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Efficient Machine Learning Approaches for Fast Risk Evaluation of VAs

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Thesis Defense, University of Waterloo

Outline

Nested Simulation Procedures

Nested simulation procedures are necessary for **complex** financial derivatives and insurance products.

$$\rho(L) = \rho(L(X)), \quad L(X) = \mathbb{E}\left[Y|X = x\right]|_{x=X}.$$

Involves two levels of Monte Carlo simulations:

- Outer level: generates underlying risk factors (outer scenarios), $X_i \sim F_X$.
- Inner level: generates scenario-wise samples of portfolio losses (inner replications), $Y_{ij} \sim F_{Y|X_i}$.
- Total simulation budget: $\Gamma = M \cdot N$.

Computationally expensive due to its nested structure.

Common Risk Measures

- Smooth h, e.g., quadratic tracking error,

$$\rho(L) = \mathbb{E}\left[(L-b)^2\right];$$

- hockey-stick h: mean excess loss,

$$\rho(L) = \mathbb{E}\left[L \cdot \mathbb{1}_{\{L \geq u\}}\right];$$

- indicator h: probability of large loss,

$$\rho(L) = \mathbb{E}\left[\mathbb{1}_{\{L \geq u\}}\right];$$

- Value at Risk (VaR),

$$\rho_\alpha(L) = Q_\alpha(L) = \inf\{u : \mathbb{P}(L \leq u) \geq \alpha\};$$

- Conditional Value at Risk (CVaR) [1],

$$\rho_\alpha(L) = \mathbb{E}\left[L | L \geq Q_\alpha(L)\right].$$

[1] Note: If $Q_\alpha(L)$ falls in a probability mass, $\rho(L) = \frac{(\beta-\alpha)Q_\alpha(L)+(1-\beta)\mathbb{E}[L|L\geq Q_\alpha(L)]}{1-\alpha}$.

Standard Nested Simulation

$$\hat{L}_{N,i} = \frac{1}{N}\sum_{j=1}^{N} Y_{ij}; \quad Y_{ij} \sim F_{Y|X_i}$$

- Uses inner sample mean to estimate $L(X_i)$.
- Proposed by Gordy and Juneja (2010); finds optimal growth order of M and N.
- Zhang et al. (2021) estimate the optimal M and N using a bootstrap method.
- Computationally expensive and potentially **wasteful** use of budget.

Other Nested Simulation Procedures

Subsequent works focus on improving the efficiency of nested simulation:

- Regression-based (Broadie et al., 2015)
- Kernel smoothing (Hong et al., 2017)
- Likelihood ratio (Feng and Song, 2020)
- Kernel ridge regression (Zhang et al., 2022)

Key ideas:

- Pool inner replications from different outer scenarios
- Use metamodeling techniques to approximate the inner simulation model

Metamodeling Approach

In this thesis, we focus on procedures that use **supervised learning metamodels** to approximate the inner simulation model.

- Treat the inner simulation as a black-box function
- Approximate $L(\cdot)$ with $\hat{L}^{\mathsf{SL}}_{M,N}(\cdot)$
- Train with a set of feature-label pairs generated from the standard procedure:

$$\{(X_i, \hat{L}_{N,i}) | i = 1, \dots, M, j = 1, \dots, N\}$$

- Use trained metamodel to make predictions for all $X \in \mathcal{X}$

There are **computational costs** associated with pooling inner replications.

Problem Statement

Minimize mean squared error (MSE) of the estimator subject to total simulation budget:

$$\min_{M,N} \quad \mathbb{E}\left[\left(\hat{\rho}_{M,N} - \rho\right)^2\right]$$

$$\text{subject to } M \cdot N = \Gamma$$

Interested in convergence order as $\Gamma \to \infty$

Asymptotic Convergence Rates of Different Procedures

Procedures	**Smooth** h	**Hockey-Stick** h	**Indicator** h
Standard Procedure	$\mathcal{O}(\Gamma^{-2/3})$	$\mathcal{O}(\Gamma^{-2/3})$	$\mathcal{O}(\Gamma^{-2/3})$
Regression	$\mathcal{O}(\Gamma^{-1})$	$\mathcal{O}(\Gamma^{-1+\delta})$	No Result
Kernel Smoothing	$\mathcal{O}(\Gamma^{-\min(1,4/(d+2))})$		
Kernel Ridge Regression	$\mathcal{O}(\Gamma^{-1})$		
Likelihood Ratio	$\mathcal{O}(\Gamma^{-1})$		

- We show the asymptotic convergence rates of the standard procedure for smooth and hockey-stick h.
- Only kernel smoothing depends on the asset dimension d.

Key Theoretical Results

Observations:

- Most literature focuses on the MSE of $\hat{\rho}$.
- Wang et al. (2022) analyze convergence of absolute error in probabilistic order.

Contribution: bridging the gap between MSE and absolute error convergence.

- Convergence in MSE:

$$\mathbb{E}\left[(\hat{\rho}_\Gamma - \rho)^2\right] = \mathcal{O}\left(\Gamma^{-\xi}\right)$$

- Convergence in Probabilistic Order:

$$|\hat{\rho}_\Gamma - \rho| = \mathcal{O}_{\mathbb{P}}(\Gamma^{-\xi})$$

Key Theoretical Results

Theorem

If $\hat{\rho}_\Gamma$ converges in MSE to ρ in order ξ, then $\hat{\rho}_\Gamma$ converges in probabilistic order to ρ in order $\frac{\xi}{2}$.

- First result to draw connection between MSE and probabilistic order convergence.
- Applicable to any nested simulation procedure.
- Convergence in MSE implies convergence in probabilistic order.

Experiment Design

We compare 5 nested simulation procedures

- Standard nested simulation
- Regression-based
- Kernel smoothing
- Likelihood ratio
- Kernel ridge regression

And their empirical convergence stable across different:

- Risk measures
- Option types
- Asset dimensions
- Asset models (GBM vs. Heston)
- Regression bases (only for the regression-based procedure)

Finite-Sample Performance

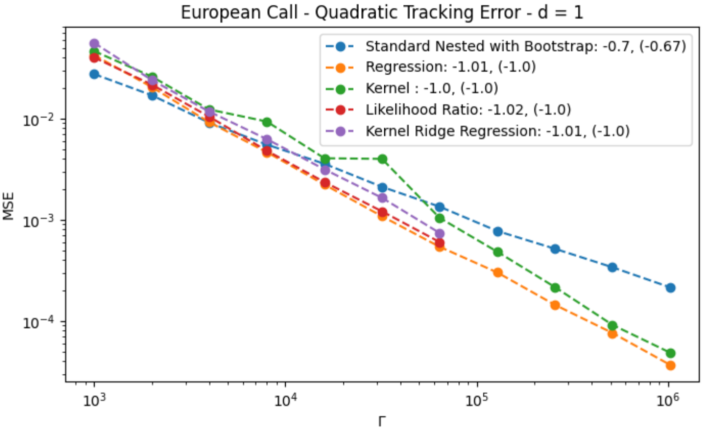
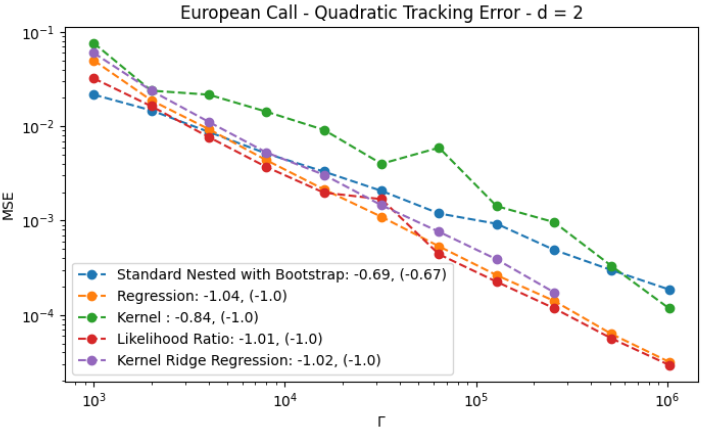


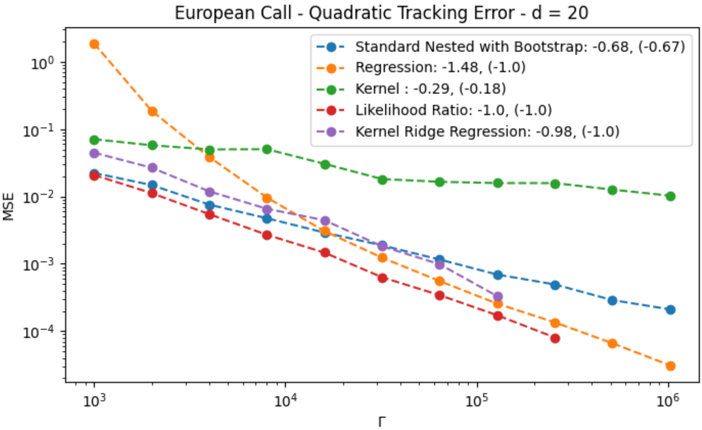
Figure: Empirical convergence rates of different procedures for the base case

Sensitivity to Asset Dimension



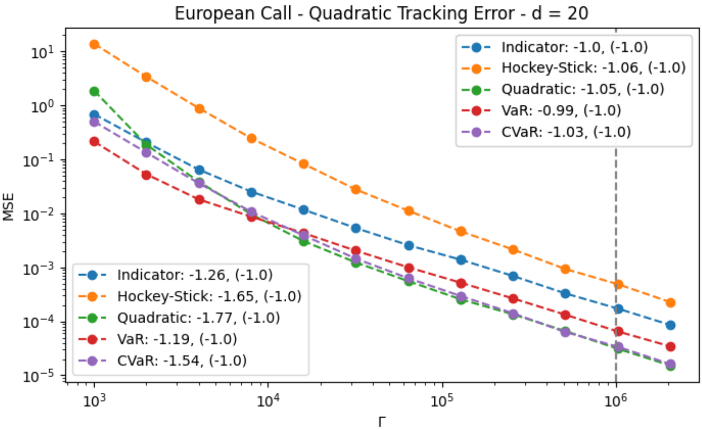
- Standard, KRR, and likelihood ratio procedures are dimension-independent
- Kernel smoothing and regression show sensitivity to dimension, but in different ways

Sensitivity to Asset Dimension



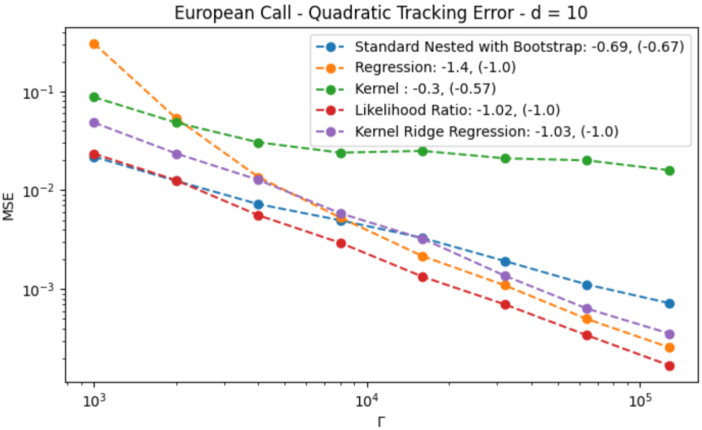
- Standard, KRR, and likelihood ratio procedures are dimension-independent
- Kernel smoothing and regression show sensitivity to dimension, but in different ways

Fast Convergence of Regression-based Procedure



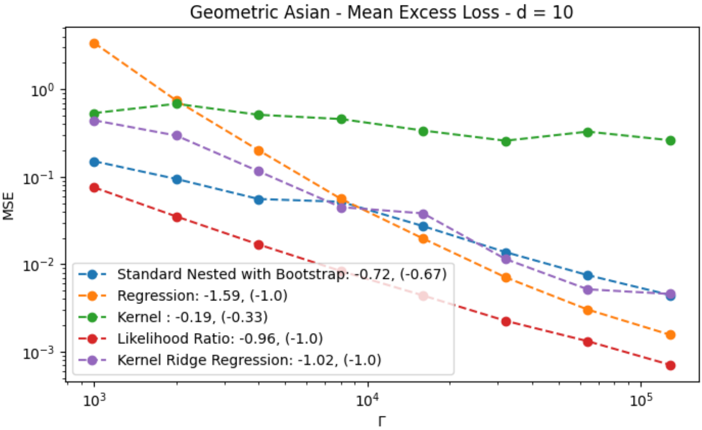
- Higher initial convergence rate
- Stabilizes to match asymptotic rate at higher budgets
- Consistent across different asset dimensions

Sensitivity to Option Type



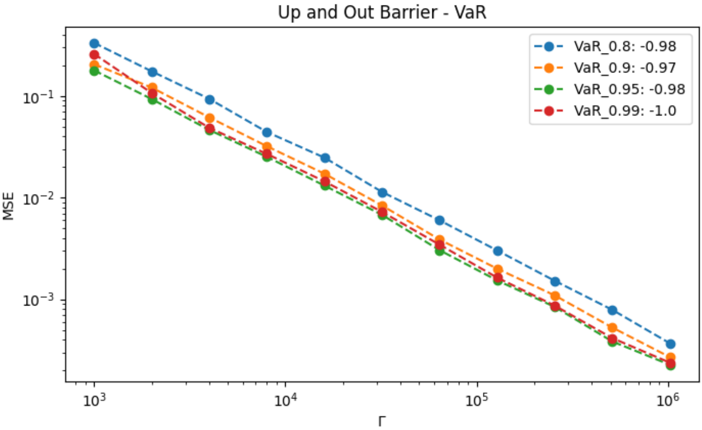
- Similar convergence patterns across different option types
- Regression and kernel smoothing show higher empirical rates for barrier options

Sensitivity to Risk Measure



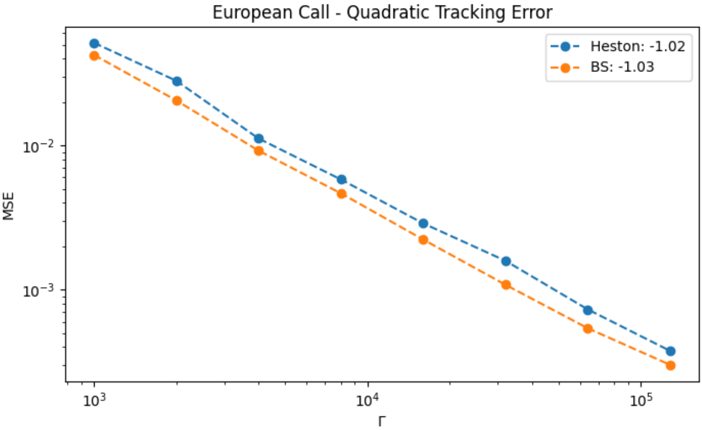
- Convergence behavior consistent across different risk measures
- Regression-based method shows highest empirical convergence rates

Sensitivity to VaR/CVaR Level



- Regression-based method not sensitive to VaR/CVaR level
- Consistent performance across different levels

Sensitivity to Asset Model



- Regression-based method insensitive to asset model (GBM vs. Heston)
- Consistent performance across different asset models

Computational Complexity

There are **computational costs** associated with pooling inner replications.

- Standard procedure: cost of estimating the optimal M and N
- Regression: most efficient among metamodel-based procedures
- Kernel smoothing: costly distance calculations and cross-validation
- Likelihood ratio: No training, but costly weight calculations
- KRR: even more expensive than kernel smoothing

Total Computation Time

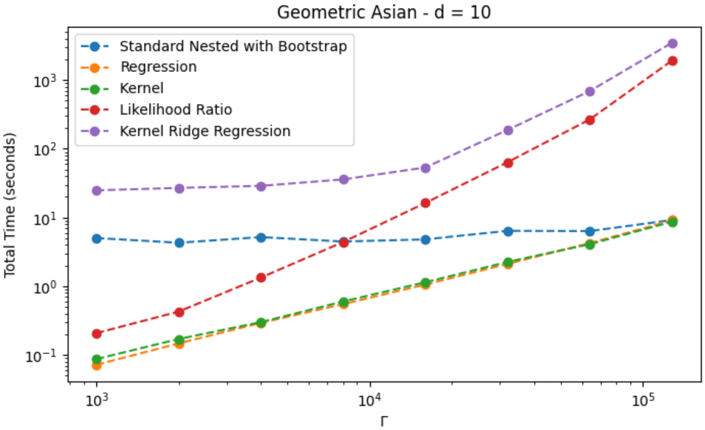


Figure: Total computation time for different procedures

Cost of Hyperparameter Tuning

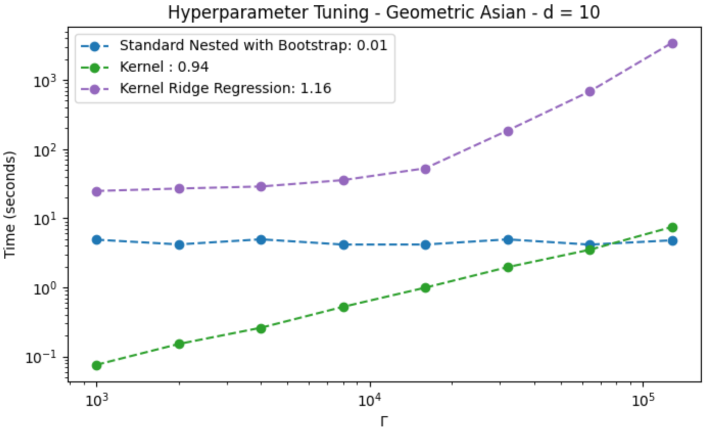


Figure: Cost of hyperparameter tuning for different procedures

Cost of Model Fitting and Validation

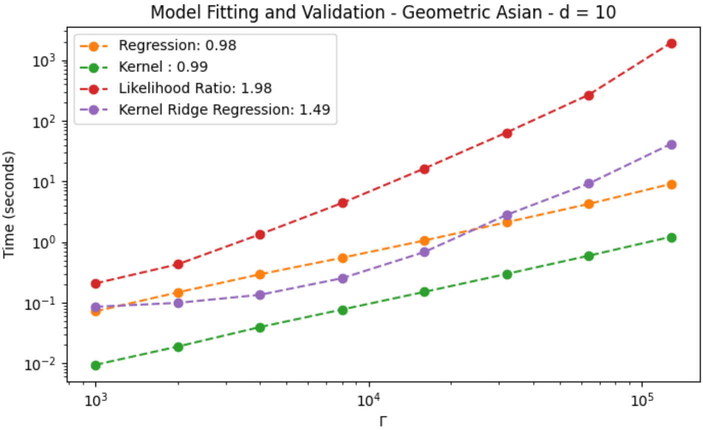


Figure: Cost of model fitting and validation for different procedures

Conclusion

Regression-based nested simulation procedure:

- Most robust and stable for limited budgets
- Efficient to implement
- Fast empirical convergence for option portfolios

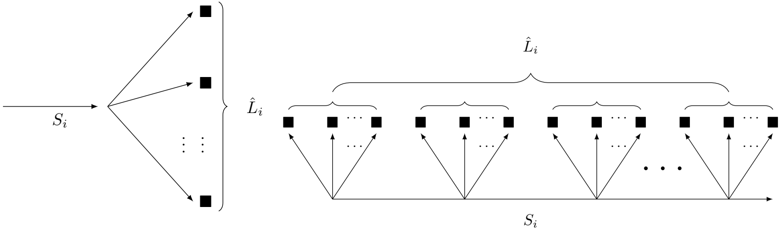
For high-dimensional or complex payoffs:

- Difficult to find a good regression basis
- Neural network-based procedures may be more suitable

Next project: examining performance of metamodel-based simulation procedures for variable annuities

From Options to Variable Annuities

Variable annuities (VAs) poses a challenge for nested simulation due to its **high-dimensional** and **complex payoff structure**.



1 Outer Path for Options

1 Outer Path for VAs

- Need to reconstruct a metamodeling-based nested simulation procedure

Nested Simulation for Risk Management of VAs

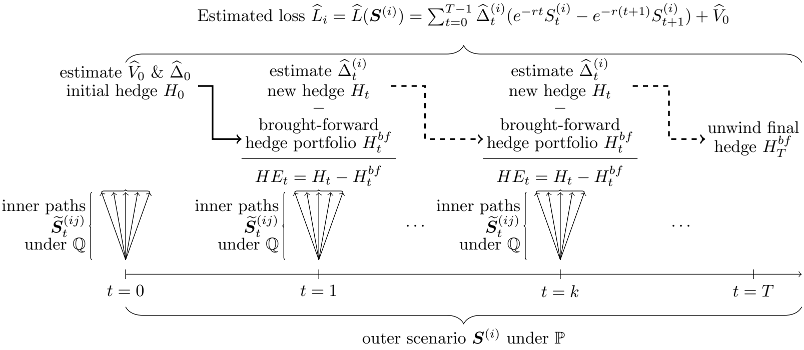


Figure: Illustration of nested simulation that estimates the P&L for one outer scenario

Standard Nested Simulation for VAs

Standard nested simulation for VAs is similar to the one for options.

- Generate M outer scenarios
- For each outer scenario,
  - Perform N inner simulations
  - Estimate hedging loss L_i with $\hat{L}_i$
- Use estimated losses to calculate tail risk measures (e.g., 95%-CVaR)

Observations:

- computational budget is limited;
- high-dimensional input space;
- only a **small** portion of scenarios are relevant when estimating **tail** risk measures.

Metamodel-based Nested Simulation

We use deep neural networks (DNNs) as metamodels

- Use LSTMs for sequential data
- **Challenge**: lack of transparency and interpretability

Research Contributions:

1. Propose two generic DNN-based nested simulation procedures
   - Accurate tail scenario identification
   - Significant computational savings by **budget concentration**
2. Study noise tolerance of DNNs using simulated data
   - **Control noise levels** by adjusting simulation parameters
   - Provide direct evidence on transparency and interpretability

Two-Stage Metamodel-based Nested Simulation

Algorithm Two-Stage Metamodel-based Nested Simulation for VAs

1: **Generate training data for metamodels:**

- Use a fraction of the simulation budget to run the standard nested simulation procedure with M outer scenarios and N' inner replications.
- Construct feature-label pairs $\{(X_i, Y_{ij}) : i = 1, \ldots, M, j = 1, \ldots, N'\}$

2: **Train metamodels:**

- Use the feature-label pairs to train a metamodel.
- Use the trained metamodel to make predictions for $\{X_i : i = 1, \ldots, M\}$.
- Sort the predicted losses to identify a predicted tail scenario set that contains the m largest predicted losses.

3: **Concentrate simulation on predicted tail scenarios:**

- Run the standard procedure on the predicted tail scenarios.
- Estimate the α-CVaR of L using the estimated losses on the predicted tail scenarios.

Benefits of a Two-Stage Procedure

Simulation budget can be saved when:

- the metamodel is accurate (a small m includes most tail scenarios)
- the metamodel can tolerate noise in training labels (a small N')

Key findings:

- Substantial computational savings ($70\% - 85\%$ reduction)
- Maintains accuracy comparable to standard procedure
- DNN metamodels can distinguish between tail and non-tail scenarios effectively
- Addresses regulatory concerns by using actual simulations for final estimates

Another finding: some DNN metamodels make **accurate loss predictions** for given scenarios.

Single-Stage Metamodel-based Nested Simulation

Algorithm Single-Stage Metamodel-based Nested Simulation for VAs

1: **Generate training data for metamodels:**
- Use the entire simulation budget to run the standard nested simulation procedure with M outer scenarios and N inner replications.
- Construct feature-label pairs $\{(X_i, Y_{ij}) : i = 1, \ldots, M, j = 1, \ldots, N\}$

2: **Train metamodels:**
- Use the feature-label pairs to train a metamodel.
- Use the trained metamodel to make predictions for $\{X_i : i = 1, \ldots, M\}$.

3: **Use metamodel predictions to estimate tail risk measures directly.**

Key advantages:
- more efficient than a two-stage procedure;
- avoids specifying m.

Experiment Setting

We estimate the 95%-CVaR of the hedging loss for a GMWB contract with 20-year maturity.

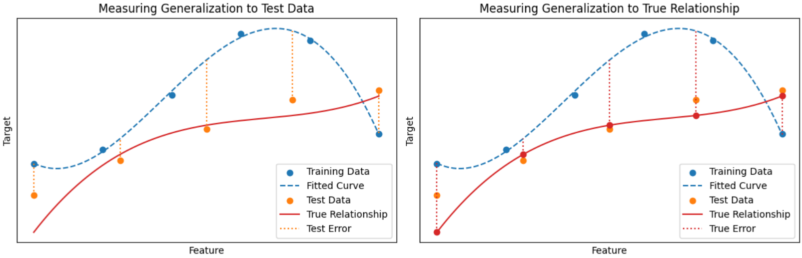
Specifications:

- The underlying asset follows a regime-switching geometric Brownian motion;
- The contract is delta-hedged monthly (240 periods);
- The true 95%-CVaR is estimated using 100,000 outer scenarios and 100,000 inner replications.
- The metamodel is trained using 90,000 outer scenarios and 100 inner replications.
- Benchmark: standard nested simulation procedure with 100,000 outer scenarios and 1,000 inner replications.

Experiment Design

Research Questions:

- What do DNNs learn from noisy data?
- How well do DNNs learn from noisy data?



- Our 90,000 training data is noisy, and the test data is also **noisy**.
- Our evaluation is based on the true (**noiseless**) feature-label relationship[2].

[2] Made possible by novel simulation design.

Experiment Setting

We consider the following metamodel architectures:

Metamodel	Abbreviation	Capacity
Multiple Linear Regression	MLR	241
Quadratic Polynomial Regression	QPR	481
Feedforward Neural Network	FNN	35,009
Recurrent Neural Network	RNN	32,021
Long Short-Term Memory	LSTM	35,729

Table: Metamodel architectures for GMWB inner simulation model

Capacity is defined as the number of parameters in the metamodel.

- Higher capacity metamodels are more flexible and expressive.
- Lower capacity metamodels are less likely to overfit.

Traditional Regression Metamodels

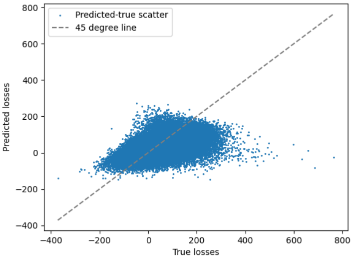


Figure: QQ plots between true and predicted loss labels for MLR and QPR metamodels

- MLR and QPR metamodels make **inaccurate** loss predictions.
- Feature engineering is hardly feasible for our 240-dimensional X.

Deep Neural Network Metamodels

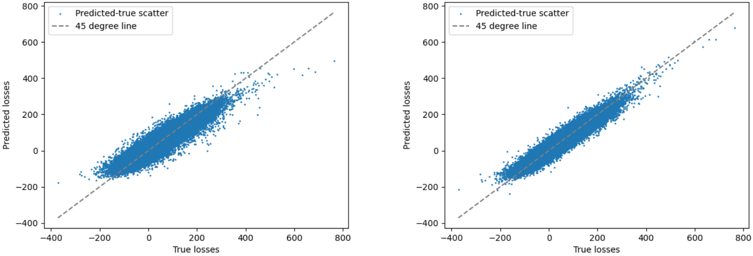


Figure: QQ plots between true and predicted loss labelsfor FNN and LSTM metamodels

- DNN metamodels are more flexible.
- Time series features prefer a LSTM metamodel over FNN.
- Network architecture serves as prior knowledge that regularizes DNNs.

Metamodel Performance on Different Datasets

Metamodel	Training error	Test error	True error
MLR	$0.706(\pm 8.3 \times 10^{-4})$	$0.713(\pm 2.7 \times 10^{-2})$	$0.706(\pm 3.4 \times 10^{-4})$
QPR	$0.543(\pm 8.3 \times 10^{-4})$	$0.554(\pm 2.7 \times 10^{-2})$	$0.544(\pm 4.1 \times 10^{-4})$
FNN	$0.129(\pm 6.0 \times 10^{-3})$	$0.240(\pm 9.8 \times 10^{-3})$	$0.132(\pm 5.8 \times 10^{-3})$
RNN	$0.132(\pm 7.5 \times 10^{-3})$	$0.137(\pm 7.6 \times 10^{-3})$	$0.119(\pm 7.5 \times 10^{-3})$
LSTM	$0.075(\pm 4.5 \times 10^{-3})$	$0.079(\pm 5.4 \times 10^{-3})$	$0.063(\pm 4.4 \times 10^{-3})$
RNN*[3]	$0.109(\pm 5.2 \times 10^{-3})$	$0.128(\pm 5.2 \times 10^{-3})$	$0.109(\pm 5.2 \times 10^{-3})$

Table: Average MSEs and 95% confidence bands of metamodels for GMWB inner simulation model.

- RNN-based metamodels have lower true errors than their training errors.
- DNN metamodels with suitable architectures **cut through the noise** in training labels.

[3] This row summarizes the results of the well-trained RNNs.

Issues with RNNs

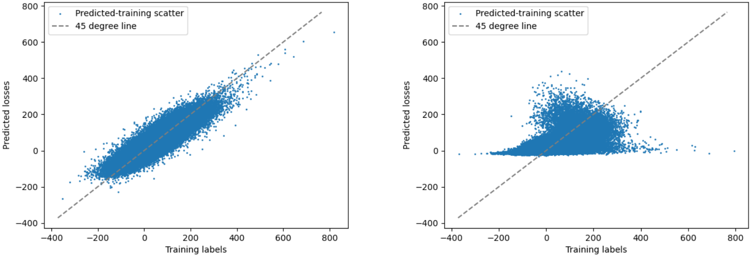


Figure: QQ plots between training and predicted loss labels for RNN metamodels

- RNN metamodels suffers from **vanishing gradient problem**.
- Ease of training (reliability) is a critical factor when choosing a DNN metamodel.

Safety Margin

Consider estimating the 95% CVaR with 100,000 outer scenarios.

Figure: Illustration: a safety margin of 4% ($m = 9000$)

Choosing a safety margin: a trade-off between accuracy and efficiency.

- A lower margin: not enough tail identified.
- A higher margin: more accurate CVaR estimate, but more budget needed to perform extensive inner simulations.

Tail Scenario Identification

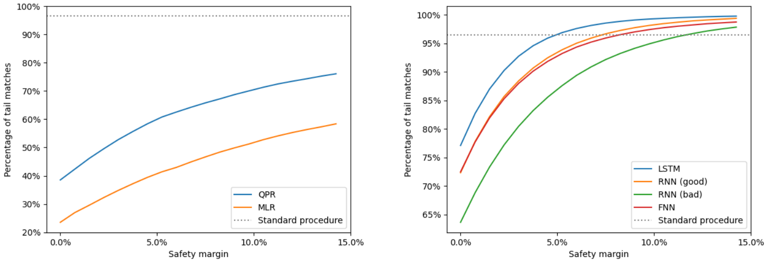


Figure: Tail scenario identification for regression and DNN metamodels

- Traditional regression metamodels are **unable** to accurately identify tail scenarios even with high safety margins.
- LSTM metamodels surpasses the standard procedure with 5% safety margin.

Estimating CVaR

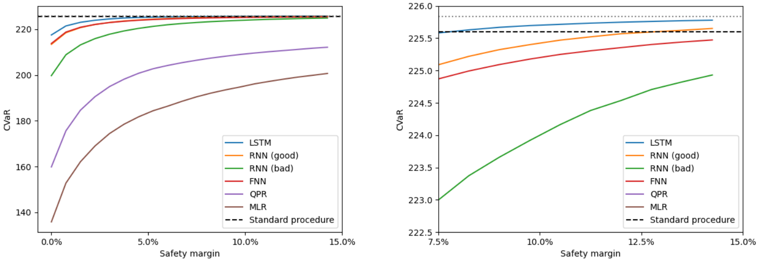


Figure: CVaR estimation for DNN metamodels

- Traditional regression metamodels are **unable** to accurately estimate CVaR even with high safety margins.
- LSTM surpasses the standard procedure with a **5% safety margin**.
- With a 95% safety margin, any two-stage procedure produce the same CVaR estimate as a standard procedure.

Sensitivity Testing for DNNs

In a simulation study, we have control over the **noise level in training labels** and the **number of training samples**.

Controling the inner replications N' varies the noise level in training labels.

- **Low noise labels**: $N' = 100$
- **Medium noise labels**: $N' = 10$
- **High noise labels**: $N' = 1$

Controling the outer scenarios M varies the number of training samples.

- $M \in \{10^2, 10^3, 10^4, 10^5\}$

2 LSTMs of **different capacities** are examined based on their MSEs.

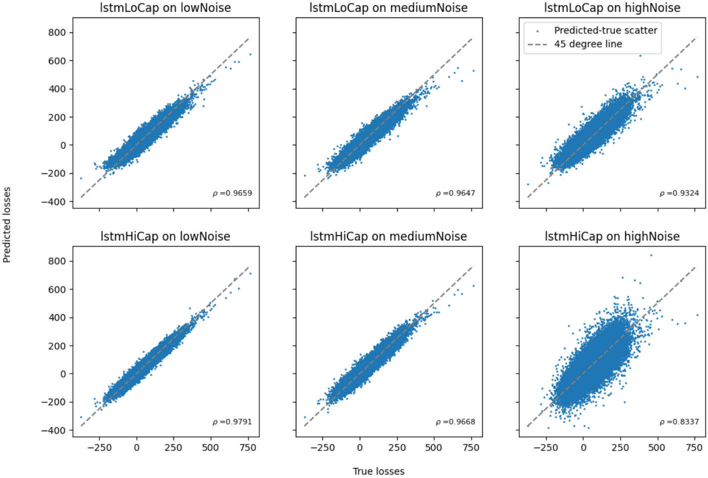
Noise Tolerance of DNNs

Model	N'	Training error	Test error	True error
LSTM	100	0.075	0.079	0.063
High-capacity LSTM	100	0.068	0.102	0.060
Average Difference	100	−0.007	0.023	−0.003
LSTM	10	0.195	0.193	0.070
High-capacity LSTM	10	0.157	0.199	0.065
Average Difference	10	−0.038	0.006	−0.005
LSTM	1	1.366	0.781	0.129
High-capacity LSTM	1	1.354	0.795	0.149
Average Difference	1	−0.012	0.014	0.020

Table: MSEs of LSTM metamodels.

- Both LSTMs cut through the noise in training labels.
- Both LSTMs deteriorate dramatically on **high-noise** labels.
- High-capacity LSTM can tolerate **low** and **medium** label noise.

Noise Tolerance of DNNs



Sensitivity of Regular LSTM

	$N' = 1$	$N' = 10$	$N' = 100$	$N' = 1000$
$M = 100$	1.139	0.229	0.167	0.158
$M = 1000$	0.559	0.173	0.123	0.127
$M = 10000$	0.283	0.115	0.099	0.097
$M = 100000$	0.129	0.070	0.063	0.063

Table: MSE between regular LSTM's predicted losses and true losses.

- Same color $\rightarrow$ same total simulation budget.
- $N' = 10$ is a reasonable budget allocation for LSTM metamodels.

Sensitivity of High-capacity LSTM

	$N' = 1$	$N' = 10$	$N' = 100$	$N' = 1000$
$M = 100$	0.764	0.408	0.131	0.087
$M = 1000$	0.878	0.367	0.156	0.087
$M = 10000$	0.351	0.147	0.064	0.063
$M = 100000$	0.149	0.065	0.060	0.038

Table: MSE between high-capacity LSTM's predicted losses and true losses.

- Same color $\rightarrow$ same total simulation budget.
- $N' = 10$ is a reasonable budget allocation for LSTM metamodels.

Single-Stage Procedure

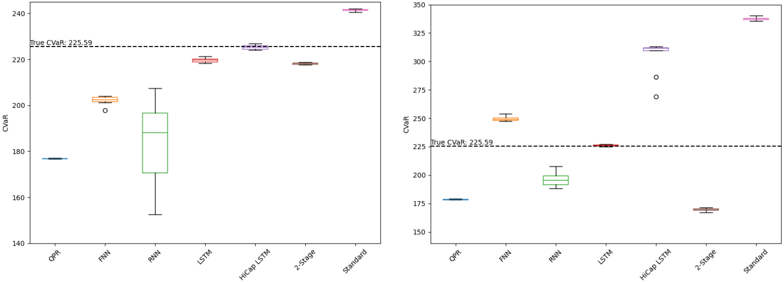


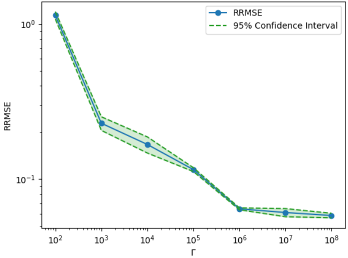
Figure: CVaR estimates of single-stage procedures (left: $N' = 10$. right: $N' = 1$).

- The single-stage procedure outperforms the two-stage procedure.
- The single-stage procedure is more efficient than the two-stage procedure.
- Setting $N' = 10$ is a reasonable budget allocation.

Convergence Analysis

For each Γ, the best performing metamodel is used.

- Maximum number of outer scenarios $M = 10^5$.



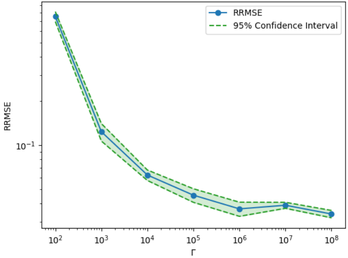


Figure: Empirical convergence of CVaR for single-stage procedures with LSTM metamodels (left: regular LSTM. right: high-capacity LSTM).

- Minimal effect of increasing N' on CVaR estimation.
- Similar behavior as regression metamodels ($d = 20$) in the previous section.

Convergence Analysis

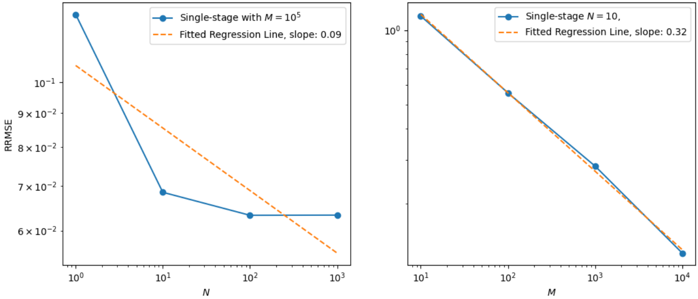


Figure: Empirical convergence of the single-stage procedure with a LSTM metamodel.

- Minimal effect of increasing N' on CVaR estimation.
- For a given Γ, set N' constant and allocate budget to outer simulations.

Conclusion

Key Findings:

- LSTMs are **resilient** to moderate levels of noise in training labels.
- Deep neural networks can learn **true** complex dynamic hedging model.
- Two-stage procedure addresses regulatory concerns by avoiding direct use of metamodel predictions.
- Single-stage procedure is **efficient** and **versatile**.
- **Increasing outer scenarios** is more beneficial.
- High-capacity LSTM requires lower noise training labels.

Future Directions:

- Apply deep neural network metamodels to other risk management tasks.
- Investigate impact of label noise on other deep learning models
- Explore optimal network architectures for different simulation models.

Transfer Learning for Rapid Adaptation of DNN Metamodels

Challenge: Adapting deep neural network metamodels to changing conditions.

- Retraining from scratch is computationally expensive.
- Efficient incorporation of new VA contract data.
- Balancing model accuracy and computational costs.

Solution: Transfer learning (TL) to develop adaptable, efficient metamodels for VA dynamic hedging.

- Pre-train deep neural network on contracts with abundant simulation data.
- Fine-tune on smaller dataset of new contracts/market conditions.
- Leverages shared features between VA contracts.
- Computational savings:
  - Reduced training time.
  - Fewer data points needed for good performance.

Transfer Learning Framework

Key Components:

- **Domain** $\mathcal{D}$: feature space $\mathcal{X}$ + probability distribution F
- **Task** $\mathcal{T}$: label space $\mathcal{Y}$ + predictive function $f : \mathcal{X} \to \mathcal{Y}$

Source vs. Target:

- **Source:** $\mathcal{D}_{\text{So}} = \{\mathcal{X}_{\text{So}}, F_{\text{So}}(X)\}$
- **Target:** $\mathcal{D}_{\text{Ta}} = \{\mathcal{X}_{\text{Ta}}, F_{\text{Ta}}(X)\}$

Our Goal:

- **Input features** X: risk factors from outer simulation
- **Output labels** L: contract losses
- **Source and target**: from VAs with abundant simulation data to new VAs with limited data
- **Goal**: improve $f_{\text{Ta}}(\cdot)$ using knowledge from $\mathcal{D}_{\text{So}}$ and $f_{\text{So}}(\cdot)$

Transfer Learning Techniques

Common Techniques:

- **Fine-tuning:** a model pre-trained on a source task is used as a starting point for a target task.
- **Layer freezing:** only part of the model is fine-tuned.
- **Multi-task learning:** perform training on multiple tasks simultaneously.

Key considerations:

- Similarity between source and target tasks
- Appropriate learning rate

Fine-tuning Algorithm for LSTM Metamodels in VA Hedging

Algorithm Fine-tuning Algorithm for LSTM Metamodels in VA Hedging

1: **Input:** $\mathcal{D}_{\mathsf{So}} = \{(X_{\mathsf{So}}^{(i)}, L_{\mathsf{So}}^{(i)})\}_{i=1}^{M_{\mathsf{So}}}$, $\mathcal{D}_{\mathsf{Ta}} = \{(X_{\mathsf{Ta}}^{(i)}, L_{\mathsf{Ta}}^{(i)})\}_{i=1}^{M_{\mathsf{Ta}}}$, α_{So}, and α_{Ta}.

2: Train a LSTM metamodel $f_{\mathsf{So}}(\cdot;\theta_{\mathsf{So}})$ on $\mathcal{D}_{\mathsf{So}}$:

$$\theta_{\mathsf{So}} = \min_{\theta} \frac{1}{M_{\mathsf{So}}} \sum_{i=1}^{M_{\mathsf{So}}} \left(f_{\mathsf{So}}(X_{\mathsf{So}}^{(i)};\theta) - L_{\mathsf{So}}^{(i)} \right)^2$$

3: Initialize the target metamodel parameters θ_{Ta} using the pre-trained metamodel parameters:

$$\theta_{\mathsf{Ta}} \leftarrow \theta_{\mathsf{So}}$$

4: Fine-tune the entire LSTM metamodel $f_{\mathsf{Ta}}(\cdot;\theta_{\mathsf{Ta}})$ on the target dataset $\mathcal{D}_{\mathsf{Ta}}$ using a smaller learning rate α_{Ta}:

$$\theta_{\mathsf{Ta}} = \min_{\theta} \frac{1}{M_{\mathsf{Ta}}} \sum_{i=1}^{M_{\mathsf{Ta}}} \left(f_{\mathsf{Ta}}(X_{\mathsf{Ta}}^{(i)};\theta) - L_{\mathsf{Ta}}^{(i)} \right)^2$$

5: **Output:** Final adapted LSTM metamodel $f_{\mathsf{Ta}}(\cdot;\theta_{\mathsf{Ta}})$ for the target task

Layer Freezing Algorithm for LSTM Metamodels in VA Hedging

Algorithm Layer Freezing Algorithm for LSTM Metamodels in VA Hedging

1: **Input:** $\mathcal{D}_{\mathsf{So}} = \{(X_{\mathsf{So}}^{(i)}, L_{\mathsf{So}}^{(i)})\}_{i=1}^{M_{\mathsf{So}}}$, $\mathcal{D}_{\mathsf{Ta}} = \{(X_{\mathsf{Ta}}^{(i)}, L_{\mathsf{Ta}}^{(i)})\}_{i=1}^{M_{\mathsf{Ta}}}$, α_{So}, and α_{Ta}.

2: Train LSTM model $f_{\mathsf{So}}(\cdot;\theta_{\mathsf{So}})$ on $\mathcal{D}_{\mathsf{So}}$.

3: Initialize the target model parameters $\theta_{\mathsf{Ta}} = [\theta_0, \theta_1]$ using the pre-trained source model parameters θ_{So}:

$$\theta_{\mathsf{Ta}} \leftarrow \theta_{\mathsf{So}} = [\theta_0, \theta_1]$$

4: Freeze the parameters of the shared layers θ_0 and fine-tune the trainable layers θ_1 on the target dataset $\mathcal{D}_{\mathsf{Ta}}$:

$$\theta_{\mathsf{Ta}} = \min_{\theta_1} \frac{1}{M_{\mathsf{Ta}}} \sum_{i=1}^{M_{\mathsf{Ta}}} \left(f_{\mathsf{Ta}}(X_{\mathsf{Ta}}^{(i)}; [\theta_0, \theta_1]) - L_{\mathsf{Ta}}^{(i)} \right)^2$$

5: **Output:** Adapted model $f_{\mathsf{Ta}}(\cdot;\theta_{\mathsf{Ta}})$ for the target task.

Multi-task Learning Algorithm for LSTM Metamodels in VA Hedging

Algorithm Multi-task Learning Algorithm for LSTM Metamodels in VA Hedging

1: **Input:** learning rate α, set of K tasks $\{\mathcal{T}_k\}_{k=1}^{K}$ with datasets $\mathcal{D}_k = \{(X_k^{(i)}, L_k^{(i)})\}_{i=1}^{M_k}$, task-specific parameters θ_k for each task k, and shared parameters θ_0.

2: Train the multi-head LSTM metamodel on all K tasks simultaneously by minimizing the multi-task loss function:

$$\min_{\theta_0, \{\theta_k\}_{k=1}^{K}} \sum_{k=1}^{K} \frac{1}{M_k} \sum_{i=1}^{M_k} \left(f_i(X_k^{(i)}; \theta_0, \theta_k) - L_k^{(i)} \right)^2 \quad (1)$$

3: Update both the shared parameters θ_0 and task-specific parameters $\{\theta_k\}_{k=1}^{K}$ simultaneously using backpropagation and gradient descent with learning rate α.

4: **Output:** Trained multi-task metamodel $f(\cdot; \theta_0, \{\theta_k\}_{k=1}^{K})$ for all K tasks

Experiment Setup

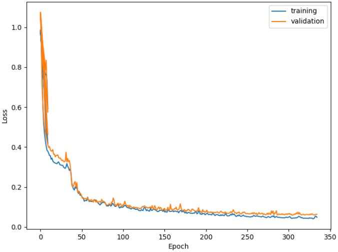
Contract	Asset Model	Lapse	M_{So}	M_{Ta}
GMMB	GBM	No lapse	50000	N/A
GMMB	RS-GBM	No lapse	50000	2000
GMMB	RS-GBM	Static lapse	50000	2000
GMMB	RS-GBM	Dynamic lapse	50000	2000
GMWB	RS-GBM	Dynamic lapse	N/A	2000

Table: VA Contracts for Transfer Learning Experiments

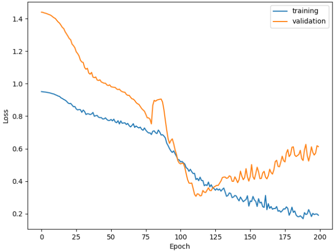
We aim to examine the performance of TL techniques

- learning the lapse features,
- learning the dynamic lapse, and
- transferring to other contract types.

The Effect of Number of Training Samples



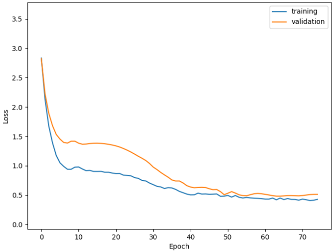
(a) 50000 samples



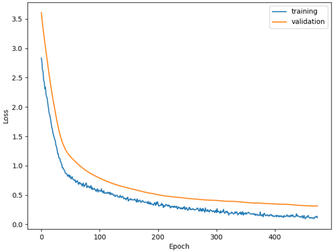
(b) 2000 samples

Figure: Learning curves of GMMB contracts with static lapse

Learning Lapse Features using Transfer Learning



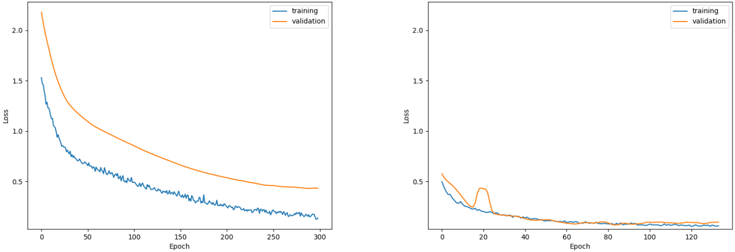
(a) Fine-tuning



(b) Layer Freezing

Figure: Learning curves of GMMB contracts with static lapse

Learning Curves: Learning Dynamic Lapse



(a) No Lapse (b) Static Lapse

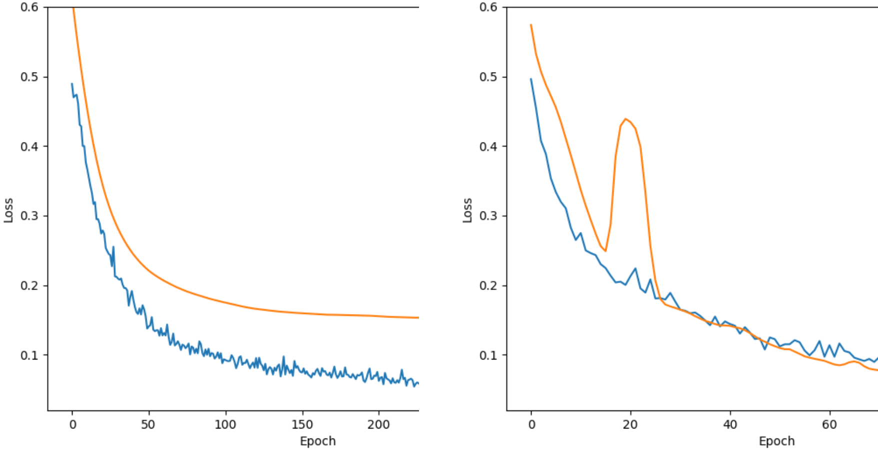
Figure: Learning curves of GMMB contracts with dynamic lapse

Results: Accuracy Comparison

Lapse Type	Extensive	Fine-tuning	Layer Freezing	Without TL
No Lapse	N/A	0.4894	**0.3361**	N/A
Static Lapse	N/A	0.0794	0.0763	N/A
Dynamic Lapse	**0.0587**	N/A	N/A	**0.2950**

Comparison of different TL methods on GMMB contracts (best MSE values)

Learning Curves: Effect of Similarity



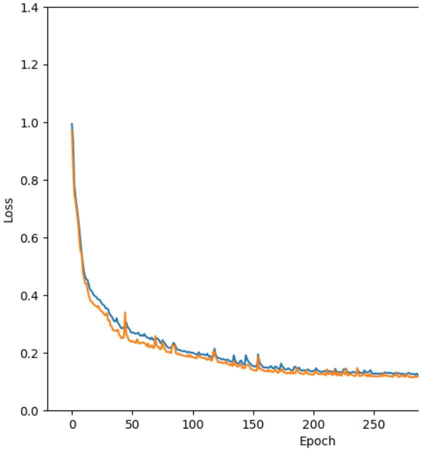
The effect of similarity between source and target on the convergence speed

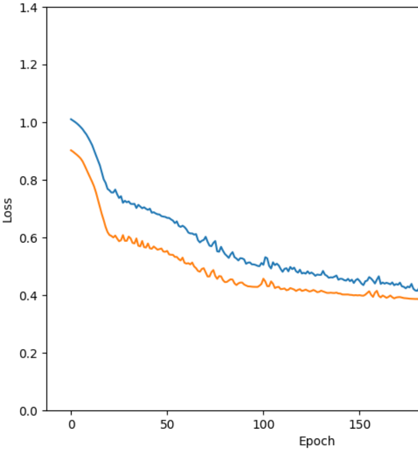
- *Left: from No Lapse; right: from Static Lapse*

Transfer Knowledge to other Contract Types

From GMMB to GMWB Contract

Learning Curves: Transfer Knowledge to other Contract Types

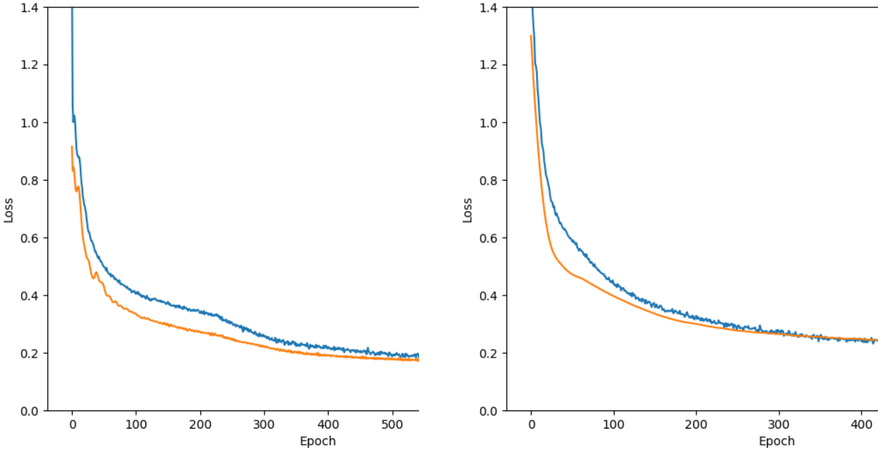




Learning curves of GMWB contracts

- *Left: 50000 samples; right: 2000 samples*

Learning Curves: Effect of Similarity



Comparison of different TL methods on GMWB contracts

- *Left: fine-tuning; right: layer freezing*

Performance Comparison (GMMB $\rightarrow$ GMWB)

Model	Training MSE	True MSE
Without TL	0.3588	0.4188
Fine Tuning	0.1690	0.1780
Layer Freezing	0.1828	0.2295
Extensive Training	0.0853	0.0726

- Fine-tuning outperforms layer freezing for dissimilar tasks
- Both TL methods better than training from scratch
- Extensive training still superior with abundant data

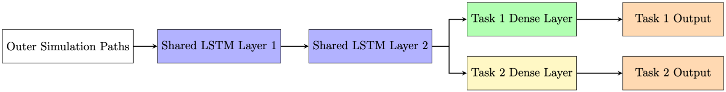
Practical Implementation

- **Framework**: TensorFlow/PyTorch implementation
- **Workflow**:
  1. Train base model on source contract
  2. Freeze early layers
  3. Fine-tune later layers on target contract
  4. Deploy for production use

PyTorch is recommended.

- It is more flexible and easier to customize
- It is more intuitive and easier to understand
- It is more efficient and easier to debug

Multi-task Learning



- LSTM layers shared across multiple tasks
- Task-specific fully connected layers
- Objective: Minimize sum of loss functions across all tasks

Multi-task Learning Framework

Input: Set of K tasks $\{\mathcal{T}_k\}_{k=1}^{K}$ with datasets $\mathcal{D}_k = \{(X_k^{(i)}, L_k^{(i)})\}_{i=1}^{M_k}$, shared parameters θ_0, and task-specific parameters θ_k for each task k
Algorithm:

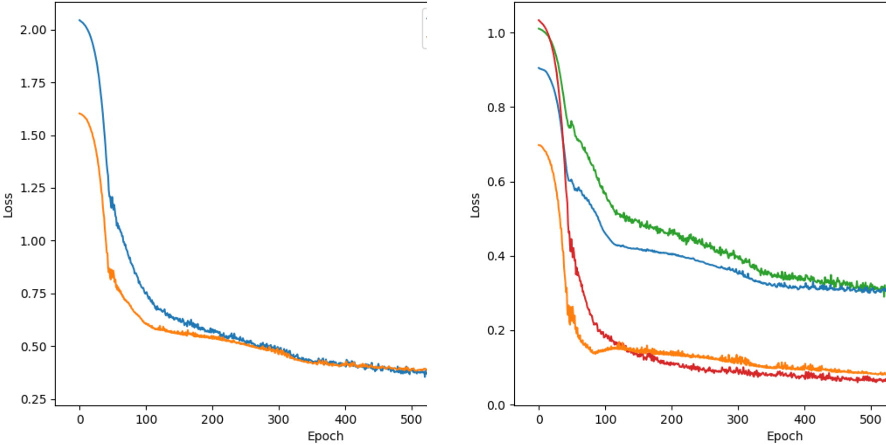
1. Train the multi-head LSTM metamodel on all K tasks simultaneously by minimizing the multi-task loss function:

$$\min_{\theta_0, \{\theta_k\}_{k=1}^{K}} \sum_{k=1}^{K} \frac{1}{M_k} \sum_{i=1}^{M_k} \left(f_i(X_k^{(i)}; \theta_0, \theta_k) - L_k^{(i)} \right)^2 \tag{2}$$

2. Update both the shared parameters θ_0 and task-specific parameters $\{\theta_k\}_{k=1}^{K}$ simultaneously using backpropagation and gradient descent with learning rate α

Output: Trained multi-task LSTM metamodel $f(\cdot; \theta_0, \{\theta_k\}_{k=1}^{K})$ for all K tasks

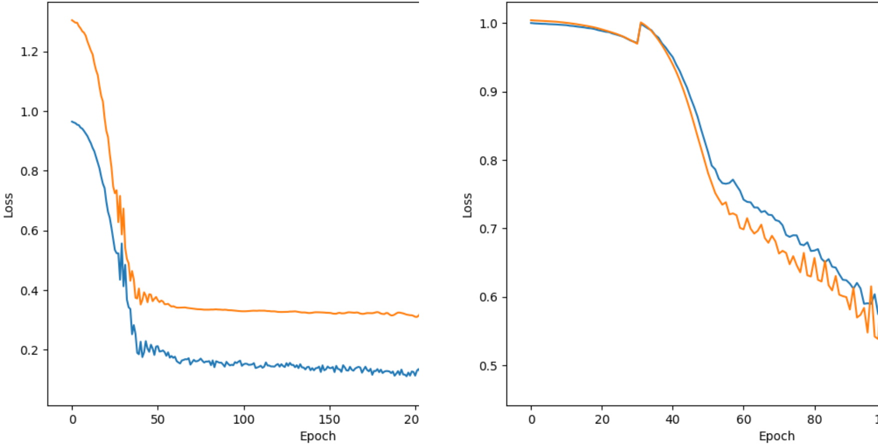
Multi-task Learning: GMMB and GMWB



Multi-task learning of GMMB and GMWB

- *Left: total MSE; right: individual MSE*

Multi-task Learning: GMMB and GMWB



Training without multi-task learning

- *Left: GMMB; right: GMWB*

Challenges & Limitations

- Determining optimal layer freezing strategy
- Handling significantly different contract structures
- Quantifying uncertainty in transfer learning predictions
- Regulatory acceptance of black-box approaches

Conclusions

- Transfer learning significantly improves metamodeling for VA contracts:
  - Faster training convergence
  - Better prediction accuracy
  - Reduced computational requirements
- Enables more frequent risk assessments and faster decision-making

Future Directions

- Incorporating domain knowledge into transfer process
- Extension to other insurance and financial products
- Multi-task learning with more than two tasks

References

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