Analysis of Algorithms

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Lecture 11 University of Southern California

Linear Programming

Reading: chapter 8

Linear Programming

DP

In this lecture we describe linear programming that is used to express a wide variety of different kinds of problems. LP can solve the max-flow problem and the shortest distance, find optimal strategies in games, and many other things.

We will primarily discuss the setting and how to code up various problems as linear programs.

Solving by Reduction

Formally, to reduce a problem Y to a problem X (we write $\boxed{Y \le_p X}$) we want a function f that maps Y to X such that:

- f is a <u>polynomial</u> time computable
- $\forall$ *instance* $y \in Y$ is solvable if and only if $f(y) \in X$ is solvable.

$Y \le_p NF$ $\quad$ $Y \le_p LP$

$\uparrow$ $\quad$ $\uparrow$

A Production Problem

A company wishes to produce two types of souvenirs: type-A will result in a profit of \$1.00, and type-B in a profit of \$1.20.

To manufacture a type-A souvenir requires 2 minutes on machine I and 1 minute on machine II.

A type-B souvenir requires 1 minute on machine I and 3 minutes on machine II.

There are 3 hours available on machine I and 5 hours available on machine II.

How many souvenirs of each type should the company make in order to maximize its profit?

A Production Problem

	Type-A	Type-B	Time Available
Profit/Unit	\$1.00	\$1.20	
Machine I	2 min	1 min	180 min
Machine II	1 min	3 min	300 min

Let $x \geq 0$ be the number of type-A,
$y \geq 0$ — " — " type-B.

Profit: $x \cdot 1 + y \cdot 1.20 \to \max$

Subject to:

A Linear Program

We want to maximize the objective function

$$\max(x + 1.2y)$$

$x + 1.2y = \text{const}$

subject to the system of inequalities:

$$\begin{aligned} 2x + y &\le 180 \\ x + 3y &\le 300 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

draw polygon

A Production Problem

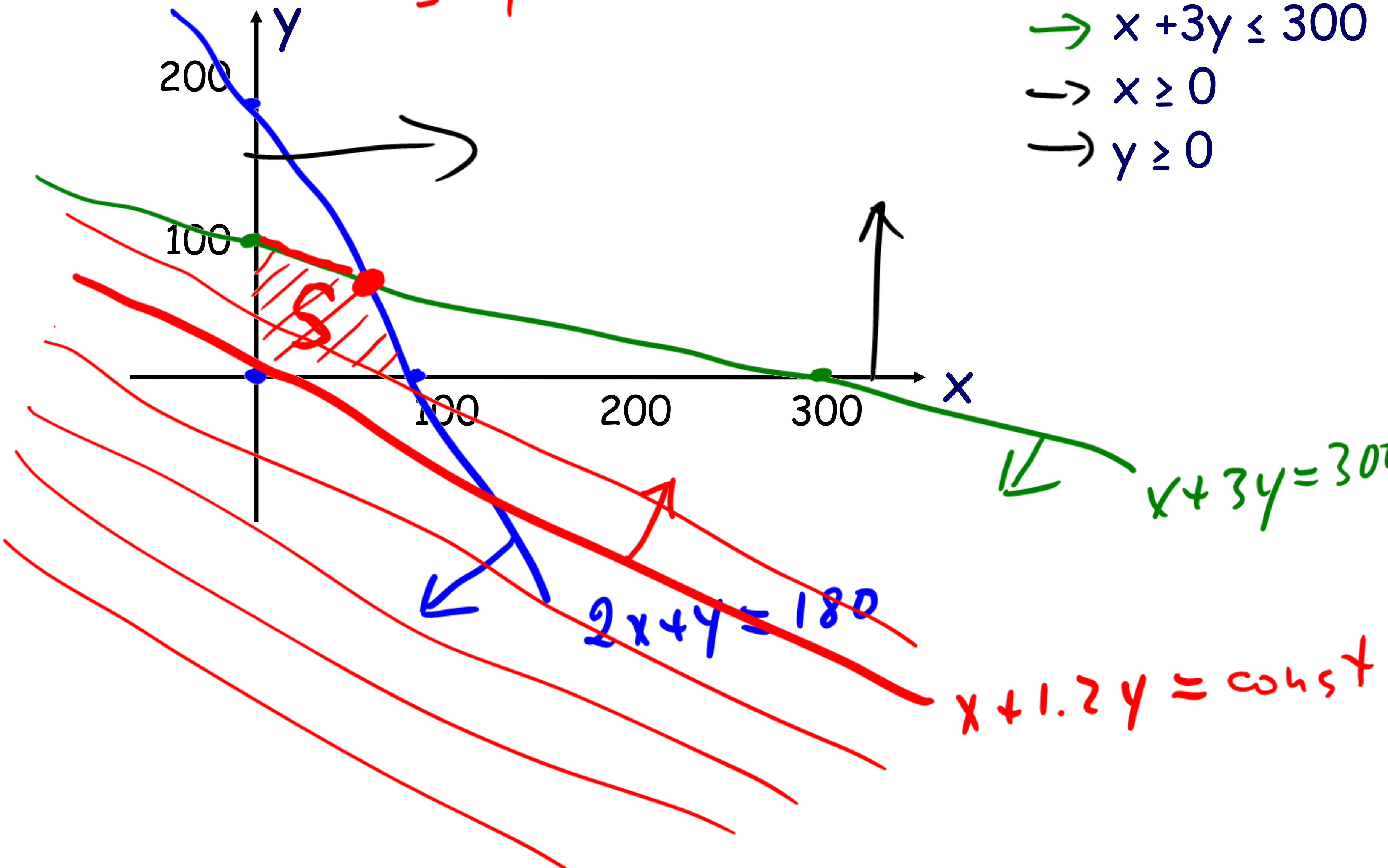
S - feasible solution

→ $2x + y \le 180$

→ $x + 3y \le 300$

→ $x \ge 0$

→ $y \ge 0$

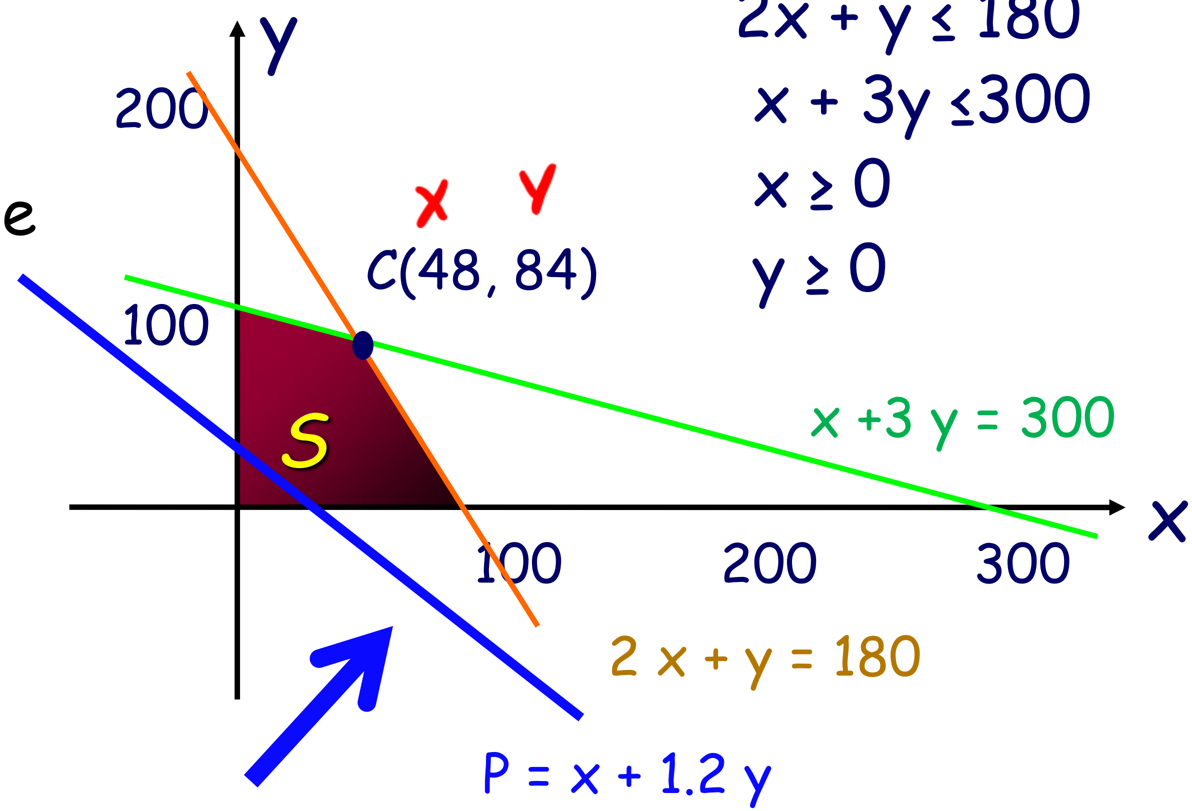


A Production Problem

We want to find the feasible point that is farthest in the "objective" direction $P = x + 1.2\,y$

$$2x + y \le 180$$
$$x + 3y \le 300$$
$$x \ge 0$$
$$y \ge 0$$

We can see that *P* is maximized at the vertex *C*(48, 84) and has a value of 148.8



Fundamental Theorem

If a linear programming problem has a solution, then it must occur at a vertex, or corner point, of the feasible set *S* associated with the problem.

If the objective function *P* is optimized at *two* adjacent vertices of *S*, then it is optimized at *every point* on the line segment joining these vertices, in which case there are infinitely many solutions to the problem.

Existence of Solution

Suppose we are given a LP problem with a feasible set S and an objective function P. There are 3 cases to consider

1. S is empty. LP has NO solution.

$$\boxed{\begin{array}{l} \max(x) \\ x \geq 0 \\ x \leq -1 \end{array}}$$

2. S is unbounded.
LP may or may not have solution.

$$\boxed{\begin{array}{l} \max(x) \\ x \geq 0 \end{array}}$$

3. S is bounded.
LP has solution(s)

$$\boxed{\begin{array}{l} \max(x) \\ x \leq 0 \end{array}}$$

Standard LP form

We say that a maximization linear program with n variables is in standard form if for every variable x_k we have the inequality $x_k \geq 0$ and all other m linear inequalities. A LP in standard form is written as

$c^T x$

$$\max\ (c_1x_1 + \ldots + c_nx_n)$$

$\geq$ or $=$

$$c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

subject to

$$a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1$$

$$\vdots$$

$$a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m$$

$Ax \leq b$

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$$

$x_1 \leq 5$

$$x_1 \geq 0, \ldots, x_n \geq 0$$

$x \geq 0$

Standard LP in Matrix Form

The vector c is the column vector $(c_1, \dots, c_n)$.
The vector x is the column vector $(x_1, \dots, x_n)$.
The matrix A is the $n \times m$ matrix of coefficients of the left-hand sides of the inequalities, and
$b = (b_1, \dots, b_m)$ is the vector of right-hand sides of the inequalities.

$$\max\ (c^T x)$$

subject to

$$A x \le b$$
$$x \ge 0$$

Exercise: Convert to Matrix Form

$\max(x_1 + 1.2\,x_2)$

$2x_1 + x_2 \leq 180$

$x_1 + 3x_2 \leq 300$

$x_1 \geq 0$

$x_2 \geq 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}, \quad b = \begin{pmatrix} 180 \\ 300 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 3x_2 \end{pmatrix} \leq \begin{pmatrix} 180 \\ 300 \end{pmatrix}$$

Algorithms for LP

The standard algorithm for solving LPs is the Simplex Algorithm, due to Dantzig, 1947.

This algorithm starts by finding a vertex of the polytope, and then moving to a neighbor with increased cost as long as this is possible. By linearity and convexity, once it gets stuck it has found the optimal solution.

Unfortunately simplex does not run in polynomial time it does well in practice, but poorly in theory.

Algorithms for LP

In 1974 Khachian has shown that LP could be done in polynomial time by something called the Ellipsoid Algorithm (but it tends to be fairly slow in practice).

In 1984, Karmarkal discovered a faster polynomial-time algorithm called "interior-point". While simplex only moves along the outer faces of the polytope, "interior-point" algorithm moves inside the polytope.

MATLAB

https://www.mathworks.com/help/optim/ug/linprog.html

linprog

min ⟷ − max

Linear programming solver

Finds the minimum of a problem specified by

$$\min_{x} f^T x \text{ such that } \begin{cases} A \cdot x \leq b, \\ Aeq \cdot x = beq, \\ lb \leq x \leq ub. \end{cases}$$

$x \geq 0$

f, *x*, *b*, *beq*, *lb*, and *ub* are vectors, and *A* and *Aeq* are matrices.

Description

`x = linprog(f,A,b)` solves min `f'*x` such that `A*x ≤ b`.

`x = linprog(f,A,b,Aeq,beq)` includes equality constraints `Aeq*x = beq`. Set `A = []` and `b = []` if no inequalities exist.

`x = linprog(f,A,b,Aeq,beq,lb,ub)` defines a set of lower and upper bounds on the design variables, `x`, so that the solution is always in the range `lb ≤ x ≤ ub`. Set `Aeq = []` and `beq = []` if no equalities exist.

Discussion Problem 1

A cargo plane can carry a maximum weight of 100 tons and a maximum volume of 60 cubic meters. There are three materials to be transported, and the cargo company may choose to carry any amount of each, up to the maximum available limits given below.

	Density	Volume	Price	
Material 1	2 tons/m^3	40 m^3	\$1,000 per m^3	x_1
Material 2	1 tons/m^3	30 m^3	\$2,000 per m^3	x_2
Material 3	3 tons/m^3	20 m^3	\$12,000 per m^3	x_3

Write a linear program that optimizes revenue within the constraints.

max

Let x_1, x_2, x_3 be the volumes...

Objective function: $\max(1000x_1 + 2000x_2 + 12000x_3)$

Subject to (constraints):

$$2x_1 + x_2 + 3x_3 \le 100$$
$$x_1 + x_2 + x_3 \le 60$$
$$0 \le x_1 \le 40$$
$$0 \le x_2 \le 30$$
$$0 \le x_3 \le 20$$

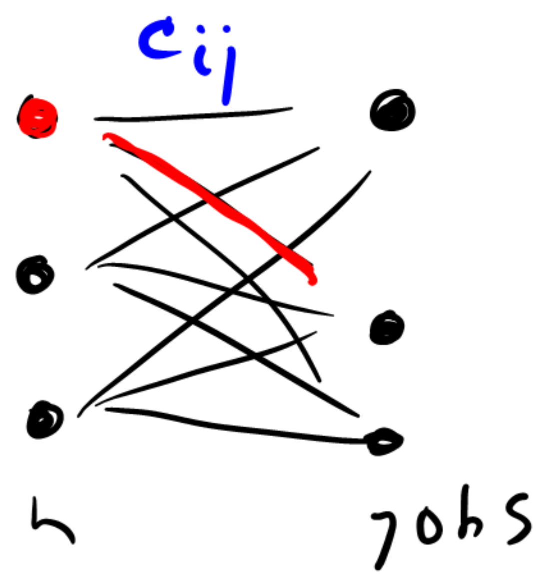
$\to A$

write down A.

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Discussion Problem 2

There are n people and n jobs. You are given a cost matrix, C, where c_{ij} represents the cost of assigning person i to do job j. You need to assign all the jobs to people and also only one job to a person. You also need to minimize the total cost of your assignment. Write a linear program that minimizes the total cost of your assignment.



matching

$$x_{ij} = \begin{cases} 1, & \text{person } i \text{ does job } j \\ 0, & \text{o.w.} \end{cases}$$

Integer LP = ILP

Objective function

$$\min\left(\sum_{i,j} x_{ij} \cdot c_{ij}\right)$$

subject to

persons

$$\sum_{j=1}^{n} x_{ij} = 1 \quad , \; i = 1, 2, \ldots, n$$

jobs

$$\sum_{i=1}^{n} x_{ij} = 1 \quad , \; j = 1, 2, \ldots, n$$

$$\cancel{x_{ij} \geq 0} \qquad x_{ij} \in \{0, 1\}$$

Discussion Problem 3

Convert the following LP to standard form

$$\max\ (5x_1 - 2x_2 + 9x_3)$$

$$3x_1 + x_2 + 4x_3 = 8$$

$$2x_1 + 7x_2 - 6x_3 \le 4$$

$$x_1 \le 0,\ x_3 \ge 1$$

x_2 - ? free

$-\infty \le x_2 \le +\infty$

$3x_1 + x_2 + 4x_2 \le 8$

$-3x_1 - x_2 - 4x_2 \le -8$

$x_1 = -z_1;\ z_1 \ge 0$

$x_3 = z_3 + 1;\ z_3 \ge 0$

$x_2 = z_2 - z_4;\ z_2 \ge 0,\ z_4 \ge 0$

Substitute $x \to z$

$$5x_1 - 2x_2 + 9x_3 = 5(-z_1) - 2(z_2 - z_4) + 9(z_3 + 1)$$

$$= -5z_1 - 2z_2 + 9z_3 + 2z_4 + 9$$

objective function: $\max(-5z_1 - 2z_2 + 9z_3 + 2z_4)$

subject to (do it ~~at home~~ yourself) $\boxed{\text{DIY}}$

$$a = b \Rightarrow \begin{cases} a \geq b \\ a \leq b \end{cases} \Rightarrow \begin{cases} -a \leq -b \\ a \leq b \end{cases}$$

Discussion Problem 4

Explain why LP cannot contain constrains in the form of $\boxed{\textit{strong}}$ inequalities.

$\geq$

$=$

$\leq$

$$\max(7x_1 - x_2 + 5x_3)$$
$$x_1 + x_2 + 4x_3 < 8$$
$$3x_1 - x_2 + 2x_3 > 3$$
$$2x_1 + 5x_2 - x_3 \leq -7$$
$$x_1, x_2, x_3 \geq 0$$

wrong!

$$\boxed{\begin{array}{l} \max(x) \\ x < 1 \\ x \geq 0 \end{array}}$$

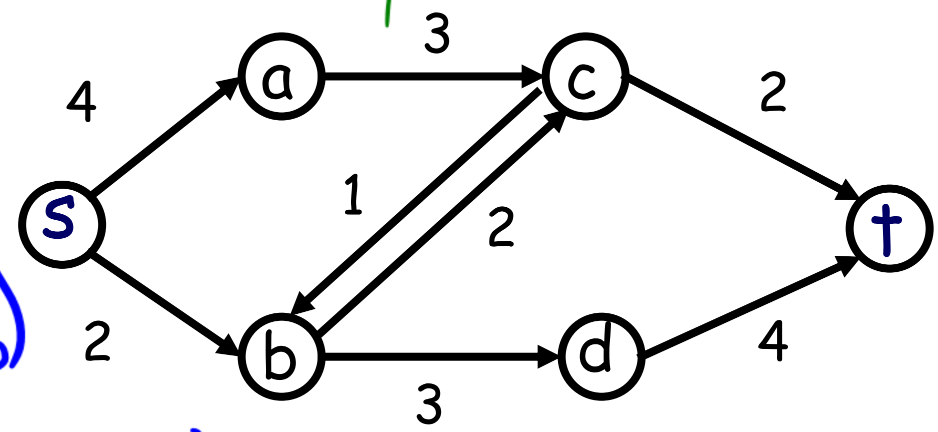
$x = 1$

$x = 0.9$

Exercise: Max-Flow as LP

Write a max-flow problem as a linear program.

$MF \le_P LP$



f_{uv} - flow on (u,v)

Obj. function: $\max(f_{sa} + f_{sb})$

or

$\max(f_{ct} + f_{dt})$

Constraints: $|E+V|$

$0 \le f_{sa} \le 4$

$0 \le f_{sb} \le 2$

for $\forall e \in E$

$f_{sa} = f_{ac}$

$f_{sb} + f_{cb} = f_{bc} + f_{bd}$

for $\forall v \in V \setminus \{s,t\}$

Exercise: Shortest Path as LP

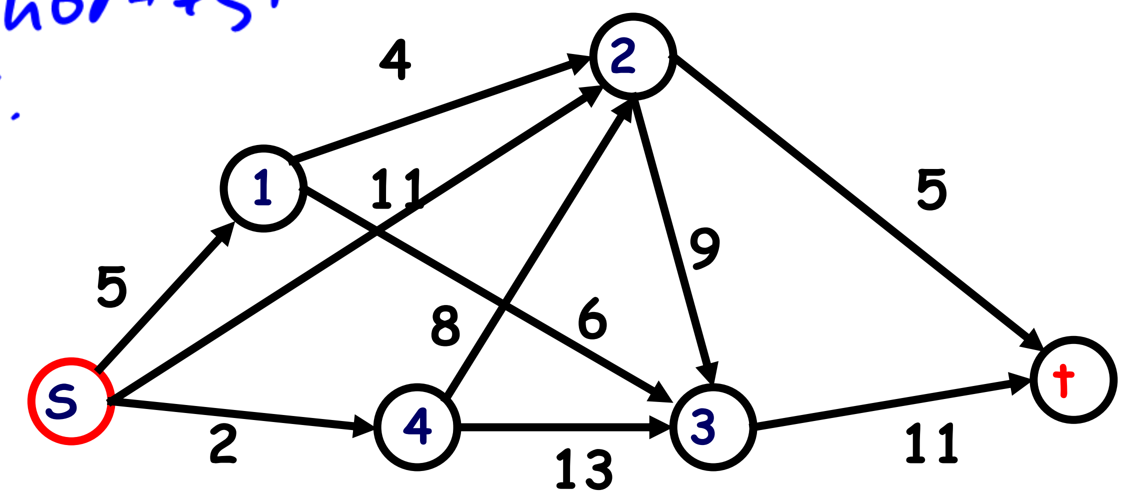
Write a shortest st-path problem as a linear program.

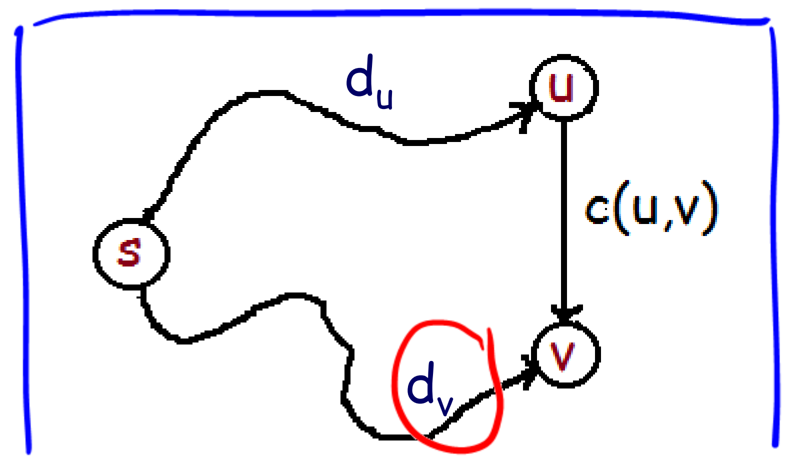
let d(v) be the shortest path from s to T.

$d(s) = 0$, $\boxed{d(v) \geq 0}$

$d(v) \leq d(u) + c(u,v)$

we use it to write constraints.

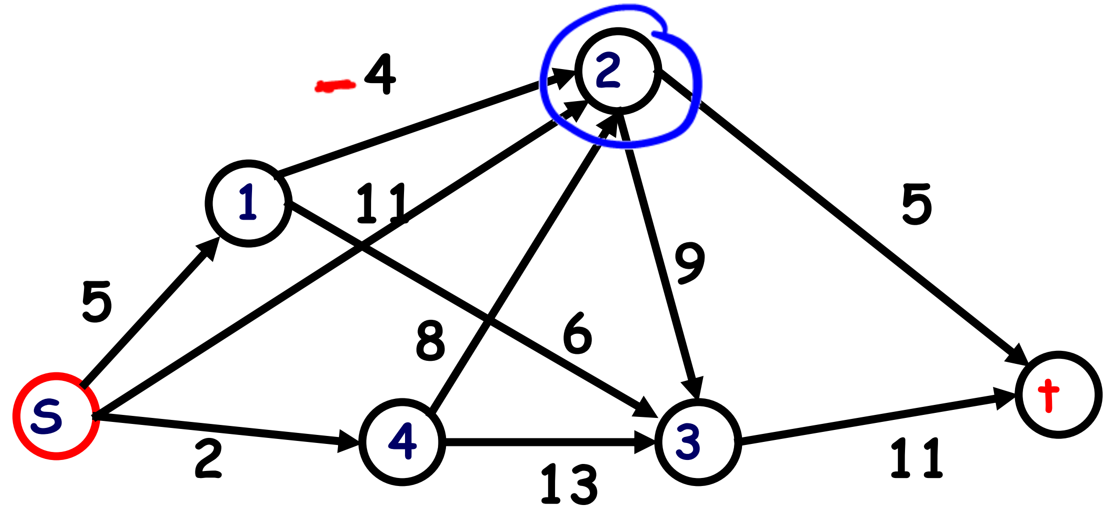




$$d(1) \le d(s) + 5$$
$$d(4) \le d(s) + 2$$
$$d(2) \le d(1) \underline{+} 4$$
$$d(2) \le d(s) + 11$$
$$d(2) \le d(4) + 8$$
$$\vdots$$
for $\forall e \in E$

$$\boxed{\begin{aligned} &\min\ d(1) \\ &d(1) \le d(s) + 5 = 5 \\ &d(1) \ge 0 \end{aligned}}$$

$d(1) = 0$



Objective function: ~~min d(t)~~ or max d(t)

$$\boxed{\begin{aligned} &\max\ d(1) \\ &d(1) \le 5 \\ &d(1) \ge 0 \end{aligned}}$$

$d(1) = 5$

Discussion Problem 5

Write a 0-1 Knapsack Problem as a linear program.

Given n items with weights $w_1, w_2, \ldots, w_n$ and values $v_1, v_2, \ldots, v_n$. Put these items in a knapsack of capacity W to get the maximum total value in the knapsack.

$$\text{Given } \sum_{k=1}^{m} w_k^{\;x_k} \leq W$$

$$\text{optimize } \sum_{k=1}^{m} v_k^{\;x_k} \to \max$$

Knapsack as LP

$$x_k = \begin{cases} 1, & \text{item } k \text{ is selected} \\ 0, & \text{o.w.} \end{cases}$$

Objective fuction: $\max \left(\sum_{k=1}^{n} x_k \cdot v_k \right)$

subject to

$$\sum_{k=1}^{n} x_k \cdot w_k \le W$$

ILP

is it good idea?

$x_k \ge 0$

$$\cancel{x_k \in \{0, 1\}}$$

1.6 -> 2

Knapsack $\le_p$ ILP

$\not\Downarrow$

polynomial

ILP is NOT polynomial

$$\boxed{BM \le_p ILP}$$

BM ≤ NF

Knapsack as LP

Dual LP

To every linear program there is a dual linear program

$$|f| \le cap(A, B)$$

primal dual

$$\max(f) \le \min cap(A, B)$$

$$\max(f) = \min cap(A, B)$$

Duality

<u>Definition.</u> The dual of the standard maximum problem

$$\max c^T x$$
$$Ax \le b \text{ and } x \ge 0$$

primal

max-flow

is defined to be the standard minimum problem

$$\min b^T y$$
$$A^T y \ge c \text{ and } y \ge 0$$

dual

min-cut

Exercise: duality

$$c = \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}$$

Consider the LP:

$$\max(7x_1 - x_2 + 5x_3)$$
$$x_1 + x_2 + 4x_3 \leq 8$$
$$3x_1 - x_2 + 2x_3 \leq 3$$
$$2x_1 + 5x_2 - x_3 \leq -7$$
$$x_1, x_2, x_3 \geq 0$$

$$\begin{pmatrix} 8 \\ 3 \\ -7 \end{pmatrix} = b$$

standard form

Write the dual problem.

primal LP	dual LP
$\max(c^T x)$	$\min(b^T y)$
$Ax \leq b$	$A^T y \geq c$
$x \geq 0$	$y \geq 0$

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 3 & -1 & 2 \\ 2 & 5 & -1 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 3 & 2 \\ 1 & -1 & 5 \\ 4 & 2 & -1 \end{pmatrix}$$

Dual

$$\min\left(8y_1 + 3y_2 - 7y_3\right)$$

$$y_1 + 3y_2 + 2y_3 \geq 7$$

$$y_1 - y_2 + 5y_3 \geq -1$$

$$4y_1 + 2y_2 - y_3 \geq 5$$

$$y_1, y_2, y_3 \geq 0$$

From Primal to Dual

Consider the max LP constrains

$$y_1 \left(a_{11}x_1 + \ldots + a_{1n}x_n \right) \le b_1 \cdot y_1$$

$$+ \qquad \vdots$$

$$y_n \left(a_{m1}x_1 + \ldots + a_{mn}x_n \right) \le b_m \cdot y_n$$

$$\ldots \le \ldots$$

1) Multiply each equation by a new variable $y_k \ge 0$.
2) Add up those m equations.
3) Collect terms wrt to x_k.
4) Choose y_k in a way that $A^T y \ge c$.

$$
+\begin{cases}
y_1(a_{11}x_1 + \dots + a_{1n}x_n) \le b_1 \cdot y_1 \\
\qquad \vdots \\
y_m(a_{m1}x_1 + \dots + a_{mn}x_n) \le b_m \cdot y_m
\end{cases}
$$

$$
y_1(a_{11}x_1 + \dots + a_{1n}x_n) + \dots + y_m(a_{m1}x_1 + \dots + a_{mn}x_n) \le b_1y_1 + \dots + b_my_m
$$

$$
x_1(\overbrace{y_1a_{11} + \dots + y_ma_{m1}}^{\geq c_1}) + \dots + x_n(\overbrace{y_1a_{1n} + \dots + y_ma_{mn}}^{\geq c_n}) \le b_1y_1 + \dots + b_my_m
$$

Dual constraints:

$$
A^Ty \geq c \qquad
\begin{array}{c}
x_1 \\ \\ x_n
\end{array}
\left[
\begin{array}{l}
y_1a_{11} + \dots + y_ma_{m1} \geq c_1 \cdot x_1 \\
\qquad \vdots \\
y_1a_{1n} + \dots + y_ma_{mn} \geq c_n \cdot x_n
\end{array}
\right.
$$

Dual objective function

$$c^T x = x_1 c_1 + \dots + x_n c_n \le x_1(y_1 a_{11} + \dots + y_m a_{m1}) + \dots + x_n(y_1 a_{1n} + \dots + y_m a_{mn})$$

collect wrt y primal

$$= y_1\overbrace{(x_1 a_{11} + \dots + x_n a_{1n})}^{\le b_1} + \dots + y_m\overbrace{(x_1 a_{m1} + \dots + x_n a_{mn})}^{\le b_m}$$

$$\le y_1 b_1 + \dots + y_m b_m = b^T y$$

$$\boxed{\text{Showed } \quad \underset{\max}{c^T x} \le \underset{\min}{b^T y}}$$

Weak Duality

max ($c^T x$) $A x \leq b$ $x \geq 0$	$\Longleftrightarrow$	min ($b^T y$) $A^T y \geq c$ $y \geq 0$
primal linear program		dual linear program

Weak Duality. The optimum of the dual is an upper bound to the optimum of the primal.

max min

$$\text{opt(primal)} \leq \text{opt(dual)}$$

Weak Duality

$$\max(c^T x)$$
$$Ax \le b$$
$$x \ge 0$$

$\Longleftrightarrow$

$$\min(b^T y)$$
$$A^T y \ge c$$
$$y \ge 0$$

Theorem (The weak duality).
Let P and D be primal and dual LP correspondingly.
If x is a feasible solution for P and y is a feasible solution for D, then $c^T x \le b^T y$.

Proof (in matrix form).

$$c^T x = x^T c \le x^T (A^T y) = (Ax)^T y \le b^T y$$

圓

Weak Duality: opt(primal) ≤ opt(dual)

Corollary 1. If a standard problem and its dual are both feasible, then both are feasible bounded.

$$\underset{\text{max}}{c^T x} \leq \underset{\text{min}}{b^T y}$$

a) y is feasible

b) x is feasible

Corollary 2. If one problem has an unbounded solution, then the dual of that problem is infeasible.

$$\underbrace{c^T x}_{\substack{\text{unbounded} \\ x = \infty}} \leq b^T y \quad , \quad \infty \leq b^T y$$

y has no solution

Strong Duality

$$\max(c^T x) \quad A x \le b,\ x \ge 0 \qquad \Longleftrightarrow \qquad \min(b^T y) \quad A^T y \ge c,\ y \ge 0$$

Theorem (The strong duality).
Let P and D be primal and dual LP correspondingly.
If P and D are feasible, then $c^T x = b^T y$.

The proof of this theorem is not as easy and beyond the scope of this course.

Possibilities for the Feasibility

$$\max(c^T x)$$
$$Ax \le b$$
$$x \ge 0$$

$$\min(b^T y)$$
$$A^T y \ge c$$
$$y \ge 0$$

P\D	F.B.	F.U.	I.
F.B.	YES	NO	NO
F.U.	NO	? NO	YES
I.	NO	YES	YES

example

DIY

feasible bounded - F.B.
feasible unbounded - F.U.
infeasible - I.

dual

$\max(x_1 + x_2)$

$x_1 - x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1 \geq 0,\ x_2 \geq 0$

unbounded

F.U.

$\min(4y_1 + 2y_2)$

$y_1 + y_2 \geq 1$

$-y_1 - y_2 \geq 1$

$y_1 \geq 0,\ y_2 \geq 0$

infeasible

Discussion Problem 6

LP → standard LP → dual LP

Consider the LP:

$x_2 = z_2 - 1;\ z_2 \geq 0$

$x_3 = 5 - z_3;\ z_3 \geq 0$

$x_1 = z_1 - z_4;\ z_1 \geq 0$
$z_4 \geq 0$

$$\max(3x_1 + 8x_2 + x_3)$$

$$x_1 + 4x_2 - 2x_3 \leq 20$$
$$x_1 + x_2 + x_3 \geq 7 \rightarrow -x_1 - x_2 - x_3 \leq -7$$
$$x_2 + x_3 = 3 \rightarrow \text{dual}$$
$$x_2 \geq -1$$
$$x_3 \leq 5$$

$x_1 - ?$

$$x_2 + x_3 \leq 3$$
$$-x_2 - x_3 \leq -3$$

Write the dual problem.

Substitution

$$\max(3(z_1 - z_4) + 8(z_2 - 1) + 5 - z_3) =$$

$$= \max(3z_1 + 8z_2 - z_3 - 3z_4)$$

$$x_1 + 4x_2 - 2x_3 \le 20$$

$$z_1 - z_4 + 4(z_2 - 1) - 2(5 - z_3) \le 20$$

$$z_1 + 4z_2 + 2z_3 - z_4 \le 34 \qquad y_1$$

$$-z_1 - z_2 + z_3 + z_4 \le -3 \qquad y_2$$

$$z_2 - z_3 \le -1 \qquad y_3$$

$$z_3 - z_2 \le 1 \qquad y_4$$

~~Knapsack as LP~~

Dual

$$\min(34y_1 - 3y_2 - y_3 + y_4)$$

$y_1 - y_2 \geq 3$ $\quad y_5 \geq 3$ $\quad y_5 = y_1 - y_2$

$4y_1 - y_2 + y_3 - y_4 \geq 8$

$2y_1 + y_2 - y_3 + y_4 \geq -1$

$-y_1 + y_2 \geq -3$ $\quad -y_5 \geq -3$, $\quad y_5 \leq 3$

$y_1, y_2, y_3, y_4 \geq 0$

Nonlinear Optimization

DIY

convex

non-con

$$\max f(x)$$

$$h_i(x) \le 0,\ i = 1,2\ ,\ \ldots,\ m.$$

$$x_k \ge 0,\ k = 1,2\ ,\ \ldots,\ n.$$

Here $x^T = (x_1, x_2, \ldots, x_n)$, f and/or h are nonlinear functions.

The problem is solved using Lagrange multipliers.

Lagrange Duality (KKT-1951)

Primal in x:

max f(x)
subject to
$h_k(x) \leq 0$

Dual in λ:

min $g(\lambda)$
subject to
$\lambda_k \geq 0$

$$L(x,\lambda) = f(x) + \sum_k \lambda_k h_k(x)$$

$$g(\lambda) = \min_x \ L(x,\lambda)$$

Weak Duality:
Let P and D be the optimum of primal and dual problems respectively. Then opt(P) ≤ opt(D).
Equality (strong duality) holds for convex functions.