{2}$ , then  $dx = \sec \theta \tan \theta d\theta$

$$\begin{aligned} RHS &= \int \frac{\sec \theta \tan \theta}{\sqrt{\sec^2 \theta - 1}} d\theta = \int \frac{\sec \theta \tan \theta}{\tan \theta} dx \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C \\ &= \ln |x+1 + \sqrt{(x+1)^2-1}| + C \\ &= \ln |x+1 + \sqrt{x^2+2x}| + C. \end{aligned}$$



- $\int (x-2)^3 \sqrt{5+4x-x^2} \, dx$  [6pt]

Note that

$$\int (x-2)^3 \sqrt{5+4x-x^2} \, dx = \int (x-2)^3 \sqrt{9-(x-2)^2} \, dx.$$

Let  $x-2 = 3 \sin \theta$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ , then  $dx = 3 \cos \theta \, d\theta$

$$RHS = \int (3 \sin \theta)^3 \cdot \sqrt{9-9 \sin^2 \theta} \cdot 3 \cos \theta \, d\theta = 3^5 \int \sin^3 \theta \cos^2 \theta \, d\theta.$$

To solve  $\int \sin^3 \theta \cos^2 \theta \, d\theta$ , apply substitution rule with  $u = \cos \theta$ ,  $du = -\sin \theta \, d\theta$ .  
So

$$\begin{aligned} \int \sin^3 \theta \cos^2 \theta \, d\theta &= \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta \, d\theta = - \int (1 - u^2) u^2 \, du \\ &= \frac{u^5}{5} - \frac{u^3}{3} + \tilde{C} = \frac{\cos^5 \theta}{5} - \frac{\cos^3 \theta}{3} + \tilde{C} \end{aligned}$$

Final answer:

$$\begin{aligned} &3^5 \left[ \frac{1}{5} \left( \frac{\sqrt{9-(x-2)^2}}{3} \right)^5 - \frac{1}{3} \left( \frac{\sqrt{9-(x-2)^2}}{3} \right)^3 \right] + C \\ &= 3^5 \left[ \frac{1}{5} \left( \frac{\sqrt{5+4x-x^2}}{3} \right)^5 - \frac{1}{3} \left( \frac{\sqrt{5+4x-x^2}}{3} \right)^3 \right] + C. \end{aligned}$$

There's another way to do this problem. We will discuss that in problem session.

## Homework 2

Due: Friday, Feb 11, by the end of the class

### 1. Partial Fractions

•  $\int \frac{2x+5}{x^2+4x+8} dx$  [5pt]

We have

$$\int \frac{2x+5}{x^2+4x+8} dx = \int \frac{2x+4}{x^2+4x+8} dx + \int \frac{1}{x^2+4x+8} dx =: I + II.$$

Solve for  $I$ : substitution rule with  $u = x^2 + 4x + 8$ ,  $du = (2x + 4) dx$ . Then

$$I = \int \frac{1}{u} du = \ln |u| + C_1 = \ln |x^2 + 4x + 8| + C_1.$$

Solve for  $II$ : the second step uses substitution rule with  $u = \frac{x+2}{2}$ ,  $du = \frac{1}{2} dx$ .

$$\begin{aligned} II &= \int \frac{1}{(x+2)^2+4} dx = \frac{1}{2} \int \frac{1}{u^2+1} du \\ &= \frac{1}{2} \arctan u + C_2 = \frac{1}{2} \arctan \frac{x+2}{2} + C_2. \end{aligned}$$

Final answer:  $\ln |x^2 + 4x + 8| + \frac{1}{2} \arctan \frac{x+2}{2} + C.$

•  $\int \frac{2x^2 - x + 4}{(x^2 + 4)(x - 1)} dx$  [5pt]

Update: To decompose that rational function, set

$$\frac{2x^2 - x + 4}{(x^2 + 4)(x - 1)} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x - 1} = \frac{(Ax + B)(x - 1) + C(x^2 + 4)}{(x^2 + 4)(x - 1)}.$$

Solving for  $A, B, C$  by comparing the coefficients gives  $A = 1, B = 0, C = 1$ .

We have

$$\int \frac{2x^2 - x + 4}{(x^2 + 4)(x - 1)} dx = \int \frac{x}{x^2 + 4} dx + \int \frac{1}{x - 1} dx =: I + II.$$

Solve for  $I$ : substitution rule with  $u = x^2 + 4$ ,  $du = 2x dx$ . Then

$$I = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C_1 = \frac{1}{2} \ln |x^2 + 4| + C_1.$$

Solve for  $II$ :

$$II = \ln |x - 1| + C_2.$$

Final answer:  $\frac{1}{2} \ln |x^2 + 4| + \ln |x - 1| + C.$

- $\int \frac{x}{x^4 + 2x^2 + 2} dx$  [4pt]

Substituting  $u = x^2$ ,  $du = 2x dx$  gives

$$\begin{aligned} \int \frac{x}{x^4 + 2x^2 + 2} dx &= \int \frac{1}{(u+1)^2 + 1} du = \frac{1}{2} \arctan(u+1) + C \\ &= \frac{1}{2} \arctan(x^2 + 1) + C. \end{aligned}$$

- $\int \ln(x^2 + 1) dx$  [4pt]

Integration by parts with  $u = \ln(x^2 + 1)$ ,  $v = x$  gives

$$\begin{aligned} \int \ln(x^2 + 1) dx &= x \ln(x^2 + 1) - \int \frac{2x^2}{x^2 + 1} dx \\ &= x \ln(x^2 + 1) - \int \frac{2x^2 + 2 - 2}{x^2 + 1} dx \\ &= x \ln(x^2 + 1) - 2 \int 1 dx + \int \frac{1}{x^2 + 1} dx \\ &= x \ln(x^2 + 1) - 2x + 2 \arctan x + C. \end{aligned}$$

- $\int \frac{1}{\sqrt{x} + x\sqrt{x}} dx$  [4pt]

Note that

$$\int \frac{1}{\sqrt{x} + x\sqrt{x}} dx = \int \frac{1}{1+x} \cdot \frac{1}{\sqrt{x}} dx.$$

Substituting  $u = \sqrt{x}$ ,  $du = \frac{1}{2\sqrt{x}} dx$  gives

$$RHS = 2 \int \frac{1}{1+u^2} du = 2 \arctan u + C = 2 \arctan(\sqrt{x}) + C.$$

- $\int \frac{1}{x + \sqrt[3]{x}} dx$  [4pt]

Note that

$$\int \frac{1}{x + \sqrt[3]{x}} dx = \int \frac{1}{x^{2/3} + 1} \cdot \frac{1}{\sqrt[3]{x}} dx.$$

Substituting  $u = x^{2/3}$ ,  $du = \frac{2}{3\sqrt[3]{x}} dx$  gives

$$RHS = \frac{3}{2} \int \frac{1}{u+1} du = \frac{3}{2} \ln|u+1| + C = \frac{3}{2} \ln|x^{2/3} + 1| + C.$$

## 2. Approximate Integration

- Use the Midpoint Rule with  $n = 5$  to approximate  $\int_0^{10} x^2 \, dx$ . [3pt]

The width of each subinterval is 2. Compute the value of  $f$  and substituting into the formula gives

$$\int_0^{10} x^2 \, dx \approx 2(1^2 + 3^2 + 5^2 + 7^2 + 9^2) = 330.$$

- Use the Trapezoidal Rule with  $n = 6$  to approximate  $\int_0^{\pi} \sin^2 x \, dx$ . [3pt]

The width of each subinterval is  $\frac{\pi - 0}{n} = \frac{\pi}{6}$ . Compute the value of  $f$  and substituting into the formula gives

$$\int_0^{\pi} \sin^2 x \, dx \approx \frac{\pi}{6 \cdot 2} \left[ 0 + 2 \cdot \frac{1}{4} + 2 \cdot \frac{3}{4} + 2 \cdot 1 + 2 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4} + 0 \right] = \frac{\pi}{2}.$$

## 3. Improper Integrals: compute the following integrals or show that it diverges.

- $\int_1^{\infty} \frac{1}{\sqrt{x}} \, dx$  [3pt]

$$\int_1^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{\sqrt{x}} \, dx = \lim_{t \rightarrow \infty} 2\sqrt{x} \Big|_1^t = \lim_{t \rightarrow \infty} (2\sqrt{t} - 2).$$

The limit goes to infinity, hence the integral diverges.

- $\int_1^{\infty} \frac{1}{1+x^2} \, dx$  [3pt]

$$\begin{aligned} \int_1^{\infty} \frac{1}{1+x^2} \, dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{1+x^2} \, dx = \lim_{t \rightarrow \infty} \arctan x \Big|_1^t \\ &= \lim_{t \rightarrow \infty} (\arctan t - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \end{aligned}$$

So the integral converges.

- $\int_{\pi}^{\infty} \sin x \, dx$  [3pt]

$$\int_{\pi}^{\infty} \sin x \, dx = \lim_{t \rightarrow \infty} \int_{\pi}^t \sin x \, dx = \lim_{t \rightarrow \infty} (-\cos x) \Big|_{\pi}^t = \lim_{t \rightarrow \infty} (\cos \pi - \cos t).$$

The limit does not exist, hence the integral diverges.

- $\int_e^\infty \frac{1}{x \ln x} \, dx$  [4pt]

Note that

$$\int_e^\infty \frac{1}{x \ln x} \, dx = \int_1^\infty \frac{1}{u} \, du = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{u} \, du = \lim_{t \rightarrow \infty} \ln u \Big|_1^t = \lim_{t \rightarrow \infty} (\ln t - \ln 1).$$

The first step uses substitution rule with  $u = \ln x$ ,  $du = \frac{1}{x} \, dx$ . The limit goes to infinity, hence the integral diverges.

- $\int_{-\infty}^\infty x e^{-x^2} \, dx$  [5pt]

Note that

$$\int_{-\infty}^\infty x e^{-x^2} \, dx = \int_{-\infty}^0 x e^{-x^2} \, dx + \int_0^\infty x e^{-x^2} \, dx =: I + II.$$

Let's compute  $II$

$$II = \lim_{t \rightarrow \infty} \int_0^t x e^{-x^2} \, dx = \lim_{t \rightarrow \infty} \left( -\frac{e^{-x^2}}{2} \right) \Big|_0^t = \lim_{t \rightarrow \infty} \left( \frac{1}{2} - \frac{e^{-t^2}}{2} \right) = \frac{1}{2}.$$

Since  $x e^{-x^2}$  is an odd function  $I = -II = -\frac{1}{2}$ . Hence the original integral converges to 0.

## Homework 3

**Due: Friday, Feb 25, by the end of the class**

1. Arclength: for the following curves write down (do not evaluate) an integral w.r.t.  $x$  representing the length. Then write down an integral w.r.t.  $y$ .

- $y = x^3$  for  $x \in [1, 2]$ . [5pt]

$$L = \int_C ds = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^2 \sqrt{1 + (3x^2)^2} dx \quad (2\text{pt})$$

$$= \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_1^8 \sqrt{1 + \left(\frac{1}{3}y^{-\frac{2}{3}}\right)^2} dy \quad (3\text{pt})$$

- $y = e^x$  for  $x \in [0, 2]$ . [5pt]

$$L = \int_C ds = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^2 \sqrt{1 + (e^x)^2} dx \quad (2\text{pt})$$

$$= \int_1^{e^2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_1^{e^2} \sqrt{1 + \left(\frac{1}{y}\right)^2} dy \quad (3\text{pt})$$

2. Arclength: compute determine the arclength of the following curves

- $y = \frac{x^3}{6} + \frac{1}{2x}$  for  $x \in [1, 3]$ . [8pt]

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2} dx. \quad (2\text{pt})$$

Then

$$L = \int_1^3 \sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2} dx$$
$$= \int_1^3 \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}} dx \quad (3\text{pt})$$

$$= \int_1^3 \frac{x^4 + 1}{2x^2} dx = \int_1^3 \frac{x^4}{2x^2} + \frac{1}{2x^2} dx \quad (2\text{pt})$$

$$= \int_1^3 \frac{1}{2}x^2 + \frac{1}{2x^2} dx = \frac{x^3}{6} - \frac{1}{2x} \Big|_1^3 = \frac{28}{6} = \frac{14}{3}. \quad (1\text{pt})$$

- $y = \cosh x$  for  $x \in [0, \ln 2]$ . [8pt]

The hyperbolic cosine function  $\cosh x$  is given by  $\cosh x = \frac{e^x + e^{-x}}{2}$ .

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} dx. \quad (2\text{pt})$$

Then

$$L = \int_0^{\ln 2} \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} dx = \int_0^{\ln 2} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{4e^{2x}}} dx \quad (3\text{pt})$$

$$\begin{aligned} &= \int_0^{\ln 2} \frac{e^{2x} + 1}{2e^x} dx = \int_0^{\ln 2} \frac{e^x}{2} + \frac{1}{2e^x} dx \\ &= \left. \frac{e^x}{2} - \frac{e^{-x}}{2} \right|_0^{\ln 2} = 1 - \frac{1}{4} = \frac{3}{4}. \end{aligned} \quad (3\text{pt})$$

- $y = \ln(\cos x)$  for  $x \in \left[0, \frac{\pi}{3}\right]$ . [7pt]

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (-\tan x)^2} dx. \quad (2\text{pt})$$

Then

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1 + (-\tan x)^2} dx = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x} dx \quad (2\text{pt})$$

$$= \int_0^{\frac{\pi}{3}} \sec x dx = \ln |\sec x + \tan x| \Big|_0^{\frac{\pi}{3}} \quad (2\text{pt})$$

$$= \ln(2 + \sqrt{3}) - \ln 1 = \ln(2 + \sqrt{3}). \quad (1\text{pt})$$

3. Area of a Surface of Revolution: determine the area of the surface obtained by rotating the curve

- $y = \sqrt{9 - x^2}$  for  $x \in [-2, 2]$ , rotating about the  $x$ -axis. [5pt]

Since we are rotating about the  $x$ -axis

$$S = \int 2\pi y ds.$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \left(-\frac{x}{\sqrt{9 - x^2}}\right)^2} dx = \frac{3}{\sqrt{9 - x^2}} dx.$$

Then

$$S = \int_{-2}^2 2\pi \sqrt{9 - x^2} \cdot \frac{3}{\sqrt{9 - x^2}} dx = \int_{-2}^2 6\pi dx = 24\pi.$$

- $y = x^2$  for  $x \in [1, 2]$ , rotating about the  $y$ -axis. [6pt]

Since we are rotating about the  $y$ -axis

$$S = \int 2\pi x \, ds.$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + (2x)^2} \, dx.$$

Then

$$S = \int_1^2 2\pi x \cdot \sqrt{1 + (2x)^2} \, dx = 2\pi \int_1^2 x \cdot \sqrt{1 + 4x^2} \, dx.$$

Substituting  $u = 1 + 4x^2$ ,  $du = 8x \, dx$  we have

$$RHS = \frac{\pi}{4} \int_5^{17} 7\sqrt{u} \, du = \frac{\pi}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_5^{17} = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5}).$$

- $y = \frac{(x^2 + 2)^{3/2}}{3}$  for  $x \in [1, 2]$ , rotating about the  $y$ -axis. [6pt]

Since we are rotating about the  $y$ -axis

$$S = \int 2\pi x \, ds.$$

$$\begin{aligned} ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \sqrt{1 + (x\sqrt{x^2 + 2})^2} \, dx \\ &= \sqrt{1 + x^2(x^2 + 2)} \, dx = (x^2 + 1) \, dx. \end{aligned}$$

Then

$$\begin{aligned} S &= \int_1^2 2\pi x \cdot (x^2 + 1) \, dx = 2\pi \left( \frac{1}{4}x^4 + \frac{1}{2}x^2 \right) \Big|_1^2 \\ &= 2\pi \left( 4 + 2 - \frac{1}{4} - \frac{1}{2} \right) = \frac{21\pi}{2}. \end{aligned}$$



## Homework 4

**Due: Friday, Mar 4, by the end of the class**

1. Determine whether the sequence converges or diverges. If it converges, find the limit.

- $a_n = \frac{3 + 5n^2}{n + n^2}$  [2pt]

The sequence converges since  $\lim_{n \rightarrow \infty} \frac{3 + 5n^2}{n + n^2} = \lim_{n \rightarrow \infty} \frac{3/n^2 + 5}{1/n + 1} = 5 < \infty$ .

- $a_n = \frac{2n^4 - 11n + 5}{4n - 1}$  [2pt]

The sequence diverges since  $\lim_{n \rightarrow \infty} \frac{2n^4 - 11n + 5}{4n - 1} = \lim_{n \rightarrow \infty} \frac{2 - 11/n^3 + 5/n^4}{4/n^3 - 1/n^4} = \infty$ .

- $a_n = \frac{n^2 - 2n - 1}{n^3 + 3}$  [2pt]

The sequence converges since  $\lim_{n \rightarrow \infty} \frac{n^2 - 2n - 1}{n^3 + 3} = \lim_{n \rightarrow \infty} \frac{1/n - 2/n^2 - 1/n^3}{1 + 3/n^3} = 0 < \infty$ .

- $a_n = \left(1 + \frac{2}{n}\right)^n$  [5pt]

The sequence converges because  $\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = e^2$ .

Solution 1: Using the fact that  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ . Let  $m = \frac{n}{2}$  then

$$\lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n = \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^{2m} = \left(\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m}\right)^m\right)^2 = e^2.$$

Solution 2: Compute the limit directly

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right)^n &= \lim_{n \rightarrow \infty} \exp \left( \ln \left(1 + \frac{2}{n}\right)^n \right) \quad (\text{exp and ln functions are inverses.}) \\ &= \exp \left( \lim_{n \rightarrow \infty} n \ln \left(1 + \frac{2}{n}\right) \right) = \exp \left( \lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{2}{n}\right)}{\frac{1}{n}} \right). \end{aligned}$$

To compute the limit inside exponential, apply L'Hopital's rule ( $x$  is used because we need the function to be differentiable, but  $n$  is discrete)

$$\lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x \rightarrow \infty} \frac{-\frac{1}{1+\frac{2}{x}} \cdot \frac{2}{x^2}}{-\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{2}{1 + \frac{1}{x}} = 2.$$

So the final answer for the limit is  $e^2$ .

## 2. Computing Series

- $\sum_{n=0}^{\infty} 9^{-\frac{n}{2}} 2^{1+n}$  [4pt]

$$\begin{aligned}\sum_{n=0}^{\infty} 9^{-\frac{n}{2}} 2^{1+n} &= \sum_{n=0}^{\infty} \left(9^{-\frac{1}{2}}\right)^n 2 \cdot 2^n = 2 \cdot \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n 2^n \\ &= 2 \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 2 \cdot \frac{1}{1 - \frac{2}{3}} = 6.\end{aligned}$$

- $\sum_{n=5}^{\infty} \frac{3}{n^2 - 7n + 12}$  [5pt]

$$\sum_{n=5}^{\infty} \frac{3}{n^2 - 7n + 12} = \sum_{n=5}^{\infty} \frac{3}{n-4} - \frac{3}{n-3} = 3 \cdot \sum_{n=5}^{\infty} \frac{1}{n-4} - \frac{1}{n-3}.$$

Similar to the example  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ , let's call  $b_n = \frac{1}{n-4} - \frac{1}{n-3}$  and compute partial sum for  $n \geq 5$ :

$$\begin{aligned}s_N &= b_5 + b_6 + b_7 + \cdots + b_N \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N-4} - \frac{1}{N-3}\right) \\ &= 1 - \frac{1}{N-4}.\end{aligned}$$

Hence

$$\sum_{n=5}^{\infty} \frac{3}{n^2 - 7n + 12} = 3 \sum_{n=5}^{\infty} b_n = 3 \lim_{n \rightarrow \infty} \left(1 - \frac{1}{N-3}\right) = 3.$$

## 3. The Divergence Test: prove the following series diverges.

- $\sum_{n=2}^{\infty} \cos\left(\frac{n\pi}{2}\right)$  [3pt]

$\lim_{n \rightarrow \infty} \cos\left(\frac{n\pi}{2}\right)$  does not have a limit (cos is oscillating). So the series diverges.

- $\sum_{n=1}^{\infty} \frac{1}{4 + e^{-n}}$  [3pt]

Here  $\lim_{n \rightarrow \infty} \frac{1}{4 + e^{-n}} = \frac{1}{4} \neq 0$ , so the series diverges.

- $\sum_{n=0}^{\infty} \frac{e^n}{n^3 + n}$  [3pt]

To compute the limit of  $a_n = \frac{e^n}{n^3 + n}$  we need to apply L'Hopital's rule. To make sense of the derivatives, we consider the function  $f(x) = \frac{e^x}{x^3 + x}$ , then

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^3 + x} = \lim_{x \rightarrow \infty} \frac{e^x}{3x^2 + 1} = \lim_{x \rightarrow \infty} \frac{e^x}{6x} = \lim_{x \rightarrow \infty} \frac{e^x}{6} = \infty \neq 0.$$

The series diverges.

4. The Integral Test: determine if the following series converges or diverges.

- $\sum_{n=1}^{\infty} \frac{n^4}{e^n}$  [5pt]

Let  $f(x) = \frac{x^4}{e^x}$ . Note that  $f$  is a positive and continuous function defined on  $[1, \infty)$  and  $\lim_{x \rightarrow \infty} f(x) = 0$ . Moreover (2pt)

$$f'(x) = \frac{x^3(4-x)}{e^x} \implies f'(x) \leq 0 \text{ when } x \geq 4, \quad (1\text{pt})$$

and

$$\int_4^{\infty} \frac{x^4}{e^x} dx \leq \int_4^{\infty} \frac{x^4}{x^6} dx = \int_4^{\infty} \frac{1}{x^2} dx \quad (\text{converges by the integral test, 1pt})$$

This shows that for  $a_n = \frac{n^4}{e^n}$ ,  $\sum_{n=4}^{\infty} a_n < \infty$ . Hence

$$\sum_{n=1}^{\infty} \frac{n^4}{e^n} = a_1 + a_2 + a_3 + \sum_{n=4}^{\infty} a_n < \infty. \quad (\text{Converges. 1pt})$$

- $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$  [5pt]

Let  $f(x) = \frac{x}{x^3 + 1}$ . Note that  $f$  is a positive and continuous function defined on  $[1, \infty)$  and  $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{1}{3x^2} = 0$ . Moreover (2pt)

$$f'(x) = \frac{1 - 2x^2}{(x^3 + 1)^2} \implies f'(x) < 0 \text{ when } x > \sqrt{\frac{1}{2}}, \quad (2\text{pt})$$

and

$$\int_1^{\infty} \frac{x}{x^3 + 1} dx \leq \int_1^{\infty} \frac{x}{x^3} dx = \int_1^{\infty} \frac{1}{x^2} dx \quad (\text{converges by the integral test, 1pt})$$

- $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$  for  $p > 1$  (and for  $p \leq 1$  respectively). [11pt]

Let  $f(x) = \frac{1}{x(\ln x)^p}$ . Note that for all  $p$ ,  $f$  is a positive and continuous function defined on  $[2, \infty)$  and  $\lim_{x \rightarrow \infty} f(x) = 0$ . Moreover (3pt)

$$f'(x) = -p(x \ln x)^{-p-1}(\ln x + 1) < 0, \text{ when } x \geq 2. \quad (3\text{pt})$$

Finally

$$\int_2^{\infty} \frac{1}{x(\ln x)^p} dx = \int_{\ln 2}^{\infty} \frac{1}{u^p} du = \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases} \quad (3\text{pt})$$

By the integral test the series converges if  $p > 1$  and diverges if  $p \leq 1$ . (1pt)