Metamodeling-Based Nested Simulation Procedures in Financial Engineering and Dynamic Hedging

by Xintong Li

Nested Simulation Procedures in Financial Engineering

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Introduction

- Nested simulation procedures are used to estimate risk measures for complex financial derivatives portfolios

$$\rho(L) = \rho(L(X)), \quad L(X) = \mathbb{E}\left[Y|X = x\right]|_{x=X}$$

- Involves two levels of Monte Carlo simulations:
  - Outer level: generates underlying risk factors (outer scenarios), $X_i \sim F_X$
  - Inner level: generates scenario-wise samples of portfolio losses (inner replications), $Y_{ij} \sim F_{Y|X_i}$
- Computationally expensive due to nested structure

Common Risk Measures

- Smooth h, e.g., quadratic tracking error

$$\rho(L) = \mathbb{E}\left[(L-b)^2\right]$$

- hockey-stick h: mean excess loss

$$\rho(L) = \mathbb{E}\left[L \cdot \mathbb{1}_{\{L \geq u\}}\right]$$

- indicator h: probability of large loss

$$\rho(L) = \mathbb{E}\left[\mathbb{1}_{\{L \geq u\}}\right]$$

- Value at Risk (VaR)
- Conditional Value at Risk (CVaR)

Standard Nested Simulation

$$\hat{L}_{N,i} = \frac{1}{N}\sum_{j=1}^{N} Y_{ij}; \quad Y_{ij} \sim F_{Y|X_i}$$

- Proposed by Gordy and Juneja (2010)
- Uses standard MC estimator (sample mean of inner replications)
- Finds optimal growth order of M and N
- Zhang et al. (2021) estimate the optimal M and N using a bootstrap method
- Computationally expensive and potentially wasteful use of budget

Nested Simulation Procedures with Metamodeling

1. Regression-based (Broadie et al., 2015)
2. Kernel smoothing (Hong et al., 2017)
3. Kernel ridge regression (Zhang et al., 2022)

Key ideas:

- Pool inner replications from different outer scenarios
- Use metamodeling techniques to approximate the inner simulation model

Metamodeling Approach

- Use supervised learning models to approximate the inner simulation model
- Treat inner simulation as a black-box function
- Approximate $L(\cdot)$ with $\hat{L}^{\mathrm{SL}}_{M,N}(\cdot)$
- Use trained model to make predictions for all $X \in \mathcal{X}$

Actual Implementation - Training

Use the standard nested simulation procedure to generate training data:

- Generate M outer scenarios ($\mathbf{X} = \{X_{(i)}, i = 1, \ldots, M\}$)
- For each outer scenario:
  - Perform N inner simulations
  - Use the sample mean of inner simulations ($\mathbf{Y} = \{\bar{Y}_{(i)}, i = 1, \ldots, M\}$) as the loss prediction of the outer scenario

$$\bar{Y}_{(i)} = \frac{1}{N} \sum_{j=1}^{N} Y_{ij}^{(i)}$$

Train a supervised learning model $\hat{f}(\cdot)$ on the training data.

Actual Implementation - Prediction

Use the trained model to make predictions **using the same outer scenarios**:

- Use the trained model to make predictions, i.e., $\hat{L}_{(i)} = \hat{f}(X_{(i)})$

Use the trained model to make predictions **using new outer scenarios**:

- Generate M outer scenarios ($\mathbf{X}^* = \{X^*_{(i)}, i = 1, \ldots, M\}$)
- Use the trained model to make predictions, i.e., $\hat{L}^*_{(i)} = \hat{f}(X^*_{(i)})$

Actual Implementation - Estimation

Use the predicted losses ($\hat{L}_{(i)}$ or $\hat{L}^*_{(i)}$) to calculate risk measures (e.g., 95%-CVaR).

A Metamodeling-Based Nested Simulation in Dynamic Hedging

by Xintong Li

Introduction

Dynamic Hedging is a risk management technique used in financial engineering to mitigate market risk in financial derivatives.

- **Objective**: Mitigate market risk in VA contracts
- **Instruments**: Stocks, bonds, futures, derivatives
- **Our Simplified Market**: Delta hedging with 1 asset + 1 bond

From an **insurer's perspective**, the goal of dynamic hedging is to mitigate market risk of a sold **index-linked** VA contracts.

Why do we need hedging?

Imagine you just wrote a **put option** on a $100 stock with maturity T and strike K.

- You may want to hedge yourself against the **market risk**.

Without hedging, your **profit and loss** (P&L) is given by:

$$L = -\max(0, K - S_T)$$

It is a **random variable** and depends on the **stock price** at maturity S_T.

However, you can **hedge** yourself against the market risk by delta hedging.

- **Delta**: $\Delta = \frac{\partial P}{\partial S}$
- This is the amount of stock you need to buy or sell to hedge the option.

Why dynamic?

- The delta of the option changes as the stock price changes.
- You need to rebalance your hedge portfolio as the stock price changes.

It is most ideal to **dynamically** adjust your hedge portfolio as the stock price changes.

- **Dynamic Hedging**: Adjust your hedge **continuously**.
- This is not possible due to
  - **transaction cost**,
  - **market impact**,
  - **time not infinitely divisible**.

Therefore, we need to **dynamically** adjust our hedge portfolio across **time periods**.

Dynamic Hedging Mechanics in Our Simplified Market

For time periods $t = 0, 1, \ldots, T-1$:

Portfolio Value at $t-1$:

$$H_{t-1} = \Delta_{t-1} S_{t-1} + B_{t-1}$$

- Δ_t: Stock units
- B_t: Bond amount

Portfolio Value at t:

$$H_t^{bf} = \Delta_{t-1} S_t + B_{t-1} e^r$$

Hedging Error Calculation

Time-t Hedging Error:

$$HE_t = H_t - H_t^{bf} \quad \text{for } t = 1, \ldots, T-1$$

Total P&L:

$$L = H_0 + \sum_{t=1}^{T-1} e^{-rt} HE_t - e^{-rT} H_T^{bf} + V_0$$

Simplifies to:

$$L = \sum_{t=0}^{T-1} \Delta_t (e^{-rt} S_t - e^{-r(t+1)} S_{t+1}) + V_0$$

Nested Simulation Framework

Key Measures:

- Real-world measure ($\mathbb{P}$): Outer scenarios
- Risk-neutral measure ($\mathbb{Q}$): Inner simulations

Data Generation:

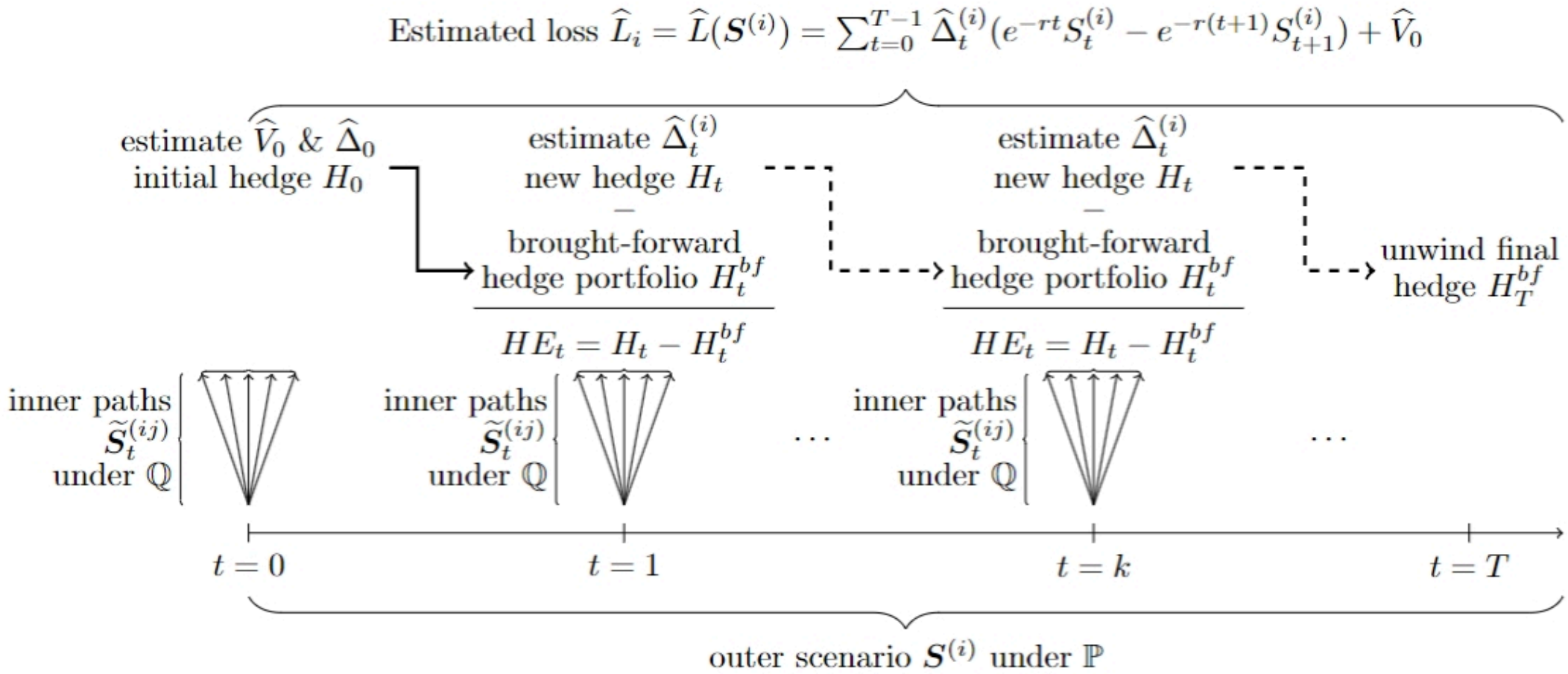
- Outer scenarios: $\mathbf{S}^{(i)} = (S_1^{(i)}, \ldots, S_T^{(i)})$
- Inner paths: $\tilde{\mathbf{S}}_t^{(j)} = (\tilde{S}_{t+1}^{(j)}, \ldots, \tilde{S}_T^{(j)})$

Standard Nested Simulation Algorithm

1. **Simulate Outer Scenarios**: enerate M paths under $\mathbb{P}$
2. **Initial Inner Simulation**: Estimate V_0 and Δ_0 at $t = 0$
3. **Scenario Processing**: For each scenario $\mathbf{S}^{(i)}$:
    - Estimate $\Delta_t^{(i)}$ at each t
    - Calculate losses $\hat{L}_i^{MC}$
4. **CVaR Estimation**: Sort losses and compute:

$$\widehat{CVaR}_\alpha = \frac{1}{(1-\alpha)M} \sum_{i=\alpha M+1}^{M} \hat{L}_{(i)}^{MC}$$

Overview of Standard Nested Simulation for Dynamic Hedging



- This is the standard nested simulation procedure for dynamic hedging of 1 outer

Overview of Standard Nested Simulation for Dynamic Hedging

1. Generate M outer scenarios ($S^{(i)}, i = 1, \ldots, M$)
    - Geometric Brownian motion with regime-switching
2. For each outer scenario:
    - Perform N inner simulations
    - Estimate hedging loss L_i for scenario i
3. Use estimated losses to calculate tail risk measures (e.g., 95%-CVaR)
    - Computational budget: $M * N$ simulations
    - Accuracy depends on both M and N

Critical Observations of the Standard Procedure

1. **Computational Intensity**
   - Each inner simulation requires T time steps
2. **Data Structure** (M outer scenarios)
   - Features: T-dimensional stock paths ($\mathbf{S}$)
   - Labels: Estimated losses ($\hat{L}_i$)
3. **Monte Carlo Properties**
   - Unbiased estimators with variance $\propto 1/N$
4. **Tail Focus**
   - Only tail scenarios ($\approx 5\%$) used for CVaR

A Two-Stage LSTM-based Nested Simulation Procedure

Intuition:

- we want to avoid the **nested** simulation of the standard procedure.
- So we replace the **inner simulation** with a **metamodel**.

Stage 1: Metamodel Training

1. Generate M outer scenarios $(S^{(i)}, i = 1, \ldots, M)$
2. For each outer scenario:
    - Perform N' inner simulations $(N' << N)$
    - Estimate noisy hedging loss $\hat{L}_i$ for scenario i
3. Train LSTM metamodel on $(S^{(i)}, \hat{L}_i)$ pairs
4. Use trained metamodel to identify m potential tail scenarios

A Two-Stage LSTM-based Nested Simulation Procedure (Continued)

Stage 2: Refined Estimation

5. For each identified tail scenario:
    - Perform N inner simulations (same as standard procedure)
    - Estimate refined hedging loss $\hat{L}_i$
6. Use refined loss estimates to calculate risk measures

- Computational budget: $M * N' + m * N$ simulations
- Typically uses $15\% - 30\%$ of standard procedure's budget
- Accuracy comparable to standard procedure with proper safety margin

Key Advantages of a Two-Stage LSTM-based Procedure

1. Substantial computational savings ($70\% - 85\%$ reduction)
2. Maintains accuracy comparable to standard procedure
3. LSTM metamodel shows resilience to noisy training data
4. Can distinguish between tail and non-tail scenarios effectively
5. Addresses regulatory concerns by using actual simulations for final estimates

A Single-Stage LSTM-based Nested Simulation Procedure

1. Train LSTM metamodel (same as Two-Stage Procedure stage 1)
2. Use trained LSTM to predict losses for all M scenarios
3. Calculate α-CVaR directly using predicted losses

Key Advantages of a Single-Stage Procedure

- Uses metamodel predictions to estimate risk measures directly
- More efficient than a two-stage procedure
- Can estimate risk measures requiring full loss distribution
- Avoids calibration of safety margin

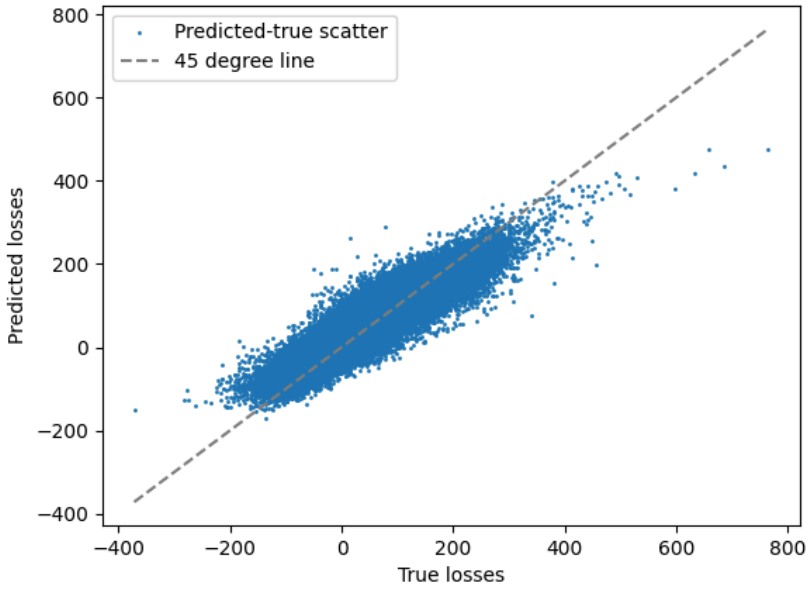
Experimental Setup

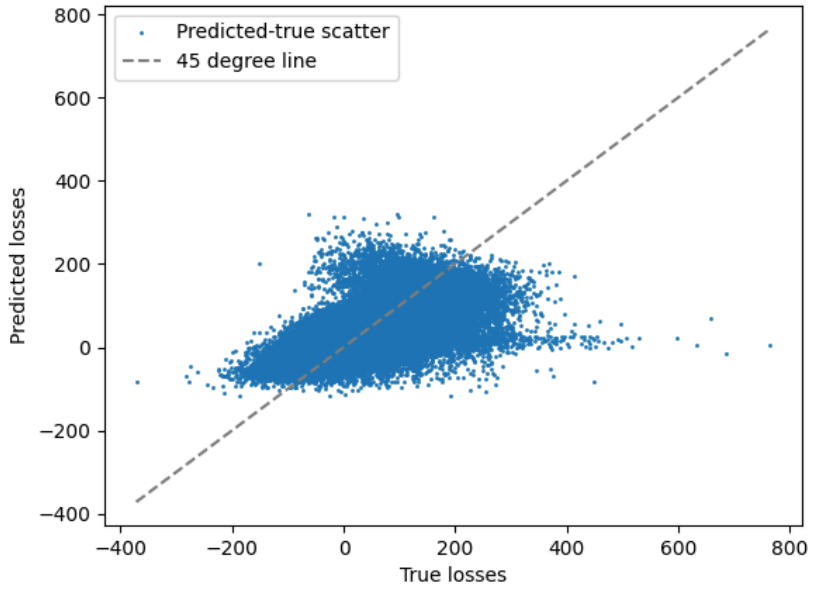
- Estimating 95% CVaR of hedging loss for GMWB contract
- 20-year maturity, monthly delta-hedging (240 periods)
- Regime-switching geometric Brownian motion for underlying asset
- Benchmark: 100,000 outer scenarios, 100,000 inner replications

Metamodel Architectures Compared

1. Regression (MLR, QPR)
2. Feedforward Neural Network (FNN)
3. Recurrent Neural Network (RNN)
4. Long Short-Term Memory (LSTM)

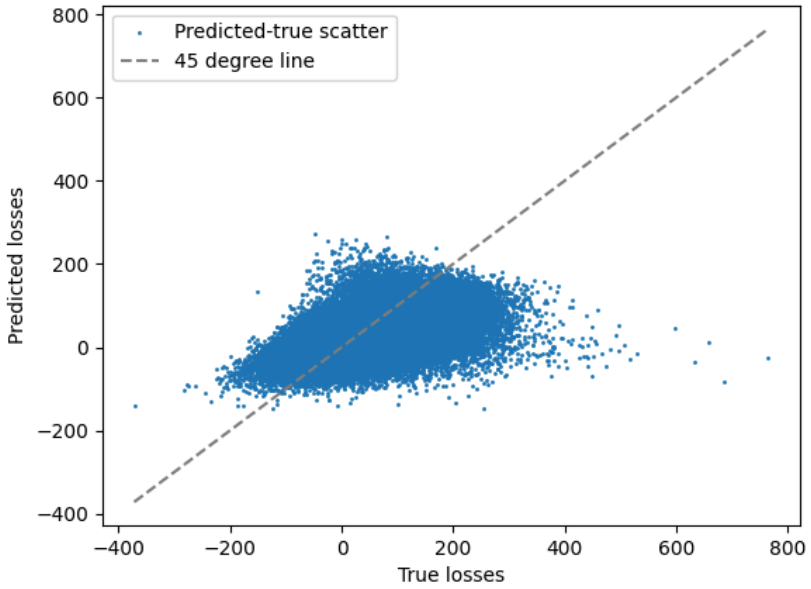
RNN vs LSTM Performance

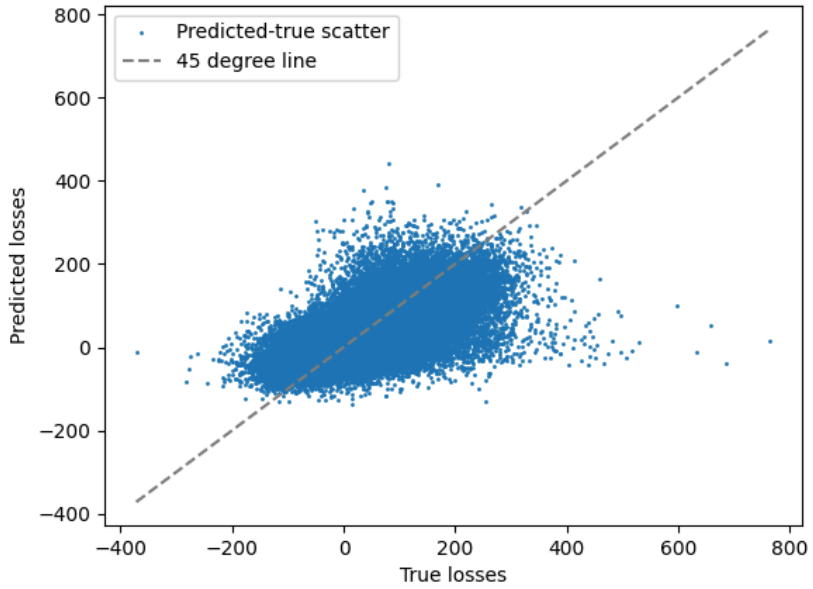




- LSTM overcomes vanishing gradient problem in RNN
- LSTM better captures long-term dependencies in 240-dimensional time series

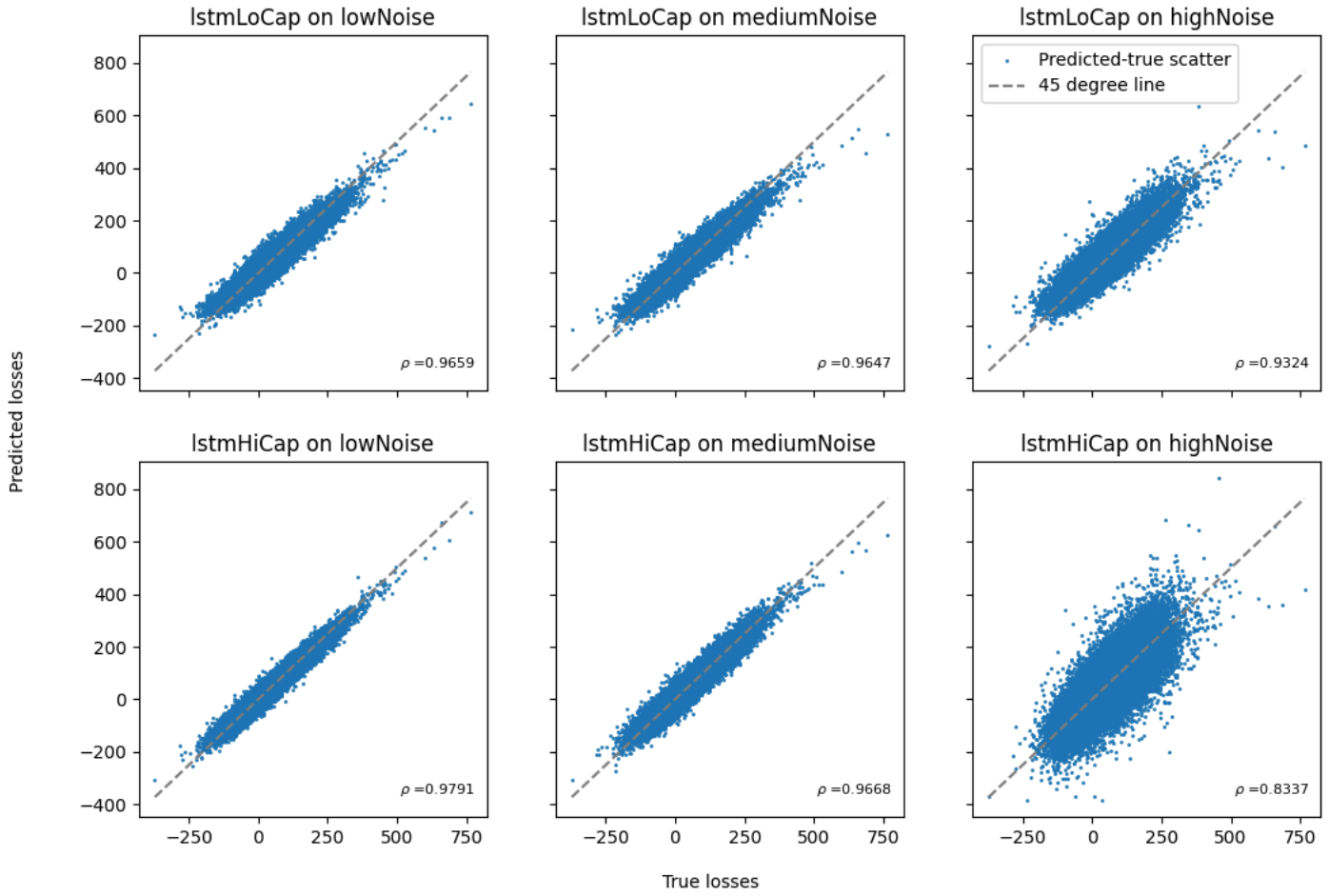
Regression vs Neural Network Metamodels



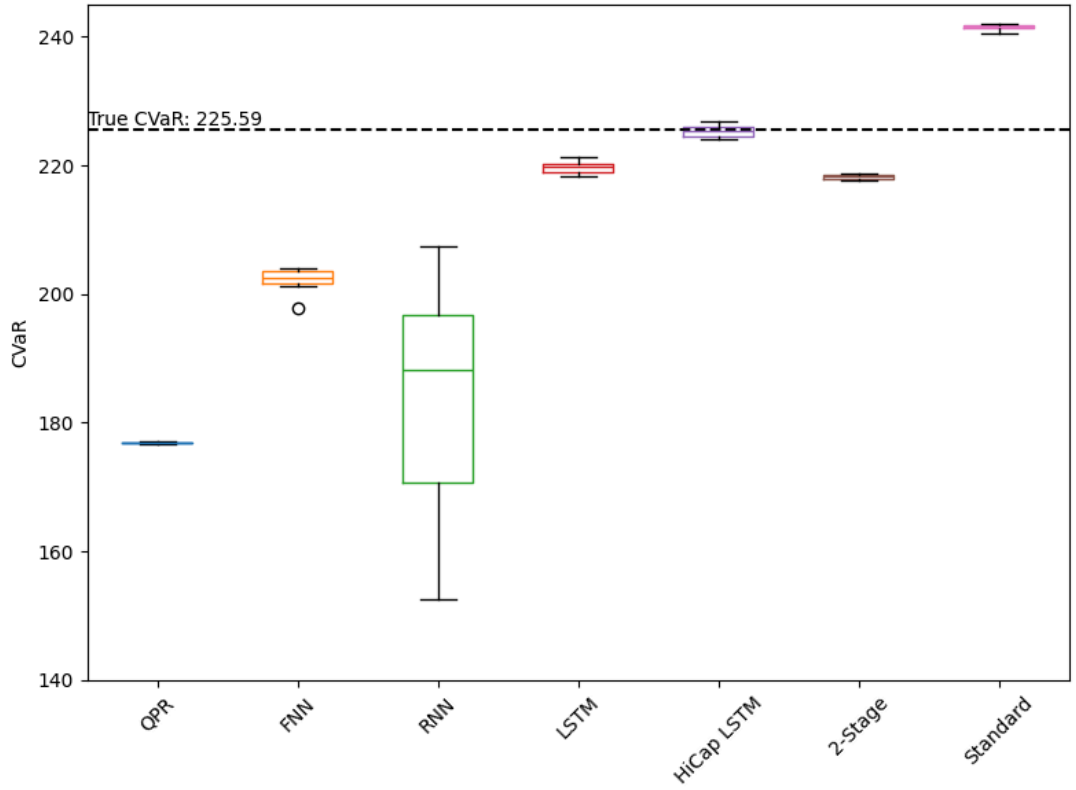


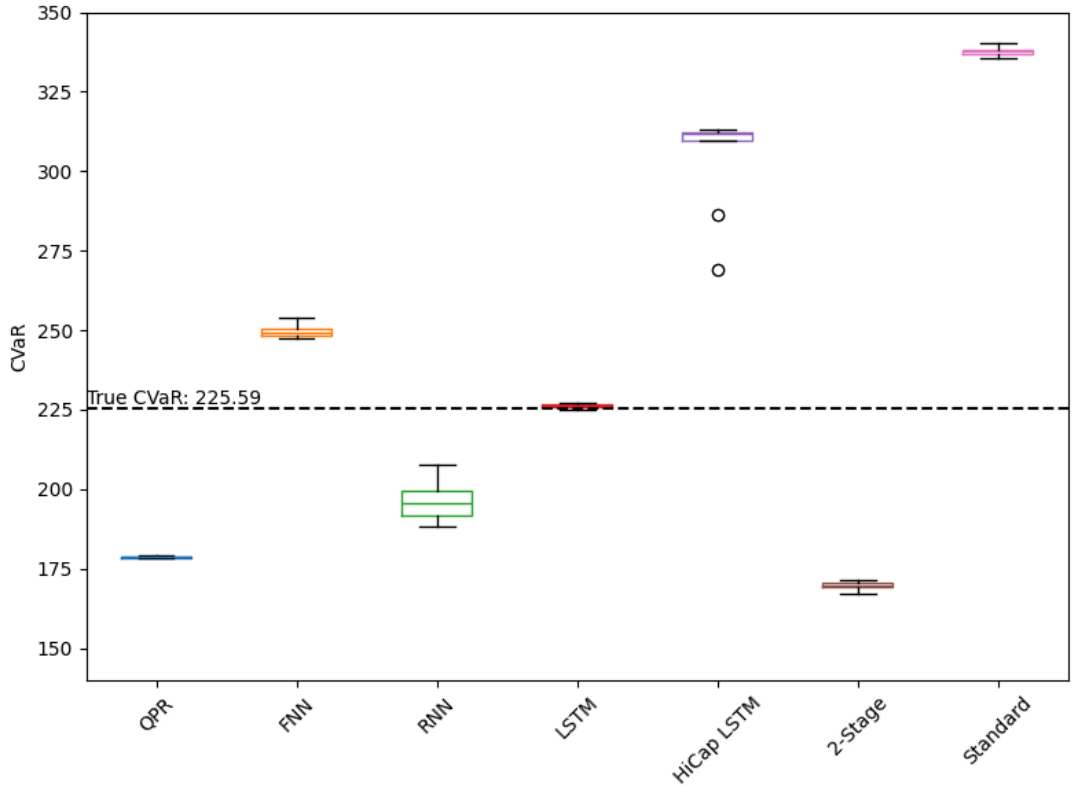
- Regression metamodels (MLR, QPR) generalize poorly to true data
- Neural network metamodels show better generalization

LSTM Performance with Different Noise Levels



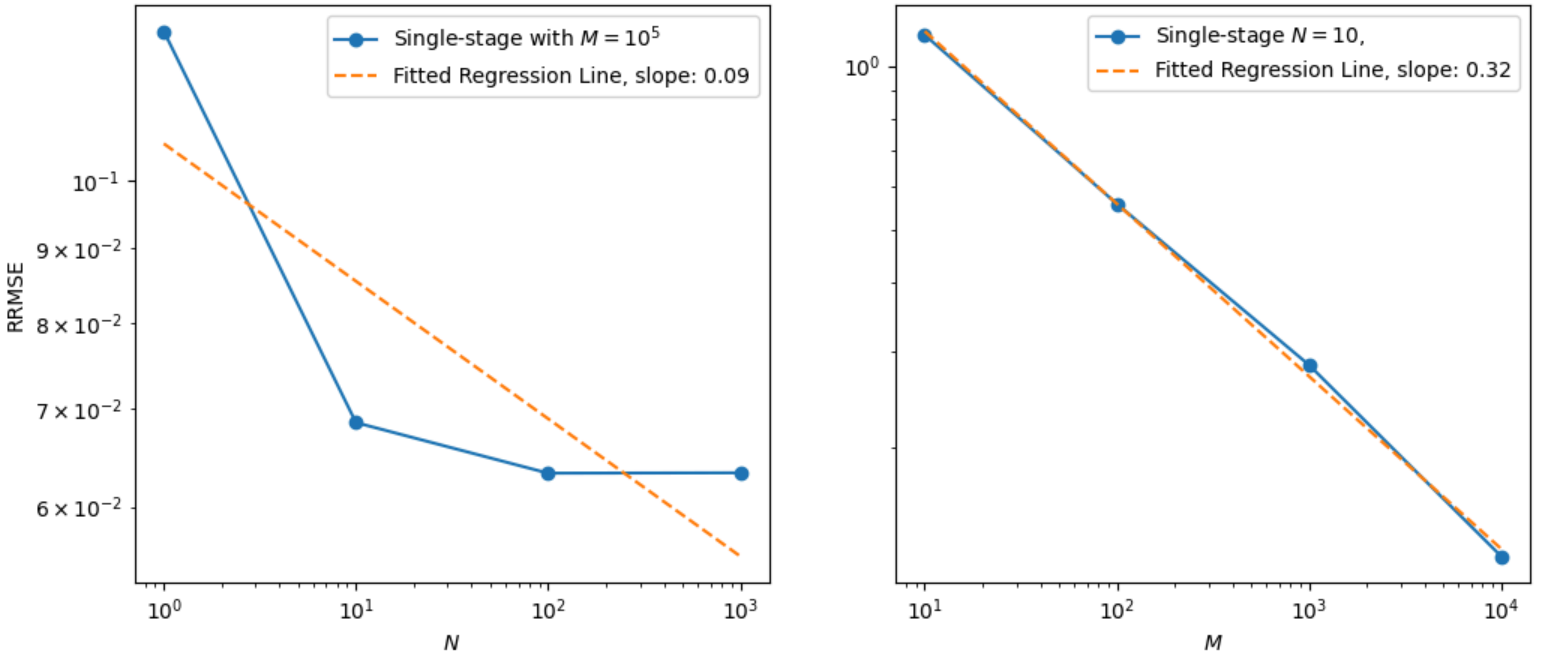
CVaR Estimates Comparison





- LSTM metamodels consistently outperform standard procedure
- High-capacity LSTM produces most accurate estimates

Single-Stage Procedure Convergence Analysis



- RRMSE decreases as simulation budget increases
- Higher convergence rate with increased data quantity
- Diminishing returns for increasing inner replications (N > 100)

Key Findings

1. LSTM metamodels show resilience to high levels of noise in training labels
2. Deep neural networks can learn true complex dynamic hedging model despite noisy data
3. Two-stage procedure addresses regulatory concerns by avoiding direct use of metamodel predictions
4. Single-stage procedure is more efficient and versatile for various risk measures
5. Increasing outer scenarios more beneficial than increasing inner replications
6. High-capacity LSTM requires training labels with lower noise

Future Directions

1. Apply deep neural network metamodels to other financial risk management tasks
2. Investigate impact of label noise on other deep learning models (CNNs, Transformers)
3. Explore optimal network architectures for different simulation models

Deep Dynamic Hedging of Variable Annuities

by Xintong Li

Introduction

- We want to use an **Reinforcement Learning** (RL)a agent to hedge VA contracts.

Instead of using a **delta** hedging strategy, there are several reasons we want to use RL:

1. **Transaction costs** are too high for delta hedging.
2. We want **flexibility** in our hedging strategy.
3. We don't want to assume any **asset model** to begin with.

Key Advantages of RL:

- Learns directly from market dynamics
- Handles discrete hedging periods
- Incorporates transaction costs natively

How to do Deep Dynamic Hedging?

Key Components:

1. A **Markov Decision Process** (MDP) that describes the dynamic hedging problem
2. A policy network that outputs a **hedging action** based on the current state
3. A **reward function** that measures the performance of the hedging action

Implementation Challenges

State Representation:

- Translated directly from the standard nested simulation procedure

Reward Engineering:

- Balancing immediate vs terminal rewards
- Penalizing excessive trading

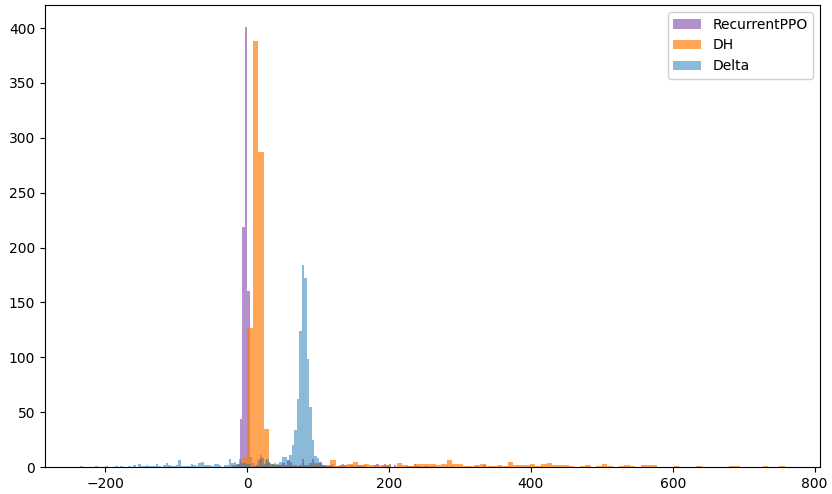
Training Considerations:

- Market regime transitions
- Rare event simulation
- Portfolio liquidity constraints

What We've Tried

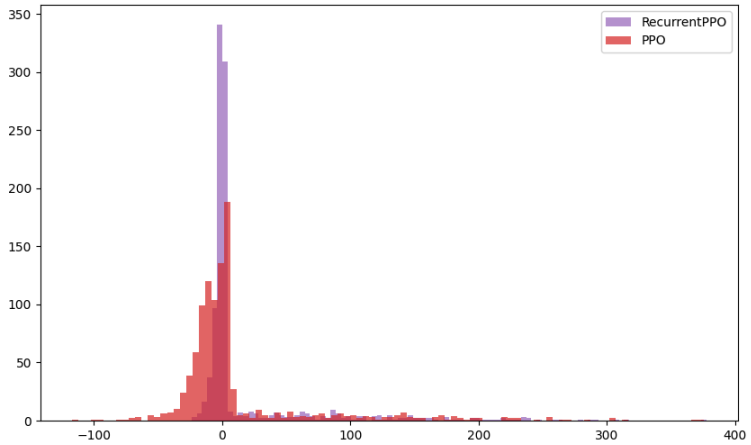
1. Applied Proximal Policy Optimization (PPO) to VA hedging
2. Compared recurrent PPO with LSTM to standard PPO
3. Tested PPO against traditional delta hedging
4. Experimented with different asset models and VA riders

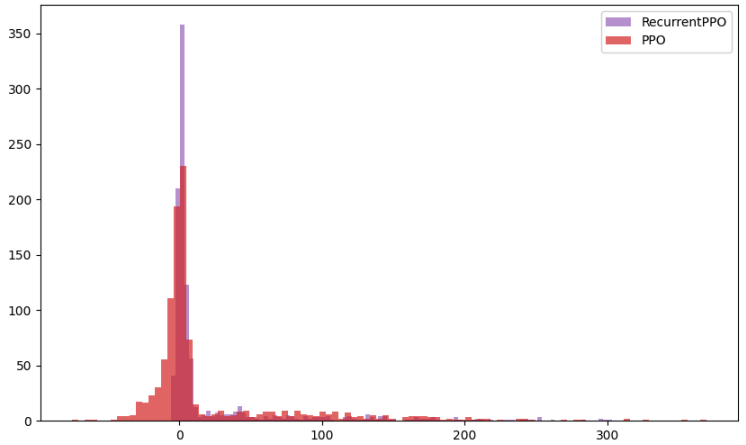
Recurrent PPO vs Deep Hedging



- Compared recurrent PPO with LSTM to deep hedging algorithm
- Both methods applied to GMMB rider
- Recurrent PPO shows competitive performance

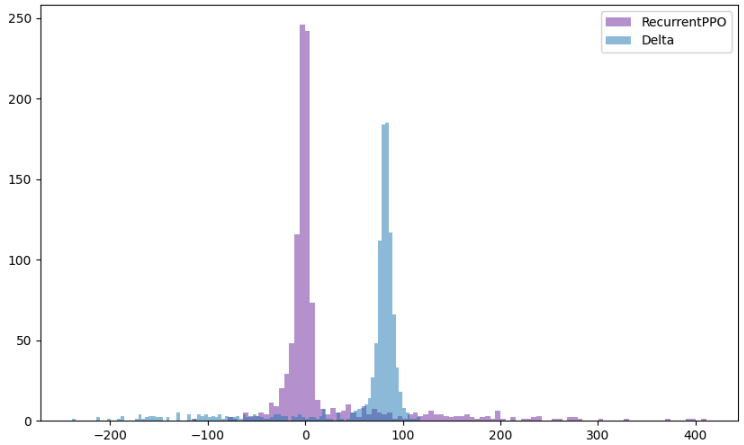
Standard PPO vs Recurrent PPO

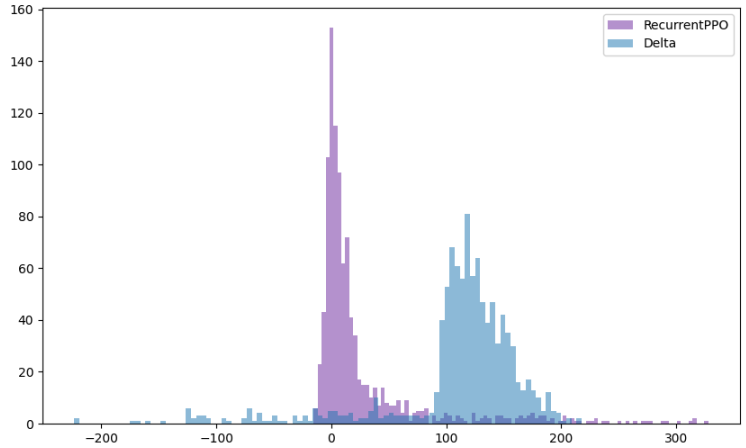




- Recurrent PPO outperforms standard PPO for both GBM and regime-switching GBM
- LSTM component helps capture historical information, improving hedging decisions

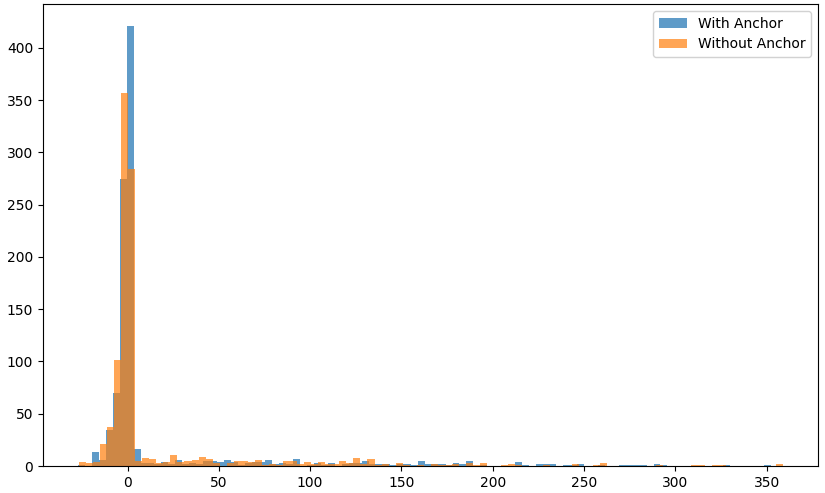
PPO vs Delta Hedging with Transaction Costs





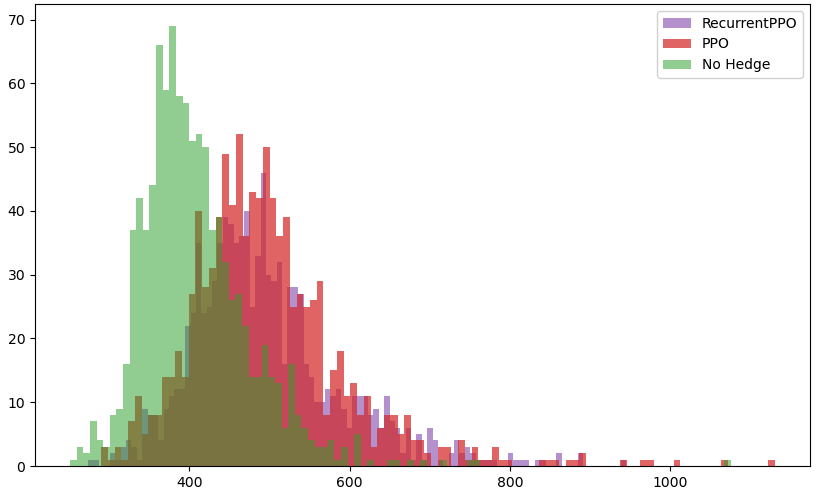
- Delta hedging outperforms PPO with low transaction costs
- PPO adapts better as transaction costs increase
- Demonstrates PPO's ability to learn from the environment

PPO Performance with Model Information



- Tested PPO with and without model information (asset model, current liability value)
- Surprisingly, additional information didn't significantly improve performance
- Suggests PPO can learn effective strategies without explicit model knowledge

Challenges: PPO with GMWB



- PPO struggled with GMWB rider
- Hedging errors more dispersed compared to GMMB
- Indicates need for more training or enhanced approaches for complex VA products

Where We've Faced Challenges

1. GMWB hedging: PPO struggled with increased complexity
2. Sample efficiency: More episodes needed for satisfactory performance on complex products
3. Generalization: Difficulty in transferring knowledge between different VA riders
4. Computational costs: High resources required for training, especially for complex scenarios
5. Model-free limitations: Current approach may not fully leverage available financial knowledge

Future Directions: Adressing the Challenges

1. Enhance sample efficiency of RL algorithms for complex VA products
2. Develop more robust transfer learning techniques between VA types
3. Explore hybrid approaches combining model-based and model-free methods
4. Investigate meta-learning for faster adaptation to new market conditions
5. Improve interpretability of RL models for regulatory considerations
6. Extend to more complex market models (e.g., stochastic volatility, jump diffusion)