Homework Solutions

MATH231

Spring 2022

Homework 0

1. Calculating Limits

   - $\lim_{x\to 2} \dfrac{x^2+x-6}{x-2}$

   $$\lim_{x\to 2} \frac{x^2+x-6}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2}(x+3) = 5.$$

   - $\lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$

     Consider x as a "constant"

     $$\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2-2xh-h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh-h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

   - $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2+x}\right)$

   $$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right) = \lim_{x\to 0} \frac{x^2+x-x}{x(x^2+x)} = \lim_{x\to 0} \frac{x^2}{x^2(x+1)} = \lim_{x\to 0} \frac{1}{x+1} = 1.$$

2. The Chain Rule

   - $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln(x+\sin x)$

     $\dfrac{\mathrm{d}}{\mathrm{d}x} \ln(x+\sin x) = (1+\cos x)\dfrac{1}{x+\sin x}.$

   - $\dfrac{\mathrm{d}}{\mathrm{d}x} \cos(x^2e^x)$

     $\dfrac{\mathrm{d}}{\mathrm{d}x} \cos(x^2e^x) = -(2xe^x + x^2e^x)\sin(x^2e^x).$

3. Implicit Differentiation: Solve for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following implicit function.

   - $x^2+y^2=r^2$, where r is a constant

     Differentiate on both sides w.r.t. x gives $2x+2yy'=0$. Hence $y' = -\dfrac{x}{y}$.

   - $\dfrac{x+y}{x-y} = x$

     The above is equivalent to $x+y = x^2-xy$. Differentiate on both sides w.r.t. x gives $1+y' = 2x-y-xy' \iff (1+x)y' = 2x-y-1$. Hence $y' = \dfrac{2x-y-1}{x+1}$.

4. Linear Approximations and Differentials: Find the Taylor polynomials of degree two approximating the given function centered at the given point.

   - $f(x) = \sin(2x)$ at $a = \frac{\pi}{2}$

     $f' = 2\cos(2x)$, $f'' = -4\sin(2x)$. So $f \sim -2\left(x - \dfrac{\pi}{2}\right)$.

   - $f(x) = e^x$ at $a = 1$

     $f' = f'' = e^x$. So $f \sim e + e(x-1) + \dfrac{e}{2}(x-1)^2$.

5. Mean Value Theorem: Determine if the Mean Value Theorem can be applied to the following function on the the given closed interval.

   Both intervals are closed. It suffices to check that these functions are continuous on the given interval. One can do this by computing the derivative exists.

   - $f(x) = 3 + \sqrt{x}, x \in [0, 4]$

     Here $f' = \frac{1}{2\sqrt{x}}$.

   - $f(x) = \dfrac{x}{1+x}, x \in [1, 3]$

     Here $f' = \frac{1}{(1+x)^2}$.

6. L'Hospital's Rule

   - $\displaystyle\lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

     Check that as $x \to 2$, $x^3 - 7x^2 + 10x \to 0$ and $x^2 + x - 6 \to 0$ so L'Hospital's rule applies. Then

     $$\lim_{x\to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x\to 2} \frac{3x^2 - 14x + 10}{2x + 1} = -\frac{6}{5}.$$

     The last step uses division property of limits.

   - $\displaystyle\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}$

     As exp and ln are continuous functions

     $$\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}} = \exp\left[\ln\left(\lim_{x\to\infty} (e^x + x)^{\frac{1}{x}}\right)\right] = \exp\left(\lim_{x\to\infty} \frac{\ln(e^x + x)}{x}\right).$$

     Check that as $x \to \infty$, $e^x + x \to \infty$ so L'Hospital's rule applies.

     $$\begin{aligned} RHS &= \exp\left(\lim_{x\to\infty} \frac{e^x + 1}{e^x + x}\right) \\ &= \exp\left(\lim_{x\to\infty} 1 + \frac{1 - x}{e^x + x}\right) && \text{(Check that L'Hospital's rule applies)} \\ &= \exp\left(1 + \lim_{x\to\infty} \frac{-1}{e^x + 1}\right) = e. \end{aligned}$$

- $\lim_{x\to\infty} x \ln\left(1+\frac{3}{x}\right)$

  $$\lim_{x\to\infty} x \ln\left(1+\frac{3}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}.$$

  Check that as $x \to \infty$, $\ln\left(1+\frac{3}{x}\right), \frac{1}{x} \to 0$ so L'Hospital's rule applies.

  $$RHS = \lim_{x\to\infty} \frac{\frac{-3/x^2}{1+3/x}}{-1/x^2} = \lim_{x\to\infty} \frac{3}{1+\frac{3}{x}} = 3.$$

7. The Fundamental Theorem of Calculus: Find the derivative of the following

- $\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t$

  Apply FTC

  $$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t = \frac{1}{x^3+1}.$$

- $\int_1^{\sqrt{x}} \sin t\,\mathrm{d}t$

  Let $u(x) = \sqrt{x}$. Apply chain rule and FTC

  $$\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \frac{1}{t^3+1}\,\mathrm{d}t = \sin u(x) \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \sin u(x) \cdot \frac{1}{2\sqrt{x}} = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

- $\int_x^{2x} t^3\,\mathrm{d}t$

  Using subtraction property of integral,

  $$\int_x^{2x} t^3\,\mathrm{d}t = \int_0^{2x} t^3\,\mathrm{d}t - \int_0^x t^3\,\mathrm{d}t.$$

  Apply FTC to each term

  $$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{2x} t^3\,\mathrm{d}t = 16x^3 - x^3 = 15x^3.$$

8. Substitution Rule

- $\displaystyle\int_{\frac{1}{2}}^{0} \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$

  Take $u = 1-4x^2$, then $\mathrm{d}u = -8x\,\mathrm{d}x$ and $\mathrm{d}x = -\dfrac{1}{8}\,\mathrm{d}u$.

$$\int_{\frac{1}{2}}^{0} \frac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x = \int_{0}^{1} -\frac{1}{8\sqrt{u}}\,\mathrm{d}u = -\frac{1}{4}\sqrt{u}\Big|_{0}^{1} = -\frac{1}{4}.$$

- $\displaystyle\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)}\,\mathrm{d}x$

  Take $u = \sin(\pi x)$, then $\mathrm{d}u = \pi\cos(\pi x)\,\mathrm{d}x$ and $\mathrm{d}x = -\dfrac{1}{8}\,\mathrm{d}u$.

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\cos(\pi x)}{\sin^2(\pi x)}\,\mathrm{d}x = \int_{\frac{\sqrt{2}}{2}}^{1} \frac{1}{\pi u^2}\,\mathrm{d}u = -\frac{1}{u}\Big|_{\frac{\sqrt{2}}{2}}^{1} = \frac{\sqrt{2}-1}{\pi}.$$

- $\displaystyle\int_{0}^{1} xe^{4x^2+3}\,\mathrm{d}x$

  Take $u = 4x^2+3$, then $\mathrm{d}u = 8x\,\mathrm{d}x$

$$\int_{0}^{1} xe^{4x^2+3}\,\mathrm{d}x = \frac{1}{8}\int_{3}^{7} e^{u}\,\mathrm{d}u = \frac{1}{8}\sqrt{u}\Big|_{3}^{7} = \frac{e^7-e^3}{8}.$$

Homework 1

The following solutions provide a possible way to solve the problems. Any other reasonable solution is accepted.

1. Integration by parts (Note that the following integrals are indefinite. You need to add constants to your final answer.) You may also need to use substitution rule.

   - $\displaystyle\int \frac{\ln x}{x^2}\,\mathrm{d}x$ [3pt]

     Take $u = \ln x, v = -\dfrac{1}{x}$. Then

     $$\int \frac{\ln x}{x^2}\,\mathrm{d}x = \int \ln x\,\mathrm{d}\Big(-\frac{1}{x}\Big) = -\frac{1}{x}\ln x - \int -\frac{1}{x}\,\mathrm{d}(\ln x) = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

   - $\displaystyle\int x^2 \sin x\,\mathrm{d}x$ [4pt]

     Take $u = x^2, v = -\cos x$. Then

     $$\int x^2 \sin x\,\mathrm{d}x = -x^2\cos x - \int -\cos x\,\mathrm{d}(x^2) = -x^2\cos x + 2\int x\cos x\,\mathrm{d}x.$$

     To evaluate $\displaystyle\int x\cos x\,\mathrm{d}x$ we use integration by parts again with $u = x, v = \sin x$.

     $$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + \tilde{C}.$$

     Final answer: $\quad -x^2\cos x + 2x\sin x + 2\cos x + C.$

   - $\displaystyle\int (\ln x)^2\,\mathrm{d}x$ [4pt]

     Take $u = (\ln x)^2, v = x$. Then

     $$\int (\ln x)^2\,\mathrm{d}x = x(\ln x)^2 - \int x\,\mathrm{d}(\ln x)^2 = x(\ln x)^2 - \int \ln x\,\mathrm{d}x.$$

     To evaluate $\displaystyle\int \ln x\,\mathrm{d}x$ we use integration by parts again, with $u = \ln x, v = x$.

     $$\int \ln x\,\mathrm{d}x = x\ln x - \int x\,\mathrm{d}(\ln x) = x\ln x - \int 1\,\mathrm{d}x = x\ln x - x + C.$$

     Final answer: $\quad x(\ln x)^2 - 2x\ln x + 2x + C.$

   - $\displaystyle\int \arccos x\,\mathrm{d}x$ [4pt]

     Take $u = \arccos x, v = x$. Then

     $$\int \arccos x\,\mathrm{d}x = x\arccos x - \int x\,\mathrm{d}(\arccos x) = x\arccos x - \int -\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

To evaluate $\int \frac{x}{\sqrt{1-x^2}}\, \mathrm{d}x$ we use substitution rule with $u = 1-x^2,\ \mathrm{d}u = -2x\, \mathrm{d}x$.

$$\int \frac{x}{\sqrt{1-x^2}}\, \mathrm{d}x = -\int \frac{1}{2\sqrt{u}}\, \mathrm{d}u = -\sqrt{u} + C.$$

Final answer: $\quad x \arccos x - \sqrt{1-x^2} + C$.

- $\int e^{\sqrt{x}}\, \mathrm{d}x$ [4pt]

  Using substitution rule with $t = \sqrt{x}$ we obtain

  $$\int e^{\sqrt{x}}\, \mathrm{d}x = 2\int te^t\, \mathrm{d}t$$

  Integration by parts: take $u = t, v = e^t$. Then

  $$RHS = 2\int t\, \mathrm{d}(e^t) = 2\left(te^t - \int e^t\, \mathrm{d}t\right) = 2(te^t - e^t + \tilde{C}) = 2(\sqrt{x}-1)e^{\sqrt{x}} + C.$$

2. Trigonometric integration: Evaluate the following integral of the form $\int \sin^n x \cos^m x\, \mathrm{d}x$.

> You need specify the values for θ, so that you can get rid of absolute values.

- $\int \sin^2 x \cos^3 x\, \mathrm{d}x$ [3pt]

  Note that

  $$\int \sin^2 x \cos^3 x\, \mathrm{d}x = \int \sin^2 x \cos^2 x \cdot \cos x\, \mathrm{d}x = \int \sin^2 x (1-\sin^2 x) \cdot \cos x\, \mathrm{d}x.$$

  Apply substitution rule with $u = \sin x,\ \mathrm{d}u = \cos x\, \mathrm{d}x$. So

  $$RHS = \int u^2(1-u^2)\, \mathrm{d}u = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

- $\int \cos^4 x\, \mathrm{d}x$ [4pt]

  Note that

  $$\begin{aligned}\cos^4 x = \cos^2 x \cos^2 x &= \frac{1}{4}(1+\cos(2x))^2 = \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\ &= \frac{1}{4}\left(1+2\cos(2x)+\frac{1}{2}(1+\cos(4x))\right) = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x).\end{aligned}$$

  Hence

  $$\begin{aligned}\int \cos^4 x\, \mathrm{d}x &= \int \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\, \mathrm{d}x \\ &= \frac{3}{8}\int 1\, \mathrm{d}x + \frac{1}{2}\int \cos(2x)\, \mathrm{d}x + \frac{1}{8}\int \cos(4x)\, \mathrm{d}x \\ &= \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C.\end{aligned}$$

3. Trigonometric substitution

- $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x$ [6pt]

  Let $x = 3\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$

$$\begin{aligned}\int \frac{x^2}{\sqrt{9-x^2}}\,\mathrm{d}x &= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\cdot 3\cos\theta\,\mathrm{d}\theta = \int \frac{27\sin^2\theta\cos\theta}{3\cos\theta}\,\mathrm{d}\theta\\ &= \int 9\sin^2\theta\,\mathrm{d}\theta = \int \frac{9(1-\cos 2\theta)}{2}\,\mathrm{d}\theta = \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C\\ &= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C = \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \frac{x}{3}\sqrt{1-\left(\frac{x}{3}\right)^2}\right) + C\\ &= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C.\end{aligned}$$

- $\displaystyle\int \frac{1}{\sqrt{25+x^2}}\,\mathrm{d}x$ [6pt]

  Let $x = 5\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $\mathrm{d}x = 5\sec^2\theta\,\mathrm{d}\theta$

$$\begin{aligned}\int \frac{1}{\sqrt{25+x^2}}\,\mathrm{d}x &= \int \frac{5\sec^2\theta}{\sqrt{25+25\tan^2\theta}}\,\mathrm{d}\theta = \int \frac{5\sec^2\theta}{\sqrt{25\sec^2\theta}}\,\mathrm{d}\theta\\ &= \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C\\ &= \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C.\end{aligned}$$

- $\displaystyle\int \frac{1}{\sqrt{x^2+2x}}\,\mathrm{d}x$ [6pt]

  Note that

$$\int \frac{1}{\sqrt{x^2+2x}}\,\mathrm{d}x = \int \frac{1}{\sqrt{(x+1)^2-1}}\,\mathrm{d}x.$$

  Let $x+1 = \sec\theta, 0 \leq \theta < \frac{\pi}{2}$, then $\mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$

$$\begin{aligned}RHS &= \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta - 1}}\,\mathrm{d}\theta = \int \frac{\sec\theta\tan\theta}{\tan\theta}\,\mathrm{d}x\\ &= \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C\\ &= \ln\left|x+1+\sqrt{(x+1)^2-1}\right| + C\\ &= \ln\left|x+1+\sqrt{x^2+2x}\right| + C.\end{aligned}$$

- $\displaystyle\int (x-2)^3\sqrt{5+4x-x^2}\,\mathrm{d}x$ [6pt]

  Note that

$$\int (x-2)^3\sqrt{5+4x-x^2}\,\mathrm{d}x = \int (x-2)^3\sqrt{9-(x-2)^2}\,\mathrm{d}x.$$

  Let $x-2 = 3\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $\mathrm{d}x = 3\cos\theta\,\mathrm{d}\theta$

$$RHS = \int (3\sin\theta)^3 \cdot \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,\mathrm{d}\theta = 3^5\int \sin^3\theta\cos^2\theta\,\mathrm{d}\theta.$$

  To solve $\displaystyle\int \sin^3\theta\cos^2\theta\,\mathrm{d}\theta$, apply substitution rule with $u=\cos\theta,\ \mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$. So

$$\begin{aligned}\int \sin^3\theta\cos^2\theta\,\mathrm{d}\theta &= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\,\mathrm{d}\theta = -\int (1-u^2)u^2\,\mathrm{d}u\\ &= \frac{u^5}{5}-\frac{u^3}{3}+\tilde{C} = \frac{\cos^5\theta}{5}-\frac{\cos^3\theta}{3}+\tilde{C}\end{aligned}$$

  Final answer:

$$\begin{aligned}&3^5\left[\frac{1}{5}\left(\frac{\sqrt{9-(x-2)^2}}{3}\right)^5-\frac{1}{3}\left(\frac{\sqrt{9-(x-2)^2}}{3}\right)^3\right]+C\\ =&3^5\left[\frac{1}{5}\left(\frac{\sqrt{5+4x-x^2}}{3}\right)^5-\frac{1}{3}\left(\frac{\sqrt{5+4x-x^2}}{3}\right)^3\right]+C.\end{aligned}$$

  There's another way to do this problem. We will discuss that in problem session.

Homework 2

Due: Friday, Feb 11, by the end of the class

1. Partial Fractions

- $\displaystyle\int \frac{2x+5}{x^2+4x+8}\,\mathrm{d}x$ [5pt]

  We have

  $$\int \frac{2x+5}{x^2+4x+8}\,\mathrm{d}x = \int \frac{2x+4}{x^2+4x+8}\,\mathrm{d}x + \int \frac{1}{x^2+4x+8}\,\mathrm{d}x =: I + II.$$

  Solve for I: substitution rule with $u = x^2+4x+8,\ \mathrm{d}u = (2x+4)\,\mathrm{d}x$. Then

  $$I = \int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C_1 = \ln|x^2+4x+8| + C_1.$$

  Solve for II: the second step uses substitution rule with $u = \dfrac{x+2}{2},\ \mathrm{d}u = \dfrac{1}{2}\,\mathrm{d}x$.

  $$\begin{aligned} II &= \int \frac{1}{(x+2)^2+4}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{u^2+1}\,\mathrm{d}u \\ &= \frac{1}{2}\arctan u + C_2 = \frac{1}{2}\arctan\frac{x+2}{2} + C_2. \end{aligned}$$

  Final answer: $\quad \ln|x^2+4x+8| + \dfrac{1}{2}\arctan\dfrac{x+2}{2} + C.$

- $\displaystyle\int \frac{2x^2-x+4}{(x^2+4)(x-1)}\,\mathrm{d}x$ [5pt]

  Update: To decompose that rational function, set

  $$\frac{2x^2-x+4}{(x^2+4)(x-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1} = \frac{(Ax+B)(x-1)+C(x^2+4)}{(x^2+4)(x-1)}.$$

  Solving for A, B, C by comparing the coefficients gives $A = 1, B = 0, C = 1$.

  We have

  $$\int \frac{2x^2-x+4}{(x^2+4)(x-1)}\,\mathrm{d}x = \int \frac{x}{x^2+4}\,\mathrm{d}x + \int \frac{1}{x-1}\,\mathrm{d}x =: I + II.$$

  Solve for I: substitution rule with $u = x^2+4,\ \mathrm{d}u = 2x\,\mathrm{d}x$. Then

  $$I = \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u = \frac{1}{2}\ln|u| + C_1 = \frac{1}{2}\ln|x^2+4| + C_1.$$

  Solve for II:

  $$II = \ln|x-1| + C_2.$$

  Final answer: $\quad \dfrac{1}{2}\ln|x^2+4| + \ln|x-1| + C.$

- $\displaystyle\int \frac{x}{x^4+2x^2+2}\,\mathrm{d}x$ [4pt]

  Substituting $u = x^2,\ \mathrm{d}u = 2x\,\mathrm{d}x$ gives

$$\begin{aligned}\int \frac{x}{x^4+2x^2+2}\,\mathrm{d}x &= \int \frac{1}{(u+1)^2+1}\,\mathrm{d}u = \frac{1}{2}\arctan(u+1)+C\\ &= \frac{1}{2}\arctan(x^2+1)+C.\end{aligned}$$

- $\displaystyle\int \ln(x^2+1)\,\mathrm{d}x$ [4pt]

  Integration by parts with $u = \ln(x^2+1), v = x$ gives

$$\begin{aligned}\int \ln(x^2+1)\,\mathrm{d}x &= x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,\mathrm{d}x\\ &= x\ln(x^2+1) - \int \frac{2x^2+2-2}{x^2+1}\,\mathrm{d}x\\ &= x\ln(x^2+1) - 2\int 1\,\mathrm{d}x + \int \frac{1}{x^2+1}\,\mathrm{d}x\\ &= x\ln(x^2+1) - 2x + 2\arctan x + C.\end{aligned}$$

- $\displaystyle\int \frac{1}{\sqrt{x}+x\sqrt{x}}\,\mathrm{d}x$ [4pt]

  Note that

$$\int \frac{1}{\sqrt{x}+x\sqrt{x}}\,\mathrm{d}x = \int \frac{1}{1+x}\cdot\frac{1}{\sqrt{x}}\,\mathrm{d}x.$$

  Substituting $u = \sqrt{x},\ \mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$ gives

$$RHS = 2\int \frac{1}{1+u^2}\,\mathrm{d}u = 2\arctan u + C = 2\arctan(\sqrt{x}) + C.$$

- $\displaystyle\int \frac{1}{x+\sqrt[3]{x}}\,\mathrm{d}x$ [4pt]

  Note that

$$\int \frac{1}{x+\sqrt[3]{x}}\,\mathrm{d}x = \int \frac{1}{x^{2/3}+1}\cdot\frac{1}{\sqrt[3]{x}}\,\mathrm{d}x.$$

  Substituting $u = x^{2/3},\ \mathrm{d}u = \dfrac{2}{3\sqrt[3]{x}}\,\mathrm{d}x$ gives

$$RHS = \frac{3}{2}\int \frac{1}{u+1}\,\mathrm{d}u = \frac{3}{2}\ln|u+1| + C = \frac{3}{2}\ln|x^{2/3}+1| + C.$$

2. Approximate Integration

- Use the Midpoint Rule with $n = 5$ to approximate $\displaystyle\int_0^{10} x^2 \, \mathrm{d}x$. [3pt]

  The width of each subinterval is 2. Compute the value of f and substituting into the formula gives

  $$\int_0^{10} x^2 \, \mathrm{d}x \approx 2\big(1^2 + 3^2 + 5^2 + 7^2 + 9^2\big) = 330.$$

- Use the Trapezoidal Rule with $n = 6$ to approximate $\displaystyle\int_0^{\pi} \sin^2 x \, \mathrm{d}x$. [3pt]

  The width of each subinterval is $\dfrac{\pi - 0}{n} = \dfrac{\pi}{6}$. Compute the value of f and substituting into the formula gives

  $$\int_0^{\pi} \sin^2 x \, \mathrm{d}x \approx \frac{\pi}{6 \cdot 2}\left[0 + 2 \cdot \frac{1}{4} + 2 \cdot \frac{3}{4} + 2 \cdot 1 + 2 \cdot \frac{3}{4} + 2 \cdot \frac{1}{4} + 0\right] = \frac{\pi}{2}.$$

3. Improper Integrals: compute the following integrals or show that it diverges.

- $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}} \, \mathrm{d}x$ [3pt]

  $$\int_1^{\infty} \frac{1}{\sqrt{x}} \, \mathrm{d}x = \lim_{t \to \infty} \int_1^{t} \frac{1}{\sqrt{x}} \, \mathrm{d}x = \lim_{t \to \infty} 2\sqrt{x} \,\Big|_1^t = \lim_{t \to \infty} (2\sqrt{t} - 2).$$

  The limit goes to infinity, hence the integral diverges.

- $\displaystyle\int_1^{\infty} \frac{1}{1 + x^2} \, \mathrm{d}x$ [3pt]

  $$\begin{aligned}\int_1^{\infty} \frac{1}{1 + x^2} \, \mathrm{d}x = \lim_{t \to \infty} \int_1^{t} \frac{1}{1 + x^2} \, \mathrm{d}x &= \lim_{t \to \infty} \arctan x \,\Big|_1^t \\ &= \lim_{t \to \infty} (\arctan t - \arctan 1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.\end{aligned}$$

  So the integral converges.

- $\displaystyle\int_{\pi}^{\infty} \sin x \, \mathrm{d}x$ [3pt]

  $$\int_{\pi}^{\infty} \sin x \, \mathrm{d}x = \lim_{t \to \infty} \int_{\pi}^{t} \sin x \, \mathrm{d}x = \lim_{t \to \infty} (-\cos x) \,\Big|_{\pi}^t = \lim_{t \to \infty} (\cos \pi - \cos t).$$

  The limit does not exists, hence the integral diverges.

- $\displaystyle\int_e^{\infty} \frac{1}{x\ln x}\,\mathrm{d}x$ [4pt]

  Note that

  $$\int_e^{\infty} \frac{1}{x\ln x}\,\mathrm{d}x = \int_1^{\infty} \frac{1}{u}\,\mathrm{d}u = \lim_{t\to\infty} \int_1^{t} \frac{1}{u}\,\mathrm{d}u = \lim_{t\to\infty} \ln u\,\Big|_1^t = \lim_{t\to\infty} (\ln t - \ln 1).$$

  The first step uses substitution rule with $u = \ln x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. The limit goes to infinity, hence the integral diverges.

- $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{d}x$ [5pt]

  Note that

  $$\int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{d}x = \int_{-\infty}^{0} xe^{-x^2}\,\mathrm{d}x + \int_0^{\infty} xe^{-x^2}\,\mathrm{d}x =: I + II.$$

  Let's compute II

  $$II = \lim_{t\to\infty} \int_0^t xe^{-x^2}\,\mathrm{d}x = \lim_{t\to\infty} \left(-\frac{e^{-x^2}}{2}\right)\Big|_0^t = \lim_{t\to\infty} \left(\frac{1}{2} - \frac{e^{-t^2}}{2}\right) = \frac{1}{2}.$$

  Since xe^{-x^2} is an odd function $I = -II = -\frac{1}{2}$. Hence the original integral converges to 0.

Homework 3

Due: Friday, Feb 25, by the end of the class

1. Arclength: for the following curves write down (do not evaluate) an integral w.r.t. x representing the length. Then write down an integral w.r.t. y.

   - $y = x^3$ for $x \in [1, 2]$. [5pt]

$$L = \int_C \mathrm{d}s = \int_1^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_1^2 \sqrt{1 + (3x^2)^2}\,\mathrm{d}x \qquad (2\text{pt})$$

$$= \int_1^8 \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y = \int_1^8 \sqrt{1 + \left(\frac{1}{3}y^{-\frac{2}{3}}\right)^2}\,\mathrm{d}y \qquad (3\text{pt})$$

   - $y = e^x$ for $x \in [0, 2]$. [5pt]

$$L = \int_C \mathrm{d}s = \int_0^2 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_0^2 \sqrt{1 + (e^x)^2}\,\mathrm{d}x \qquad (2\text{pt})$$

$$= \int_1^{e^2} \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y = \int_1^{e^2} \sqrt{1 + \left(\frac{1}{y}\right)^2}\,\mathrm{d}y \qquad (3\text{pt})$$

2. Arclength: compute determine the arclength of the following curves

   - $y = \frac{x^3}{6} + \frac{1}{2x} =$ for $x \in [1, 3]$. [8pt]

$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2}\,\mathrm{d}x. \qquad (2\text{pt})$$

   Then

$$\begin{aligned}
L &= \int_1^3 \sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2}\,\mathrm{d}x \\
&= \int_1^3 \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}\,\mathrm{d}x && (3\text{pt}) \\
&= \int_1^3 \frac{x^4 + 1}{2x^2}\,\mathrm{d}x = \int_1^3 \frac{x^4}{2x^2} + \frac{1}{2x^2}\,\mathrm{d}x && (2\text{pt}) \\
&= \int_1^3 \frac{1}{2}x^2 + \frac{1}{2x^2}\,\mathrm{d}x = \frac{x^3}{6} - \frac{1}{2x}\Big|_1^3 = \frac{28}{6} = \frac{14}{3}. && (1\text{pt})
\end{aligned}$$

- $y = \cosh x$ for $x \in [0, \ln 2]$. [8pt]

  The hyperbolic cosine function $\cosh x$ is given by $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

  $$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\,\mathrm{d}x. \qquad (2\text{pt})$$

  Then

  $$\begin{aligned} L &= \int_0^{\ln 2} \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\,\mathrm{d}x = \int_0^{\ln 2} \sqrt{\frac{e^{4x} + 2e^{2x} + 1}{4e^{2x}}}\,\mathrm{d}x & (3\text{pt}) \\ &= \int_0^{\ln 2} \frac{e^{2x} + 1}{2e^x}\,\mathrm{d}x = \int_0^{\ln 2} \frac{e^x}{2} + \frac{1}{2e^x}\,\mathrm{d}x \\ &= \frac{e^x}{2} - \frac{e^{-x}}{2}\Big|_0^{\ln 2} = 1 - \frac{1}{4} = \frac{3}{4}. & (3\text{pt}) \end{aligned}$$

- $y = \ln(\cos x)$ for $x \in \left[0, \dfrac{\pi}{3}\right]$. [7pt]

  $$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + (-\tan x)^2}\,\mathrm{d}x. \qquad (2\text{pt})$$

  Then

  $$\begin{aligned} L &= \int_0^{\frac{\pi}{3}} \sqrt{1 + (-\tan x)^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\,\mathrm{d}x & (2\text{pt}) \\ &= \int_0^{\frac{\pi}{3}} \sec x\,\mathrm{d}x = \ln|\sec x + \tan x|\,\Big|_0^{\frac{\pi}{3}} & (2\text{pt}) \\ &= \ln(2 + \sqrt{3}) - \ln 1 = \ln(2 + \sqrt{3}). & (1\text{pt}) \end{aligned}$$

3. Area of a Surface of Revolution: determine the area of the surface obtained by rotating the curve

   - $y = \sqrt{9 - x^2}$ for $x \in [-2, 2]$, rotating about the x-axis. [5pt]

     Since we are rotating about the x-axis

     $$S = \int 2\pi y\,\mathrm{d}s.$$

     $$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \sqrt{1 + \left(-\frac{x}{\sqrt{9 - x^2}}\right)^2}\,\mathrm{d}x = \frac{3}{\sqrt{9 - x^2}}\,\mathrm{d}x.$$

     Then

     $$S = \int_{-2}^{2} 2\pi\sqrt{9 - x^2} \cdot \frac{3}{\sqrt{9 - x^2}}\,\mathrm{d}x = \int_{-2}^{2} 6\pi\,\mathrm{d}x = 24\pi.$$

- $y = x^2$ for $x \in [1, 2]$, rotating about the y-axis. [6pt]

  Since we are rotating about the y-axis
$$S = \int 2\pi x \, \mathrm{d}s.$$
$$\mathrm{d}s = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x = \sqrt{1 + (2x)^2} \, \mathrm{d}x.$$
  Then
$$S = \int_1^2 2\pi x \cdot \sqrt{1 + (2x)^2} \, \mathrm{d}x = 2\pi \int_1^2 x \cdot \sqrt{1 + 4x^2} \, \mathrm{d}x.$$
  Substituting $u = 1 + 4x^2, \ \mathrm{d}u = 8x \, \mathrm{d}x$ we have
$$RHS = \frac{\pi}{4} \int_5^1 7\sqrt{u} \, \mathrm{d}u = \frac{\pi}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_5^{17} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}).$$

- $y = \dfrac{(x^2 + 2)^{3/2}}{3}$ for $x \in [1, 2]$, rotating about the y-axis. [6pt]

  Since we are rotating about the y-axis
$$S = \int 2\pi x \, \mathrm{d}s.$$
$$\begin{aligned} \mathrm{d}s &= \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x = \sqrt{1 + (x\sqrt{x^2 + 2})^2} \, \mathrm{d}x \\ &= \sqrt{1 + x^2(x^2 + 2)} \, \mathrm{d}x = (x^2 + 1) \, \mathrm{d}x. \end{aligned}$$
  Then
$$\begin{aligned} S &= \int_1^2 2\pi x \cdot (x^2 + 1) \, \mathrm{d}x = 2\pi \left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right)\Big|_1^2 \\ &= 2\pi \left(4 + 2 - \frac{1}{4} - \frac{1}{2}\right) = \frac{21\pi}{2}. \end{aligned}$$

Homework 4

Due: Friday, Mar 4, by the end of the class

1. Determine whether the sequence converges or diverges. If it converges, find the limit.

   - $a_n = \dfrac{3+5n^2}{n+n^2}$ [2pt]

     The sequence converges since $\lim\limits_{n\to\infty} \dfrac{3+5n^2}{n+n^2} = \lim\limits_{n\to\infty} \dfrac{3/n^2+5}{1/n+1} = 5 < \infty$.

   - $a_n = \dfrac{2n^4 - 11n + 5}{4n-1}$ [2pt]

     The sequence diverges since $\lim\limits_{n\to\infty} \dfrac{2n^4-11n+5}{4n-1} = \lim\limits_{n\to\infty} \dfrac{2-11/n^3+5/n^4}{4/n^3-1/n^4} = \infty$.

   - $a_n = \dfrac{n^2-2n-1}{n^3+3}$ [2pt]

     The sequence converges since $\lim\limits_{n\to\infty} \dfrac{n^2-2n-1}{n^3+3} = \lim\limits_{n\to\infty} \dfrac{1/n-2/n^2-1/n^3}{1+3/n^3} = 0 < \infty$.

   - $a_n = \left(1+\dfrac{2}{n}\right)^n$ [5pt]

     The sequence converges because $\lim\limits_{n\to\infty} \left(1+\dfrac{2}{n}\right)^n = e^2$.

     Solution 1: Using the fact that $\lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n = e$. Let $m = \frac{n}{2}$ then

     $$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{m\to\infty} \left(1+\frac{1}{m}\right)^{2m} = \left(\lim_{m\to\infty} \left(1+\frac{1}{m}\right)^m\right)^2 = e^2.$$

     Solution 2: Compute the limit directly

     $$\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty} \exp\left(\ln\left(1+\frac{2}{n}\right)^n\right) \quad \text{(exp and ln functions are inverses.)}$$

     $$= \exp\left(\lim_{n\to\infty} n\ln\left(1+\frac{2}{n}\right)\right) = \exp\left(\lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}\right).$$

     To compute the limit inside exponential, apply L'Hopital's rule (x is used because we need the function to be differentiable, but n is discrete)

     $$\lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{-\frac{1}{1+\frac{2}{x}}\cdot\frac{2}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{2}{1+\frac{1}{x}} = 2.$$

     So the final answer for the limit is e^2.

2. Computing Series

- $\displaystyle\sum_{n=0}^{\infty} 9^{-\frac{n}{2}} 2^{1+n}$ [4pt]

$$\sum_{n=0}^{\infty} 9^{-\frac{n}{2}} 2^{1-n} = \sum_{n=0}^{\infty} \left(9^{-\frac{1}{2}}\right)^n 2 \cdot 2^n = 2 \cdot \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n 2^n$$
$$= 2 \cdot \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 2 \cdot \frac{1}{1-\frac{2}{3}} = 6.$$

- $\displaystyle\sum_{n=5}^{\infty} \frac{3}{n^2-7n+12}$ [5pt]

$$\sum_{n=5}^{\infty} \frac{3}{n^2-7n+12} = \sum_{n=5}^{\infty} \frac{3}{n-4} - \frac{3}{n-3} = 3 \cdot \sum_{n=5}^{\infty} \frac{1}{n-4} - \frac{1}{n-3}.$$

Similar to the example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, let's call $b_n = \dfrac{1}{n-4} - \dfrac{1}{n-3}$ and compute partial sum for $n \geq 5$:

$$\begin{aligned} s_N &= b_5 + b_6 + b_7 + \cdots + b_N \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{N-4} - \frac{1}{N-3}\right) \\ &= 1 - \frac{1}{N-4}. \end{aligned}$$

Hence

$$\sum_{n=5}^{\infty} \frac{3}{n^2-7n+12} = 3 \sum_{n=5}^{\infty} b_n = 3 \lim_{n\to\infty} \left(1 - \frac{1}{N-3}\right) = 3.$$

3. The Divergence Test: prove the following series diverges.

- $\displaystyle\sum_{n=2}^{\infty} \cos\left(\frac{n\pi}{2}\right)$ [3pt]

$\displaystyle\lim_{n\to\infty} \cos\left(\frac{n\pi}{2}\right)$ does not have a limit (cos is oscillating). So the series diverges.

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$ [3pt]

Here $\displaystyle\lim_{n\to\infty} \frac{1}{4+e^{-n}} = \frac{1}{4} \neq 0$, so the series diverges.

- $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n^3+n}$ [3pt]

  To compute the limit of $a_n = \dfrac{e^n}{n^3+n}$ we need to apply L'Hopital's rule. To make sense of the derivatives, we consider the function $f(x) = \dfrac{e^x}{x^3+x}$, then

$$\lim_{x\to\infty} \frac{e^x}{x^3+x} = \lim_{x\to\infty} \frac{e^x}{3x^2+1} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty \neq 0.$$

  The series diverges.

4. The Integral Test: determine if the following series converges or diverges.

- $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{e^n}$ [5pt]

  Let $f(x) = \dfrac{x^4}{e^x}$. Note that f is a positive and continuous function defined on $[1, \infty)$ and $\displaystyle\lim_{x\to\infty} f(x) = 0$. Moreover (2pt)

$$f'(x) = \frac{x^3(4-x)}{e^x} \qquad \Longrightarrow \qquad f'(x) \leq 0 \text{ when } x \geq 4, \qquad \text{(1pt)}$$

  and

$$\int_4^\infty \frac{x^4}{e^x}\, \mathrm{d}x \leq \int_4^\infty \frac{x^4}{x^6}\, \mathrm{d}x = \int_4^\infty \frac{1}{x^2}\, \mathrm{d}x \qquad \text{(converges by the integral test, 1pt)}$$

  This shows that for $a_n = \dfrac{n^4}{e^n}$, $\displaystyle\sum_{n=4}^{\infty} a_n < \infty$. Hence

$$\sum_{n=1}^{\infty} \frac{n^4}{e^n} = a_1 + a_2 + a_3 \sum_{n=4}^{\infty} a_n < \infty. \qquad \text{(Converges. 1pt)}$$

- $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ [5pt]

  Let $f(x) = \dfrac{x}{x^3+1}$. Note that f is a positive and continuous function defined on $[1, \infty)$ and $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{3x^2} = 0$. Moreover (2pt)

$$f'(x) = \frac{1-2x^2}{(x^3+1)^2} \qquad \Longrightarrow \qquad f'(x) < 0 \text{ when } x > \sqrt{\frac{1}{2}}, \qquad \text{(2pt)}$$

  and

$$\int_1^\infty \frac{x}{x^3+1}\, \mathrm{d}x \leq \int_1^\infty \frac{x}{x^3}\, \mathrm{d}x = \int_1^\infty \frac{1}{x^2}\, \mathrm{d}x \qquad \text{(converges by the integral test, 1pt)}$$

- $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ for $p > 1$ (and for $p \leq 1$ respectively). [11pt]

Let $f(x) = \dfrac{1}{x(\ln x)^p}$. Note that for all p, f is a positive and continuous function defined on $[2, \infty)$ and $\lim_{x\to\infty} f(x) = 0$. Moreover (3pt)

$$f'(x) = -p(x \ln x)^{-p-1}(\ln x + 1) < 0, \text{ when } x \geq 2. \qquad (3pt)$$

Finally

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,\mathrm{d}x = \int_{\ln 2}^{\infty} \frac{1}{u^p}\,\mathrm{d}u = \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases} \qquad (3pt)$$

By the integral test the series converges if $p > 1$ and diverges if $p \leq 1$. (1pt)

Homework 5

Due: Friday, Apr 1, by the end of the class

1. Comparison Test for Sequence

   - $\displaystyle\sum_{n=1}^{\infty} \frac{3n-2}{2n^3+5}$ [5pt]

     Comparison test: let $a_n = \dfrac{3n-2}{2n^3+5}$ and $b_n = \dfrac{3n}{n^3} = \dfrac{3}{n^2}$ so that $a_n \le b_n$ for all $n > 0$. Then comparison test says (3pt)

     $$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{n^2} \text{ converges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.} \qquad (2\text{pt})$$

   - $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n+n^2}$ [5pt]

     Comparison test: let $a_n = \dfrac{1}{e^n+n^2}$ and $b_n = \dfrac{1}{n^2}$ so that $a_n \le b_n$ for all $n > 0$. Then comparison test says (3pt)

     $$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ converges.} \qquad (2\text{pt})$$

2. Limit Comparison Test

   - $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-n+5}{n^3-3n+6}$ [6pt]

     Limit comparison test: let $a_n = \dfrac{n^2-n+5}{n^3-3n+6}$ and $b_n = \dfrac{1}{n}$. Then (2pt)

     $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3-n^2+5n}{n^3-3n+6} = \lim_{n\to\infty} \frac{1-\frac{1}{n}+\frac{5}{n^2}}{1-\frac{3}{n^2}+\frac{6}{n^3}} = 1. \qquad (2\text{pt})$$

     The limit is finite and not zero, so comparison test applies. Hence

     $$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n \text{ diverges.} \qquad (2\text{pt})$$

   - $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$ [6pt]

     Limit comparison test: let $a_n = \dfrac{1}{n\sqrt{n+1}}$ and $b_n = \dfrac{1}{n\sqrt{n}}$. Then (2pt)

     $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n\sqrt{n}}{n\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1. \qquad (2\text{pt})$$

The limit is finite and not zero, so comparison test applies. Hence

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} \text{ converges} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \text{ converges.} \tag{2pt}$$

Caution: The limit comparison test requests the limit to be positive. We need to take the dominated part of the sequence. Consider $\tilde{b}_n = \frac{1}{n}$, you can check that $\lim_{n\to\infty} \frac{a_n}{\tilde{b}_n} = 0$. We may not apply the limit comparison test with $\tilde{b}_n$.

3. Alternating Series: determine the following series converges absolutely, conditionally or diverges.

- $\sum_{n=1}^{\infty} \frac{(-3)^n n^2}{n!}$ [6pt]

  Ratio test:

$$L = \lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}(n+1)^2}{(n+1)!}}{\frac{(-3)^n n^2}{n!}} \right| = \lim_{n\to\infty} \frac{3(n+1)^2}{n^2(n+1)} = 0 < 1.$$

  By the ratio test, the series converges absolutely.

- $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$ [6pt]

  Root test:

$$L = \lim_{n\to\infty} \left(\frac{(-2)^{2n}}{n^n} \right)^{1/n} = \lim_{n\to\infty} \frac{4}{n} = 0 < 1.$$

  By the root test, the series converges absolutely.

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2/3}}$ [8pt]

  Consider $b_n := |a_n| = \frac{1}{n^{2/3}}$. By p-test with $p = \frac{2}{3} \le 1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges.

  Hence the original series $\sum_{n=1}^{\infty} a_n$ does not converge absolutely. (3pt)

  Now using the alternating series test to check conditional convergence:

  - $b_n \ge 0$.
  - $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^{2/3}} = 0$.
  - $n^{2/3} < (n+1)^{2/3}$ implies $\{b_n\}$ is a decreasing sequence for all n.

  By alternating series test, the original series $\sum_{n=1}^{\infty} a_n$ converges conditionally. (5pt)

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (\ln n)^2}{n}$ [8pt]

  Consider $b_n := |a_n| = \dfrac{(\ln n)^2}{n}$. Since $|a_n| > \dfrac{1}{n}$ for $n > e$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Break the series into two parts gives

$$\sum_{n=1}^{\infty} b_n = b_1 + \sum_{n=2}^{\infty} b_n.$$

  The first term is finite and the second term is divergent. So the series with absolute value is divergent. We conclude $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (\ln n)^2}{n}$ does not converges absolutely. (3pt)

  Now consider the alternating series test:

  - $b_n \geq 0$.
  - $0 \leq \lim\limits_{n\to\infty} b_n \leq \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ implies $\lim\limits_{n\to\infty} b_n = 0$.
  - To show $\{b_n\}$ is a decreasing sequence for $n > e^2$: check

$$\left(\frac{(\ln x)^2}{x}\right)' = \frac{2(\ln x)(1/x)\cdot x - (\ln x)^2}{x} = \frac{(2 - \ln x)\cdot(\ln x)}{x} < 0 \implies \ln x > 2.$$

    Now using the fact that $e^2 < 9$, we can break the series into two parts

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{8} a_n + \sum_{n=9}^{\infty} a_n.$$

    The first term is finite and the second term is convergent by alternating series test. This implies the alternating series is convergent.

  By alternating series test, the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges conditionally. (5pt)

Homework 6

Due: Friday, Apr 8, by the end of the class

1. Power Series: determine the radius of convergence R and interval of convergence I.

   - $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(x-3)^n}{n\cdot 5^n}$ [5pt]

     Compute the limit

     $$L=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(x-3)^{n+1}}{(n+1)\cdot 5^{n+1}}}{(-1)^n\frac{(x-3)^n}{n\cdot 5^n}}\right|=\lim_{n\to\infty}\left|\frac{n(x-3)}{5(n+1)}\right|=\frac{|x-3|}{5}.$$

     The inequality $L<1$ gives $|x-3|<5$. Hence $R=5$ and the series converges when $-2<x<8$. (3pt)
     Now consider the boundary cases: $x=-2$ and $x=8$:

     – When $x=-2$, we have $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(-5)^n}{n\cdot 5^n}=\sum_{n=1}^{\infty}\frac{1}{n}$, which is divergent.

     – When $x=8$, we have $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{5^n}{n\cdot 5^n}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$, which is convergent.

     This implies $I=(-2,8]$. (2pt)

   - $\displaystyle\sum_{n=2}^{\infty}\frac{(x+2)^n}{2^n\ln n}$ [7pt]

     $$\begin{aligned}L&=\lim_{n\to\infty}\left|\frac{\frac{(x+2)^{n+1}}{2^{n+1}\ln(n+1)}}{\frac{(x+2)^n}{2^n\ln n}}\right|=\frac{|x+2|}{2}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \qquad \text{(apply L'Hopital's rule)}\\&=\frac{|x+2|}{2}\lim_{n\to\infty}\frac{1/n}{1/(n+1)}=\frac{|x+2|}{2}\lim_{n\to\infty}\frac{n+1}{n}=\frac{|x+2|}{2}.\end{aligned}$$

     The inequality $L<1$ gives $|x+2|<2$. Hence $R=2$ and the series converges when $-4<x<0$. (4pt)
     Now consider the boundary cases: $x=-4$ and $x=0$:

     – When $x=-4$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{2^n\ln n}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{\ln n}$, which is convergent. To see why, apply the alternating series test: $\dfrac{1}{\ln n}$ is a positive, continuous and decreasing sequence for $n\geq 2$.

     – When $x=0$, we have $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{2^n\ln n}=\sum_{n=1}^{\infty}\frac{1}{\ln n}$, which is divergent by the comparsion test with $b_n=\dfrac{1}{n}$.

     This implies $I=[-4,0)$. (3pt)

- $\sum_{n=1}^{\infty} \frac{n^2 x^{2n}}{(2n)!}$ [5pt]

$$L = \lim_{n\to\infty} \left| \frac{\frac{(n+1)x^{2n+2}}{(n+1)!}}{\frac{n^2x^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2x^2}{n^2(2n+2)(2n+1)} \right| = x^2 \cdot 0 = 0 < 1.$$

Note that the limit is identically zero, i.e. does not depend on x. By ratio test it is converges for all x. Hence $R = \infty$, and $I = (-\infty, \infty)$.

2. Functions as Power Series: find the power series representations using substitution, term-by-term integration and differentiation.

- $\arctan(x)$ [6pt]

We know derivative of $\arctan x$ is $\frac{1}{1+x^2}$. There are two ways to do the problem.
Solution 1: Using FTC, so that the constant term is given by $\arctan(0)$.

$$\arctan x - \arctan(0) = \int_0^x \frac{1}{1+t^2}\,\mathrm{d}t = \int_0^x \frac{1}{1-(-t^2)}\,\mathrm{d}t \qquad \text{(FTC, 1pt)}$$

$$\arctan x = \int_0^x \sum_{n=0}^{\infty} (-t^2)^n\,\mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^{2n}\,\mathrm{d}t$$

(substituting $u = -t^2$)

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \qquad \text{(4pt)}$$

Note that substitution may change the range whereas integration will not. Here we used $u = -x^2$. Then $|u| = |-x^2| < 1$ gives $|x| < 1$. So the above expression is valid only for $x \in (-1, 1)$. (1pt)

Final answer: $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ when $|x| < 1$.

Solution 2: Using indefinite integral, then we will have to determine the constant term later.

$$\arctan x = \int \frac{1}{1+x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C.$$

(Similar computation as above, but we have to add a constant C.)

To figure out what the constant is: take $x = 0$, then $LHS = \arctan 0 = 0$ and $RHS = C$, so $C = 0$.
Computing the range for x is similar as above and we get the same answer.

- $\ln\left(\dfrac{1+x}{1-x}\right)$ [7pt]

  Again, there are two ways to do the problem.
  Solution 1: Using FTC, so that the constant term is given by $\ln(1)$.

$$\ln\left(\frac{1+x}{1-x}\right)-\ln(1)=\ln(1+x)-\ln(1-x)$$

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \int_0^x \frac{1}{1+t}\,\mathrm{d}t-\int_0^x \frac{1}{1-t}\,\mathrm{d}t \qquad \text{(FTC 1pt, and substitution)}\\
&= \int_0^x \sum_{n=0}^{\infty}(-t)^n\,\mathrm{d}t+\int_0^x \sum_{n=0}^{\infty} t^n\,\mathrm{d}t\\
&= \sum_{n=0}^{\infty}(-1)^n\int_0^x t^n\,\mathrm{d}t+\sum_{n=0}^{\infty}\int_0^x t^n\,\mathrm{d}t
\end{aligned}$$

  (Note that odd terms cancelled with each other, even terms are the same. 3pt)

$$= \sum_{n=0}^{\infty} 2\int_0^x t^{2n}\,\mathrm{d}t=\sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}. \qquad \text{(2pt)}$$

  Note that substitution may change the range whereas integration will not. Here we used $u=-x$. Then $|u|=|-x|<1$ gives $|x|<1$. So the above expression is valid only for $x\in(-1,1)$. (1pt)

  Final answer: $\ln\left(\dfrac{1+x}{1-x}\right)=\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ when $|x|<1$.

  Solution 2: Using indefinite integral, then we will have to determine the constant term later. Similar computation gives

$$\ln\left(\frac{1+x}{1-x}\right)=\int\frac{1}{1+x}\,\mathrm{d}x-\int\frac{1}{1-x}\,\mathrm{d}x=\sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1}+C.$$

  To figure out what the constant is: take $x=0$, then $LHS=\ln 1=0$ and $RHS=C$, so $C=0$. Substitute C into the power series gives the same answer.

- $\dfrac{7x-x}{3x^2+2x-1}$ [6pt]

$$\begin{aligned}
\frac{7x-x}{3x^2+2x-1} &= \frac{3}{2}\left(\frac{1}{1+x}-\frac{1}{1-3x}\right) \qquad \text{(substitute } u_1=-x \text{ and } u_2=3x\text{)}\\
&= \frac{3}{2}\left(\sum_{n=0}^{\infty}(-x)^n-\sum_{n=0}^{\infty}(3x)^n\right)\\
&= \frac{3}{2}\sum_{n=0}^{\infty}\Big((-1)^n-3^n\Big)x^n. \qquad \text{(3pt)}
\end{aligned}$$

  To find the range of x: Note that we used $u=-x$ and $u=3x$. So there are two inequalities $|u_1|=|-x|<1$ and $|u_2|=|3x|<1$. Both of them should be satisfied. So the above expression is valid only for $x\in\left(-\dfrac{1}{3},\dfrac{1}{3}\right)$. (3pt)

  Final answer: $\dfrac{7x-x}{3x^2+2x-1}=\dfrac{3}{2}\displaystyle\sum_{n=0}^{\infty}\Big((-1)^n-3^n\Big)x^n$ when $|x|<\dfrac{1}{3}$.

3. Taylor and Maclaurin Series: find the power series representation for by computing the n-th derivative $f^{(n)}(0)$.

- $f(x) = \sin(x)$ centered at $\frac{\pi}{2}$. [4pt]

$$f' = \cos x,\ f'' = -\sin x,\ f^{(3)} = -\cos x,\ f^{(4)} = \sin x,\quad \text{and the sequence repeats}$$

Using the formula for Taylor series with $a = \frac{\pi}{2}$, we have $\cos\frac{\pi}{2} = 0, \pm\sin\frac{\pi}{2} = \pm 1$. Hence

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!} \text{ for all } x.$$

- $f(x) = (1+x)^k$ centered at 0. [4pt]

$$f' = k(1+x)^{k-1},\ f'' = k(k-1)(1+x)^{k-2},\ f^{(3)} = k(k-1)(k-2)(1+x)^{k-3},$$
$$\cdots \quad f^{(i)} = \frac{k!}{(k-i)!}(1+x)^{k-i}$$

Here the series actually is a finite sum since the $k+1$-th derivative is zero and all higher order derivatives are zero. Using the formula for Taylor series with $a = 0$, we have

$$f(x) = \sum_{i=0}^{k} \frac{k!}{(k-i)!\,i!} x^{k-i} = \sum_{i=0}^{k} \binom{k}{i} (1+x)^{k-i}.$$

Now $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|1-k/n|}{1+1/n}|x| = |x|$ suggests that $|x| < 1$.

4. Applications:

- Using the Maclaurin series for $f(x) = e^x$ and the alternating series estimation theorem in 11.5 to approximate $\int_0^1 e^{-x^2}\,\mathrm{d}x$ with error $R < 0.04$. [6pt]

Power series expansion of e^x implies

$$\int e^{-x^2}\,\mathrm{d}x = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + C.$$

So the definite integral is just the power series evaluated at $x = 0$ and $x = 1$. That is

$$\int_0^1 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}. \qquad \text{(3pt)}$$

To ensure $R < 0.04$, we need the estimation for alternating series:

$$|R_N| \le |a_N| = \frac{1}{(2n+1)n!} < 0.04.$$

Check that

$$|R_1| \le \frac{1}{3}, \quad |R_2| \le \frac{1}{5\cdot 2} = \frac{1}{10}, \quad |R_3| \le \frac{1}{7\cdot 6} = \frac{1}{42} < \frac{1}{25}$$

Hence we can take $N = 3$, and

$$\int_0^1 e^{-x^2}\,\mathrm{d}x \approx \sum_{n=0}^{3} \frac{(-1)^n}{(2n+1)n!} = 1 - \frac{1}{3} + \frac{1}{10} = \frac{30-1-3}{30} = \frac{26}{30} = \frac{13}{15}. \qquad \text{(3pt)}$$