

Cutting Through the Noise: Using Deep Neural Network Metamodels for High Dimensional Nested Simulation

Tony Wirjanto

Mingbin Feng

Xintong Li

Abstract

Deep neural network models have gained great success in many applications, but their adoption in financial and actuarial applications has been received by regulators with some trepidation. The lack of transparency and interpretability of these models leads to skepticism about their resilience and reliability, which are important factors to ensure financial stability and insurance benefit fulfillment. In this study, we use stochastic simulation as a data generator to examine deep neural networks under controlled settings. Our study shows interesting findings in fundamental questions like “What do deep neural networks learn from noisy data?” and “How well do they learn from noisy data?”, and the findings provides justifications for using deep neural networks in actuarial applications. Based on our findings, we propose an efficient nested simulation procedure that uses deep neural networks as metamodels to estimate risk measures of hedging errors for variable annuities. The proposed procedures use deep neural networks as metamodels to provide accurate loss predictions and concentrate simulation budget on tail scenarios while maintaining transparency in the estimation step, and we provide practical guidelines to extend our generic procedures for other financial and actuarial applications.

1 Introduction

Deep neural networks (??) have attracted attentions of researchers and practitioners due to their success in solving real-world supervised learning tasks such as AlphaGo (?) and ChatGPT (?). Since the first artificial neural network model (?) and the first algorithm for training a perceptron (?), especially after the introduction of backpropagation (?) and the growth of high-performance computing, the field of artificial neural network and deep learning in general has grown rapidly. Two specialized neural network architectures that are relevant to our study are recurrent neural networks (RNNs) (??) and long-short-term memory (LSTM) (??), as we need to train metamodels that take sequential observations as input. Despite their success, deep neural network models are often criticized for their lack of transparency and interpretability, which hinders their adoption in financial and actuarial applications. Enormous research efforts are spent to test and improve the robustness of deep neural network models with carefully designed noise injection methods. ? show that injecting synthetic noise before and after hidden unit activations during training improves the performance of autoencoders. ? improve learning for deep neural networks by injecting synthetic noise to the gradients during backpropagations. A branch of research has been devoted to understanding the resilience of neural network models to noise in training labels. For example, ? show that adding synthetic label noise to the convolutional neural network(CNN) can improve its ability to capture global features. ? quantify the error tolerance by injecting synthetic label noise with a custom Boltzmann machine hardware. ? find that neural networks are vulnerable to adversarial examples, and ? design an efficient method to generate such noisy examples to exploit the vulnerability to adversarial perturbations. ? design targeted attacks to training labels to test the robustness of neural networks. Instead of using synthetic noise, ? inject real-world label noise and examine noise tolerance of neural networks with controlled noise levels. The aforementioned studies use real-world data, as is typically the case for many neural network studies, where noise is already present in the training labels before any noise injection. Users of real-world data have little control over the noise level of the original training labels and usually examine the effect of noisy data by injecting noise, but it is unclear whether a neural network model trained on noisy data actually learns the real, i.e., noiseless, feature-label relationship. Due to their lack of transparency and interpretability, the adoption of deep neural networks in financial and actuarial applications has been received by regulators with some skepticism.

The contributions of our study are two-fold:

1. We study what deep neural networks learn from noisy data by training them using simulated data based on well-designed simulation experiments. This is a novel way to study the effect of noisy data and error tolerance of neural network models as one can *reduce noise* in the data by increasing the number of replications in a simulation model. This new way of studying neural network models can provide more direct evidence on their transparency and interpretability.
2. We propose two generic nested simulation procedures that uses deep neural networks as metamodels to improve its efficiency while maintaining transparency. In essence, a pilot stage simulation is used to generate a large number of noisy data, which are then used to train a metamodel. Depending on the application, a trained metamodel can serve two purposes: (1) to identify a set of tail scenarios, and (2) to estimate risk measures directly. The first procedure uses a metamodel to identify a set of potential tail scenarios on which computations are performed in the second stage, while the second procedure uses metamodel predictions to estimate risk measures directly. Our numerical results show that deep neural network metamodels can identify the tail scenarios accurately and so the proposed procedures can estimate tail risk measures with similar accuracy while, at the same time, using less simulation budget.

We are curious about fundamental questions like “What do deep neural networks learn from noisy data?” and “How well do neural networks learn from noisy data?”. Data-driven answers to these questions prevail in the existing literature. In supervised learning, deep neural networks are believed to learn from the given data about the feature-label relationship to predict new labels for unseen features. Cross validation using to assess a subset, i.e., the validation set, of the original data, is a common way to assess the quality of learning. Generalization error on the test labels is another popular assessment metric. But the test set is also a subset of the original data. In this study, we revisit these questions in a simulation context and propose an alternative approach to answer them. Instead of relying solely on real-data (splitting it into multiple subsets), we propose using stochastic simulation outputs as training labels for deep neural network models. By controlling the simulation design parameters, such as the number of independent replications, we can control the quality (and also the quantity) of the training labels fed into the neural networks. In such a controlled environment, we obtain more clear-cut answers to the above fundamental questions.

In nested simulation, a simulation model is used to generate a large number of outer scenarios, and each scenario is then used as an input to another simulation model. Borrowing terminologies from machine learning research, we can view a set of simulated outer scenarios and the estimated hedging errors for those scenarios as the *features* and (noisy) *labels*. One can train supervised learning models using these simulated features and labels. They are then used to replace the time-consuming inner simulations by the trained model. We refer to the trained supervised learning models as *metamodels* of the inner simulation, which is also known as the *surrogate models*. Metamodeling is a popular approach to reduce the computational burden of simulation-based applications by replacing the time-consuming simulation with a metamodel. The metamodel is trained using a set of simulated data, and it is used to predict the simulation output for new inputs. The study of metamodeling is an active research area in the simulation literature, and using deep neural networks as metamodels is a relatively new development. ? provide general guidelines for simulation metamodeling with neural networks, ? use deep neural networks as metamodels of a simulation model for structural reliability analysis, and ? show that neural network metamodels help achieve higher prediction accuracy than other metamodels in approximating agent-based simulation models. A popular metamodel in nested simulation procedures is stochastic kriging. ? use stochastic kriging as a metamodel of Monte Carlo simulations to estimate the Conditional Value-at-Risk (CVaR) of a portfolio of derivative securities, and ? use stochastic kriging for an efficient valuation of large portfolios of variable annuity (VA) contracts. Other studies, such as ?, ?, and ? use regression, kernel smoothing, and the likelihood ratio method, respectively. Our study has three key distinctions over the existing ones:

1. our metamodel has high-dimensional inputs. In machine learning terminology, the features are high-dimensional vectors. To estimate the hedging error of a typical VA contract, the number of features is in the order of hundreds, which is at least one order of magnitude larger than the number of features in the aforementioned studies,
2. for estimating tail risk measures, our metamodel is only used for tail scenario identification but is *not* used in the estimation of the tail risk measures. This is a feature designed particularly to

convince regulators that the losses used in estimating the risk measure are based on a transparent inner simulation model rather than on some black-box metamodels, and

3. using simulation models as data generators, we can decrease the noise level and get arbitrarily close to the true labels by increasing the number of replications in the simulation model. This design allows a systematic study of the effect of noisy training labels on the performance of neural network models in predicting the noiseless labels.

The rest of this article is organized as follows: Section ?? presents the problem settings for tail risk measures and dynamic hedging of VAs. Section ?? proposes an efficient two-stage nested simulation procedure that uses deep neural networks as metamodels to help reduce simulation budget by only performing computations on identified tail scenarios. Section ?? proposes a one-stage nested simulation procedure that estimates risk measures directly with metamodel predictions. Section ?? demonstrates the efficiency of deep learning proxies and examines error tolerance of two LSTM proxy models with different numbers of trainable parameters. Practical suggestions are provided for the choice of suitable metamodels and simulation settings.

2 Problem Formulation

In this section we present notations, problem settings, and a simulation model for risk estimation for hedging errors of variable annuities. A main goal of the section is to showcase the complexity of the simulation model, which we use as a data generator to train deep neural network metamodels (Section ?? and Section ??). For readers who are interested in the examination of a neural network metamodel, it is sufficient to understand that our simulation model generates data with 240 features and 1 real-value label and our metamodels are generally applicable to any simulation model that generates data with similar characteristics.

2.1 Tail Risk Measures: VaR and CVaR

Measuring and monitoring risks, particularly tail risks, are important risk management tasks for financial institutions like banks and insurance companies. Two most popular tail risk measures are Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) (?). Other names of CVaR include Conditional Tail Expectation (CTE), Tail Value-at-Risk (TailVaR), and Expected Shortfall (ES).

Consider a loss random variable L whose losses and gains lie in the right and left tails, respectively, of its distribution. For a given confidence level $\alpha \in [0, 1]$, the α -VaR is defined as the α -quantile of L : $\text{VaR}_\alpha = q_\alpha = \inf \{q : \Pr(L \leq q) \geq \alpha\}$. The α -CVaR of L is defined as $\text{CVaR}_\alpha = \frac{1}{1-\alpha} \int_{v=\alpha}^1 q_v dv$. Tail risk measures like VaR and CVaR are widely used for setting regulatory and economic capital, which is the amount of capital a financial institution holds to cover its risk. For example, European insurers set regulatory capital at 99.5%-VaR according to Solvency II ?. In Canada, the regulatory capital requirement for VAs is set based on CVaRs as prescribed in ?.

Let L_1, L_2, \dots, L_M be M independent and identically distributed (i.i.d.) simulated losses of L and let $L_{(1)} \leq L_{(2)} \leq \dots \leq L_{(M)}$ be the corresponding ordered losses. For a given confidence level α (assume that αM is an integer for simplicity), α -VaR can be estimated by the sample quantile $\widehat{\text{VaR}}_\alpha = L_{(\alpha M)}$. Also, α -CVaR can be estimated by

$$\widehat{\text{CVaR}}_\alpha = \frac{1}{(1-\alpha)M} \sum_{i=\alpha M+1}^M L_{(i)} = \frac{1}{(1-\alpha)M} \sum_{i \in \mathcal{T}_{(1-\alpha)M}} L_i,$$

where we define a *true tail scenario set* of size k as $\mathcal{T}_k = \{i : L_i > L_{(M-k)}\}$. In this study, the loss random variable of interest is the hedging error for VA.

2.2 Simulation Model for Variable Annuity Payouts

Variable annuity contracts offer different types of guarantees. Generally speaking, a portion of the VA premium is invested in a sub-account whose return is linked to some stock indices.

Two relevant types of guarantees in our studies are:

- **Guaranteed Minimum Maturity Benefit (GMMB):** A GMMB contract pays a maturity benefit equal to the greater of the sub-account value and a fixed guarantee value. The guarantee value is often set as a percentage, e.g., 75% or 100%, of the initial premium.
- **Guaranteed Minimum Withdrawal Benefit (GMWB):** A GMWB contract guarantees the minimum amount of periodic withdrawal the policyholder can take from the sub-account until maturity, even if the sub-account value reduces to zero. The minimum withdrawal benefit is typically a fixed percentage of the guarantee value. The guarantee value will decrease if the withdrawal exceeds the guaranteed minimum. The GMWB is typically offered with an accumulation period, during which no withdrawals are made but a GMMB is usually offered. Additional features offered with the GMWB include roll-up, ratchet, and reset (?).

For a comprehensive review of other types of VA contracts such as Guaranteed Minimum Death Benefit (GMMB), Guaranteed Minimum Accumulation Benefit (GMAB) and Guaranteed Lifetime Withdrawal Benefit (GLWB), we refer readers to ?. Next we present a summary of dynamic hedging for VA contracts. We refer readers to ? for detailed modeling of insurer liabilities in different VA contracts and Greek estimation.

Consider a generic VA contract with maturity $T > 0$ periods, e.g., $T = 240$ months. Denote the policyholder's (random) time of death by $\tau > 0$. Then the contract expires at $T' = \min\{T, \tau\}$, i.e., the earlier of the contract maturity and the death of the policyholder. Let S_t , F_t , and G_t be the indexed stock price, the subaccount value and the guarantee value, respectively, at time $t = 1, 2, \dots, T$. Evolution of the subaccount value and the guarantee value of a VA contract affect the contract payout. Note that the policyholder's (random) time of death also affects the timing of the benefit payout for certain types of VA such as GMMB, but this is not considered in our study for simplicity. For clarity, we use F_t and F_{t+} to denote the sub-account value just before and just after the withdrawal at time t , if any. Let η_g be the gross rate of management fee that is deducted from the fund value at each period and let $\eta_n < \eta_g$ be the net rate of management fee income to the insurer. The difference between the gross management fee and the net management fee income represents the incurred investment expenses.

At the inception of the contract, i.e., $t = 0$, we assume that the whole premium is invested in the stock index and the guarantee base is set to the sub-account value:

$$S_0 = F_0 = G_0.$$

At each time $t = 1, \dots, T$, the following events take place in the following order:

1. The sub-account value changes according to the growth of the underlying stock and the (gross) management fee is deducted. That is,

$$F_t = F_{(t-1)+} \cdot \frac{S_t}{S_{t-1}} \cdot (1 - \eta_g),$$

where $(x)^+ = \max\{x, 0\}$ and $F_{(t-1)+}$ will be defined later. The insurer's income at time t is the net management fee, i.e., $F_t \eta_n$.

2. The guarantee value ratchets up (ratcheting is a common feature in GMWB) if the sub-account value exceeds the previous guarantee value, i.e.,

$$G_t = \max\{G_{t-1}, F_t\}.$$

3. The withdrawal is made (for GMWB) and is deducted from the sub-account value, i.e.,

$$F_{t+} = (F_t - I_t)^+,$$

where $I_t = \gamma G_t$. A GMMB can be modeled with $\gamma = 0$.

We see from the above modeling steps that the status of a generic VA contract is summarized by a triplet (S_t, F_t, G_t) whose evolution is driven by the stochasticity of S_t . In practice, the simulation model may also incorporate additional complications like mortality, lapse, and excess withdrawal, etc.

At any time $t = 1, \dots, T$, the insurer's liability in a VA contract is the present value of all payments, net of the fee income. For example, suppose that the per-period risk-free rate is r , then the insurer's time- t liability for a GMMB contract is $V_t = e^{-r(T-t)} \cdot (G_T - F_T)^+ - \sum_{s=t+1}^T e^{-r(T-s)} F_s \eta_n$. Also, the insurer's time- t liability for a GMWB contract is $V_t = \sum_{s=t+1}^T e^{-r(T-s)} [(I_s - F_s)^+ - \eta_n F_s]$.

For example, consider the time- t liability V_t of a GMWB: Suppose that given the stock sample path, e.g., an outer path S_1, \dots, S_t , one can simulate future stock prices $\tilde{S}_{t+1}, \dots, \tilde{S}_T$, e.g., inner sample paths, based on some asset model such as a Black-Scholes model. The tilde symbol (\sim) over a quantity denotes its association with the inner simulation. Given the time t state (S_t, F_t, G_t) , following ? the sensitivity of V_t with respect to S_t can be estimated by a pathwise estimator (?):

$$\Delta_t(\tilde{S}_{t+1}, \dots, \tilde{S}_T | S_t) = \frac{\partial V_t}{\partial S_t} = \sum_{s=t+1}^T e^{-r(T-s)} \left[\mathbf{1}\{\tilde{I}_s > \tilde{F}_s\} \cdot \left(\frac{\partial \tilde{I}_s}{\partial S_t} - \frac{\partial \tilde{F}_s}{\partial S_t} \right) - \eta_n \frac{\partial \tilde{F}_s}{\partial S_t} \right], \quad t = 0, \dots, T-1, \quad (1)$$

where $\mathbf{1}\{\cdot\}$ is an indicator function and

$$\begin{aligned} \frac{\partial \tilde{F}_s}{\partial S_t} &= \mathbf{1}\{\tilde{I}_{s-1} < \tilde{F}_{s-1}\} \cdot \left(\frac{\partial \tilde{F}_{s-1}}{\partial S_t} - \frac{\partial \tilde{I}_{s-1}}{\partial S_t} \right) \cdot \frac{\tilde{S}_s}{\tilde{S}_{s-1}} \cdot (1 - \eta_g), \\ \frac{\partial \tilde{G}_s}{\partial S_t} &= \mathbf{1}\{\tilde{G}_{s-1} < \tilde{F}_s\} \cdot \frac{\partial \tilde{F}_s}{\partial S_t} + \mathbf{1}\{\tilde{G}_{s-1} \geq \tilde{F}_s\} \cdot \frac{\partial \tilde{G}_{s-1}}{\partial S_t}, \\ \frac{\partial \tilde{I}_s}{\partial S_t} &= \gamma \frac{\partial \tilde{G}_s}{\partial S_t}. \end{aligned}$$

The recursion is initialized with $(\tilde{S}_t, \tilde{F}_t, \tilde{G}_t) = (S_t, F_t, G_t)$, $\frac{\partial \tilde{F}_s}{\partial S_t} = \frac{\tilde{F}_t}{S_t}$, and $\frac{\partial \tilde{G}_s}{\partial S_t} = \frac{\partial \tilde{I}_s}{\partial S_t} = 0$.

2.3 Dynamic Hedging for Variable Annuities

Below we provide a scheme used to perform a multi-period nested simulation in estimating (profit and loss) P&L for one outer scenario.

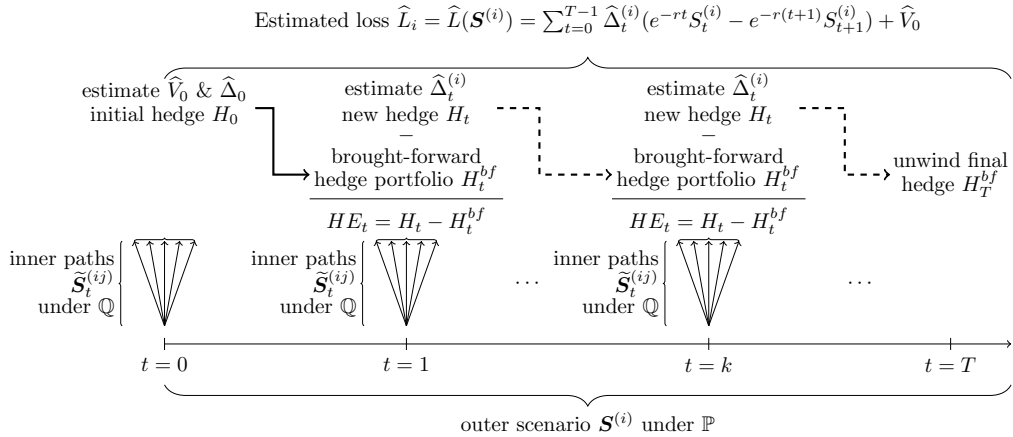


Figure 1: Illustration of multi-period nested simulation that estimates the P&L for one outer scenario.

Insurers commonly use dynamic hedging to mitigate a market risk exposure in VA contract's embedded options. In a dynamic hedging program, a hedge portfolio is set up and periodically rebalanced for a portfolio of VA contracts using stocks, bonds, futures, and other derivatives. For simplicity, in this study we consider delta hedging for a generic VA liability using one stock and one bond. The metamodeling procedures in Section ?? and Section ?? can be trivially adapted to more general hedging strategies.

Consider a generic VA contract whose delta hedge portfolio at any time t , $t = 0, 1, \dots, T-1$, consists of Δ_t units in the underlying stock and B_t amount of a risk-free zero-coupon bond maturing at time T . The value of the hedge portfolio at time $(t-1)$ is:

$$H_{t-1} = \Delta_{t-1} S_{t-1} + B_{t-1},$$

where S_t is the underlying stock price and any time $t > 0$. This hedge portfolio is brought forward to the next rebalancing time t , when its value becomes:

$$H_t^{bf} = \Delta_{t-1} S_t + B_{t-1} e^r.$$

Therefore, the time t hedging error, i.e., the cash flow incurred by the insurer due to rebalancing at time t , is

$$HE_t = H_t - H_t^{bf}, \quad t = 1, \dots, T-1. \quad (2)$$

The P&L of the VA contract includes the cost of the initial hedge (H_0), the hedging errors (??), the unwinding of the hedge at maturity (H_T^{bf}), and the unhedged liability (V_0). Mathematically, the present value of these cash flows is given by

$$L = H_0 + \sum_{t=1}^{T-1} e^{-rt} HE_t - e^{-rT} H_T^{bf} + V_0 = \sum_{t=0}^{T-1} \Delta_t (e^{-rt} S_t - e^{-r(t+1)} S_{t+1}) + V_0, \quad (3)$$

where the second equality holds by a telescopic sum simplification of $e^{-rt} B_t$, $t = 0, \dots, T-1$.

In (??), Δ_t and V_0 are determined by using a risk-neutral measure \mathbb{Q} while the distribution of L is under a real-world measure \mathbb{P} . If Δ_t and V_0 cannot be calculated analytically, a nested simulation is required to estimate the tail risk measure of L . Recall from Section ?? that the stock sample path, regardless of the inner or outer simulation or a combination of both, determines the evolution of the triplet (S_t, F_t, G_t) . Specifically, the outer scenarios $\mathbf{S}^{(i)} = (S_1^{(i)}, \dots, S_T^{(i)})$, $i = 1, \dots, M$ are generated under \mathbb{P} . At each time $t = 1, \dots, T-1$ of a given outer scenario $\mathbf{S}^{(i)}$, inner sample paths $\tilde{\mathbf{S}}_t^{(j)} = (\tilde{S}_{t+1}^{(j)}, \dots, \tilde{S}_T^{(j)})$, $j = 1, \dots, N$ are generated under \mathbb{Q} to estimate $\Delta_t^{(i)}$, $i = 1, \dots, M$. Also, $V_0^{(i)}$, $i = 1, \dots, M$ are estimated under \mathbb{Q} via inner simulations at time 0. Recall from Section ?? that the stock's sample path, regardless of inner or outer simulation or a combination of both, determines the evolution of the triplet (S_t, F_t, G_t) . For GMWB, a standard nested simulation procedure to estimate the α -CVaR of L is described in Algorithm ??.

Algorithm 1 Standard Nested Simulation Procedure for Estimating CVaR for GMWB Hedging Losses

- 1: For $i = 1, \dots, M$, simulate outer scenarios $\mathbf{S}^{(i)} = (S_1^{(i)}, \dots, S_T^{(i)})$ under the real-world measure \mathbb{P} .
 - 2: For $t = 0$, simulate time-0 inner paths $\tilde{\mathbf{S}}_0^{(j)} = (\tilde{S}_1^{(j)}, \tilde{S}_2^{(j)}, \dots, \tilde{S}_T^{(j)})$, $j = 1, \dots, N$ under \mathbb{Q} , then estimate V_0 by $\hat{V}_0 = \sum_{s=1}^T e^{-r(T-s)} [(I_s - F_s)^+ - \eta_n F_s]$ and $\hat{\Delta}_0 = \Delta_0(\tilde{S}_1^{(j)}, \dots, \tilde{S}_T^{(j)} | S_0)$.
 - 3: Given each scenario $\mathbf{S}^{(i)}$:
 - a. At each time $t = 1, \dots, T-1$, simulate inner paths $\tilde{\mathbf{S}}_t^{(ij)} = (\tilde{S}_{t+1}^{(ij)}, \dots, \tilde{S}_T^{(ij)})$, $j = 1, \dots, N$ under \mathbb{Q} , then estimate Δ_t by $\hat{\Delta}_t^{(i)} = \Delta_t(\tilde{S}_{t+1}^{(ij)}, \dots, \tilde{S}_T^{(ij)} | \mathbf{S}^{(i)})$.
 - b. Use scenarios $\mathbf{S}^{(i)}$ and \hat{V}_0 and $\hat{\Delta}_t^{(i)}$ to calculate losses \hat{L}_i^{MC} , $t = 0, \dots, T-1$, then sort them as $\hat{L}_{(1)}^{MC} \leq \hat{L}_{(2)}^{MC} \leq \dots \leq \hat{L}_{(M)}^{MC}$.
 - 4: Estimate α -CVaR of L by $\widehat{\text{CVaR}}_\alpha^{MC} = \frac{1}{(1-\alpha)M} \sum_{i=\alpha M+1}^M \hat{L}_{(i)}^{MC} = \frac{1}{(1-\alpha)M} \sum_{i \in \hat{\mathcal{T}}_{(1-\alpha)M}^{MC}} \hat{L}_i^{MC}$.
-

We refer to the collection of experiments needed conditional on one scenario $\mathbf{S}^{(i)}$ to estimate L_i , that is, all upward arrows in Figure ??, as one inner simulation experiment. We make four observations:

- each inner simulation is time-consuming, as it includes T simulation experiments, one at each time $t = 0, \dots, T-1$,
- after running inner simulations for M scenarios, we obtain simulated data, that is, feature-label pairs, $(\mathbf{S}^{(i)}, \hat{L}_i)$, $i = 1, \dots, M$; the feature vector \mathbf{S} is T dimensional,
- \hat{L}_i^{MC} is a standard Monte Carlo estimator of the true loss for scenario $\mathbf{S}^{(i)}$. It is an unbiased estimator and its variance is inversely proportional to the number of inner replications N . As N approaches infinity, \hat{L}_i^{MC} converges to the true loss L_i , and
- most importantly, when estimating tail risk measures such as α -CVaR, only a small number of estimated losses, that is, those associated with the set of tail scenarios $\hat{\mathcal{T}}_k$ are used in the estimator.

3 Two-Stage Nested Simulation with Metamodels

Based on the three observations above and inspired by ?, we propose a two-stage nested simulation procedure which uses a deep neural network metamodel to identify potential tail scenarios. We present our proposed procedure as a competitor to the standard procedure with M outer scenarios and N inner

replications for each outer scenario, as described in Section ?? . We propose a two-stage procedure with a neural network metamodel that aims to produce a CVaR estimate that's as accurate as that of the standard procedure, but uses fewer computations as than latter.

Algorithm 2 Two-Stage Metamodeling Nested Simulation Procedure for Estimating CVaR

1: **Train a neural network metamodel using simulation data:**

- a. Use a fraction of the total simulation budget to run Steps 1, 2, and 3 in the standard procedure, i.e., Algorithm ??, with the same number of outer scenarios, M , but a much smaller number of inner replications, $N' \ll N$, in each scenario. Obtain M simulated samples (feature-label pairs), $(\mathbf{S}^{(i)}, \widehat{L}_i)$, $i = 1, \dots, M$. Note that N' may be much smaller than N , so \widehat{L}_i are expected to have larger variance.
- b. Use the simulated data, $(\mathbf{S}^{(i)}, \widehat{L}_i)$, $i = 1, \dots, M$ to train a neural network. Refer to the trained model as a metamodel and denote it by $\widehat{L}^{PD}(\mathbf{S})$. Denote the predicted losses for the outer scenarios by $\widehat{L}_i^{PD} = \widehat{L}^{PD}(\mathbf{S}^{(i)})$, $i = 1, \dots, M$.
- c. Sort the predicted losses $\widehat{L}_{(1)}^{PD} \leq \widehat{L}_{(2)}^{PD} \leq \dots \leq \widehat{L}_{(M)}^{PD}$ to identify a predicted tail scenario set, $\widehat{\mathcal{T}}_m^{PD}$, associated with the largest predicted losses. The number of predicted tail scenarios, m , is a user's choice.

2: **Concentrate simulation on predicted tail scenarios:**

- a. Run Steps 2 and 3 of Algorithm ?? with the same number of inner replications, N , but only on the predicted tail scenarios, i.e., scenarios in $\widehat{\mathcal{T}}_m^{PD}$. Denote the standard procedure's estimated losses and sorted losses by \widehat{L}_i^{ML} and $\widehat{L}_{(i)}^{ML}$, respectively, $i = 1, \dots, m$.
 - b. Estimate the α -CVaR of L by $\widehat{\text{CVaR}}_\alpha^{ML} = \frac{1}{(1-\alpha)M} \sum_{i=\alpha M+1}^M \widehat{L}_{(i)}^{ML} = \frac{1}{(1-\alpha)M} \sum_{i \in \widehat{\mathcal{T}}_{(1-\alpha)M}^{ML}} \widehat{L}_i^{ML}$, where $\widehat{\mathcal{T}}_k^{ML}$ denotes a predicted tail scenario set associated with the largest k estimated losses.
-

Similar to ?, the proposed two-stage procedure in Algorithm ?? uses the metamodel predictions to identify the predicted tail scenario set in Stage 1. But, different from their fixed-budget simulation design, we attempt to achieve a target accuracy. Specifically, in Stage 2 we propose using a standard procedure with the same number of inner replications, N , as a competing simulation procedure (or a benchmark). There are two different experiment designs for nested simulation procedures: fixed-budget design and fixed-accuracy design. In a fixed-budget design, the simulation budget is fixed and the goal is to achieve the highest accuracy possible within the budget. Let $\Gamma = MN$ be the simulation budget for the standard procedure, where each scenario receives $\frac{\Gamma}{M}$ inner replications. In the proposed two-stage procedure, suppose Stage 2 uses 1% of the simulation budget, $\alpha = 95\%$, and $m = (1 - \alpha)M$, then 99% of the simulation budget is concentrated on $5\%M$ predicted tail scenarios in Stage 2. In other words, each predicted tail scenario receives $\frac{99\% \Gamma}{5\% M}$ inner replications, almost 20 times more than that in the standard procedure. This budget concentration is expected to improve the estimation accuracy of the two-stage procedure compared to a standard procedure with the same budget. If the metamodel is accurate in predicting true tail scenarios, then the two-stage procedure is expected to achieve higher accuracy than the standard procedure with the same budget. However, we believe that the goal of designing an efficient simulation procedure is to solve practical problems faster, so a target-accuracy design is more suitable, which refers to obtaining a similar level of accuracy as the standard procedure but with much less simulation budget. One other reason for this fixed-accuracy design is to investigate whether deep neural network metamodels trained with much noisier labels can identify true tail scenarios with similar accuracy as the standard procedure. The size of the predicted tail scenario set in Stage 1, m , is an important experiment design parameter that affects the correct identification of true tail scenarios and ultimately affects the estimation accuracy for CVaR. Clearly, there is a lower bound $m \geq (1 - \alpha)M$ because the α -CVaR is estimated by the average of $(1 - \alpha)M$ largest losses at the end of Stage 2. For ease of reference, we call the additional percentage of predicted tail scenarios above this lower bound, i.e., $\epsilon = \frac{m - (1 - \alpha)M}{M}$, as a *safety margin*. On one hand, large ϵ is not desirable because it increases computations in Stage 2. On the other hand, ϵ should be set reasonably large so more true tail scenarios are included in the predicted tail scenario set $\widehat{\mathcal{T}}_m^{PD}$ and are ultimately included in $\widehat{\mathcal{T}}_{(1-\alpha)M}^{PD}$ at the end of Stage 2. The selection of m is highly dependent on the choice of the metamodel. Due to the simulation errors and approximation error in the metamodel in Stage 1, we do not expect perfect match between the true tail scenario set \mathcal{T}_k and the proxy tail scenario set $\widehat{\mathcal{T}}_k^{PD}$ for any size k . This means that we should not set m at its lower bound: Some safety margin ϵM should be added to the proxy tail scenario set, i.e., $m = (1 - \alpha)M + \epsilon M$, to increase the likelihood that $\mathcal{T}_k \subseteq \widehat{\mathcal{T}}_m^{PD}$ and that the true tail

scenarios are included in estimating α -CVaR at the end of Stage 2. In the numerical experiments, we examine the relationship between the safety margin and the correct identification of true tail scenarios for different metamodels.

4 Single-Stage Nested Simulation with Neural Network Metamodels

In our exploratory experiment with the two-stage procedure, we observe that a suitable metamodel trained with noisy labels is accurate enough to identify true tail scenarios with a relatively small safety margin. This observation motivates us to propose a single-stage procedure that uses the identical neural network metamodel, not for the purpose of differentiating between tail and non-tail scenarios, but rather to estimate the CVaR directly using the predicted losses.

Algorithm 3 Single-Stage Metamodeling Nested Simulation Procedure for Estimating CVaR

- 1: Run Step 1 of Algorithm ??, and denote the predicted losses by $\hat{L}_i^{PD} = \hat{L}^{PD}(\mathbf{S}^{(i)})$, $i = 1, \dots, M$.
 - 2: Sort the predicted losses $\hat{L}_{(1)}^{PD} \leq \hat{L}_{(2)}^{PD} \leq \dots \leq \hat{L}_{(M)}^{PD}$ to identify the largest predicted losses.
 - 3: Directly estimate the risk measure (e.g., α -CVaR) of L using the metamodel predictions: Calculate
$$\widehat{\text{CVaR}}_{\alpha}^{PD} = \frac{1}{(1-\alpha)M} \sum_{i=\alpha M+1}^M \hat{L}_{(i)}^{PD}.$$
-

The single-stage procedure has three major advantages over the two-stage procedure: (1) the single-stage procedure is expected to be more efficient than the two-stage procedure because it does not require running the standard procedure in Stage 2, (2) the safety margin ϵ is not needed in the single-stage procedure, and (3) most importantly, the single-stage procedure is not limited to just estimating tail risk measures and can be extended to provide a broader assessment of risk. It can be naturally adapted to estimate risk measures that require the knowing of the entire loss distribution, such as the standard deviation or the squared tracking error. In our numerical experiments, we will compare the single-stage procedure to the two-stage procedure and the standard procedure in estimating the α -CVaR of the hedging losses for GMWB, and we will also examine the single-stage procedure's performance in estimating the standard deviation of the hedging losses.

5 Numerical Results

We conduct a series of simulation experiments to (1) demonstrate the efficiency of the proposed metamodeling procedures and (2) examine the error tolerance to noisy training data in deep learning models. The problem settings in our experiments are identical to those in ??: we consider estimating the 95% CVaR of the hedging loss of a GMWB contract, which is one of the most complex VA contracts in the market. The VA contracts have a 20-year maturity and are delta-hedged with monthly rebalancing, i.e., $T = 240$ rebalancing periods. The gross and net management fees are $\eta_g = 0.2\%$ and $\eta_n = 0.1\%$, respectively. The withdrawal rate for GMWB is 0.375% per month. The risk-free rate is 0.2% per period and the underlying asset S_t is modeled by a regime-switching geometric Brownian motion with parameters specified in Table 2 of ?.

To compare the numerical performances of different simulation procedures, we create a benchmark dataset with a large-scale nested simulation: We first simulate $M = 100,000$ outer scenarios, i.e., 240-periods stock paths $\mathbf{S}^{(1)}, \dots, \mathbf{S}^{(M)}$ under \mathbb{P} and used these outer scenarios in all further experiments. Note that the 5% tail scenario set includes 5,000 scenarios. As the hedging losses for these scenarios cannot be calculated analytically, we run inner simulations with a large number of replications, $N = 100,000$, conditional on each of the M scenarios. We denote these losses by L_1, \dots, L_M and will refer to them as *true losses*. We also use these true losses to estimate $\widehat{\text{CVaR}}_{95\%}$ and denote the corresponding *true tail scenario set* by \mathcal{T}_{5000} . Lastly, we refer to the set of feature-label pairs $\{(\mathbf{S}^{(i)}, L_i) : i = 1, \dots, M\}$ as a *true dataset*. Note that the feature vector \mathbf{S} is a 240-dimension stock path.

We compare our two-stage procedure to a standard nested simulation procedure that runs $N = 1,000$ inner replications for each of the $M = 100,000$ outer scenarios. In Stage 1 of the proposed procedure, we first run inner simulations with $N' = 100$ inner replications for each of the $M = 100,000$ outer scenarios. So, Stage 1's simulation budget is 10% of the standard procedure's. The resulting feature-label pairs $\{(\mathbf{S}^{(i)}, \tilde{L}_i) : i = 1, \dots, M\}$ is used for training different metamodels. Specifically, following the convention

in machine learning research, we split this dataset into three parts: The training, validation, and test sets have 90,000, 5,000, and 5,000 data points (90%, 5%, and 5% of the dataset), respectively. At the end of Stage 1, m predicted tail scenarios, are identified by the trained metamodels. In Stage 2, $N = 1,000$ inner replications are run for all predicted tail scenarios. Stage 2’s simulation budget is $\frac{m}{M}$ of the standard procedure’s. In short, the two-stage procedure uses 15% – 30% of the standard procedure’s budget for a safety margin between 0% – 15%.

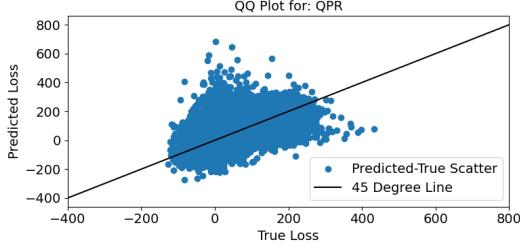
Five metamodels are considered in this experiment: multiple linear regression (MLR), quadratic polynomial regression (QPR) without interaction terms, feed-forward neural network (FNN), recurrent neural network (RNN), and long short-term memory (LSTM) network. MLR and QPR are considered as extensions of regression metamodels in the nested simulation literature. FNN is a generic neural network while RNN and LSTM are specialized models to accommodate the sequential structure of our time-series features. A tanh activation function is used for RNN and LSTM layers, and a Rectified Linear Unit (ReLU) activation function is used for the fully-connected layers. All neural network metamodels are trained by the Adam optimizer (?) with an initial learning rate of 0.001 and an exponential learning rate decay schedule. FNN is trained with a dropout rate of 20%. RNN and LSTM are trained with a dropout rate of 10%. The architectures and training settings are typical choices in the deep learning literature. The training labels are normalized to have zero mean and unit standard deviation. The architectures and the numbers of trainable parameters are shown in the first two columns of Table ?? . We see that the three deep neural network metamodels’ have orders of magnitudes more trainable parameters than the two regression models. In machine-learning terminologies, the three deep neural network models have much higher *model capacities*.

Model	Layer size	# parameters	Training errors	Test errors	True errors
MLR	N/A	241	0.708(± 0.001)	0.691(± 0.003)	0.694(± 0.001)
QPR	N/A	481	0.543(± 0.001)	0.544(± 0.003)	0.530(± 0.001)
FNN	240, 128, 16	35,009	0.138(± 0.010)	0.291(± 0.006)	0.134(± 0.009)
RNN	(240, 32), (240, 4), 32	32,021	0.109(± 0.010)	0.143(± 0.013)	0.100(± 0.010)
LSTM	(240, 32), (240, 4), 32	35,729	0.086(± 0.004)	0.093(± 0.004)	0.074(± 0.004)

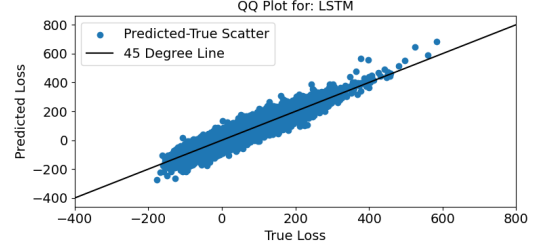
Table 1: Architectures and MSEs (standard errors) of metamodels for GMWB inner simulation model.

The last three columns in Table ?? display the average squared errors between the predicted losses and the losses (labels) in different datasets. The standard errors of these quantities are also reported in parentheses. We first observe that the two regression metamodels’ errors are larger than those of three deep neural network models. This is because model capacities of the regression metamodels are too low to learn the complex dynamic hedging simulation model. There are also differences among the three deep neural network models. The FNN is a generic neural network, its test error is larger than the training error, which is a sign of over-fitting and poor generalization. In contrast, RNN and LSTM are networks designed to capture temporal relationship in the high-dimensional feature. As a result, they have lower training errors, i.e., they fit the data better, and have lower testing error, i.e., they generalize better. Notably, the true errors for the deep neural network metamodels are lower than the test errors. Recall that all three data sets are generated from the same simulation model, but the true data set is less noisy than the training and test data. Part of the test error is due to simulation noise, and this noise is lower in the true error. We observe that the two regression metamodels not only generalize to the test data poorly but also generalize to the true data poorly. In contrast, the deep neural network metamodels generalize better to the true data than to the test data. Figure ?? shows the quantile-quantile (QQ) plots between the true losses and the proxy losses. We see that the LSTM proxy models’ QQ-plots closely follow that 45-degree line. Again, these proxy models are trained on noisy data, so the good fit to the true data should not be taken for granted. This implies that these models indeed learn the true feature-label relationship in the dynamic hedging simulation model (i.e., true data) even though they are trained on noisy observations (e.g., training data) of the model.

From a unique analytical standpoint, our numerical study offers a methodical exploration into the robustness of deep neural network models against noise in training labels. By employing the standard nested simulation procedure in Algorithm ?? as a data generator, we gain the ability to manipulate the noise level through adjusting the number of inner replications while keeping the same simulation procedure. As a result, this approach provides a controlled environment to examine the impact of label noise on neural network model. It allows us to generate our true dataset that approximates the true hedging losses with a high degree of precision, and, as a result, it enables us to explore the crucial



(a) QPR metamodel



(b) LSTM metamodel

Figure 2: QQ-plots between predicted losses (x-axis) and true losses (y-axis) for the same outer scenarios.

question of whether deep learning models are capable of learning the true feature-label relationship from noisy training labels. Our numerical results in Table ?? provide direct evidence that deep neural network models are indeed able to cut through the noise in the training labels and learn the true feature-label relationship.

5.1 Two-Stage Procedure

In our proposed two-stage procedure, the metamodel is used to identify the predicted tail scenario set, on which the standard nested simulation procedure is run in Stage 2 to estimate the 95%-CVaR.

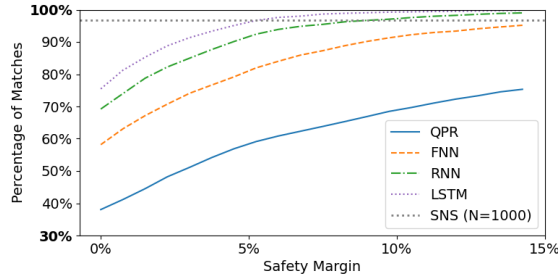
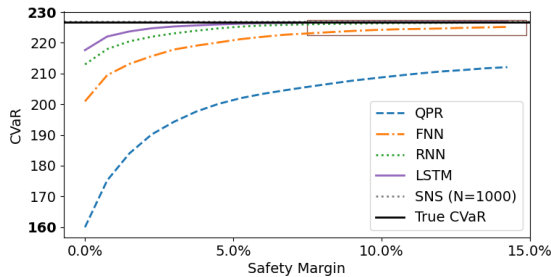
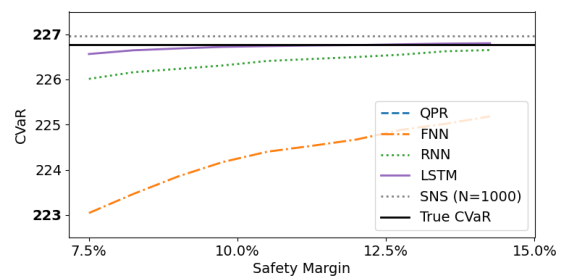


Figure 3: Percentage of correctly identified true tail scenarios by different proxy models.

We examine the metamodels' abilities to correctly identify true tail scenarios, which is crucial for the two-stage procedure to produce accurate CVaR estimates. Figure ?? depicts the percentage of correctly identified true tail scenarios by different metamodels for different safety margins. We see that a poor metamodel like the QPR identifies less than 40% of the true tail scenario without any safety margin. In contrast, a good proxy model like the LSTM identifies more than 75% of the true tail scenarios without any safety margin and more than the QPR proxy does with 15% of safety margin. For comparison, the horizontal line shows the true tail identification percentage for the standard nested simulation procedure. The LSTM proxy reaches similar percentage with 5% of safety margin, which corresponds to 5 times less simulation budget than the standard procedure's.



(a) Safety margin 0% - 15%.



(b) Safety margin 7.5% - 15%.

Figure 4: 95%-CVaR estimates by different procedures. Right figure is a zoomed-in version of left figure.

Lastly, we return to our original goal of estimating the 95%-CVaR of the dynamic hedging error. Figure ?? shows the 95%-CVaR estimates for different proxy models with different safety margins. Figure ?? is a zoomed version of Figure ?. Because the safety margin only affects the two-stage procedure, the true 95%-CVaR and the one estimated by the standard procedure are horizontal (solid and dotted, respectively) lines in Figure ?. We see that, with reasonable safety margins, the two-stage procedures with RNN and LSTM metamodels produce estimates that are closer to the true value than the standard procedure’s estimate. The LSTM metamodel is particularly superior as it accurately identifies the true tail scenarios and produces highly accurate CVaR estimates with small safety margins. For our two-stage procedure, achieving a similar accuracy as the standard procedure with a safety margin of 7.5% corresponds to a 77.5% reduction in the total simulation budget. By concentrating the simulation budget on the predicted tail scenarios, the two-stage procedure with the LSTM metamodel is able to achieve a similar accuracy as the standard procedure with a much smaller simulation budget.

5.2 Single-Stage Procedure

In our single-stage procedure, the metamodel is used to estimate the 95%-CVaR directly.

6 Conclusion

The proposed nested simulation procedures with deep neural network metamodels result in substantial computational savings in estimating CVaR of the hedging loss of a VA contract from accurately predicting the hedging losses and identifying the tail scenarios. When new outer scenarios are generated, a trained LSTM metamodel can distinguish between tail and non-tail scenarios and make accurate predictions without the need to run new inner simulations. Our novel experiment design allows us to examine the impact of label noise on deep neural network models. We find that a deep neural network with a suitable architecture is able to cut through the noise in training labels and learn the true complex dynamic hedging model. By showcasing the resilience of these models, our study aims to influence regulatory bodies towards recognizing the value and applicability of deep learning metamodels in financial risk management, and it provides informed suggestions and guidance for the incorporation and oversight of advanced deep learning metamodel in Monte Carlo Simulation in financial applications. Our findings are particularly insightful in this context. In our numerical experiments, a LSTM metamodel is resilient to a high level of noise in training labels and is able to make accurate predictions.

References

- Mark Broadie, Yiping Du, and Ciamac C Moallemi. Risk estimation via regression. *Operations Research*, 63(5):1077–1097, 2015.
- Nicholas Carlini and David Wagner. Towards evaluating the robustness of neural networks. In *2017 IEEE Symposium on Security and Privacy (SP)*, pages 39–57. Ieee, 2017.
- Mark J Cathcart, Hsiao Yen Lok, Alexander J McNeil, and Steven Morrison. Calculating variable annuity liability “greeks” using monte carlo simulation. *ASTIN Bulletin: The Journal of the IAA*, 45(2):239–266, 2015.
- Junyoung Chung, Caglar Gulcehre, Kyunghyun Cho, and Yoshua Bengio. Empirical evaluation of gated recurrent neural networks on sequence modeling. 2014. <https://arxiv.org/abs/1412.3555>. Accessed 7th Sep 2023.
- Ou Dang. *Efficient Nested Simulation of Tail Risk Measures for Variable Annuities*. Ph.D. thesis, Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, Canada, 2021. <https://uwaterloo.ca/handle/10012/17084>.
- Ou Dang, Mingbin Feng, and Mary R Hardy. Efficient nested simulation for conditional tail expectation of variable annuities. *North American Actuarial Journal*, 24(2):187–210, 2020.
- EIOPA. The underlying assumptions in the standard formula for the solvency capital requirement calculation. Technical report, The European Insurance and Occupational Pensions Authority, 2014.
- Daniel J Fonseca, Daniel O Navarrese, and Gary P Moynihan. Simulation metamodeling through artificial neural networks. *Engineering applications of artificial intelligence*, 16(3):177–183, 2003.
- Guojun Gan and Sheldon X Lin. Valuation of large variable annuity portfolios under nested simulation: A functional data approach. *Insurance: Mathematics and Economics*, 62:138–150, 2015.
- Paul Glasserman. *Monte Carlo Methods in Financial Engineering*, volume 53. Springer, 2004.
- Ian J Goodfellow, Jonathon Shlens, and Christian Szegedy. Explaining and harnessing adversarial examples. *arXiv preprint arXiv:1412.6572*, 2014. <https://arxiv.org/abs/1412.6572>. Accessed 1st Jan 2024.
- Mary R Hardy. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*, volume 168. John Wiley & Sons, 2003.
- Trevor Hastie, Robert Tibshirani, and Jerome H Friedman. *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, volume 2. Springer, 2009.
- Sepp Hochreiter and Jürgen Schmidhuber. Long short-term memory. *Neural Computation*, 9(8):1735–1780, 1997.
- Jeff L Hong, Sandeep Juneja, and Guangwu Liu. Kernel smoothing for nested estimation with application to portfolio risk measurement. *Operations Research*, 65(3):657–673, 2017.
- Lu Jiang, Di Huang, Mason Liu, and Weilong Yang. Beyond synthetic noise: Deep learning on controlled noisy labels. In *International conference on machine learning*, pages 4804–4815. PMLR, 2020.
- Diederik P Kingma and Jimmy Ba. Adam: A method for stochastic optimization. *arXiv preprint arXiv:1412.6980*, 2014. <https://arxiv.org/pdf/1412.6980.pdf>. Accessed 7th Sep 2023.
- Yann LeCun, Yoshua Bengio, and Geoffrey E Hinton. Deep learning. *Nature*, 521(7553):436–444, 2015.
- Qui X Lieu, Khoa T Nguyen, Khanh D Dang, Seunghye Lee, Joowon Kang, and Jaehong Lee. An adaptive surrogate model to structural reliability analysis using deep neural network. *Expert Systems with Applications*, 189:116104, 2022.
- Ming Liu and Jeremy Staum. Stochastic kriging for efficient nested simulation of expected shortfall. *Journal of Risk*, 12(3):3, 2010.
- Wenjie Luo, Yujia Li, Raquel Urtasun, and Richard Zemel. Understanding the effective receptive field in deep convolutional neural networks. *Advances in Neural Information Processing Systems*, 29, 2016.
- Warren S McCulloch and Walter Pitts. A logical calculus of the ideas immanent in nervous activity. *The Bulletin of Mathematical Biophysics*, 5:115–133, 1943.
- Arvind Neelakantan, Luke Vilnis, Quoc V Le, Ilya Sutskever, Lukasz Kaiser, Karol Kurach, and James Martens. Adding gradient noise improves learning for very deep networks. *arXiv preprint arXiv:1511.06807*, 2015. <https://arxiv.org/abs/1511.06807>. Accessed 1st Jan 2024.
- OpenAI. Chatgpt, 2023. <https://chat.openai.com/chat>. Accessed 7th Sep 2023.
- OSFI. Life insurance capital adequacy test. Technical report, Office of the Superintendent of Financial Institutions Canada, 2017.
- Ben Poole, Jascha Sohl-Dickstein, and Surya Ganguli. Analyzing noise in autoencoders and deep networks. *arXiv preprint arXiv:1406.1831*, 2014. <https://arxiv.org/abs/1406.1831>. Accessed 1st Jan 2024.

- R Tyrrell Rockafellar and Stanislav Uryasev. Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26(7):1443–1471, 2002.
- Frank Rosenblatt. The perceptron: A probabilistic model for information storage and organization in the brain. *Psychological Review*, 65(6):386, 1958.
- David E Rumelhart, Geoffrey E Hinton, and Ronald J Williams. Learning internal representations by error propagation. Technical report, California University San Diego La Jolla Institute for Cognitive Science, 1985.
- Isabelle Salle and Murat Yıldızoğlu. Efficient eampling and meta-modeling for computational economic models. *Computational Economics*, 44:507–536, 2014.
- David Silver, Aja Huang, Chris J Maddison, Arthur Guez, Laurent Sifre, George Van Den Driessche, Julian Schrittwieser, Ioannis Antonoglou, Veda Panneershelvam, Marc Lanctot, et al. Mastering the game of go with deep neural networks and tree search. *Nature*, 529(7587):484–489, 2016.
- Nitish Srivastava, Geoffrey Hinton, Alex Krizhevsky, Ilya Sutskever, and Ruslan Salakhutdinov. Dropout: A simple way to prevent neural networks from overfitting. *The Journal of Machine Learning Research*, 15(1):1929–1958, 2014.
- Ilya Sutskever, Oriol Vinyals, and Quoc V Le. Sequence to sequence learning with neural networks. In Z. Ghahramani, M. Welling, C. Cortes, N. Lawrence, and K.Q. Weinberger, editors, *Advances in Neural Information Processing Systems*, volume 27. Curran Associates, Inc., 2014. URL https://proceedings.neurips.cc/paper_files/paper/2014/file/a14ac55a4f27472c5d894ec1c3c743d2-Paper.pdf.
- Christian Szegedy, Wojciech Zaremba, Ilya Sutskever, Joan Bruna, Dumitru Erhan, Ian Goodfellow, and Rob Fergus. Intriguing properties of neural networks. *arXiv preprint arXiv:1312.6199*, 2013. <https://arxiv.org/abs/1312.6199>. Accessed 1st Jan 2024.
- The Geneva Association. Variable annuities—an analysis of financial stability. Technical report, 2013. https://www.genevaassociation.org/sites/default/files/research-topics-document-type/pdf_public/ga2013-variable-annuities.0. Accessed 7th Sep 2023.
- Ronald J Williams and David Zipser. A learning algorithm for continually running fully recurrent neural networks. *Neural Computation*, 1(2):270–280, 1989.
- Kun Zhang, Mingbin Ben Feng, Guangwu Liu, and Shiyu Wang. Sample recycling for nested simulation with application in portfolio risk measurement. 2022. <https://arxiv.org/abs/2203.15929>. Accessed 2nd May 2023.