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Valuation of large variable annuity portfolios under nested simulation: A functional data approach

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ABSTRACT

A variable annuity (VA) is equity-linked annuity product that has rapidly grown in popularity around the world in recent years. Research up to date on VA largely focuses on the valuation of guarantees embedded in a single VA contract. However, methods developed for individual VA contracts based on option pricing theory cannot be extended to large VA portfolios. Insurance companies currently use nested simulation to valuate guarantees for VA portfolios but efficient valuation under nested simulation for a large VA portfolio has been a real challenge. The computation in nested simulation is highly intensive and often prohibitive. In this paper, we propose a novel approach that combines a clustering technique with a functional data analysis technique to address the issue. We create a highly non-homogeneous synthetic VA portfolio of 100,000 contracts and use it to estimate the dollar Delta of the portfolio at each time step of outer loop scenarios under the nested simulation framework over a period of 25 years. Our test results show that the proposed approach performs well in terms of accuracy and efficiency.

1. Introduction

A variable annuity (VA) is a deferred annuity that allows an annuitant to invest his/her contributions into one mutual fund or a basket of mutual funds. A separate account termed as sub-account is set up for the investment. Unlike mutual funds, a VA provides a downside protection from the fluctuation of the financial markets in the form of minimum guarantee. When an annuitant enters a VA contract with an insurance company, the annuitant agrees to make one lump-sum or a series of purchase payments to the insurance company (Chi and Lin, 2012). The insurance company in return offers a guaranteed minimum death benefit (GMDB) and often offers the guaranteed minimum maturity benefit (GMAB) or the guaranteed minimum income benefit (GMIB) as a rider. GMDB guarantees that the beneficiary of a VA holder receives the greater of (a) the sub-account value or (b) the total purchase payments, upon the death of the VA holder. GMAB and GMIB provide accumulation and income protection for a fixed number of years contingent on survival of the policyholder, respectively. Recently many insurance companies offer the guaranteed minimum withdrawal benefit (GMWB) that allows an annuitant to withdraw a maximum percentage of his/her total purchase payments each year until the total purchase payments have been recouped.

Due to the innovative guarantee features, VAs have rapidly grown in popularity around the world in recent years. In the US, 2012 new sales were 147 billion (IRI, 2013). In the UK, the sales increased from 539 million pounds in 2007 to 1.42 billion pounds in 2012. Canada, Japan, and South Korea also experienced a similar rapid growth in VA sales. As a result, almost all the major insurance companies in these countries are managing very large VA portfolios, each of which might have well over 100,000 VA contracts. Furthermore, a typical VA portfolio is highly non-homogeneous in that every VA contract is unique in terms of gender, age, time to maturity, guarantee type, and fund type. How to hedge and manage the risks of a large VA portfolio and how to determine required capitals are a real challenge to insurance companies (Bauer et al., 2012).

Research up to date on valuation, hedging, and risk management of VA guarantees largely focuses on individual VA contracts. Milevsky and Posner (2001) used risk-neutral option pricing theory to price GMDB in a VA contract. Gerber and Shiu (2003) used the idea of European lookback options to derive closed-form formulas to price complex guarantees embedded in some equity-linked

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annuities. Coleman et al. (2006) used local risk minimization to study the discrete hedging of the guarantees embedded in a VA contract with both equity risk and interest rate risk and concluded that hedging with standard options is superior to hedging with the underlying. Milevsky and Salisbury (2006) studied the impact of policyholder behavior on the cost and value of the GMWB rider and argued that the current pricing is not sustainable. Boyle and Tian (2008) analyzed the design of general equity-indexed annuities from the investor's perspective and proposed a generalization of the conventional design. Lin et al. (2009) used Esscher transform to determine an equivalent martingale measure for the fair valuation of a VA contract under a regime-switching model in the incomplete market setting. Bélanger et al. (2009) proposed a method to compute the fair insurance fee of the GMDB rider under partial withdrawals. Jiang and Chang (2010) used the Black–Scholes model to derive an analytical solution of the cost of the GMAB rider. Bacinello et al. (2011) proposed a unifying framework to price various types of guarantees. Gao and Ulm (2012) used a utility-based approach to study the valuation of the GMDB rider. Kolkiewicz and Liu (2012) proposed a method to hedge the GMWB rider by constructing a portfolio of traded European options that approximates the closest path-independent option to the guarantee. Gerber et al. (2013) proposed a method based on discounted joint density function to price the GMDB rider. Deelstra and Rayée (2013) studied the pricing of GMIB under a local volatility framework and argued that an appropriate volatility modeling is important to the long-dated guarantees. Yang and Dai (2013) proposed a tree model to price the GMWB rider embedded in deferred life annuity contracts.

Unfortunately, the aforementioned studies for individual VA contracts cannot be extended to large VA portfolios for (i) the complexity of the payoff function of the guarantees results in no closed-form formulas for the liability value of the guarantees and (ii) valuation and sensitivity calculation must be done for each contract individually and the calculation becomes extremely challenging when the number of contracts is very large. In practice, insurance companies follow a market-to-model approach and rely heavily on Monte Carlo simulation. In particular, they use nested simulations to determine the probability distribution of losses from mismatching in order to calculate required capitals (Bauer et al., 2012; Reynolds and Man, 2008).

As we will demonstrate in the next section, nested simulation of a large VA portfolio is extremely time consuming. The main contribution of this paper is to present a novel approach to speed up the nested simulation by reducing the number of VA contracts that go through the nested simulation. This approach combines a clustering technique with a functional data analysis technique. First, we use a clustering technique to select a small set of representative VA contracts from a large VA portfolio and calculate quantities of interest of the representative contracts under nested simulation. Then we apply a functional data analysis technique called universal kriging to calculate those quantities of interest for the remaining VA contracts in the portfolio. The efficiency of the universal kriging method allows the remaining VA contracts to be calculated very quickly.

The remaining of the paper is structured as follows. Section 2 gives a brief introduction to the nested simulation of VA contracts and several existing approaches that address computational issues arising from the valuation of large VA portfolios. Section 3 gives a brief description of functional data analysis and the universal kriging method for spatial functional data. Section 4 presents some test results of applying the universal kriging method to VA dollar Delta calculation under the nested simulation framework. Section 5 discusses several aspects of the proposed approach. Section 6 concludes the paper with some discussion on future work.

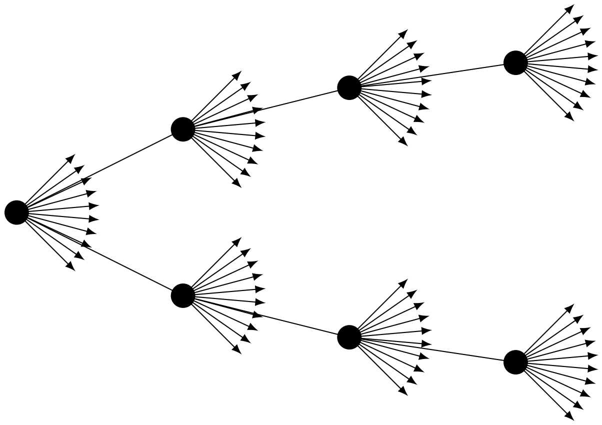

Fig. 1. Nested simulation has two loops: an outer loop and an inner loop.

2. Nested simulation and VA portfolio valuation

As mentioned earlier, nested simulation is used to determine the probability distribution of losses from mismatching in order to calculate required capitals or other quantities of interest (Bauer et al., 2012; Reynolds and Man, 2008). Nested simulation is a two-level procedure. At the first level, the so-called outer loop is simulated and at the second level, the inner loop is simulated (Bauer et al., 2012; Reynolds and Man, 2008; Fox, 2013). Fig. 1 shows the basic structure of nested simulation. The outer loop involves projecting the VA liabilities along real world scenarios. At each node of an outer loop scenario, the liabilities are projected using a large number of simulated risk-neutral paths.

The computation of nested simulation of a large VA portfolio is highly intensive and often prohibitive. For example, if we use 1000 real world scenarios in the outer loop and 1000 risk-neutral paths in the inner loop, and project the liabilities at yearly steps for 25 years, then the total number of projections for each contract is

$$1000 \times 1000 \times 25 \times 26/2 = 3.25 \times 10^8,$$

which is a very big number. For a portfolio of 100,000 contracts, the number of projections would be 3.25×10^{13}.

To reduce the computation time of a nested simulation for a large VA portfolio, one may reduce the number of VA contracts that go thorough the nested simulation, reduce the number of outer loop scenarios, or reduce the number of inner loop paths. Here both scenarios and paths represent the economic scenarios simulated from an asset model such as the Black–Scholes model or more generally a regime-switching model under the real world measure and risk-neutral probability measure, respectively. The words "scenario" and "path" are used to differentiate the levels of the nested simulation. The time saved from reducing outer loop scenarios or inner loop paths is limited because the accuracy of Monte Carlo heavily depends on the number of simulation runs. As a result, too few outer loop scenarios or inner loop paths would affect the accuracy of the assessment. For example, using too few inner loop paths may lead to volatile fair market value estimation of the guarantees embedded in a VA portfolio, and using too few outer loop paths may result in a biased loss distribution.

Several approaches have been developed to deal with large portfolios of VA contracts. One approach involves replicating portfolios, which consist of standard financial instruments such as futures, European options, and swaps. Daul and Vidal (2009) studied the quadratic programming method to replicate cash flows of life insurance liabilities in general. Oechslin et al. (2007) applied the quadratic programming method to replicate embedded options in insurance contracts. Liability cash flows depend on actuarial factors (e.g. mortality) as well as financial factors. In this approach, that actuarial factors are assumed to simply follow their expectations given a large pool of policyholders. The replication method requires economic scenarios and cash flows projected by an actuarial projection system in order to construct a replicating portfolio. Dembo and Rosen (1999) presented a portfolio replication framework that minimizes the sum of absolute differences instead of the sum of squared differences. Dembo and Rosen (1999) formed the cash flow replication problem as a linear programming problem. Compared to the quadratic programming method, the linear programming approach can incorporate linear constraints easily. The replicating portfolio method allows to calculate the Greeks of the liability portfolio by closed-form formulas. However, constructing a replicating portfolio of a large VA portfolio of is time consuming because the cash flow of the portfolio at each time step and each scenario must be projected by an actuarial projection system.

Another approach involves a statistical model that regresses the liability value on some key economic factors (Cathcart and Morrison, 2009) and the least-square Monte Carlo method (Carriere, 1996; Longstaff and Schwartz, 2001) is used to approximate the future liabilities at each time step. The approximation is achieved by a linear combination of basis functions, such as powers of the state variables. The approach can significantly reduce the number of inner paths used in nested simulation.

Other approaches involve reducing the number of VA contracts that go through Monte Carlo simulation. Vadiveloo (2012) proposed a method based on replicated stratified sampling and only these sample policies are valuated. Gan (2013) proposed a method based on data clustering and machine learning that selects a small set of representative VA policies and prices the representative policies. The clustering technique and the machine learning technique used by Gan (2013) are k-prototypes and ordinary kriging, respectively.

The replicating portfolio approach cannot be used to speed up nested simulation of large VA portfolios because replicating portfolios are constructed from a one-level simulation. The least-square Monte Carlo method works only for nested simulation with outer loops of one time step. The policy reduction methods proposed by Vadiveloo (2012) and Gan (2013) were developed for one-level simulation rather than nested simulation. In this paper, we propose a policy reduction method for speeding up nested simulation of large VA portfolios.

Other relevant works on portfolio valuation include Chen et al. (2012), Broadie et al. (2011a), and Broadie et al. (2011b). Chen et al. (2012) proposed a stochastic kriging metamodel to compute the VaR of a portfolio of options. Broadie et al. (2011a) proposed a nonuniform nested simulation algorithm for estimating the probability of a loss. Broadie et al. (2011b) proposed a weighted regression method to estimate risk measures in nested Monte Carlo simulation. For stochastic kriging for simulation metamodeling, readers are referred to Ankenman et al. (2010).

3. A spatial functional data analysis method

Functional data analysis as a branch of statistics focuses on analyzing functional data, which includes curves and surfaces (Ramsay and Silverman, 2002, 2005). In many cases, underlying functional data involve a time argument t. In such cases, the data are organized in terms of time. In other cases, underlying functional data involve a spatial argument such as locations. Spatial functional data analysis, as its name indicates, aims to analyze data organized by space, time, or both.

In spatial functional data analysis, we study functional random processes $\{\mathcal{X}_s : s \in D \subseteq \mathbb{R}^d\}$, where $\mathbb{R}$ is the set of real numbers. Each functional random process $\mathcal{X}_s$ is assumed to be a function in $L^2([a, b])$ defined as:

$$L^2([a,b]) = \left\{ f : [a,b] \to \mathbb{R}, \int_a^b f(t)^2 dt < \infty \right\}.$$

In addition, the functional random processes $\{\mathcal{X}_s : s \in D \subseteq \mathbb{R}^d\}$ are assumed to have the following stationary properties (Caballero et al., 2013):

$$E[\mathcal{X}_s(t)] = m(t), \quad t \in [a,b],\ s \in D. \tag{1a}$$

$$\mathrm{Var}[\mathcal{X}_s(t)] = \sigma^2(t), \quad t \in [a,b],\ s \in D. \tag{1b}$$

$$\mathrm{Cov}(\mathcal{X}_{s_i}(t), \mathcal{X}_{s_j}(u)) = C(h; t, u), \quad t, u \in [a,b],\ s_i, s_j \in D, \tag{1c}$$

where $h = \|s_i - s_j\|$ is the Euclidean distance between s_i and s_j. The function

$$\frac{1}{2}\mathrm{Var}[\mathcal{X}_{s_i}(t) - \mathcal{X}_{s_j}(u)] = \gamma(h; t, u), \quad t, u \in [a,b],\ s_i, s_j \in D \tag{1d}$$

is called semivariogram and plays an important role in the subsequent universal kriging analysis.

As discussed earlier, the purpose of nested simulation is to calculate the quantities of interest such as Greeks (e.g., dollar Delta, dollar Rho) of a liability portfolio along outer loop scenarios in order to construct a hedging portfolio and in turn to determine required capitals (regulatory capital and economic capital for example) for the future mismatches between the liability portfolio and the hedging portfolio. In this section, we propose a method that combines a clustering technique (Gan et al., 2007) and a universal kriging method for spatial functional data (Caballero et al., 2013) to calculate dollar Deltas along the outer loop scenarios. Note that with minor adjustments, the method is applicable to calculate other quantities of interest. We now describe the proposed method in the following subsections.

3.1. Selection of representative VA contracts

The k-prototypes algorithm (Huang, 1998; Gan et al., 2007) is a clustering algorithm that was developed to cluster mixed-type data. We use the k-prototypes algorithm to select representative VA contracts for several reasons: first, a VA contract is characterized by both numerical and categorical attributes; second, the k-prototypes algorithm was developed from the popular k-means algorithm and thus is faster than other clustering methods (e.g., Ji et al., 2012) for mixed-type data; third, the k-prototypes algorithm is a center-based clustering algorithm and produces sphere-shaped clusters, each of which has a center that can be selected as a representative point. In this subsection, we give a brief description of the k-prototypes algorithm.

Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ denote the portfolio of VA contracts, where n is the number of VA contracts and $\mathbf{x}_i$ represents the ith VA contract. Without loss of generality, we assume that every VA contract is characterized by d attributes (e.g., gender, age, account value, etc.) and that the first d_1 attributes are numeric and the remaining $d_2 = d - d_1$ attributes are categorical. The distance between two contracts $\mathbf{x}$ and $\mathbf{y}$ in X can be defined as (Huang, 1998; Huang et al., 2005):

$$D(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{h=1}^{d_1} w_h (x_h - y_h)^2 + \sum_{h=d_1+1}^{d} w_h \delta(x_h, y_h)}, \tag{2}$$

where x_h and y_h are the hth component of $\mathbf{x}$ and $\mathbf{y}$, respectively, $w_h > 0$ is a weight assigned to the hth component, and $\delta(\cdot, \cdot)$ is the simple matching distance defined as

$$\delta(x_h, y_h) = \begin{cases} 0, & \text{if } x_h = y_h, \\ 1, & \text{if } x_h \neq y_h. \end{cases}$$

In the above distance calculation, there are many ways to assign the weights. For the numeric attributes, we may assume the weights to take the form

$$w_h = \frac{1}{R_h^2}, \tag{3}$$

where R_h^2 measures the degree of variability of the data of the h-th attribute. The most sensible choice for R_h^2 would be the corresponding sample variance:

$$R_h^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ih} - \bar{x}_h)^2,$$

where $\bar{x}_h$ is the sample mean. We conduct our numerical experiment in Section 4 based on this approach. We note that the distance calculated in this way is equivalent to that one first converts each x_{ih} to its z-score and then calculate an unweighted distance, i.e. $w_h = 1$ for $h = 1, \ldots, d_1$.

Another choice is to use the range of the data set. In this case, we may define

$$R_h = \max_{1\le i\le n} x_{ih} - \min_{1\le i\le n} x_{ih}.$$

However, this choice would give a small weight to a numeric variable with a large extreme value. In other words, the approach is more sensitive to outliers. In Section 5, a comparison study is conducted to see whether or not this choice of weights would outperform the sample variance approach.

Since there is not apparent reason that the categorical attributes need to be treated differently from each other, we may assign a common weight, say α, to all the categorical attributes. By varying the value of α (e.g. $\alpha = 0.5$, 1, or 2), we can examine the impact of categorical variables on cluster selection.

The k-prototypes algorithm aims to minimize the following objective function:

$$P = \sum_{j=1}^{k}\sum_{\mathbf{x}\in C_j} D^2(\mathbf{x}, \boldsymbol{\mu}_j), \tag{4}$$

where $D(\cdot,\cdot)$ is defined in Eq. (2), k is the number of clusters, C_j is the jth cluster, and $\boldsymbol{\mu}_j$ is the center or prototype of cluster C_j. To do that, the k-prototypes algorithm proceeds iteratively. In other words, the k-prototypes algorithm repeats updating the cluster memberships given the cluster centers and updating the cluster centers given the cluster memberships until some stop condition is satisfied.

Mathematically, the k-prototypes algorithm can be described as follows:

1. **Initialize cluster center**. At this step, the algorithm initializes the k cluster centers by selecting k distinct contracts from portfolio X randomly. Suppose $\boldsymbol{\mu}_1^{(0)}, \boldsymbol{\mu}_2^{(0)}, \ldots, \boldsymbol{\mu}_k^{(0)}$ are the k initial cluster centers.
2. **Update cluster memberships**. At this step, the algorithm updates the cluster memberships $\gamma_1, \gamma_2, \ldots, \gamma_n$ as follows:

$$\gamma_i^{(0)} = \operatorname*{argmin}_{1\le j\le k} D(\mathbf{x}_i, \boldsymbol{\mu}_j^{(0)}), \tag{5}$$

where $D(\cdot,\cdot)$ is defined in Eq. (2).

3. **Update cluster centers**. At this step, the algorithm updates the cluster centers as follows:

$$\mu_{jh}^{(1)} = \frac{1}{|C_j|}\sum_{\mathbf{x}\in C_j} x_h, \quad h = 1, 2, \ldots, d_1, \tag{6a}$$

$$\mu_{jh}^{(1)} = \text{mode}_h(C_j), \quad h = d_1 + 1, \ldots, d, \tag{6b}$$

where $C_j = \left\{\mathbf{x}_i \in X : \gamma_i^{(0)} = j\right\}$ for $j = 1, 2, \ldots, k$, and $\text{mode}_h(C_j)$ is the most frequent categorical value of the h-th attribute in cluster C_j. Let $A_{h1}, A_{h2}, \ldots, A_{h,m_h}$ be the distinct values the h-th attribute can take, where m_h is the number of distinct values the h-th attribute can take. Let $f_{ht}(C_j)$ be the number of records in cluster C_j, whose h-th attribute takes value A_{ht} for $t = 1, 2, \ldots, m_h$. That is,

$$f_{ht}(C_j) = |\{\mathbf{x} \in C_j : x_h = A_{ht}\}|, \quad t = 1, 2, \ldots, m_h.$$

Then

$$\text{mode}_h(C_j) = \operatorname*{argmax}_{1\le t\le m_h} f_{ht}(C_j), \quad h = d_1 + 1, \ldots, d.$$

4. Repeat Step 2 and Step 3 until the cluster memberships do not change between two iterations or the maximum number of iterations is reached.

Suppose that the cluster centers obtained from the k-prototypes algorithm are denoted by $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \ldots, \boldsymbol{\mu}_k$. Then we select the representative VA contracts $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ as follows:

$$\mathbf{z}_j = \operatorname*{argmin}_{\mathbf{x}\in X} D(\mathbf{x}, \boldsymbol{\mu}_j).$$

That is, the representative VA contract $\mathbf{z}_j$ is the VA contract in X that is closest to the cluster center $\boldsymbol{\mu}_j$. We assume that these k representative VA contracts are mutually distinct, that is,

$$D(\mathbf{z}_r, \mathbf{z}_s) > 0$$

for all $1 \le r < s \le k$.

3.2. Universal kriging for functional data

In this subsection, we show how to use universal kriging to accurately estimate the dollar Deltas of any contract in the VA portfolio using the representative VA contracts. This idea is similar to that in Fourier analysis in which the representative contracts may be viewed as basis functions and any other contract is projected onto the subspace spanned by the representative contracts. The approach described below is a slight modification of the universal kriging proposed in Caballero et al. (2013). The distance between two VA contracts is not the Euclidean distance used in the universal kriging method proposed by Caballero et al. (2013) because VA contracts contain categorical attributes. Other than the distance, the universal kriging method used in this paper is the same as the one proposed by Caballero et al. (2013). As a result, the derivation of the formulas is similar and hence omitted here.

Let $Z = \{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k\}$ now be the k distinct representative VA contracts selected in Section 3.1. Further let $\mathcal{X}_{\mathbf{z}_j}(t)$ be the dollar Delta at time t of the representative contract $\mathbf{z}_j$ along an outer loop scenario that is calculated in a nested simulation model. Denote

$$\mathcal{X}_Z(t) = \left(\mathcal{X}_{\mathbf{z}_1}(t), \mathcal{X}_{\mathbf{z}_2}(t), \ldots, \mathcal{X}_{\mathbf{z}_k}(t)\right)^T,$$

a k-dimensional column vector, where T is the transpose operator.

The dollar Delta at time t of an arbitrary contract $\mathbf{x}_i$ in the VA portfolio is expressed linearly as

$$\hat{\mathcal{X}}_{\mathbf{x}_i}(t) = \boldsymbol{\lambda}_i^T \mathcal{X}_Z(t) = \sum_{j=1}^{k}\lambda_{ij}\mathcal{X}_{\mathbf{z}_j}(t), \tag{7}$$

where $\boldsymbol{\lambda}_i = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik})^T$ is a vector of weights determined by the following system of linear equations:

$$\begin{pmatrix} \mathbf{A}(Z) & \mathbf{B}(Z) \\ \mathbf{B}^T(Z) & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\lambda}_i \\ \mathbf{v}_i \end{pmatrix} = \begin{pmatrix} \mathbf{A}(Z, \mathbf{x}_i) \\ \mathbf{B}(\mathbf{x}_i)^T \end{pmatrix}. \tag{8}$$

Here, $\mathbf{A}(Z)$ is a $k \times k$ matrix defined as

$$\mathbf{A}(Z) = \begin{pmatrix} \gamma(\mathbf{z}_1, \mathbf{z}_1) & \gamma(\mathbf{z}_1, \mathbf{z}_2) & \cdots & \gamma(\mathbf{z}_1, \mathbf{z}_k) \\ \gamma(\mathbf{z}_2, \mathbf{z}_1) & \gamma(\mathbf{z}_2, \mathbf{z}_2) & \cdots & \gamma(\mathbf{z}_2, \mathbf{z}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(\mathbf{z}_k, \mathbf{z}_1) & \gamma(\mathbf{z}_k, \mathbf{z}_2) & \cdots & \gamma(\mathbf{z}_k, \mathbf{z}_k) \end{pmatrix}, \tag{9}$$

where $\gamma(\cdot,\cdot)$ is a kriging kernel function to be determined. The column vector $\mathbf{A}(Z, \mathbf{x}_i)$ is defined as

$$\mathbf{A}(Z, \mathbf{x}_i) = \begin{pmatrix} \gamma(\mathbf{z}_1, \mathbf{x}_i) \\ \gamma(\mathbf{z}_2, \mathbf{x}_i) \\ \vdots \\ \gamma(\mathbf{z}_k, \mathbf{x}_i) \end{pmatrix}. \tag{10}$$

Note that in Eqs. (9) and (10), the kriging kernel function $\gamma(\cdot,\cdot)$ can be isotropic or anisotropic. In our implementation, we use an isotropic kriging kernel defined as

$$\gamma(\mathbf{x}, \mathbf{y}) = \gamma(D(\mathbf{x}, \mathbf{y})),$$

where $\gamma(\cdot)$ is a semivariogram to be defined later and $D(\cdot,\cdot)$ is the distance function defined in Eq. (2). The use of the isotropic kriging kernel leads to a linear system that can be solved easily.

Furthermore, the $k \times q$ matrix $\mathbf{B}(Z)$ is defined as

$$\mathbf{B}(Z) = \begin{pmatrix} f_0(\mathbf{z}_1) & f_1(\mathbf{z}_1) & \cdots & f_d(\mathbf{z}_1) \\ f_0(\mathbf{z}_2) & f_1(\mathbf{z}_2) & \cdots & f_d(\mathbf{z}_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(\mathbf{z}_k) & f_1(\mathbf{z}_k) & \cdots & f_d(\mathbf{z}_k) \end{pmatrix}, \tag{11}$$

where $f_1(\cdot), \cdots, f_d(\cdot)$ are d attribute functions, each of which quantifies an attribute of a VA contract, and $f_0(\cdot)$ may be a constant or an additional attribute function. See Section 4 for a detailed description. The row vector $\mathbf{B}(\mathbf{x}_i)$ is defined as

$$\mathbf{B}(\mathbf{x}_i) = \begin{pmatrix} f_0(\mathbf{x}_i) & f_1(\mathbf{x}_i) & \cdots & f_d(\mathbf{x}_i) \end{pmatrix}. \tag{12}$$

There are several choices for the semivariogram function $\gamma(h)$. In a typical application of a universal kriging method, the locations (i.e., the representative contracts in our context) are predetermined and h is often the Euclidean distance between two locations. Other choices include the exponential function, the Gaussian function, and spherical function. We tested all three semivariogram functions and found that the spherical function works well for VA portfolios. The spherical function has the following form:

$$\gamma(h; a, b, c) = \begin{cases} 0, & \text{if } h = 0; \\ a + b\left[\dfrac{3h}{2c} - \dfrac{1}{2}\left(\dfrac{h}{c}\right)^3\right], & \text{if } 0 < h \le c; \\ a + b, & \text{if } h > c. \end{cases} \tag{13}$$

To estimate the parameters in the spherical function, we minimize the objective function

$$P(a, b, c) = \sum_{l=1}^{L} \left[\gamma(h_l; a, b, c) - \hat{\gamma}(h_l)\right]^2, \tag{14}$$

where the empirical semivariogram function is defined as

$$\hat{\gamma}(h) = \frac{1}{2|N(h)|} \sum_{(\mathbf{x},\mathbf{y}) \in N(h)} \sum_{s=1}^{S} \left(\mathcal{X}_{\mathbf{x}}(t_s) - \mathcal{X}_{\mathbf{y}}(t_s)\right)^2, \tag{15}$$

where S is the number of time steps,

$$N(h) = \{(\mathbf{x}, \mathbf{y}) : D(\mathbf{x}, \mathbf{y}) = h, : \mathbf{x} \in Z, \mathbf{y} \in Z\},$$

and $|N(h)|$ is the number of elements in $N(h)$.

With the estimated semivariogram function $\gamma(h)$, the system of linear equations (i.e., Eq. (8)) is completely identified and ready to be solved. As a result, the weights λ_{ij}, $j = 1, \ldots, k$, of contract $\mathbf{x}_i$, $i = 1, \ldots, n$, are obtained. The dollar Delta of the portfolio at time t can be calculated as

$$\hat{\mathcal{X}}_X(t) = \sum_{i=1}^{n} \boldsymbol{\lambda}_i^T \mathcal{X}_Z(t) = \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_{ij} \mathcal{X}_{\mathbf{z}_j}(t). \tag{16}$$

From the above equation it is clear that the weights λ_{ij} are independent of t, a key advantage that makes the estimation efficient. However, we would like to emphasize that from Eq. (15) the weights λ_{ij} depend on the dollar Deltas of the selected representative VA contracts at all times t.

If we are interested in only the aggregated dollar Delta, then we can calculate $\hat{\mathcal{X}}_X(t)$ in Eq. (16) efficiently by observing that

$$\begin{pmatrix} \mathbf{A}(Z) & \mathbf{B}(Z) \\ \mathbf{B}^T(Z) & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \sum_{i=1}^{n} \boldsymbol{\lambda}_i \\ \sum_{i=1}^{n} \mathbf{v}_i \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} \mathbf{A}(Z, \mathbf{x}_i) \\ \sum_{i=1}^{n} \mathbf{B}(\mathbf{x}_i)^T \end{pmatrix}. \tag{17}$$

In this way, we only need to solve the linear equation system once.

We remark that one assumption of the universal kriging for functional data (UKFD) method is that the random variable $\mathcal{X}_{\mathbf{x}}(t)$ is weakly stationary (Caballero et al., 2013). The guarantees embedded in VAs are not stationary because the value of the guarantee embedded in a VA contract is linearly dependent on the account value of the contract. To use the UKFD method for VA valuation, we scale all the VA contracts such that they have approximately the same account value. In other words, we determine the scaling factor θ_i for contract $\mathbf{x}_i$ such that the following objective function

$$\sum_{j \in A} (\theta_i x_{ij} - c_j)^2$$

is minimized, where A is the set of dollar-valued attributes (e.g., account values) and c_j is the average of the jth attribute values of all contracts in the portfolio. Thus, each scaled contract has approximately the same account value. In particular, if the contracts have only one dollar-valued attribute, then all scaled contracts have the same account value. After the quantity of a scaled contract is estimated by the UKFD method, we scale the quantity back to be the estimation of the original contract.

4. Numerical experiments

To evaluate the effectiveness of the proposed method, we generate 100,000 synthetic VA contracts with GMDB and GMWB riders and use the method to calculate the dollar Deltas at each time step along several real world scenarios. Note that the proposed method can also be used for other quantities of interest such as dollar Rhos (sensitivity to interest rates), economic capital (Zhuo and Park, 2006; IAA, 2010), etc.

Dynamic hedging (see Boyle and Hardy, 1997 for an example) is a popular risk management approach for VA and is adopted by many insurance companies. To reflect the hedging in quarterly financial statements, insurance companies use nested simulation models to calculated the net positions by offsetting the liabilities with the hedge portfolio. In particular, insurance companies calculate the dollar Delta of the VA liabilities at each time step along many real world scenarios and then determine the Futures position at each time step that can offset the dollar Delta appropriately. In this section, we present some test results of applying the UKFD method (cf. Section 3.2) to estimate the dollar Deltas along real world scenarios for a synthetic VA portfolio of 100,000 contracts generated randomly.

The attributes of a VA and their ranges of values are given in Table 1. We consider only two types of guarantees: GMDB and GMWB. Every contract contains a GMDB rider. However, the GMWB rider is not included in all contracts. The total account value of this portfolio is 25,552,267,607 dollars.

Table 1
Distributions and ranges of VA attributes.

Attribute	Values	Distribution
Guarantee type	{GMDB, GMDB + GMWB}	50%, 50%
Gender	{Male, Female}	50%, 50%
Age	{20, 21, 22, . . . , 60}	Uniform
Account value	[10 000, 500 000]	Uniform
GMWB withdrawal rate	{0.04, 0.05, 0.06, 0.07, 0.08}	Uniform
Maturity	{10, 11, 12, . . . , 25}	Uniform

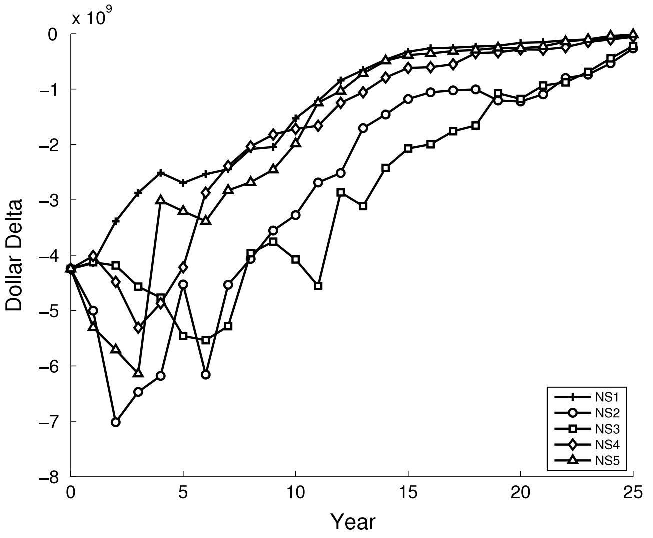

Fig. 2. The annual dollar Deltas of the portfolio along five outer loop scenarios calculated by the nested simulation model.

Let $\mathbf{x}$ be a scaled VA contract. Then we use 6 attribute functions (i.e., $d = 5$) defined as follows:

$$f_0(\mathbf{x}) = 1, \quad f_1(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \text{ is male;} \\ 1, & \text{if } \mathbf{x} \text{ is female,} \end{cases}$$

$$f_2(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \text{ contains GMDB only;} \\ 1, & \text{if } \mathbf{x} \text{ contains GMWB,} \end{cases}$$

$$f_3(\mathbf{x}) = \text{normalized age of } \mathbf{x},$$

$$f_4(\mathbf{x}) = \text{normalized guaranteed withdrawal rate of } \mathbf{x},$$

$$f_5(\mathbf{x}) = \text{normalized maturity of } \mathbf{x}.$$

Since all contracts are scaled to have the same account value, we do not need to include the account value variable in the UKFD method. But the scaling factors are used to scale the dollar Deltas estimated by the UKFD method back to reflect the account value of the original contracts.

Since the annual dollar Deltas along each outer loop scenario can be estimated in the same way, we illustrate the UKFD method for only five outer loop scenarios. To do that, we first use a nested simulation model (cf. Appendix) to calculate the dollar Deltas at all annual time steps of the five outer loop scenarios. The annual dollar Deltas of the portfolio along five outer loop scenarios are plotted in Fig. 2. Note that we used the regime-switching lognormal model of Hardy (2001) to generate outer loop scenarios. The resulting annual dollar Deltas of the portfolio are plotted in Fig. 2. It took a single CPU core 32 520.68 s (or 9 h 2 min) to calculate these dollar Deltas.

From Fig. 2 we see that the dollar Deltas along all outer loop scenarios decrease to zero in magnitude. Since all the synthetic VA contract have maturities between 10 and 25 years, the dollar Deltas of the portfolio will be zero after 25 years. In addition, the dollar Deltas decrease in magnitude along outer loop scenarios because about half of the contracts in the portfolio have the GMWB feature.

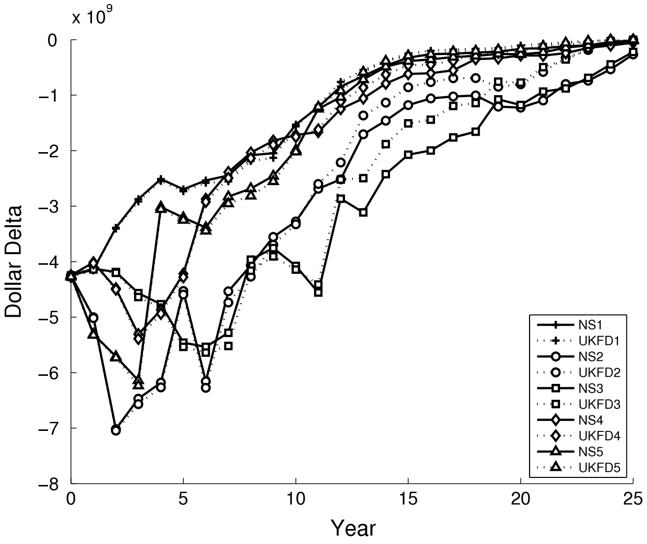

Fig. 3. The annual dollar Delta along five outer loop scenarios estimated by the UKFD method with 512 representative contracts.

We tested the UKFD method for different numbers of representative contracts. In particular, we tested the UKFD method with the target number of representative contracts being 550, 1100 and 2200, respectively. These numbers represents roughly 0.5%, 1%, and 2% of the total number of contracts in the VA portfolio. In our first test case, we run the k-prototypes algorithm with 550 clusters. Since all the 100,000 contracts are scaled to have the same account value, some scaled contracts are identical. As a result, some of the 550 representative contracts are identical. After the duplicate contracts from the 550 representative contracts are removed, the number of mutually distinct representative contracts is 512. We calculate the annual dollar Deltas along the five outer loop real world scenarios for each of the 512 representative contracts using the nested simulation model. Then we estimate the annual dollar Deltas of every scaled contract in the portfolio using the UKFD method. Note that the tests in this section were conducted with all the weights set to 1 in the distance calculation.

Fig. 3 plots the annual dollar Deltas along the five outer loop scenarios estimated by the UKFD method with 512 representative contracts. To compare the numbers, we also plotted the annual dollar Deltas calculated by the nested simulation model in Fig. 3.

From Fig. 3 we see that along the second and the third outer loop scenarios, the annual dollar Deltas estimated by the UKFD method do not match well those calculated by the nested simulation model. Some of the differences along the second and the third outer loop scenarios are large. However, the differences along the other three outer loop scenarios are relatively small. Fig. 4(a) shows the mean squared errors (MSE) at each point in time along the five outer loop scenarios when 512 representative policies were selected. We see that the MSE increases in time. Fig. 4(b) shows the histogram of the differences of dollar Deltas at time 0 across all the 100,000 policies in the portfolio under the first outer loop scenario. The histogram exhibits a normal behavior. From Fig. 4(b) we see that most of the differences of the dollar Deltas are close to zero. To see whether the account value has an impact on the accuracy of the UKFD method, we plotted the account values against the differences of the dollar Deltas calculated from the simulation model and the UKFD method. The resulting scatter plot is shown in Fig. 5(a).

From Fig. 5(a), we see that variance of the differences increases with the account values. In addition, some differences corresponding to contracts with large account values are close to zero. If we normalize the differences by the corresponding account

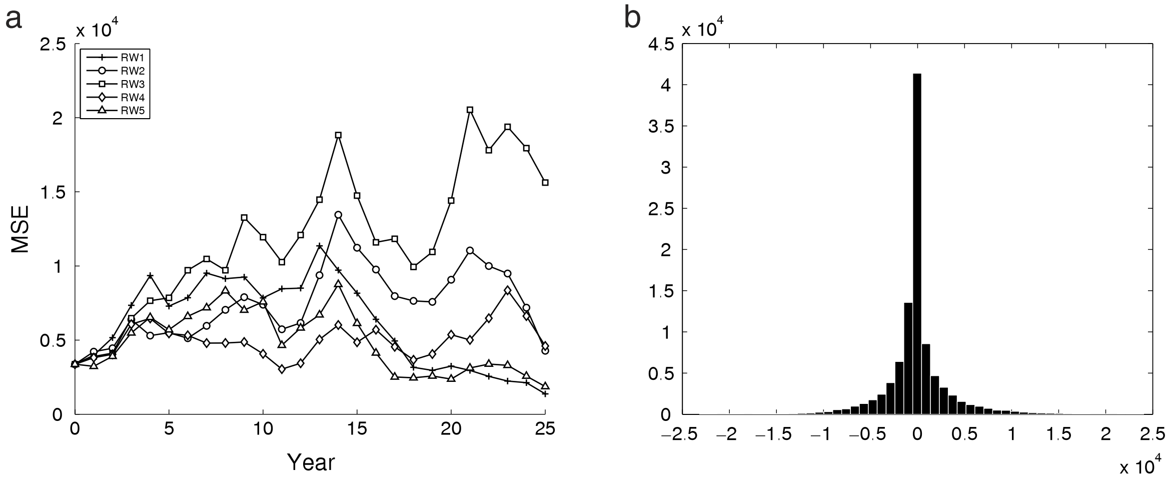

Fig. 4. Figure (a) shows the mean squared errors (MSE) at each point in time along the five outer loop scenarios. Figure (b) shows the histogram of the differences of dollar Deltas at time 0. These results were based on 512 representative policies.

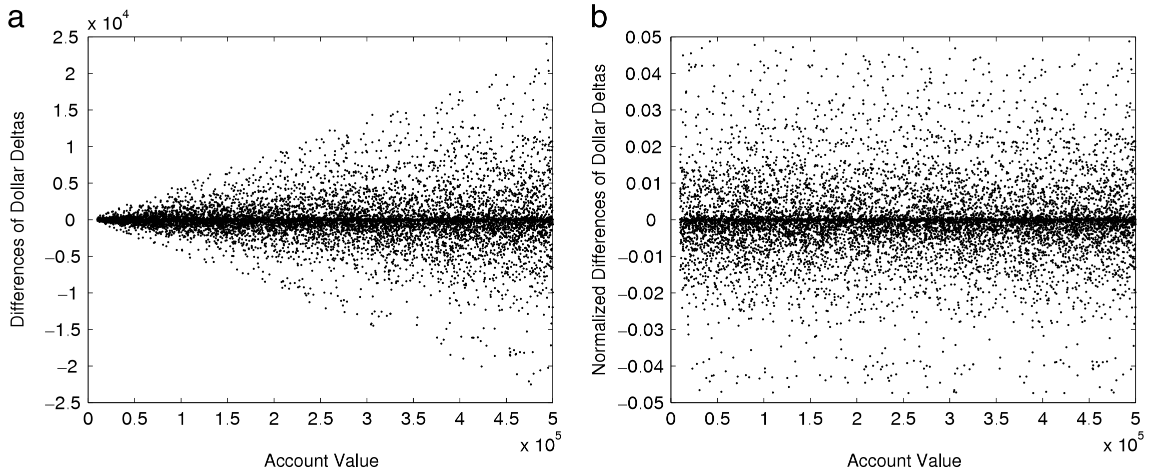

Fig. 5. Figure (a) is the scatter plot between the account values and the differences of the dollar Deltas calculated by the simulation model and those estimated by the UKDF model. Figure (b) is the scatter plot between the account values and the normalized differences of the dollar Deltas. These results were based on 512 representative policies.

values, the variance of the normalized differences is almost the same for different account values as shown in Fig. 5(b). The reason is that we scaled the contracts by account values before applying the UKFD method and re-scaled the estimated dollar Deltas back afterwards.

In our second test case, we run the *k*-prototypes algorithm with 1100 clusters and obtained 1100 representative contracts. Some of the 1100 representative contracts are identical. After the duplicate contracts from the 1100 representative contracts are removed, the number of mutually distinct representative contracts is 987. We calculate the annual dollar Deltas along the five outer loop real world scenarios for the 987 representative contracts using the nested simulation model. Then we estimate the annual dollar Deltas of every scaled contract in the portfolio using the UKFD method. Fig. 6 shows the annual dollar Deltas along the five outer loop real world scenarios estimated by the UKFD method with the 987 representative VA contracts.

From Fig. 6 we see that the annual dollar Deltas estimated by the UKFD method approximate those calculated by the nested simulation model well. Comparing Figs. 6 and 3, we further see that the differences in this test case are smaller than those in the first test case. Again, we compute the MSEs and plot the histogram of the dollar Delta differences at time 0 across the entire portfolio under the first outer loop scenario. The results are presented in Fig. 7.

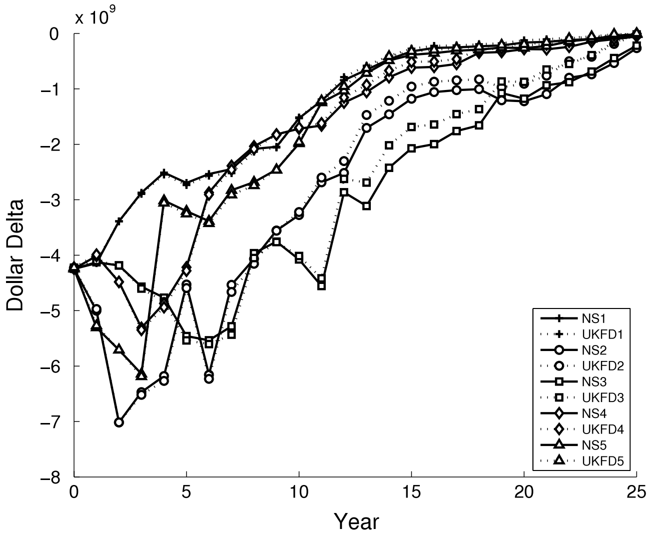

Fig. 6. The annual dollar Delta along five outer loop scenarios estimated by the UKFD method with 987 representative contracts.

Comparing Figs. 7(a) to 4(a), it is clear that the MSEs produced by the 987 representative policies and by the 512 representative

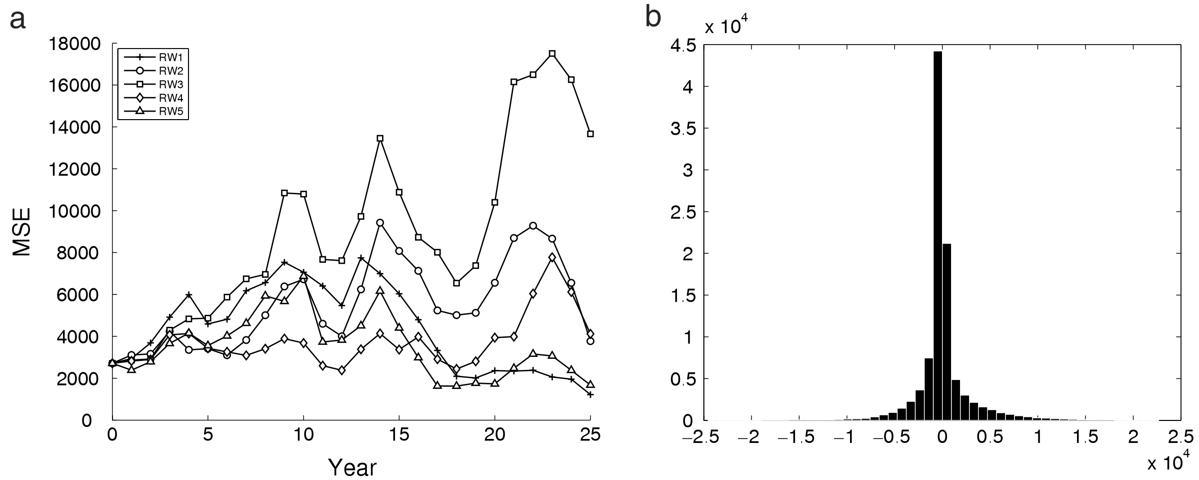

Fig. 7. Figure (a) shows the mean squared errors (MSE) at each point in time along the five outer loop scenarios. Figure (b) shows the histogram of the differences of dollar Deltas at time 0. These results were based on 987 representative policies.

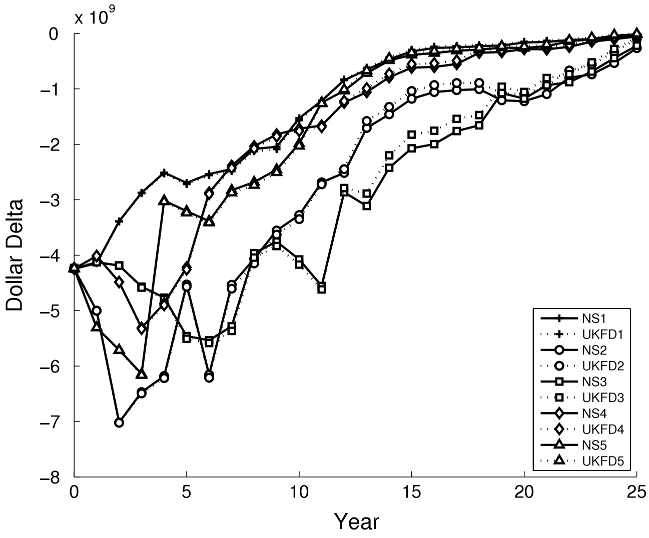

Fig. 8. The annual dollar Delta along five outer loop scenarios estimated by the UKFD method with 1845 representative contracts.

policies have similar patterns, but the MSEs produced by the former are about 30% lower than those by the latter. Improving in accuracy is further confirmed when comparing Figs. 7(b) to 4(b).

In our third test, we run the k-prototypes algorithm with 2200 clusters. The number of mutually distinct representative VA contract is 1845. In this case, the annual dollar Deltas along the five outer loop scenarios estimated by the UKFD method are shown in Fig. 8. Comparing Figs. 6 and 8, we see that the differences of the test with 1845 representative contracts are smaller than those of the test with 987 representative contracts.

We again produce the MSEs and the histogram of the dollar Delta differences at time 0 across the entire portfolio under the first outer loop scenario.

Fig. 9(a) shows the further decrease in MSE by about 25%. The histogram in Fig. 9(b) has a smaller standard deviation. From Figs. 4, 7, and 9, we see that the higher the number of representative policies is, the more accurate approximation we produce.

In order to further show the impact of the number of representative contracts on the accuracy of the UKFD method, we also calculate the mean absolute percentage error of the annual dollar Deltas along the five outer loop scenarios from the three test cases. The mean absolute percentage error is defined as the weighted average of the absolute percentage errors between the annual dollar Deltas estimated by the UKFD method and the corresponding ones calculated by the simulation model at all the 30 time steps. Mathematically, it is given by:

Table 2
The mean absolute percentage errors of annual dollar Deltas estimated by the UKFD method and those calculated by the simulation model. RW1 to RW5 denote the five outer loop real world scenarios.

Number of contracts	Mean absolute percentage error				
	RW1	RW2	RW3	RW4	RW5
512	4.49%	8.75%	10.17%	5.42%	4.18%
987	2.82%	5.75%	6.67%	3.52%	2.62%
1845	1.72%	3.31%	3.90%	2.07%	1.59%

Table 3
Computing times used by the clustering algorithm, the nested simulation model, and the UKFD method. The numbers are in seconds.

	Number of contracts			Entire portfolio
	512	987	1845	100,000
k-prototypes	4.52	3.83	4.22	NA
Nested simulation	166.50	317.75	580.18	32 520.68
UKFD	48.16	100.71	199.05	NA
Total	219.18	422.29	783.45	32 520.68

$$MAPE = \sum_{t=1}^{T} \frac{|\Delta_t|}{\sum_{s=1}^{T} |\Delta_s|} \frac{|\Delta_t - \hat{\Delta}_t|}{|\Delta_t|} = \frac{\sum_{t=1}^{T} |\Delta_t - \hat{\Delta}_t|}{\sum_{s=1}^{T} |\Delta_s|},$$

where Δ_t and $\hat{\Delta}_t$ are the dollar Deltas at time t calculated from the simulation model and the UKFD method, respectively. The numerical results are presented in Table 2.

From Table 2 we see that the mean absolute percentage error decreases when we increase the number of representative VA contracts. For example, the mean absolute percentage error for the first outer loop is 4.49% when we use 512 representative contracts. If we use 987 representative contracts, the mean absolute percentage error decreases to 2.82%.

In the next table we show the time used by the UKFD method when different numbers of representative contracts are used.

From Table 3 we see that it took the k-prototypes algorithm 3.84 s to select 987 representative VA contracts. It took the nested simulation model 317.75 s (or 5 min 17.75 s) to calculate the dollar Deltas along five outer loop scenarios for the 987 representative contracts. To estimate the dollar Deltas of the whole portfolio

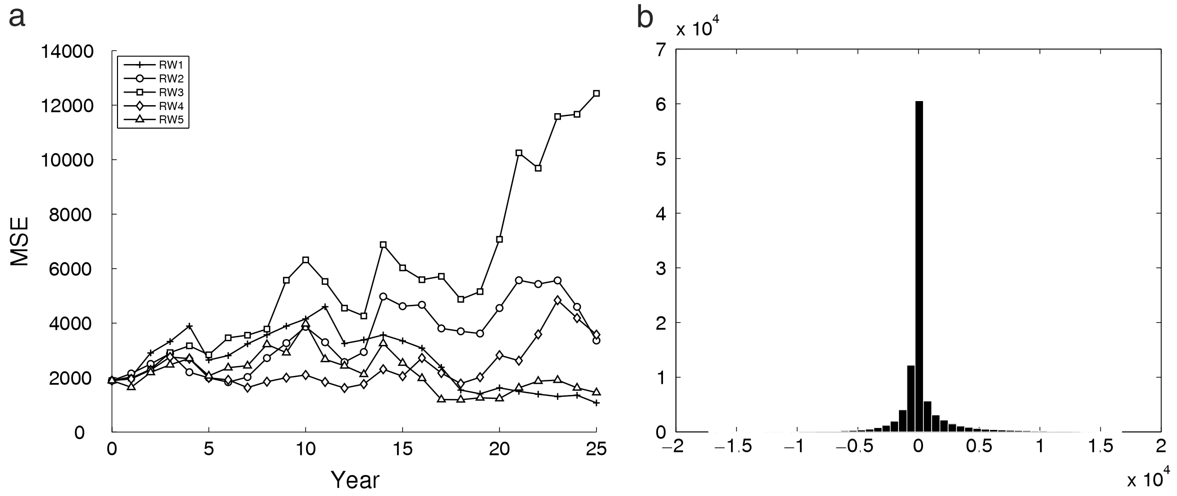

Fig. 9. Figure (a) shows the mean squared errors (MSE) at each point in time along the five outer loop scenarios. Figure (b) shows the histogram of the differences of dollar Deltas at time 0.These results were based on 1845 representative policies.

of 100,000 contracts, the UKFD method used 100.71 s (or 1 min 40.71 s). The total time used by the process is 422.29 s, which is 1.3% of the time used by the nested simulation model to calculate the dollar Deltas of the whole portfolio. When 1845 representative contracts are used, the total time used by the process is 783.45 s, which is 2.4% of the time used by the nested simulation model to calculate the dollar Deltas of the whole portfolio.

Our test results above show that the UKFD method is able to efficiently approximate the annual dollar Deltas along the outer loop scenarios under nested simulation. When the number of representative contracts increases, the absolute differences between the annual dollar Deltas estimated by the UKFD method and those calculated by the simulation model decrease.

5. Further numerical experiments and discussion

Our numerical results in Section 4 show that using higher number of clusters produces more accurate approximation. However, it also requires longer running time. There is a tradeoff between how many clusters to use and how much running time to spend. One practical approach may be to try different numbers of clusters to reach a desirable approximation accuracy at time 0 of the outer loop. For examples, insurance companies may try 100, 200, 400, 800, 1600, 3200 clusters and choose the number of clusters that can produce the acceptable approximations at time 0 of the outer loop.

The kriging method in this paper requires calculating the distance not only between all representative policies but also the distance between the representation policies and other policies in the portfolio. If we use 1000 representative policies for a portfolio of 100,000 policies, then we need to calculate 10^8 distances. In our models, we used a weighted distance and hence the computing time is linear to the number of variables. The weighted distance allows insurance companies to assign a weight to a variable to indicate the importance of the variable. As discussed in Section 3.1, two approaches are proposed to choose weights automatically. In the first approach, we choose the weight of each numerical variable to be the reciprocal of the sample variance and the weights of all categorical variables to be a constant α. To examine the impact of the weight on the categorical variables, we rerun the test in Section 4 with 2200 initial representative policies and let $\alpha = 0.5$ or 2.

Figs. 10 and 11 show the MSEs and histograms from these two test cases. When we change the weight of the categorical variables to be smaller or larger, the number of representative policies remains almost the same. Comparing the three Figs. 9–11, we see that the estimation accuracy does not change either. We may conclude that the UKFD method is insensitive to the weights of the categorical variables.

In the second approach, we assign weights based on their ranges. We tested the approach and found that if we assign weights to numerical variables based on their ranges, the clustering algorithm produces a smaller number of distinct representative policies because such a weighting method gives more weights to categorical variables. For example, if we specify 2200 clusters in the clustering algorithm with this weighting method, the clustering algorithm produces only 1410 distinct representative policies. Fig. 12(a) shows the MSEs along the five outer loop scenarios produced by the 1410 representative policies. Comparing Figs. 7(a) and 12(a), we see that the MSEs produced by the 1410 representative policies with this weighting method are similar to those produced by the 987 representative policies with the equal weights. As a result, the range-based approach seems to be not as efficient as the sample variance based approach.

We may further extend the sample variance based distance by incorporating co-variances and use the so-called Mahalanobis distance:

$$\sqrt{(\mathbf{x}-\mathbf{y})'\Sigma^{-1}(\mathbf{x}-\mathbf{y})},$$

where Σ is the covariance matrix. In this situation, the computing time is quadratic to the number of variables. As a result, the number of operations required to calculate the Mahalanobis distance is quadratic to the dimension of the data points. Such a distance might not be efficient in practice because real variable annuities have 50 to 150 variables.

6. Concluding remarks

When the underlying portfolio of VA contracts is large, calculating the dollar Deltas (or other quantities of interest) at all the time steps along outer loop scenarios under nested simulation is very time consuming (Reynolds and Man, 2008; Fox, 2013). In this paper, we proposed a method based on functional data analysis to estimate the annual dollar Deltas along outer loop scenarios under the nested simulation framework. The proposed method consists of three components:

1. Using a clustering technique to select a small set of representative VA contracts from a large VA portfolio;

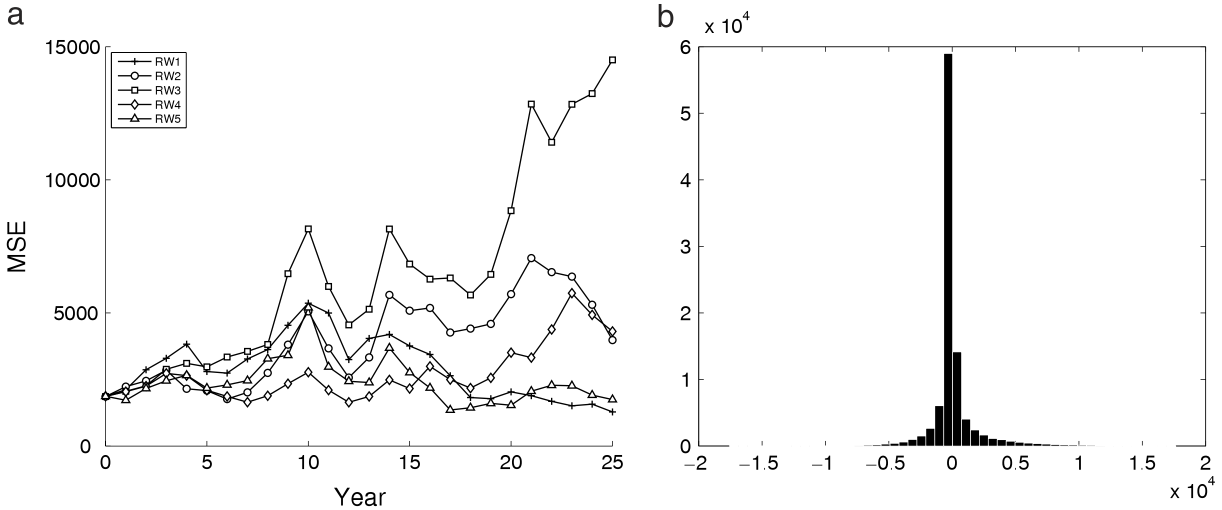

Fig. 10. Figure (a) shows the mean squared errors (MSE) at each point in time along the five outer loop scenarios. Figure (b) shows the histogram of the differences of dollar Deltas at time 0. These results were based on 1842 representative policies, the sample variance based distance and a weight of 0.5 to all categorical variables.

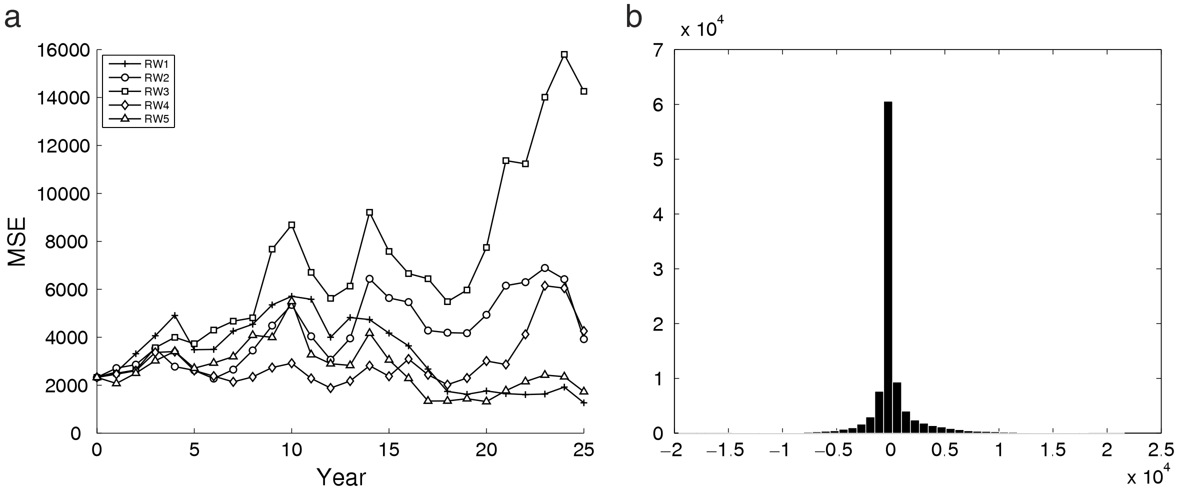

Fig. 11. Figure (a) shows the mean squared errors (MSE) at each point in time along the five outer loop scenarios. Figure (b) shows the histogram of the differences of dollar Deltas at time 0. These results were based on 1802 representative policies, the sample variance based distance and a weight of 2 to all categorical variables.

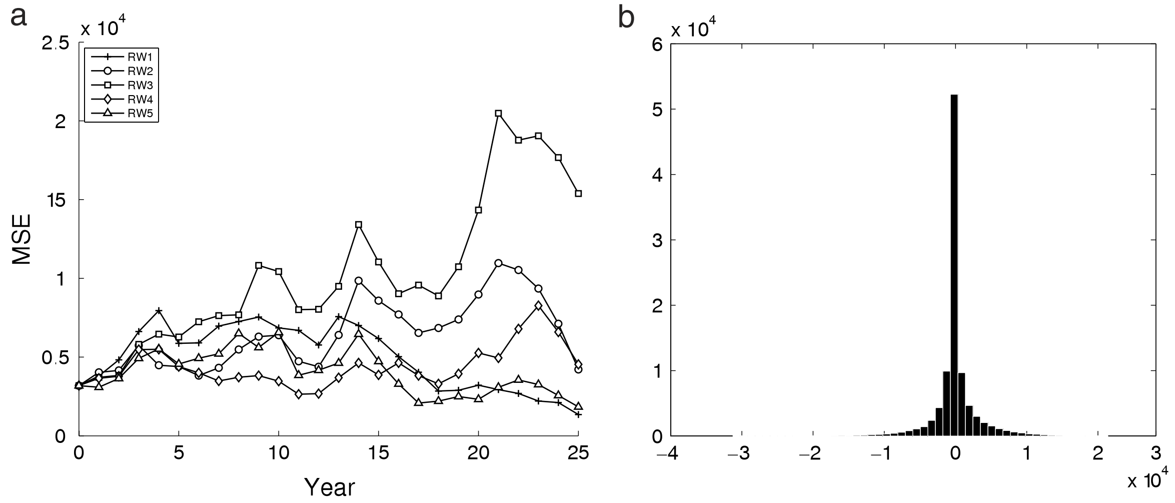

Fig. 12. Figure (a) shows the mean squared errors (MSE) at each point in time along the five outer loop scenarios. Figure (b) shows the histogram of the differences of dollar Deltas at time 0. These results were based on 1410 representative policies and the range-based distance.

2. Calculating the Greeks along outer loop scenarios for every representative VA contract under a nested simulation model;
3. Estimating the Greeks along outer loop scenarios for every VA contract in the large VA portfolio.

We used the *k*-prototypes algorithm to select representative VA contracts from a large VA portfolio and used the universal kriging for functional data (UKFD) to estimate the Greeks along outer loop scenarios for all contracts in the VA portfolio. Since the time-consuming nested simulation is only applied to a small number of representative VA contracts and the UKFD method is fast, the proposed method can reduce the computation time significantly.

Our test results show that the proposed method works well in terms of accuracy and speed. In particular, they show that we may use a very small subset, such as 1%, of VA contracts in a VA portfolio as representative contracts to achieve a fairly satisfactory accuracy. As we increase the number of representative contracts, the accuracy can be drastically improved but at the expense of computation time.

In our test cases, we used annual time steps in the nested simulation model. The method can be also applied to cases when monthly steps or variable-time steps are used. When the method is applied to monthly data, the gain in computation time will be even more significant because the UKFD method's performance does not depend on the number of time steps. Although we used dollar Deltas to illustrate the effectiveness of the proposed method, we can apply the same method to estimate other quantities of interest such as dollar Rho, cash flows, and risk measures.

All components of the proposed method can be changed or modified. As a result, there may be several directions for future research. One would be to explore other methods for selecting representative VA contracts. For example, we can use a different clustering method with a different distance (e.g., Ji et al., 2012) to select representative VA policies. Under the current clustering method, the set of representative contracts does not include the boundary contracts that have extreme attribute values. The boundary contracts might be critical to the accuracy of the kriging method (Kleijnen and van Beers, 2004).

Another direction may be to test the proposed method under a realistic nested simulation model. In the nested simulation model used in this paper, we considered only one investment fund and simple guarantee features. Real VA contracts have several investment funds and contain complex guarantee features.

Acknowledgments

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Appendix. A nested simulation model

In this appendix, we briefly describe a nested simulation used to test the effectiveness of the proposed method. In the nested simulation model, we use two loops: an outer loop and an inner loop. The outer loop uses real world scenarios, which are generated by the regime switching lognormal model (Hardy, 2001, 2003; Hardy et al., 2006; Cheung and Yang, 2005; Lin et al., 2009; Augustyniak and Boudreault, 2012; Hartman and Groendyke, 2013). The inner loop uses risk neutral scenarios, which are generated by the lognormal model (Black and Scholes, 1973). To determine the value of guarantees at a time step along an outer scenario, we follow the Monte Carlo method proposed by Bauer et al. (2008). To simplify our discussion, we consider only the GMDB rider and the GMWB rider.

Table A.4
A list of symbols used to describe the nested simulation model.

Symbol	Description
S_t	The underlying mutual fund at time t of the VA
A_t	The account value at time t
W_t	The withdrawal benefit at time t
D_t	The death benefit at time t
G_t^W	The remaining total amount that can be withdrawn after time t
G_t^E	The maximum amount that can be withdrawn annually
G_t^D	The guaranteed minimum death benefit at time t
x_W	The proportion of the premium that can be withdrawn annually
T	The maturity of the contract

For VA contracts with the GMDB rider and GMWB rider, there are two possible types of events (Bauer et al., 2008):

- the policyholder withdraws money as a guaranteed withdrawal of the GMWB rider;
- the policyholder dies.

We use $(\cdot)_t^-$ and $(\cdot)_t^+$ to denote the value of a state variable (e.g., A_t) immediately before and after the occurrence of such events, respectively. Table A.4 shows a list of symbols used to describe the nested simulation model.

In our Monte Carlo valuation of the VA contracts, we assume the following:

- a real world scenario is simulated by a two-regime switching lognormal model $(\mu_1, \mu_2, \sigma_1, \sigma_2, p_{12}, p_{21})$ (Hardy, 2001). We used the parameters from (Hardy, 2001):

  $$(\mu_1, \mu_2, \sigma_1, \sigma_2, p_{12}, p_{21}) = (0.0126, -0.0185, 0.0350, 0.0748, 0.0398, 0.3798).$$

  Mathematically, a real world scenario is simulated as

  $$R_0 = 1, \qquad R_t = R_{t-1} \exp\left(\left[\mu - \frac{1}{2}\sigma^2\right] + \sigma Z\right),$$

  where the values of μ and σ are determined according to the transition probabilities p_{12} and p_{21}. Since the parameter values were estimated from monthly data, we generate monthly real world scenarios and then convert them to annual scenarios;
- a risk-neutral path is simulated as

  $$S_0 = 1, \qquad S_t = S_{t-1} \exp\left(\left[r - \frac{1}{2}\sigma^2\right] + \sigma Z\right)$$

  for $t = 1, 2, \ldots, 40$, where r is the interest rate, σ is the volatility of the underlying mutual fund, and Z is a standard normal random variable. In our tests, we use $r = 3\%$ and $\sigma = 20\%$ and simulate 1000 paths;
- for a contract with the GMWB rider, the policyholder takes maximum annual withdrawals;
- all the events happen only at anniversary date;
- there are no fees;
- there are no lapses;
- the mortality follows the 1996 IAM mortality tables provided by the Society of Actuaries.

At time step s along an outer loop, we have

$$G_s^W = A_s, \qquad G_s^E = x_W A_0, \qquad G_s^D = A_s,$$

where A_s is the account value at time s that is projected based on the real world scenario from time 0 to time s. The projection is done recursively as follows. At the beginning, A_0 is the money deposited by a policyholder. At the end of year 1, the account value after withdrawal is $A_1 = \max(A_0 R_1 - x_W A_0, 0)$. At the end of year 2, the account value after withdrawal is $A_2 = \max(A_1 * R_2/R_1 - A_0 x_W, 0)$. Repeating this procedure, we can obtain A_s.

At time step s along the outer loop, we need to calculate the value of the guarantee using risk-neutral paths. In our nested simulation model, we use truncated risk-neutral scenarios. That is, at time step s along the outer loop, we use the risk-neutral paths from time $s+1$ to T. For $t = s+1, s+2, \ldots, T-1$, the evolution of the state variables between t^+ and $(t+1)^-$ is described as follows. The account value evolves as

$$A_{t+1}^- = A_t^+ \frac{S_{t+1}}{S_t}.$$

The guaranteed minimum death benefit, the maximum amount that can be withdrawn annually, and the remaining total amount that can be withdrawn do not change, i.e.,

$$G_{t+1}^{D-} = G_t^{D+}, \qquad G_{t+1}^{E-} = G_t^{E+}, \qquad G_{t+1}^{W-} = G_t^{W+}.$$

The evolution of the state variables between $(t+1)^-$ and $(t+1)^+$ is described as follows. The death benefit at time $t+1$ is calculated as

$$D_{t+1} = \max\left(0, G_{t+1}^{D-} - A_{t+1}^-\right).$$

Since we assume the policyholder takes maximally available withdrawal annually, the withdrawal amount at year $t+1$ is given by

$$E = \min\left(G_{t+1}^{E-}, G_{t+1}^{W-}\right),$$

and the maximum amount that can be withdrawn annually does not change, i.e., $G_{t+1}^{E+} = G_{t+1}^{E-}$. The withdrawal benefit at time $t+1$ is given by

$$W_{t+1} = \max\left(0, E - A_{t+1}^-\right).$$

The account value becomes

$$A_{t+1}^+ = \max\left(0, A_{t+1}^- - E\right).$$

The remaining total amount that can be withdrawn after time $t+1$ becomes

$$G_{t+1}^{W+} = \max\left(0, G_{t+1}^{W-} - E\right).$$

The guaranteed minimum death benefit will be adjusted pro rata as follows

$$G_{t+1}^{D+} = \frac{A_{t+1}^+}{A_{t+1}^-} G_{t+1}^{D-}.$$

Then the present value of the GMDB and the GMWB benefits at time s is given by

$$V(S_1, S_2, \ldots, S_{40}) = \sum_{t=s+1}^{T+1} {}_{t-1}p_{x_0}(1 - q_{x_0+t-1}) W_t e^{-rt} + \sum_{t=s+1}^{T+1} {}_{t-1}p_{x_0} q_{x_0+t-1} D_t e^{-rt}, \tag{A.1}$$

where x_0 is the age of the policyholder. The value of the GMDB rider and the GMWB rider is the average of $V(S_1, S_2, \ldots, S_{40})$ along all risk-neutral paths. If a contract does not have the GMWB rider, the above formula still applies by letting the withdrawal rate to be zero.

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