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Valuation of a Mixture of GMIB and GMDB Variable Annuity

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Abstract: The Guaranteed Minimum Income Benefit (GMIB) and Guaranteed Minimum Death Benefit (GMDB) are options that may be included at the inception of a variable annuity (VA) contract. In exchange for small fees charged by the insurer, they give the policyholder a right to receive a guaranteed minimum level of annuity payment (GMIB) and a guaranteed minimum level of payment when the policyholder dies (GMDB), respectively. A combination of these two options may be attractive since it protects the policyholder's investment from potential poor market behavior as well as mortality risk during the accumulation phase. This study examined the pricing of a composite variable annuity incorporating both the GMIB and GMDB options (a Guaranteed Minimum Income–Death Benefit, notated GMIDB). We used a non-arbitrage valuation method, decomposed the GMIDB value into two parts, and derived an analytical pricing formula based on a constant fee structure. The formula can be used to determine the fair fee to be charged. We conducted comprehensive sensitivity analyses on critical parameters to determine what drives the value of a GMIDB option. Our approach offers a simple and deterministic way to price a VA embedded with the GMIDB option. Our numerical findings suggested that the annuity conversion rate, age of the policyholder, and volatility of risky investments are significant in the valuation of a GMIDB option.

Keywords: variable annuity; Guaranteed Minimum Income–Death Benefit (GMIDB); mortality; dichotomy algorithm

MSC: 91G05

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1. Introduction

In 1952, the American Teachers' Insurance and Annuity Association (TIAA) revolutionized the financial landscape by introducing a groundbreaking variable annuity product tailored exclusively for university educators. This innovative financial instrument became an integral component of educators' prospective pension arrangements. A variable annuity (VA) operates as an insurance product where the policyholder initiates the contract by remitting an initial premium to the insurance company. This capital is then strategically invested within financial markets for a designated period, commonly known as the accumulation phase. The pivotal moment arrives when the policyholder decides to convert the investment into a series of variable annuity payments.

The realm of variable annuities offers policyholders various embedded options designed to "safeguard" their benefits, albeit accompanied by additional fees. These options include Guaranteed Minimum Withdrawal Benefits (GMWBs), Guaranteed Minimum Death Benefits (GMDBs), Guaranteed Minimum Accumulation Benefits (GMABs), and Guaranteed Minimum Income Benefits (GMIBs), each providing unique features and risk management strategies. For an in-depth understanding of these options, interested readers are encouraged to explore Boyle and Hardy's comprehensive work from 2003.

In the United States, the sales of variable annuities have witnessed a remarkable surge, reaching USD 102.6 billion in 2022, constituting a substantial 33% of the US annuity market according to LIMRA data. The prudential pricing and effective risk management of these products carry significant implications, crucially informing the assessment of guarantee costs and furnishing valuable insights for the prospective development of such offerings.

Within this context, it becomes evident that elderly policyholders constitute a substantial portion of the demographic interested in GMIB VAs. For these individuals, the risk of mortality during the accumulation phase is a critical consideration. Consequently, we propose an embedded combination of the GMIB and GMDB options, referred to as the GMIDB. This hybrid structure adeptly addresses both the risk of mortality during the accumulation phase and the appeal of a lifelong, guaranteed, and stable income. By seamlessly integrating the advantages of both options, the GMIDB emerges as a compelling and more attractive choice for the elderly when compared to the singular options available in the market.

Our work can be summarized as follows. First of all, we propose the product, an embedded combination of the GMIB and GMDB options. While closed-form formulas are typically unavailable in much of the literature on pricing-related products, we derived an analytical valuation outcome in a complete market and identified the appropriate fee rate that should be imposed. Secondly, we employed Vasicek's classical model to characterize the stochastic interest rate, with the fee being assessed as a fixed percentage of the investment account. In our model, the premium is dynamically allocated to both risky and risk-free assets, accounting for the correlation between the short rate and underlying assets. Lastly, we present a comprehensive sensitivity analysis of our model's parameters, which can serve as valuable guidance for insurance companies when valuing and managing risks associated with analogous financial options in their product portfolio.

1.1. *Related Literature*

GMDBs have garnered significant attention within the realm of variable annuities, particularly in academic research. Milevsky and Posner (2001) [1] comprehensively analyzed diverse death benefit clauses encompassed within GMDBs, such as the premium refund clause, cumulative clause, and maximum clause. They derived analytical pricing under the exponential mortality model. Ulm (2006) [2] proposed a comprehensive framework for pricing various guarantee clauses within GMDBs and embedded variable annuities.

Further advancements in GMDB analysis were made by Gerber et al. (2012) [3], who introduced a discount density method to evaluate GMDBs for various variable- and equity-linked annuities within the Brownian motion model. This result was extended under the double-exponential jump diffusion model (Gerber et al., 2013) [4] and discrete model (Gerber et al., 2015) [5]. Fan et al. (2015) [6] extended Gerber et al.'s (2012) [3] work on GMDBs into a regime-switching environment. Ignatieva et al. (2016) [7] introduced a framework for the pricing and hedging of GMMBs, GMDBs, and GMIBs under the regime-switching model, utilizing a Fourier space–time stepping (FST) algorithm for their computations.

Wang et al. (2021) [8] further extended this work under the regime-switch model, employing a complex Fourier series expansion. Feng et al. (2019) [9] explored the use of an exponential Lévy process instead of geometric Brownian motion, as seen in the classical Black–Scholes model, for valuing a GMDB option based on equity returns. Yu et al. (2019) [10] and Zhang and Yong (2019) [11] expanded Gerber et al.'s (2012) [3] work by approximating the density discount function using a Fourier cosine series expansion (COS) method and Laguerre basis time-until-death variable, respectively. Ai and Zhang (2022) [12] also utilized COS to study the valuation of life-contingent lookback options embedded in GMDB variable annuities. Wang and Liu (2022) [13] considered a novel and efficient method to price equity-linked GMDBs with European-style geometric Asian and arithmetic Asian payoffs. They assumed that the underlying asset price process followed the regime-switching Lévy model and used the complex Fourier series (CFS) expansion method to derive the approximate value of GMDB products.

In terms of computational techniques, Zhang et al. (2020) [14] applied the Fast Fourier Transform (FFT) for efficient analysis. Kirkby and Nguyen (2021) [15] analyzed GMDB forms with payoffs dependent on a dollar-cost averaging (DCA)-style periodic investment in the risky index. Ulm (2022) [16] obtained analytic solutions for allocation and GMDB pricing when the market is incomplete and the policyholder seeks to maximize the lifetime utility of the VA/GMDB combination. Chong et al. (2023) [17] proposed a two-phase deep reinforcement learning approach for hedging variable annuity contracts with both GMMB and GMDB riders, which could address model miscalibration in Black–Scholes financial and constant-force-of-mortality actuarial market environments.

While comprehensive research on GMIBs in the academic domain appears to be relatively limited compared to research on GMDBs, there exists notable literature on similar guarantee structures in Europe, particularly concerning a variable annuity guarantee known as the guaranteed annuities option (GAO). Boyle and Hardy (2003) [18], Biffis and Millosovich (2006) [19], Ballotta and Haberman (2003) [20], and Pelsser (2003) [21] have explored the pricing and hedging of the GAO within the European context. However, it is important to note that findings related to guaranteed annuity options may not be directly applicable to GMIBs due to inherent differences in the income structure of these two products.

Marshall et al. (2010) [22] extended the results obtained by Bauer et al. (2008) [23] to accommodate interest rates following a stochastic process using Hull and White's interest rate model. Deelstra and Rayée (2013) [24] proposed pricing methodologies for both GAOs and GMIBs, incorporating the Hull–White Gaussian interest rate model. They further considered scenarios where the volatility of the underlying asset becomes a stochastic process dependent on the asset price itself, often referred to as the local volatility model.

Kling et al. (2014) [25] expanded upon Bauer et al.'s work on GAOs and GMIBs by utilizing a Cox–Ingersoll–Ross (CIR) model for the interest rate and a modified forward mortality model based on mortality assumptions. Kalife et al. (2018) [26] delved into policyholder behavior and optimal strategies concerning GMIB variable annuities, specifically considering potential partial-withdrawal policies. Jones and Ocejo (2019) [27] simulated the pricing value of a GMIB with a reset option, introducing an additional option enabling the policyholder to defer the annuitization date to a later time. Bing et al. (2020) [28] proposed an efficient willow tree method to evaluate VAs embedded with GMIBs and GMDBs on the market, considering various stochastic models. Moreover, their tree structure is also applicable for computing the dollar delta, value at risk (VaR), and conditional tail expectation (CTE) in hedging and risk-based capital calculation. Sharma et al. (2022) [29] proposed a mixed FBM (MFBM) model with jumps. They evaluated variable annuities with different riders and analyzed the proposed model numerically to determine the impact of mortality risk. They performed a comparison between eight stochastic models to obtain the most suitable mortality model for a dataset of the US male population and obtained the price of GMIBs and GMDBs using the forecasted values from the fitted mortality model.

For a broader perspective on the valuation of guaranteed benefit variable annuities, readers can refer to Feng and Jing (2017) [30], Feng and Yi (2019) [31], Liu (2021) [32], Moenig (2021) [33], and Gweon and Li (2023) [34] for additional insights and research outcomes.

1.2. Outline

The subsequent sections of this paper are organized as follows. Section 2 elucidates the payoff nature of a GMIDB and outlines the assumptions made. Section 3 expounds on the model utilized to evaluate a GMIDB. In Section 4, we introduce the analytical formula for pricing a GMIDB. In Section 5, we present a sensitivity analysis of the model parameters. Finally, Section 6 provides concluding remarks. We include proofs of the analytical formula in Appendix A for reference.

2. GMIDB Payoff

In this section, we elucidate the policyholder's payoff within a variable annuity incorporating the GMIDB option. This involves elucidating the operational framework of variable annuities, GMIBs, and GMDBs. Variable annuities are financial products that allow individuals to invest funds, typically in both risky and risk-free assets, to accumulate a retirement nest egg. The GMDB component introduces a guaranteed minimum death benefit, ensuring that if the policyholder passes away during the accumulation phase, the beneficiary receives the greater form of the account value or a guaranteed benefit base. The base often increases annually based on a certain policy during the accumulation phase, while the account value is determined by the performance of the investment assets. Upon surviving the accumulation phase, the policyholder is presented with the choice of either receiving the account value immediately or converting the guaranteed benefit base into a lifelong annuity income (GMIB component). The annuity phase may vary, and the income benefit is determined by factors such as the annuity conversion rate and the policyholder's choice between the account value and the guaranteed benefit base. The GMIDB structure, a combination of the GMIB and GMDB, aims to provide policyholders with a combination of growth potential through investment and a safety net in the form of guaranteed benefits, addressing both market risks and longevity concerns.

As introduced above, the GMIDB option payoff consists of two components:

- In the event of the policyholder's demise during the accumulation phase, as per the guaranteed minimum death benefit policy, the policyholder is entitled to the greater form of the account value (denoted as $A(t)$, representing the account value at time t) and the guaranteed benefit base (denoted as $B(t)$), both assessed at the time of death. This component constitutes the guaranteed minimum death benefit, denoted as $Y_D(t)$.
- If the policyholder survives the accumulation phase, they possess the option to either receive the account value instantly or convert the benefit base into a life-long annuity income. This component represents the guaranteed income benefit, denoted as $Y_I(T)$, where T signifies the duration of the accumulation phase.

In practical terms, the calculation of the guaranteed benefit base often follows the roll-up policy, where the guaranteed death benefit is incremented at an annual rate denoted by the constant roll-up rate r_g, i.e.,

$$B(t) = A_0(1 + r_g)^t, \tag{1}$$

where r_g is the constant roll-up rate specified by the insurer, and A_0 is the initial account value. In practice, r_g may take values varying from 4% to 6%, according to the expected performance of the market.

If we denote by τ the survival span of the policyholder (from the time of option purchase), and T the accumulation phase, the death benefit component is defined as

$$Y_D(\tau) = I_{\{\tau \leq T\}} max\{A(\tau), B(\tau)\}, \tag{2}$$

where $I_{\{\tau \leq T\}}$ is the indicator function, assuming a value of 1 when $\tau \leq T$ is true and 0 otherwise.

Once the policyholder surpasses the accumulation phase, they have the option to either instantly receive the account value, $A(T)$, or convert the guaranteed benefit base into a life-long annuity. The annuity phase duration can vary in practice, with some insurers offering a fixed period such as 20 or 15 years instead of a life-long annuity. If we additionally assume that the policyholder will choose the greater of these two values, the income benefit part $Y_I(T)$ may then be written as

$$Y_I(T) = I_{\{\tau > T\}} max\{A(T), B(T)g\ddot{a}_{x+T}\}, \tag{3}$$

where g represents the annuity conversion rate and $\ddot{a}_{x+T}$ is the discounted market price at time T of a one-dollar annuity payment at time $x+T$. For instance, if the policyholder chooses to convert their guaranteed benefit base into an annuity, they will receive g times the benefit base each year until their death. The value of g is determined by the insurance company based on considerations of mortality, investment risk, and future interest rate assumptions, which will be further discussed in subsequent sections. The formula for $\ddot{a}_{x+T}$ is given by

$$\ddot{a}_{x+T} = \sum_{j=0}^{\infty} {}_jP_{x+T} \cdot e^{-\int_T^{T+j} r_s ds}, \tag{4}$$

where P_{x+T} is the probability that the policyholder survives the $x+T$ years after the option purchase, and r_s is the instantaneous interest rate at time s.

There are several points to note regarding this payment equation:

- $B(T)$, the guaranteed benefit base, cannot be taken out in cash. If the policyholder wishes to receive the guaranteed benefit base instead of the account value $A(T)$, their only choice is to convert this payment into an annuity.
- In practice, even if the guaranteed benefit base $B(T)$ is greater than the account value $A(T)$, the policyholder may still choose to receive the account value in cash, since the insurance company may set the conversion rate g conservatively.
- The conversion rate g may have a large impact on the value of a GMIDB, which we will address further in later sections.
- In most cases, if a GMIDB option is included, the policyholder cannot withdraw the option during the accumulation phase, unless they are willing to forfeit this option.

In the valuation of a GMIDB, the following key assumptions are made:

- Choice of Benefit: Based on their expectations, the policyholder will opt for the more favorable option between the account value $A(T)$ and the annuity determined by the guaranteed benefit base $B(T)g\ddot{a}_{x+T}$. The specifics of these expectations will be discussed in subsequent sections.
- Life-Long Payment: If the policyholder survives the accumulation phase and opts to convert the guaranteed benefit base into an annuity, they will receive payments annually until their passing, i.e., the total discounted payment received can be written as

$$B(T)g\ddot{a}_{x+T} = B(T)g \sum_{j=0}^{\infty} {}_jP_{x+T} \cdot e^{-\int_T^{T+j} r_s ds}. \tag{5}$$

  This annuity is life-related, aligning with common insurance practices. Although there may be options for payments over a specific term (e.g., a 20-year term, in which case the formula of $\ddot{a}_{x+T}$ may be conservatively different), in practice we assume a life-related annuity for simplicity.
- Premium Payment and Withdrawals: The policyholder makes a single premium payment $A(0)$ at the outset and refrains from making subsequent withdrawals. While, in practice, policyholders might have the option for additional premiums or withdrawals with potential penalties, we consider a one-time premium payment without withdrawals for this valuation.
- Annuity Payment Frequency: Despite annuities typically being paid monthly in practice, we assume an annual annuity payment frequency in this paper. This assumption is not anticipated to significantly impact the GMIDB value.
- Exclusion of Certain Fees: Administrative fees, investment management fees, and other similar fees, apart from the fair fee rate, are not factored into the valuation. These fees typically range from 0.5% to 3% of the policyholder's account value annually during the accumulation phase. A consideration of these fees could be an extension of this paper, assuming reasonable certainty regarding their impact.

3. Valuation Model

In this section, we proceed to evaluate the GMIDB option using the no-arbitrage pricing methodology. Within the framework of a probability space (Ω, F, P) and the respective sigma algebra filtration $\mathbf{F}_t$, where P is the risk-neutral measure, in a perfectly competitive market, the dynamics of the investment account value can be expressed as the following stochastic differential equation (SDE):

$$dS(t) = r_t S(t)dt + \sigma_s S(t)dW_s^P(t). \tag{6}$$

Here:

- $S(t)$ represents the value of risky assets.
- r_t is the free interest rate at time t.
- σ_s denotes the annualized instantaneous volatility of the investment account (σ_s is a constant).
- $W_s^P(t)$ stands for a standard Brownian motion under the risk-neutral measure P.

For modeling interest rates, the Hull–White model, an extension of the Vasicek model, is often employed. Namely, the free rate is modeled under the risk-neutral measure P by the following SDE:

$$dr_t = k(\theta - r_t)dt + \sigma_r dW_r^P(t). \tag{7}$$

Here:

- k, θ, and σ_r are constants representing the average regression speed, average long-term rate, and fluctuation standard deviation, respectively.
- $W_r^P(t)$ denotes another standard Brownian motion under the risk-neutral measure P.

In our model, we assume that these two processes are correlated according to the following correlation structure:

$$Cov(dW_s^P(t), dW_r^P(t)) = \rho dt. \tag{8}$$

Regarding the mortality assumption, we will employ the model of Benjamin Gompertz (1825). The mortality force μ_x is given by

$$\mu_x = \frac{1}{b} e^{\frac{x-m}{b}}, \tag{9}$$

where b and m are constant parameters that can be calibrated by empirical data.

In this context, the insurance company does not collect an additional option premium at the initiation of the GMIDB option. Instead, fees are directly deducted from the investment account. For this paper, we assume that the fee is charged at a constant rate. Consequently, the change in the value of the investment account can be described as follows:

$$dA(t) = A(t)(r_t - \alpha)dt + \sigma_s \pi A(t) dW_s^P(t), \tag{10}$$

where

- α is the fee rate. One of the main purposes of this paper is to give a reasonable suggestion for a fair fee rate.
- π is the ratio of investment on risky assets, and $1 - \pi$ is the ratio of investment on risk-free assets. π is a constant parameter.

Throughout this paper, we will denote by $V(\alpha)$ the GMIDB value (discounted at time 0) of a certain fee rate level α:

$$\begin{aligned} V(\alpha) &= \mathbf{E_p}\left[(e^{-\int_0^\tau r_s ds} Y_D(\tau) + e^{-\int_0^T r_s ds} Y_I(T)\right] \\ &= \mathbf{E_p}\left[e^{-\int_0^\tau r_s ds} I_{\{\tau \le T\}} max\{A(\tau), B(\tau)\} + e^{-\int_0^T r_s ds} I_{\{\tau > T\}} max\{A(T), B(T) g \ddot{a}_{x+T}\}\right] \end{aligned} \tag{11}$$

When all other parameters remain constant, it is important to note that a higher fee rate will result in a faster depletion of the account value. The decrease in the account value will subsequently lead to a reduction in the GMIDB value, at least within a certain range. However, once the fee rate surpasses a particular threshold, the GMIDB value will no longer decrease with further increases in the fee rate. This is because the policyholder has the option to opt for an annuity based on the guaranteed benefit base, which remains independent of the fee rate.

4. Analytical Formula for GMIDB Value

In this section, we present our analytical formula for a variable annuity with a GMIDB option. As discussed in previous sections, the GMIDB value can be decomposed into two components: death benefit and income benefit. We will discuss them in terms of their discounted values (at the time of option purchase). Detailed proofs of the lemmas and theorems are provided in Appendix A. We will begin by addressing the income benefit:

$$\begin{aligned} V_I(\alpha) &= \mathbf{E_p}\left[e^{-\int_0^T r_s ds} I_{\{\tau>T\}} max\{A(T), B(T)g\ddot{a}_{x+T}\}\right] \\ &= \underbrace{E_P[e^{-\int_0^T r_s ds} A(T) I_{(A(T)>B(T)g\ddot{a}_{x+T})]}}_{(1)} P(\tau>T) \\ &+ \underbrace{E_P[e^{-\int_0^T r_s ds} B(T)g\ddot{a}_{x+T} I_{(A(T)\le B(T)g\ddot{a}_{x+T})]}}_{(2)} P(\tau>T). \end{aligned} \tag{12}$$

Note that

$$B(T) = A_0(1+r_g)^T, \tag{13}$$

and T, r_g, and g are constants; therefore, $B(T)$ has a constant value if parameters are given. Taking discounted factor $\ddot{a}_{x+T}$ into consideration, we may determine the value of $B(T)g\ddot{a}_{x+T}$.

Lemma 1. *If the policyholder survives the accumulation phase T and chooses to convert the guaranteed benefit base $B(T)$ into a life-long annuity income, the discounted value of this annuity (discounted at the end of the accumulation phase) can be written as*

$$\begin{aligned} &B(T)g\ddot{a}_{x+T} = A(0)(1+r_g)^T g \sum_{j=0}^{\infty} {}_jP_{x+T} \cdot \\ &\exp\Big((\tfrac{\sigma_r^2}{2k^2}-\theta)j - (\tfrac{r_0-\theta}{k}e^{-kT}+\tfrac{\sigma_r^2}{k^3})(1-e^{-kj}) + \tfrac{\sigma_r^2}{4k^3}(1-e^{-2kj}) + \tfrac{\sigma_r^2}{4k^3}(1-e^{-kj})^2(1-e^{-2kT}) \Big). \end{aligned} \tag{14}$$

The proof is achieved by applying Ito's integral and Fubini's theorem to Vasicek's interest model. Readers may find a detailed proof in Appendix A. In our discussions, we will use the DBB (discounted benefit base) to represent this value. It is essential to note that while the guaranteed benefit base $B(T)$ remains a constant with specific parameters, the DBB is not a constant. Its randomness arises from the stochastic interest rate encapsulated within the discounted factor.

To handle the account value, Girsanov's theorem for measure transformation is applied. Let Q_1 be a new risk-neutral measure. The Radon–Nikodym derivative between Q_1 and the original risk-neutral measure P is given by

$$\frac{dQ_1}{dP} = \frac{e^{\sigma_s \pi W_s^P(T)}}{E[e^{\sigma_s \pi W_s^P(T)}]} = e^{\sigma_s \pi W_s^P(T) - \frac{\sigma_s{}^2\pi^2}{2}T}. \tag{15}$$

Additionally,

$$W_s^{Q_1}(T) = W_s^P(T) - \sigma_s \pi T \tag{16}$$

is a standard Brownian motion under the risk-neutral measure Q_1. With this transformation, we derive the subsequent results.

Lemma 2. *Under the risk-neutral Q_1 measure, the account value $A(T)$ follows the log-normal distribution, and its exponential part obeys the following normal distribution:*

$$\int_0^T (r_s + \frac{\sigma_s^2\pi^2}{2} - \alpha)ds + \sigma_s \pi W_s^{Q_1}(T) \sim N\Big((\frac{r_0-\theta}{k} - \frac{\sigma_s\sigma_r\rho\pi}{k^2})(1-e^{-kT}) + (\theta + \frac{\sigma_s\sigma_r\rho\pi}{k} + \frac{\sigma_s^2\pi^2}{2} - \alpha)T, \frac{\sigma_r^2}{k^2}\Big[T - \frac{2}{k}(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})\Big] + \frac{2\sigma_s\sigma_r\rho\pi}{k}\Big[T - \frac{1-e^{-kT}}{k}\Big] + \sigma_s^2\pi^2 T\Big). \tag{17}$$

The proof is achieved by a change of measure, using Girsanov's theorem, Ito's integral, and Fubini's theorem. Readers may find a detailed proof in Appendix A. Another measure of transformation is obtained similarly. Let Q_2 stand for a new risk-neutral measure. The Radon–Nikodym derivative between Q_2 and the original risk-neutral measure P is given by

$$\frac{dQ_2}{dP} = \frac{e^{-\int_0^T r_s ds}}{E[e^{-\int_0^T r_s ds}]} = e^{-\frac{\sigma_r}{k}\int_0^T (1-e^{ks-kT})dW_r^P(s) - \frac{1}{2}\frac{\sigma_r^2}{k^2}\int_0^T (1-e^{ks-kT})^2 ds}. \tag{18}$$

Additionally,

$$W_r^{Q_2}(t) = W_r^P(t) + \frac{\sigma_r}{k}\int_0^T (1-e^{ks-kT})ds \tag{19}$$

is a standard Brownian motion under the risk-neutral measure Q_2. A similar lemma follows.

Lemma 3. *Under the risk-neutral Q_2 measure, the account value $A(T)$ follows the log-normal distribution, and its exponential part obeys the following normal distribution:*

$$\int_0^T (r_s - \frac{\sigma_s^2\pi^2}{2} - \alpha)ds + \sigma_s\pi W_s^P(T) \sim N\Big(\Big(\frac{(r_0-\theta)}{k} + \frac{2\sigma_r^2}{k^3} + \frac{\sigma_r\sigma_s\pi}{\rho k^2}\Big)(1-e^{-kT}) + \Big(\theta - \frac{\sigma_s^2\pi^2}{2} - \alpha - \frac{\sigma_r\sigma_s\pi}{\rho k}\Big)T - \frac{\sigma_r^2}{k^2}T(1+e^{-kT}), \frac{\sigma_r^2}{k^2}\Big[T - \frac{2}{k}(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})\Big] + \frac{2\sigma_r\sigma_s\rho\pi}{k}\Big[T - \frac{1-e^{-kT}}{k}\Big] + \sigma_s^2\pi^2 T\Big). \tag{20}$$

The proof of Lemma 3 is similar to the steps in Lemma 2. Finally, we may consider the mortality risk. When the number of insurance policies issued grows, the mortality risk accumulated in the insurance pool will be closer to the real probability of mortality. Therefore, in this paper, we use the real mortality data listed in the life table for calculation:

$$P(\tau > T) = {}_T P_x(\text{empirical data used}). \tag{21}$$

With the lemmas presented earlier, we can proceed to derive the valuation of the income benefit component of a VA embedded with the GMIDB. Let us formulate this in the following theorem.

Theorem 1. *The non-arbitrage value formula of the income benefit component of a GMIDB at time zero can be written as*

$$V_I(\alpha) = {}_T P_x \cdot \Bigg\{ A(0)e^{-\alpha T}\Bigg[1 - \Phi\Bigg(\frac{\ln\Big(\frac{DBB}{A(0)}\Big) - \mu_{Q_1}^I}{\sigma_{Q_1}^I}\Bigg)\Bigg] + e^{-\left(\frac{r_0-\theta}{k}\right)(1-e^{-kT}) - \theta T + \frac{\sigma_r^2}{2k^2}\left(T - \frac{2}{k}(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})\right)} \Phi\Bigg(\frac{\ln\Big(\frac{DBB}{A(0)}\Big) - \mu_{Q_2}^I}{\sigma_{Q_2}^I}\Bigg) \cdot DBB \Bigg\}, \tag{22}$$

where:

- *ϕ is the standard normal curve.*
- *DBB is given by Lemma 1,*

$$\begin{aligned} DBB = \quad & A(0)(1+r_g)^T g \sum_{j=0}^{\infty} {}_j P_{x+T} \cdot \\ & \exp\Big((\frac{\sigma_r^2}{2k^2} - \theta)j - (\frac{r_0-\theta}{k}e^{-kT} + \frac{\sigma_r^2}{k^3})(1-e^{-kj}) + \frac{\sigma_r^2}{4k^3}(1-e^{-2kj}) + \frac{\sigma_r^2}{4k^3}(1-e^{-kj})^2(1-e^{-2kT})\Big). \end{aligned} \tag{23}$$

- $\mu_{Q_1}^{I}$ *is given by*

$$\mu_{Q_1}^{I} = (\frac{r_0-\theta}{k} - \frac{\sigma_s\sigma_r\rho\pi}{k^2})(1-e^{-kT}) + (\theta + \frac{\sigma_s\sigma_r\rho\pi}{k} + \frac{\sigma_s^2\pi^2}{2} - \alpha)T.$$

- $\sigma_{Q_1}^{I}$ *is given by*

$$\left(\sigma_{Q_1}^{I}\right)^2 = \frac{\sigma_r^2}{k^2}\left[T - \frac{2}{k}(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})\right] + \frac{2\sigma_s\sigma_r\rho\pi}{k}\left[T - \frac{1-e^{-kT}}{k}\right] + \sigma_s^2\pi^2 T.$$

- $\mu_{Q_2}^{I}$ *is given by*

$$\mu_{Q_2}^{I} = \left(\frac{(r_0-\theta)}{k} + \frac{2\sigma_r^2}{k^3} + \frac{\sigma_r\sigma_s\rho\pi}{k^2}\right)(1-e^{-kT}) + \left(\theta - \frac{\sigma_s^2\pi^2}{2} - \alpha - \frac{\sigma_r\sigma_s\rho\pi}{k}\right)T - \frac{\sigma_r^2}{k^2}T(1+e^{-kT}).$$

- $\sigma_{Q_2}^{I}$ *is given by*

$$\left(\sigma_{Q_2}^{I}\right)^2 = \frac{\sigma_r^2}{k^2}\left[T - \frac{2}{k}(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})\right] + \frac{2\sigma_r\sigma_s\rho\pi}{k}\left[T - \frac{1-e^{-kT}}{k}\right] + \sigma_s^2\pi^2 T.$$

The proof of Theorem 1 is just a straightforward corollary, since we have already clarified the distribution of benefits in the Lemmas above. Likewise, we can divide the death benefit portion of a GMIDB into two scenarios: when the policyholder passes away during the accumulation phase at time t, the account value $A(t)$ is greater than the guaranteed benefit base B(t), and vice versa. Therefore, the death benefit can be written as

$$V^D(x,\alpha,0) = \int_0^T Ee^{-\int_0^t r_s ds}\left[\underbrace{A(t)I(A(t) > B(t))}_{(1)} + \underbrace{B(t)I(A(t) \leq B(t))}_{(2)}\right] f_\tau(t)dt. \tag{24}$$

We can now present the following result for the valuation of the death benefit part of a VA embedded with the GMIDB.

Theorem 2. *The non-arbitrage value formula of the death benefit part of a GMIDB at time zero can be written as*

$$V^D(x,\alpha,0) = \int_0^T \left(A(0)e^{-\alpha t}\Phi\left(\frac{\mu_{Q_1}^D}{\sigma_{Q_1}^D}\right) + A(0)(1+r_g)^t e^{-\theta t - \frac{r_0-\theta}{k}(1-e^{-kt}) + \frac{\sigma_r^2}{2k^2}\left(t - \frac{2}{k}(1-e^{-kt}) + \frac{1}{2k}(1-e^{-2kt})\right)}\Phi\left(-\frac{\mu_{Q_2}^D}{\sigma_{Q_2}^D}\right)\right) \cdot \frac{1}{b}e^{\left(e^{\frac{x-m}{b}}(1-e^{\frac{t}{b}}) + \frac{x+t-m}{b}\right)}dt, \tag{25}$$

where:

- $\mu_{Q_1}^{D}$ *is given by*

$$\mu_{Q_1}^{D} = \left(\theta + \frac{\sigma_r\sigma_s\rho\pi}{k} + \frac{\sigma_s^2\pi^2}{2} - \alpha - \ln(1+r_g)\right)t + (\frac{r_0-\theta}{k} - \frac{\sigma_r\sigma_s\rho\pi}{k^2})(1-e^{-kt}).$$

- $\sigma_{Q_1}^{D}$ *is given by*

$$\sigma_{Q_1}^{D} = \sqrt{\frac{\sigma_r^2}{k^2}\left[t - \frac{2}{k}(1-e^{-kt}) + \frac{1}{2k}(1-e^{-2kt})\right] + \frac{2\sigma_s\sigma_r\rho\pi}{k}\left[t - \frac{1-e^{-kt}}{k}\right] + \sigma_s^2\pi^2 t}.$$

The proof of Theorem 2 is similar to the steps in the income benefit component. Combining the two theorems provided above yields the following analytical solution for the valuation of the whole VA embedded with a GMIDB option.

Theorem 3. *The non-arbitrage value formula of a GMIDB at time zero can be written as*

$$V(x,\alpha,0) = V^I(x,\alpha,0) + V^D(x,\alpha,0) \tag{26}$$

$$\begin{aligned}
&= {}_T P_x \cdot \left\{ A(0)e^{-\alpha T}\left[1 - \Phi\left(\frac{\ln\left(\frac{a}{A(0)}\right) - \mu^I_{Q_1}}{\sigma^I_{Q_1}}\right)\right]\right. \\
&\left. + e^{-\left(\frac{r_0-\theta}{k}\right)(1-e^{-kT}) - \theta T + \frac{\sigma_r^2}{2k^2}\left(T - \frac{2}{k}(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})\right)} \Phi\left(\frac{\ln\left(\frac{a}{A(0)}\right) - \mu^I_{Q_2}}{\sigma^I_{Q_2}}\right) \cdot a \right\} + \\
&\int_0^T \left(A(0)e^{-\alpha t}\Phi\left(\frac{\mu^D_{Q_1}}{\sigma^D_{Q_1}}\right) + A(0)(1+r_g)^t e^{-\theta t - \frac{r_0-\theta}{k}(1-e^{-kt}) + \frac{\sigma_r^2}{2k^2}\left(t - \frac{2}{k}(1-e^{-kt}) + \frac{1}{2k}(1-e^{-2kt})\right)} \Phi\left(-\frac{\mu^D_{Q_2}}{\sigma^D_{Q_2}}\right)\right) \\
&\frac{1}{b} e^{\left(e^{\frac{x-m}{b}}(1-e^{\frac{t}{b}}) + \frac{x+t-m}{b}\right)} dt.
\end{aligned} \tag{27}$$

It is important to note that the extensive expression above is a function of the fee rate charged, α. In the subsequent section, we will employ numerical methods to determine the fair fee rate and analyze its sensitivity to various parameters.

5. Numerical Results

In this section, we present a numerical analysis to value VAs embedded with a GMIDB option. We specifically focused on assessing the influence of the following factors:

- Fluctuation in the standard deviation of risky assets, σ_s.
- Average long-term rate, θ.
- Correlation between assets and interests, ρ.
- Length of accumulation phase, T.
- Ratio of risky investments, π.

In our discussions, we will illustrate the GMIDB value as a function of the fee rate and determine the fair fee rate for a realistic range of g values. Furthermore, we explore how the model responds to varying parameter values for the underlying processes. The analytical formula presented in Section 4 was employed for pricing the option.

Adhering to the actuarial principle of insurance, under the fair fee rate, the GMIDB value discounted at time zero should equate to the initial premium paid by the policyholder when the insurance contract is signed. At this juncture, the cost borne by the insurance company due to the guarantee of interest is zero. The parameters provided in the Table 1 above were utilized in this analysis unless stated otherwise. Note that, although different choices of g (the annuity conversion rate) are discussed below, we denote a standard case as $g = 0.05$, given its wide utilization in the industry. As shown in Figure 1 below, the fair fee rate given the standard parameters is approximately $\alpha = 1.7\%$.

Table 1. Parameters used.

Parameter	Symbol	Value	Unit (if AP)
Age of policy holder	x	60	years
Accumulation phase	T	10	years
Initial premium (= initial account value)	$A(0)$	1000	CNY
Annuity conversion rate	g	0.05	
Guaranteed roll-up rate	r_g	0.05	
Initial interest rate	r_0	0.05	
Ratio of Risky Investment	π	0.5	
Average regression speed of interest rate	k	0.1001	
Average long-term rate	θ	0.0215	
Fluctuation in standard deviation of interest rate	σ_r	0.0018	
Fluctuation in standard deviation of risky assets	σ_s	0.35	
Correlation between assets and interest	ρ	0.6	
Gompertz model dispersion coefficient	b	9.645	
Gompertz model death age	m	87.43	

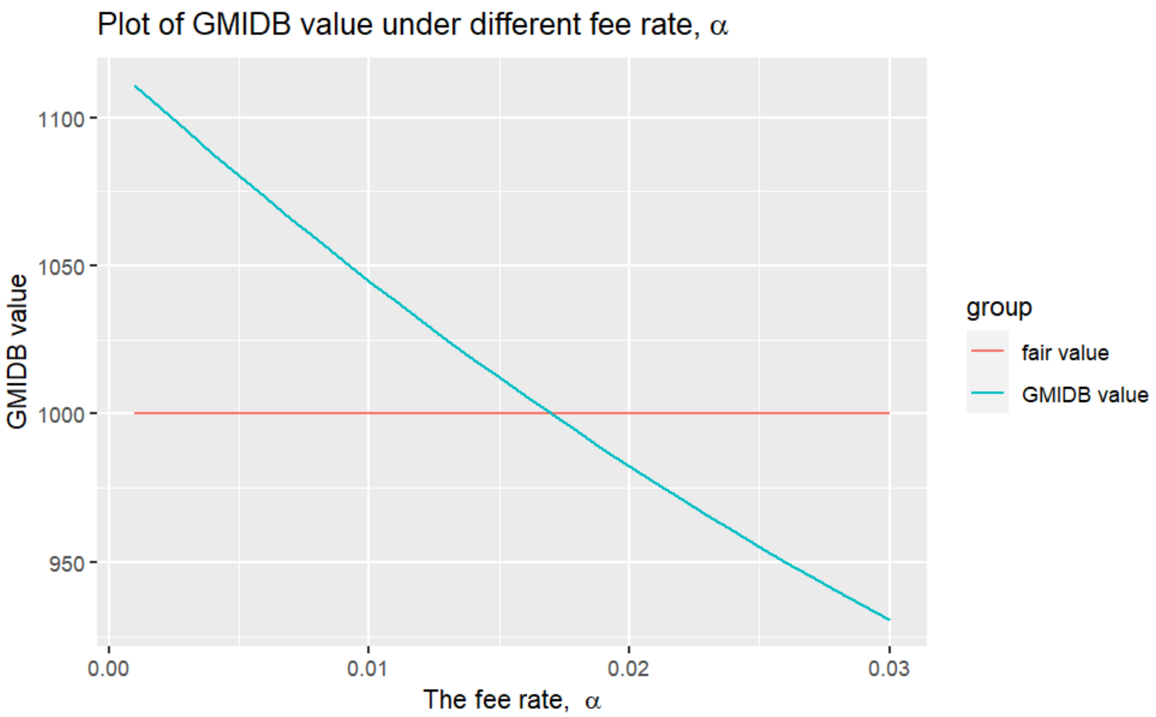

Figure 1. Prices of variable annuities with GMIDB option.

5.1. Choices of g

Figure 2 below illustrates how the GMIDB value is affected by changes in the conversion rate g and the fee rate. The red dotted line represents the fair discounted value, equal to the initial premium. The figure demonstrates how the value fluctuates from 4.5% to 6.5%, and the corresponding fair fee level is obtained at the point where the curves intersect with the fair horizontal line.

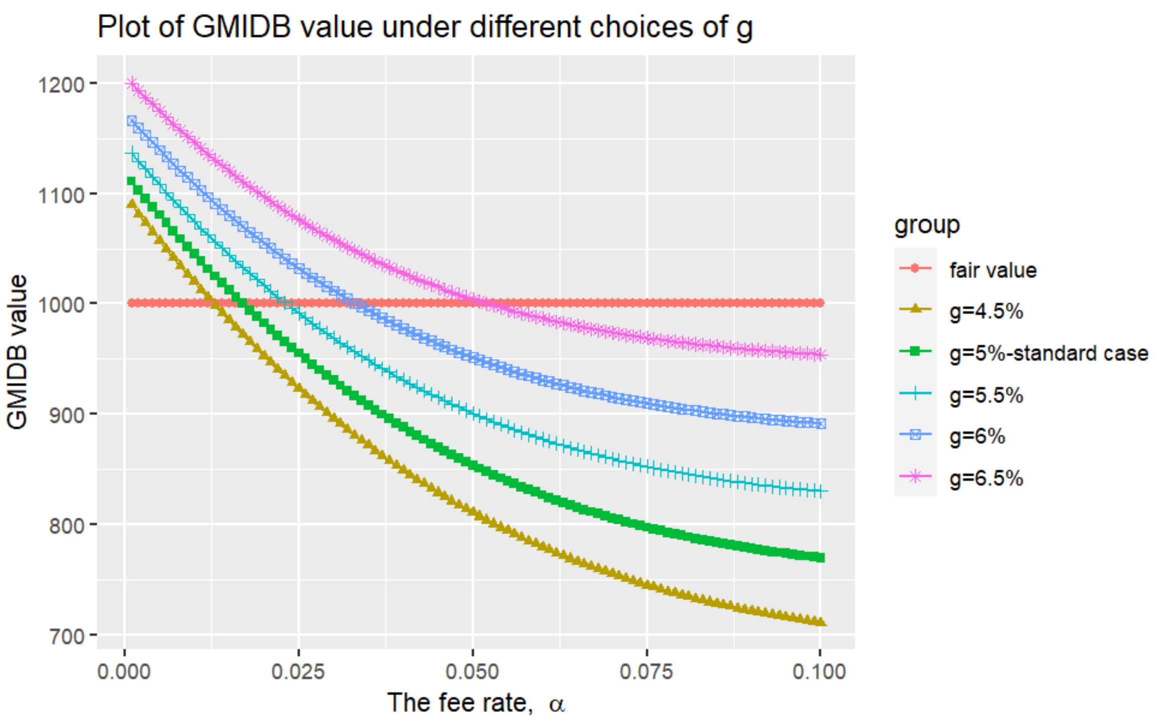

Figure 2. Choices of g.

The efficacy of a GMIDB option is significantly influenced by the annuity conversion rate g, since the income component of the annuity payoff is proportional to the value of g. Determining an appropriate and competitive g is essential, although in practice the value is often selected relatively conservatively. More fees must be charged to strike a balance with a higher conversion rate. When g exceeds 6.5%, the fair fee rate experiences a dramatic increase, rendering such a high fee rate unacceptable to the majority of potential policyholders. In the following analysis, we considered g in the range of 5% to 6.5%, deeming these reasonable values.

5.2. Age of the Policyholder

Figure 3 below presents the GMIDB value according to the age of the policyholder at the time of product purchase. Given that the GMIDB annuity provides life-long payments, the policyholder's age, or, more precisely, their expected remaining lifetime, is a critical factor in GMIDB valuation. The longer a policyholder survives after the accumulation phase, the higher the potential payout they may receive if they opt to convert the benefit into an annuity, with the fee level remaining constant. The figure illustrates how the GMIDB value changes as the policyholder's age at purchase ranges from 50 to 65 years old. If the policyholder purchases the product at an age younger than 50, a fee exceeding 10% must be applied to strike a balance, as in such cases, the guaranteed benefit base becomes relatively costly.

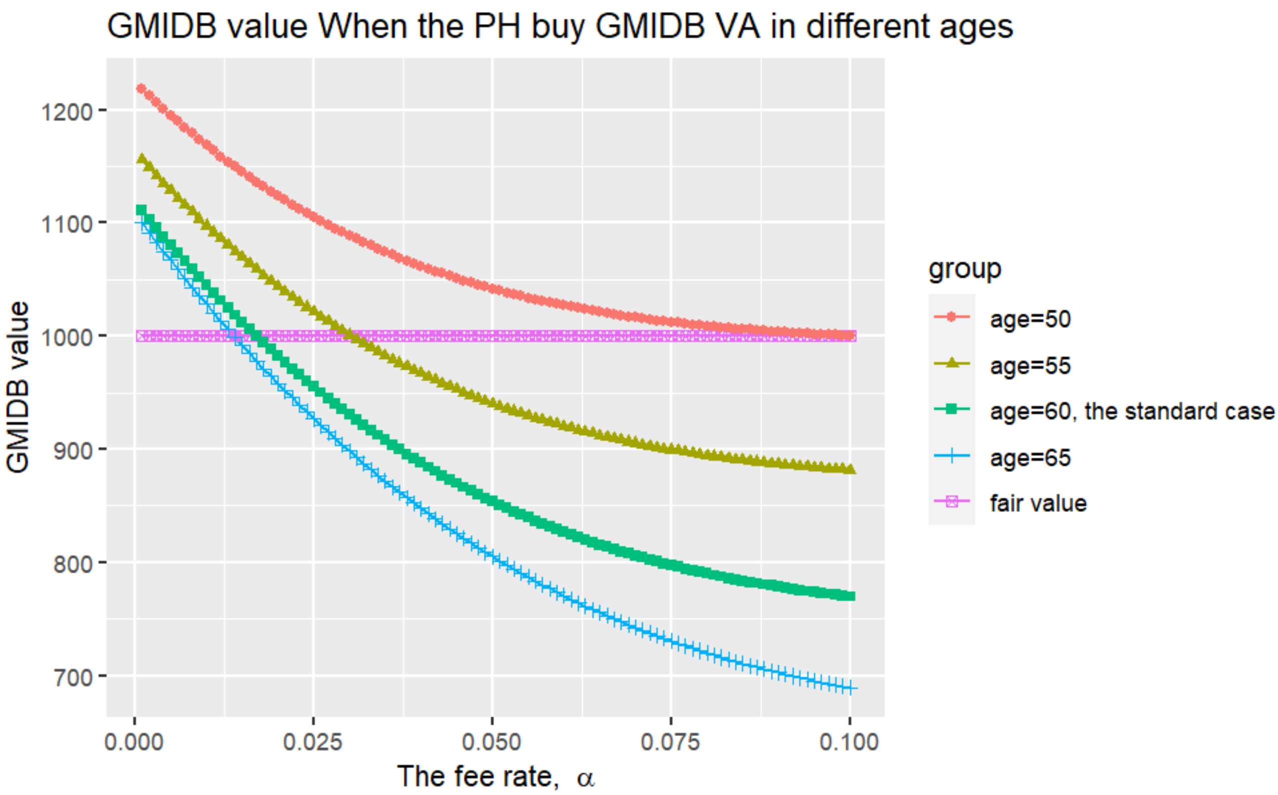

Figure 3. GMIDB value for different policyholder ages at GMIDB VA purchase.

5.3. Volatility of the Risky Investments

Figure 4 presents the variation in the price of a VA with the GMIDB option as the volatility of risky assets fluctuates from 0 to 0.5 for different g values. The analysis considered diverse values of g while keeping the other parameters constant. The upper plot showcases the scenario without any fees, while the plot on the right illustrates the situation with a fair fee applied. The lower figure illustrates the intersection of curves at a risky investment volatility of 0.35, the designated standard in our model. As anticipated, both plots indicate a consistent rise in the GMIDB value with the escalating volatility of risky assets. This pattern aligns with our expectations, signifying that a higher volatility allows the policyholder to anticipate a superior account value during market upswings, with the added advantage of no risk during market downturns, as they have the option to opt for a guaranteed base benefit instead of relying on the account value.

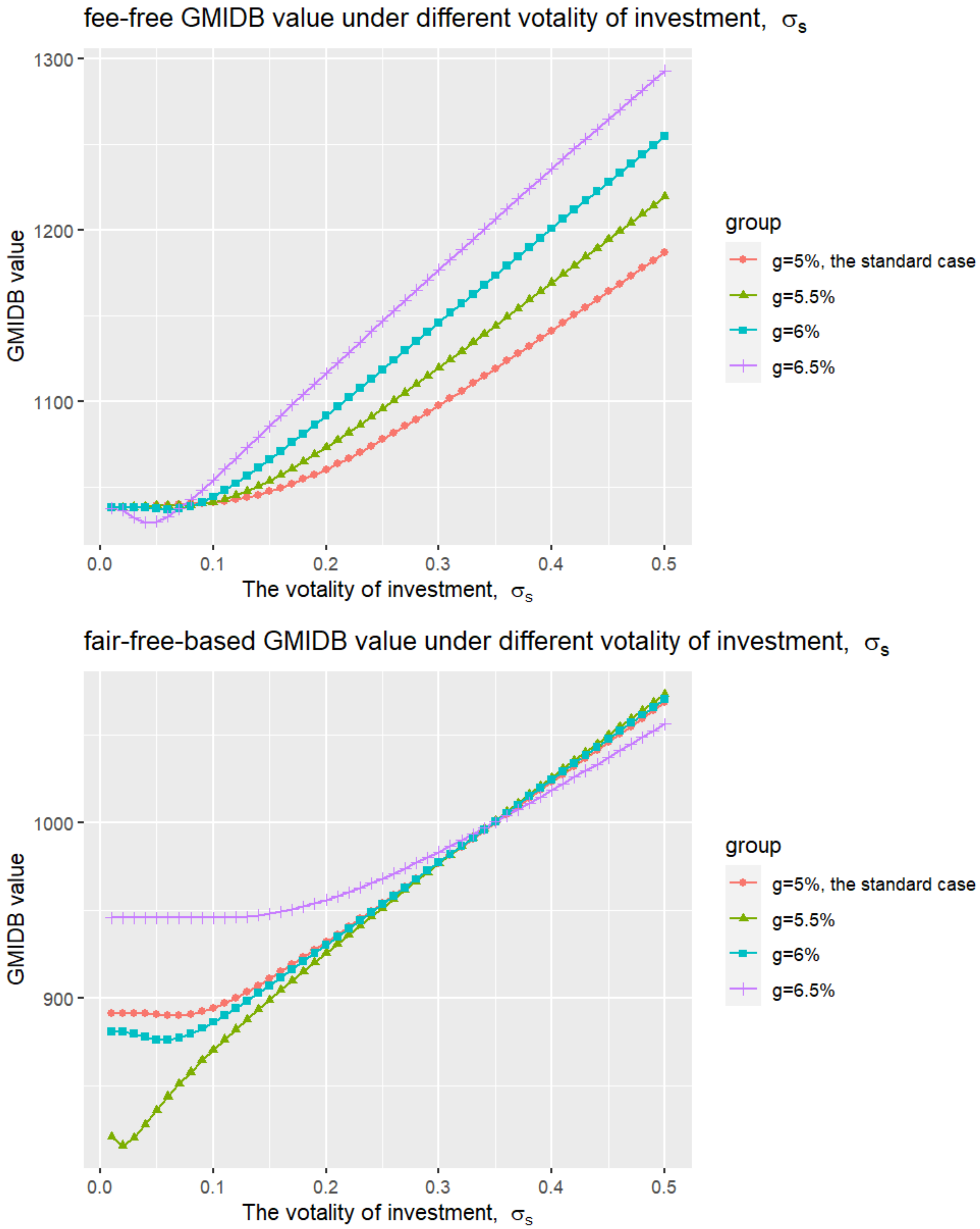

Figure 4. Effect of volatility of investment, σ_s.

5.4. Correlation between Assets and Interest Rate

In Figure 5, under the assumption of constant parameters except for the correlation coefficient between the interest rate process and the risky asset process, the correlation coefficient ranges from −1 to 1. The observed trend indicates a positive correlation between the fair fee rate and the correlation coefficient. However, compared to the parameters discussed earlier, the impact of the correlation on the GMIDB value appears less pronounced. The range of GMIDB variation is relatively minor as the correlation undergoes changes.

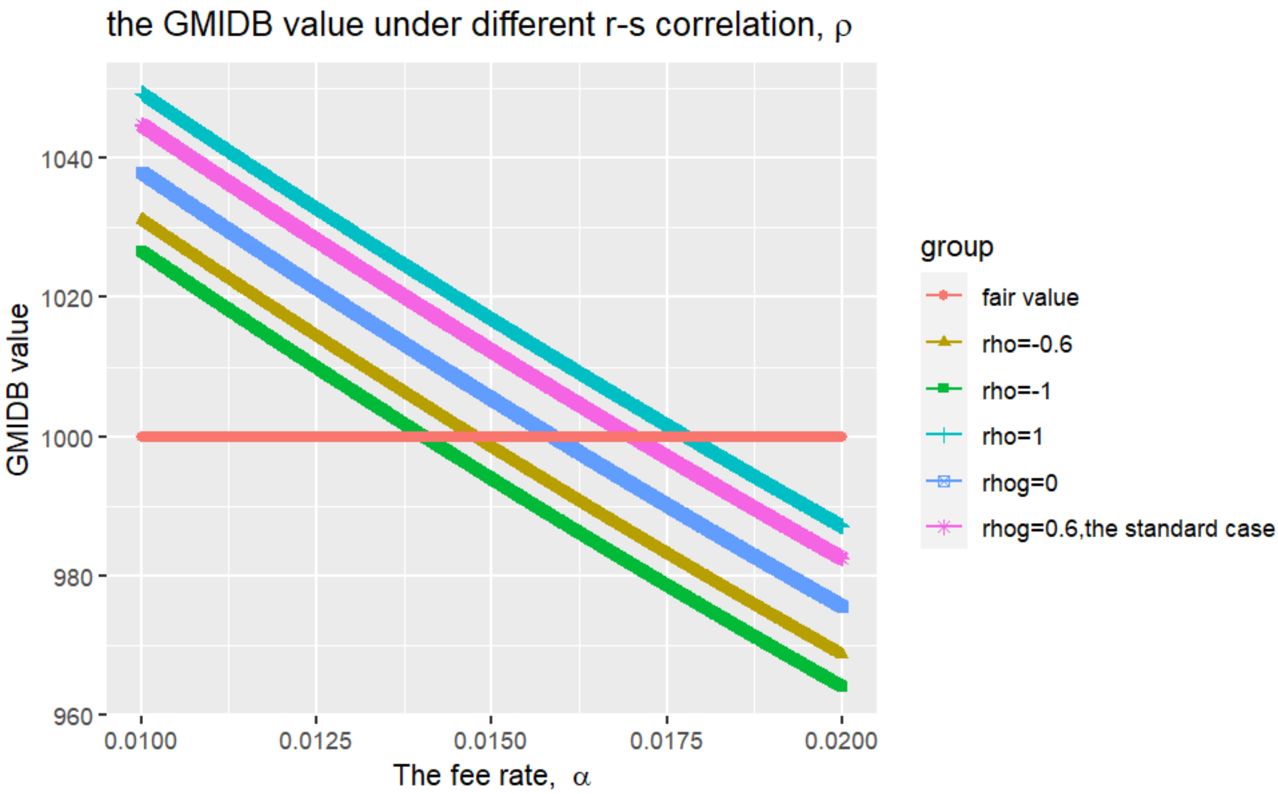

Figure 5. Effect of correlation between interest rate process and risky asset process, ρ.

5.5. Impact of Death Benefit for Older Policyholders

In the preceding numerical analyses, we illustrated the significant impact of critical parameters on the valuation of a GMIDB option. As highlighted by the motivations of this paper, the GMIDB was designed to offer more appealing choices to elderly policyholders. To further underline the advantages of the GMIDB over a VA with only the GMIB option, we introduce a comparison wherein the death benefit is not included. For simplicity, we assume that policyholders receive nothing in the event of death during the accumulation phase.

Figure 6 presents the outcomes of this comparison. Notably, we observe a substantial reduction in the option value for each age group when the death benefit is excluded. Moreover, the decrease in value is more pronounced for policyholders with advanced ages. This suggests that, in the absence of a guaranteed death benefit, policyholders stand to lose significant benefits. Specifically, for policyholders aged 65 or older, the option becomes a less cost-effective product, even under a fee-free structure.

The underlying reason for this observation is straightforward—the risk of policyholders passing away during the accumulation phase becomes more significant. In such instances, our GMIDB structure proves to be a more robust solution, effectively safeguarding the benefits of elderly policyholders and representing a more attractive option for this demographic.

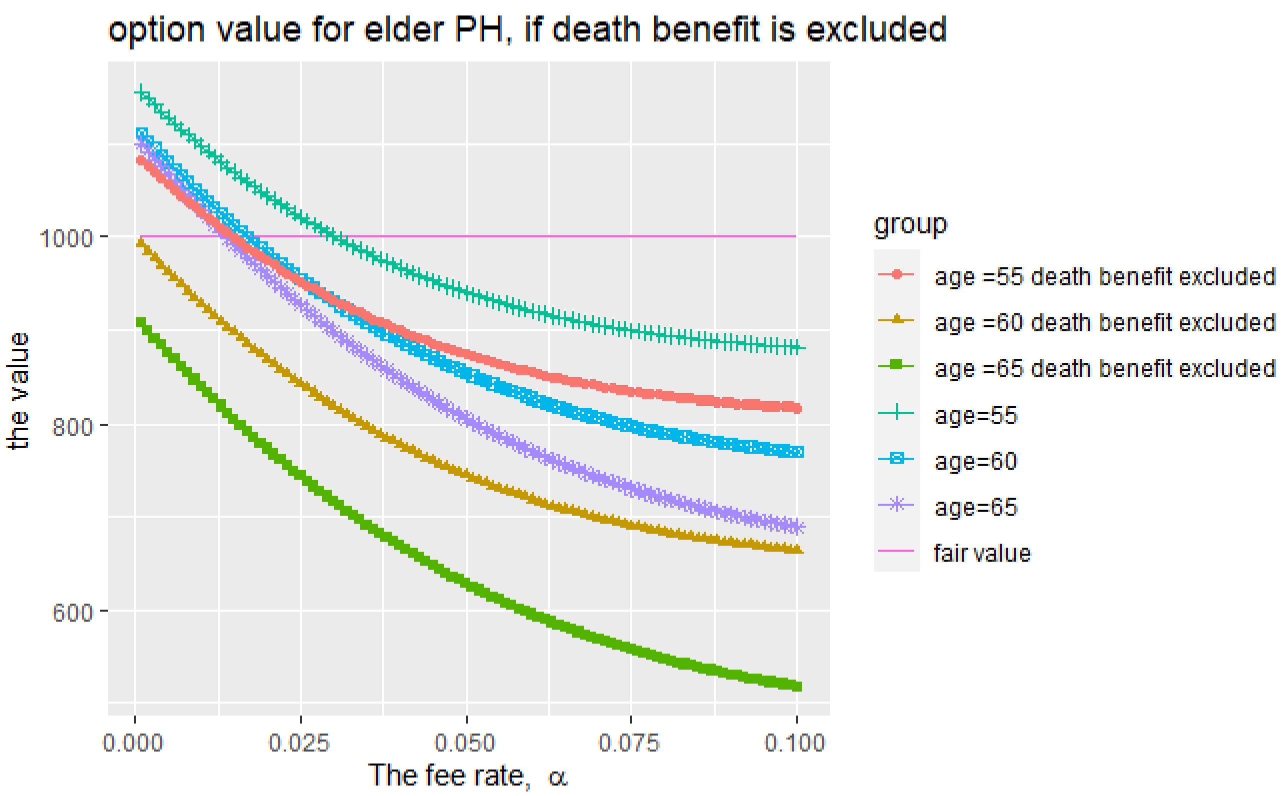

Figure 6. Importance of death benefit for older policyholders, ρ.

6. Concluding Remarks

This paper focused on the valuation of a Guaranteed Minimum Income Death Benefit (GMIDB) associated with a life-long variable annuity within a complete market framework.

In examining the potential impact of the Guaranteed Minimum Income Death Benefit (GMIDB) framework on the insurance sector, our study suggests that its introduction holds promise for positive transformations within the industry. GMIDB policies offer additional options to address concerns related to longevity risk and market volatility for policyholders. However, for a more comprehensive understanding, further research and empirical validation are imperative to ascertain the successful application of the pricing formula in diverse market and economic scenarios.

Moving beyond theoretical considerations, our study offers practical policy recommendations for industry practitioners. Insurers are advised to closely monitor market and interest rate fluctuations, adjusting pricing strategies to adapt to changing market conditions. Furthermore, we encourage insurance professionals to actively explore the market potential of GMIDB policies, seeking innovative sales channels and marketing strategies.

The primary contribution of this paper lies in providing a comprehensive valuation model for GMIDB policies. Leveraging Vasicek's stochastic interest rate model and Gompertz's mortality rate model, our study introduced a robust valuation approach. The application of the dichotomy algorithm to derive the fair fee rate under stochastic interest rate conditions enhanced the reliability of our valuation model. Moreover, our sensitivity analysis shed light on the nuanced impact of various parameters on the fair fee rate, contributing valuable insights to the existing literature.

While acknowledging the strengths of our study, it is essential to recognize its limitations. The reliance on specific parameters for empirical interest and mortality rates may not have captured all real-world complexities. Additionally, the study assumed certain market conditions, potentially limiting its generalizability. Future research endeavors could address these limitations by incorporating more sophisticated models and expanding the scope of market scenarios considered.

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Conflicts of Interest: The authors declare no conflict of interest.The funders had no role in the design of the study.

Appendix A. Proof of Results

Proof of Lemma 1. Concerning the interest rate r_t, Vasicek's model gives

$$dr_t = k(\theta - r_t)dt + \sigma_r dW_r^P(t). \tag{A1}$$

Hence,

$$d(e^{kt}r_t) \quad = ke^{kt}r_t dt + e^{kt}dr_t = k\theta e^{kt}dt + e^{kt}\sigma_r dW_r^P(t). \tag{A2}$$

Insert r_t and take integrals on both sides; then, we have

$$e^{kt}r_t - r_0 = k\theta \int_0^t e^{ks}ds + \sigma_r \int_0^t e^{ks}dW_r^P(s). \tag{A3}$$

That is,

$$r_t = \theta + (r_0 - \theta)e^{-kt} + \sigma_r \int_0^t e^{ks-kt}dW_r^P(s). \tag{A4}$$

Then,

$$\begin{aligned} \int_T^{T+j} r_s ds \quad &= \int_T^{T+j} \left(\theta + (r_0 - \theta)e^{-ks}\right)ds + \sigma_r \int_T^{T+j} \int_0^s e^{ku-ks}dW_r^P(u)ds \\ &= \theta j + \tfrac{r_0-\theta}{k}e^{-kT}(1-e^{-kj}) + \sigma_r \int_T^{T+j}\int_u^{T+j} e^{ku-ks}dsdW_r^P(u) + \sigma_r \int_0^T \int_T^{T+j} e^{ku-ks}dsdW_r^P(u) \\ &= \theta j + \tfrac{r_0-\theta}{k}e^{-kT}(1-e^{-kj}) + \tfrac{\sigma_r}{k}\int_T^{T+j}(1-e^{ku-kT-kj})dW_r^P(u) + \tfrac{\sigma_r}{k}\int_0^T e^{ku-kT}(1-e^{-kj})dW_r^P(u), \end{aligned} \tag{A5}$$

where Fubini's theorem is applied.

By the independent and stationary increments property of Brownian motion, the last two terms of the equation are independent and normally distributed:

$$\frac{\sigma_r}{k}\int_T^{T+j}(1-e^{ku-kT-kj})dW_r^P(u) \sim N(0, \frac{\sigma_r^2}{k^2}\int_T^{T+j}(1-e^{ku-kT-kj})^2du), \tag{A6}$$

$$\frac{\sigma_r}{k}\int_0^T e^{ku-kT}(1-e^{-kj})dW_r^P(u) \sim N(0, \frac{\sigma_r^2}{k^2}\int_0^T e^{2ku-2kT}(1-e^{-kj})^2du). \tag{A7}$$

Therefore,

$$E(e^{-\int_T^{T+j} r_s ds}) = e^{-\theta j - \frac{r_0-\theta}{k}e^{-kT}(1-e^{-kj}) + \frac{1}{2}\frac{\sigma_r^2}{k^2}\int_T^{T+j}(1-e^{ku-kT-kj})^2du + \frac{1}{2}\frac{\sigma_r^2}{k^2}\int_0^T e^{2ku-2kT}(1-e^{-kj})^2du} \tag{A8}$$

$$= e^{-\theta j - \frac{r_0-\theta}{k}e^{-kT}(1-e^{-kj}) + \frac{\sigma_r^2}{2k^2}\left(j - \frac{2}{k}(1-e^{-kj}) + \frac{1-e^{-2kT}}{2k}\right) + \frac{\sigma_r^2}{4k^3}(1-e^{-kj})^2(1-e^{-2kT})} \tag{A9}$$

$$= e^{(\frac{\sigma_r^2}{2k^2}-\theta)j - (\frac{r_0-\theta}{k}e^{-kT} + \frac{\sigma_r^2}{k^3})(1-e^{-kj}) + \frac{\sigma_r^2}{4k^3}(1-e^{-2kj}) + \frac{\sigma_r^2}{4k^3}(1-e^{-kj})^2(1-e^{-2kT})}, \tag{A10}$$

and the result follows. □

Proof of Lemma 2. Recall that

$$\frac{dQ_1}{dP} = \frac{e^{\sigma_s \pi W_s^P(T)}}{E[e^{\sigma_s \pi W_s^P(T)}]} = e^{\sigma_s \pi W_s^P(T) - \frac{\sigma_s{}^2 \pi^2}{2} T}, \tag{A11}$$

$$W_s^{Q_1}(T) = W_s^P(T) - \sigma_s \pi T. \tag{A12}$$

The account value can then be written as

$$A(T) = A(0) \exp\left(\int_0^T (r_s - \frac{\sigma_s^2 \pi^2}{2} - \alpha) ds + \sigma_s \pi W_s^P(T) \right) \tag{A13}$$

$$= A(0) \exp\left(\int_0^T (r_s + \frac{\sigma_s^2 \pi^2}{2} - \alpha) ds + \sigma_s \pi \int_0^T dW_s^{Q_1}(s) \right). \tag{A14}$$

Taking an integral on r_t and simplifying gives

$$\int_0^T r_s ds = \frac{(r_0 - \theta)}{k}(1 - e^{-kT}) + \theta T + \sigma_r \int_0^T \int_0^s e^{ku - ks} dW_r^P(u) ds. \tag{A15}$$

By Fubini's theorem,

$$\int_0^T \int_0^s e^{ku-ks} dW_r^P(u) ds = \int_0^T (\int_u^T e^{ku-ks} ds) dW_r^P(u) = \frac{1}{k} \int_0^T (1 - e^{ku-kT}) dW_r^P(u). \tag{A16}$$

Therefore,

$$\int_0^T r_s ds = \frac{(r_0 - \theta)}{k}(1 - e^{-kT}) + \theta T + \frac{\sigma_r}{k} \int_0^T (1 - e^{ks-kT}) dW_r^P(s). \tag{A17}$$

Recall that there are correlations between underlying processes:

$$W_r^P(t) = \rho W_s^P(t) + \sqrt{1 - \rho^2} W_2^P(t). \tag{A18}$$

This leads to

$$dW_r^P(s) = \rho dW_s^{Q_1}(s) + \sigma_s \rho \pi ds + \sqrt{1 - \rho^2} dW_2^P(s). \tag{A19}$$

Inserting this, we have

$$\begin{aligned} &\textstyle\int_0^T r_s ds = \left(\frac{(r_0 - \theta)}{k} - \frac{\sigma_r \sigma_s \rho \pi}{k^2} \right)(1 - e^{-kT}) + \left(\theta + \frac{\sigma_r \sigma_s \rho \pi}{k}\right) T + \frac{\sigma_r \rho}{k} \int_0^T (1 - e^{ks-kT}) dW_s^{Q_1}(s) \\ &+ \textstyle\frac{\sigma_r \sqrt{1-\rho^2}}{k} \int_0^T (1 - e^{ks-kT}) dW_2^P(s). \end{aligned} \tag{A20}$$

Recall that

$$A(T) = A(0) \exp\left(\int_0^T (r_s + \frac{\sigma_s^2 \pi^2}{2} - \alpha) ds + \sigma_s \pi \int_0^T dW_s^{Q_1}(s) \right). \tag{A21}$$

Hence, the exponential part of the f account value is given by

$$\begin{aligned} &\textstyle\int_0^T (r_s + \frac{\sigma_s^2 \pi^2}{2} - \alpha) ds + \sigma_s \pi \int_0^T dW_s^{Q_1}(s) = \left(\frac{(r_0 - \theta)}{k} - \frac{\sigma_r \sigma_s \rho \pi}{k^2} \right)(1 - e^{-kT}) + \left(\theta + \frac{\sigma_r \sigma_s \rho \pi}{k} + \frac{\sigma_s^2 \pi^2}{2} - \alpha \right) T \\ &+ \textstyle\int_0^T \left(\frac{\sigma_r \rho}{k}(1 - e^{ks-kT}) + \sigma_s \pi \right) dW_s^{Q_1}(s) + \frac{\sigma_r \sqrt{1-\rho^2}}{k} \int_0^T (1 - e^{ks-kT}) dW_2^P(s), \end{aligned} \tag{A22}$$

where the last two terms are independent and normally distributed:

$$\int_0^T \left(\frac{\sigma_r\rho}{k}(1-e^{ks-kT})+\sigma_s\pi\right)dW_s^{Q_1}(s) \sim N\left(0, \int_0^T \left(\frac{\sigma_r\rho}{k}(1-e^{ks-kT})+\sigma_s\pi\right)^2 ds\right), \quad \text{(A23)}$$

$$\frac{\sigma_r\sqrt{1-\rho^2}}{k}\int_0^T (1-e^{ks-kT})dW_2^P(s) \sim N\left(0, \left(\frac{\sigma_r\sqrt{1-\rho^2}}{k}\right)^2 \int_0^T (1-e^{ks-kT})^2 ds\right). \quad \text{(A24)}$$

Hence,

$$\begin{aligned}&\textstyle\int_0^T (r_s+\frac{\sigma_s^2\pi^2}{2}-\alpha)ds+\sigma_s\pi W_s^{Q_1}(T) \sim N\Big((\frac{r_0-\theta}{k}-\frac{\sigma_s\sigma_r\rho\pi}{k^2})(1-e^{-kT})+(\theta+\frac{\sigma_s\sigma_r\rho\pi}{k}+\frac{\sigma_s^2\pi^2}{2}-\alpha)T,\\ &\textstyle\frac{\sigma_r^2}{k^2}\Big[T-\frac{2}{k}(1-e^{-kT})+\frac{1}{2k}(1-e^{-2kT})\Big]+\frac{2\sigma_s\sigma_r\rho\pi}{k}\Big[T-\frac{1-e^{-kT}}{k}\Big]+\sigma_s^2\pi^2T\Big),\end{aligned} \quad \text{(A25)}$$

and the result follows. □

Proof of Lemma 3. Recall that

$$\frac{dQ_2}{dP}=\frac{e^{-\int_0^T r_s ds}}{E[e^{-\int_0^T r_s ds}]}=e^{-\frac{\sigma_r}{k}\int_0^T (1-e^{ks-kT})dW_r^P(s)-\frac{1}{2}\frac{\sigma_r^2}{k^2}\int_0^T (1-e^{ks-kT})^2 ds}, \quad \text{(A26)}$$

and

$$W_r^{Q_2}(t)=W_r^P(t)+\frac{\sigma_r}{k}\int_0^T (1-e^{ks-kT})ds. \quad \text{(A27)}$$

Therefore,

$$W_r^P(t)=W_r^{Q_2}(t)-\frac{\sigma_r}{k}t+\frac{\sigma_r}{k^2}(1-e^{-kt}). \quad \text{(A28)}$$

Since

$$\int_0^T r_s ds = \frac{(r_0-\theta)}{k}(1-e^{-kT})+\theta T+\frac{\sigma_r}{k}\int_0^T (1-e^{ks-kT})d\Big[W_r^{Q_2}(s)-\frac{\sigma_r}{k}s+\frac{\sigma_r}{k^2}(1-e^{-ks})\Big] \quad \text{(A29)}$$

$$= \frac{(r_0-\theta)}{k}(1-e^{-kT})+\theta T+\frac{\sigma_r}{k}\int_0^T (1-e^{ks-kT})dW_r^{Q_2}(s)-\frac{\sigma_r^2}{k^2}\int_0^T (1-e^{ks-kT})(1-e^{-ks})ds \quad \text{(A30)}$$

$$= \frac{(r_0-\theta)}{k}(1-e^{-kT})+\theta T+\frac{\sigma_r}{k}\int_0^T (1-e^{ks-kT})dW_r^{Q_2}(s)-\frac{\sigma_r^2}{k^2}\Big[T(1+e^{-kT})-\frac{2}{k}(1-e^{-kT})\Big], \quad \text{(A31)}$$

consider the correlation between underlying processes:

$$W_s^P(t)=\rho W_r^P(t)+\sqrt{1-\rho^2}W_3^P(t). \quad \text{(A32)}$$

Thus,

$$dW_s^P(s)=\rho dW_r^{Q_2}(s)-\frac{\sigma_r\rho}{k}(1-e^{-ks})ds+\sqrt{1-\rho^2}dW_3^P(s), \quad \text{(A33)}$$

and we have

$$\begin{aligned}&\textstyle\int_0^T (r_s-\frac{\sigma_s^2\pi^2}{2}-\alpha)ds+\sigma_s\pi W_s(T)=\Big(\frac{(r_0-\theta)}{k}+\frac{2\sigma_r^2}{k^3}+\frac{\sigma_r\sigma_s\rho\pi}{k^2}\Big)(1-e^{-kT})+\Big(\theta-\frac{\sigma_s^2\pi^2}{2}-\alpha-\frac{\sigma_r\sigma_s\rho\pi}{k}\Big)T\\ &\textstyle-\frac{\sigma_r^2}{k^2}T(1+e^{-kT})+\int_0^T \Big(\frac{\sigma_r}{k}(1-e^{ks-kT})+\sigma_s\pi\rho\Big)dW_r^{Q_2}(s)+\sigma_s\pi\sqrt{1-\rho^2}\int_0^T dW_3^P(s).\end{aligned} \quad \text{(A34)}$$

Again, the last two terms are independent and normally distributed:

$$\int_0^T \Big(\frac{\sigma_r}{k}(1-e^{ks-kT})+\sigma_s\pi\rho\Big)dW_r^{Q_2}(s) \sim N\bigg(0, \int_0^T \Big(\frac{\sigma_r}{k}(1-e^{ks-kT})+\sigma_s\pi\rho\Big)^2 ds\bigg), \quad \text{(A35)}$$

$$\sigma_s\pi\sqrt{1-\rho^2}\int_0^T dW_3^P(s) \sim N\Big(0, \sigma_s^2\pi^2(1-\rho^2)T\Big). \quad \text{(A36)}$$

Hence,

$$\begin{aligned}&\textstyle\int_0^T (r_s - \frac{\sigma_s^2\pi^2}{2}-\alpha)ds + \sigma_s\pi W_s(T) \sim N\Big(\Big(\frac{(r_0-\theta)}{k}+\frac{2\sigma_r^2}{k^3}+\frac{\sigma_r\sigma_s\rho\pi}{k^2}\Big)(1-e^{-kT})+\Big(\theta-\frac{\sigma_s^2\pi^2}{2}-\alpha-\frac{\sigma_r\sigma_s\rho\pi}{k}\Big)T \\ &\textstyle -\frac{\sigma_r^2}{k^2}T(1+e^{-kT}), \frac{\sigma_r^2}{k^2}\Big[T-\frac{2}{k}(1-e^{-kT})+\frac{1}{2k}(1-e^{-2kT})\Big]+\frac{2\sigma_r\sigma_s\rho\pi}{k}\Big[T-\frac{1-e^{-kT}}{k}\Big]+\sigma_s^2\pi^2T\Big),\end{aligned} \quad \text{(A37)}$$

and the result follows. □

Proof of Theorem 1. With Lemmas 1–3, we clarified the distribution of the account value $A(T)$ under measures Q_1 and Q_2. Therefore, the probability that the account value is greater than the discounted benefit base (DBB) can be easily given by

$$Q_1(A(T) > DBB) = Q_1\bigg(\int_0^T (r_s + \frac{\sigma_s^2\pi^2}{2}-\alpha)ds + \sigma_s\pi W_s^{Q_1}(T) > \ln\Big(\frac{DBB}{A(0)}\Big)\bigg) = 1-\Phi\left(\frac{\ln\Big(\frac{DBB}{A(0)}\Big)-\mu_{Q1}^I}{\sigma_{Q1}^I}\right), \quad \text{(A38)}$$

$$Q_2(A(T) \le DBB) = Q_2\bigg(\int_0^T (r_s - \frac{\sigma_s^2\pi^2}{2}-\alpha)ds + \sigma_s\pi W_s^{Q_2}(T) \le \ln\Big(\frac{DBB}{A(0)}\Big)\bigg) = \Phi\left(\frac{\ln\Big(\frac{DBB}{A(0)}\Big)-\mu_{Q2}^I}{\sigma_{Q2}^I}\right). \quad \text{(A39)}$$

Insert the equations above into Equation (12), and the result follows. □

Proof of Theorem 2. We may separate Equation (24) into two cases: $A(t) \ge B(t)$ and $A(t) \le B(t)$.

For the first part,

$$E_P\Big(e^{-\int_0^t r_s ds}A(t)I(A(t)>B(t))\Big) = E_P\Big(e^{-\int_0^t r_s ds}A(0)e^{\int_0^t (r_s-\frac{\sigma_s^2\pi^2}{2}-\alpha)ds+\sigma_s\pi W_s(t)}I(A(t)>B(t))\Big) \quad \text{(A40)}$$

$$= E_P\Big(A(0)e^{-(\frac{\sigma_s^2\pi^2}{2}+\alpha)t+\sigma_s\pi W_s(t)}I(A(t)>B(t))\Big). \quad \text{(A41)}$$

By a similar change of measure in Lemma 2,

$$\frac{dQ_1}{dP} = \frac{e^{\sigma_s\pi W_s(t)}}{E\big[e^{\sigma_s\pi W_s(t)}\big]} = e^{\sigma_s\pi W_s(t)-\frac{\sigma_s^2\pi^2 t}{2}}, \quad \text{(A42)}$$

where $W_s^{Q_1}(t) = W_s(t) - \sigma_s\pi t$ is a standard Brownian motion under the Q_1 measure, since

$$\begin{aligned} A(t) &= A(0)\exp\Big(\int_0^t (r_s - \frac{\sigma_s^2\pi^2}{2}-\alpha)ds + \sigma_s\pi W_s(t)\Big) && \text{(A43)}\\ &= A(0)\exp\Big(\int_0^t (r_s + \frac{\sigma_s^2\pi^2}{2}-\alpha)ds + \sigma_s\pi W_s^{Q_1}(t)\Big), && \text{(A44)}\end{aligned}$$

and

$$Q_1(A(t) > B(t)) = Q_1\left(A(0)\exp\left(\int_0^t (r_s + \frac{\sigma_s^2\pi^2}{2} - \alpha)ds + \sigma_s\pi W_s^{Q_1}(t)\right) > A(0)(1+r_g)^t\right) \tag{A45}$$

$$= Q_1\left(\int_0^t (r_s + \frac{\sigma_s^2\pi^2}{2} - \alpha)ds + \sigma_s\pi W_s^{Q_1}(t) > t\ln(1+r_g)\right). \tag{A46}$$

As in Lemma 2,

$$\int_0^t (r_s + \frac{\sigma_s^2\pi^2}{2} - \alpha)ds + \sigma_s\pi W_s^{Q_1}(t) - t\ln(1+r_g) \sim N\left(\mu_{Q_1}^D, \left(\sigma_{Q_1}^D\right)^2\right), \tag{A47}$$

where

$$\mu_{Q_1}^D = \left(\theta + \frac{\sigma_r\sigma_s\rho\pi}{k} + \frac{\sigma_s^2\pi^2}{2} - \alpha - \ln(1+r_g)\right)t + (\frac{r_0-\theta}{k} - \frac{\sigma_r\sigma_s\rho\pi}{k^2})(1-e^{-kt}) \tag{A48}$$

and

$$\sigma_{Q_1}^D = \sqrt{\frac{\sigma_r^2}{k^2}\left[t - \frac{2}{k}(1-e^{-kt}) + \frac{1}{2k}(1-e^{-2kt})\right] + \frac{2\sigma_s\sigma_r\rho\pi}{k}\left[t - \frac{1-e^{-kt}}{k}\right] + \sigma_s^2\pi^2 t}. \tag{A49}$$

For the second part, by changing the measure,

$$\frac{dQ_2}{dP} = \frac{e^{-\int_0^t r_s ds}}{E\left(e^{-\int_0^t r_s ds}\right)} = e^{-\frac{\sigma_r}{k}\int_0^t (1-e^{ku-kt})dW_r(u) - \frac{1}{2}\frac{\sigma_r^2}{k^2}\int_0^t (1-e^{ku-kt})^2 du}, \tag{A50}$$

where $W_r^{Q_2}(t) = W_r(t) + \frac{\sigma_r}{k}\int_0^t (1-e^{ku-kt})du$ is a standard Brownian motion under the Q_2 measure. Then, the second part can be written as

$$\begin{aligned} &E_P\left(e^{-\int_0^t r_s ds}B(t)I(A(t) \le B(t))\right) = E_{Q_2}\left(A(0)(1+r_g)^t \exp\{-\theta t - \tfrac{r_0-\theta}{k}(1-e^{-kt}) + \tfrac{1}{2}\tfrac{\sigma_r^2}{k^2}\int_0^t (1-e^{ks-kt})^2 ds\}\cdot\right. \\ &\left. I\left(\int_0^t (r_s - \tfrac{\sigma_s^2\pi^2}{2} - \alpha)ds + \sigma_s\pi\tilde{W}_s(t) \le t\ln(1+r_g)\right)\right) \end{aligned} \tag{A51}$$

As in Lemma 3,

$$\int_0^t (r_s - \frac{\sigma_s^2\pi^2}{2} - \alpha)ds + \sigma_s\pi W_s(t) - t\ln(1+r_g) \sim N\left(\mu_{Q_2}^D, \left(\sigma_{Q2}^D\right)^2\right), \tag{A52}$$

where

$$\mu_{Q_2}^D = \left(\frac{(r_0-\theta)}{k} + \frac{2\sigma_r^2}{k^3} + \frac{\sigma_r\sigma_s\rho\pi}{k^2}\right)(1-e^{-kt}) + \left(\theta - \frac{\sigma_s^2\pi^2}{2} - \alpha - \frac{\sigma_r\sigma_s\rho\pi}{k} - \ln(1+r_g)\right)t - \frac{\sigma_r^2}{k^2}t(1+e^{-kt}) \tag{A53}$$

and

$$\sigma_{Q_2}^D = \sqrt{\frac{\sigma_r^2}{k^2}\left[t - \frac{2}{k}(1-e^{-kt}) + \frac{1}{2k}(1-e^{-2kt})\right] + \frac{2\sigma_r\sigma_s\rho\pi}{k}\left[t - \frac{1-e^{-kt}}{k}\right] + \sigma_s^2\pi^2 t}. \tag{A54}$$

Consider the mortality force under the Gompertz curve

$$f_x(t) = {}_tp_x \cdot \mu_{x+t} = \frac{1}{b}e^{\left(e^{\frac{x-m}{b}}(1-e^{\frac{t}{b}}) + \frac{x+t-m}{b}\right)}, \tag{A55}$$

and the result follows. □

The proof of Theorem 3 is straightforward following Theorems 1 and 2.

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