Notes on Many Things

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1 On a Differentiable Manifold - to be finished

2 On a Vector Bundle

On a general vector bundle (E, V, M), we can define *vector-valued p-form field* on $\mathcal{M}$ to be a section of the bundle $E \otimes \bigwedge^p \mathcal{M}$.

We can endow the vector bundle with an additional structure called the *affine* (This word needs review) *conneciton* which is a map $\nabla : \mathscr{X}[E] \to \mathscr{X}[E \otimes T^*M]$, note T^*M is the same as $\bigwedge^1 M$. After choosing a local frame $\{e_\alpha\}$ the connection can be represented by a set of 1-forms ${\omega_\alpha}^\beta$ which is defined by

$$\nabla \mathbf{e}_\alpha = {\omega_\alpha}^\beta \mathbf{e}_\beta \tag{2.1}$$

. (Note that strictly speaking the product on the right-hand-side is to be understood as tensor product, because the vector spaces they belong are distinct so no confusion is to be raised by writing them commutatively)

Now we can define for vector-valued forms the similar but "hybrid" operations such as exterior multiplication and exterior differentiation. Exterior multiplication is between an ordinary p-form and a vector-valued q-form, and it is defined rather ordinarily by first tensor producting and then antisymmetrizing all the base vector slots with suitable normalizations.

Exterior differentiation is a map $\mathscr{D} : \mathscr{X}[E \otimes \bigwedge^p M] \to \mathscr{X}[E \otimes \bigwedge^{p+1} M]$, which is determined by requiring $\mathscr{D}(\Psi\omega) = \mathscr{D}\Psi \wedge \omega + \Psi d\omega = d\omega\Psi + (-1)^p \omega \wedge \mathscr{D}\Psi$ for section $\Psi \in \mathscr{X}[E]$ and $\omega \in \bigwedge^p M$.

Then we can observe that for function $f \in \mathscr{F}[M]$ we have

$$\mathscr{D}^2(f\Psi) = \mathscr{D}(df\Psi + f\mathscr{D}\Psi) = -df \wedge \mathscr{D}\Psi + df \wedge \mathscr{D}\Psi + f\mathscr{D}^2\Psi = f\mathscr{D}^2\Psi \tag{2.2}$$

thus in fact $\mathscr{D}^2$ on $\mathscr{X}[E]$ is a point-wise antisymmetric linear map from $T_xM \otimes T_xM$ to $\mathfrak{gl}(V)$, so we can define the *curvature* θ of the connection ∇ (or $\mathscr{D}$) to be $\theta\Psi = \mathscr{D}^2\Psi$, in a local frame we can write

$${\theta_\alpha}^\beta \mathbf{e}_\beta = \theta \mathbf{e}_\alpha = \mathscr{D}^2 \mathbf{e}_\alpha = \mathscr{D}({\omega_\alpha}^\beta \mathbf{e}_\beta) = d{\omega_\alpha}^\beta \mathbf{e}_\beta - {\omega_\alpha}^\beta \wedge {\omega_\beta}^\gamma \mathbf{e}_\gamma = (d{\omega_\alpha}^\beta - {\omega_\alpha}^\gamma \wedge {\omega_\gamma}^\beta)\mathbf{e}_\beta \tag{2.3}$$

So we have ${\theta_\alpha}^\beta = d{\omega_\alpha}^\beta - {\omega_\alpha}^\gamma \wedge {\omega_\gamma}^\beta$.

The dual bundle (E^*, V^*, M) is constructed as an associate bundle of the principal bundle, and the connection can be transferred to it by requiring in a local frame $(\nabla\Psi, \Omega) = d(\Psi, \Omega) - (\Psi, \nabla\Omega)$.

3 On the Tangent Bundle - to be finished

An affine connection can be given to the tangent bundle. Tangent bundle is a very special vector bundle in that it inherits the manifold structure in some way, thus a connection of such a bundle can have more properties to study, such as its *torsion tensor*, defined as

$$\tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \tag{3.1}$$

note that its establishment relies on the fact that it's a connection on the tangent bundle.

Choose a tetrad $\{\mathbf{e}_\alpha\}$ and a co-tetrad $\{\mathbf{e}^\alpha\}$

4 On a (pseudo-)Riemannian Manifold

For an n-dimensional oriented manifold with a nondegenerate metric g_{ab} whose number of negative eigenvalues is x, the volume pseudo-form is defined as

$$Vol_I = \sqrt{(-1)^x \det g}\ \epsilon_I \tag{4.1}$$

in a coordinate system compatible with the orientation, where ϵ is the *totally antisymmetric symbol.* Note this form is indeed pseudo.

We will also use in this section the *antisymmetric identity tensor* δ^I_J (see Section 11), whose value is 1 if J is an even permutation of I and -1 if odd and 0 otherwise. This tensor is naturally defined for any rank on any generic manifold without additional structure.

We also have the *contravariant volume form* whose definition and properties are

$$Vol^I = Vol_J\ g^{IJ} = \frac{(-1)^x}{\sqrt{(-1)^x \det g}} \epsilon^I \tag{4.2a}$$

$$Vol^{IK}\ Vol_{J\underline{K}} = (-1)^x \delta^I_J \tag{4.2b}$$

where the underlined $\underline{K}$ indicates nonrepeating of antisymmetric contraction, namely, for rank-p antisymmetric contraction we have $\phi^I \psi_{\underline{I}} = \frac{1}{p!}\phi^I \psi_I$.

Hodge duality is a pointwise map denoted by $* : \bigwedge^p T^*_x M \to \bigwedge^{n-p} T^*_x M$, and it is defined by the relation

$$\phi \wedge *\psi = \frac{1}{p!}\ \langle \phi, \psi \rangle\ Vol \tag{4.3}$$

where $\langle\rangle$ denotes inner product induced from that of $\bigotimes^p T^*_x M$, which in turn is the tensor product metric of the metric g^{ab}. Explicitly

$$(*\phi)_I = \phi^J\, Vol_{\underline{J}I} \tag{4.4}$$

for p-form we have the relation

$$*\,(*\phi) = (-1)^{p(n-p)+x}\phi \tag{4.5}$$

we also have

$$*\,1 = Vol \tag{4.6}$$

The *codifferential operator* is defined for p-form to be

$$d^*\phi = (-1)^{n(p-1)+x+1} * (d(*\phi)) \tag{4.7}$$

on a closed (compact without boundary) manifold it is adjoint to the *exterior differential* d with respect to the nondegenerate metric $(,) = \int \langle , \rangle\ Vol$ defined on the algebra $\bigwedge T^*M$, namely

$$\int \frac{1}{(p+1)!} \left\langle d\phi^p, \psi^{p+1} \right\rangle\ Vol = \int \frac{1}{p!} \left\langle \phi^p, d^*\psi^{p+1} \right\rangle\ Vol \tag{4.8}$$

we can calculate the component expression of codifferential for a p-form, let ∇ denote the torsion-free connection compatible with the metric, then

$$
\begin{aligned}
(d^*\phi)_I &= (-1)^{n(p-1)+x+1}\frac{1}{(n-p+1)!}(d(*\phi))^J\, Vol_{JI} \\
&= (-1)^{n(p-1)+x+1}\frac{n-p+1}{(n-p+1)!}\nabla^j(*\phi)^{J'}\, Vol_{jJ'I} \\
&= (-1)^{n(p-1)+x+1}\frac{1}{(n-p)!}\nabla^j\left[\phi_K\, Vol^{\underline{K}J'}\right] Vol_{jJ'I} \\
&= (-1)^{x+1}\left(\nabla^j\phi_K\right) Vol^{\underline{K}J'}\, Vol_{jI\underline{J'}} \\
&= -\left(\nabla^j\phi_K\right)\delta^{\underline{K}}_{jI} \\
&= -\nabla^j\phi_{jI}
\end{aligned} \tag{4.9}
$$

just inverse the usual divergence formula that we are familiar with. Note at the fourth equality we have used another good property of the Levi-civita connection, that is $\nabla\, Vol = 0$, this can be easily seen by choosing gauss coordinates for any point so that all connection coefficients vanish, explicitly,

$$
\nabla_a\, Vol_I = \partial_a\left(\sqrt{|\det g|}\right)\epsilon_I = \sqrt{|\det g|}\left(\mathit{\Gamma}_{ab}{}^b\right)\epsilon_I = 0 \tag{4.10}
$$

Let's derive *Gauss's divergence formula* as the end of this section

$$
\begin{aligned}
\int_D \left(\nabla^a\Phi_a\right)\, Vol &= -\int_D d^*\Phi\cdot *1 \\
&= -(-1)^{x+1}\int_D *(*d*\Phi) \\
&= \int_D d(*\Phi) \\
&= \int_{\partial D}\Phi^b\, Vol_{ba_1\cdots a_{n-1}}
\end{aligned} \tag{4.11}
$$

5 Riemannian Curvature Tensor - to be finished

Note that in this section n is the space(-time) dimension!

6 About Weyl Tensor

In this section the Levi-Civita connection is chosen.

Define

$$
V_{ab}{}^{cd} \equiv R_{ab}{}^{cd} - \frac{2}{n(n-1)}R\cdot\delta^c_{[a}\delta^d_{b]} \tag{6.1}
$$

then we have

$$
V_{ab}{}^{ab} = 0
$$

. Define

$$
V_a{}^c \equiv V_{ab}{}^{cb} = R_a{}^c - \frac{1}{n}R\cdot\delta^c_a \tag{6.2}
$$

and then define the *Weyl Tensor*

$$
\begin{aligned}
W_{ab}{}^{cd} &\equiv V_{ab}{}^{cd} - \frac{4}{n-2} V_{[a}{}^{[c}\delta_{b]}^{d]} \\
&= R_{ab}{}^{cd} - \frac{2}{n(n-1)} R \cdot \delta_{[a}^{c}\delta_{b]}^{d} - \frac{4}{n-2} R_{[a}{}^{[c}\delta_{b]}^{d]} + \frac{4}{n(n-2)} R \cdot \delta_{[a}{}^{[c}\delta_{b]}^{d]} \\
&= R_{ab}{}^{cd} - \frac{4}{n-2} R_{[a}{}^{[c}\delta_{b]}^{d]} + \frac{2}{(n-1)(n-2)} R \cdot \delta_{[a}{}^{[c}\delta_{b]}^{d]} \qquad (6.3)
\end{aligned}
$$

then we can see that $W_{ab}{}^{cb} = 0$ so that $W_{ab}{}^{cd}$ represents the totally traceless part of the Riemannian tensor.

If a Riemannian tensor is of the form

$$R_{ab}{}^{cd} = L_{[a}{}^{[c}\delta_{b]}^{d]} \qquad (6.4)$$

for some symmetric tensor field L_{ab}, then the Weyl tensor vanishes.

In 3-dimensions

Using Hodge Duality R_{ab}^{cd} can be represented by a symmetric 2-rank tensor

$$\rho_a{}^b \equiv Vol_{acd} Vol^{bef} R_{ef}{}^{cd} \qquad (6.5)$$

, the inverse formula being

$$
\begin{aligned}
R_{ab}{}^{cd} &= \frac{1}{4} \rho_e{}^f \, Vol^{ecd} \, Vol_{fab} \\
&= -\frac{1}{4} \rho_e{}^f \delta_{fab}^{ecd} \\
&= -\frac{1}{4} (\rho_e{}^e \delta_{ab}^{cd} + \rho_a{}^e \delta_{be}^{cd} + \rho_b{}^e \delta_{ea}^{cd}) \\
&= -\frac{1}{2} (\rho_e{}^e \delta_{[a}^{c}\delta_{b]}^{d} + \delta_{[b}^{c}\rho_{a]}{}^d + \rho_{[b}{}^c\delta_{a]}^{d}) \\
&= -\frac{1}{2} (\rho_e{}^e \delta_{[a}^{c}\delta_{b]}^{d} - 2\rho_{[a}{}^{[c}\delta_{b]}^{d]}) \qquad (6.6)
\end{aligned}
$$

Riemannian tensors of this form has vanishing Weyl tensor.

Weyl Transformation

Under the transformation

$$\widetilde{g}_{ab} = \Omega^2 g_{ab} \quad \text{or} \quad \widetilde{g}^{ab} = \frac{1}{\Omega^2} g^{ab}$$

, Define tilded quantities to be those corresponding to the new metric $\widetilde{g}$, define

$$C_{ab}^c \equiv \widetilde{\Gamma}_{ab}{}^c - \Gamma_{ab}{}^c$$

, Note that it is a tensor, so

$$
\begin{aligned}
C_{ab}^c &= \frac{1}{2} \widetilde{g}^{cd} (\nabla_a \widetilde{g}_{bd} + \nabla_b \widetilde{g}_{ad} - \nabla_d \widetilde{g}_{ab}) \\
&= (\delta_b^c \partial_a + \delta_a^c \partial_b - g_{ab} \partial^c) \ln \Omega \qquad (6.7)
\end{aligned}
$$

Riemannian tensor transforms as

$$\begin{aligned}
\widetilde{R}_{abc}{}^{d} - R_{abc}{}^{d} &= 2\nabla_{[a}C_{b]c}{}^{d} - 2C_{[a|c}{}^{e}C_{b]e}{}^{d} \\
&= (\delta^d_c\nabla_a\nabla_b + \delta^d_b\nabla_a\nabla_c - g_{bc}\nabla_a\nabla^d)\ln\Omega \\
&\qquad - (\delta^e_c\nabla_a + \delta^e_a\nabla_c - g_{ac}\nabla^e)\ln\Omega\cdot(\delta^d_e\nabla_b + \delta^d_b\nabla_e - g_{be}\nabla^d)\ln\Omega \\
&\qquad - exchange(a,b)
\end{aligned} \tag{6.8}$$

Define

$$\begin{aligned}
\omega_{ab} &= \nabla_a\nabla_b\ln\Omega \\
\Omega_{ab} &= (\nabla_a\ln\Omega)(\nabla_b\ln\Omega)
\end{aligned} \tag{6.9}$$

both are symmetric. So that

$$\begin{aligned}
\widetilde{R}_{abc}{}^{d} - R_{abc}{}^{d} &= (\delta^d_c\omega_{ab} + \delta^d_b\omega_{ac} - g_{bc}\omega^d_a) \\
&\quad - (\delta^d_c\Omega_{ab} + 2\delta^d_b\Omega_{ac} - g_{bc}\Omega^d_a + \delta^d_a\Omega_{bc} - g_{ab}\Omega^d_c - g_{ac}\delta^d_b\Omega^e_e) \\
&\quad - exchange(a,b) \\
&= (\delta^d_b\omega_{ac} - g_{bc}\omega^d_a) - (2\delta^d_b\Omega_{ac} - g_{bc}\Omega^d_a + \delta^d_a\Omega_{bc} - g_{ac}\delta^d_b\Omega^e_e) - exchange(a,b)
\end{aligned} \tag{6.10}$$

$$(\widetilde{R}_{abe}{}^{d} - R_{abe}{}^{d})g^{ec} = 4\omega^{[c}_{[a}\delta^{d]}_{b]} - 4\Omega^{[c}_{[a}\delta^{d]}_{b]} + 2\Omega^e_e\delta^c_{[a}\delta^d_{b]} \tag{6.11}$$

$$(\widetilde{R}_{ae} - R_{ae})g^{ec} = (n-2)(\omega^c_a - \Omega^c_a) + \delta^c_a\left[\omega^b_b + (n-2)\Omega^b_b\right] \tag{6.12}$$

$$\begin{aligned}
\widetilde{R} &= (\widetilde{R}_{ac} - R_{ac})\widetilde{g}^{ac} + R_{ac}\widetilde{g}^{ac} \\
&= \frac{1}{\Omega^2}\left[(n-2)(\omega^a_a - \Omega^a_a) + n(\omega^a_a + (n-2)\Omega^a_a)\right] + \frac{1}{\Omega^2}R \\
&= \frac{1}{\Omega^2}\left[R + 2(n-1)\omega^a_a + (n-1)(n-2)\Omega^a_a\right]
\end{aligned} \tag{6.13}$$

In the end we prove Weyl Invariance of the Weyl tensor. Note that $W_{ab}{}^{cd}$ is constructed linearly and solely from $R_{ab}{}^{cd}$, notate this as $W_{ab}{}^{cd} = \mathcal{P}(R_{ab}{}^{cd})$, then

$$\begin{aligned}
\widetilde{W}_{abc}{}^{d} - W_{abc}{}^{d} &= \widetilde{g}_{ce}\mathcal{P}(\widetilde{R}_{abf}{}^{d}\widetilde{g}^{fe}) - g_{ce}\mathcal{P}(R_{abf}{}^{d}g^{fe}) \\
&= g_{ce}\mathcal{P}(\widetilde{R}_{abf}{}^{d}g^{fe} - R_{abf}{}^{d}g^{fe})
\end{aligned} \tag{6.14}$$

now compare (6.11) and (6.4) it is almost trivial to see that this vanishes.

7 Hypersurface : Extrinsic Curvature and Gauss Codazzi Equation

On a (pseudo-)Riemannian manifold, suppose there's a hypersurface whose normal vector field is n^a (space-like or time-like), we'll define the spatial metric as

$$h_{ab} = g_{ab} - (n_en^e)n_an_b \tag{7.1}$$

which always has zero contraction with n^a. Tensors that have zero contraction with n^a are called *spatial tensors*, which consists a subspace of the space of all tensors, luckily h^a_b acts as a projection operator onto spatial vectors. We'll adopt the following "tilde" notation for indices:

$$T^{d\tilde{e}f\ldots}_{\tilde{a}bc\ldots} = h^{a'}_a h^e_{e'} T^{de'f\ldots}_{a'bc\ldots} \tag{7.2}$$

just keep in mind that the h^b_a is always multiplied on the outmost of the expression, see e.g. $\nabla_a(\nabla_b(T_{\tilde{c}})) = h^{c'}_c \nabla_a(\nabla_b(T_{c'}))$.

Define D_a operating on spatial tensors:

$$D_a T^{de}_{bc} = \nabla_{\tilde{a}} T^{\tilde{d}\tilde{e}}_{\tilde{b}\tilde{c}} \tag{7.3}$$

It is easily verified that this satisfies all the requirements that defines the Levi-Civita connection on the hypersurface compatible with the induced metric.

Gauss-Codazzi equation is the relation between Riemannian tensor of the full manifold and that of the hypersurface, the nexus being the extrinsic curvature (spatial) tensor defined as

$$\begin{aligned}
K_{ab} &= \frac{1}{2}\pounds_n h_{ab} \\
&= \frac{1}{2}\pounds_n (g_{ab} - (n_e n^e) n_a n_b) \\
&= \frac{1}{2}\left[\nabla_a n_b + \nabla_b n_a - (n_e n^e)(n_b n^c \nabla_c n_a + n_a n^c \nabla_c n_b)\right] \\
&= \frac{1}{2}(\nabla_{\tilde{a}} n_b + \nabla_{\tilde{b}} n_a)
\end{aligned} \tag{7.4}$$

Note that the first line requires that the n field be extended outside of the hypersurface, while the last line involves only derivative along spatial direction and thus demonstrates the extension independence of K_{ab}. But also note that the n field is hypersurface orthogonal, thus we have that it satisfies Fubini integrability condition:

$$n_{[a}\nabla_b n_{c]} = 0 \tag{7.5}$$

contracting with n^a we get

$$\nabla_{\tilde{b}} n_c - \nabla_{\tilde{c}} n_b = 0 \tag{7.6}$$

with which the extrinsic curvature becomes simply

$$K_{ab} = \nabla_{\tilde{a}} n_b \tag{7.7}$$

Now the Gauss-Codazzi equation itself, for a spatial vector X^a we have

$$\begin{aligned}
\widetilde{R}_{abc}{}^d X^c &= 2D_{[a}D_{b]}X^d \\
&= 2\nabla_{[\tilde{a}}(h^{b'}_{\tilde{b}]} h^{\tilde{d}}_{d'} \nabla_{b'} X^{d'}) \\
&= 2\left(\nabla_{[\tilde{a}} h^{b'}_{\tilde{b}]}\right) h^{\tilde{d}}_{d'} \nabla_{b'} X^{d'} + 2h^{b'}_{[\tilde{b}}\left(\nabla_{\tilde{a}]} h^{\tilde{d}}_{d'}\right)\nabla_{b'} X^{d'} + 2\nabla_{[\tilde{a}}\nabla_{\tilde{b}]} X^{\tilde{d}} \\
&= -2(n_e n^e) h^{b'}_{[\tilde{b}}\left(\nabla_{\tilde{a}]} n^{\tilde{d}}\right) n_{d'} \nabla_{b'} X^{d'} + R_{\tilde{a}\tilde{b}\tilde{c}}{}^{\tilde{d}} X^c \\
&= 2(n_e n^e)\left(\nabla_{[\tilde{a}} n^{\tilde{d}}\right)\left(\nabla_{\tilde{b}]} n_c\right) X^c + R_{\tilde{a}\tilde{b}\tilde{c}}{}^{\tilde{d}} X^c \\
&= \left[R_{\tilde{a}\tilde{b}\tilde{c}}{}^{\tilde{d}} + 2(n_e n^e) K^d_{[a} K_{b]c}\right] X^c
\end{aligned} \tag{7.8}$$

Since X is arbitrary thus comes the *Gauss-Codazzi Equation*

$$\widetilde{R}_{abc}{}^{d} = R_{\tilde{a}\tilde{b}\tilde{c}}{}^{\tilde{d}} + 2(n_e n^e) K^d_{[a} K_{b]c} \tag{7.9}$$

In the following we'll begin to use the symbol $\widetilde{n}^a = (n_e n^e) n^a$.

The full Ricci scalar can also be decomposed

$$\begin{aligned}
R &= g^{ac} g^{bd} R_{abcd} \\
&= (h^{ac} + n^a \widetilde{n}^c)(h^{bd} + n^b \widetilde{n}^d) R_{abcd} \\
&= R_{\tilde{a}\tilde{b}}{}^{\tilde{a}\tilde{b}} + 2\widetilde{n}^a n^c h^b_d R_{abc}{}^{d} \\
&= \widetilde{R} + 2(n_e n^e) K^a_{[a} K^b_{b]} + 4\nabla_{[a}\left(\widetilde{n}^a \nabla_{b]} n^b\right) - 4\left(\nabla_{[a}\widetilde{n}^a\right)\left(\nabla_{b]} n^b\right) \\
&= \widetilde{R} - 2(n_e n^e) K^a_{[a} K^b_{b]} + 4\nabla_{[a}\left(\widetilde{n}^a \nabla_{b]} n^b\right)
\end{aligned} \tag{7.10}$$

Again the expression is independent of the extension. This formula will be of use when we do $d+1$ decomposition of Einstein-Hilbert action.

8 On Flat Spacetime

8.1 Lorentz Boost In An Arbitrary Direction

$$\begin{aligned}
L(\vec{u}) &= \exp\left[\sinh^{-1}(|\vec{\mathbf{u}}|)\begin{pmatrix} 0 & \vec{\mathbf{u}}^0 \\ \vec{\mathbf{u}}^{0T} & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix} \exp\left[\sinh^{-1}(|\vec{\mathbf{u}}|)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R}^T \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R} \end{pmatrix} \begin{pmatrix} \sqrt{1+|\vec{\mathbf{u}}|^2} & |\vec{\mathbf{u}}| & 0 & 0 \\ |\vec{\mathbf{u}}| & \sqrt{1+|\vec{\mathbf{u}}|^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathbf{R}^T \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{1+|\vec{\mathbf{u}}|^2} & \vec{\mathbf{u}} \\ \vec{\mathbf{u}} & \boldsymbol{\delta} + (\sqrt{1+|\vec{\mathbf{u}}|^2} - 1)\vec{\mathbf{u}}^T \vec{\mathbf{u}} \end{pmatrix}
\end{aligned} \tag{8.1}$$

8.2 Lorentz Group Reps - not reviewed

$$\begin{aligned}
J_i &= \epsilon_{ijk} x^j \partial_k \\
K_i &= x^0 \partial_i + x^i \partial_0
\end{aligned} \tag{8.2}$$

we can see that J_i is the rotation generator around positive x^i axis with periodicity 2π, and K_i is the boost generator along the positive x^i direction.

Their commutators are

$$\begin{aligned}[J_i, J_j] &= -\epsilon_{ijk} J_k \\ [J_i, K_j] &= -\epsilon_{ijk} K_k \\ [K_i, K_j] &= \epsilon_{ijk} J_k \end{aligned} \tag{8.3}$$

Note that the index of J_i and K_i both transform in the same behavior under the action of J_i, namely, that of a 3-vector.

By defining

$$A_i = \frac{J_i + iK_i}{2} \quad and \quad B_i = \frac{J_i - iK_i}{2} \tag{8.4}$$

we have decomposed the lorentz algebra into a direct sum of two $\mathfrak{so}(3)$ algebras (direct product in the level of groups). There seems to be a theorem stating that all irreps of a direct product group arises from external direct products of factor groups. Thus in order to list all the irreps of Lorentz group, we need only to know all irreps of $\mathfrak{so}(3)$, which we already know. Thus two basic irreps are $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ and all others are constructed by direct production.

Our convention for Pauli matrices are such that

$$x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{8.5}$$

thus we have

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k \tag{8.6}$$

and the basic irreps of lorentz group are:

$$\begin{aligned} (\frac{1}{2}, 0) &: A_i = \frac{i\sigma_i}{2}, B_i = 0 \\ (0, \frac{1}{2}) &: A_i = 0, B_i = \frac{i\sigma_i}{2} \end{aligned} \tag{8.7}$$

8.3 Action For Free Spin≤2 Field - Gotta finish this

Method of construction! Finish this immediately when you have time!! Because you forget very fast man!!

9 Relation between the noetherian EM tensor and the gravitational(covariant) EM tensor

Finish this immediately when you have time!! Because you forget very fast man!!

10 Cosmological Perturbations - Writing

In this section we deal 3+1 cosmology.

Action:

$$S = -\frac{1}{16\pi\mathbb{G}}\int\sqrt{-g}R + \int\sqrt{-g}\mathcal{L}_M \tag{10.1}$$

Equation of motion

$$\frac{1}{16\pi\mathbb{G}}G^{\mu\nu} + \frac{1}{2}\mathcal{L}_M g^{\mu\nu} + \frac{\partial\mathcal{L}_M}{\partial g_{\mu\nu}} = 0 \tag{10.2}$$

Define

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}} = \mathcal{L}_M g^{\mu\nu} + 2\frac{\partial\mathcal{L}_M}{\partial g_{\mu\nu}} \tag{10.3}$$

then

$$G_{\mu\nu} + 8\pi\mathbb{G}T_{\mu\nu} = 0 \tag{10.4}$$

With perfect fluid we can write

$$T_{\mu\nu} = \rho u_\mu u_\nu + p(u_\mu u_\nu + g_{\mu\nu}) \tag{10.5}$$

where u_μ is the unit tangent to the flow lines and thus is future-timelike. Note the definitions for ρ, p and u^μ are intrinsic, ρ and p being the eigenvalues of the symmetric operator T_μ^ν while u^μ being the eigenvector thereof (corresponding to ρ), thus they transform as scalar and vectors under diffeomorphism.

Generics

It turns out computation of perturbation is greatly simplified in conformal coordinates, so we'll switch there. In conformal gauge,

$$g_{\mu\nu} = a^2(h_{\mu\nu} + \eta_{\mu\nu}) = a^2\widetilde{g}_{\mu\nu} \tag{10.6}$$

where $h_{\mu\nu}$ is perturbation quantity, We calculate taking advantage of the conformal flatness of FRW

$$R_{\mu\nu} = \widetilde{R}_{\mu\nu} + 2\widetilde{\nabla}_\mu\partial_\nu\ln a - 2\partial_\mu\ln a\partial_\nu\ln a + \widetilde{g}_{\mu\nu}(\widetilde{g}^{\rho\sigma}\widetilde{\nabla}_\rho\partial_\sigma\ln a + 2\widetilde{g}^{\rho\sigma}\partial_\rho\ln a\partial_\sigma\ln a) \tag{10.7}$$

because $h_{\mu\nu}$ is perturbation quantity, $\widetilde{R}_{\mu\nu}$ is easily calculated to be

$$\widetilde{R}_{\mu\nu} = -\partial_{(\mu}\partial^\rho h_{\nu)\rho} + \frac{1}{2}\partial_\mu\partial_\nu h + \frac{1}{2}\partial^2 h_{\mu\nu} \tag{10.8}$$

here we convent that the indices of ∂_μ and $h_{\mu\nu}$ are raised by $\eta^{\mu\nu}$. The perturbations thereof

$$\begin{aligned}\delta R_{\mu\nu} &= \widetilde{R}_{\mu\nu} - 2\widetilde{\Gamma}_{\mu\nu}{}^\rho\partial_\rho\ln a + h_{\mu\nu}(\partial^2\ln a + 2\frac{\partial_\rho a\partial^\rho a}{a^2}) \\ &\quad + \eta_{\mu\nu}(-\eta^{\rho\sigma}\widetilde{\Gamma}_{\rho\sigma}{}^\gamma\partial_\gamma\ln a - h^{\rho\sigma}\partial_\rho\partial_\sigma\ln a - 2h^{\rho\sigma}\partial_\rho\ln a\partial_\sigma\ln a) \\ &= -\partial_{(\mu}\partial^\rho h_{\nu)\rho} + \frac{1}{2}\partial_\mu\partial_\nu h + \frac{1}{2}\partial^2 h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{H}h' + 2\mathcal{H}\partial_{(\mu}h_{\nu)0} - \mathcal{H}h'_{\mu\nu} \\ &\quad - (\mathcal{H}' + 2\mathcal{H}^2)h_{\mu\nu} - \eta_{\mu\nu}(\mathcal{H}' + 2\mathcal{H}^2)h_{00} - \eta_{\mu\nu}\mathcal{H}\partial_\rho h^{\rho0}\end{aligned} \tag{10.9}$$

↓↓Below are scripts↓↓

The gauge transformation property of $\delta R_{\mu\nu}$ should be

$$\begin{aligned}\delta_g R_{\mu\nu} &= T(\bar{R}_{\mu\nu})' + \bar{R}_{\rho\nu}\partial_\mu X^\rho + \bar{R}_{\mu\rho}\partial_\nu X^\rho \\ \bar{R}_{\mu\nu} &= \delta_\mu^0\delta_\nu^0(2\mathcal{H}' - 2\mathcal{H}^2) - (\mathcal{H}' + 2\mathcal{H}^2)\eta_{\mu\nu} \\ \bar{R}'_{\mu\nu} &= \delta_\mu^0\delta_\nu^0(2\mathcal{H}'' - 4\mathcal{H}\mathcal{H}') - (\mathcal{H}'' + 4\mathcal{H}\mathcal{H}')\eta_{\mu\nu}\end{aligned} \tag{10.10}$$

Let's check this by calculating

$$\begin{aligned}
\delta_g R_{\mu\nu}(a,h) &= 2\partial_\mu\partial_\nu(T\mathcal{H}) + \eta_{\mu\nu}\partial^2(T\mathcal{H}) - 4\mathcal{H}\delta^0_{(\nu}\partial_{\mu)}(T\mathcal{H}) - 2\mathcal{H}\partial_\mu\partial_\nu T \\
&\quad - 2(\mathcal{H}' + 2\mathcal{H}^2)\partial_{(\mu}X_{\nu)} + 2\eta_{\mu\nu}(\mathcal{H}' + 2\mathcal{H}^2)T' - \eta_{\mu\nu}\mathcal{H}(\partial^2 T) - 4\eta_{\mu\nu}\mathcal{H}(T\mathcal{H})' \\
&= T\left[+\delta^0_\mu\delta^0_\nu(2\mathcal{H}'' - 4\mathcal{H}\mathcal{H}') - (\mathcal{H}'' + 4\mathcal{H}\mathcal{H}')\eta_{\mu\nu}\right] \\
&\quad + 2\left[+2(\mathcal{H}' - \mathcal{H}^2)\delta^0_\rho\delta^0_{(\nu}\bar{\partial}_{\mu)}X^\rho - (\mathcal{H}' + 2\mathcal{H}^2)\eta_{\rho(\nu}\partial_{\mu)}X^\rho\right] \\
&= T(\bar{R}_{\mu\nu})' + 2\bar{R}_{\rho(\mu}\partial_{\nu)}X^\rho
\end{aligned} \tag{10.11}$$

The transformation property is indeed correct! Calculated using different routes.

$$\delta_g h_{\mu\nu} = 2T\mathcal{H}\eta_{\mu\nu} + 2\partial_{(\nu}X_{\mu)} \tag{10.12}$$

$$\delta_g h = 8T\mathcal{H} + 2\partial_\rho X^\rho \tag{10.13}$$

$$\begin{aligned}
\delta_g R_{\mu\nu}(a^2 h) &= 2\partial_\mu\partial_\nu(T\mathcal{H}) + \eta_{\mu\nu}\partial^2(T\mathcal{H}) - 4\mathcal{H}\delta^0_{(\nu}\partial_{\mu)}(T\mathcal{H}) - 2\mathcal{H}\partial_\mu\partial_\nu T \\
&\quad - 2(\mathcal{H}' + 2\mathcal{H}^2)\partial_{(\nu}X_{\mu)} + 2\eta_{\mu\nu}(\mathcal{H}' + 2\mathcal{H}^2)T' - \eta_{\mu\nu}\mathcal{H}\partial^2 T - 4\mathcal{H}(T\mathcal{H})'\eta_{\mu\nu}
\end{aligned} \tag{10.14}$$

↑↑Above are scripts↑↑

We'll prefer δG^μ_ν because non derivative terms are often auto canceled.

$$\begin{aligned}
a^2\delta G^\mu_\nu &= -h^{\mu\rho}\bar{R}_{\rho\nu} + \eta^{\mu\rho}\delta R_{\rho\nu} - \frac{1}{2}\left(\eta^{\rho\sigma}\delta R_{\rho\sigma} - \bar{R}_{\rho\sigma}h^{\rho\sigma}\right)\delta^\mu_\nu \\
&= -\frac{1}{2}\partial^\mu\partial^\rho h_{\nu\rho} - \frac{1}{2}\partial_\nu\partial_\rho h^{\mu\rho} + \frac{1}{2}\partial^\mu\partial_\nu h + \frac{1}{2}\partial^2 h^\mu_\nu + \mathcal{H}\partial^\mu h_{\nu 0} - \mathcal{H}\partial_\nu h^{\mu 0} - \mathcal{H}(h^\mu_\nu)' \\
&\quad - (\mathcal{H}' + 2\mathcal{H}^2)h^\mu_\nu - \bar{R}_{\nu\rho}h^{\rho\mu} + \delta^\mu_\nu\frac{1}{2}\left[\bar{R}_{\rho\sigma}h^{\rho\sigma} + \partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^2 h + 2\mathcal{H}h'\right. \\
&\quad \left. + (\mathcal{H}' + 2\mathcal{H}^2)h + 2(\mathcal{H}' + 2\mathcal{H}^2)h_{00} + 4\mathcal{H}\partial_\rho h^{\rho 0}\right]
\end{aligned} \tag{10.15}$$

Cosmological Background

In flat FLRW universe, Background Ricci and Einstein tensors are

$$\bar{G}_{\mu\nu} = \begin{pmatrix} -3\mathcal{H}^2 & \\ & (2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} \end{pmatrix} \qquad \bar{R}_{\mu\nu} = \begin{pmatrix} 3\mathcal{H}' & \\ & -(\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij} \end{pmatrix} \tag{10.16}$$

where $\mathcal{H} = a'/a$ and prime means differentiation with conformal time. Background Stress-Energy is

$$T_{\mu\nu} = \begin{pmatrix} \rho a^2 & \\ & pa^2\delta ij \end{pmatrix} \tag{10.17}$$

field equation reads

$$H^2 = \frac{8\pi\mathbb{G}}{3}\rho \tag{10.18}$$

$$\dot{H} = -4\pi\mathbb{G}(\rho + p) \tag{10.19}$$

or alternatively

$$\mathcal{H} = \frac{8\pi\mathbb{G}}{3}\rho a^2 \tag{10.20}$$

$$H' = -\frac{4\pi\mathbb{G}}{3}(\rho + 3p)a^2 \tag{10.21}$$

where $\mathcal{H} = aH$ is the hubble constant defined with conformal time and $\mathcal{H}' = a\dot{\mathcal{H}} = a^2(H^2 + \dot{H})$ is derivative of $\mathcal{H}$ with respect to conformal time.

Nonzero connection coefficients

$$\Gamma_{ij}^{\ 0} = a^2 H\delta_{ij} \qquad \Gamma_{0i}^{\ j} = H\delta_i^j \tag{10.22}$$

Gauge Transformation of Perturbations

Under infinitesimal diffeomorphism of vector field $X = T\partial_\eta + L_i\delta^{ij}\partial_j$,

$$\delta_g h_{\mu\nu} = a^{-2}\pounds_X\bar{g} = a^{-2}\pounds_X(a^2\eta) = 2T\mathcal{H}\eta_{\mu\nu} + 2\eta_{\rho(\mu}\partial_{\nu)}X^\rho \tag{10.23}$$

In components

$$\delta_g h_{00} = -2T\mathcal{H} - 2T' \tag{10.24}$$

$$\delta_g h_{0i} = L_i' - \partial_i T \tag{10.25}$$

$$\delta_g h_{ij} = 2T\mathcal{H}\delta_{ij} + 2\partial_{(i}L_{j)} \tag{10.26}$$

Let's also consider the gauge tranformation of the perturbation of a tensor of the form G^μ_ν (assuming it has the FRW symmetry). See

$$\begin{aligned}\delta_g G^\mu_\nu &= \pounds_X(\bar{G}^\mu_\nu dx^\nu \otimes \partial_\mu) \\ &= T(\bar{G}^\mu_\nu)' + \bar{G}^\mu_\rho(\partial_\nu X^\rho) - \bar{G}^\sigma_\nu(\partial_\sigma X^\mu)\end{aligned} \tag{10.27}$$

In components

$$\delta_g G^0_0 = T(\bar{G}^0_0)' \tag{10.28}$$

$$\delta_g G^0_i = T(\bar{G}^0_i)' + \bar{G}^0_0\partial_i T - \bar{G}^k_i\partial_k T \tag{10.29}$$

$$\delta_g G^i_0 = T(\bar{G}^i_0)' + \bar{G}^i_k L_k' - \bar{G}^0_0 L_i' \tag{10.30}$$

$$\delta_g G^i_j = T(\bar{G}^i_j)' + \bar{G}^i_k\partial_j L^k - \bar{G}^k_j\partial_k L_i \tag{10.31}$$

Knowing these properties will be beneficial later.

Tensor Perturbations

Not really much to calculate in this sector, all but a few terms are gone due to symmetry reasons. We don't even need a new symbol. In this case

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix} \tag{10.32}$$

$$h^i_i = \partial^i h_{ij} = 0 \tag{10.33}$$

There are no gauge transformation of the tensor type, so it is automatically gauge invariant.

$$a^2 \delta G^i_j = -\frac{1}{2} h''_{ij} + \frac{1}{2}\nabla^2 h_{ij} - \mathcal{H} h'_{ij} \tag{10.34}$$

Vector Perturbations

In this case

$$h_{\mu\nu} = \begin{pmatrix} 0 & S_j \\ S_i & 2\partial_{(i}F_{j)} \end{pmatrix} \tag{10.35}$$

gauge transformation variables are

$$X^\mu = (0, M_i) \tag{10.36}$$

they satisfy

$$\partial^i S_i = \partial^i F_i = \partial^i M_i = 0 \tag{10.37}$$

gauge transformations are

$$S_i \to S_i + M'_i \tag{10.38}$$

$$F_i \to F_i + M_i \tag{10.39}$$

One obvious invariant is constructed

$$\mathbf{V}_i = S_i - F'_i \tag{10.40}$$

Note that we calculate δG^0_i instead of δG^i_0 because the former is invariant under vector gauge transformations.

$$\begin{aligned} a^2 \delta G^0_i &= -\frac{1}{2}\partial^0 \partial^\rho h_{i\rho} - \frac{1}{2}\partial^2 h_{0i} + (\mathcal{H}' + 2\mathcal{H}^2) h_{0i} - \bar{R}_{ik} h^{0k} \\ &= \frac{1}{2}\nabla (F'_i - S_i) \end{aligned} \tag{10.41}$$

It's indeed gauge invariant.

In order to extract vector information in δG^j_i we perform ∂_j to it. Note that it is again a gauge invariant vector quantity.

$$\begin{aligned} a^2 \partial_j \delta G^j_i &= -\frac{1}{2}\nabla^2 \partial^\rho h_{i\rho} - \frac{1}{2}\partial_i \partial_\rho \partial_j h^{j\rho} + \frac{1}{2}\partial^2 \partial^j h_{ji} + \mathcal{H}\nabla^2 h_{i0} - \mathcal{H}\partial^j h'_{ji} - (\mathcal{H}' + 2\mathcal{H}^2)\partial^j h_{ji} - \bar{R}_{ik}\partial_j h^{jk} \\ &= \frac{1}{2}\nabla^2 (S_i - F'_i)' + \mathcal{H}\nabla^2 (S_i - F'_i) \end{aligned} \tag{10.42}$$

Gauge invariant indeed.

Scalar Perturbations - Needs to be checked

This is the most complicated sector of the calculation, not a single term is eliminated by symmetry.

In this case

$$h_{\mu\nu} = \begin{pmatrix} -2\phi & \partial_j B \\ \partial_i B & -2\psi\delta_{ij} + 2\partial_i\partial_j E \end{pmatrix} \tag{10.43}$$

also gauge transformation variables

$$X^\mu = (T, \partial_i L) \tag{10.44}$$

then it is straight forward that

$$\phi \to \phi + T\mathcal{H} + T' \tag{10.45}$$

$$\psi \to \psi - T\mathcal{H} \tag{10.46}$$

$$B \to B + L' - T \tag{10.47}$$

$$E \to E + L \tag{10.48}$$

Two independent gauge invariant quantities can be constructed

$$\Phi = \phi + \frac{1}{a}\left[a(B - E')\right]' \qquad \Psi = \psi - \mathcal{H}(B - E') \tag{10.49}$$

There are four scalar components in the Einstein tensor listed below

$$\begin{aligned} a^2\delta G^0_0 &= \partial_0\partial^\rho h_{0\rho} - \frac{1}{2}h'' - \frac{1}{2}\partial^2 h_{00} - \mathcal{H}h'_{00} \\ &\quad + (\mathcal{H}' + 2\mathcal{H}^2)h_{00} - \bar{R}_{0\rho}h^{\rho 0} + (TR) \\ &= 2\nabla^2\psi - 6\mathcal{H}\psi' - 6\mathcal{H}^2\phi + 2\mathcal{H}\nabla^2 E' - 2\mathcal{H}\nabla^2 B \end{aligned} \tag{10.50}$$

$$\tag{10.51}$$

$$\begin{aligned} a^2\partial^i\delta G^0_i &= -\frac{1}{2}(\partial^i h_{i0})'' + \frac{1}{2}(\partial^j\partial^i h_{ij})' + \frac{1}{2}\nabla^2(\partial^i h_{0i}) - \frac{1}{2}\nabla^2(h^i_i)' - \frac{1}{2}\partial^2\partial^i h_{0i} - \mathcal{H}\nabla^2 h_{00} \\ &\quad + (\mathcal{H}' + 2\mathcal{H}^2)\partial^i h_{0i} - \bar{R}_{ij}\partial^i h^{j0} \\ &= 2\nabla^2(\psi' + \mathcal{H}\phi) \end{aligned} \tag{10.52}$$

$$\begin{aligned} a^2\delta G^i_i &= -6\psi'' + 2\nabla^2\psi - 12\mathcal{H}\psi' - 2\nabla^2\phi - 6(2\mathcal{H}' + \mathcal{H}^2)\phi - 6\mathcal{H}\phi' \\ &\quad - 2\nabla^2 B' - 4\mathcal{H}\nabla^2 B + 2\nabla^2 E'' + 4\mathcal{H}\nabla^2 E' \end{aligned} \tag{10.53}$$

$$a^2\nabla^{-2}\partial_i\partial^j\delta G^i_j = -2\psi'' - 4\mathcal{H}\psi' - 2(2\mathcal{H}' + \mathcal{H}^2)\phi - 2\mathcal{H}\phi' \tag{10.54}$$

↓↓Below are scripts↓↓

The following part is calculated prior to the above

$$
\begin{aligned}
TR &= \frac{1}{2}\left[\bar{R}_{\rho\sigma}h^{\rho\sigma} + \partial^\rho\partial^\sigma h_{\rho\sigma} - \partial^2 h + 2\mathcal{H}h' + (\mathcal{H}' + 2\mathcal{H}^2)h + 2(\mathcal{H}' + 2\mathcal{H}^2)h_{00} + 4\mathcal{H}\partial_\rho h^{\rho 0}\right] \\
&= -3\psi'' + 2\nabla^2\psi - 6\mathcal{H}\psi' - \nabla^2\phi - 2(\mathcal{H}' - \mathcal{H}^2)\phi + \nabla^2 E'' + 2\mathcal{H}\nabla^2 E' - \nabla^2 B' \\
&\quad - 2(\mathcal{H}' + 2\mathcal{H}^2)\phi - 2\mathcal{H}\phi' - 2\mathcal{H}\nabla^2 B \\
&= -3\psi'' + 2\nabla^2\psi - 6\mathcal{H}\psi' - \nabla^2\phi - 2(2\mathcal{H}' + \mathcal{H}^2)\phi + \nabla^2 E'' + 2\mathcal{H}\nabla^2 E' - \nabla^2 B' - 2\mathcal{H}\phi' - 2\mathcal{H}\nabla^2 B
\end{aligned}
\tag{10.55}
$$

↑↑Above are scripts↑↑

11 Antisymmetric Identity Tensor

Introduce the antisymmetric identity tensor

$$\delta^{i_1 i_2 \dots i_s}_{j_1 j_2 \dots j_s} = s! \delta^{i_1}_{[j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_s}_{j_s]}$$

It has the property that

$$\delta^{I_s J_t}_{I_s K_t} = \frac{(D-t)!}{(D-t-s)!} \delta^{J_t}_{K_t}$$

where s, t indicates how many indices the composite index I, J, K contains. It also has the compisition formula

$$\delta^{K_s L_t}_{I_s J_t} = \frac{(s+t)!}{s!t!} \delta^{K_s}_{[I_s} \delta^{L_t}_{J_t]}$$

which is straight forward.

Note that here D is the space(-time) dimension.

12 Electromagnetic Field Strengths in the Covariant Language

Adopting the convention -1,1,1,1 for $g_{\mu\nu}$. This has the advantage that you don't need an extra minus to represent the dot product of spatial vectors, namely $\vec{A} \cdot \vec{B} = A^\mu B_\mu$. The author found this convenient.

$\vec{E}$ and $\vec{B}$ can be defined as Observer-Dependent Tensors, for an observer whose 4-velocity is n^μ (directed to the future), we have

$$E^\mu = -n_\rho F^{\rho\mu}, \; B_\mu = \frac{1}{2} n^\rho \, Vol_{\rho\mu\nu\sigma} F^{\nu\sigma} \tag{12.1}$$

We can work out the inverse formula

$$F_{\mu\nu} = 2 n_{[\mu} E_{\nu]} + n^\rho \, Vol_{\rho\sigma\mu\nu} B^\sigma \tag{12.2}$$

There're (at least) two scalars (one scalar and one pseudo-scalar to be precise) that we can construct out of the field strength tensor. Namely

$$\begin{aligned} F^{\mu\nu} F_{\mu\nu} = \langle F, F \rangle &= 4 n^{[\mu} E^{\nu]} n_\mu E_\nu + n^\rho \, Vol_{\rho\sigma\mu\nu} B^\sigma n_\gamma \, Vol^{\gamma\delta\mu\nu} B_\delta + 4 n^\mu E^\nu n^\rho \, Vol_{\rho\sigma\mu\nu} B^\sigma \\ &= 2 n^\mu n_\mu E^\nu E_\nu - 2 n^\rho B^\sigma n_\gamma B_\delta \delta^{\gamma\delta}_{\rho\sigma} \\ &= 2 (n^\rho n_\rho)(\vec{E}^2 - \vec{B}^2) \\ &= 2(\vec{B}^2 - \vec{E}^2) \end{aligned} \tag{12.3}$$

and

$$\begin{aligned} Vol_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = \langle F, *F \rangle &= 4 n^\mu E^\nu n_\gamma B_\delta \, Vol^{\gamma\delta\rho\sigma} \, Vol_{\mu\nu\rho\sigma} \\ &= -8 n^\mu E^\nu n_\gamma B_\delta \delta^{\gamma\delta}_{\mu\nu} \\ &= 8 \vec{E} \cdot \vec{B} \end{aligned} \tag{12.4}$$

13 Proof of Conservation of the SEM Tensor

Suppose an action $S[\psi, g] = \int \mathcal{L}$ of which g is an arbitrary non-dynamical background. consider an arbitrary vector field X which is sufficiently zero on the boundary of a domain D, consider the flow generated by it, and invariance of integral under diffeomorphism, assume the EOM of ψ is satisfied in D, we get

$$\begin{aligned}
0 &= \int_D L_X \mathcal{L} \\
&= \int_D (E_\psi L_X \psi + T^{ab} L_X g_{ab}) Vol \\
&= \int_D (2T^{ab} \nabla_a X_b) Vol \\
&= \int_{\partial D} boundaryterms - 2 \int_D (\nabla_a T^{ab}) X_b \cdot Vol
\end{aligned} \tag{13.1}$$

The boundary term is proportional to X, therefore zero. Because X is arbitrary inside D, so

$$\nabla_a T^{ab} = 0 \tag{13.2}$$

14 The GHY Term

We'll Motivate the GHY boundary term as follows, suppose we have a manifold with boundary whose topology is $C^d \times [0, 1]$, where C^d is a closed manifold (compact without boundary), by doing this foliation, a "normal" vector field n^a is defined all over the manifold. With the help of (7.10), it is immediately seen that Einstein-Hilbert action becomes

$$\begin{aligned}
\int_M R\ Vol &= \int_M \left[\widetilde{R} - 2(n_e n^e) K^a_{[a} K^b_{b]} + 4\nabla_{[a} \left(\widetilde{n}^a \nabla_{b]} n^b \right) \right] Vol \\
&= \int_M \left[\widetilde{R} - 2(n_e n^e) K^a_{[a} K^b_{b]} \right] Vol + \oint_{\partial M} 4 \left(n_a \widetilde{n}^{[a} \nabla_b n^{b]} \right) Vol^3 \\
&= \int_M \left[\widetilde{R} - 2(n_e n^e) K^a_{[a} K^b_{b]} \right] Vol + \oint_{\partial M} 2 \left(n_a \widetilde{n}^a \nabla_b n^b \right) Vol^3 \\
&= \int_M \left[\widetilde{R} - 2(n_e n^e) K^a_{[a} K^b_{b]} \right] Vol + 2 \oint_{\partial M} K\ Vol^3
\end{aligned} \tag{14.1}$$

where Vol^3 is the spatial volume form and K is the trace of extrinsic curvature, PLEASE pay attention to the tilded n in the Vol^3 definition.

$$Vol^3 = i_{\widetilde{n}} Vol$$

$$K = \nabla_a n^a$$

if we know the fact that extrinsic curvature involves only first derivative of the metric, we won't stop ourselves from adding an extra term to the E-H action in order to make it a first order action (the necessity of this is called for by well posedness of fixed-boundary-value variational problem and path-integration), the term being the E-H term. We now suspect that if we start with the action

$$S = \int_M R\ Vol - 2 \oint_{\partial M} K\ Vol^3 \tag{14.2}$$

we will get a well posed fixed-boundary-value variational problem. We now check this explicitly by calculating its variation without requiring boundary conditions.

First we need to define the variation of some quantities. Before that note that whether a *contravariant tensor* is spatial or not is well defined even without a metric (a vector can be tangent to the surface), whereas to define whether a differential form is spatial or not we need a metric. Hard not to mention that the notion of *normal to the surface* is naturally defined for forms and for vectors we need a metric to tell. Keeping this in mind it becomes clear what quantities to define, that is, γ^{ab} which is the contravariant metric tensor of the boundary, and n^a which is the normal vector to the boundary, the full metric is uniquely defined (for points on the boundary) once the above two quantities are specified. All these quantities can always be extended a bit off the boundary into the bulk, in order for some expressions to make sense, but again the final results are always independent on these arbitrary extensions. the Full metric is expressed

$$g^{ab} = \gamma^{ab} + n^a \tilde{n}^b \tag{14.3}$$

at first sight you might think whether n is space-like or time-like is ambiguous, but it is in fact not, because we know the signature of the boundary metric γ, the signature of n is thus determined by requiring g to be Lorentzian. Here we simply ignore the fact that the boundary might be somewhere spacelike and somewhere timelike, or even somewhere lightlike (in which case our construction fails), because the author still can't understand this.

Next step is to define the variations of these quantity. Define δg_{ab} to be the variation of g_{ab} and $\delta\gamma^{ab}$ to be the variation of γ^{ab} (which is automatically a boundary tensor) and δn^a to be that of n^a, the position of the indices counts very much because for example we can't determine whether $\delta(\gamma_{ab})$ is still spatial (with respect to g). After this we claim that all index manipulations of these quantities are done with g. Now δg is expressible by $\delta\gamma$ and δn

$$\delta g_{ab} = -(\delta\gamma_{ab} + \delta n_a \tilde{n}_b + \tilde{n}_a \delta n_b) \tag{14.4}$$

you'll be careful enough to notice that the minus sign is a result of the conventions for the variated quantities.

Now it's time to take out the calculation itself.

$$\begin{aligned}
\delta S &= \int (-G^{ab}\delta g_{ab}) + 2\oint \left(n_a \nabla^{[a}\delta g_c{}^{c]}\right) Vol^3 - 2\oint \delta K \; Vol^3 + \oint K\gamma_{ab}\delta\gamma^{ab} \; Vol^3 \\
&= \int (-G^{ab}\delta g_{ab}) + \oint K\gamma_{ab}\delta\gamma^{ab} \; Vol^3 + \oint \left(2n_a \nabla^{[a}\delta g_c{}^{c]} - 2\delta\nabla_a n^a - 2\nabla_a \delta n^a\right) Vol^3
\end{aligned} \tag{14.5}$$

the first two terms are already well arranged for our purposes in that the first is the equation of motion term and the second is a boundary term that only depends on the variation of the boundary metric which immediately becomes zero once fixed-boundary-value condition is imposed. The remaining terms are

$$
\begin{aligned}
&2n_a\nabla^{[a}\delta g_c{}^{c]}-2(\delta\nabla)_a n^a-2\nabla_a\delta n^a\\
=&n_a\nabla^a\delta g_b{}^b-n_a\nabla^b\delta g_b{}^a-2\Gamma_{ab}{}^a n^b-2\nabla_a\delta n^a\\
=&n_a\nabla^a\delta g_b{}^b+n_a\nabla_b\left(\delta\gamma^{ab}+\delta n^a\widetilde{n}^b+\widetilde{n}^a\delta n^b\right)-n^b\nabla_b\delta g_a{}^a-2\nabla_a\delta n^a\\
=&n_a\nabla_b\left(\delta\gamma^{ab}+\delta n^a\widetilde{n}^b+\widetilde{n}^a\delta n^b\right)-2\nabla_a\delta n^a\\
=&n_a\nabla_b\delta\gamma^{ab}+\widetilde{n}_a\delta n^a\nabla_b n^b+n_a\widetilde{n}^b\nabla_b\delta n^a+\nabla_b\delta n^b+\widetilde{n}_a\delta n^b\nabla_b n^a-2\nabla_a\delta n^a\\
=&n_a\nabla_b\delta\gamma^{ab}+\widetilde{n}_a\delta n^a\nabla_b n^b-\gamma_a{}^b\nabla_b\delta n^a\\
=&n_a\nabla_b\delta\gamma^{ab}+\widetilde{n}_a\delta n^a\nabla_b n^b-\gamma_a{}^b\nabla_b(\gamma_c{}^a\delta n^c+\widetilde{n}^a n_c\delta n^c)\\
=&-\delta\gamma^{ab}\nabla_b n_a+\widetilde{n}_a\delta n^a\nabla_b n^b-D_a(\gamma_c{}^a\delta n^c)-\widetilde{n}_c\delta n^c\gamma_a{}^b\nabla_b n^a\\
=&-K_{ab}\delta\gamma^{ab}-D_a(\gamma_b{}^a\delta n^b)
\end{aligned}
\tag{14.6}
$$

Now the famous fact that a boundary has no boundary eliminates the second term, thus,

$$
\delta S=\int\left(-G^{ab}\delta g_{ab}\right)Vol+\oint\left(K\gamma_{ab}-K_{ab}\right)\delta\gamma^{ab}\;Vol^3 \tag{14.7}
$$

This formula is beautiful needless to say. Please see Brown & York for its relation with a generalization of Hamilton-Jacobi equation.

15 $d+1$ Decomposition of the Gravity Action - to be finished

I'll finish this sooner or later!

Autocheckin

16 Cosmological Background of dRGT Mass Term

In $d+1=D$ dimensions, convention for quantities are

$$
\mathcal{U}=\sum_{s=0}^{D}\alpha_s\mathcal{U}_s(\mathcal{K}) \tag{16.1}
$$

$$
\mathcal{K}=\sqrt{g^{-1}f} \tag{16.2}
$$

$\mathcal{U}_s$ is such that

$$
\det(1+\lambda A)=\sum_{s=0}^{D}\lambda^s\mathcal{U}_s(A) \tag{16.3}
$$

Minisuperspace

Consider $d+1=D$ dimensional case.

$$g = -N^2 dt^2 + a^2 dx^2 \tag{16.4}$$
$$f = -\mathcal{N}^2 dt^2 + b^2 dx^2 \tag{16.5}$$

where $N, \mathcal{N}, a, b$ are functions only of t, and dx^2 stands for $\sum (dx^i)^2$. Then

$$\mathcal{K} = g^{-1} f = \begin{pmatrix} \frac{\mathcal{N}}{N} & 0 \\ 0 & \frac{b}{a}\boldsymbol{\delta} \end{pmatrix} \tag{16.6}$$

and

$$\mathcal{U}_s(\mathcal{K}) = \frac{\mathcal{N}}{N}\left[C_d^{s-1} (\frac{b}{a})^{s-1} \right] + C_d^s (\frac{b}{a})^s \tag{16.7}$$

thus

$$\mathcal{U} = \sum \alpha_s \mathcal{U}_s = \frac{\mathcal{N}}{N} \cdot A + B \tag{16.8}$$

Where

$$\begin{cases} A = \sum_{s=0}^{d} \alpha_{s+1} C_d^s (\frac{b}{a})^s \\ B = \sum_{s=0}^{d} \alpha_s C_d^s (\frac{b}{a})^s \end{cases} \tag{16.9}$$

where $C_m^n = \frac{m!}{n!(m-n)!}$.

For later use

$$\sqrt{-g}\mathcal{U} = \mathcal{N}\mathbb{A} + N\mathbb{B} \tag{16.10}$$

where

$$\mathbb{A} = a^d A \quad \text{and} \quad \mathbb{B} = a^d B \tag{16.11}$$

So we have

$$\begin{cases} X^{00} = \frac{1}{Na^d} \frac{\partial(\sqrt{-g}\mathcal{U})}{\partial(-N^2)} = -\frac{\mathbb{B}}{2N^2 a^d} \\ X^{ij} = X\delta^{ij} \ , \ X = \frac{1}{d}\frac{1}{Na^d}\frac{\partial(\sqrt{-g}\mathcal{U})}{\partial(a^2)} = \frac{1}{2a^{d+1}Nd}(\mathcal{N}\frac{\partial \mathbb{A}}{\partial a} + N\frac{\partial \mathbb{B}}{\partial a}) \end{cases} \tag{16.12}$$

The Bianchi Constraint

For normal matter couplings, the dRGT part as a separate matter sector must satisfy the constraint

$$\nabla_\mu X^{\mu\nu} = 0 \tag{16.13}$$

where

$$X^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g}\mathcal{U})}{\partial(g_{\mu\nu})} \tag{16.14}$$

for homogeneous background only one component is effective

$$\begin{aligned}
\nabla_\mu X^{\mu 0} &= \partial_0 X^{00} + \Gamma_{\mu 0}{}^{\mu} X^{00} + \Gamma_{\mu\nu}{}^{0} X^{\mu\nu} \\
&= \dot{X}^{00} + X^{00}\partial_0 \ln\sqrt{-g} + X^{00}\Gamma_{00}{}^{0} + X\delta^{ij}\Gamma_{ij}^{\ 0} \\
&= X^{00}\left(\partial_0(\ln X^{00}) + 2\frac{\dot{N}}{N} + \frac{\dot{a}}{a}d\right) + X\frac{a\dot{a}}{N^2}d \\
&= -\frac{\dot{\mathbb{B}}}{2N^2a^d} + \frac{1}{2a^dN}(\mathcal{N}\frac{\partial\mathbb{A}}{\partial a} + N\frac{\partial\mathbb{B}}{\partial a})\frac{\dot{a}}{N^2} \\
&= \frac{\mathcal{N}}{2a^dN^2}\left[\frac{\dot{a}}{N}\frac{\partial\mathbb{A}}{\partial a} + \frac{\dot{a}}{\mathcal{N}}\frac{\partial\mathbb{B}}{\partial a} - \frac{\dot{\mathbb{B}}}{\mathcal{N}}\right] \\
&= \frac{\mathcal{N}}{2a^dN^2}\left[\frac{\dot{a}}{N}\frac{\partial\mathbb{A}}{\partial a} - \frac{\dot{b}}{\mathcal{N}}\frac{\partial\mathbb{B}}{\partial b}\right] \\
&= \frac{\mathcal{N}}{2a^dN^2}\left(\frac{\dot{a}}{N} - \frac{\dot{b}}{\mathcal{N}}\right)\frac{\partial\mathbb{A}}{\partial a}
\end{aligned} \tag{16.15}$$

We've used the miraculous $\frac{\partial\mathbb{A}}{\partial a} = \frac{\partial\mathbb{B}}{\partial b}$, about which I need to understand more!

17 Linear Perturbation of dRGT Mass Term

The First Order Matrix Square Root Formula which I get from quantum mechanical perturbation theory. In a basis where $A^m{}_n = diag[\lambda_1, \lambda_2, \ldots, \lambda_D]$, and $h^m{}_n$ is a very small matrix, we have

$$(\sqrt{A+h})^m{}_n - (\sqrt{A})^m{}_n \approx \frac{h^m{}_n}{\sqrt{\lambda_m} + \sqrt{\lambda_n}} \tag{17.1}$$

it is easily verified

$$\begin{aligned}
&\left((\sqrt{A})^m{}_q + \frac{h^m{}_q}{\sqrt{\lambda_m} + \sqrt{\lambda_q}}\right)\left((\sqrt{A})^q{}_n + \frac{h^q{}_n}{\sqrt{\lambda_q} + \sqrt{\lambda_n}}\right) \\
=&\delta^m_n\lambda_n + \frac{h^m{}_q}{\sqrt{\lambda_m} + \sqrt{\lambda_q}}\delta^q_n\sqrt{\lambda_n} + \sqrt{\lambda_m}\delta^m_q\frac{h^q{}_n}{\sqrt{\lambda_q} + \sqrt{\lambda_n}} \\
=&\delta^m_n\lambda_n + \frac{h^m{}_n}{\sqrt{\lambda_m} + \sqrt{\lambda_n}}\sqrt{\lambda_n} + \sqrt{\lambda_m}\frac{h^m{}_n}{\sqrt{\lambda_m} + \sqrt{\lambda_n}} \\
=&\delta^m_n\lambda_n + h^m{}_n
\end{aligned} \tag{17.2}$$

The following formula is what we need, strictly speaking, this is luck that the above formula worked for our case (non symmetric matrix square root)

$$
\begin{aligned}
&\sqrt{\bar{g}^{\mu\rho} f_{\rho\nu} - \bar{g}^{\mu\sigma} h_{\sigma\gamma} \bar{g}^{\gamma\rho} f_{\rho\nu}} \\
=&\sqrt{\begin{pmatrix} M^2/N^2 & \\ & \delta^i_j b^2/a^2 \end{pmatrix} - \begin{pmatrix} -1/N^2 & \\ & \delta^i_j/a^2 \end{pmatrix}\begin{pmatrix} h_{00} & h_{0j} \\ h_{0i} & h_{ij} \end{pmatrix}\begin{pmatrix} M^2/N^2 & \\ & \delta^i_j b^2/a^2 \end{pmatrix}} \\
=&\sqrt{\begin{pmatrix} M^2/N^2 & \\ & \delta^i_j b^2/a^2 \end{pmatrix} - \begin{pmatrix} -h_{00}M^2/N^4 & -h_{0j}b^2/(N^2a^2) \\ h_{0i}M^2/(N^2a^2) & h_{ij}b^2/a^4 \end{pmatrix}} \\
\approx&\begin{pmatrix} M/N & \\ & \delta^i_j b/a \end{pmatrix} - \begin{pmatrix} \frac{-h_{00}M}{2N^3} & \frac{-h_{0j}b^2}{N^2a^2(M/N+b/a)} \\ \frac{h_{0i}M^2}{N^2a^2(M/N+b/a)} & \frac{h_{ij}b}{2a^3} \end{pmatrix}
\end{aligned}
\tag{17.3}
$$

We could verify that this is symmetric with respect to f and thus meets the definition of the square rooting in dRGT theory (I need to check if this is correct)

$$
\begin{aligned}
&\begin{pmatrix} -M^2 & \\ & \delta^i_j b^2 \end{pmatrix}\begin{pmatrix} \frac{-h_{00}M}{2N^3} & \frac{-h_{0j}b^2}{N^2a^2(M/N+b/a)} \\ \frac{h_{0i}M^2}{N^2a^2(M/N+b/a)} & \frac{h_{ij}b}{2a^3} \end{pmatrix} \\
=&\begin{pmatrix} \frac{h_{00}M^3}{2N^3} & \frac{h_{0j}M^2b^2}{N^2a^2(M/N+b/a)} \\ \frac{h_{0i}M^2b^2}{N^2a^2(M/N+b/a)} & \frac{h_{ij}b^3}{2a^3} \end{pmatrix}
\end{aligned}
\tag{17.4}
$$

17.1 Variation Formulas

The convention is, as usual, such that

$$
\det(\delta + \mu A) = \sum_{\alpha=0}^{D} \mu^\alpha \mathcal{U}_\alpha(A) \tag{17.5}
$$

then

$$
\delta\mathcal{U}_\alpha(\mathcal{K}) = \sum_{k=0}^{\alpha} (-1)^k \left\langle \mathcal{K}^k \delta\mathcal{K} \right\rangle \mathcal{U}_{\alpha-k-1}(\mathcal{K}) \tag{17.6}
$$

here $\langle\rangle$ means tracing. And

$$
\begin{aligned}
\left\langle \mathcal{K}^k \delta\mathcal{K} \right\rangle &= \frac{1}{2}\left\langle \mathcal{K}^{k-1}(\delta\mathcal{K}\,\mathcal{K} + \mathcal{K}\,\delta\mathcal{K}) \right\rangle \\
&= -\frac{1}{2}\left\langle \mathcal{K}^{k-1} g^{-1} h g^{-1} f \right\rangle \\
&= -\frac{1}{2}\left\langle \mathcal{K}^{k+1} g^{-1} h \right\rangle
\end{aligned}
\tag{17.7}
$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$.

18 Characteristic Polynomials

Suppose

$$
S_n(A) = A^{\mu_1}_{[\mu_1} A^{\mu_2}_{\mu_2} \cdots A^{\mu_n}_{\mu_n]} \tag{18.1}
$$

then we have the induction formula for characteristic polynomials

$$S_n(A) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}Tr(A^k)S_{n-k}(A) \tag{18.2}$$

which could be proved by direct calculation [?]. We also have for an N dimensional matrix

$$S_N(A) = \det(A) \tag{18.3}$$

Now suppose

$$\sum_0^N \lambda^n \mathcal{U}_n(A) = \det(I + \lambda A) = p(\lambda) \tag{18.4}$$

take the derivative of both sides w.r.t. λ, we get

$$\begin{aligned}\sum_1^N \lambda^{n-1} n\mathcal{U}_n(A) &= p(\lambda)Tr(p(\lambda)^{-1}A) = (\sum_0^N \lambda^p \mathcal{U}_p(A))Tr[\sum_0^\infty (-\lambda)^q A^{q+1}] \\ &= \sum_1^N \lambda^{n-1}\sum_{k=1}^{n}\mathcal{U}_{n-k}(A)(-1)^{k-1}Tr(A^k)\end{aligned} \tag{18.5}$$

so we get exactly the same recursion

$$\mathcal{U}_n(A) = \frac{1}{n}\sum_{k=1}^{n}\mathcal{U}_{n-k}(A)(-1)^{k-1}Tr(A^k) \tag{18.6}$$

now we already know that$S_0(A) = \mathcal{U}_0(A) = 1$, so we conclude that $S_n(A)$ and $\mathcal{U}_n(A)$ are the same.

19 Unsettled Question : Relation of Laplacian On S^2 and Angular Momentum

The angular momentum operators are

$$\hat{L}_x = -\sin\phi\partial_\theta - \cos\phi\cot\theta\partial_\phi \tag{19.1}$$

$$\hat{L}_y = \cos\phi\partial_\theta - \sin\phi\cot\theta\partial_\phi \tag{19.2}$$

$$\hat{L}_z = \partial_\phi \tag{19.3}$$

Its length squared is

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \tag{19.4}$$

$$= \partial_\theta^2 + \cot^2\theta\partial_\phi^2 + \cot\theta\partial_\theta + \partial_\phi^2 \tag{19.5}$$

$$= \partial_\theta^2 + \frac{1}{\sin^2\theta}\partial_\phi^2 + \cot\theta\partial_\theta \tag{19.6}$$

Now the Laplacian on S^2 (for 0-forms) is

$$\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b) = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin\theta}\partial_\phi(\sin\theta\frac{1}{\sin^2\theta}\partial_\phi) \tag{19.7}$$

$$= \partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2 \tag{19.8}$$

They are exactly the same, why? There must have been a very simple reason which I didn't grasp!

20 Conformal scalar field (not enough argument)

(6.13) can be used to construct conformal scalar field theory on arbitrary conformal background. Namely, if the action is invariant under

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \tag{20.1}$$

$$\phi \to f(\Omega, \phi) \tag{20.2}$$

$$\tag{20.3}$$

then ϕ is always rescalable to a constant, we could solve for $\Omega = \Omega(\phi)$ from the equation $f(\Omega,\phi) =$ Const. Under this transformation, the action becomes covariant in terms of $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, also, if we further assume second order of the EOM, then (in four dimensions) we can only have two possible terms in the Lagrangian

$$S = \int \sqrt{-\tilde{g}}(\alpha R[\tilde{g}] + \beta) \tag{20.4}$$

Let's set $\alpha = 1$ preserving generality, and insert $\tilde{g}_{\mu\nu}$, we get the conformal scalar action

$$S = \int \sqrt{-g}\left[\Omega^{n-2}R[g] - (n-2)(n-1)\Omega^{n-4}(\partial\Omega)^2 + \beta\Omega^n + 2(n-1)\nabla_a(\Omega^{n-3}\nabla^a\Omega)\right] \tag{20.5}$$

21 Central Limit Theorem

the transform

We define the transform $\phi(s)$ of a function $f(x)$ as

$$\phi(s) = \mathfrak{T}[f] = \ln\int f(x)\exp(-sx)dx \tag{21.1}$$

this transform which we didn't name has very good property, namely,

$$\mathfrak{T}[\lambda f(x)] = \ln\lambda + \mathfrak{T}[f(x)] \tag{21.2a}$$

$$\mathfrak{T}[f(kx)](s) = \mathfrak{T}[f(x)](s/k) - \ln k \tag{21.2b}$$

$$\mathfrak{T}[f \star g] = \mathfrak{T}[f] + \mathfrak{T}[g] \tag{21.2c}$$

where $\star$ means convolution, which enjoys associability and commutability. Also, combine (21.2a) and (21.2b) to get

$$\mathfrak{T}[\frac{1}{k}f(kx)](s) = \mathfrak{T}[f(x)](\frac{s}{k}) \tag{21.3}$$

the theorem

There's an infinite sequence of independent zero-mean random variables X_n, with square variance σ_n^2, with distributions $f_n(x)$, which we assume the transform

$$\phi_n(s) = \ln E[\exp(-sX_n)] = \ln \int f_n(x) \exp(-sx) dx = \mathfrak{T}[f_n] \tag{21.4}$$

exists and is analytic. This puts certain finiteness assumptions on the distribution functions. Let's see its schematic form:

$$\phi_n(0) = \ln E[1] = 0 \tag{21.5a}$$
$$\phi_n'(0) = -E[X_n] = 0 \tag{21.5b}$$
$$\phi_n''(0) = E[X_n^2] - E[X_n]^2 = \sigma_n^2 \tag{21.5c}$$

then we know

$$\phi_n(s) = \frac{\sigma_n^2}{2} s^2 + O(s^3) \tag{21.6}$$

Next, note the famous fact that the distribution of the sum of independent random variables is the convolution of the distribution of the individual variables. Keeping this in mind, let's examine the following random variable:

$$S_N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \tag{21.7}$$

its distribution is

$$F(x) = \frac{1}{\sqrt{N}} [f_1 \star f_2 \star \cdots \star f_N](\sqrt{N} x) \tag{21.8}$$

its transform is

$$\begin{aligned}
\Phi_N &= \mathfrak{T}[f_1 \star f_2 \star \cdots \star f_N](\frac{s}{\sqrt{N}}) \\
&= \sum_{n=1}^{N} \mathfrak{T}[f_n](\frac{s}{\sqrt{N}}) \\
&= \sum_{n=1}^{N} \left[\frac{\sigma_n^2}{2} (\frac{s}{\sqrt{N}})^2 + O\left((\frac{s}{\sqrt{N}})^3 \right) \right] \\
&= \frac{\Sigma^2}{2} s^2 + \frac{1}{\sqrt{N}} \cdot O(s^3)
\end{aligned} \tag{21.9}$$

We could now clearly see the importance and exactness of the $1/\sqrt{N}$ scaling before S_N, had we chosen slightly more or less power of N would make the variance of the limit random variable infinite or zero. Now we can take the limit, and get

$$\Phi(s) = \frac{\Sigma^2}{2} s^2 \tag{21.10}$$

Now we see: as in the law of large numbers, where in the limiting process, we lose all the details of individual summands except their mean value; in the above analysis, the limiting process $N \to \infty$

washes off all the information except the mean square variance $\Sigma^2 = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\sigma_n^2$, that is to say, we get a perfect Gaussian distribution

$$F(x) = \frac{1}{\sqrt{2\pi}\Sigma}\exp(-\frac{x^2}{2\Sigma^2}) \tag{21.11}$$

Considering your cleverness you can think of ways to construct a sequence of new so-called central limiting theorems.

22 Wavelets

Make the following observation

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(i2\pi f x)df = \frac{1}{i2\pi x}\left[\exp(i\pi x) - \exp(-i\pi x)\right] = \frac{\sin(\pi x)}{\pi x} = \operatorname{sinc} x \tag{22.1}$$

this means that the Fourier transformation of $\operatorname{sinc} x$ is

$$\mathcal{F}[\operatorname{sinc} x] = \begin{cases} 1, & f\in[-\frac{1}{2},\frac{1}{2}] \\ 0, & \text{otherwise} \end{cases} \tag{22.2}$$

this is the starting point of Wavelets, we can also know that

$$\operatorname{sinc}(x-\tau) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp[i2\pi f\cdot(x-\tau)]df \tag{22.3}$$

its Fourier transform is thus

$$\mathcal{F}[\operatorname{sinc}(x-\tau)] = \begin{cases} \exp(i2\pi\tau f) & , \quad f\in[-\frac{1}{2},\frac{1}{2}] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{22.4}$$

we note that the function series $\mathcal{F}[\operatorname{sinc}(x-n)]$, $n\in\mathbb{Z}$ makes an orthonormal basis for complex functions on Fourier space with support $[-\frac{1}{2},\frac{1}{2}]$. We could thus expand the $[-\frac{1}{2},\frac{1}{2}]$ part of any spectrum function in terms of this basis, the expansion coefficients could be calculated with the usual inner product, which is the same in freq space as in the original space. Moreover, every function in the series is square-integrable and thus localized, this is why wavelet transformation is a more appropriate way of analysis in situations where the signals are localized.

Keeping the philosophy in mind we can construct more species of "wavelets" , just choose a frequency range (an interval or a union of several intervals) as the focus, and choose an orthonormal basis on that range, get the inverse transformation of the basis to get the desired wavelet. What we have given above is just a preliminary example.

23 Constrained Systems

We use extensively the legendre transformation technique in this section. Galileon serves as a very illustrative example. Starting with the simplest case:

$$\mathcal{L} = (\Box\phi)(\partial\phi)^2 \tag{23.1}$$

Define

$$\pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} = \eta^{\mu\nu}(\partial\phi)^2 \tag{23.2}$$

its totally constrained (meaning $\pi^{\mu\nu}$ is independent of $\partial_\mu\partial_\nu\phi$), then the transformed Lagrangian is

$$\mathcal{L}_1 = -\partial_\mu \pi^{\mu\nu}\partial_\nu\phi - (zero) - \lambda_{\mu\nu}(\pi^{\mu\nu} - \eta^{\mu\nu}(\partial\phi)^2) \tag{23.3}$$

One could check that plugging the EOM of λ and π gives the original action.

Now define

$$A^\mu = \frac{\partial \mathcal{L}_1}{\partial(\partial_\mu \phi)} = -\partial_\nu \pi^{\mu\nu} + 2\lambda_\nu{}^\nu \partial^\mu \phi \tag{23.4}$$

reverse it to get

$$\partial_\mu \phi = \frac{1}{2\lambda}(A_\mu + \partial^\nu \pi_{\mu\nu}) \tag{23.5}$$

then the next stage of the Lagrangian

$$\begin{aligned}
\mathcal{L}_2 = & A^\mu \partial_\mu \phi - \frac{1}{2\lambda} A^\mu (A_\mu + \partial^\nu \pi_{\mu\nu}) - \frac{1}{2\lambda}\partial_\mu \pi^{\mu\nu}(A_\mu + \partial^\nu \pi_{\mu\nu}) - \lambda_{\mu\nu}\pi^{\mu\nu} \\
& + \frac{1}{4\lambda}(A^\mu + \partial_\nu \pi^{\mu\nu})(A_\mu + \partial^\nu \pi_{\mu\nu}) \\
= & A^\mu \partial_\mu \phi - \frac{1}{4\lambda}(A^\mu + \partial_\nu \pi^{\mu\nu})^2 - \lambda_{\mu\nu}\pi^{\mu\nu}
\end{aligned} \tag{23.6}$$

To proceed further we define

$$\begin{aligned}
\Sigma^{\mu\nu\rho} = \frac{\partial \mathcal{L}_2}{\partial(\partial_\mu \pi_{\nu\rho})} &= -\frac{1}{2\lambda}(A^\sigma + \partial_\gamma \pi^{\gamma\sigma})\frac{\partial(\eta^{st}\partial_s \pi_{t\sigma})}{\partial(\partial_\mu \pi_{\nu\rho})} \\
&= -\frac{1}{2\lambda}(A^\sigma + \partial_\gamma \pi^{\gamma\sigma})\eta^{\mu\nu}\delta^\rho_\sigma
\end{aligned} \tag{23.7}$$

to be continued

24 Power Expansion Of Geometrical Quantities

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\nabla^2 h_{\mu\nu} + \nabla_\mu \nabla_\nu h - 2\nabla_\rho \nabla_{(\mu} h_{\nu)}{}^\rho\right) \tag{24.1}$$

$$\begin{aligned}
R^{(2)}_{\mu\nu} = & \frac{1}{2}\delta R^{(1)}_{\mu\nu} \\
= & \frac{1}{4}\Big[\cancel{-2\nabla^\rho(\delta\Gamma_{\rho(\mu}{}^\sigma h_{\nu)\sigma})} - 2\delta\Gamma_{\rho(\mu}{}^\sigma \nabla^\rho h_{\nu)\sigma} - g^{\rho\sigma}\delta\Gamma_{\rho\sigma}{}^\delta \nabla_\delta h_{\mu\nu} - h^{\rho\sigma}\nabla_\rho\nabla_\sigma h_{\mu\nu} \\
& - \nabla_\mu \nabla_\nu (h^{\rho\sigma}h_{\rho\sigma}) - \delta\Gamma_{\mu\nu}{}^\rho \nabla_\rho h \\
& + 2h^{\rho\sigma}\nabla_\rho \nabla_{(\mu} h_{\nu)\sigma} + 2\delta\Gamma_{\rho(\mu}{}^\sigma \nabla_{|\sigma|} h_{\nu)}{}^\rho + 2\delta\Gamma_{\rho(\mu}{}^\sigma \nabla_{\nu)} h_\sigma{}^\rho + 2g^{\rho\sigma}\delta\Gamma_{\rho\sigma}{}^\delta \nabla_{(\mu} h_{\nu)\delta} \\
& \cancel{+2\nabla^\sigma(\delta\Gamma_{\sigma(\mu}{}^\rho h_{\nu)\rho})} + 2\nabla^\sigma(\delta\Gamma_{\mu\nu}{}^\rho h_{\rho\sigma})\Big] \\
= & h^{\rho\sigma}\nabla_\rho \nabla_{(\mu} h_{\nu)\sigma} - \frac{1}{2}h^{\rho\sigma}\nabla_\rho\nabla_\sigma h_{\mu\nu} - \frac{1}{2}h^{\rho\sigma}\nabla_\mu\nabla_\nu h_{\rho\sigma} - \frac{1}{2}\nabla_\rho h^{\rho\sigma}\nabla_\sigma h_{\mu\nu} - \frac{1}{2}\nabla_{(\mu} h_{\nu)\rho}\nabla^\rho h \\
& + \frac{1}{4}\nabla^\rho h_{\mu\nu}\nabla_\rho h - \frac{1}{4}\nabla_\mu h^{\rho\sigma}\nabla_\nu h_{\rho\sigma} + \nabla_{(\mu} h_{\nu)}{}^\sigma \nabla^\rho h_{\rho\sigma} + \frac{1}{2}\nabla^\sigma h_{\rho\mu}\nabla^\rho h_{\nu\sigma} - \frac{1}{2}\nabla_\rho h_{\mu\sigma}\nabla^\rho h_\nu{}^\sigma
\end{aligned} \tag{24.2}$$

$$
\begin{aligned}
R^{(2)} =& g^{\mu\nu} R^{(2)}_{\mu\nu} - h^{\mu\nu} R^{(1)}_{\mu\nu} + h^{\mu}{}_{\rho} h^{\rho\nu} R_{\mu\nu} \\
=& h^{\rho\sigma}\nabla_\rho\nabla^\mu h_{\mu\sigma} - h^{\rho\sigma}\nabla_\rho\nabla_\sigma h - h^{\rho\sigma}\nabla^2 h_{\rho\sigma} + h^{\mu\nu}\nabla_\rho\nabla_\mu h_\nu{}^\rho (= h^{\mu\nu}\nabla_\mu\nabla_\rho h_\nu{}^\rho - h^{\mu\nu}R_{\rho\mu\nu}{}^\sigma h_\sigma{}^\rho + h^{\mu\nu}R_{\rho\mu\sigma}{}^\rho h_\nu{}^\sigma) \\
&- \nabla_\rho h^{\rho\sigma}\nabla_\sigma h + \frac{1}{4}\nabla^\rho h\nabla_\rho h - \frac{3}{4}\nabla_\mu h^{\rho\sigma}\nabla^\mu h_{\rho\sigma} + \nabla_\mu h^{\mu\sigma}\nabla^\rho h_{\rho\sigma} + \frac{1}{2}\nabla_\sigma h_{\rho\mu}\nabla^\rho h^{\mu\sigma} + h^\mu{}_\rho h^{\rho\nu}R_{\mu\nu} \\
=& 2h^{\rho\sigma}\nabla_\rho\nabla^\mu h_{\mu\sigma} - h^{\rho\sigma}\nabla_\rho\nabla_\sigma h - h^{\rho\sigma}\nabla^2 h_{\rho\sigma} + h^{\mu\nu}R_{\mu\rho\nu\sigma}h^{\rho\sigma} \\
&- \nabla_\rho h^{\rho\sigma}\nabla_\sigma h + \frac{1}{4}\nabla^\rho h\nabla_\rho h - \frac{3}{4}\nabla_\mu h^{\rho\sigma}\nabla^\mu h_{\rho\sigma} + \nabla_\mu h^{\mu\sigma}\nabla^\rho h_{\rho\sigma} + \frac{1}{2}\nabla_\sigma h_{\rho\mu}\nabla^\rho h^{\mu\sigma}
\end{aligned}
\tag{24.3}
$$

$$
G^{(1)}_{\mu\nu} = \frac{1}{2}\left(\nabla^2 h_{\mu\nu} + \nabla_\mu\nabla_\nu h - 2\nabla_\rho\nabla_{(\mu}h_{\nu)}{}^\rho\right) - \frac{1}{2}h_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\left[-h^{\rho\sigma}R_{\rho\sigma} + \nabla^2 h - \nabla_\rho\nabla_\sigma h^{\rho\sigma}\right] \tag{24.4}
$$

$$
\begin{aligned}
G^{(2)}_{\mu\nu} =& R^{(2)}_{\mu\nu} - \frac{1}{2}h_{\mu\nu}R^{(1)} - \frac{1}{2}g_{\mu\nu}R^{(2)} \\
=& h^{\rho\sigma}\nabla_\rho\nabla_{(\mu}h_{\nu)\sigma} - \frac{1}{2}h^{\rho\sigma}\nabla_\rho\nabla_\sigma h_{\mu\nu} - \frac{1}{2}h^{\rho\sigma}\nabla_\mu\nabla_\nu h_{\rho\sigma} - \frac{1}{2}\nabla_\rho h^{\rho\sigma}\nabla_\sigma h_{\mu\nu} - \frac{1}{2}\nabla_{(\mu}h_{\nu)\rho}\nabla^\rho h \\
&+ \frac{1}{4}\nabla^\rho h_{\mu\nu}\nabla_\rho h - \frac{1}{4}\nabla_\mu h^{\rho\sigma}\nabla_\nu h_{\rho\sigma} + \nabla_{(\mu}h_{\nu)}{}^\sigma\nabla^\rho h_{\rho\sigma} + \frac{1}{2}\nabla^\sigma h_{\rho\mu}\nabla^\rho h_{\nu\sigma} - \frac{1}{2}\nabla_\rho h_{\mu\sigma}\nabla^\rho h_\nu{}^\sigma \\
&- \frac{1}{2}g_{\mu\nu}\Big[2h^{\rho\sigma}\nabla_\rho\nabla^\mu h_{\mu\sigma} - h^{\rho\sigma}\nabla_\rho\nabla_\sigma h - h^{\rho\sigma}\nabla^2 h_{\rho\sigma} + h^{\mu\nu}R_{\mu\rho\nu\sigma}h^{\rho\sigma} \\
&- \nabla_\rho h^{\rho\sigma}\nabla_\sigma h + \frac{1}{4}\nabla^\rho h\nabla_\rho h - \frac{3}{4}\nabla_\mu h^{\rho\sigma}\nabla^\mu h_{\rho\sigma} + \nabla_\mu h^{\mu\sigma}\nabla^\rho h_{\rho\sigma} + \frac{1}{2}\nabla_\sigma h_{\rho\mu}\nabla^\rho h^{\mu\sigma}\Big] \\
&- \frac{1}{2}h_{\mu\nu}\left[-h^{\rho\sigma}R_{\rho\sigma} + \nabla^2 h - \nabla_\rho\nabla_\sigma h^{\rho\sigma}\right]
\end{aligned}
\tag{24.5}
$$

25 power expansion of

26 Machine Learning Related

26.1 Bayesian Linear Regression

Under the assumption that:

$$
t = \vec{\omega}\cdot\vec{x} + \epsilon \tag{26.1}
$$

where ϵ follows $\mathcal{N}(0,\beta)$, with β a hyperparameter; $\vec{\omega}$ is assumed to have a prior distribution of multidimensional gaussian $\mathcal{N}(\vec{\mu}_0, \widehat{\alpha}_0)$. (Usually we choose $\vec{\mu}_0 = 0$ and $\widehat{\alpha}_0 = \alpha\hat{I}$)

$$
\begin{aligned}
& P\left(\vec{\omega}\,|\,(t_{(1\ldots N)},\vec{x}_{(1\ldots N)}),\widehat{\alpha}_0,\vec{\mu}_0,\beta\right)\\
=& P\left((t_{(1\ldots N)},\vec{x}_{(1\ldots N)})\,|\,\vec{\omega},\beta\right)\cdot P\left(\vec{\omega}|\widehat{\alpha}_0,\vec{\mu}_0,\beta\right)\cdot\mathcal{N}(\not\vec{\omega})\\
=& \exp\left[-\beta\sum_{i=1}^{N}(t_i-\vec{\omega}\cdot\vec{x}_i)^2\right]\exp\left[-(\vec{\omega}-\vec{\mu}_0)\cdot\widehat{\alpha}\cdot(\vec{\omega}-\vec{\mu}_0)\right]\cdot\mathcal{N}(\not\vec{\omega})\\
=& \exp\left[-(\vec{\omega}-\vec{\mu}_N)\cdot\widehat{\alpha}_N\cdot(\vec{\omega}-\vec{\mu}_N)\right]\cdot\mathcal{N}(\not\vec{\omega})
\end{aligned}
\tag{26.2}
$$

where $\mathcal{N}(\not\vec{\omega})$ means normalization constant that is independent of $\vec{\omega}$, and

$$
\begin{aligned}
\widehat{\alpha}_N =& \widehat{\alpha}_0+\beta\sum_{i=1}^{N}\vec{x}_i\vec{x}_i\\
\vec{\mu}_N\cdot\widehat{\alpha}_N =& \vec{\mu}_0\cdot\widehat{\alpha}_0+\beta\sum_{i=1}^{N}t_i\vec{x}_i
\end{aligned}
\tag{26.3}
$$

For the determination of hyper-parameters we need to calculate the expectance:

$$
\begin{aligned}
& P\left((t_{(1\ldots N)},\vec{x}_{(1\ldots N)})\,|\,\widehat{\alpha}_0,\vec{\mu}_0,\beta\right)\\
=& \sum_{\vec{\omega}} P\left((t_{(1\ldots N)},\vec{x}_{(1\ldots N)}),\vec{\omega}\,|\,\widehat{\alpha}_0,\vec{\mu}_0,\beta\right)\\
=& \int d^m\vec{\omega}\exp\left[-\vec{\omega}\cdot\widehat{\alpha}_N\cdot\vec{\omega}+2\vec{\mu}_N\cdot\widehat{\alpha}_N\cdot\vec{\omega}-\beta\sum t_i^2-\vec{\mu}_0\cdot\widehat{\alpha}_0\cdot\vec{\mu}_0\right]\cdot\frac{\beta^{N/2}||\widehat{\alpha}_0||^{1/2}}{\pi^{(N+m)/2}}\\
=& \exp\left[\vec{\mu}_N\cdot\widehat{\alpha}_N\cdot\vec{\mu}_N-\beta\sum t_i^2-\vec{\mu}_0\cdot\widehat{\alpha}_0\cdot\vec{\mu}_0\right]\cdot\frac{\beta^{N/2}||\widehat{\alpha}_0||^{1/2}}{\pi^{N/2}||\widehat{\alpha}_N||^{1/2}}
\end{aligned}
\tag{26.4}
$$

Now we discuss the simplifying assumption $\vec{\mu}_0=0,\widehat{\alpha}_0=\alpha\widehat{I}$. In this case

$$
\begin{aligned}
& E(\alpha,\beta)\\
=& \log P\left((t_{(1\ldots N)},\vec{x}_{(1\ldots N)})\,|\,\alpha,\beta\right)\\
=& \vec{\mu}_N\cdot\widehat{\alpha}_N\cdot\vec{\mu}_N-\beta\sum t_i^2+\frac{N}{2}(\log\beta-\log\pi)+\frac{M}{2}\log\alpha-\frac{1}{2}\log||\widehat{\alpha}_N||\\
=& -\beta\sum(t_i-\vec{x}_i\cdot\vec{\mu}_N)^2-\alpha\vec{\mu}_N\cdot\vec{\mu}_N+\frac{N}{2}(\log\beta-\log\pi)+\frac{M}{2}\log\alpha-\frac{1}{2}\log||\widehat{\alpha}_N||
\end{aligned}
\tag{26.5}
$$

Their derivatives w.r.t. the hyper-parameters needs recheck:

$$
\begin{aligned}
\frac{\partial E}{\partial\alpha} =& \vec{\mu}_N\cdot\widehat{\alpha}_N\cdot\frac{\partial\widehat{\alpha}_N^{-1}}{\partial\alpha}\cdot\widehat{\alpha}_N\cdot\vec{\mu}_N+\frac{M}{2\alpha}-\frac{1}{2}\langle\widehat{\alpha}_N^{-1}\frac{\partial\widehat{\alpha}_N}{\partial\alpha}\rangle\\
=& -|\vec{\mu}_N|^2+\frac{M}{2\alpha}-\frac{1}{2}\langle\widehat{\alpha}_N\rangle
\end{aligned}
\tag{26.6}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial\beta} =& \vec{\mu}_N\cdot\widehat{\alpha}_N\cdot\frac{\partial\widehat{\alpha}_N^{-1}}{\partial\beta}\cdot\widehat{\alpha}_N\cdot\vec{\mu}_N+2\frac{\partial}{\beta}(\vec{\mu}_N\cdot\widehat{\alpha}_N)\cdot\vec{\mu}_N-\sum t_i^2+\frac{N}{2\beta}-\frac{1}{2}\langle\widehat{\alpha}_N^{-1}\sum\vec{x}_i\vec{x}_i\rangle\\
=& -\sum(\vec{x}_i\cdot\vec{\mu}_N)^2+2\sum t_i\vec{x}_i\cdot\vec{\mu}_N-\sum t_i^2+\frac{N}{2\beta}-\frac{1}{2}\langle\widehat{\alpha}_N^{-1}\sum\vec{x}_i\vec{x}_i\rangle
\end{aligned}
\tag{26.7}
$$

27 Two Different Representations of General Covariance in Terms of Vector Fields

By means of Lie derivative

By means of Gauss normal map