

Data Science with Actuarial Applications

Week 7

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- ► Claim severity: Gamma regression

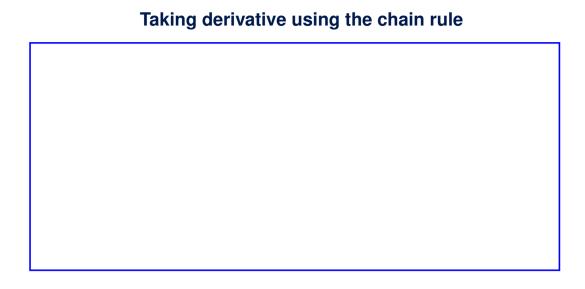
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- ► Solving for the maximum likelihood estimators (MLE)

2.6 Parameter Estimation (MLE)



Evaluating each partial derivative

Remark: saturated model

Corresponding GLM overfits. However, it is useful in the definition of deviance.

Multiplicative Poisson frequency model

Multiplicative gamma severity model

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- ► Example in R: Moped dataset

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- ► Testing hierarchical models
- Confidence intervals based on Fisher's information

2.7 GLM Model Building

Null hypothesis:

H₀: fitted GLM model

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Under general conditions, the deviance statistic D^* is approximately chi-squared distributed with n-p degrees of freedom, where

- n is the number of observations,
- ightharpoonup and p = r + 1 is the number of parameters in the model.

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Compare the deviance statistic with $c_{1-\alpha}$,

- $ightharpoonup c_{1-\alpha}$ is the $1-\alpha$ quantile of the chi-squared distribution with n-p degrees of freedom.
- ► Reject if $D^* > c_{1-\alpha}$, accept otherwise.

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Using p-values, reject if p-value $< \alpha$, accept otherwise.

Pearson's Goodness-of-Fit Test

Null hypothesis: H₀: fitted GLM model

$$X^{2} = \sum_{i=1}^{n} \frac{(y_{i} - \hat{\mu}_{i})^{2}}{\mathsf{Var}(Y_{i})} = \frac{1}{\Phi} \sum_{i=1}^{n} w_{i} \frac{(y_{i} - \hat{\mu}_{i})^{2}}{\nu(\hat{\mu}_{i})}.$$

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Estimation of ϕ

▶ **Method 1**: Matching moments with the Pearson statistic

$$\hat{\Phi} = \frac{1}{n - (r + 1)} \sum_{i=1}^{n} w_i \frac{(y_i - \hat{\mu}_i)^2}{v(\hat{\mu}_i)}.$$

where we set $X^2 = n - (r + 1)$ by matching the expected value of X^2

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▶ **Method 2**: Matching moments with the deviance statistic

$$\hat{\Phi}_{D} = \frac{D}{n - (r+1)}.$$

where we set $D^* = n - (r + 1)$ by matching the expected value of D^*

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- Using p-values, reject if p-value $< \alpha$, accept otherwise.

To get the confidence intervals, we need the Fisher's information matrix:

$$[I]_{jk} = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \beta_i \partial \beta_k}\right], \quad j, k = 0, 1, \dots, r.$$

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Asymptotically, we have $\hat{\beta} \sim N(\beta, I^{-1})$. Thus we can find the confidence intervals for β_j :

$$\hat{\beta}_{\mathfrak{j}} \pm z_{1-\alpha/2} \sqrt{[\mathrm{I}^{-1}]_{\mathfrak{j}\mathfrak{j}}}.$$

where $z_{1-\alpha/2}$ is the $(1-\alpha/2)$ -quantile of the standard normal distribution.

In practice, we also want to find the confidence intervals for μ_i .

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We do this by finding the confidence intervals for η_i :

$$\hat{\eta}_{i} \pm z_{1-\alpha/2} \sqrt{[I_{\eta}^{-1}]_{jj}}.$$

where $I_n^{-1} = (x_{i0}, x_{i1}, \dots x_{ir})I^{-1}(x_{i0}, x_{i1}, \dots x_{ir})^T$ is the variance of $\hat{\eta}_i$.

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Then we can find the confidence intervals for μ_i :

$$(e^{\hat{\eta}_{\mathfrak{i}}-z_{1-\alpha/2}\sqrt{[I_{\eta}^{-1}]_{jj}}},e^{\hat{\eta}_{\mathfrak{i}}+z_{1-\alpha/2}\sqrt{[I_{\eta}^{-1}]_{jj}}}).$$

Example: Moped dataset

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► Tasks:

- 1. Test the frequency model using deviance test using its p-value.
- 2. Test the severity model using deviance test using its p-value.
- 3. Test the frequency model using Pearson's chi-squared test using its p-value.
- 4. Test the severity model using Pearson's chi-squared test using its p-value.
- 5. Estimate ϕ for the severity model.
- 6. Combine zone 5, 6, 7 into zone 4. Fit a new frequency model with the new zone 4.
- 7. Test the new frequency model against the previous one.
- 8. Repeat step 6 and 7 for the severity model.
- 9. Find the 95% confidence intervals for the parameters and the relativities of the frequency model.
- 10. Repeat step 9 for the severity model.