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Efficient Machine Learning Approaches for Fast Risk Evaluation of VAs

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Thesis Defense, University of Waterloo

- 1 Introduction
- 2 Nested Simulation Procedures in Financial Engineering: A Selected Review
 - Theoretical Results
 - Finite-Sample Analysis
- 3 Using Deep Neural Network Metamodels for High-Dimensional Nested Simulation
- 4 Transfer Learning for Dynamic Hedging

Nested Simulation Procedures

Nested simulation procedures are necessary for **complex** financial derivatives and insurance products.

$$\rho(L) = \rho(L(X)), \quad L(X) = \mathbb{E}[Y|X = x]_{x=X}$$

Involves two levels of Monte Carlo simulations:

- ❖ Outer level: generates underlying risk factors (outer scenarios), $X_i \sim F_X$
- ❖ Inner level: generates scenario-wise samples of portfolio losses (inner replications), $Y_{ij} \sim F_{Y|X_i}$

Computationally expensive due to its nested structure.

Common Risk Measures

- Smooth h , e.g., quadratic tracking error

$$\rho(L) = \mathbb{E}[(L - b)^2]$$

- hockey-stick h : mean excess loss

$$\rho(L) = \mathbb{E}[L \cdot \mathbb{1}_{\{L \geq u\}}]$$

- indicator h : probability of large loss

$$\rho(L) = \mathbb{E}[\mathbb{1}_{\{L \geq u\}}]$$

- Value at Risk (VaR)

$$\rho_\alpha(L) = Q_\alpha(L) = \inf\{u : \mathbb{P}(L \leq u) \geq \alpha\}$$

- Conditional Value at Risk (CVaR)¹

$$\rho_\alpha(L) = \mathbb{E}[L | L \geq Q_\alpha(L)]$$

¹Note: If $Q_\alpha(L)$ falls in a probability mass, $\rho(L) = \frac{(\beta - \alpha)Q_\alpha(L) + (1 - \beta)\mathbb{E}[L | L \geq Q_\alpha(L)]}{1 - \alpha}$.

Standard Nested Simulation

$$\hat{L}_{N,i} = \frac{1}{N} \sum_{j=1}^N Y_{ij}; \quad Y_{ij} \sim F_{Y|X_i}$$

- ❖ Uses inner sample mean to estimate $L(X_i)$.
- ❖ Proposed by Gordy and Juneja (2010); finds optimal growth order of M and N .
- ❖ Zhang et al. (2021) estimate the optimal M and N using a bootstrap method.
- ❖ Computationally expensive and potentially **wasteful** use of budget.

Other Nested Simulation Procedures

Subsequent works focus on improving the efficiency of nested simulation:

- ❖ Regression-based (Broadie et al., 2015)
- ❖ Kernel smoothing (Hong et al., 2017)
- ❖ Likelihood ratio (Feng and Song, 2020)
- ❖ Kernel ridge regression (Zhang et al., 2022)

Key ideas:

- ❖ Pool inner replications from different outer scenarios
- ❖ Use metamodeling techniques to approximate the inner simulation model

Metamodeling Approach

In this thesis, we focus on procedures that use **supervised learning metamodels** to approximate the inner simulation model.

- ❖ Treat the inner simulation as a black-box function
- ❖ Approximate $L(\cdot)$ with $\hat{L}_{M,N}^{\text{SL}}(\cdot)$
- ❖ Train with a set of feature-label pairs generated from the standard procedure:

$$\{(X_i, \hat{L}_{N,i}) | i = 1, \dots, M, j = 1, \dots, N\}$$

- ❖ Use trained metamodel to make predictions for all $X \in \mathcal{X}$

There are **computational costs** associated with pooling inner replications.

Problem Statement

Minimize mean squared error (MSE) of the estimator subject to total simulation budget:

$$\begin{aligned} \min_{M,N} \quad & \mathbb{E} [(\hat{\rho}_{M,N} - \rho)^2] \\ \text{subject to} \quad & M \cdot N = \Gamma \end{aligned}$$

Interested in convergence order as $\Gamma \rightarrow \infty$

Asymptotic Convergence Rates of Different Procedures

Procedures	Smooth h	Hockey-Stick h	Indicator h
Standard Procedure	$\mathcal{O}(\Gamma^{-2/3})$	$\mathcal{O}(\Gamma^{-2/3})$	$\mathcal{O}(\Gamma^{-2/3})$
Regression	$\mathcal{O}(\Gamma^{-1})$	$\mathcal{O}(\Gamma^{-1+\delta})$	No Result
Kernel Smoothing	$\mathcal{O}(\Gamma^{-\min(1, 4/(d+2))})$		
Kernel Ridge Regression	$\mathcal{O}(\Gamma^{-1})$		
Likelihood Ratio	$\mathcal{O}(\Gamma^{-1})$		

- ✦ We show the asymptotic convergence rates of the standard procedure for smooth and hockey-stick h .
- ✦ Only kernel smoothing depends on the asset dimension d .

Key Theoretical Results

Observations:

- ❖ Most literature focuses on the MSE of $\hat{\rho}$.
- ❖ Wang et al. (2022) analyze convergence of absolute error in probabilistic order.

Contribution: bridging the gap between MSE and absolute error convergence.

- ❖ Convergence in MSE:

$$\mathbb{E} [(\hat{\rho}_{\Gamma} - \rho)^2] = \mathcal{O}(\Gamma^{-\xi})$$

- ❖ Convergence in Probabilistic Order:

$$|\hat{\rho}_{\Gamma} - \rho| = \mathcal{O}_{\mathbb{P}}(\Gamma^{-\xi})$$

Key Theoretical Results

Theorem

If $\hat{\rho}_T$ converges in MSE to ρ in order ξ , then $\hat{\rho}_T$ converges in probabilistic order to ρ in order $\frac{\xi}{2}$.

- ❖ First result to draw connection between MSE and probabilistic order convergence.
- ❖ Applicable to any nested simulation procedure.
- ❖ Convergence in MSE implies convergence in probabilistic order.

Experiment Design

We compare 5 nested simulation procedures

- ❖ Standard nested simulation
- ❖ Regression-based
- ❖ Kernel smoothing
- ❖ Likelihood ratio
- ❖ Kernel ridge regression

And their empirical convergence stable across different:

- ❖ Risk measures
- ❖ Option types
- ❖ Asset dimensions
- ❖ Asset models (GBM vs. Heston)
- ❖ Regression bases (only for the regression-based procedure)

Finite-Sample Performance

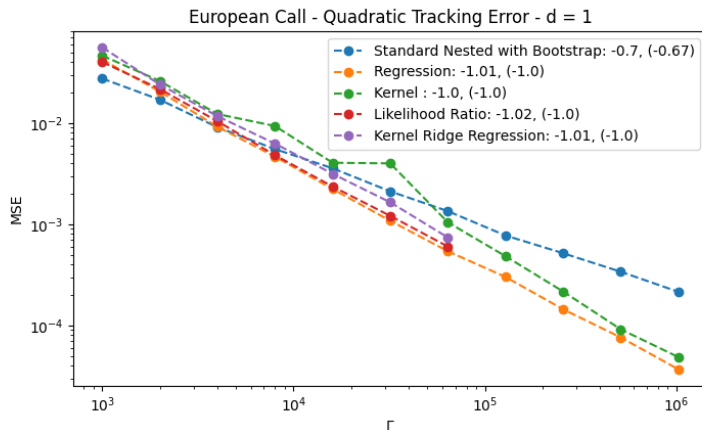
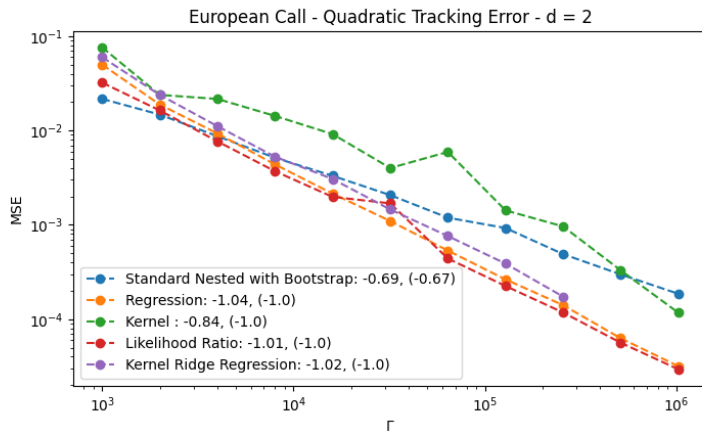


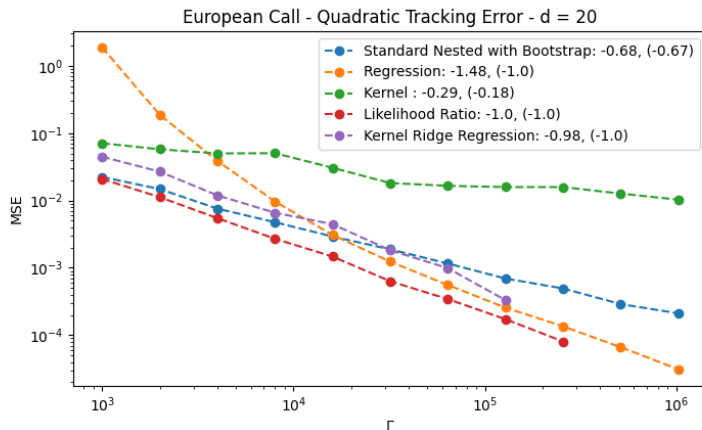
Figure: Empirical convergence rates of different procedures for the base case

Sensitivity to Asset Dimension



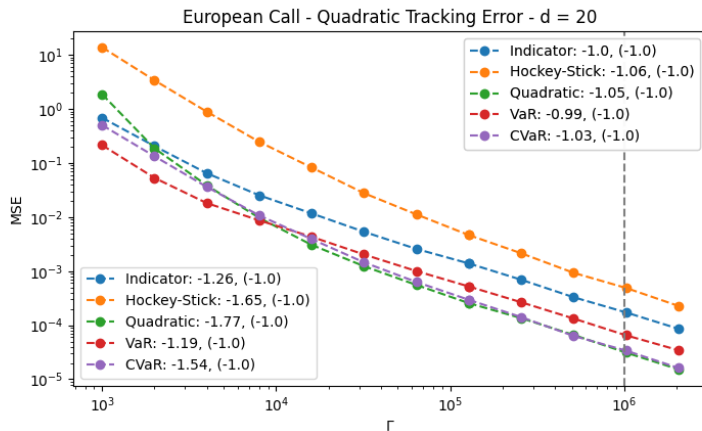
- Standard, KRR, and likelihood ratio procedures are dimension-independent
- Kernel smoothing and regression show sensitivity to dimension, but in different ways

Sensitivity to Asset Dimension



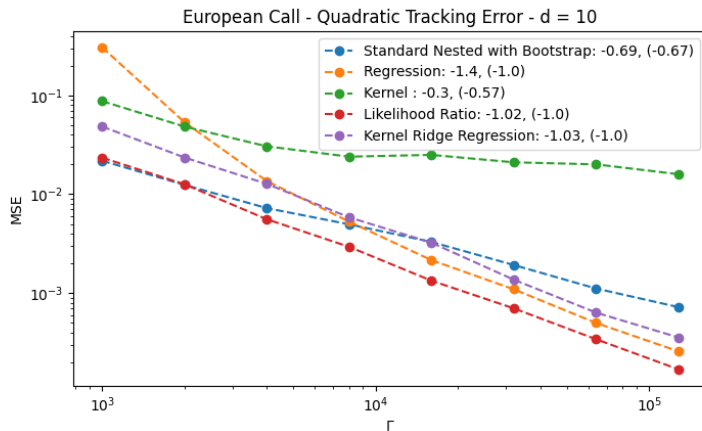
- Standard, KRR, and likelihood ratio procedures are dimension-independent
- Kernel smoothing and regression show sensitivity to dimension, but in different ways

Fast Convergence of Regression-based Procedure



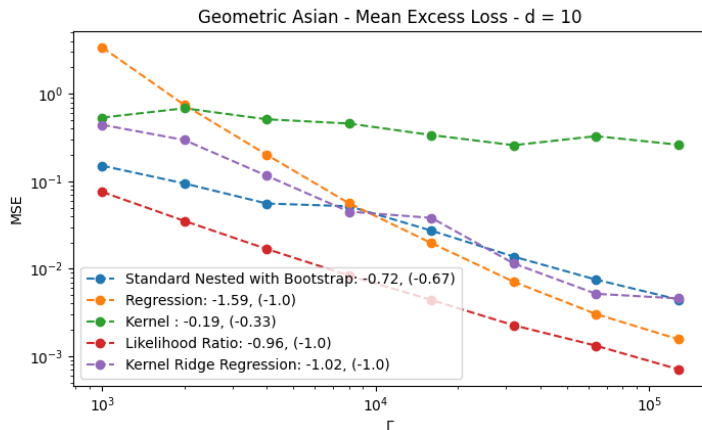
- Higher initial convergence rate
- Stabilizes to match asymptotic rate at higher budgets
- Consistent across different asset dimensions

Sensitivity to Option Type



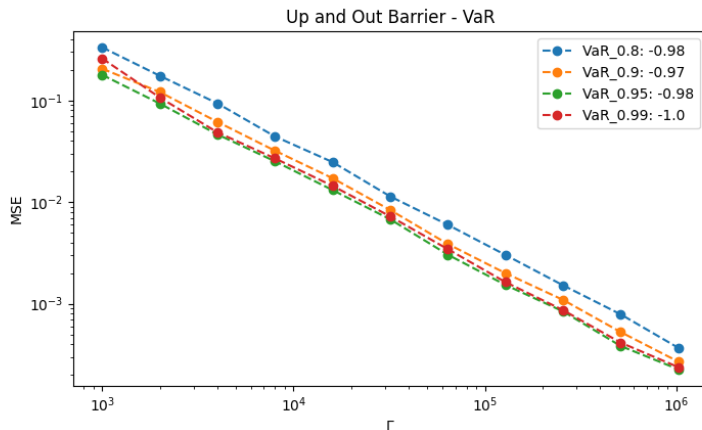
- Similar convergence patterns across different option types
- Regression and kernel smoothing show higher empirical rates for barrier options

Sensitivity to Risk Measure



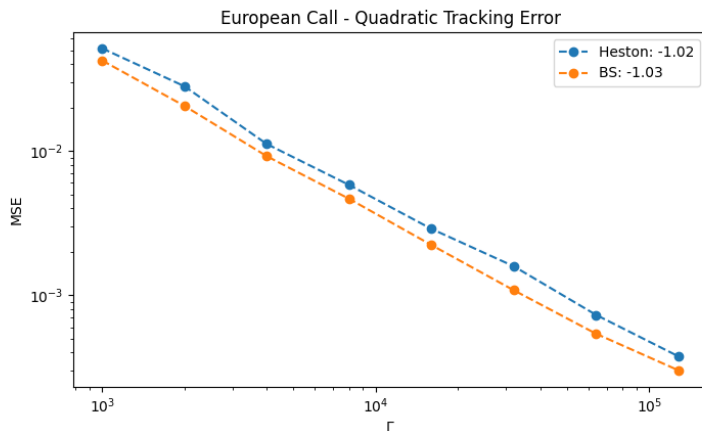
- Convergence behavior consistent across different risk measures
- Regression-based method shows highest empirical convergence rates

Sensitivity to VaR/CVaR Level



- Regression-based method not sensitive to VaR/CVaR level
- Consistent performance across different levels

Sensitivity to Asset Model



- ❖ Regression-based method insensitive to asset model (GBM vs. Heston)
- ❖ Consistent performance across different asset models

Computational Complexity

There are **computational costs** associated with pooling inner replications.

- ❖ Standard procedure: cost of estimating the optimal M and N
- ❖ Regression: most efficient among metamodel-based procedures
- ❖ Kernel smoothing: costly distance calculations and cross-validation
- ❖ Likelihood ratio: No training, but costly weight calculations
- ❖ KRR: even more expensive than kernel smoothing

Total Computation Time

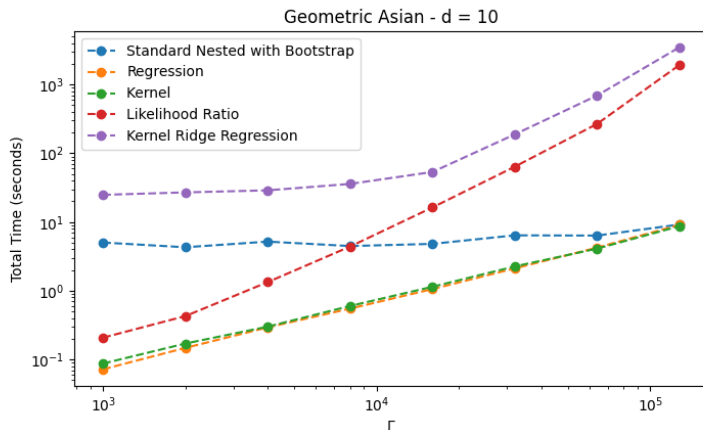


Figure: Total computation time for different procedures

Cost of Hyperparameter Tuning

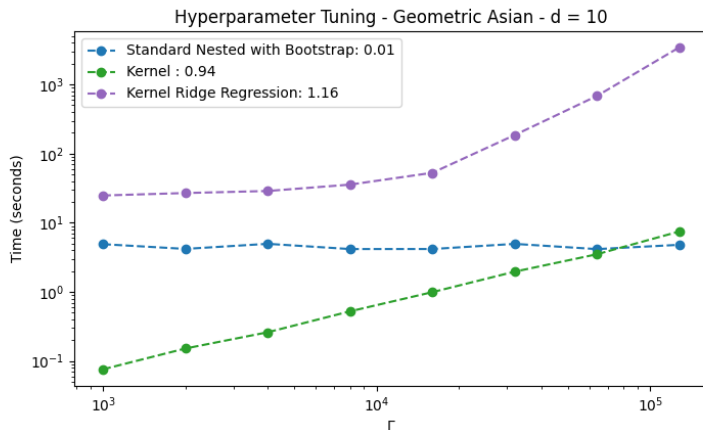


Figure: Cost of hyperparameter tuning for different procedures

Cost of Model Fitting and Validation

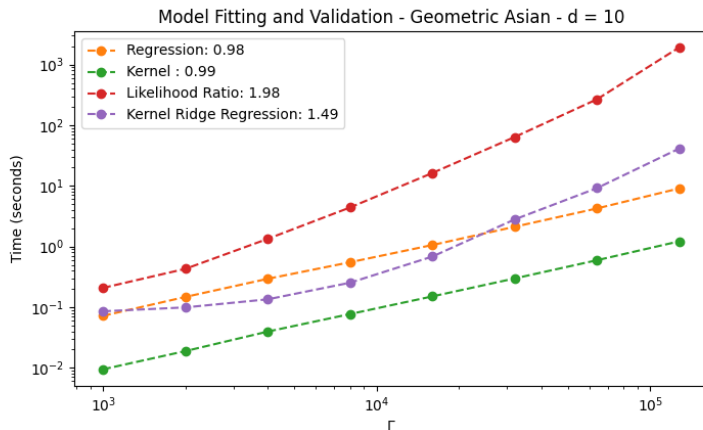


Figure: Cost of model fitting and validation for different procedures

Conclusion

Regression-based nested simulation procedure:

- ❖ Most robust and stable for limited budgets
- ❖ Efficient to implement
- ❖ Fast empirical convergence for option portfolios

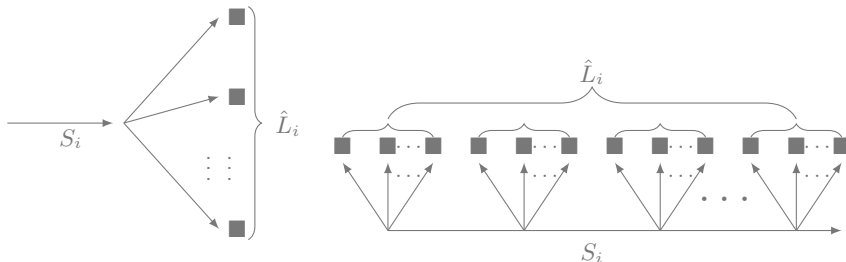
For high-dimensional or complex payoffs:

- ❖ Difficult to find a good regression basis
- ❖ Neural network-based procedures may be more suitable

Next project: examining performance of metamodel-based simulation procedures for variable annuities

From Options to Variable Annuities

Variable annuities (VAs) poses a challenge for nested simulation due to its **high-dimensional** and **complex payoff structure**.



1 Outer Path for Options

1 Outer Path for VAs

- ❖ Need to reconstruct a metamodeling-based nested simulation procedure

Nested Simulation for Risk Management of VAs

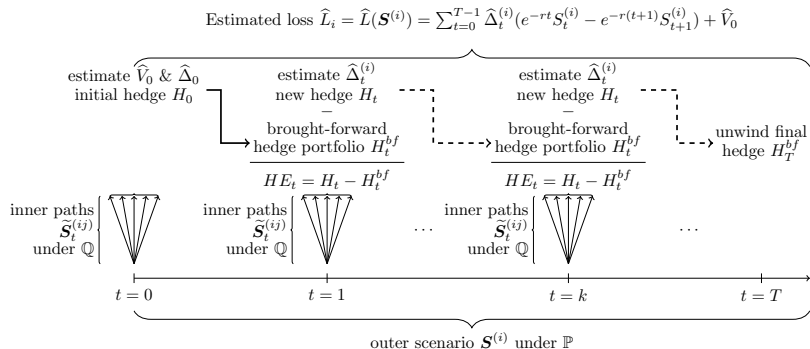


Figure: Illustration of nested simulation that estimates the P&L for one outer scenario

Standard Nested Simulation for VAs

Standard nested simulation for VAs is similar to the one for options.

- ❖ Generate M outer scenarios
- ❖ For each outer scenario,
 - ❖ Perform N inner simulations
 - ❖ Estimate hedging loss L_i with \hat{L}_i
- ❖ Use estimated losses to calculate tail risk measures (e.g., 95%-CVaR)

Observations:

- ❖ computational budget is limited;
- ❖ high-dimensional input space;
- ❖ only a **small** portion of scenarios are relevant when estimating **tail** risk measures.

Metamodel-based Nested Simulation

We use deep neural networks (DNNs) as metamodels

- ❖ Use LSTMs for sequential data
- ❖ **Challenge:** lack of transparency and interpretability

Research Contributions:

1. Propose two generic DNN-based nested simulation procedures
 - ❖ Accurate tail scenario identification
 - ❖ Significant computational savings by **budget concentration**
2. Study noise tolerance of DNNs using simulated data
 - ❖ **Control noise levels** by adjusting simulation parameters
 - ❖ Provide direct evidence on transparency and interpretability

Two-Stage Metamodel-based Nested Simulation

Algorithm Two-Stage Metamodel-based Nested Simulation for VAs

1: **Generate training data for metamodels:**

- ❖ Use a fraction of the simulation budget to run the standard nested simulation procedure with M outer scenarios and N' inner replications.
- ❖ Construct feature-label pairs $\{(X_i, Y_{ij}) : i = 1, \dots, M, j = 1, \dots, N'\}$

2: **Train metamodels:**

- ❖ Use the feature-label pairs to train a metamodel.
- ❖ Use the trained metamodel to make predictions for $\{X_i : i = 1, \dots, M\}$.
- ❖ Sort the predicted losses to identify a predicted tail scenario set that contains the m largest predicted losses.

3: **Concentrate simulation on predicted tail scenarios:**

- ❖ Run the standard procedure on the predicted tail scenarios.
 - ❖ Estimate the α -CVaR of L using the estimated losses on the predicted tail scenarios.
-

Benefits of a Two-Stage Procedure

Simulation budget can be saved when:

- ❖ the metamodel is accurate (a small m includes most tail scenarios)
- ❖ the metamodel can tolerate noise in training labels (a small N')

Key findings:

- ❖ Substantial computational savings (70% – 85% reduction)
- ❖ Maintains accuracy comparable to standard procedure
- ❖ DNN metamodels can distinguish between tail and non-tail scenarios effectively
- ❖ Addresses regulatory concerns by using actual simulations for final estimates

Another finding: some DNN metamodels make **accurate loss predictions** for given scenarios.

Single-Stage Metamodel-based Nested Simulation

Algorithm Single-Stage Metamodel-based Nested Simulation for VAs

1: **Generate training data for metamodels:**

- ❖ Use the entire simulation budget to run the standard nested simulation procedure with M outer scenarios and N inner replications.
- ❖ Construct feature-label pairs $\{(X_i, Y_{ij}) : i = 1, \dots, M, j = 1, \dots, N\}$

2: **Train metamodels:**

- ❖ Use the feature-label pairs to train a metamodel.
- ❖ Use the trained metamodel to make predictions for $\{X_i : i = 1, \dots, M\}$.

3: **Use metamodel predictions to estimate tail risk measures directly.**

Key advantages:

- ❖ more efficient than a two-stage procedure;
- ❖ avoids specifying m ;

Experiment Setting

We estimate the 95%-CVaR of the hedging loss for a GMWB contract with 20-year maturity.

Specifications:

- ❖ The underlying asset follows a regime-switching geometric Brownian motion;
- ❖ The contract is delta-hedged monthly (240 periods);
- ❖ The true 95%-CVaR is estimated using 100,000 outer scenarios and 100,000 inner replications.
- ❖ The metamodel is trained using 90,000 outer scenarios and 100 inner replications.
- ❖ Benchmark: standard nested simulation procedure with 100,000 outer scenarios and 1,000 inner replications.

Experiment Design

Research Questions:

- ❖ What do DNNs learn from noisy data?
- ❖ How well do DNNs learn from noisy data?



- ❖ Our 90,000 training data is noisy, and the test data is also **noisy**.
- ❖ Our evaluation is based on the true (**noiseless**) feature-label relationship².

²Made possible by novel simulation design.

Experiment Setting

We consider the following metamodel architectures:

Metamodel	Abbreviation	Capacity
Multiple Linear Regression	MLR	241
Quadratic Polynomial Regression	QPR	481
Feedforward Neural Network	FNN	35,009
Recurrent Neural Network	RNN	32,021
Long Short-Term Memory	LSTM	35,729

Table: Metamodel architectures for GMWB inner simulation model

Capacity is defined as the number of parameters in the metamodel.

- ✦ Higher capacity metamodels are more flexible and expressive.
- ✦ Lower capacity metamodels are less likely to overfit.

Traditional Regression Metamodels

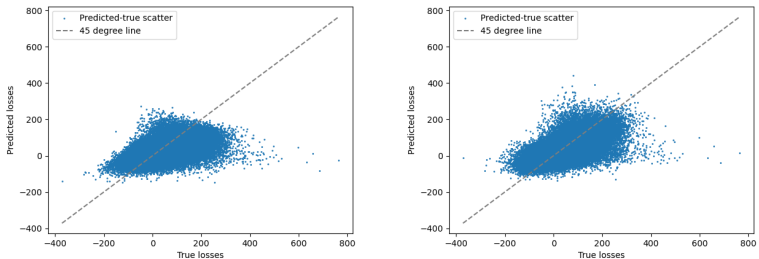


Figure: QQ plots between true and predicted loss labels for MLR and QPR metamodels

- ✚ MLR and QPR metamodels make **inaccurate** loss predictions.
- ✚ Feature engineering is hardly feasible for our 240-dimensional X .

Deep Neural Network Metamodels

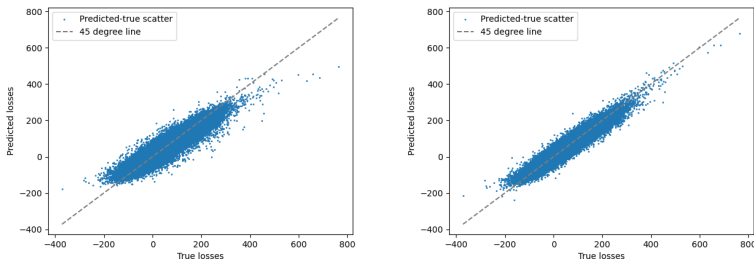


Figure: QQ plots between true and predicted loss labels for FNN and LSTM metamodels

- ❖ DNN metamodels are more flexible.
- ❖ Time series features prefer a LSTM metamodel over FNN.
- ❖ Network architecture serves as prior knowledge that regularizes DNNs.

Metamodel Performance on Different Datasets

Metamodel	Training error	Test error	True error
MLR	$0.706(\pm 8.34 \times 10^{-4})$	$0.713(\pm 2.67 \times 10^{-2})$	$0.706(\pm 3.44 \times 10^{-4})$
QPR	$0.543(\pm 8.27 \times 10^{-4})$	$0.554(\pm 2.32 \times 10^{-2})$	$0.544(\pm 4.12 \times 10^{-4})$
FNN	$0.129(\pm 5.95 \times 10^{-3})$	$0.240(\pm 9.82 \times 10^{-3})$	$0.132(\pm 5.82 \times 10^{-3})$
RNN	$0.132(\pm 7.53 \times 10^{-3})$	$0.137(\pm 7.62 \times 10^{-3})$	$0.119(\pm 7.51 \times 10^{-3})$
LSTM	$0.075(\pm 4.48 \times 10^{-3})$	$0.079(\pm 5.35 \times 10^{-3})$	$0.063(\pm 4.43 \times 10^{-3})$
RNN ^{*3}	$0.109(\pm 5.20 \times 10^{-3})$	$0.128(\pm 5.22 \times 10^{-3})$	$0.109(\pm 5.20 \times 10^{-3})$

Table: MSEs of metamodels for GMWB inner simulation model.

❖ DNN metamodels with suitable architectures **cut through the noise** in training labels.



³This row summarizes the results of the well-trained RNNs.

Issues with RNNs

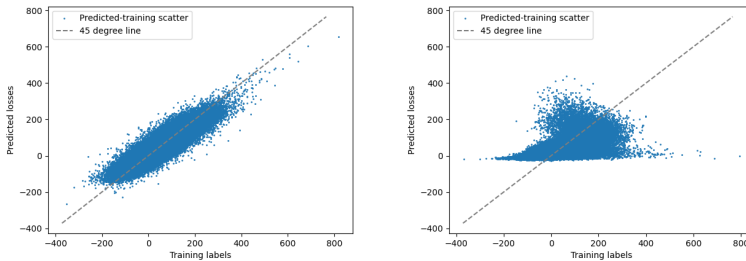
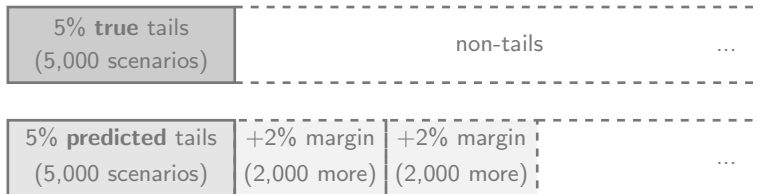


Figure: QQ plots between training and predicted loss labels for RNN metamodels

- ❖ RNN metamodels suffers from **vanishing gradient problem**.
- ❖ Ease of training (reliability) is a critical factor when choosing a DNN metamodel.

Safety Margin

Consider estimating the 95% CVaR with 100,000 outer scenarios.



Choosing a safety margin: a trade-off between accuracy and efficiency.

- ❖ A lower margin: not enough tail identified.
- ❖ A higher margin: more accurate CVaR estimate, but more budget needed to perform extensive inner simulations.

Tail Scenario Identification

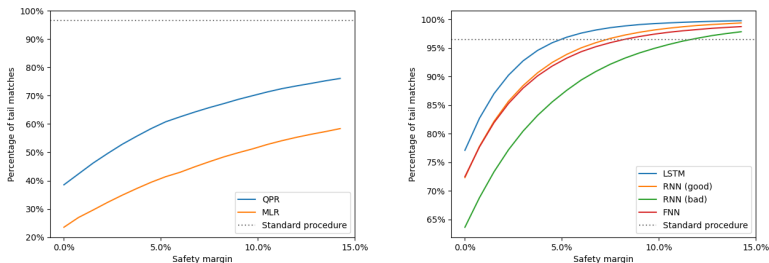


Figure: Tail scenario identification for regression and DNN metamodels

- ❖ Traditional regression metamodels are **unable** to accurately identify tail scenarios even with high safety margins.
- ❖ LSTM metamodels surpasses the standard procedure with 5% safety margin.

Estimating CVaR

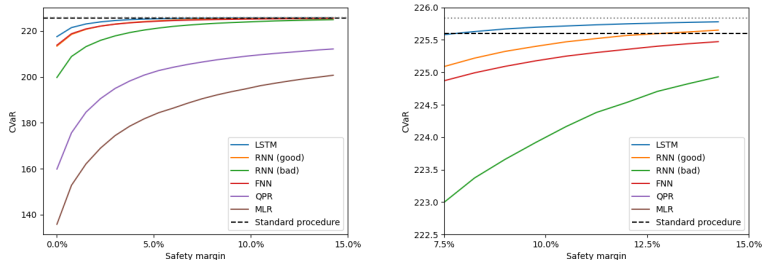


Figure: CVaR estimation for DNN metamodels

- ❖ Traditional regression metamodels are **unable** to accurately estimate CVaR even with high safety margins.
- ❖ LSTM surpasses the standard procedure with a **5% safety margin**.
- ❖ With a 95% safety margin, any two-stage procedure produce the same CVaR estimate as a standard procedure.

Sensitivity Testing for DNNs

In a simulation study, we have control over the **noise level in training labels** and the **number of training samples**.

Controlling the inner replications N' varies the noise level in training labels.

- ❖ **Low noise labels:** $N' = 100$
- ❖ **Medium noise labels:** $N' = 10$
- ❖ **High noise labels:** $N' = 1$

Controlling the outer scenarios M varies the number of training samples.

- ❖ $M \in \{10^2, 10^3, 10^4, 10^5\}$

2 LSTMs of **different capacities** are examined based on their MSEs.

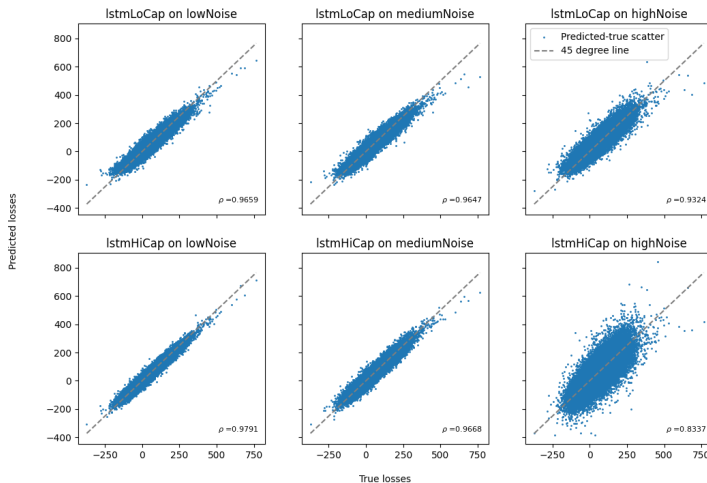
Noise Tolerance of DNNs

Model	N'	Training error	Test error	True error
LSTM	100	0.075	0.079	0.063
High-capacity LSTM	100	0.068	0.102	0.060
Average Difference	100	-0.007	0.023	-0.003
LSTM	10	0.195	0.193	0.070
High-capacity LSTM	10	0.157	0.199	0.065
Average Difference	10	-0.038	0.006	-0.005
LSTM	1	1.366	0.781	0.129
High-capacity LSTM	1	1.354	0.795	0.149
Average Difference	1	-0.012	0.014	0.020

Table: MSEs of LSTM metamodels.

- Both LSTMs cut through the noise in training labels.
- Both LSTMs deteriorate dramatically on **high-noise** labels.
- High-capacity LSTM can tolerate **low** and **medium** label noise.

Noise Tolerance of DNNs



Sensitivity of Regular LSTM

	$N' = 1$	$N' = 10$	$N' = 100$	$N' = 1000$
$M = 100$	1.139	0.229	0.167	0.158
$M = 1000$	0.559	0.173	0.123	0.127
$M = 10000$	0.283	0.115	0.099	0.097
$M = 100000$	0.129	0.070	0.063	0.063

Table: MSE between regular LSTM's predicted losses and true losses.

- ❖ Same color \rightarrow same total simulation budget.
- ❖ $N = 10$ is a reasonable budget allocation for LSTM metamodels.

Sensitivity of High-capacity LSTM

	$N' = 1$	$N' = 10$	$N' = 100$	$N' = 1000$
$M = 100$	0.764	0.408	0.131	0.087
$M = 1000$	0.878	0.367	0.156	0.087
$M = 10000$	0.351	0.147	0.064	0.063
$M = 100000$	0.149	0.065	0.060	0.038

Table: MSE between high-capacity LSTM's predicted losses and true losses.

- ❖ Same color \rightarrow same total simulation budget.
- ❖ $N' = 10$ is a reasonable budget allocation for LSTM metamodels.

Single-Stage Procedure

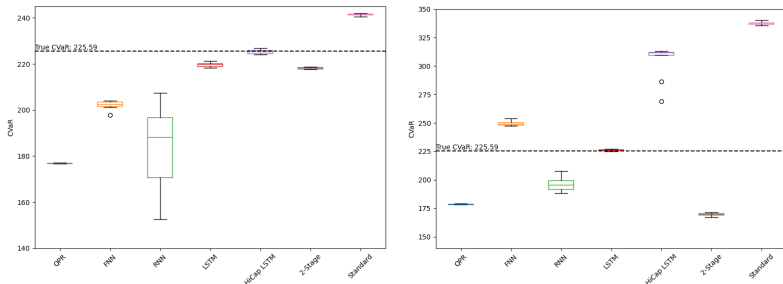


Figure: CVaR estimates of single-stage procedures (left: $N' = 10$. right: $N' = 1$).

- ❖ The single-stage procedure outperforms the two-stage procedure.
- ❖ The single-stage procedure is more efficient than the two-stage procedure.
- ❖ Setting $N' = 10$ is a reasonable budget allocation.

Convergence Analysis

For each Γ , the best performing metamodel is used.

- Maximum number of outer scenarios $M = 10^5$.

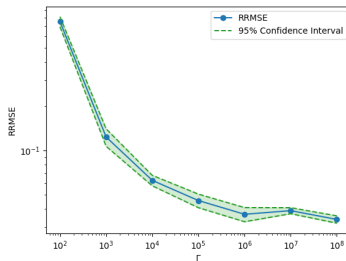
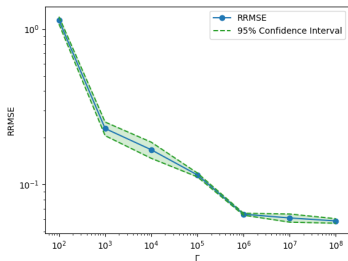


Figure: Empirical convergence of CVaR for single-stage procedures with LSTM metamodels (left: regular LSTM. right: high-capacity LSTM).

- Minimal effect of increasing N' on CVaR estimation.
- Similar behavior as regression metamodels ($d = 20$) in the previous section.

Convergence Analysis

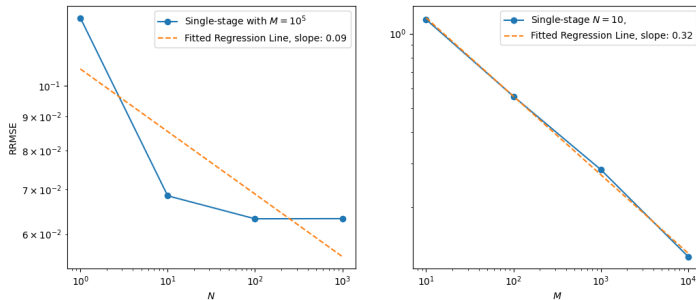


Figure: Empirical convergence of the single-stage procedure with a LSTM metamodel.

- Minimal effect of increasing N' on CVaR estimation.
- For a given Γ , set N' constant and allocate budget to outer simulations.

Conclusion

Key Findings:

- ❖ LSTMs are **resilient** to moderate levels of noise in training labels.
- ❖ Deep neural networks can learn **true** complex dynamic hedging model.
- ❖ Two-stage procedure addresses regulatory concerns by avoiding direct use of metamodel predictions.
- ❖ Single-stage procedure is **efficient** and **versatile**.
- ❖ **Increasing outer scenarios** is more beneficial.
- ❖ High-capacity LSTM requires lower noise training labels.

Future Directions:

- ❖ Apply deep neural network metamodels to other risk management tasks.
- ❖ Investigate impact of label noise on other deep learning models
- ❖ Explore optimal network architectures for different simulation models.

Transfer Learning for Dynamic Hedging

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