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Efficient Machine Learning Approaches for Fast Risk Evaluation of VAs

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Thesis Defense, University of Waterloo

Outline

Nested Simulation Procedures

Nested simulation procedures are necessary for **complex** financial derivatives and insurance products.

$$\rho(L) = \rho(L(X)), \quad L(X) = \mathbb{E}\left[Y|X=x\right]|_{x=X}$$

Involves two levels of Monte Carlo simulations:

- Outer level: generates underlying risk factors (outer scenarios), $X_i \sim F_X$
- Inner level: generates scenario-wise samples of portfolio losses (inner replications), $Y_{ij} \sim F_{Y|X_i}$

Computationally expensive due to its nested structure.

Common Risk Measures

- Smooth h, e.g., quadratic tracking error

$$\rho(L) = \mathbb{E}\left[(L-b)^2\right]$$

- hockey-stick h: mean excess loss

$$\rho(L) = \mathbb{E}\left[L \cdot \mathbb{1}_{\{L \geq u\}}\right]$$

- indicator h: probability of large loss

$$\rho(L) = \mathbb{E}\left[\mathbb{1}_{\{L \geq u\}}\right]$$

- Value at Risk (VaR)

$$\rho_\alpha(L) = Q_\alpha(L) = \inf\{u : \mathbb{P}(L \leq u) \geq \alpha\}$$

- Conditional Value at Risk (CVaR) [1]

$$\rho_\alpha(L) = \mathbb{E}\left[L | L \geq Q_\alpha(L)\right]$$

[1] Note: If $Q_\alpha(L)$ falls in a probability mass, $\rho(L) = \frac{(\beta-\alpha)Q_\alpha(L)+(1-\beta)\mathbb{E}[L|L\geq Q_\alpha(L)]}{1-\alpha}$.

Standard Nested Simulation

$$\hat{L}_{N,i} = \frac{1}{N}\sum_{j=1}^{N} Y_{ij}; \quad Y_{ij} \sim F_{Y|X_i}$$

- Uses inner sample mean to estimate $L(X_i)$.
- Proposed by Gordy and Juneja (2010); finds optimal growth order of M and N.
- Zhang et al. (2021) estimate the optimal M and N using a bootstrap method.
- Computationally expensive and potentially **wasteful** use of budget.

Other Nested Simulation Procedures

Subsequent works focus on improving the efficiency of nested simulation:

- Regression-based (Broadie et al., 2015)
- Kernel smoothing (Hong et al., 2017)
- Likelihood ratio (Feng and Song, 2020)
- Kernel ridge regression (Zhang et al., 2022)

Key ideas:

- Pool inner replications from different outer scenarios
- Use metamodeling techniques to approximate the inner simulation model

Metamodeling Approach

In this thesis, we focus on procedures that use **supervised learning metamodels** to approximate the inner simulation model.

- Treat the inner simulation as a black-box function
- Approximate $L(\cdot)$ with $\hat{L}^{\mathsf{SL}}_{M,N}(\cdot)$
- Train with a set of feature-label pairs generated from the standard procedure:

$$\{(X_i, \hat{L}_{N,i}) | i = 1, \ldots, M, j = 1, \ldots, N\}$$

- Use trained metamodel to make predictions for all $X \in \mathcal{X}$

There are **computational costs** associated with pooling inner replications.

Problem Statement

Minimize mean squared error (MSE) of the estimator subject to total simulation budget:

$$\min_{M,N} \quad \mathbb{E}\left[(\hat{\rho}_{M,N} - \rho)^2\right]$$
$$\text{subject to } M \cdot N = \Gamma$$

Interested in convergence order as $\Gamma \to \infty$

Asymptotic Convergence Rates of Different Procedures

Procedures	**Smooth** h	**Hockey-Stick** h	**Indicator** h
Standard Procedure	$\mathcal{O}(\Gamma^{-2/3})$	$\mathcal{O}(\Gamma^{-2/3})$	$\mathcal{O}(\Gamma^{-2/3})$
Regression	$\mathcal{O}(\Gamma^{-1})$	$\mathcal{O}(\Gamma^{-1+\delta})$	No Result
Kernel Smoothing	$\mathcal{O}(\Gamma^{-\min(1,4/(d+2))})$		
Kernel Ridge Regression	$\mathcal{O}(\Gamma^{-1})$		
Likelihood Ratio	$\mathcal{O}(\Gamma^{-1})$		

- We show the asymptotic convergence rates of the standard procedure for smooth and hockey-stick h.
- Only kernel smoothing depends on the asset dimension d.

Key Theoretical Results

Observations:

- Most literature focuses on the MSE of $\hat{\rho}$.
- Wang et al. (2022) analyze convergence of absolute error in probabilistic order.

Contribution: bridging the gap between MSE and absolute error convergence.

- Convergence in MSE:

$$\mathbb{E}\left[(\hat{\rho}_\Gamma - \rho)^2\right] = \mathcal{O}\left(\Gamma^{-\xi}\right)$$

- Convergence in Probabilistic Order:

$$|\hat{\rho}_\Gamma - \rho| = \mathcal{O}_{\mathbb{P}}(\Gamma^{-\xi})$$

Key Theoretical Results

Theorem

If $\hat{\rho}_\Gamma$ converges in MSE to ρ in order ξ, then $\hat{\rho}_\Gamma$ converges in probabilistic order to ρ in order $\frac{\xi}{2}$.

- First result to draw connection between MSE and probabilistic order convergence.
- Applicable to any nested simulation procedure.
- Convergence in MSE implies convergence in probabilistic order.

Experiment Design

We compare 5 nested simulation procedures

- Standard nested simulation
- Regression-based
- Kernel smoothing
- Likelihood ratio
- Kernel ridge regression

And their empirical convergence stable across different:

- Risk measures
- Option types
- Asset dimensions
- Asset models (GBM vs. Heston)
- Regression bases (only for the regression-based procedure)

Finite-Sample Performance

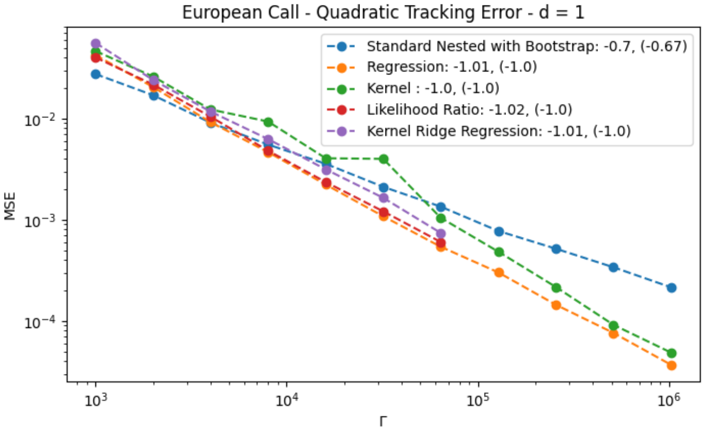
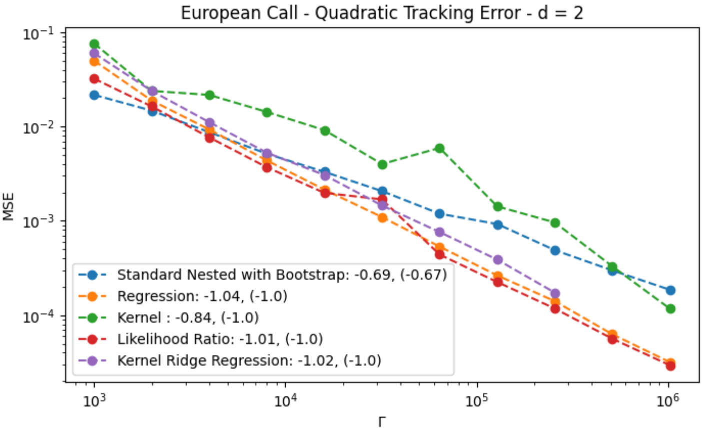


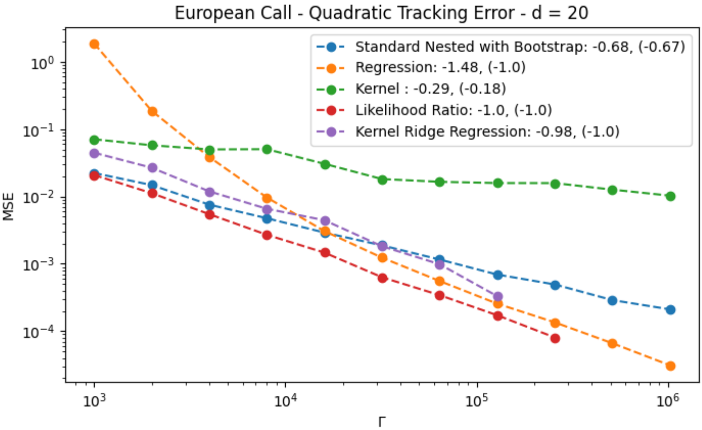
Figure: Empirical convergence rates of different procedures for the base case

Sensitivity to Asset Dimension



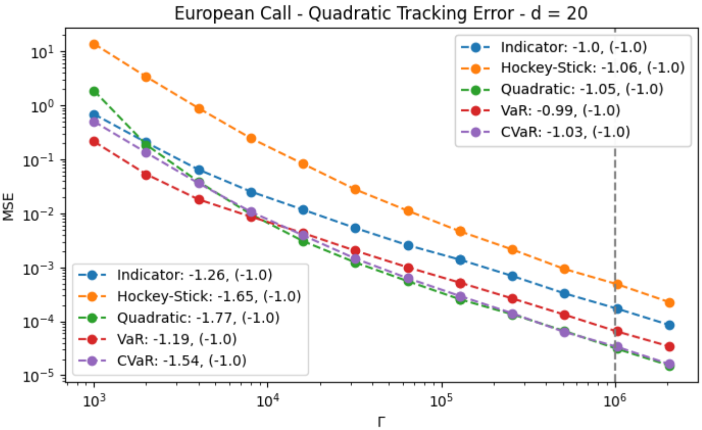
- Standard, KRR, and likelihood ratio procedures are dimension-independent
- Kernel smoothing and regression show sensitivity to dimension, but in different ways

Sensitivity to Asset Dimension



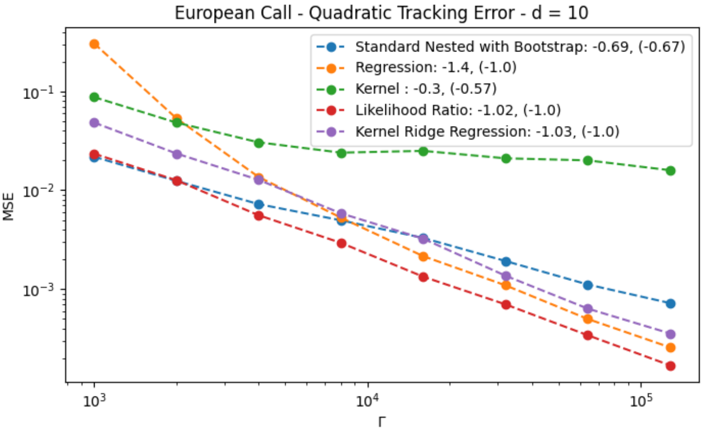
- Standard, KRR, and likelihood ratio procedures are dimension-independent
- Kernel smoothing and regression show sensitivity to dimension, but in different ways

Fast Convergence of Regression-based Procedure



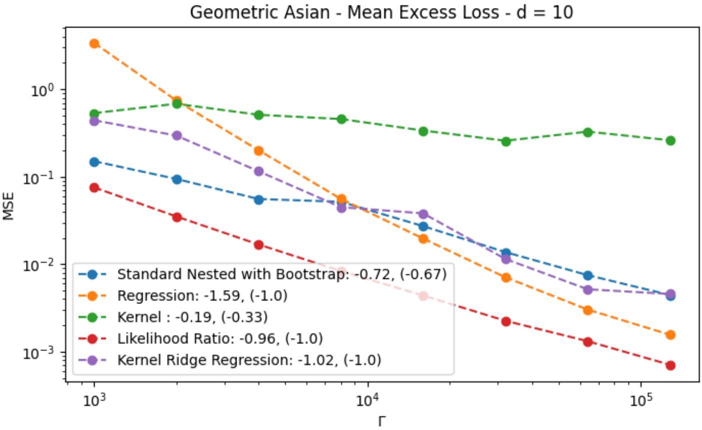
- Higher initial convergence rate
- Stabilizes to match asymptotic rate at higher budgets
- Consistent across different asset dimensions

Sensitivity to Option Type



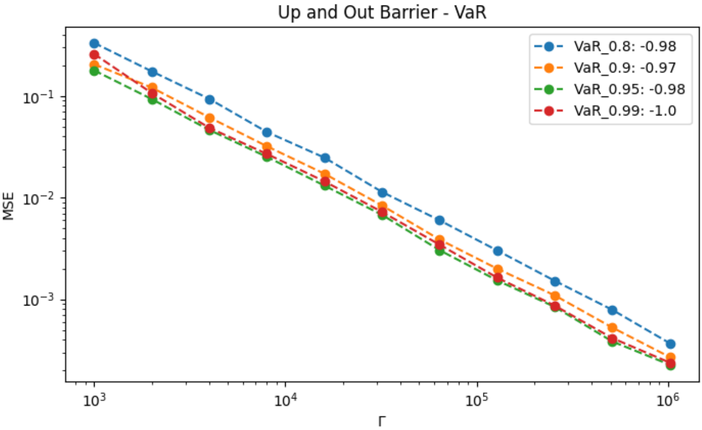
- Similar convergence patterns across different option types
- Regression and kernel smoothing show higher empirical rates for barrier options

Sensitivity to Risk Measure



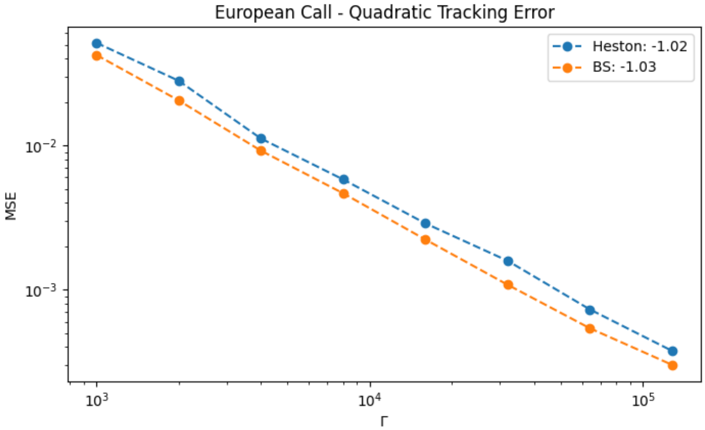
- Convergence behavior consistent across different risk measures
- Regression-based method shows highest empirical convergence rates

Sensitivity to VaR/CVaR Level



- Regression-based method not sensitive to VaR/CVaR level
- Consistent performance across different levels

Sensitivity to Asset Model



- Regression-based method insensitive to asset model (GBM vs. Heston)
- Consistent performance across different asset models

Computational Complexity

There are **computational costs** associated with pooling inner replications.

- Standard procedure: cost of estimating the optimal M and N
- Regression: most efficient among metamodel-based procedures
- Kernel smoothing: costly distance calculations and cross-validation
- Likelihood ratio: No training, but costly weight calculations
- KRR: even more expensive than kernel smoothing

Total Computation Time

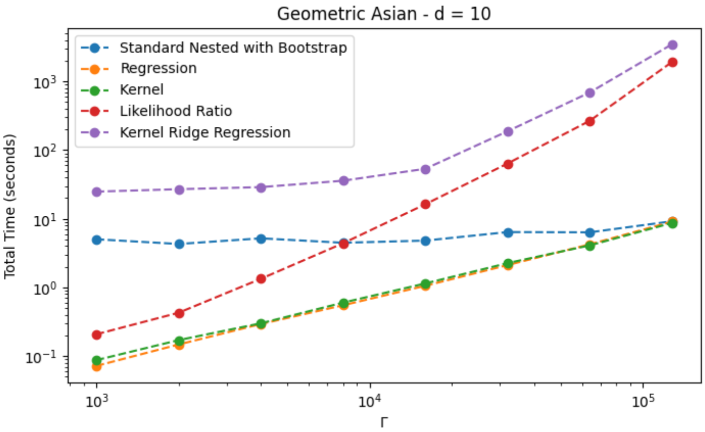


Figure: Total computation time for different procedures

Cost of Hyperparameter Tuning

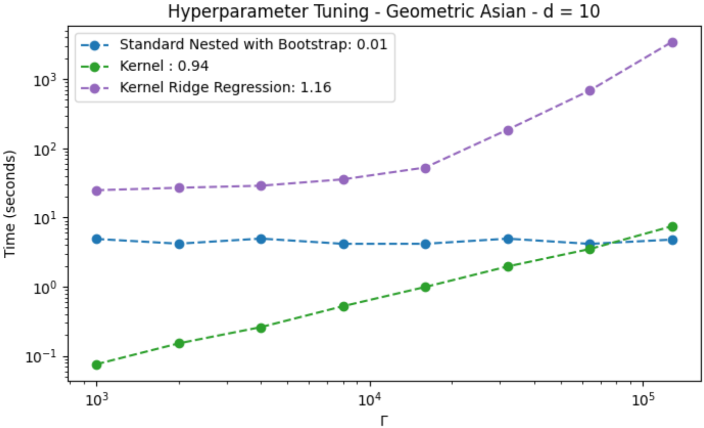


Figure: Cost of hyperparameter tuning for different procedures

Cost of Model Fitting and Validation

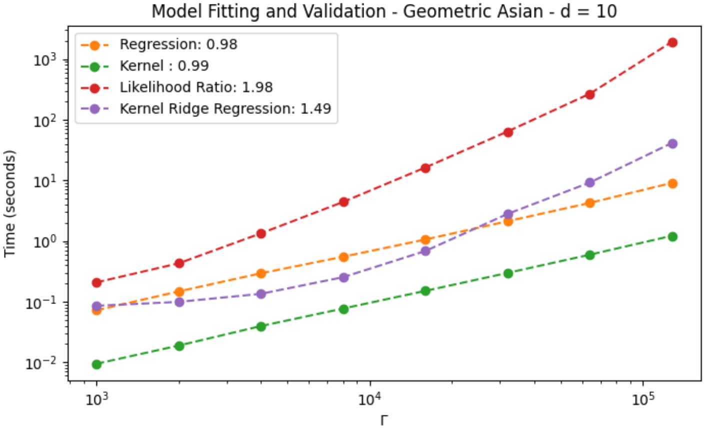


Figure: Cost of model fitting and validation for different procedures

Conclusion

Regression-based nested simulation procedure:

- Most robust and stable for limited budgets
- Efficient to implement
- Fast empirical convergence for option portfolios

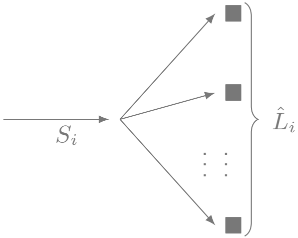
For high-dimensional or complex payoffs:

- Difficult to find a good regression basis
- Neural network-based procedures may be more suitable

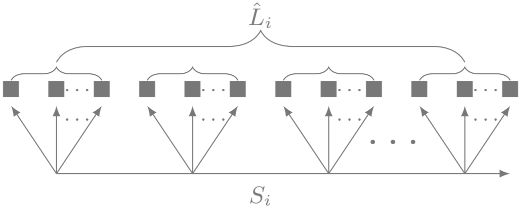
Next project: examining performance of metamodel-based simulation procedures for variable annuities

From Options to Variable Annuities

Variable annuities (VAs) poses a challenge for nested simulation due to its **high-dimensional** and **complex payoff structure**.



1 Outer Path for Options



1 Outer Path for VAs

- Need to reconstruct a metamodeling-based nested simulation procedure

Nested Simulation for Risk Management of VAs

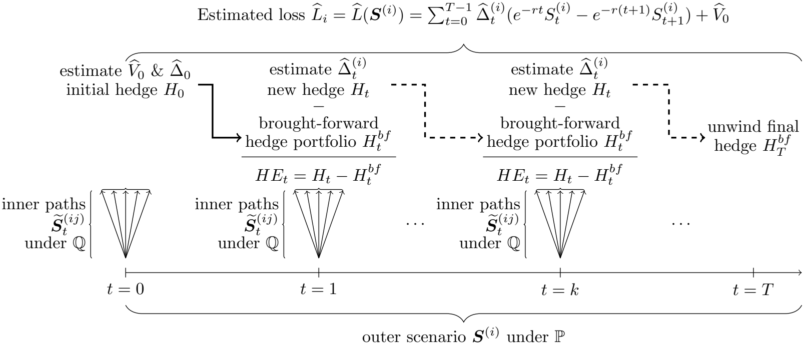


Figure: Illustration of nested simulation that estimates the P&L for one outer scenario

Standard Nested Simulation for VAs

Standard nested simulation for VAs is similar to the one for options.

- Generate M outer scenarios
- For each outer scenario,
  - Perform N inner simulations
  - Estimate hedging loss L_i with $\hat{L}_i$
- Use estimated losses to calculate tail risk measures (e.g., 95%-CVaR)

Main challenge:

- Computational budget is limited.
- Accuracy of the estimation depends on both M and N.

Two-Stage LSTM-based Nested Simulation for VAs

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