# Supporting information for the connection between convergence in MSE and AE

This document provides the supporting information for the comment made on Page 29-30 of the manuscript.
It uses two examples to show that the convergence in MSE in order of $\mathcal{O}(\Gamma^{-k})$ does not imply the convergence in probability in order of $\mathcal{O}(\Gamma^{-k/2})$.
For the case of a Pareto random variable and a normal random variable, we are not able to prove the convergence in probability based on the convergence in MSE.
Further assumptions are needed to bridge the gap.

## Pareto random variable

Consider a random variable $(\rho_\Gamma - \rho)$ that follows a Pareto distribution with parameters $\Gamma^{-1}$ and $\alpha$.

Its survival function is given by:

$$\frac{1}{(x\Gamma)^\alpha} \text{ for } x \geq \Gamma^{-1}$$

The variance of the random variable is given by:

$$\Gamma^{-2}\frac{\alpha^2}{(\alpha-1)^2(\alpha-2)}$$

Let's state the definition of convergence in MSE:

**Definition 4 (convergence in MSE)**:

If there exists a constant $C$ such that:

$$\limsup_{\Gamma\to\infty} \ \mathbb{E}[\frac{(\rho_\Gamma-\rho)^2}{\Gamma^{-2}}] \leq C$$

The MSE converges in the order of $\mathcal{O}(\Gamma^{-2})$.

Hence, it can be observed that above Pareto random variable converges in MSE in the order of $\mathcal{O}(\Gamma^{-2})$.

Let's state the definition of convergence in probability:

**Definition 5 (Convergence in probability)**:

For any $\epsilon > 0$, there exists a constant $C$ such that:

$$\mathbb{P}\left(|\rho_\Gamma - \rho| \geq C\Gamma^{-1}\right) \leq \epsilon$$

For this Pareto random variable, we are able to compute the probability using the survival function:

$$\mathbb{P}\left(|\rho_\Gamma - \rho| \geq C\Gamma^{-1}\right) = \frac{1}{(C\Gamma^{-1}\Gamma)^\alpha} = C^{-\alpha}$$

The right hand side of the above equation is a constant, so it can not be smaller than any $\epsilon > 0$.

Hence, **Definition 5** is not satisfied.

Therefore, the above Pareto random variable does not converge in probability in the order of $\mathcal{O}(\Gamma^{-1})$.

Since Professor Weng's comment is related to the convergence in probability, we can easily verify that the above Pareto random variable converges in probability in the order of $\mathcal{O}(\Gamma^{-1-\delta})$ for any $\delta > 0$.

## Normal random variable

If we try to show the same result for a normal random variable, we can observe that the normal random variable satisfies **Definition 4** with $C = 1$.
Therefore, we have:

$$\mathbb{E}[(\rho_\Gamma - \rho)^2] = \mathcal{O}(\Gamma^{-2})$$

We observe that:

$$\mathbb{P}(|\rho_\Gamma - \rho| \geq C\Gamma^{-1}) = 2\phi\left(\frac{C\Gamma^{-1}}{\Gamma^{-1}}\right) \tag{1}$$

$$= 2\phi\left(C\right) \tag{2}$$

Hence, **Definition 5** is not satisfied for the normal random variable.