1.3 - Rational Indices

Review - Negative Indices

$$a^{-n} = \frac{1}{a^n}$$

- Example: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
- Exercise: Simplify the following expressions.
  i. 3^{-2}
  ii. $\left(\frac{2}{3}\right)^{-3}$

Rational Number

Rational numbers are numbers that can be expressed as a fraction of two integers.

$$\frac{p}{q}$$

where p and q are integers and $q \neq 0$.

- Example 1: $\frac{1}{2}$ is a rational number because it can be expressed as $\frac{p}{q}$, where $p = 1$ and $q = 2$.
- Example 2: Is π a rational number? Is $\sqrt{2}$ a rational number?

Could you give an example like the above?

Roots

The nth root of a number a is a number b such that $b^n = a$.

$$\sqrt[n]{a} = b$$

when n is 2, we usually omit the index.

$$\sqrt{a} = b$$

- Example 1: $\sqrt{9} = 3$ because $3^2 = 9$.
- Example 2: $\sqrt[3]{8} = 2$ because $2^3 = 8$.

Could you give an example like the above?

Converting Rational Indices to Roots

We can extend the definition of indices to rational numbers.

The $1/n$-th power of a number a is equal to the n-th root of a.

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

- Example 1: $2^{\frac{1}{2}} = \sqrt{2}$
- Example 2: $2^{\frac{1}{3}} = \sqrt[3]{2}$
- Example 3: $49^{-\frac{1}{2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

Could you give an example like the above?

Converting Roots to Rational Indices

It also works the other way round.

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

- Example 1: $\sqrt{2} = 2^{\frac{1}{2}}$
- Example 2: $\sqrt[3]{2} = 2^{\frac{1}{3}}$
- Example 3: $\frac{1}{\sqrt{49}} = 49^{-\frac{1}{2}}$

Could you give an example like the above?

Unifying Laws in Square Roots and Rational Indices

$$\sqrt{2} \times \sqrt{2} = 2$$

is equivalent to

$$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^1 = 2$$

Could you give an example like the above?

Example 4 Self Tutor

Simplify:

a $49^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $49^{-\frac{1}{2}}$ **d** $27^{-\frac{1}{3}}$

a $$\begin{aligned} &49^{\frac{1}{2}} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

b $$\begin{aligned} &27^{\frac{1}{3}} \\ &= \sqrt[3]{27} \\ &= 3 \end{aligned}$$

c $$\begin{aligned} &49^{-\frac{1}{2}} \\ &= \frac{1}{49^{\frac{1}{2}}} \\ &= \frac{1}{\sqrt{49}} \\ &= \frac{1}{7} \end{aligned}$$

d $$\begin{aligned} &27^{-\frac{1}{3}} \\ &= \frac{1}{27^{\frac{1}{3}}} \\ &= \frac{1}{\sqrt[3]{27}} \\ &= \frac{1}{3} \end{aligned}$$

Universal Laws of Rational Indices

Positive Rational Indices

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

In order to understand this, we need to understand the meaning of $a^{\frac{m}{n}}$.

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{\underbrace{a \times a \times \ldots \times a}_{m \text{ times}}} = \sqrt[n]{a^m}$$

Another way to understand this:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \underbrace{(\sqrt[n]{a}) \times (\sqrt[n]{a}) \times \ldots \times (\sqrt[n]{a})}_{m \text{ times}} = \sqrt[n]{a^m}$$

Negative Rational Indices

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}} = \frac{1}{\sqrt[n]{a^m}}$$

- Example 1: $2^{\frac{2}{3}} = \sqrt[3]{2^2} = \sqrt[3]{4}$
- Example 2: $2^{-\frac{2}{3}} = \frac{1}{2^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{2^2}} = \frac{1}{\sqrt[3]{4}}$

INVESTIGATION 1 RATIONAL INDICES

This Investigation will help you discover the meaning of numbers raised to rational indices of the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$, $n \neq 0$.

For example, what does $8^{\frac{2}{3}}$ mean?

What to do:

1 Use the rule $(a^m)^n = a^{mn}$ to simplify $\left(8^2\right)^{\frac{1}{3}}$ and $\left(8^{\frac{1}{3}}\right)^2$.

2 Simplify: **a** $\left(8^2\right)^{\frac{1}{3}} = \sqrt[3]{8^2} = ...$ **b** $\left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = ...$

3 Hence write $a^{\frac{m}{n}}$ in two different forms.

From the **Investigation** you should have discovered that $\boldsymbol{a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}}$.

When dealing with indices of this form, it is often easiest to write the base number as a prime raised to a power. We simplify the result using the index laws.

Example 5 **Self Tutor**

Evaluate without using a calculator:

a $8^{\frac{4}{3}}$ **b** $32^{-\frac{2}{5}}$

a $8^{\frac{4}{3}}$
$= (2^3)^{\frac{4}{3}}$
$= 2^{3\times\frac{4}{3}}$
$= 2^4$
$= 16$

b $32^{-\frac{2}{5}}$
$= (2^5)^{-\frac{2}{5}}$
$= 2^{5\times-\frac{2}{5}}$
$= 2^{-2}$
$= \frac{1}{4}$

EXERCISE 1B

1 Evaluate without using your calculator:

a $16^{\frac{1}{2}}$ **b** $16^{-\frac{1}{2}}$ **c** $25^{\frac{1}{2}}$ **d** $25^{-\frac{1}{2}}$

e $8^{\frac{1}{3}}$ **f** $8^{-\frac{1}{3}}$ **g** $(-8)^{\frac{1}{3}}$ **h** $(-8)^{-\frac{1}{3}}$

i $81^{\frac{1}{4}}$ **j** $81^{-\frac{1}{4}}$ **k** $32^{\frac{1}{5}}$ **l** $32^{-\frac{1}{5}}$

2 Evaluate if possible:

a $(-1)^{\frac{1}{2}}$ **b** $(-1)^{\frac{1}{3}}$ **c** $(-27)^{-\frac{1}{3}}$ **d** $(-64)^{-\frac{1}{2}}$

3 Write the following in index form:

a $\sqrt{10}$ b $\frac{1}{\sqrt{10}}$ c $\sqrt[3]{15}$ d $\frac{1}{\sqrt[3]{15}}$

e $\sqrt[4]{19}$ f $\frac{1}{\sqrt[4]{19}}$ g $\sqrt[5]{13}$ h $\frac{1}{\sqrt[5]{13}}$

4 Evaluate without using a calculator:

a $8^{\frac{2}{3}}$ b $4^{\frac{3}{2}}$ c $4^{\frac{5}{2}}$ d $8^{\frac{5}{3}}$

e $16^{\frac{3}{4}}$ f $9^{\frac{3}{2}}$ g $9^{-\frac{3}{2}}$ h $4^{-\frac{1}{2}}$

i $32^{\frac{1}{5}}$ j $32^{\frac{2}{5}}$ k $32^{\frac{3}{5}}$ l $16^{-\frac{3}{4}}$

m $8^{-\frac{2}{3}}$ n $27^{-\frac{4}{3}}$ o $25^{-\frac{3}{2}}$

5 Write the following as powers of 2:

a $\sqrt{8}$ b $\sqrt[3]{32}$ c $\sqrt[4]{4}$ d $\sqrt[3]{16}$

e $\frac{1}{\sqrt[4]{8}}$ f $\frac{1}{\sqrt[3]{16}}$ g $\frac{1}{\sqrt[7]{8}}$ h $\frac{1}{\sqrt[5]{64}}$

i $8\sqrt{2}$ j $4\sqrt{32}$ k $\frac{2}{\sqrt[3]{4}}$ l $\frac{4\sqrt{32}}{8}$

6 Write the following as powers of 3:

a $\sqrt[3]{9}$ b $\sqrt{27}$ c $\frac{1}{\sqrt[4]{27}}$ d $\frac{1}{\sqrt[5]{81}}$

e $9\sqrt{3}$ f $3\sqrt{27}$ g $\frac{9}{\sqrt[5]{3}}$ h $\frac{\sqrt[3]{81}}{9}$

7 Write with a prime number base:

a $\sqrt[3]{25}$ b $\sqrt[4]{32}$ c $\sqrt[5]{125}$ d $\sqrt[7]{121}$

e $\frac{1}{\sqrt[3]{49}}$ f $\sqrt[5]{64}$ g $\frac{1}{\sqrt[7]{625}}$ h $\frac{1}{\sqrt[6]{243}}$

i $16\sqrt{8}$ j $25\sqrt{125}$ k $\frac{13}{\sqrt[3]{169}}$ l $\frac{81}{\sqrt{27}}$

8 Use your calculator to evaluate, rounded to 3 significant figures where necessary:

a $25^{\frac{3}{2}}$ b $27^{\frac{2}{3}}$ c $8^{\frac{7}{3}}$ d $9^{\frac{2}{5}}$

e $10^{\frac{3}{7}}$ f $15^{\frac{5}{3}}$ g $10^{\frac{2}{7}}$ h $18^{\frac{7}{3}}$

i $16^{\frac{3}{11}}$ j $146^{\frac{4}{9}}$ k $4^{-\frac{5}{2}}$ l $27^{-\frac{5}{3}}$

m $15^{-\frac{2}{5}}$ n $53^{-\frac{3}{7}}$ o $3^{-\frac{7}{5}}$

9 Without using your calculator, evaluate $\frac{\sqrt[3]{9} \times \sqrt[4]{27}}{\sqrt[12]{243}}$.