UNIVERSITY OF WATERLOO

ACTSC 632 Spring 2023

Data Science with Actuarial Applications

Week 7

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Last Week

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- Claim frequency: Poisson regression
- Claim severity: Gamma regression

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- Estimating the parameters of a GLM from:

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- Solving for the maximum likelihood estimators (MLE)

2.6 Parameter Estimation (MLE)

Taking derivative using the chain rule

Evaluating each partial derivative

Remark: satured model

Corresponding GLM overfits. However, it is useful in the definition of deviance.

Multiplicative Poisson frequency model

Multiplicative gamma severity model

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2.7 GLM Model Building

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Compare the deviance statistic with $c_{1-\alpha}$,

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- Reject if $D^* > c_{1-\alpha}$, accept otherwise.

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Pearson's Goodness-of-Fit Test

Null hypothesis: H_0 : fitted GLM model

$$X^2 = \sum_{i=1}^{n} \frac{(y_i - \hat{\mu}_i)^2}{\text{Var}(Y_i)} = \frac{1}{\phi} \sum_{i=1}^{n} w_i \frac{(y_i - \hat{\mu}_i)^2}{v(\hat{\mu}_i)}.$$

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Estimation of ϕ

- **Method 1**: Matching moments with the Pearson statistic

$$\hat{\phi} = \frac{1}{n - (r+1)} \sum_{i=1}^{n} w_i \frac{(y_i - \hat{\mu}_i)^2}{\nu(\hat{\mu}_i)}.$$

where we set $X^2 = n - (r+1)$ by matching the expected value of X^2

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- **Method 2**: Matching moments with the deviance statistic

$$\hat{\phi}_D = \frac{D}{n-(r+1)}.$$

where we set $D^* = n-(r+1)$ by matching the expected value of D^*

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- Using p-values, reject if p-value $< \alpha$, accept otherwise.

Confidence intervals based on Fisher's information

To get the confidence intervals, we need the Fisher's information matrix:

$$[I]_{jk} = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \beta_j \partial \beta_k}\right], \qquad j, k = 0, 1, \ldots, r.$$

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Asymptotically, we have $\hat{\beta} \sim \mathrm{N}(\beta, \mathrm{I}^{-1})$. Thus we can find the confidence intervals for β_j:

$$\hat{\beta}_j \pm z_{1-\alpha/2}\sqrt{[\mathrm{I}^{-1}]_{jj}}.$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of the standard normal distribution.

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where $I_\eta^{-1} = (x_{i0}, x_{i1}, \ldots x_{ir}) I^{-1} (x_{i0}, x_{i1}, \ldots x_{ir})^T$ is the variance of $\hat{\eta}_i$.

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Then we can find the confidence intervals for μ_i:

$$\left(e^{\hat{\eta}_i - z_{1-\alpha/2}\sqrt{[I_\eta^{-1}]_{jj}}}, e^{\hat{\eta}_i + z_{1-\alpha/2}\sqrt{[I_\eta^{-1}]_{jj}}}\right).$$

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  2. Test the severity model using deviance test using its p-value.
  3. Test the frequency model using Pearson's chi-squared test using its p-value.
  4. Test the severity model using Pearson's chi-squared test using its p-value.
  5. Estimate ϕ for the severity model.
  6. Combine zone 5, 6, 7 into zone 4. Fit a new frequency model with the new zone 4.
  7. Test the new frequency model against the previous one.
  8. Repeat step 6 and 7 for the severity model.
  9. Find the 95% confidence intervals for the parameters and the relativities of the frequency model.
  10. Repeat step 9 for the severity model.