Time Value of Money - How to be a mortgage expert

Example

Assume an annual interest rate of 5%.

1. If you have $100 today, how much will it be worth in one year?

$$100 \times (1 + 0.05) = 105$$

2. If you need $100 in one year, how much should you save today?

$$\frac{100}{1 + 0.05} = 95.24$$

Some questions to think about:

- What if you save $100 today, how much will it be worth in 10 years?
- What if you need $100 in 10 years, how much should you save today?

Interest Rate

- **Interest rate: 5% pa**
  - **5%**: the interest is 5% of principal
  - **pa** (per annum): the interest is paid annually

Interest Rate Example

An investor deposits $10,000 in a bank account that pays interest at a rate of 5% *pa*.

- After 1 year, the interest earned by the investor is:

$$\$10,000 \times 0.05 = \$500$$

- The investor's bank balance after 1 year is:

$$\$10,000 + \$500 = \$10,500$$

Interest Rate Example

Year	Principal	Interest	Total
0	$10,000	$0	$10,000
1	$10,000	$500	$10,500

What happens after 2 years?

- **Simple interest**: Interest does not earn interest
  - Interest is calculated on the principal amount only
- **Compound interest**: Interest itself earns interest
  - Interest is calculated on the current total amount (principal + interest)

Simple Interest

- **Simple interest**: interest does not earn interest

An investor deposits C in a bank account that pays simple interest at a rate of $i\%$ *pa*

- After n years, the interest earned by the investor is:

$$\underbrace{C \times i + C \times i + \cdots + C \times i}_{n} = nCi$$

- Accumulated value of C today at time n is:

$$AV = C + nCi = C(1 + ni)$$

- n can be a non-integer

Compound Interest

- **Compound interest**: interest itself earns interest

An investor deposits $10,000$ in a bank account that pays compound interest at a rate of 5% pa.

- After 1 year, the investor's bank balance is:
  $10,000 \times (1 + 0.05) = 10,500$
- After 2 years, the investor's bank balance is:
  $10,500 \times (1 + 0.05) = 11,025$
  $= 10,000 \times (1 + 0.05) \times (1 + 0.05)$
  $= 10,000 \times 1.05^2$

Compound Interest

- After 3 years, the investor's bank balance is: $10,000 \times (1+0.05)^2 \times (1+0.05) = 11,576.25$

Year	Principal	Interest	Total
0	$10,000	$0	$10,000
1	$10,000	$500	$10,500
2	$10,500	$525	$11,025
3	$11,025	$551.25	$11,576.25
...	...	...	...

Compound Interest

An investor deposits C in a bank account that pays compound interest at a rate of $i\%$ *pa*

- After n years, the investor's bank balance is:

$$AV = C\underbrace{\times(1+i)\times(1+i)\times\cdots\times(1+i)}_{n} = C\times(1+i)^n$$

- After n years, the interest earned by the investor is:

$$AV - C = C\times(1+i)^n - C$$

- n can be a non-integer

Simple vs. Compound Interests

- Accumulated value of C with simple interest rate i pa after n years:

$$C(1 + ni)$$

- Accumulated value with compound interest rate i pa after n years:

$$C(1 + i)^n$$

- Actively reinvest interest yields a higher accumulated deposit!

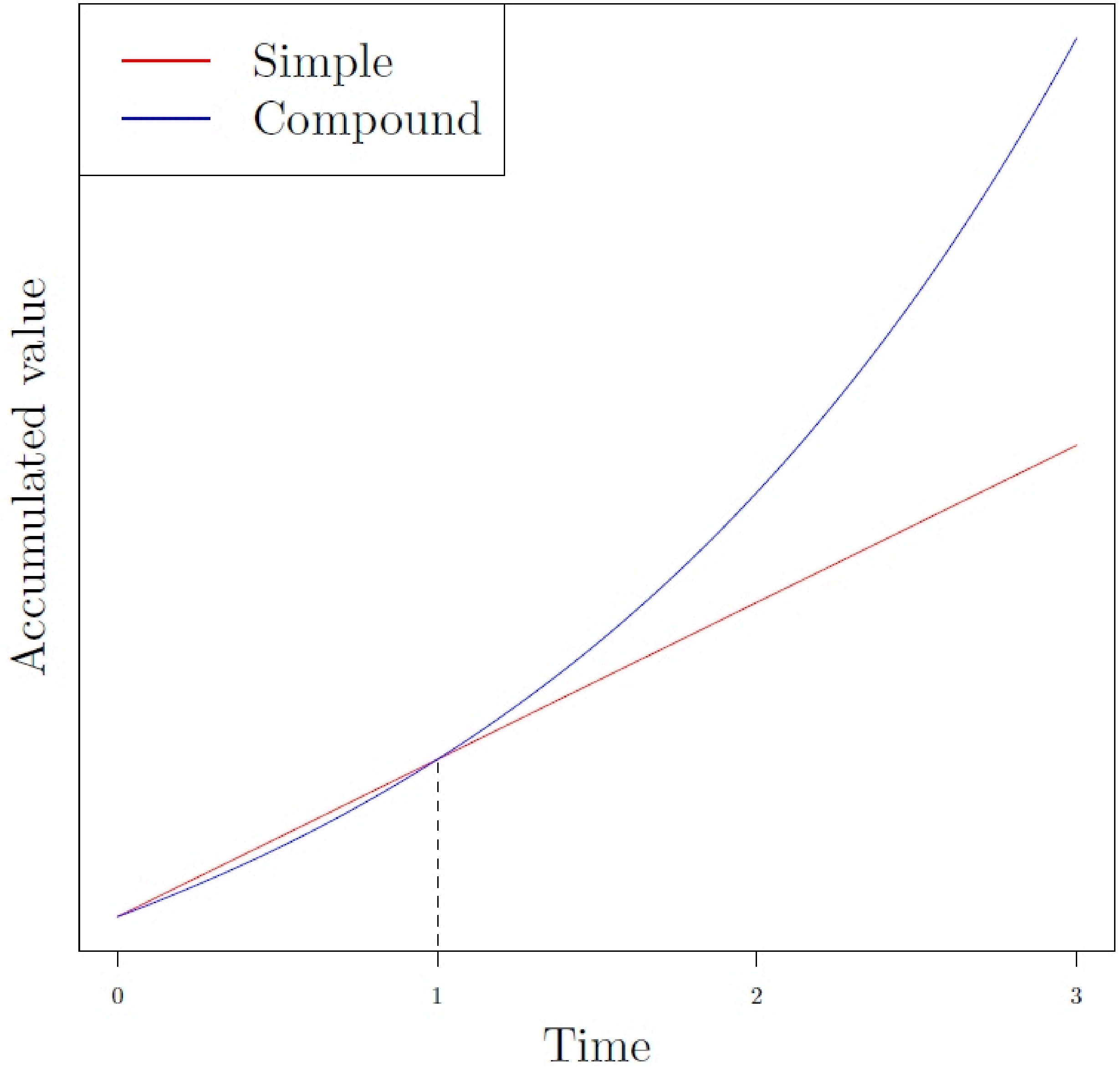
$$11,576.25 > 11,500$$

Simple vs. Compound Interests

For the same principal, interest rate, and time period:

- Simple interest follows a straight line
- Compound interest follows an exponential curve

Compound interest grows faster **after the first period** (Why?).



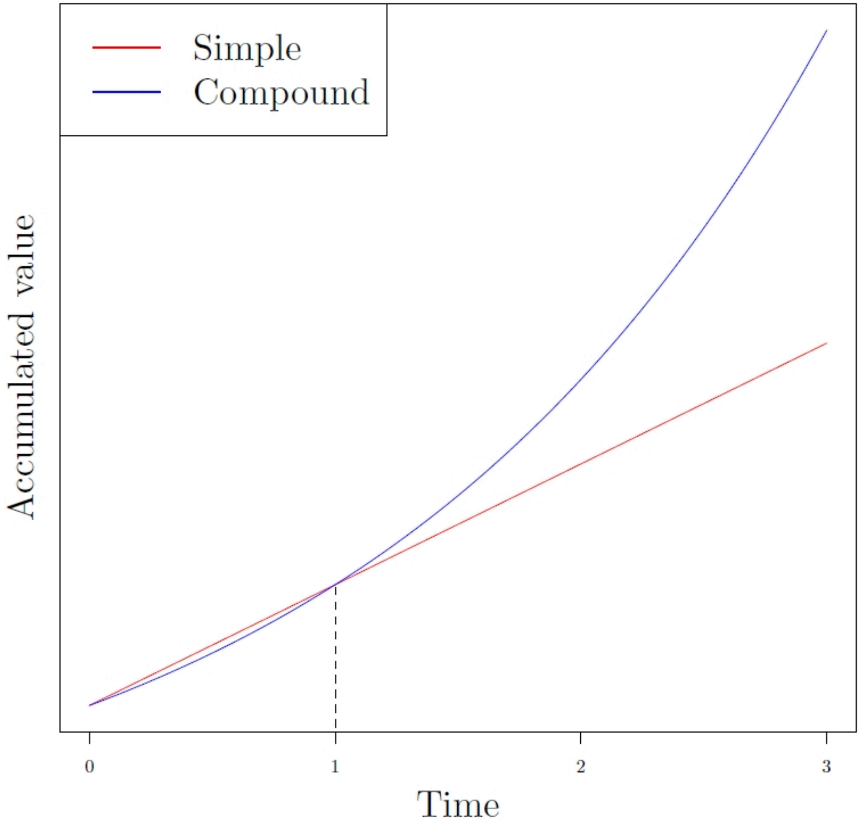
Account Value as a Function of Time

Let $A(t)$ be the account value at time t for a deposit at time 0. Here:

- t is the time in years
- $A(t)$ is a value that depends on t

Note: Usually we write $y = f(x)$ to indicate

- y is a dependent variable that depends on x
- f is a function, or a relationship to get y from x
- x is a independent variable



Accumulation Factor

- For $t_1 < t_2$, we define $A(t_1, t_2)$ to be the value at time t_2 of an investment of 1 at time t_1

$$t_1 \to t_2$$
$$1 \to A(t_1, t_2)$$

- By proportion, the accumulated value of deposit C from time t_1 to time t_2 is $C \cdot A(t_1, t_2)$

$$t_1 \to t_2$$
$$C \to C \cdot A(t_1, t_2)$$

- $A(t_1, t_2)$ is a function of two variables: t_1 and t_2, that is, its value depends on the two times, t_1 (the starting time) and t_2 (the ending time).

Accumulation Factor

In other words,

$$A(t_1, t_2) = \frac{A(t_2)}{A(t_1)}$$

Where:

- $A(t_1, t_2)$ is the accumulation factor
- $A(t_1)$ is the amount at time t_1
- $A(t_2)$ is the amount at time t_2

Accumulation Factor

The accumulation factor is a "black box" to calculate the money received by customers:

$$C \text{ at } t_1 \xrightarrow{A(t_1,t_2)} CA(t_1,t_2) \text{ at } t_2$$

- As an mortgage expert, we should design the "black box"
- What is the functional form of $A(t_1, t_2)$ and $A(n)$ using **Simple interest**?
- What is the functional form of $A(t_1, t_2)$ and $A(n)$ using **Compound interest**?

Accumulation Factor - Simple Interest

Time:

$$t_1 \rightarrow t_2$$

Account Value:

$$1 \rightarrow 1 + (t_2 - t_1)i$$

The accumulation factor from time t_1 to time t_2 for simple interest:

$$A(t_1, t_2) = 1 + (t_2 - t_1)i$$

Be careful!

The accumulation of $(1 + t_1 i)$ at t_1 with simple interest is not $(1 + t_2 i)$ at time t_2.

Accumulation Factor - Compound Interest

Time:

$$t_1 \to t_2$$

Account Value:

$$1 \to (1+i)^{t_2 - t_1}$$

The accumulation factor from time t_1 to time t_2 for compound interest:

$$A(t_1, t_2) = (1+i)^{t_2 - t_1}$$

Accumulation Factor

- The abbreviated notation $A(n)$ is used for $A(0, n)$
- The accumulation factor from time 0 to time n for simple interest:

$$A(n) = \frac{1 + ni}{1} = 1 + ni$$

- The accumulation factor from time 0 to time n for compound interest:

$$A(n) = \frac{(1 + i)^n}{1} = (1 + i)^n$$

Example

The accumulation factor $A(5) = 2.5$. Calculate the accumulated value of an investment of $1,000$ at time 0 after 5 years.

- $t_1 = 0$ and $t_2 = 5$
- $A(5) = 2.5$
- Amount at time 0 = $1,000$

Answer

$$\text{Amount at time 5} = A(5) \times \text{Amount at time 0} = 2.5 \times 1,000 = 2,500$$

Example

An investment of $1,000$ accumulates to $2,500$ after 5 years. Calculate the accumulation factor $A(5)$.

Answer

$$A(5) = \frac{\text{Amount at time 5}}{\text{Amount at time 0}} = \frac{2,500}{1,000} = 2.5$$

Python Implementation

We will use Python functions to implement the accumulation factor.

Fill in the blanks:

```python
def acc_factor(t1, t2, i, simple=True):
    if simple:
        return ___
    else:
        return ___
```

Python Implementation

We will use Python functions to implement the accumulation factor.

```python
def acc_factor(t1, t2, i, simple=True):
    if simple:
        return 1 + (t2 - t1) * i
    else:
        return (1 + i) ** (t2 - t1)
```

Practice Problems

Basic

1. Using the accumulation factor, calculate the accumulated value of an investment of $1,000$ at time 0 after 5 years using simple interest of 5% pa.
2. Using the accumulation factor, calculate the accumulated value of an investment of $1,000$ at time 0 after 5 years using compound interest of 5% pa.

Intermediate

3. Using the accumulation factor, calculate the accumulated value at 5.5 years of an investment of $1,000$ at year 3 using simple interest of 1.5% pa.
4. Using the accumulation factor, calculate the accumulated value at 5.5 years of an investment of $1,000$ at year 3 using compound interest of 1.5% pa.

Advanced

5. If you need $1,000$ at time 5, how much should you deposit at time 0 using simple interest of 5% pa?
6. If you need $1,000$ at time 5, how much should you deposit at time 0 using compound interest of 5% pa?