Xintong Li
xintong.li1@uwaterloo.ca

Dept. Statistics and Actuarial Science
University of Waterloo

UNIVERSITY OF WATERLOO

Resilient Machine Learning Approaches for Fast Risk Evaluation and Management of Financial Portfolios and Variable Annuities

Supervised by Prof. Tony Wirjanto and Prof. Mingbin Feng

Thesis Defense,
University of Waterloo

Outline

Nested Simulation Procedures

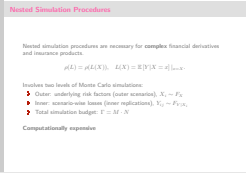
Nested simulation procedures are necessary for **complex** financial derivatives and insurance products.

$$\rho(L) = \rho(L(X)), \quad L(X) = \mathbb{E}\left[Y|X = x\right]|_{x=X}.$$

Involves two levels of Monte Carlo simulations:

- Outer: underlying risk factors (outer scenarios), $X_i \sim F_X$
- Inner: scenario-wise losses (inner replications), $Y_{ij} \sim F_{Y|X_i}$
- Total simulation budget: $\Gamma = M \cdot N$

Computationally expensive



In the context of quantitative risk management, nested simulation is used for risk evaluation of financial derivatives and insurance products.
The nested structure is due to the fact that the loss random variable is in the form of a conditional expectation of the portfolio loss given the underlying risk factors.
In our problems, the loss random variable L depends on the risk factors X and is the conditional expectation of the portfolio loss given X.
More specifically, X is d-dimensional and L is 1-dimensional.
Estimating a risk measure, $\rho(L)$, involves two levels of Monte Carlo simulations: outer level generates underlying risk factors, and inner level generates scenario-wise samples of portfolio losses.
The nested structure makes the estimation computationally expensive.
In this thesis, we focus on examining and improving the efficiency of nested simulation procedures mostly using supervised learning metamodels.

Common Risk Measures

- Smooth h, e.g., quadratic tracking error,

$$\rho(L) = \mathbb{E}\left[(L-b)^2\right];$$

- hockey-stick h: mean excess loss,

$$\rho(L) = \mathbb{E}\left[L \cdot \mathbb{1}_{\{L \geq u\}}\right];$$

- indicator h: probability of large loss,

$$\rho(L) = \mathbb{E}\left[\mathbb{1}_{\{L \geq u\}}\right];$$

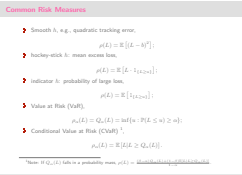
- Value at Risk (VaR),

$$\rho_\alpha(L) = Q_\alpha(L) = \inf\{u : \mathbb{P}(L \leq u) \geq \alpha\};$$

- Conditional Value at Risk (CVaR) [1],

$$\rho_\alpha(L) = \mathbb{E}\left[L | L \geq Q_\alpha(L)\right].$$

[1] Note: If $Q_\alpha(L)$ falls in a probability mass, $\rho(L) = \frac{(\beta-\alpha)Q_\alpha(L)+(1-\beta)\mathbb{E}[L|L\geq Q_\alpha(L)]}{1-\alpha}$.



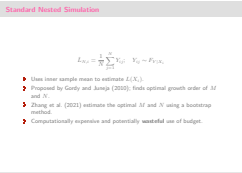
We consider different types of risk measures in this thesis.
The first three are in expectation form, and the last two are quantile-based risk measures.
The difference between the first three is the choice of the function h.
h can be a smooth function, a hockey-stick function, or an indicator function.
The examples of risk measures are quadratic tracking error, mean excess loss, and probability of large loss.
The last two are value at risk and conditional value at risk, which are widely used in practice.

Note that the definition of CVaR requires the knowledge of the Value at Risk.

Standard Nested Simulation

$$\hat{L}_{N,i} = \frac{1}{N}\sum_{j=1}^{N} Y_{ij}; \quad Y_{ij} \sim F_{Y|X_i}$$

- Uses inner sample mean to estimate $L(X_i)$.
- Proposed by Gordy and Juneja (2010); finds optimal growth order of M and N.
- Zhang et al. (2021) estimate the optimal M and N using a bootstrap method.
- Computationally expensive and potentially **wasteful** use of budget.



In our first project, we focus on comparing existing nested simulation procedures for different types of risk measures.
They differ in the way they estimate $L(X_i)$, the portfolio loss conditional on the risk factors X_i.
The standard nested simulation procedure uses the inner sample mean under that scenarioto estimate $L(X_i)$.
The optimal growth order of M and N is found by Gordy and Juneja (2010).
For a growing total simulation budget Γ, M should grow in the order of $\Gamma^{2/3}$, and N should grow in the order of $\Gamma^{1/3}$.
For a finite-sample case, the bootstrap method is used to estimate the optimal M and N by Zhang et al. (2021).
Since the standard procedure only uses the information from the same scenario, it is potentially **wasteful** use of budget.
Subsequent works focus on improving the efficiency of nested simulation by pooling inner replications from different outer scenarios.

Metamodeling Approach

We focus on procedures that **pool** with **supervised learning metamodels**.

- Treat inner simulation as a black-box function
- Approximate $L(\cdot)$ with $\hat{L}^{\mathsf{SL}}_{M,N}(\cdot)$
- Train with feature-label pairs generated by simulation:

$$\{(X_i, \hat{L}_{N,i}) | i = 1, \ldots, M, j = 1, \ldots, N\}$$

- Use metamodel predictions to estimate risk measures

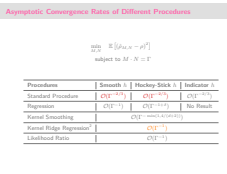
There are **computational costs** associated with pooling inner replications.

A supervised learning based nested simulation procedure is implemented following these steps.
First, the user runs the standard nested simulation procedure to generate a set of outer scenarios and inner replications.
Then, the outer scenarios and inner sample means can be treated as features and labels.
Their relationship is then approximated by a supervised learning metamodel.
Finally, the trained metamodel is used to make predictions for all $X \in \mathcal{X}$.
In other words, instead of using the inner sample mean to estimate $L(X_i)$, we use the trained metamodel to make predictions of the portfolio loss.
These loss predictions are then used to calculate the risk measures of interest.
For most research in this area, the focus is on the convergence rate of the risk measure estimator in terms of the simulation budget.
However, in this thesis, we note that there are other important aspects to consider, such as the cost of pooling that may come from metamodel training and validation.

Asymptotic Convergence Rates of Different Procedures

$$\min_{M,N} \quad \mathbb{E}\left[(\hat{\rho}_{M,N} - \rho)^2\right]$$
$$\text{subject to } M \cdot N = \Gamma$$

Procedures	Smooth h	Hockey-Stick h	Indicator h
Standard Procedure	$\mathcal{O}(\Gamma^{-2/3})$	$\mathcal{O}(\Gamma^{-2/3})$	$\mathcal{O}(\Gamma^{-2/3})$
Regression	$\mathcal{O}(\Gamma^{-1})$	$\mathcal{O}(\Gamma^{-1+\delta})$	No Result
Kernel Smoothing	$\mathcal{O}(\Gamma^{-\min(1,4/(d+2))})$		
Kernel Ridge Regression[2]	$\mathcal{O}(\Gamma^{-1})$		
Likelihood Ratio	$\mathcal{O}(\Gamma^{-1})$		



We start by comparing the asymptotic convergence rates of different nested simulation procedures.
These are the convergence rates of the MSE of the risk measure estimator when the total simulation budget Γ goes to infinity, and when one follows the optimal growth order of M and N.
The standard procedure has the same convergence rate of $\mathcal{O}(\Gamma^{-2/3})$ for all types of h considered. The rates in red are not available in the original works but are derived in this thesis.
The regression-based procedure has $\mathcal{O}(\Gamma^{-1})$ convergence rate for smooth and hockey-stick h.
It is achieved by only allowing M to grow in the order of Γ^1. In their experiments, N is set to equal a value of 1.
The kernel smoothing procedure is the only procedure that depends on the asset dimension d.
When risk factor X is high-dimensional, the kernel smoothing procedure may converge slower than the standard procedure.
The kernel ridge regression procedure and likelihood ratio procedure have the same convergence rate as the regression-based procedure.
However, as we will see in the numerical experiments, their additional cost of pooling inner replications may offset their benefit of fast convergence rate.
Additionally for KRR, the authors analyze convergence of absolute error in probabilistic order instead of MSE.

Key Theoretical Results

Contribution: bridging the gap between MSE and absolute error convergence.

- Convergence in MSE:

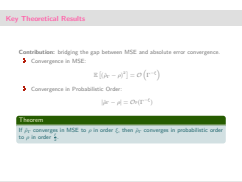
$$\mathbb{E}\left[(\hat{\rho}_\Gamma - \rho)^2\right] = \mathcal{O}\left(\Gamma^{-\xi}\right)$$

- Convergence in Probabilistic Order:

$$|\hat{\rho}_\Gamma - \rho| = \mathcal{O}_{\mathbb{P}}(\Gamma^{-\xi})$$

Theorem

If $\hat{\rho}_\Gamma$ converges in MSE to ρ in order ξ, then $\hat{\rho}_\Gamma$ converges in probabilistic order to ρ in order $\frac{\xi}{2}$.



In most theoretical works, the focus is on the convergence rate of the MSE of the risk measure estimator.
However, for KRR, the authors analyze convergence of absolute error in probabilistic order.
One of the contributions of this thesis is to bridge the gap between MSE and absolute error convergence.
Let's first define the convergence in MSE and the convergence in probabilistic order.
The convergence in MSE is the convergence of the MSE of the risk measure estimator to the true risk measure, that is, $\hat{\rho}_\Gamma$ minus ρ squared.
The convergence in probabilistic order is the convergence of the absolute error of the risk measure estimator to the true risk measure, that is, $|\hat{\rho}_\Gamma - \rho|$ to 0 in probability.
We show that the convergence rate of the MSE of the risk measure estimator implies the convergence rate of the absolute error in probabilistic order.

We will also show that the converse is not necessarily true.

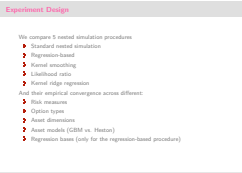
Experiment Design

We compare 5 nested simulation procedures

- Standard nested simulation
- Regression-based
- Kernel smoothing
- Likelihood ratio
- Kernel ridge regression

And their empirical convergence across different:

- Risk measures
- Option types
- Asset dimensions
- Asset models (GBM vs. Heston)
- Regression bases (only for the regression-based procedure)



Previously, we worked on the asymptotic convergence rates of different nested simulation procedures.
In practice, the total simulation budget is finite.When the budget is finite, the theoretical convergence rate may not be the only thing we should consider.
We should also consider the cost of pooling itself.
The additional cost of pooling inner replications is not considered in the theoretical analysis of the previous works.
Neither it is mentioned in their numerical experiments.
More specifically, we have found that different research uses different numerical experiments to show the empirical rate of different procedures.
We aim to provide a fair and comprehensive analysis of the finite-sample performance of different nested simulation procedures.
The following numerical experiments test the empirical convergence rates of different nested simulation procedures across different risk measures, option types, asset dimensions, and asset models.

Finite-Sample Performance

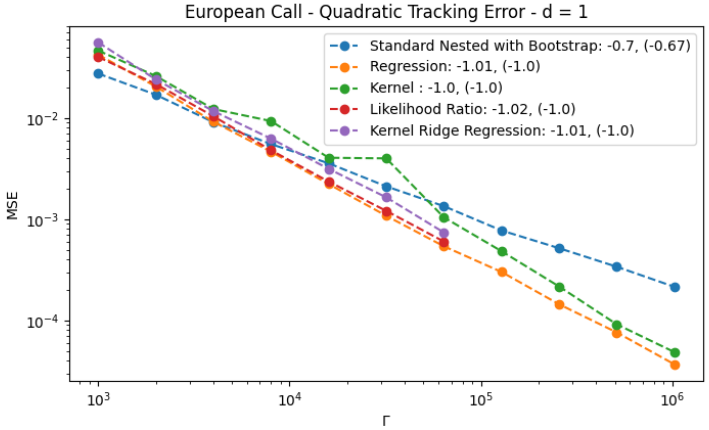
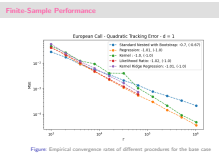
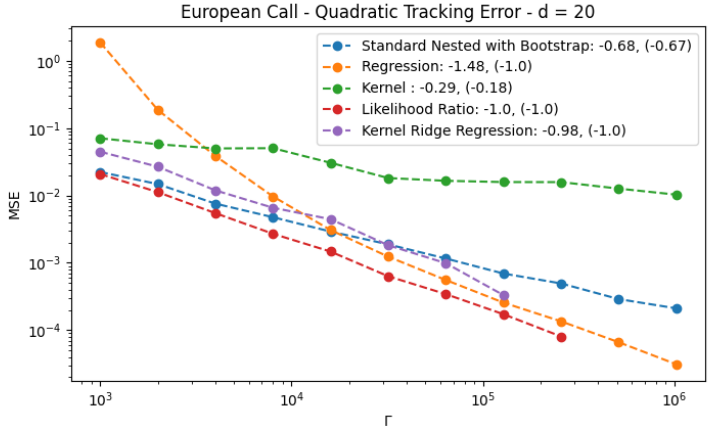


Figure: Empirical convergence rates of different procedures for the base case

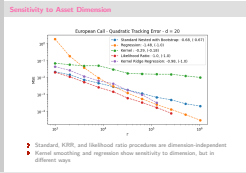


This figure shows the performance of all 5 procedures in their estimation of the quadratic tracking error for the portfolio of European call options.It is a log-log plot of the MSE of the estimator versus the total simulation budget Γ.For each procedure, we run 1000 macro replications of Monte Carlo simulations with different total simulation budgets. For each data point on the plot, 1000 independent replications of the procedure are used to estimate the MSE.
The slopes of the lines estimates the empirical convergence rates of the procedures.
In the legend records the slopes of the lines and the corresponding asymptotic convergence rates of the procedures.
In the most basic case, most procedures have their empirical convergence rates match their asymptotic convergence rates.
However, as we will see after, this is not the case for all experiments.
The following slides will show the sensitivity of the procedures to different factors.

Sensitivity to Asset Dimension

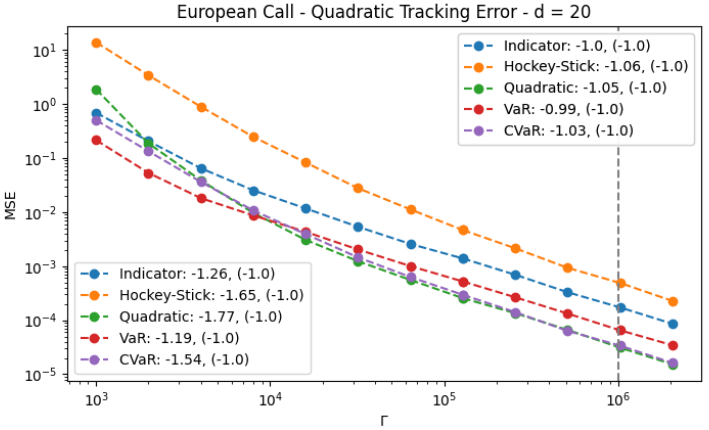


- Standard, KRR, and likelihood ratio procedures are dimension-independent
- Kernel smoothing and regression show sensitivity to dimension, but in different ways

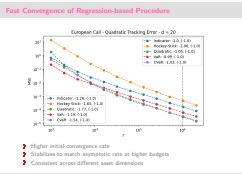


This figure shows the same experiment as the previous figure, but we further increase the asset dimension to 20.
The standard procedure is not affected by the asset dimension.
The kernel smoothing procedure actually converges faster than what its asymptotic convergence rate suggests.
The regression-based procedure is the only procedure that actually converges faster when the asset dimension is high. We will do a deeper dive into it in the following experiment.
It is also worth noting that the likelihood ratio and KRR procedures are dimension-independent.
While it is not shown in their empirical rate, due to their additional cost of pooling inner replications, we are not able to run these procedures when Γ is large.

Fast Convergence of Regression-based Procedure



- Higher initial convergence rate
- Stabilizes to match asymptotic rate at higher budgets
- Consistent across different asset dimensions



To investigate the regression-based procedure, we further increase the total simulation budget Γ.
The figure shows the performance of the regression-based procedure in the estimation of the quadratic tracking error for the portfolio of European call options.
We separate different values of Γ into two different parts.
The first part is when Γ is small.
The second part is when Γ is large.
The first part is when the regression-based procedure has a higher convergence rate than its asymptotic convergence rate.
The second part is when the regression-based procedure stabilizes and matches its asymptotic convergence rate.
In our experiments, we find that the regression-based procedure is very poor when Γ is small.
It means that when we don't have enough data points, the regression-based procedure is particularly bad.
As we increase the total simulation budget Γ, the regression-based procedure improves fast, and eventually matches the asymptotic convergence rate.

Total Computation Time

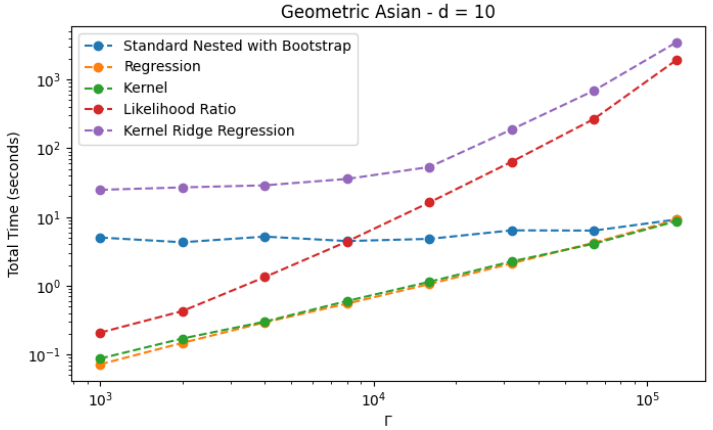
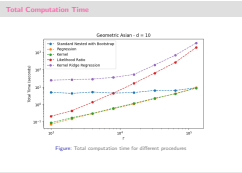


Figure: Total computation time for different procedures



This figure shows the total computation time for different procedures.
It includes the time for both the simulation, training, validation, and everything else.
The total computation time of the standard procedure does not increase fast with Γ.
It is because the computation time of estimating the optimal M and N does not increase for larger Γ.
The regression procedure and kernel smoothing procedure are the most efficient among metamodel-based procedures.
Our implementation of the kernel smoothing procedure is efficient because we only run the cross-validation once for all macro replications.
The likelihood ratio procedure and KRR procedure are more expensive due to their additional costsThe likelihood ratio weight calculations increase quadratically with Γ.
For KRR, the cost of cross-validation is so high that running it once still takes a very long time.

In the thesis includes a detailed analysis of the computational complexity of different procedures in terms of Γ.

Conclusion

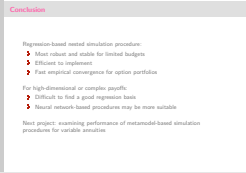
Regression-based nested simulation procedure:

- Most robust and stable for limited budgets
- Efficient to implement
- Fast empirical convergence for option portfolios

For high-dimensional or complex payoffs:

- Difficult to find a good regression basis
- Neural network-based procedures may be more suitable

Next project: examining performance of metamodel-based simulation procedures for variable annuities



In our numerical experiemenct, we have shown that the regression-based procedure is the most cost-effective among all the procedures.
It is robust and efficient to implement for limited budget sizes, and has fast empirical convergence for option portfolios.
However, the regression-based procedure is not suitable for extremely high-dimensional risk factors or complex payoffs.
The first reason is that it is hard to find a good regression basis for high-dimensional risk factors. It will involve a lot of feature engineering and expert knowledge.
The second reason is that the regression-based procedure is not able to capture the nonlinearity in the payoff.
In the next project, we will examine the performance of metamodel-based simulation procedures for variable annuities.
This is where a neural network-based procedure can be more suitable as a form of automated feature engineering.

Nested Simulation for Risk Management of VAs

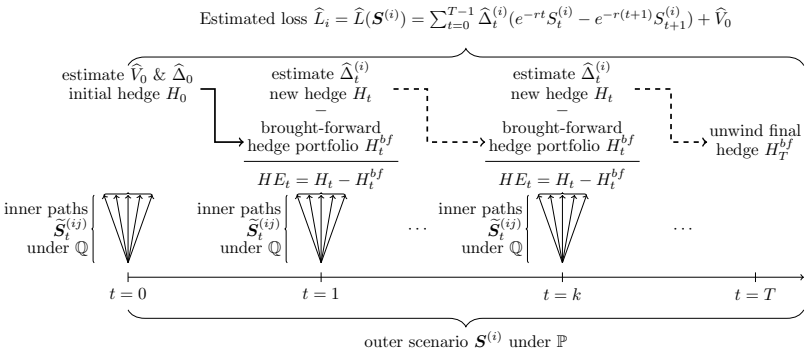
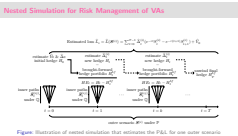


Figure: Illustration of nested simulation that estimates the P&L for one outer scenario



The main difference of the standard nested simulation of VAs can be explained by this illustration.
This is how the P&L is estimated for one outer scenario.
It starts at $t = 0$ with a simulation to estimate the initial value of the VA contract and the initial hedge. Here we are using delta-hedging.
This hedge gets carried forward and updated at each time period with another simulation.
The P&L is then estimated by adding up the hedging error at each time period to the initial value of the VA contract.
All the simulation mentioned is a single set of inner simulations for one outer scenario.
It is very expensive to run this standard nested simulation for VAs.

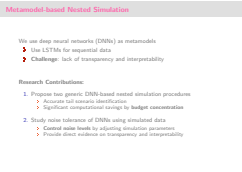
Metamodel-based Nested Simulation

We use deep neural networks (DNNs) as metamodels

- Use LSTMs for sequential data
- **Challenge**: lack of transparency and interpretability

Research Contributions:

1. Propose two generic DNN-based nested simulation procedures
   - Accurate tail scenario identification
   - Significant computational savings by **budget concentration**
2. Study noise tolerance of DNNs using simulated data
   - **Control noise levels** by adjusting simulation parameters
   - Provide direct evidence on transparency and interpretability



After numerical experiments, we find that the deep neural network metamodels are suitable for this problem.
We use LSTM for the sequential data. It provides accurate tail scenario identification and significant computational savings by budget concentration.
However, it is not transparent and interpretable. Many researchers are working on this topic.
This is where we can contribute to the research.
By using nested simulation as a data generating process, we can study the noise tolerance of DNNs.
The noise level in the training labels can be controlled by adjusting the simulation parameters.
By examining the performance of DNNs with different noise levels, we can provide direct evidence on the transparency and interpretability of DNNs.

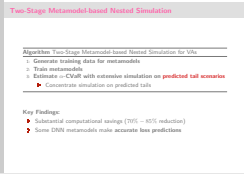
Two-Stage Metamodel-based Nested Simulation

Algorithm Two-Stage Metamodel-based Nested Simulation for VAs

1: **Generate training data for metamodels**
2: **Train metamodels**
3: **Estimate α-CVaR with extensive simulation on predicted tail scenarios**
- Concentrate simulation on predicted tails

Key Findings:

- Substantial computational savings ($70\% - 85\%$ reduction)
- Some DNN metamodels make **accurate loss predictions**



The two-stage procedure is a two-step process that is based on the idea of budget concentration.
First, we generate training data for the metamodel.
We use a fraction of the simulation budget to run the standard nested simulation procedure with M outer scenarios and N' inner replications.
We then train a metamodel using the feature-label pairs.
We use the trained metamodel to make predictions for all outer scenarios.
We then identify the predicted tail scenario set that contains the m largest predicted losses.
Finally, we run the standard nested simulation procedure on the predicted tail scenarios.

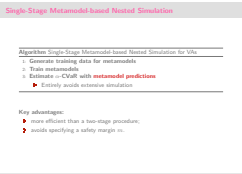
Single-Stage Metamodel-based Nested Simulation

Algorithm Single-Stage Metamodel-based Nested Simulation for VAs

1: **Generate training data for metamodels**
2: **Train metamodels**
3: **Estimate α-CVaR with metamodel predictions**
   - Entirely avoids extensive simulation

Key advantages:

- more efficient than a two-stage procedure;
- avoids specifying a safety margin m.



The single-stage procedure shares the same idea of running the standard nested simulation procedure to generate training data for the metamodel.
After a metamodel is trained, we use it to make predictions for VA contract losses accross all outer scenarios.
The risk measure is estimated using the predicted losses of the metamodel.
It is more efficient than the two-stage procedure because it avoids the second stage of the two-stage procedure.
It also avoids specifying a safety margin m, which is a feature of the two-stage procedure that we will discuss later.

Experiment Setting

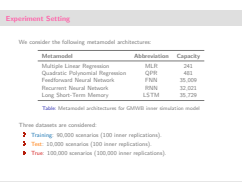
We consider the following metamodel architectures:

Metamodel	Abbreviation	Capacity
Multiple Linear Regression	MLR	241
Quadratic Polynomial Regression	QPR	481
Feedforward Neural Network	FNN	35,009
Recurrent Neural Network	RNN	32,021
Long Short-Term Memory	LSTM	35,729

Table: Metamodel architectures for GMWB inner simulation model

Three datasets are considered:

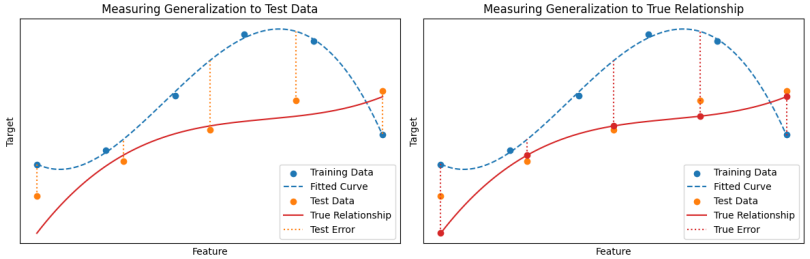
- Training: 90,000 scenarios (100 inner replications).
- Test: 10,000 scenarios (100 inner replications).
- True: 100,000 scenarios (100,000 inner replications).



We consider the following metamodel architectures:
Multiple Linear Regression, Quadratic Polynomial Regression, Feedforward Neural Network, Recurrent Neural Network, and Long Short-Term Memory Network.
Capacity is defined as the number of parameters in the metamodel.
Higher capacity metamodels are more flexible and expressive.
Lower capacity metamodels are less likely to overfit.
The deep neural network metamodels are more flexible and expressive, but they are said to be more prone to overfitting.
We will see whether this is indeed the case in our experiments.

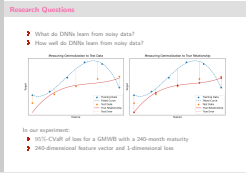
Research Questions

- What do DNNs learn from noisy data?
- How well do DNNs learn from noisy data?



In our experiment:

- 95%-CVaR of loss for a GMWB with a 240-month maturity
- 240-dimensional feature vector and 1-dimensional loss



To further clarify our research questions, we consider the following scenario:
We have a noisy training data and a noisy test data.
We want to know what the metamodel learns from the noisy data.
We also want to know how well the metamodel can make accurate loss predictions for given scenarios.
In a simulation experiment, we can control the noise level in the training data and the test data.
We also have access to the true feature-label relationship when we increase the number of replications.

We can use this true feature-label relationship to evaluate the performance of deep neural networks, which is usually not possible in practice.

Metamodel Performance

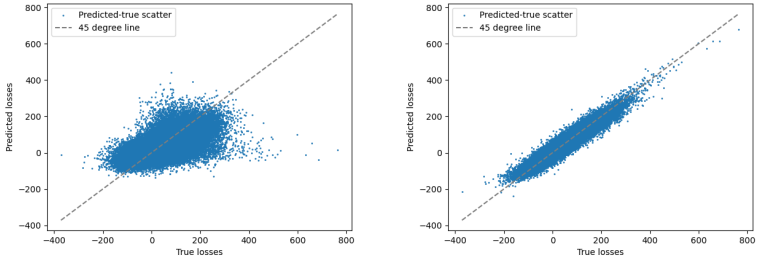
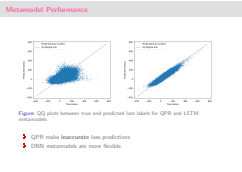


Figure: QQ plots between true and predicted loss labels for QPR and LSTM metamodels

- QPR make **inaccurate** loss predictions
- DNN metamodels are more flexible.



We first consider the traditional regression metamodels.
On the left is the QQ plot between the true loss labels and the predicted loss labels for MLR, and on the right is the QQ plot for QPR.
MLR and QPR metamodels make inaccurate loss predictions.
Feature engineering is hardly feasible for our 240-dimensional X.
We will see whether deep neural networks can make more accurate loss predictions.

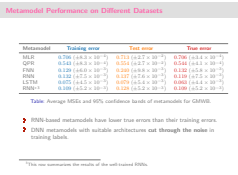
Metamodel Performance on Different Datasets

Metamodel	Training error	Test error	True error
MLR	0.706 $(\pm 8.3 \times 10^{-4})$	0.713 $(\pm 2.7 \times 10^{-2})$	0.706 $(\pm 3.4 \times 10^{-4})$
QPR	0.543 $(\pm 8.3 \times 10^{-4})$	0.554 $(\pm 2.7 \times 10^{-2})$	0.544 $(\pm 4.1 \times 10^{-4})$
FNN	0.129 $(\pm 6.0 \times 10^{-3})$	0.240 $(\pm 9.8 \times 10^{-3})$	0.132 $(\pm 5.8 \times 10^{-3})$
RNN	0.132 $(\pm 7.5 \times 10^{-3})$	0.137 $(\pm 7.6 \times 10^{-3})$	0.119 $(\pm 7.5 \times 10^{-3})$
LSTM	0.075 $(\pm 4.5 \times 10^{-3})$	0.079 $(\pm 5.4 \times 10^{-3})$	0.063 $(\pm 4.4 \times 10^{-3})$
RNN*[3]	0.109 $(\pm 5.2 \times 10^{-3})$	0.128 $(\pm 5.2 \times 10^{-3})$	0.109 $(\pm 5.2 \times 10^{-3})$

Table: Average MSEs and 95% confidence bands of metamodels for GMWB.

- RNN-based metamodels have lower true errors than their training errors.
- DNN metamodels with suitable architectures **cut through the noise** in training labels.

[3] This row summarizes the results of the well-trained RNNs.



This table summarizes the results of our experiments.
It is the most interesting finding that DNN metamodels with suitable architectures can **cut through the noise** in training labels.
The average MSEs of metamodels on the training, test, and true data are reported in the three columns, respectively.
Note that the training and test labels are noisy. The simulation noise is what makes them noisy, which is less present in the true data.
First, we note that DNN metamodels has lower training errors than the traditional regression metamodels.
This is expected because DNNs are more flexible and expressive than traditional regression metamodels.
What is more interesting is the comparison between the test and true errors.
We find that the true errors of RNN-based metamodels are lower than their test errors.
This suggests that the RNN and LSTM metamodels are able to cut through the simulation noise and learn the true feature-label relationship.
When we take another look at the table, we find that for LSTM, its true error is even lower than its training error.
This is another evidence that LSTM metamodels are able to cut through the noise in training labels and fit to the true relationship.
We also find that the LSTM metamodel is able to cut through the noise in training labels and fit to the true relationship.
We have another finding for RNN. RNN metamodels suffer from the vanishing gradient

Safety Margin

Consider estimating the 95% CVaR with 100,000 outer scenarios.

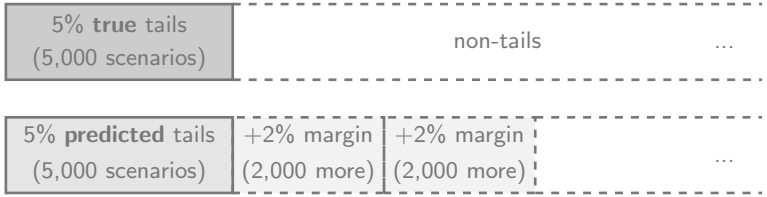
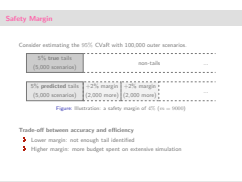


Figure: Illustration: a safety margin of 4% ($m = 9000$)

Trade-off between accuracy and efficiency

- Lower margin: not enough tail identified
- Higher margin: more budget spent on extensive simulation



For a two-stage procedure, we need to choose a safety margin m to identify the tail scenarios.
Consider estimating the 95% CVaR with 100,000 outer scenarios.
On the top is the true tail and non-tail scenarios, and on the bottom is the predicted tail and non-tail scenarios.
The 5% predicted tail scenarios are not guaranteed to contain all the true tail scenarios.
Therefore, we need to add a safety margin m to the predicted tail scenarios.
On the bottom, we add a 2% safety margin to the predicted tail scenarios.
This adds 2,000 more scenarios to the inner simulations.
We may add another 2% safety margin to be more confident.
This introduce a safety margin of 4%. 9000 predicted tail scenarios are used for extensive inner simulations in stage 2.
This is a trade-off between accuracy and efficiency.
For a good metamodel, we only need a small safety margin to be accurate in tail identification.
However, for a bad metamodel, the safety margin needs to be large to identify all the tail scenarios.
This comes at the cost of more inner simulations, which is computationally expensive.
So, one of the limitations of the two-stage procedure is the choice of safety margin.
It can be partially mitigated by using a good metamodel.

Metamodel Performance

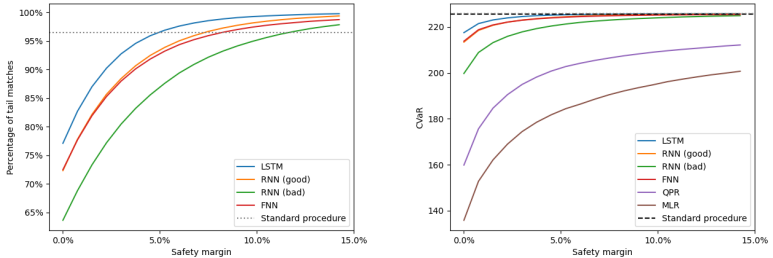
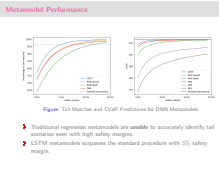


Figure: Tail Matches and CVaR Predictions for DNN Metamodels

- Traditional regression metamodels are **unable** to accurately identify tail scenarios even with high safety margins.
- LSTM metamodels surpasses the standard procedure with 5% safety margin.



We now consider the tail scenario identification for the two-stage procedure.
On the left is the tail scenario identification for the traditional regression metamodel, and on the right is the tail scenario identification for the LSTM metamodel.
We compare the two-stage procedure with the standard procedure.
When colored lines cross the grey dashed line, it means that the metamodel is able to identify the tail scenarios as well as the standard procedure.
We find that the LSTM metamodel is able to surpass the standard procedure on tail identification with a less than 5% safety margin.
However, the traditional regression metamodel is unable to identify the tail scenarios even with a high safety margin.
A well-trained RNN is better than a FNN, but the poorly-trained RNN is far worse.This further motivates the use of LSTM metamodels over RNN metamodels.

Sensitivity Testing for DNNs

Simulation controls **noise level in training labels** and **number of training samples**.

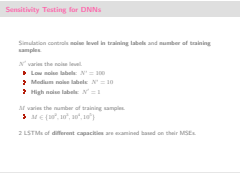
N' varies the noise level.

- **Low noise labels**: $N' = 100$
- **Medium noise labels**: $N' = 10$
- **High noise labels**: $N' = 1$

M varies the number of training samples.

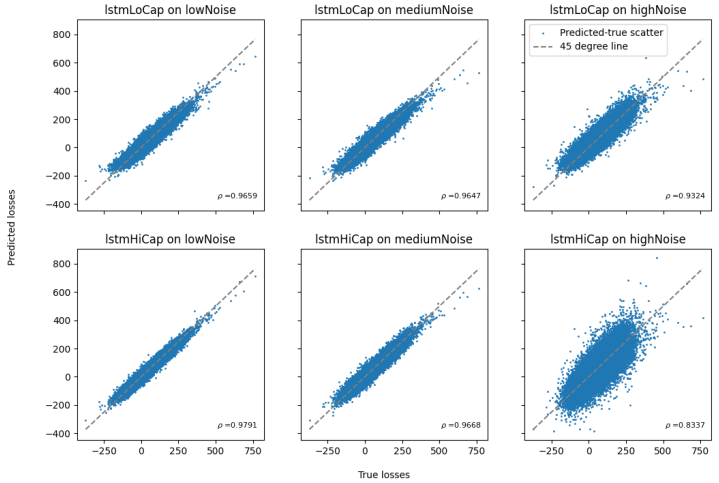
- $M \in \{10^2, 10^3, 10^4, 10^5\}$

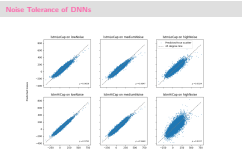
2 LSTMs of **different capacities** are examined based on their MSEs.



We now consider the sensitivity testing for the DNN metamodels.
We are interested in the sensitivity of the DNN metamodels to the noise level in training labels and the number of training samples.
In a simulation experiment, we can vary them to see how the DNN metamodels perform.
We consider the noise level in training labels by varying the number of inner replications N'.
We consider the number of training samples by varying the number of outer scenarios M.
We consider two LSTM metamodels of different capacities because we want to see if the capacity of the metamodel matters for overfitting.
The high-capacity LSTM metamodel has 10 times more parameters than the regular LSTM metamodel

Noise Tolerance of DNNs





This figure shows the QQ plots of the two LSTM metamodels.
We plot the predicted losses versus the true losses for both LSTMs and different noise levels.
The closer the points are to the line, the better the model is.
The high-capacity LSTM is better at low and medium noise levels, while the regular LSTM is better at high noise levels.

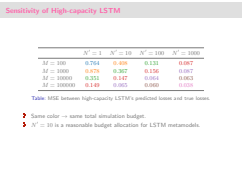
We can see on the bottom right that the high-capacity LSTM has very poor performance.

Sensitivity of High-capacity LSTM

	$N' = 1$	$N' = 10$	$N' = 100$	$N' = 1000$
$M = 100$	0.764	0.408	0.131	0.087
$M = 1000$	0.878	0.367	0.156	0.087
$M = 10000$	0.351	0.147	0.064	0.063
$M = 100000$	0.149	0.065	0.060	0.038

Table: MSE between high-capacity LSTM's predicted losses and true losses.

- Same color → same total simulation budget.
- $N' = 10$ is a reasonable budget allocation for LSTM metamodels.



The findings are similar for the high-capacity LSTM metamodel.
We can find similar patterns as in the previous table.
The model is more sensitive to N' than M, unless N' is very small.

For a practical application, we recommend using the high-capacity LSTM metamodel with $N' = 10$ for almost all simulation budgets.

Single-Stage Procedure

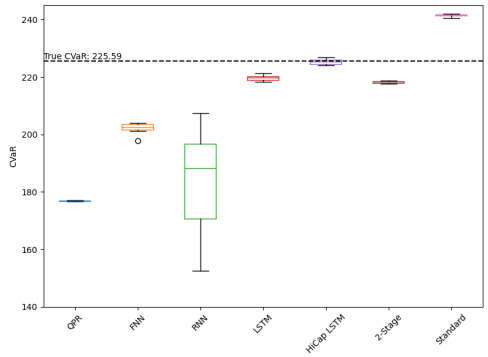


Figure: CVaR estimates of single-stage procedures with $N' = 10$.

- The single-stage procedure outperforms the two-stage procedure.
- $N' = 10$ is a reasonable budget allocation.

This figure shows the CVaR estimates of the single-stage procedure.
We compare it with the two-stage procedure using regular LSTM metamodel with no safety margin. So we are using the 5% metamodel predicted tails to estimate the 95% CVaR with extensive simulation in stage 2.
The single-stage procedure with LSTM metamodels outperforms the two-stage procedure when noise level is moderate.
Due to its design, the single-stage procedure is more efficient than the two-stage procedure.
Setting $N' = 10$ is a reasonable budget allocation for the single-stage procedure.
Further decreasing N' to 1 deteriorates the performance of the single-stage procedure.
But it is worth noting that the 2-stage procedure can overcome this issue by increasing the safety margin.

Convergence Analysis

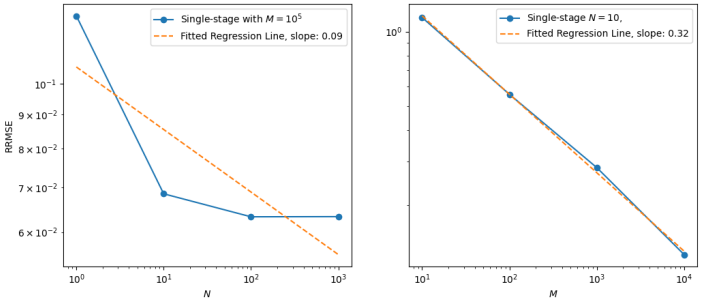
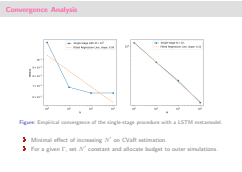


Figure: Empirical convergence of the single-stage procedure with a LSTM metamodel.

- Minimal effect of increasing N' on CVaR estimation.
- For a given Γ, set N' constant and allocate budget to outer simulations.



This figure shows the empirical convergence of the single-stage procedure with LSTM metamodels when only M or N' is varied.
On the left, we fix $M = 10^5$ and vary N'.
There is no significant difference to further increase N' when $N' \geq 10$.
On the right, we fix $N' = 10$ and vary M.
We can see that the convergence rate is around $O(M^{-2/3})$.
This finding is consistent with our previous analysis on the sensitivity of LSTM metamodels to data quality and data quantity.
Once the quality is good enough, the quantity is more important.

Conclusion

Key Findings:

- LSTMs are **resilient** to moderate levels of noise in training labels.
- DNNs can learn **true** complex dynamic hedging model.
- Two-stage procedure addresses regulatory concerns.
- Single-stage procedure is **efficient**.

Future Directions:

- Apply DNNs to other risk management tasks.
- Fast adaptation to new contracts/market conditions.

In this project, we present our extensive research into LSTM metamodels in a simulation setting.
Our findings demonstrate that LSTMs can effectively learn complex hedging strategies even with noisy training data.
The single-stage procedure offers an excellent balance of efficiency and accuracy for practical applications.
We've shown that allocating computational budget to increase outer scenarios rather than inner simulations yields better results.
These insights provide valuable guidance for implementing machine learning in financial risk management.
For the research directions. In the future, we can apply deep neural network metamodels to other actuarial applications.
In the future, we can also apply transfer learning to speed up the training process for deep neural network metamodels.

References

Gordy, M. B. and Juneja, S. (2010). Nested simulation in portfolio risk measurement. *Management Science*, 56(10):1833–1848.

Zhang, K., Liu, G., and Wang, S. (2021). Bootstrap-based budget allocation for nested simulation. *Operations Research*.

References

Gordy, M. B. and Juneja, S. (2010). Nested simulation in portfolio risk measurement. *Management Science*, 56(10):1833–1848.
Zhang, K., Liu, G., and Wang, S. (2021). Bootstrap-based budget allocation for nested simulation. *Operations Research*.