The Yamabe Problem

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Outline

Motivation

In 2D case

Uniformazation Theorem

Every simply connected Riemann surface S is conformally equivalent to

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- *the complex plane*
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The theorem is consequence of the fact that every Riemann surface has a conformal metric with constant Gaussian curvature.

Definition

Two Riemannian metrics g and h are **conformal** if there exists positive function $f \in C^\infty(M)$ such that $h = e^{2f} g$.

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- scalar curvature

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- Riemannian curvature tensor, n^4 components
- Ricci curvature, n^2 components
- scalar curvature, 1 component

Question: Which curvature to choose?

The Yamabe Problem

Given a compact Riemannian manifold (M, g) with $n = \dim M \geq 3$, find a metric conformal to g with constant scalar curvature.

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Given two metrics g and $\tilde{g}$, the transformation law between the scalar curvatures S and $\tilde{S}$,

$$\tilde{S} = \varphi^{1-p}(a\Delta\varphi + S\varphi).$$

Here φ satisfies $\tilde{g} = \varphi^{p-2}g$ and $a = \dfrac{4(n-1)}{n-2}$, $p = \dfrac{2n}{n-2}$ are constants.

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Define $\Box = a\Delta + S$ and call it the **conformal Laplacian**. Let $\tilde{S} = \lambda =$ const. Then

$$\Box\varphi = \lambda\varphi^{p-1}. \tag{$\star$}$$

Equation $(\star)$ is the Euler-Lagrange equation for the **Yamabe functional**

$$Q_g(\varphi) = \frac{\int_M a|\nabla\varphi|^2 + S\varphi^2 \,\mathrm{d}V_g}{\left(\int_M |\varphi|^p \,\mathrm{d}V_g\right)^{2/p}} = \frac{E(\varphi)}{\|\varphi\|_p^2}.$$

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By Hölder's inequality $Q_g(\varphi)$ is bounded below so we can take the infimum

Definition

The **Yamabe invariant** is the constant

$$\begin{aligned}\lambda(M) &= \inf\{Q_g(\varphi) \mid \varphi \in C^\infty(M) \text{ and positive}\} \\ &= \inf\{Q_g(\varphi) \mid \varphi \in L_1^2(M)\}.\end{aligned}$$

$\lambda(M)$ is an invariant of the conformal class of (M, g).

Main Results

Theorem A (Yamabe, Trudinger, Aubin)

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If M has dimension $n \geq 6$ and M is not locally conformally flat, then $\lambda(M) < \lambda(S^n)$.

Theorem C (Schoen)

If M has dimension $n = 3, 4, 5$ or M is locally conformally flat, then either $\lambda(M) < \lambda(S^n)$ or M is conformal to the n-sphere.

Definition

A map $F : (M, g) \to (N, h)$ is **conformal** if the induced metric F^*h is conformal to the original metric g on M. If F is a diffeomorphism, then we call F a **conformal diffeomorphism**.

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Example

- The stereographic map σ is a conformal diffeomorphism.
- Rotations, $\sigma^{-1}\tau_v\sigma$ and $\sigma^{-1}\delta_\alpha\sigma$ are conformal diffeomorphisms.

The Yamabe Problem on the Sphere

Let $(S^n, \bar{g})$ be the n-sphere with standard metric, then $S = \frac{n(n-1)}{r^2}$. So the Yamabe problem is solvable on the sphere. Moreover, one can prove the following.

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Theorem

The Yamabe functional $Q_g(\varphi)$ on $(S^n, \bar{g})$ is minimized by constant multiples of $\bar{g}$ and its images under conformal diffeomorphisms. These are the only metrics conformal to $\bar{g}$ with constant scalar curvature.

An Upper Bound for $\lambda(M)$

Lemma (Aubin)

For any compact Riemannian manifold (M, g) of dimension $n \geq 3$, $\lambda(M) \leq \lambda(S^n) = \Lambda$.

Proof. Step 1. Consider the function $\varphi = \eta \cdot u_\alpha(x)$ where

- η is a radical cut off function, $\operatorname{supp} \eta = B_{2\epsilon}$, $\eta|_{B_\epsilon} = 1$;
- $u_\alpha(x) = \left(\frac{|x|^2+\alpha^2}{\alpha}\right)^{(n-2)/2}$.

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Direct computation gives $\int_{\mathbb{R}^n} a|\nabla\varphi|^2 \, \mathrm{d}x \leq \Lambda\|\varphi\|_p^2 + O(\alpha^{n-2})$.

- $\Lambda = \inf\limits_{\varphi \in C^\infty(\mathbb{R}^n)} \dfrac{\int_{\mathbb{R}^n} a|\nabla\varphi|^2 \, \mathrm{d}x}{\|\varphi\|_p^2}$;
- $u_\alpha \leq \alpha^{(n-2)/2} \cdot r^{2-n}$ and $|\partial_r u_\alpha| \leq (n-2)\alpha^{(n-2)/2} \cdot r^{1-n}$.

Step 2. On a compact manifold M,

- choose normal coordinates $\{x^i\}$ in a neighbourhood of $P \in M$, then $\mathrm{d}V_g = (1 + O(r))\,\mathrm{d}x$;
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So

$$\int_{B_{2\epsilon}} a|\nabla\varphi|^2\,\mathrm{d}V_g \le (1 + C\epsilon)(\Lambda\|\varphi\|_p^2 + C\alpha^{n-2}),$$

and

$$Q_g(\varphi) = \frac{\int_M a|\nabla\varphi|^2 + S\varphi^2\,\mathrm{d}V_g}{\|\varphi\|_p^2} \le (1 + C\epsilon)(\Lambda + C\alpha).$$

Choosing ϵ and α small, then $\lambda(M) \le \Lambda$.

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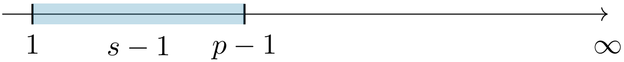
- Instead we seek for a subcritical solution. The following equation is call **subcritical equation**

$$\Box\varphi = \lambda_s\, \varphi^{s-1}. \qquad (\star')$$



$Q^s(\varphi) = \frac{E(\varphi)}{\|\varphi\|_s^2}$, $\lambda_s = \inf\{Q^s(\varphi) : \varphi \in C^\infty(M)\}$.

Proof of Thm A.

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Multiply the metric g by a constant to get $\int_M \mathrm{d}V_g = 1$.

To show $|\lambda_s|$ is non-increasing,

- $E(u) = \int_M a|\nabla u|^2 + Su^2 \,\mathrm{d}V_g$ does not depend on s;
- $Q^s(u) = \frac{\|u\|_s^2}{\|u\|_{s'}^2} Q^{s'}(u)$;
- $s \le s'$, $\|u\|_s \le \|u\|_{s'} \implies Q^s(u) < Q^{s'}(u)$.

If $\lambda(M) \geq 0$, then $\lambda_s \geq 0$ for $2 \leq s \leq p$.

- Suppose $\lambda_s < 0$ for some s;
- $\exists u \in C^\infty$, such that $Q^s(u) < 0$
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To show λ_s is continuous from the left,

- By definition $\forall \epsilon > 0$, $\exists u \in C^\infty(M)$ s.t. $Q^s(u) < \lambda_s + \epsilon$;
- $\|u\|_s$ is continuous as a function of s;
- when $s' \leq s$ and s' close to s,

$$\lambda_{s'} \leq Q^{s'}(u) < \lambda_s + 2\epsilon.$$

Step 3. Suppose $\lambda(M) < \Lambda$, and let φ_s be the subcritical solution. Then there exists $C > 0$, s_0 and r with $s_0 < p < r$ such that $\|\varphi_s\|_r \leq C$ for all $s \geq s_0$.

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For $\delta > 0$ and $w = \varphi_s^{1+\delta}$, the sharp Sobolev inequality and Hölder's inequality imply

$$\|w\|_p^2 \leq (1+\epsilon)\frac{(1+\delta)^2}{1+2\delta} \cdot \frac{\lambda_s}{\Lambda} \cdot \|\varphi_s\|_{(s-2)n/2}^{s-2} \cdot \|w\|_p^2 + C'_\epsilon \cdot \|w\|_2^2.$$

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- Need $\lambda(M) < \Lambda$ to make the coefficient less than 1;
- Choose δ and ϵ, so that $\|w\|_p^2 \leq C\|w\|_2^2$;
- Then $\|w\|_2 = 1 \implies \|w\|_p = \left\|\varphi_s^{1+\delta}\right\|_{p(1+\delta)}^{1+\delta} \leq \tilde{C}$.

Step 4. As $s \to p$, there is a subsequence of subcritical solutions that converges uniformly. So the limiting function φ is the solution of $\Box\varphi = \lambda(M)\varphi^{p-1}$.

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The functions $\{\varphi_s\}$ are uniformly bounded in $L^r(M)$, and thus in $C^{2,\alpha}(M)$. Apply Arzela-Ascoli Theorem, to obtain a subsequence in C^2 that converges to $\varphi \in C^2(M)$.

One can check that φ solves the equation above (needs Step 2), and $\varphi \in C^\infty(M)$ (ellptic regularity).

In Step 3, we assumed $\lambda \geq 0$. The fact that $\Lambda = \lambda(S^n) > 0$ completes the proof.

Remarks on Theorem B and C

Theorem B (Aubin)

If M has dimension $n \geq 6$ and M is not locally conformally flat, then $\lambda(M) < \lambda(S^n)$.

Theorem C (Schoen)

If M has dimension $n = 3, 4, 5$ or M is locally conformally flat, then either $\lambda(M) < \lambda(S^n)$ or M is conformal to the n-sphere.

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If M has dimension $n \geq 6$ and M is not locally conformally flat, then $\lambda(M) < \lambda(S^n)$.

In this case the estimation of $E(\varphi)$ gives:

$$E(\varphi) = \begin{cases} \Lambda\|\varphi\|_p^2 - C|W(P)|^2\alpha^4 + o(\alpha^4) & n > 6 \\ \Lambda\|\varphi\|_p^2 - C|W(P)|^2\alpha^4 \ln(1/\alpha) + O(\alpha^4) & n = 6 \end{cases}$$

- $\varphi = \eta u_\alpha$ as before;
- M locally conformally flat $\iff$ Weyl tensor $W \equiv 0$;
- If not, choose $P \in M$ with $|W(P)|^2 > 0$, so $\lambda(M) < \Lambda$.

Theorem C (Schoen)

If M has dimension $n = 3, 4, 5$ or M is locally conformally flat, then either $\lambda(M) < \lambda(S^n)$ or M is conformal to the n-sphere.

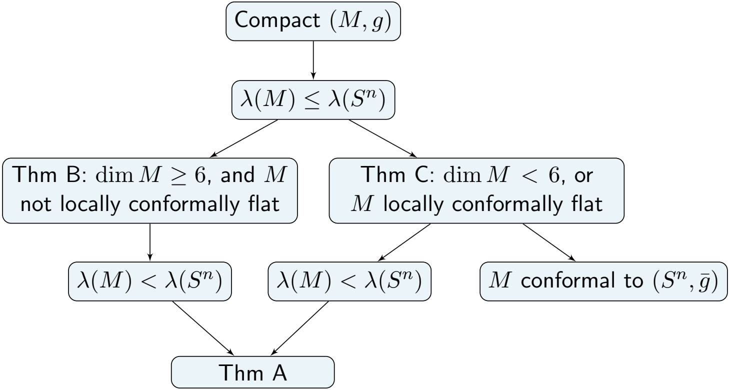
In this case the estimation of $E(\varphi)$ gives:

$$E(\varphi) \leq \Lambda \|\varphi\|_p^2 - C\mu\alpha^{-k} + O(\alpha^{-k-1}),$$

if $n \neq 6$ or M locally conformally flat.

- If the distortion coefficient $\mu > 0$, then $\lambda(M) < \Lambda$.
- The positive mass theorem gives $\mu > 0$.

Summary



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