Pareto random variable

Consider a random variable $(\rho_\Gamma - \rho)$ that follows a Pareto distribution with parameters Γ^{-1} and α.

Its survival function is given by:

$$\frac{1}{(x\Gamma)^\alpha} \text{ for } x \geq \Gamma^{-1}$$

The variance of the random variable is given by:

$$\Gamma^{-2}\frac{\alpha^2}{(\alpha-1)^2(\alpha-2)}$$

Let's state the definition of convergence in MSE:

Definition 4 (convergence in MSE):

If there exists a constant C such that:

$$\limsup_{\Gamma\to\infty} \ \mathbb{E}[\frac{(\rho_\Gamma-\rho)^2}{\Gamma^{-2}}] \leq C$$

The MSE converges in the order of $\mathcal{O}(\Gamma^{-2})$.

Hence, it can be observed that above Pareto random variable converges in MSE in the order of $\mathcal{O}(\Gamma^{-2})$.

Let's state the definition of convergence in probability:

Definition 5 (Convergence in probability):

For any $\epsilon > 0$, there exists a constant C such that:

$$\mathbb{P}\left(|\rho_\Gamma - \rho| \geq C\Gamma^{-1}\right) \leq \epsilon$$

For this Pareto random variable, we are able to compute the probability using the survival function:

$$\mathbb{P}\left(|\rho_\Gamma - \rho| \geq C\Gamma^{-1}\right) = \frac{1}{(C\Gamma^{-1}\Gamma)^\alpha} = C^{-\alpha}$$

The right hand side of the above equation is a constant, so it can not be smaller than any $\epsilon > 0$.

Hence, **Definition 5** is not satisfied.

Therefore, the above Pareto random variable does not converge in probability in the order of $\mathcal{O}(\Gamma^{-1})$.

Since Professor Weng's comment is related to the convergence in probability, we can easily verify that the above Pareto random variable converges in probability in the order of $\mathcal{O}(\Gamma^{-1-\delta})$ for any $\delta > 0$.

Normal random variable

If we try to show the same result for a normal random variable, we can observe that the normal random variable satisfies **Definition 4** with $C = 1$. Therefore, we have:

$$\mathbb{E}[(\rho_\Gamma - \rho)^2] = \mathcal{O}(\Gamma^{-2})$$

We observe that:

$$\begin{aligned} \mathbb{P}(|\rho_\Gamma - \rho| \geq C\Gamma^{-1}) &= 2\phi\left(\frac{C\Gamma^{-1}}{\Gamma^{-1}}\right) &(1)\\ &= 2\phi\left(C\right) &(2) \end{aligned}$$

Hence, **Definition 5** is not satisfied for the normal random variable.