∞-Categorical Generalized Langlands Program I: Mixed-Parity Modules and Sheaves

Xin Tong

Abstract

Mixed-parity module emerges for instance when a de Rham Galois representation is being tensored with a square root of cyclotomic character, which produces half odd integers as the corresponding Hodge-Tate weights. We build the whole foundation on the p-adic Hodge theory in this setting over small v-stacks after Scholze and we also consider certain moduli v-stack which parametrizes families of mixed-parity Hodge modules. Examples of the small v-stacks in our mind are rigid analytic spaces over p-adic fields and moduli v-stack of vector bundles over Fargues-Fontaine curves. The preparation implemented at this level will be expected to provide further essential foundationalization for generalized Langlands program after Langlands, Drinfeld, Fargues-Scholze. One side of the generalized Langlands correspondence in the geometric setting is the perverse motivic derived ∞-category over Moduli_G related to smooth representations of reductive groups, while the other side of the generalized Langlands correspondence in the geometric setting is the corresponding derived ∞-category over the stack of mixed-parity L-parametrizations (i.e. from two-fold covering of the Weil group into ℓ-adic groups) related to the representations of Weil group in our setting into Langlands dual groups. Although after Scholze and Fargues-Scholze our generalized Langlands program can go along ℓ-adic cohomologicalization to immediately achieve various solid derived ∞-categories $\mathrm{DerCat}_{\text{ét}}(\mathrm{Moduli}_G, \square)$, $\mathrm{DerCat}_{\text{lisse},\blacksquare}(\mathrm{Moduli}_G, \square)$, $\mathrm{DerCat}_{\blacksquare}(\mathrm{Moduli}_G, \square)$ and so on with well-established formalism regarding 6-functors, we already provide certain p-adic cohomologicalization of the story over Moduli_G.

Contents

Reference 1.

- Chapter 1 Main References: [Sch1], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [B], [Shi], [AI1], [AI2], [AI3], [AB1], [AB2], [Fon2], [Fon3], [Fa1], [M], [Fa2], [Fa3], [Fon4], [Fon5], [Fon6];

- Chapter 2 Main References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [B], [Shi], [AI1], [AI2], [AI3], [AB1], [AB2], [Fon2], [Fon3], [Fa1], [M], [Fa2], [Fa3], [Fon4], [Fon5], [Fon6];

- Chapter 3 Main References: [Sch1], [Sch2], [FS], [FF], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LBV], [B], [SW]; [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [B], [Shi], [AI1], [AI2], [AI3], [AB1], [AB2], [Fon2], [Fon3], [Fa1], [M], [Fa2], [Fa3], [Fon4], [Fon5], [Fon6];

- Chapter 4 Main References: [FS], [FF], [Sch1], [Sch2], [KL1], [KL2], [LBV], [B], [SW], [BS], [Lan1], [Drin1], [Drin2], [Zhu], [DHKM];

- More References: [Lan1], [Drin1], [Drin2], [Zhu], [DHKM].

Notations:

- Chapter 1: The period sheaves in the pro-étale topology in this chapter are assumed to be already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, although we do write that notation in an explicit way. We assume the corresonding interval I contains 1.

$$\Gamma_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \tag{1}$$
$$\Gamma_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \tag{2}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{3}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2},\log(t)\}; \tag{4}$$

$$\Gamma_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \tag{5}$$
$$\Gamma_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \tag{6}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{7}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2},\log(t)\}. \tag{8}$$

- Chapter 2: The period sheaves in the v-topology in this chapter are assumed to be already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, although we do write that notation in an explicit way. We assume the corresonding interval I contains 1.

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}, \tag{9}$$
$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}, \tag{10}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{11}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}; \tag{12}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2}\}, \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}, \tag{13}$$
$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}, \tag{14}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{15}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{16}$$

- Chapter 3: The period rings in 3.1, 3.2 are assumed to be not already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, we do write that notation in an explicit way; Then the period sheaves in the v-topology in 3.3, 3.4, 3.5 are assumed to be already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, although we do write that notation in an explicit way. We assume the corresonding interval I contains 1.

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}, \tag{17}$$
$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}, \tag{18}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{19}$$
$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}; \tag{20}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}, \tag{21}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}, \tag{22}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{23}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{24}$$

□ Chapter 4: The period rings in this chapter are assumed to be not already tensored with a finite extension of $\mathbb{Q}_p$ containing square roots of p, we do write that notation in an explicit way.

Chapter 1

Mixed-Parity p-adic Hodge Modules over Pro-Étale Sites

1.1 Geometric Family of Mixed-Parity Hodge Modules I: de Rham Situations

1.1.1 Period Rings and Sheaves

Reference 2. [Sch1], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [M].

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_{\text{proét}}, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_{\text{proét}} \longrightarrow X_{\text{ét}}$. Then we have the corresponding de Rham period rings and sheaves from [Sch1]:

$$\Gamma_{\text{deRham},X,\text{proét}}, \Gamma^{O}_{\text{deRham},X,\text{proét}}. \tag{1.1}$$

Our notations are different from [Sch1], we use Γ to mean B in [Sch1], while Γ^{O} will be the corresponding OB ring in [Sch1].

Definition 1. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{deRham},X,\text{proét}}$ which forms the sheaves:

$$\Gamma_{\text{deRham},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{deRham},X,\text{proét}}\{t^{1/2}\}. \tag{1.2}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{deRham},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{deRham},X,\text{proét}}\{t^{1/2}, \log(t)\}. \tag{1.3}$$

Definition 2. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I} \tag{1.4}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}. \tag{1.5}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}, \log(t)\}. \tag{1.6}$$

Definition 3. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{deRham},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{deRham},X,\text{proét}}\{t^{1/2}\}. \tag{1.7}$$

$$\Gamma_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\},\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.8}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}. \tag{1.9}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2},\log(t)\}. \tag{1.10}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 4. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.11}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 5. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}}, \tag{1.12}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.13}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.14}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 6. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.15}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 7. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.16}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \tag{1.17}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.18}$$

We call F mixed-parity de Rham if we have the following isomorphism[1]:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.19}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.20}$$

Definition 8. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.21}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}) \tag{1.22}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}). \tag{1.23}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.24}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.25}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.26}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.27}$$

[1] As in [KL, Definition 10.10], when we consider the corresponding de Rham, cristalline, semi-stable functors we will assume 1 is belonging to the interval I in all the following corresponding discussion.

We now define the $(\infty,1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 9. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.28}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.29}$$

Definition 10. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.30}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.31}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.32}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.33}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.34}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 1. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 11. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.35}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \tag{1.36}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.37}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \xrightarrow{\sim} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.38}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \xrightarrow{\sim} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.39}$$

Definition 12. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.40}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \tag{1.41}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}). \tag{1.42}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.43}$$

$$\xrightarrow{\sim} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.44}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\} \tag{1.45}$$

$$\xrightarrow{\sim} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}. \tag{1.46}$$

We now define the $(\infty,1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 13. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$-category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.47}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.48}$$

Definition 14. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\tag{1.49}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}.\tag{1.50}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}},\tag{1.51}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\tag{1.52}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}.\tag{1.53}$$

1.1.2 Mixed-Parity de Rham Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 15. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}},\tag{1.54}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\tag{1.55}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\tag{1.56}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}}\tag{1.57}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F):=f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}),\tag{1.58}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F):=f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}),\tag{1.59}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F):=f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}),\tag{1.60}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F):=f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}),\tag{1.61}$$

$$\tag{1.62}$$

respectively.

Definition 16. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.63}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.64}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.65}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.66}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.67}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.68}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.69}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.70}$$

$$\tag{1.71}$$

respectively.

1.2 Geometric Family of Mixed-Parity Hodge Modules II: Cristalline Situations

1.2.1 Period Rings and Sheaves

Reference 3. [Sch1], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [M].

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_{\text{proét}}, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_{\text{proét}} \longrightarrow X_{\text{ét}}$. Then we have the corresponding cristalline period rings and sheaves from [TT]:

$$\Gamma_{\text{cristalline},X,\text{proét}}, \Gamma^{O}_{\text{cristalline},X,\text{proét}}. \tag{1.72}$$

Our notations are different from [TT], we use Γ to mean B in [TT], while Γ^O will be the corresponding OB ring in [TT].

Definition 17. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{cristalline},X,\text{proét}}$ which forms the sheaves:

$$\Gamma_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}. \tag{1.73}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{cristalline},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}, \log(t)\}. \tag{1.74}$$

Definition 18. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I} \tag{1.75}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}. \tag{1.76}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}, \log(t)\}. \tag{1.77}$$

Definition 19. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}. \tag{1.78}$$

$$\Gamma_{\mathrm{cristalline},X,\mathrm{proét}}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{cristalline},X,\mathrm{proét}}\{t^{1/2},\log(t)\}. \tag{1.79}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2}\}. \tag{1.80}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2},\log(t)\}. \tag{1.81}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 20. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2}\}} \tag{1.82}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 21. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2}\}}, \tag{1.83}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2}\}}, \tag{1.84}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2}\}} \tag{1.85}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 22. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2}\}} \tag{1.86}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 23. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\},\tag{1.87}$$

we consider the following functor cristalline sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\tag{1.88}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}).\tag{1.89}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.90}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.91}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.92}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}.\tag{1.93}$$

Definition 24. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\},\tag{1.94}$$

we consider the following functor cristalline$^{\text{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\tag{1.95}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}).\tag{1.96}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.97}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.98}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.99}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}.\tag{1.100}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 25. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.101}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.102}$$

Definition 26. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.103}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.104}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.105}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.106}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.107}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.108}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.109}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 2. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi: F \overset{\sim}{\longrightarrow} \varphi^*F$.

Definition 27. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\},\tag{1.110}$$

we consider the following functor cristalline sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\tag{1.111}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}).\tag{1.112}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.113}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.114}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}\tag{1.115}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2}\}.\tag{1.116}$$

Definition 28. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\},\tag{1.117}$$

we consider the following functor cristalline$^{\text{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\tag{1.118}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}).\tag{1.119}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.120}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.121}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.122}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{cristalline},X,\text{proét}}\{t^{1/2},\log(t)\}.\tag{1.123}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 29. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.124}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.125}$$

Definition 30. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.126}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.127}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.128}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.129}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.130}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.131}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.132}$$

1.2.2 Mixed-Parity cristalline Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 31. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.133}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.134}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.135}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.136}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.137}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.138}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.139}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.140}$$

$$\tag{1.141}$$

respectively.

Definition 32. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.142}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.143}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.144}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.145}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.146}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.147}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.148}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}), \tag{1.149}$$

$$\tag{1.150}$$

respectively.

1.3 Geometric Family of Mixed-Parity Hodge Modules III: Semi-Stable Situations

Reference 4. [Sch1], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [Shi], [M].

1.3.1 Period Rings and Sheaves

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_{\text{proét}}, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_{\text{proét}} \longrightarrow X_{\text{ét}}$. Then we have the corresponding semi-stable period rings and sheaves from [Shi]:

$$\Gamma_{\text{semistable},X,\text{proét}}, \Gamma^{O}_{\text{semistable},X,\text{proét}}. \tag{1.151}$$

Our notations are different from [Shi], we use Γ to mean B in [Shi], while Γ^O will be the corresponding OB ring in [Shi].

Definition 33. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{semistable},X,\text{proét}}$ which forms the sheaves:

$$\Gamma_{\text{semistable},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}. \tag{1.152}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}. \tag{1.153}$$

Definition 34. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I} \tag{1.154}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}. \tag{1.155}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét}}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}, \log(t)\}. \tag{1.156}$$

Definition 35. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{semistable},X,\text{proét}}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}. \tag{1.157}$$

$$\Gamma_{\mathrm{semistable},X,\mathrm{proét}}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{semistable},X,\mathrm{proét}}\{t^{1/2},\log(t)\}. \tag{1.158}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2}\}. \tag{1.159}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2},\log(t)\}. \tag{1.160}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 36. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid},\mathrm{quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid},\mathrm{quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid},\mathrm{quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2}\}} \tag{1.161}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 37. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach},\mathrm{quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2}\}}, \tag{1.162}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach},\mathrm{quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2}\}}, \tag{1.163}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach},\mathrm{quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2}\}} \tag{1.164}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 38. We use the notation:

$$\mathrm{Module}^{\mathrm{solid},\mathrm{quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét}}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid},\mathrm{quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid},\mathrm{quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{proét},I}\{t^{1/2}\}} \tag{1.165}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 39. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}, \tag{1.166}$$

we consider the following functor semistable sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}) \tag{1.167}$$

or

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}). \tag{1.168}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}) \otimes \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\} \tag{1.169}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\} \tag{1.170}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}) \otimes \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\} \tag{1.171}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}. \tag{1.172}$$

Definition 40. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}, \tag{1.173}$$

we consider the following functor semistable$^{\text{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}) \tag{1.174}$$

or

$$f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}). \tag{1.175}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\} \tag{1.176}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\} \tag{1.177}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\} \tag{1.178}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}. \tag{1.179}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 41. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.180}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.181}$$

Definition 42. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.182}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.183}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.184}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.185}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.186}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.187}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.188}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 3. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \xrightarrow{\sim} \varphi^* F$.

Definition 43. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\},\tag{1.189}$$

we consider the following functor semistable sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\})\tag{1.190}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}).\tag{1.191}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\})\otimes\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}\tag{1.192}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}\tag{1.193}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\})\otimes\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}\tag{1.194}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}.\tag{1.195}$$

Definition 44. For any locally free coherent sheaf F over

$$\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\},\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\},\tag{1.196}$$

we consider the following functor semistable$^{\text{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2},\log(t)\})\tag{1.197}$$

or

$$f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2},\log(t)\}).\tag{1.198}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.199}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.200}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2},\log(t)\}\tag{1.201}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\text{semistable},X,\text{proét}}\{t^{1/2},\log(t)\}.\tag{1.202}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 45. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.203}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.204}$$

Definition 46. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.205}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.206}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.207}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.208}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.209}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}. \tag{1.210}$$

1.3.2 Mixed-Parity semi-stable Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 47. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.211}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.212}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.213}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.214}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.215}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}\}), \tag{1.216}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}), \tag{1.217}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,\mathrm{pro\acute{e}t}}\{t^{1/2}, \log(t)\}), \tag{1.218}$$

$$\tag{1.219}$$

respectively.

Definition 48. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}, \tag{1.220}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \tag{1.221}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.222}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{1.223}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}), \tag{1.224}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}\}), \tag{1.225}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}), \tag{1.226}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,\text{proét},I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{semistable},X,\text{proét}}\{t^{1/2}, \log(t)\}), \tag{1.227}$$

$$\tag{1.228}$$

respectively.

1.4 Localizations

Reference 5. [AI1], [AI2], [AB1], [AB2], [Fon2], [Fon3], [Fa1].

1.4.1 Extension of Fundamental Groups

In the local setting setting in fact we can have more thorough understanding of more structures. Locally we can have the Galois group of $\mathbb{Q}_p \left<T_1,...,T_n\right>$ for some $n > 0$ in the smooth situation for instance. Our current discussion will be in the following situation:

Definition 49. We define the corresponding two fold covering of the Galois group:

$$\mathrm{Gal}(\overline{\mathbb{Q}_p \left<T_1,...,T_n\right>}^\wedge / \mathbb{Q}_p \left<T_1,...,T_n\right>)_2 \tag{1.229}$$

by taking the product of

$$\mathrm{Gal}(\overline{\mathbb{Q}_p \left<T_1,...,T_n\right>}^\wedge / \mathbb{Q}_p \left<T_1,...,T_n\right>), \mathrm{Gal}(\overline{Q}_p/\mathbb{Q}_p)_2 \tag{1.230}$$

where the latter is the group defined in [BS, Just before Lemma 7.5]. This group admits an action on the element $t^{1/2}$ through the action of the group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)_2$.

1.4.2 Modules

We consider the following definition of modules with $(\varphi, \mathrm{Gal}(\overline{\mathbb{Q}_p \left<T_1,...,T_n\right>}^\wedge / \mathbb{Q}_p \left<T_1,...,T_n\right>)_2)$-structure.

Definition 50. Let $R := \mathbb{Q}_p \left<T_1,...,T_n\right>$. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}}, \mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.231}$$

to denote the $(\infty, 1)$-categories of solid modules over the corresonding Robba rings in the local setting namey associated to:

$$R^{\mathrm{perf}\flat} := \mathbb{Q}_p(p^{1/p^\infty}) \left<T_1^{1/p^\infty},...,T_n^{1/p^\infty}\right>^{\wedge\flat}. \tag{1.232}$$

Then we consider all the modules as such carrying commuting operations from φ and

$$\Sigma := \mathrm{Gal}(\overline{\mathbb{Q}_p \left<T_1,...,T_n\right>}^\wedge / \mathbb{Q}_p \left<T_1,...,T_n\right>)_2, \tag{1.233}$$

which is assumed to be semilinear. We use the notation

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\varphi,\Sigma,\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}}, \mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\varphi,\Sigma,\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}, \mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\varphi,\Sigma,\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}} \tag{1.234}$$

to denote the categories.

Definition 51. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},I}\{t^{1/2}\},\tag{1.235}$$

carrying the structure of (φ,Σ)-action, we consider the following functor dR sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\})^{\Sigma}\tag{1.236}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\}).\tag{1.237}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\}\tag{1.238}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\}.\tag{1.239}$$

Definition 52. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},I}\{t^{1/2}\},\tag{1.240}$$

carrying the structure of (φ,Σ)-action, we consider the following functor dR sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2},\log(t)\})^{\Sigma}\tag{1.241}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2}\}).\tag{1.242}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2},\log(t)\}\tag{1.243}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2},\log(t)\}\tag{1.244}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{proét},I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2},\log(t)\}\tag{1.245}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},R,\mathrm{proét}}\{t^{1/2},\log(t)\}.\tag{1.246}$$

Definition 53. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.247}$$

carrying the structure of (φ,Σ)-action, we consider the following functor cristalline sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^\Sigma \tag{1.248}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.249}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^\Sigma\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\} \tag{1.250}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^\Sigma\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}. \tag{1.251}$$

Definition 54. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}, \tag{1.252}$$

carrying the structure of (φ,Σ)-action, we consider the following functor cristalline$^{\mathrm{almost}}$ sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^\Sigma \tag{1.253}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}). \tag{1.254}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^\Sigma\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.255}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.256}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^\Sigma\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\} \tag{1.257}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}. \tag{1.258}$$

Definition 55. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\},\tag{1.259}$$

carrying the structure of (φ,Σ)-action, we consider the following functor semistable sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma}\tag{1.260}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{deRham},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}).\tag{1.261}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\tag{1.262}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\})^{\Sigma}\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}.\tag{1.263}$$

Definition 56. For any module F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\},\tag{1.264}$$

carrying the structure of (φ,Σ)-action, we consider the following functor semistable$^{\mathrm{almost}}$ sending F to the following object:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma}\tag{1.265}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2}\}).\tag{1.266}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}\tag{1.267}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}\tag{1.268}$$

or

$$(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},R,\mathrm{pro\acute{e}t},I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\})^{\Sigma}\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}\tag{1.269}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},R,\mathrm{pro\acute{e}t}}\{t^{1/2},\log(t)\}.\tag{1.270}$$

Chapter 2

Mixed-Parity p-adic Hodge Modules in v-Topology

2.1 Geometric Family of Mixed-Parity Hodge Modules I: de Rham Situations

Reference 6. [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [M].

2.1.1 Period Rings and Sheaves

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f: X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding de Rham period rings and sheaves from [Sch1]:

$$\Gamma_{\text{deRham},X,v}, \Gamma^{O}_{\text{deRham},X,v}. \tag{2.1}$$

Our notations are different from [Sch1], we use Γ to mean B in [Sch1], while Γ^O will be the corresponding OB ring in [Sch1].

Definition 57. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{deRham},X,v}$ which forms the sheaves:

$$\Gamma_{\text{deRham},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}. \tag{2.2}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{2.3}$$

Definition 58. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{2.4}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{2.5}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{2.6}$$

Definition 59. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{deRham},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{deRham},X,v}\{t^{1/2}\}. \tag{2.7}$$

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}. \tag{2.8}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{2.9}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{2.10}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 60. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.11}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 61. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{2.12}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.13}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.14}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 62. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.15}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 63. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{2.16}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\})\tag{2.17}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}).\tag{2.18}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}\tag{2.19}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}.\tag{2.20}$$

Definition 64. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{2.21}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\})\tag{2.22}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}).\tag{2.23}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}\tag{2.24}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}\tag{2.25}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}\tag{2.26}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}.\tag{2.27}$$

We now define the $(\infty,1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 65. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.28}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.29}$$

Definition 66. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.30}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.31}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.32}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.33}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.34}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.35}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.36}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 4. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \xrightarrow{\sim} \varphi^* F$.

Definition 67. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.37}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \tag{2.38}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}). \tag{2.39}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \tag{2.40}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}. \tag{2.41}$$

Definition 68. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.42}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \tag{2.43}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}). \tag{2.44}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.45}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.46}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{2.47}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{2.48}$$

We now define the $(\infty, 1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 69. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.49}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.50}$$

Definition 70. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.51}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.52}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.53}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.54}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.55}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.56}$$

2.1.2 Mixed-Parity de Rham Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 71. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.57}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.58}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.59}$$

to $(\infty, 1)$-categories in image denoted by:

$$\text{preModule}_{X,\text{ét}} \tag{2.60}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{2.61}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{2.62}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{2.63}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{2.64}$$

$$\tag{2.65}$$

respectively.

Definition 72. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed}-\text{paritydeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\text{preModule}^{\text{solid,quasicoherent,mixed}-\text{paritydeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.66}$$

and

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed}-\text{parityalmostdeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.67}$$

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed}-\text{parityalmostdeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \tag{2.68}$$

to $(\infty, 1)$-categories in image denoted by:

$$\text{preModule}_{X,\text{ét}} \tag{2.69}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{2.70}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{2.71}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{2.72}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{2.73}$$

$$\tag{2.74}$$

respectively.

2.2 Geometric Family of Mixed-Parity Hodge Modules II: Cristalline Situations

References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [M].

2.2.1 Period Rings and Sheaves

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding cristalline period rings and sheaves from [TT]:

$$\Gamma_{\text{cristalline},X,v}, \Gamma^{O}_{\text{cristalline},X,v}. \tag{2.75}$$

Our notations are different from [TT], we use Γ to mean B in [TT], while Γ^{O} will be the corresponding OB ring in [TT].

Definition 73. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{cristalline},X,v}$ which forms the sheaves:

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}. \tag{2.76}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}. \tag{2.77}$$

Definition 74. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{2.78}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{2.79}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{2.80}$$

Definition 75. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}. \tag{2.81}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}. \tag{2.82}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{2.83}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{2.84}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- The solid quasicoherent modules from [CS1], [CS2];
- The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 76. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.85}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 77. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{2.86}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.87}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.88}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 78. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.89}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 79. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.90}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}) \tag{2.91}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}). \tag{2.92}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\} \tag{2.93}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}. \tag{2.94}$$

Definition 80. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.95}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}) \tag{2.96}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}). \tag{2.97}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\} \tag{2.98}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\} \tag{2.99}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\} \tag{2.100}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}. \tag{2.101}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 81. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.102}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.103}$$

Definition 82. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.104}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.105}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.106}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.107}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.108}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.109}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.110}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 5. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^*F$.

Definition 83. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.111}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}) \tag{2.112}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}). \tag{2.113}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\} \xrightarrow{\sim} F \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\} \tag{2.114}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\} \xrightarrow{\sim} F \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}. \tag{2.115}$$

Definition 84. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.116}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}) \tag{2.117}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}). \tag{2.118}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\} \tag{2.119}$$

$$\xrightarrow{\sim} F \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\} \tag{2.120}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\} \tag{2.121}$$

$$\xrightarrow{\sim} F \otimes \Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}, \log(t)\}. \tag{2.122}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 85. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \quad (2.123)$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \quad (2.124)$$

Definition 86. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \quad (2.125)$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \quad (2.126)$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \quad (2.127)$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \quad (2.128)$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \quad (2.129)$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \quad (2.130)$$

2.2.2 Mixed-Parity Cristalline Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 87. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \quad (2.131)$$

and

$$\text{preModule}^{\text{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}},\tag{2.132}$$

$$\text{preModule}^{\text{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\tag{2.133}$$

to $(\infty,1)$-categories in image denoted by:

$$\text{preModule}_{X,\text{ét}}\tag{2.134}$$

to be the following functors sending each F in the domain to:

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}),\tag{2.135}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}),\tag{2.136}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2},\log(t)\}),\tag{2.137}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2},\log(t)\}),\tag{2.138}$$

$$\tag{2.139}$$

respectively.

Definition 88. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\text{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\text{preModule}^{\text{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}},\tag{2.140}$$

and

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}},\tag{2.141}$$

$$\varphi\text{preModule}^{\text{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\tag{2.142}$$

to $(\infty,1)$-categories in image denoted by:

$$\text{preModule}_{X,\text{ét}}\tag{2.143}$$

to be the following functors sending each F in the domain to:

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}),\tag{2.144}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}),\tag{2.145}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2},\log(t)\}),\tag{2.146}$$

$$\text{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2},\log(t)\}),\tag{2.147}$$

$$\tag{2.148}$$

respectively.

2.3 Geometric Family of Mixed-Parity Hodge Modules III: Semi-Stable Situations

References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [Shi], [M].

2.3.1 Period Rings and Sheaves

Rings

Let X be a rigid analytic space over $\mathbb{Q}_p$. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding semi-stable period rings and sheaves from [Shi]:

$$\Gamma_{\text{semistable},X,v}, \Gamma^{O}_{\text{semistable},X,v}. \tag{2.149}$$

Our notations are different from [Shi], we use Γ to mean B in [Shi], while Γ^O will be the corresponding OB ring in [Shi].

Definition 89. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{semistable},X,v}$ which forms the sheaves:

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}. \tag{2.150}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}, \log(t)\}. \tag{2.151}$$

Definition 90. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{2.152}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{2.153}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{2.154}$$

Definition 91. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}. \tag{2.155}$$

$$\Gamma_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}. \tag{2.156}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{2.157}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{2.158}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- The solid quasicoherent modules from [CS1], [CS2];
- The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 92. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.159}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 93. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{2.160}$$
$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.161}$$
$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.162}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 94. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.163}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 95. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.164}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \tag{2.165}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}). \tag{2.166}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \tag{2.167}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}. \tag{2.168}$$

Definition 96. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{2.169}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \tag{2.170}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}). \tag{2.171}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.172}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.173}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\} \tag{2.174}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}, \log(t)\}. \tag{2.175}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 97. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.176}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.177}$$

Definition 98. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.178}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.179}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.180}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.181}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.182}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.183}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 6. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 99. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{2.184}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\})\tag{2.185}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}).\tag{2.186}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}\tag{2.187}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}.\tag{2.188}$$

Definition 100. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{2.189}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\tag{2.190}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}).\tag{2.191}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{2.192}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{2.193}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{2.194}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}.\tag{2.195}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 101. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.196}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.197}$$

Definition 102. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty, 1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.198}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty, 1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.199}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.200}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.201}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.202}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{2.203}$$

2.3.2 Mixed-Parity semi-stable Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 103. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.204}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.205}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.206}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{2.207}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{2.208}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{2.209}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{2.210}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{2.211}$$

$$\tag{2.212}$$

respectively.

Definition 104. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{2.213}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{2.214}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{2.215}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{2.216}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{2.217}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{2.218}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{2.219}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{2.220}$$

$$\tag{2.221}$$

respectively.

Remark 7. We now have discussed the corresponding two different morphisms:

$$f : X_{\text{proét}} \longrightarrow X_{\text{ét}}; \tag{2.222}$$
$$f' : X_{\text{v}} \longrightarrow X_{\text{ét}}. \tag{2.223}$$

One can consider the following relation among the sites:

$$X_{\text{v}} \longrightarrow X_{\text{proét}} \longrightarrow X_{\text{ét}} \tag{2.224}$$

which produces f'. The map:

$$g : X_{\text{v}} \longrightarrow X_{\text{proét}} \tag{2.225}$$

can help us relate the corresponding constructions above as in [B, Proposition 2.37]. Namely we have:

$$\mathrm{dR}_{v} = \mathrm{dR}_{\text{proét}} g_*; \tag{2.226}$$
$$\mathrm{dR}_{v,\text{almost}} = \mathrm{dR}_{\text{proét,almost}} g_*; \tag{2.227}$$
$$\text{cristalline}_{v} = \text{cristalline}_{\text{proét}} g_*; \tag{2.228}$$
$$\text{cristalline}_{v,\text{almost}} = \text{cristalline}_{\text{proét,almost}} g_*; \tag{2.229}$$
$$\text{semistable}_{v} = \text{semistable}_{\text{proét}} g_*; \tag{2.230}$$
$$\text{semistable}_{v,\text{almost}} = \text{semistable}_{\text{proét,almost}} g_*. \tag{2.231}$$

Chapter 3

Mixed-Parity Hodge Modules over v-Stacks

3.1 $(\infty,1)$-Quasicoherent Sheaves over Extended Fargues-Fontaine Curves I

We now consider the sheaves over extended Fargues-Fontain stacks:

Remark 8. Let X be a general small v-stack over $\mathbb{Q}_p$ (as a v-stack[1]).

Definition 105.

$$\mathrm{FF}_X := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{3.1}$$

which has the corresonding structure map as in the following:

$$\begin{tikzcd} \mathrm{FF}_X \arrow[d] \\ \mathrm{FF}_{\mathrm{Spd}^\diamond(\mathbb{Q}_p)}. \end{tikzcd}$$

Definition 106. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.2}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack FF_X. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.3}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack FF_X which are perfect complexes. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 107. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.4}$$

to denote $(\infty,1)$-category of all the ind-Banach quasicoherent sheaves over the stack FF_X. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.5}$$

to denote $(\infty,1)$-category of all the indBanach quasicoherent sheaves over the stack FF_X which are perfect complexes. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

[1] All v-stacks in this chapter are assumed to be over a v-stack associated to $\mathbb{Q}_p$ like this.

Definition 108. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \tag{3.6}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I\subset J\subset K$. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \tag{3.7}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I\subset J\subset K$. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 109. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \tag{3.8}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I\subset J\subset K$. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{\underset{\mathrm{indBanach,perfectcomplexes},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)} \tag{3.9}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I\subset J\subset K$. For any local perfectoid $Y\in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Proposition 1. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \arrow[r] & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 2. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \arrow[r] & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 3. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \arrow[r] & \{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 4. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \left\{\begin{matrix}\varphi\mathrm{Module}\\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E\end{matrix}\right\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \arrow[r] & \left\{\begin{matrix}\varphi\mathrm{Module}\\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E\end{matrix}\right\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Taking the corresponding simplicial commutative object we have the following propositions:

Proposition 5. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[d] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[d] \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}.
\end{tikzcd}$$

Proposition 6. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[d] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[d] \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}.
\end{tikzcd}$$

Proposition 7. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[u] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}. \arrow[u]
\end{tikzcd}$$

Proposition 8. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[u] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[u]
\end{tikzcd}.
$$

3.2 $(\infty, 1)$-Quasicoherent Sheaves over Extended Fargues-Fontaine Curves II

We now consider the sheaves over extended Fargues-Fontain stacks:

Remark 9. Let X be a general small v-stack over $\mathbb{Q}_p$ (as a v-stack[2]). Spa will denote Clausen-Scholze analytic space in [CS2].

Definition 110.

$$\mathrm{FF}_X := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2},\log(t)\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2},\log(t)\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{3.10}$$

[3]which has the corresonding structure map as in the following:

$$\begin{tikzcd} \mathrm{FF}_X \arrow[d] \\ \mathrm{FF}_{\mathrm{Spd}^\diamond(\mathbb{Q}_p)}. \end{tikzcd}$$

Definition 111. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.12}$$

to denote $(\infty, 1)$-category of all the solid quasicoherent sheaves over the stack FF_X. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty, 1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.13}$$

to denote $(\infty, 1)$-category of all the solid quasicoherent sheaves over the stack FF_X which are perfect complexes. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty, 1)$-category in the local sense.

Definition 112. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.14}$$

---

[2]All v-stacks in this chapter are assumed to be over a v-stack associated to $\mathbb{Q}_p$ like this.

[3]Here the ring $\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2},\log(t)\}$ is defined to be just:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2}\}[\log(t)] \tag{3.11}$$

which carries the corresponding adic topology from the corresponding Banach ring $\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I\subset(0,\infty)}\{t^{1/2}\}$, which induces a topological adic ring structure (therefore a corresponding condensed animated ring structure in [CS2]). Then the corresponding spectrum will be defined to be the correponding analytic spectrum from Clausen-Scholze.

to denote $(\infty,1)$-category of all the ind-Banach quasicoherent sheaves over the stack FF_X. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \tag{3.15}$$

to denote $(\infty,1)$-category of all the indBanach quasicoherent sheaves over the stack FF_X which are perfect complexes. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 113. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2},\log(t)\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{3.16}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{\underset{\mathrm{solid,perfectcomplexes},(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)} \tag{3.17}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 114. We use the notation

$$\{\underset{\mathrm{indBanach}}{\varphi\mathrm{Module}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2},\log(t)\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{3.18}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{\underset{\mathrm{indBanach,perfectcomplexes},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)} \tag{3.19}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in X_v$ we define the corresponding $(\infty,1)$-category in the local sense.

Proposition 9. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \arrow[r] & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}
$$

Proposition 10. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \left\{\underset{\mathrm{solid,perfectcomplexes},(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \arrow[r] & \left\{\underset{\mathrm{solid,perfectcomplexes},(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)}.
\end{tikzcd}
$$

Proposition 11. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\underset{\mathrm{indBanach}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \arrow[r] & \{\underset{\mathrm{indBanach}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}
$$

Proposition 12. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \left\{\begin{matrix}\varphi\mathrm{Module}\\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E\end{matrix}\right\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}} \arrow[r] & \left\{\begin{matrix}\varphi\mathrm{Module}\\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E\end{matrix}\right\}_{I\subset(0,\infty)}
\end{tikzcd}.
$$

Taking the corresponding simplicial commutative object we have the following propositions:

Proposition 13. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[d] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[d] \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{tikzcd}.
$$

Proposition 14. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[d] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[d] \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{tikzcd}.$$

Proposition 15. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[u] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[u]
\end{tikzcd}.$$

Proposition 16. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_X,\mathcal{O}_{\mathrm{FF}_X}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,X,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[u] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2},\log(t)\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[u]
\end{tikzcd}.$$

3.3 Geometric Family of Mixed-Parity Hodge Modules I: de Rham Situations

Reference 7. [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [M].

3.3.1 Period Rings and Sheaves

Rings

Let X be a v-stack over $\mathrm{Spd}\mathbb{Q}_p$, which is required to be restricted to be a diamond which is further assumed to be spacial in the local setting. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding de Rham period rings and sheaves from [Sch1]:

$$\Gamma_{\mathrm{deRham},X,v}, \Gamma^{O}_{\mathrm{deRham},X,v}. \tag{3.20}$$

Our notations are different from [Sch1], we use Γ to mean B in [Sch1], while Γ^O will be the corresponding OB ring in [Sch1].

Definition 115. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\mathrm{deRham},X,v}$ which forms the sheaves:

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}\}, \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}. \tag{3.21}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{3.22}$$

Definition 116. We use the notations:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I} \tag{3.23}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{3.24}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{3.25}$$

Definition 117. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2}\}, \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}. \tag{3.26}$$

$$\Gamma_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2},\log(t)\}. \tag{3.27}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{3.28}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{3.29}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 118. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.30}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 119. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{3.31}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.32}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.33}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 120. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.34}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 121. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{3.35}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \tag{3.36}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}). \tag{3.37}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \tag{3.38}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}\}. \tag{3.39}$$

Definition 122. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{3.40}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \tag{3.41}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}). \tag{3.42}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.43}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.44}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.45}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{O}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{3.46}$$

We now define the $(\infty,1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 123. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.47}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.48}$$

Definition 124. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.49}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.50}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.51}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.52}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.53}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.54}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.55}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 10. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^*F$.

Definition 125. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{3.56}$$

we consider the following functor dR sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \tag{3.57}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}). \tag{3.58}$$

We call F mixed-parity de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\} \tag{3.59}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\} \overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}\}. \tag{3.60}$$

Definition 126. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{3.61}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \tag{3.62}$$

or

$$f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}). \tag{3.63}$$

We call F mixed-parity almost de Rham if we have the following isomorphism:

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.64}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.65}$$

or

$$f^*f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}) \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\} \tag{3.66}$$

$$\overset{\sim}{\longrightarrow} F \otimes \Gamma^{\mathcal{O}}_{\mathrm{deRham},X,v}\{t^{1/2}, \log(t)\}. \tag{3.67}$$

We now define the $(\infty,1)$-categories of mixed-parity de Rham modules and he corresponding mixed-parity almost de Rham modules by using the objects involved to generated these categories:

Definition 127. Considering all the mixed parity de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.68}$$

generated by the mixed-parity de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.69}$$

Definition 128. Considering all the mixed parity almost de Rham bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.70}$$

generated by the mixed-parity almost de Rham bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity de Rham complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.71}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.72}$$

Then the corresponding mixed-parity de Rham functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.73}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.74}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.75}$$

3.3.2 Mixed-Parity de Rham Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 129. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritydeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.76}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.77}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostdeRham}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.78}$$

to $(\infty, 1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{3.79}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{3.80}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{3.81}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{3.82}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{3.83}$$

$$\tag{3.84}$$

respectively.

Definition 130. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed}-\text{paritydeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed}-\text{paritydeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.85}$$

and

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed}-\text{parityalmostdeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.86}$$

$$\varphi\mathrm{preModule}^{\text{solid,quasicoherent,mixed}-\text{parityalmostdeRham}}_{\square,\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \tag{3.87}$$

to $(\infty, 1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{3.88}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{3.89}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}\}), \tag{3.90}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{3.91}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\text{deRham},X,v}\{t^{1/2}, \log(t)\}), \tag{3.92}$$

$$\tag{3.93}$$

respectively.

3.4 Geometric Family of Mixed-Parity Hodge Modules II: Cristalline Situations

References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [TT], [M].

3.4.1 Period Rings and Sheaves

Rings

Let X be a v-stack over $\mathrm{Spd}\mathbb{Q}_p$, which is required to be restricted to be a diamond which is further assumed to be spacial in the local setting. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by X_v, $X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding cristalline period rings and sheaves from [TT]:

$$\Gamma_{\text{cristalline},X,v}, \Gamma^{O}_{\text{cristalline},X,v}. \tag{3.94}$$

Our notations are different from [TT], we use Γ to mean B in [TT], while Γ^{O} will be the corresponding OB ring in [TT].

Definition 131. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{cristalline},X,v}$ which forms the sheaves:

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}. \tag{3.95}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}, \log(t)\}. \tag{3.96}$$

Definition 132. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{3.97}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{3.98}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{3.99}$$

Definition 133. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{cristalline},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{cristalline},X,v}\{t^{1/2}\}. \tag{3.100}$$

$$\Gamma_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}. \tag{3.101}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{3.102}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{3.103}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 134. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.104}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 135. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{3.105}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.106}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.107}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 136. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.108}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 137. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.109}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\tag{3.110}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}).\tag{3.111}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\tag{3.112}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}.\tag{3.113}$$

Definition 138. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.114}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\tag{3.115}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}).\tag{3.116}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.117}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.118}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}\tag{3.119}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}.\tag{3.120}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 139. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.121}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.122}$$

Definition 140. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.123}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.124}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.125}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.126}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.127}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.128}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.129}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 11. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 141. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{3.130}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}) \tag{3.131}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}). \tag{3.132}$$

We call F mixed-parity cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\} \tag{3.133}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2}\}. \tag{3.134}$$

Definition 142. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}, \tag{3.135}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}) \tag{3.136}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}). \tag{3.137}$$

We call F mixed-parity almost cristalline if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\} \tag{3.138}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\} \tag{3.139}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\} \tag{3.140}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}. \tag{3.141}$$

We now define the $(\infty,1)$-categories of mixed-parity cristalline modules and he corresponding mixed-parity almost cristalline modules by using the objects involved to generated these categories:

Definition 143. Considering all the mixed parity cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.142}$$

generated by the mixed-parity cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.143}$$

Definition 144. Considering all the mixed parity almost cristalline bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.144}$$

generated by the mixed-parity almost cristalline bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity cristalline complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.145}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.146}$$

Then the corresponding mixed-parity cristalline functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.147}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.148}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.149}$$

3.4.2 Mixed-Parity Cristalline Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 145. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.150}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.151}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.152}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{3.153}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{3.154}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{3.155}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}), \tag{3.156}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}), \tag{3.157}$$

$$\tag{3.158}$$

respectively.

Definition 146. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritycristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.159}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.160}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostcristalline}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.161}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\mathrm{\acute{e}t}} \tag{3.162}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{3.163}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2}\}), \tag{3.164}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}), \tag{3.165}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F \otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \Gamma^{\mathcal{O}}_{\mathrm{cristalline},X,v}\{t^{1/2},\log(t)\}), \tag{3.166}$$

$$\tag{3.167}$$

respectively.

3.5 Geometric Family of Mixed-Parity Hodge Modules III: Semi-Stable Situations

References: [Sch1], [Sch2], [FS], [KL1], [KL2], [BL1], [BL2], [BS], [BHS], [Fon1], [CS1], [CS2], [BK], [BBK], [BBBK], [KKM], [BBM], [LZ], [Shi], [M].

3.5.1 Period Rings and Sheaves

Rings

Let X be a v-stack over $\mathrm{Spd}\mathbb{Q}_p$, which is required to be restricted to be a diamond which is further assumed to be spacial in the local setting. We have the corresponding étale site and the corresponding pro-étale site of X, which we denote them by $X_v, X_{\text{ét}}$. The relationship of the two sites can be reflected by the corresponding morphism $f : X_v \longrightarrow X_{\text{ét}}$. Then we have the corresponding semi-stable period rings and sheaves from [Shi]:

$$\Gamma_{\text{semistable},X,v}, \Gamma^{O}_{\text{semistable},X,v}. \tag{3.168}$$

Our notations are different from [Shi], we use Γ to mean B in [Shi], while Γ^{O} will be the corresponding OB ring in [Shi].

Definition 147. Now we assume that $p > 2$, following [BS] we join the square root of t element in $\Gamma_{\text{semistable},X,v}$ which forms the sheaves:

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}. \tag{3.169}$$

And following [BL1], [BL2], [Fon1], [BHS] we further have the following sheaves of rings:

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}, \log(t)\}. \tag{3.170}$$

Definition 148. We use the notations:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I} \tag{3.171}$$

to denote the perfect Robba rings from [KL1], [KL2], where $I \subset (0, \infty)$. Then we join $t^{1/2}$ to these sheaves we have:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}\}. \tag{3.172}$$

And following [BL1], [BL2], [Fon1], [BHS] we have the following larger sheaves:

$$\Gamma^{\text{perfect}}_{\text{Robba},X,v}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,\infty}\{t^{1/2}, \log(t)\}, \Gamma^{\text{perfect}}_{\text{Robba},X,v,I}\{t^{1/2}, \log(t)\}. \tag{3.173}$$

Definition 149. From now on, we use the same notation to denote the period rings involved tensored with a finite extension of $\mathbb{Q}_p$ containing square root of p as in [BS].

$$\Gamma_{\text{semistable},X,v}\{t^{1/2}\}, \Gamma^{O}_{\text{semistable},X,v}\{t^{1/2}\}. \tag{3.174}$$

$$\Gamma_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\},\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}. \tag{3.175}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}. \tag{3.176}$$

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2},\log(t)\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2},\log(t)\}. \tag{3.177}$$

This is necessary since we to extend the action of φ to the period rings by $\varphi(t^{1/2}\otimes 1)=\varphi(t)^{1/2}\otimes 1$.

Modules

We consider quasicoherent presheaves in the following two situation:

- □ The solid quasicoherent modules from [CS1], [CS2];
- □ The ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM] with the corresponding monomorphic ind-Banach quasicoherent modules from [BK], [BBK], [BBBK], [KKM], [BBM].

Definition 150. We use the notation:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.178}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Definition 151. We use the notation:

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}}, \tag{3.179}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.180}$$

$$\mathrm{preModule}^{\mathrm{ind-Banach,quasicoherent}}_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.181}$$

to denote the $(\infty,1)$-categories of solid quasicoherent presheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of inductive Banach modules.

Definition 152. We use the notation:

$$\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}},\mathrm{Module}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.182}$$

to denote the $(\infty,1)$-categories of solid quasicoherent sheaves over the corresonding Robba sheaves. Locally the section is defined by taking the corresponding $(\infty,1)$-categories of solid modules.

Mixed-Parity Hodge Modules without Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Definition 153. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.183}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\})\tag{3.184}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}).\tag{3.185}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}\tag{3.186}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}.\tag{3.187}$$

Definition 154. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.188}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\tag{3.189}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}).\tag{3.190}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.191}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.192}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.193}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}.\tag{3.194}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 155. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.195}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.196}$$

Definition 156. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.197}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.198}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.199}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.200}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.201}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.202}$$

Mixed-Parity Hodge Modules with Frobenius

Now we consider the key objects in our study namely those complexes generated by certain mixed-parity Hodge modules. We start from the following definition.

Remark 12. All the coherent sheaves over mixed-parity Robba sheaves in this section will carry the corresponding Frobenius morphism $\varphi : F \overset{\sim}{\longrightarrow} \varphi^* F$.

Definition 157. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.203}$$

we consider the following functor dR sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\})\tag{3.204}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}).\tag{3.205}$$

We call F mixed-parity semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}\tag{3.206}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\})\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2}\}.\tag{3.207}$$

Definition 158. For any locally free coherent sheaf F over

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\},\tag{3.208}$$

we consider the following functor $\mathrm{dR}^{\mathrm{almost}}$ sending F to the following object:

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\tag{3.209}$$

or

$$f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}).\tag{3.210}$$

We call F mixed-parity almost semi-stable if we have the following isomorphism:

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.211}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.212}$$

or

$$f^*f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\})\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}\tag{3.213}$$

$$\overset{\sim}{\longrightarrow}F\otimes\Gamma^{O}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}.\tag{3.214}$$

We now define the $(\infty,1)$-categories of mixed-parity semi-stable modules and he corresponding mixed-parity almost semi-stable modules by using the objects involved to generated these categories:

Definition 159. Considering all the mixed parity semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.215}$$

generated by the mixed-parity semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.216}$$

Definition 160. Considering all the mixed parity almost semi-stable bundles (locally finite free) as defined above, we consider the sub-$(\infty,1)$ category of

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.217}$$

generated by the mixed-parity almost semi-stable bundles (locally finite free ones). These are defined to be the $(\infty,1)$-categories of mixed-parity semi-stable complexes:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.218}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.219}$$

Then the corresponding mixed-parity semi-stable functors can be extended to these categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.220}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.221}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}. \tag{3.222}$$

3.5.2 Mixed-Parity semi-stable Riemann-Hilbert Correspondence

This chapter will extend the corresponding Riemann-Hilbert correspondence from [Sch1], [LZ], [BL1], [BL2], [M] to the mixed-parity setting.

Definition 161. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.223}$$

and

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.224}$$

$$\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.225}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{3.226}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{3.227}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{3.228}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{3.229}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{3.230}$$

$$\tag{3.231}$$

respectively.

Definition 162. In the situation where we have the Frobenius action we consider the follwing. We define the following Riemann-Hilbert functor $\mathrm{RH}_{\text{mixed-parity}}$ from the one of categories:

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-paritysemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}, \tag{3.232}$$

and

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}, \tag{3.233}$$

$$\varphi\mathrm{preModule}^{\mathrm{solid,quasicoherent,mixed-parityalmostsemistable}}_{\square,\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}} \tag{3.234}$$

to $(\infty,1)$-categories in image denoted by:

$$\mathrm{preModule}_{X,\text{ét}} \tag{3.235}$$

to be the following functors sending each F in the domain to:

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{3.236}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2}\}), \tag{3.237}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,\infty}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{3.238}$$

$$\mathrm{RH}_{\text{mixed-parity}}(F) := f_*(F\otimes_{\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,v,I}\{t^{1/2}\}}\Gamma^{\mathcal{O}}_{\mathrm{semistable},X,v}\{t^{1/2},\log(t)\}), \tag{3.239}$$

$$\tag{3.240}$$

respectively.

Remark 13. We now have discussed the corresponding two different morphisms:

$$f : X_{\text{proét}} \longrightarrow X_{\text{ét}}; \tag{3.241}$$
$$f' : X_{\text{v}} \longrightarrow X_{\text{ét}}. \tag{3.242}$$

One can consider the following relation among the sites:

$$X_{\text{v}} \longrightarrow X_{\text{proét}} \longrightarrow X_{\text{ét}} \tag{3.243}$$

which produces f'. The map:

$$g : X_{\text{v}} \longrightarrow X_{\text{proét}} \tag{3.244}$$

can help us relate the corresponding constructions above as in [B, Proposition 2.37]. Namely we have:

$$\mathrm{dR}_{v} = \mathrm{dR}_{\text{proét}} g_*; \tag{3.245}$$
$$\mathrm{dR}_{v,\text{almost}} = \mathrm{dR}_{\text{proét,almost}} g_*; \tag{3.246}$$
$$\text{cristalline}_{v} = \text{cristalline}_{\text{proét}} g_*; \tag{3.247}$$
$$\text{cristalline}_{v,\text{almost}} = \text{cristalline}_{\text{proét,almost}} g_*; \tag{3.248}$$
$$\text{semistable}_{v} = \text{semistable}_{\text{proét}} g_*; \tag{3.249}$$
$$\text{semistable}_{v,\text{almost}} = \text{semistable}_{\text{proét,almost}} g_*. \tag{3.250}$$

Chapter 4

Discussion for Generalized Langlands Program

4.1 Moduli v-Stack

References: [FS], [FF], [Sch1],[Sch2], [KL1], [KL2];
Further References:[Lan1], [Drin1], [Drin2], [Zhu], [DHKM].

We consider the category of all the perfectoid spaces over $\overline{\mathbb{Q}_p((\mu_{p^\infty}))^\wedge}^{\wedge,\flat}$ as in [FS]. We use the notation Perfectoid_v to denote the associated v-site after [FS], [Sch2]. Let $p > 2$. For any $\mathrm{Spa}(A, A^+) \in \mathrm{perfectoid}_v$ we have the perfect Robba rings from [KL1], [KL2]:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}. \tag{4.1}$$

We also have the corresponding de Rham period rings:

$$\Gamma^{+}_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}, \Gamma_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}. \tag{4.2}$$

In the first filtration of this first de Rham period ring we have the generator t, we now extend the corresponding rings above by adding the square root of t, $t^{1/2}$ following [BS]. We then have the extended rings:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\}, \tag{4.3}$$

$$\Gamma^{+}_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}\{t^{1/2}\}, \Gamma_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}\{t^{1/2}\}. \tag{4.4}$$

Then we form the corresponding extended Fargues-Fontaine curve (after choosing a large finite extension E of $\mathbb{Q}_p$ containing $\varphi(t)^{1/2}$):

$$\mathrm{FF}_A := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{4.5}$$

where the Frobenius is extended to $t^{1/2}\otimes 1$ by acting $\varphi(t)^{1/2}\otimes 1$.

Definition 163. Let G be any p-adic group as in [FS][1]. We now define the pre-v-stack Moduli_G to be a presheaf valued in the groupoid over

$$\mathrm{perfectoid}_v \tag{4.6}$$

sendind each $\mathrm{Spa}(A, A^+)$ perfectoid in the site to the groupoid of all the locally finite free coherent sheaves carrying G-bundle structure over

$$\mathrm{FF}_A := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}. \tag{4.7}$$

Proposition 17. *This prestack is a small v-stack in the v-topology.*

Proof. The proof will be the same as in [FS, Proposition III.1.3]. Our stack can also be regarded as a two components extension of the original stack in [FS]. □

[1]That is to say the group G is defined over $\mathbb{Q}_p$. And the Robba rings are defined over $\mathbb{Q}_p$ as well, which strictly speaking are generated from Witt vectors in [KL1], but one can generalize this directly to the level of [KL2] by replacing the field $\overline{\mathbb{Q}_p((\mu_{p^\infty}))^\wedge}^{\wedge,\flat}$ with some larger field $\overline{F((\mu_{p^\infty}))^\wedge}^{\wedge,\flat}$, where $F/\mathbb{Q}_p$ is finite extension of $\mathbb{Q}_p$.

4.2 Motives over Moduli_G

With the notation in the previous section, we now consider the sheaves over extended Fargues-Fontain stacks:

Definition 164.

$$\mathrm{FF}_{\mathrm{Moduli}_G} := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Moduli}_G,I\subset(0,\infty)}\{t^{1/2}\}\otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{4.8}$$

which has the corresonding structure map as in the following:

$$\begin{tikzcd}
\mathrm{FF}_{\mathrm{Moduli}_G} \arrow[d] \\
\mathrm{FF}_{\mathrm{FF}_*} \arrow[d] \\
\mathrm{FF}_{\mathrm{Spd}^\diamond(\mathbb{Q}_p)} \arrow[d] \\
\mathrm{Spd}^\diamond(\mathbb{Q}_p).
\end{tikzcd}$$

Definition 165. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.9}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid},\mathrm{perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.10}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$ which are perfect complexes. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 166. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.11}$$

to denote $(\infty,1)$-category of all the ind-Banach quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local

sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.12}$$

to denote $(\infty,1)$-category of all the indBanach quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$ which are perfect complexes. For any local perfectoid $Y \in \mathrm{Moduli}_{G^v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 167. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.13}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G^v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.14}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G^v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 168. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.15}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G^v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{ind-Banach,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.16}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G^v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 169. We use the notation

$$\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_G,\infty}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E) \tag{4.17}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y \in \mathrm{Moduli}_{G^v}$ we define the

corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.18}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 170. We use the notation

$$\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba,Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.19}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{\varphi\mathrm{Module}_{\mathrm{indBanach,perfectcomplexes},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}\right\}_{I\subset(0,\infty)} \tag{4.20}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Proposition 18. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \arrow[r] & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 19. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \arrow[r] & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 20. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \arrow[r] & \{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 21. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \left\{\underset{\mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \arrow[r] & \left\{\underset{\mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}{\varphi\mathrm{Module}}\right\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Taking the corresponding simplicial commutative object we have the following propositions:

Proposition 22. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[d] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[d] \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}.
\end{tikzcd}$$

Proposition 23. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[d] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[d] \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}.
\end{tikzcd}$$

Proposition 24. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$
\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[u] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[u]
\end{tikzcd}.
$$

Proposition 25. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$
\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[u] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[u]
\end{tikzcd}.
$$

4.3 Moduli v-Stack in More General Setting

References: [FS], [FF], [Sch1],[Sch2], [KL1], [KL2];
Further References: [Lan1], [Drin1], [Drin2], [Zhu], [DHKM].

We consider the category of all the perfectoid spaces over $\mathrm{Spd}\overline{\mathbb{F}}_p$ as in [FS]. We use the notation $\mathrm{perfectoid}_v$ to denote the associated v-site after [FS], [Sch2]. Let $p > 2$. Now we fix a finite extension K of $\mathbb{Q}_p$. And the Robba rings are defined over K as well, namely we consider the generalized Witt vector over O_K as in [KL2][2]. For any $\mathrm{Spa}(A, A^+) \in \mathrm{perfectoid}_v$ we have the perfect Robba rings from [KL1], [KL2]:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}. \tag{4.21}$$

We also have the corresponding de Rham period rings:

$$\Gamma^{+}_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}, \Gamma_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}. \tag{4.22}$$

In the first filtration of this first de Rham period ring we have the generator t, we now extend the corresponding rings above by adding the square root of t, $t^{1/2}$ following [BS]. We then have the extended rings:

$$\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\}, \tag{4.23}$$

$$\Gamma^{+}_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}\{t^{1/2}\}, \Gamma_{\mathrm{deRham},\mathrm{Spa}(A,A^+)}\{t^{1/2}\}. \tag{4.24}$$

Then we form the corresponding extended Fargues-Fontaine curve (after choosing a large finite extension E of $\mathbb{Q}_p$ containing $\varphi(t)^{1/2}$):

$$\mathrm{FF}_A := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{4.25}$$

where the Frobenius is extended to $t^{1/2} \otimes 1$ by acting $\varphi(t)^{1/2} \otimes 1$.

Definition 171. Let G/K be any p-adic group as in [FS]. That is to say the group G is defined over K, where K is some finite extension of $\mathbb{Q}_p$, defined as above. We now define the pre-v-stack Moduli_G to be a presheaf valued in the groupoid over

$$\mathrm{perfectoid}_v \tag{4.26}$$

sendind each $\mathrm{Spa}(A, A^+)$ perfectoid in the site to the groupoid of all the locally finite free coherent sheaves carrying G-bundle structure over

$$\mathrm{FF}_A := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Spa}(A,A^+),I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}. \tag{4.27}$$

[2]In [FS] and [KL2], this field is denoted by E where the relative p-adic Hodge theory in [KL2] and the Langlands correspondence in [FS] both happen over this field E. To be more precise relative p-adic Hodge theory studies p-adic cohomologization over analytic stacks over E, while the Langlands correspondences relates derived ∞-categories of $\mathrm{Moduli}_{G/E}$ and derived ∞-categories of moduli stack of representations of $W_{E,2}$ into the Langlands dual groups.

Proposition 26. *This prestack is a small v-stack in the v-topology.*

Proof. The proof will be the same as in [FS, Proposition III.1.3]. Our stack can also be regarded as a two components extension of the original stack in [FS]. $\square$

4.4 Motives over Moduli_G in More General Setting

Keeping the generality in the previous section, we now consider the sheaves over extended Fargues-Fontain stacks:

Definition 172.

$$\mathrm{FF}_{\mathrm{Moduli}_G} := \bigcup_{I\subset(0,\infty)} \mathrm{Spa}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E, \Gamma^{\mathrm{perfect},+}_{\mathrm{Robba},\mathrm{Moduli}_G,I\subset(0,\infty)}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)/\varphi^{\mathbb{Z}}, \tag{4.28}$$

which has the corresonding structure map as in the following:

$$\begin{tikzcd}
\mathrm{FF}_{\mathrm{Moduli}_G} \arrow[d] \\
\mathrm{FF}_{\mathrm{FF}_*} \arrow[d] \\
\mathrm{FF}_{\mathrm{Spd}^\diamond(\mathbb{Q}_p)} \arrow[d] \\
\mathrm{Spd}^\diamond(\mathbb{Q}_p).
\end{tikzcd}$$

Definition 173. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.29}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.30}$$

to denote $(\infty,1)$-category of all the solid quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$ which are perfect complexes. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 174. We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.31}$$

to denote $(\infty,1)$-category of all the ind-Banach quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

We use the notation

$$\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \tag{4.32}$$

to denote $(\infty,1)$-category of all the indBanach quasicoherent sheaves over the stack $\mathrm{FF}_{\mathrm{Moduli}_G}$ which are perfect complexes. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 175. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.33}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.34}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 176. We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.35}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

We use the notation

$$\{\varphi\mathrm{Module}^{\mathrm{ind-Banach,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_G,I}\{t^{1/2}\} \otimes_{\mathbb{Q}_p} E)\}_{I\subset(0,\infty)} \tag{4.36}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition and glueing condition for overlapped intervals $I \subset J \subset K$. For any local perfectoid $Y \in \mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 177. We use the notation

$$\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.37}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.38}$$

to denote $(\infty,1)$-category of all the solid φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Definition 178. We use the notation

$$\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},\mathrm{Moduli}_{G},\infty}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E) \tag{4.39}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.
We use the notation

$$\left\{\varphi\mathrm{Module}_{\mathrm{indBanach,perfectcomplexes},\Gamma^{\mathrm{perfect}}_{\mathrm{Robba},X,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E}\right\}_{I\subset(0,\infty)} \tag{4.40}$$

to denote $(\infty,1)$-category of all the ind-Banach φ-modules over the extended Robba ring which are perfect complexes. The modules satisfy the Frobenius pullback condition. For any local perfectoid $Y\in\mathrm{Moduli}_{G_v}$ we define the corresponding $(\infty,1)$-category in the local sense.

Proposition 27. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \arrow[r] & \{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 28. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \arrow[r] & \{\underset{\mathrm{solid,perfectcomplexes}}{\varphi\mathrm{Module}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 29. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \arrow[r] & \{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}.
\end{tikzcd}$$

Proposition 30. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$\begin{tikzcd}
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}} \arrow[r] \arrow[d] & \left\{\begin{matrix}\varphi\mathrm{Module}\\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E\end{matrix}\right\}_{I\subset(0,\infty)} \arrow[d] \\
\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}} \arrow[r] & \left\{\begin{matrix}\varphi\mathrm{Module}\\ \mathrm{indBanach,perfectcomplexes},\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E\end{matrix}\right\}_{I\subset(0,\infty)}
\end{tikzcd}.$$

Taking the corresponding simplicial commutative object we have the following propositions:

Proposition 31. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[d] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[d] \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{tikzcd}.$$

Proposition 32. *We have the following commutative diagram by taking the global section functor*

in the horizontal rows:

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[d] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[d] \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{solid,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{solid,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}}
\end{tikzcd}.$$

Proposition 33. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[u] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{indBanach}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[u]
\end{tikzcd}.$$

Proposition 34. *We have the following commutative diagram by taking the global section functor in the horizontal rows:*

$$\begin{tikzcd}
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Moduli}_G},\mathcal{O}_{\mathrm{FF}_{\mathrm{Moduli}_G}}}} \arrow[r] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Moduli}_G,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \\
\mathrm{SimplicialRings}_{\mathrm{Quasicoherent}^{\mathrm{indBanach,perfectcomplexes}}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond},\mathcal{O}_{\mathrm{FF}_{\mathrm{Spd}(\mathbb{Q}_p)^\diamond}}}} \arrow[r] \arrow[u] & \mathrm{SimplicialRings}_{\{\varphi\mathrm{Module}^{\mathrm{indBanach,perfectcomplexes}}(\Gamma^{perfect}_{Robba,\mathrm{Spd}(\mathbb{Q}_p)^\diamond,I}\{t^{1/2}\}\otimes_{\mathbb{Q}_p}E)\}_{I\subset(0,\infty)}} \arrow[u]
\end{tikzcd}.$$

Acknowledgements

The author thanks Professor Kedlaya for all those suggestions around the corresponding mixed-parity theoreticalization of the work of Kedlaya, Kedlaya-Liu, Kedlaya-Pottharst-Xiao. The author thanks Professor Sorensen for suggestions on the mixed-parity representation theoretic perspectives.

Bibliography

[Sch1] Scholze, Peter. ”p-adic Hodge Theory for Rigid-Analytic Varieties.” Forum of Mathematics. Pi, vol. 1, 2013, https://doi.org/10.1017/fmp.2013.1.

[KL1] Kedlaya, Kiran Sridhara, and Ruochuan Liu. ”Relative p-Adic Hodge Theory: Foundations.” Société mathématique de France, 2015.

[KL2] Kedlaya, Kiran S., and Ruochuan Liu. ”Relative p-Adic Hodge Theory, II: Imperfect Period Rings.” 2016, https://doi.org/10.48550/arxiv.1602.06899.

[BL1] Bhatt, Bhargav and Jacob Lurie. ”A p-adic Riemann-Hilbert functor I: torsion coefficients.”

[BL2] Bhatt, Bhargav and Jacob Lurie. ”A p-adic Riemann-Hilbert functor II: $\mathbb{Q}_p$-coefficients.”

[BS] Breuil, Christophe, and Peter Schneider. ”First Steps Towards p-Adic Langlands Functoriality.” Journal Für Die Reine Und Angewandte Mathematik, vol. 2007, no. 610, 2007, pp. 149-80, https://doi.org/10.1515/CRELLE.2007.070.

[Fon1] Fontaine, Jean-Marc. ”Arithmétique des représentations galoisiennes p-adiques.” In Cohomologie p-adiques et applications arithmétiques (III), Astérisque, no. 295 (2004), pp. 1-115. http://www.numdam.org/item/AST_2004_295_1_0/.

[BHS] Breuil, Christophe, et al. ”A Local Model for the Trianguline Variety and Applications.” Publications Mathématiques. Institut Des Hautes Études Scientifiques, vol. 130, no. 1, 2019, pp. 299-412, https://doi.org/10.1007/s10240-019-00111-y.

[M] Mann, Lucas. ”A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry.” 2022, https://doi.org/10.48550/arxiv.2206.02022.

[CS1] Clausen, Dustin and Peter Scholze. ”Lectures on Condensed Mathematics.” https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.

[CS2] Clausen, Dustin and Peter Scholze. ”Lectures on Analytic Geometry.” https://www.math.uni-bonn.de/people/scholze/Analytic.pdf.

[BK] Bambozzi, Federico, and Kobi Kremnizer. ”On the Sheafyness Property of Spectra of Banach Rings.” 2020, https://doi.org/10.48550/arxiv.2009.13926.

[BBK] Ben-Bassat, Oren, and Kobi Kremnizer. ”Fréchet Modules and Descent.” Theory and Applications of Categories, vol. 39, no. 9, 2023.

[BBBK] Bambozzi, Federico, et al. ”Analytic Geometry over F_1 and the Fargues-Fontaine Curve.” Advances in Mathematics (New York. 1965), vol. 356, 2019, https://doi.org/10.1016/j.aim.2019.106815.

[BBM] Ben-Bassat, Oren, and Devarshi Mukherjee. ”Analytification, Localization and Homotopy Epimorphisms.” Bulletin Des Sciences Mathématiques, vol. 176, 2022, https://doi.org/10.1016/j.bulsci.2022.103129.

[KKM] Kelly, Jack, Kobi Kremnizer, and Devarshi Mukherjee. 2021. ”Analytic Hochschild-Kostant-Rosenberg Theorem.” Advances in Mathematics (New York. 1965), vol. 410, 2022, https://doi.org/10.1016/j.aim.2022.108694.

[LZ] Liu, Ruochuan, and Xinwen Zhu. ”Rigidity and a Riemann-Hilbert Correspondence for p-Adic Local Systems.” Inventiones Mathematicae, vol. 207, no. 1, 2017, pp. 291-343, https://doi.org/10.1007/s00222-016-0671-7.

[FS] Fargues, Laurent, and Peter Scholze. ”Geometrization of the Local Langlands Correspondence.” 2021, https://doi.org/10.48550/arxiv.2102.13459.

[FF] Fargues, Laurent, Jean Marc Fontaine. ”Courbes et fibrés vectoriels en théorie de Hodge p-adique.” Astérisque 406 (2018): 1-382.

[Sch2] Scholze, Peter. ”Etale Cohomology of Diamonds.” 2017, https://doi.org/10.48550/arxiv.1709.07343.

[Sch1] Scholze, Peter. ”p-adic Hodge Theory for Rigid-Analytic Varieties.” Forum of Mathematics. Pi, vol. 1, 2013, https://doi.org/10.1017/fmp.2013.1.

[KL1] Kedlaya, Kiran Sridhara, and Ruochuan Liu. ”Relative p-Adic Hodge Theory: Foundations.” Société mathématique de France, 2015.

[KL3] Kedlaya, Kiran, and Ruochuan Liu. ”On Families of (φ, Γ)-Modules.” Algebra and Number Theory, vol. 4, no. 7, 2010, pp. 943-967, https://doi.org/10.2140/ant.2010.4.943.

[Ked1] Kedlaya, Kiran S. ”A p-Adic Local Monodromy Theorem.” Annals of Mathematics, vol. 160, no. 1, 2004, pp. 93-184, https://doi.org/10.4007/annals.2004.160.93.

[KPX] Kedlaya, Kiran S, Jonathan Pottharst, and Liang Xiao. 2012. ”Cohomology of Arithmetic Families of (φ, Γ)-Modules.” Journal of the American Mathematical Society, vol. 27, no. 4, 2014, pp. 1043-1115, https://doi.org/10.1090/S0894-0347-2014-00794-3.

[KL2] Kedlaya, Kiran S., and Ruochuan Liu. ”Relative p-Adic Hodge Theory, II: Imperfect Period Rings.” 2016, https://doi.org/10.48550/arxiv.1602.06899.

[BS] Breuil, Christophe, and Peter Schneider. ”First Steps Towards p-Adic Langlands Functoriality.” Journal Für Die Reine Und Angewandte Mathematik, vol. 2007, no. 610, 2007, pp. 149-80, https://doi.org/10.1515/CRELLE.2007.070.

[CS1] Clausen, Dustin and Peter Scholze. ”Lectures on Condensed Mathematics.” https://www.math.uni-bonn.de/people/scholze/Condensed.pdf.

[CS2] Clausen, Dustin and Peter Scholze. "Lectures on Analytic Geometry." https://www.math.uni-bonn.de/people/scholze/Analytic.pdf.

[TT] Tan, Fucheng, and Jilong Tong. "Crystalline Comparison Isomorphisms in p-Adic Hodge Theory: the Absolutely Unramified Case." Algebra and Number Theory, vol. 13, no. 7, 2019, pp. 1509-81, https://doi.org/10.2140/ant.2019.13.1509.

[LBV] Bras, Arthur-César Le, and Alberto Vezzani. "The de Rham-Fargues-Fontaine Cohomology." 2021, https://doi.org/10.48550/arxiv.2105.13028.

[B] Bosco, Guido. "Rational p-adic Hodge Theory for Rigid Analytic Varieties." arXiv:2306.06100.

[Shi] Shimizu, K. (2022). "A p-adic monodromy theorem for de Rham local systems." Compositio Mathematica, 158(12), 2157-2205. doi:10.1112/S0010437X2200776X.

[AI1] Andreatta, Fabrizio, and Adrian Iovita. "Global Applications of Relative (φ, Γ)-Modules I." Astérisque, Volume 319, 339-420, 2008.

[AB1] Andreatta, Fabrizio; Brinon, Olivier. "B_{dR}-représentations dans le cas relatif." Annales scientifiques de l'École Normale Supérieure, Serie 4, Volume 43 (2010) no. 2, pp. 279-339. doi : 10.24033/asens.2121. http://www.numdam.org/articles/10.24033/asens.2121/.

[AB2] Andreatta, Fabrizio, and Olivier Brinon. 2013. "Acyclicité Géométrique de Bcris." Commentarii Mathematici Helvetici 88 (4): 965-1022. https://doi.org/10.4171/CMH/309.

[AI2] Andreatta, Fabrizio, and Adrian Iovita. 2012. "Semistable Sheaves and Comparison Isomorphisms in the Semistable Case." Rendiconti - Seminario Matematico Della Università Di Padova 128: 131-285. https://doi.org/10.4171/RSMUP/128-7.

[Fon2] Fontaine, Jean Marc. "Cohomologie de De Rham, cohomologie cristalline et representations p-adiques." (1983). Algebraic Geometry, Proceedings of the Japan-France Conference, Tokyo/Kyoto, 1982.

[Fon3] Fontaine, Jean-Marc. 1982. "Sur Certains Types de Representations p-Adiques Du Groupe de Galois D'un Corps Local; Construction D'un Anneau de Barsotti-Tate." Annals of Mathematics 115 (3): 529-577. https://doi.org/10.2307/2007012.

[Fa1] Faltings, Gerd. 1988. "p-Adic Hodge Theory." Journal of the American Mathematical Society 1 (1): 255-299. https://doi.org/10.1090/S0894-0347-1988-0924705-1.

[KL] Kedlaya, Kiran S, and Ruochuan Liu. 2016. "Finiteness of Cohomology of Local Systems on Rigid Analytic Spaces." https://doi.org/10.48550/arxiv.1611.06930.

[Fa2] Faltings, Gerd. "Crystalline cohomology and p-adic Galois-representations." (1988). Algebraic Analysis, Geometry and Number Theory, Baltimore MD 1988, page 25-80 JHU press, Baltimore MD 1989.

[Fa3] FALTINGS, Gerd. "Almost Étale Extensions." Astérisque, Société mathématique de France, 2002, pp. 185-270.

[Fon4] Fontaine, Jean-Marc. Exposé II : Le corps des périodes p-adiques, in Périodes p-adiques - Séminaire de Bures, 1988, Astérisque, no. 223 (1994), Talk no. 2, 43 p. http://www.numdam.org/item/AST_1994_223_59_0/.

[Fon5] Fontaine, Jean-Marc. ”Représentations p-adiques semi-stables.” Astérisque, 223, 1994, page 113-184.

[Fon6] Fontaine, Jean-Marc. ”Représentations ℓ-adiques potentiellement semi-stables.” Astérisque, 223, 1994, page 321-347.

[SW] Scholze, Peter, and Jared Weinstein. Berkeley Lectures on p-Adic Geometry. Princeton University Press, 2020.

[AI3] Andreatta, Fabrizio, and Adrian Iovita. 2013. ”Comparison Isomorphisms for Smooth Formal Schemes.” Journal of the Institute of Mathematics of Jussieu 12 (1): 77-151. https://doi.org/10.1017/S1474748012000643.

[Lan1] Robert Langlands. 1967. ”Letter to Weil.”

[Drin1] Drinfel'd, Vladimir G. ”Elliptic modules.” Mathematics of the USSR-Sbornik 23, no. 4 (1974): 561.

[Drin2] Drinfeld, Vladimir Gershonovich. ”Langlands' conjecture for GL(2) over functional fields.” In Proceedings of the International Congress of Mathematicians (Helsinki, 1978), vol. 2, pp. 565-574. 1980.

[Zhu] Zhu, Xinwen. ”Coherent sheaves on the stack of Langlands parameters.” arXiv preprint arXiv:2008.02998 (2020).

[DHKM] Dat, Jean-François, David Helm, Robert Kurinczuk, and Gilbert Moss. ”Moduli of Langlands parameters.” arXiv preprint arXiv:2009.06708 (2020).