# Transfer Learning for Metamodeling Nested Simulation in VA Contracts

## Outline

1. Introduction to Nested Simulation
2. Challenges in VA Contract Valuation
3. Metamodeling Approach
4. Transfer Learning Fundamentals
5. Application to VA Contracts
6. Experimental Results
7. Conclusions

## Introduction to Nested Simulation

- **Nested simulation**: A computational approach involving two layers of simulations
  - Outer scenarios: Represent possible future states
  - Inner scenarios: Estimate conditional expectations (e.g., losses) at each outer scenario
- **Applications**: Risk management, dynamic hedging, and regulatory compliance
- **Challenge**: Computational intensity due to multiple layers of simulations

## Variable Annuity (VA) Contracts

- Financial products offering investment returns with insurance guarantees
- Complex financial instruments with long-term horizons
    - Guaranteed minimum maturity benefits (GMMB)
    - Guaranteed minimum accumulation benefits (GMAB)
    - Guaranteed minimum withdrawal benefits (GMWB)

## Computational Challenges for Simulating VA Contracts

- Nested simulation for VA contracts is computationally intensive:
    - Thousands of outer scenarios
    - Hundreds/thousands of inner scenarios per outer scenario
    - Complex contract features and market dynamics
- **Real-world impact**: Delays in risk assessment and decision-making

## Metamodeling Approach

- **Metamodel**: A model that approximates the relationship between inputs and outputs of a complex simulation
- **Goal**: Replace expensive inner simulations with a fast approximation
- **Benefits**:
    - Significant reduction in computational time
    - Enables more frequent risk assessments
    - Supports real-time decision making (?)

## Traditional Metamodeling Approaches

- Multiple linear regression
- Quadratic polynomial regression
- Feedforward neural networks
- Recurrent neural networks
- Long short-term memory (LSTM) networks

**Limitation**:

- Each new VA contract or valuation date requires building a new metamodel from scratch

## Transfer Learning: Fundamentals

- **Core idea**: Knowledge gained from solving one problem can be applied to a different but related problem
- **Advantages**:
  - Reduces training data requirements
  - Accelerates learning process
  - Improves model performance

# Neural Networks & Transfer Learning

## Transfer Learning: Approaches

- **Feature-based Transfer**: Extract and transfer knowledge at feature representation level
  - Example: Domain adaptation via feature alignment (Tzeng et al., 2017)
- **Instance-based Transfer**: Reweight source domain instances for target domain relevance
  - Example: TrAdaBoost for cross-domain classification (Dai et al., 2007)
- **Parameter-based Transfer**: Share and fine-tune model parameters between domains
  - Example: Pre-trained neural networks in computer vision (Yosinski et al., 2014)

## Mathematical Formulation of Transfer Learning

In supervised learning, a **domain** $\mathcal{D}$ consists of:

- Feature space $\mathcal{X}$
- Marginal probability distribution $F$

A **task** $\mathcal{T}$ consists of:

- Label space $\mathcal{Y}$
- Predictive function $f : \mathcal{X} \to \mathcal{Y}$

## Transfer Learning Framework

- **Source domain**: $\mathcal{D}_{\mathrm{So}} = \{\mathcal{X}_{\mathrm{So}}, F_{\mathrm{So}}(\mathbf{X})\}$
- **Source task**: $\mathcal{T}_{\mathrm{So}} = \{\mathcal{Y}_{\mathrm{So}}, f_{\mathrm{So}}(\cdot)\}$
- **Target domain**: $\mathcal{D}_{\mathrm{Ta}} = \{\mathcal{X}_{\mathrm{Ta}}, F_{\mathrm{Ta}}(\cdot)\}$
- **Target task**: $\mathcal{T}_{\mathrm{Ta}} = \{\mathcal{Y}_{\mathrm{Ta}}, f_{\mathrm{Ta}}(\cdot)\}$

In our context of metamodeling nested simulation for VA contracts:

- $\mathcal{X}_{\mathrm{So}}$ and $\mathcal{X}_{\mathrm{Ta}}$ include input features derived from the outer simulation
- $F_{\mathrm{So}}$ and $F_{\mathrm{Ta}}$ are the outer simulation model that simulates the outer scenarios
- $\mathcal{Y}_{\mathrm{So}}$ and $\mathcal{Y}_{\mathrm{Ta}}$ include the VA contract losses
- $f_{\mathrm{So}}$ and $f_{\mathrm{Ta}}$ are the true contract losses that we want to approximate

## Relating to Previous Work

In the previous chapter, we have $\mathcal{D}_{\mathrm{So}} = \mathcal{D}_{\mathrm{Ta}}$ and $\mathcal{T}_{\mathrm{So}} = \mathcal{T}_{\mathrm{Ta}}$.

- We trained the LSTM network on a simulation dataset
- We used the same LSTM network to approximate the contract losses on the same data generating process

However, as new contract features are introduced, we want to transfer previous knowledge when

- we have **limited simulation budget** (data is scarce)
- we want to save the cost of **building a new metamodel from scratch**

## Transfer Learning Implementation

**Goal**: Improve learning when $\mathcal{D}_{\mathrm{So}} \neq \mathcal{D}_{\mathrm{Ta}}$ or $\mathcal{T}_{\mathrm{So}} \neq \mathcal{T}_{\mathrm{Ta}}$

1. Train source model $f_{\mathrm{So}}(\cdot;\theta_{\mathrm{So}})$ on dataset $\mathcal{D}_{\mathrm{So}} = \{(X_{\mathrm{So}}^{(i)}, L_{\mathrm{So}}^{(i)})\}_{i=1}^{M_{\mathrm{So}}}$
2. Transfer knowledge by initializing target model with $\theta_{\mathrm{So}}$
3. Fine-tune $f_{\mathrm{Ta}}(\cdot;\theta_{\mathrm{Ta}})$ on target domain $\mathcal{D}_{\mathrm{Ta}}$

## Fine-tuning Algorithm for LSTM Metamodels in VA Hedging

**Input**:

- Source dataset: $\mathcal{D}_{\mathrm{So}} = \{(X_{\mathrm{So}}^{(i)}, L_{\mathrm{So}}^{(i)})\}_{i=1}^{M_{\mathrm{So}}}$, Target dataset: $\mathcal{D}_{\mathrm{Ta}} = \{(X_{\mathrm{Ta}}^{(i)}, L_{\mathrm{Ta}}^{(i)})\}_{i=1}^{M_{\mathrm{Ta}}}$

**Algorithm**:

1. Train source LSTM metamodel $f_{\mathrm{So}}(\cdot;\theta_{\mathrm{So}})$ on $\mathcal{D}_{\mathrm{So}}$:

$$\theta_{\mathrm{So}} = \arg\min_{\theta} \frac{1}{M_{\mathrm{So}}} \sum_{i=1}^{M_{\mathrm{So}}} (f_{\mathrm{So}}(X_{\mathrm{So}}^{(i)};\theta) - L_{\mathrm{So}}^{(i)})^2.$$

2. Initialize target model: $\theta_{\mathrm{Ta}} \leftarrow \theta_{\mathrm{So}}$.

3. Fine-tune $f_{\mathrm{Ta}}(\cdot;\theta_{\mathrm{Ta}})$ on $\mathcal{D}_{\mathrm{Ta}}$:

$$\theta_{\mathrm{Ta}} = \arg\min_{\theta} \frac{1}{M_{\mathrm{Ta}}} \sum_{i=1}^{M_{\mathrm{Ta}}} (f_{\mathrm{Ta}}(X_{\mathrm{Ta}}^{(i)};\theta) - L_{\mathrm{Ta}}^{(i)})^2.$$

## Layer Freezing

**Input**:

- Source dataset: $\mathcal{D}_{\mathrm{So}} = \{(X_{\mathrm{So}}^{(i)}, L_{\mathrm{So}}^{(i)})\}_{i=1}^{M_{\mathrm{So}}}$, Target dataset: $\mathcal{D}_{\mathrm{Ta}} = \{(X_{\mathrm{Ta}}^{(i)}, L_{\mathrm{Ta}}^{(i)})\}_{i=1}^{M_{\mathrm{Ta}}}$

**Algorithm**:

1. Train source model $f_{\mathrm{So}}(\cdot; \theta_{\mathrm{So}})$ on $\mathcal{D}_{\mathrm{So}}$
2. Initialize $\theta_{\mathrm{Ta}} \leftarrow \theta_{\mathrm{So}}$
3. Freeze LSTM layers in $\theta_{\mathrm{Ta}}$
4. Fine-tune unfrozen layers of $f_{\mathrm{Ta}}(\cdot; \theta_{\mathrm{Ta}})$ on $\mathcal{D}_{\mathrm{Ta}}$

**Output**: Adapted model $f_{\mathrm{Ta}}(\cdot; \theta_{\mathrm{Ta}})$ with frozen LSTM layers

**Note**: Choice of layers to freeze depends on similarity between source and target tasks

## Transfer Learning for VA Contracts

**Key insight**: Different VA contracts share underlying financial and mathematical structures

**Application approaches**:

- Transfer between different contract types
- Transfer between different valuation dates
- Transfer between different market conditions

## VA Contracts and Asset Models

| Contract | Asset Model | Lapse | $M_{\mathrm{So}}$ | $M_{\mathrm{Ta}}$ |
|---|---|---|---|---|
| GMMB | GBM | No | 50,000 | N/A |
| GMMB | RS-GBM | No | 50,000 | 2,000 |
| GMMB | RS-GBM | Static | 50,000 | 2,000 |
| GMMB | RS-GBM | Dynamic | 50,000 | 2,000 |
| GMWB | RS-GBM | Dynamic | N/A | 2,000 |

All training data is generated by the standard procedure with 100 inner replications.

# Learning Lapse Features

## From No Lapse to Static Lapse

# Learning Curves: Without Transfer Learning

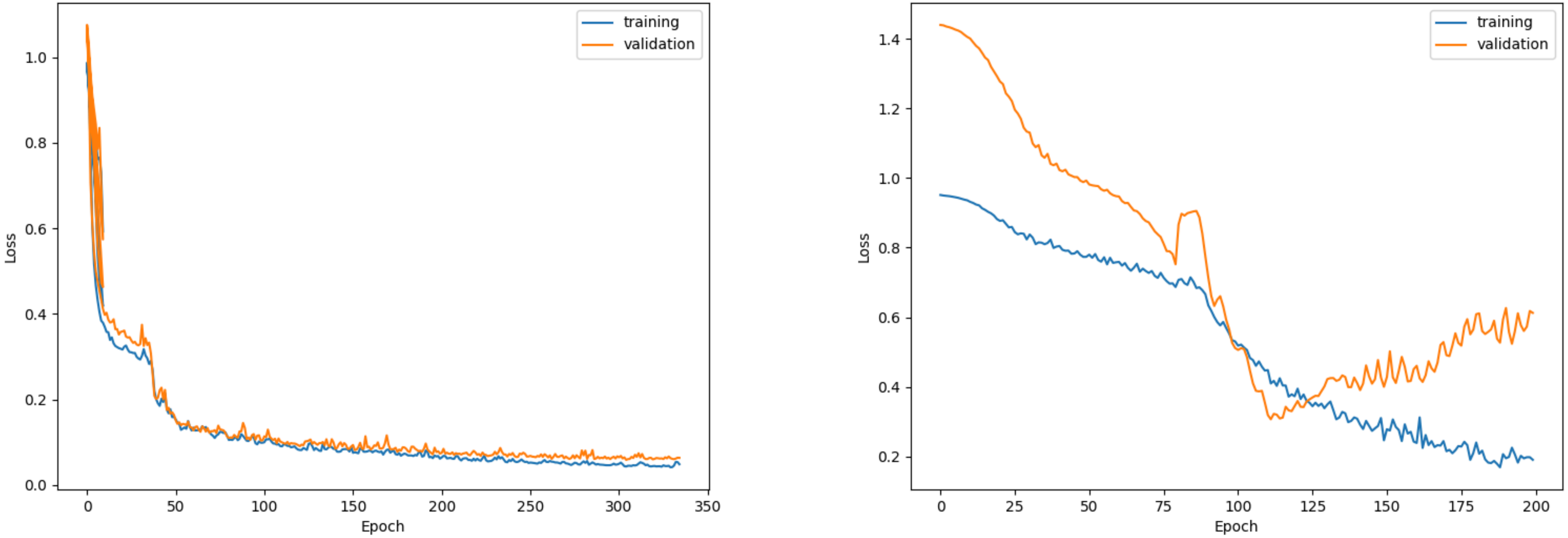



*Figure 1: The effect of number of training samples on the performance of the metamodel*

- *Left: 50000 samples; right: 2000 samples*

# Learning Curves: With Transfer Learning

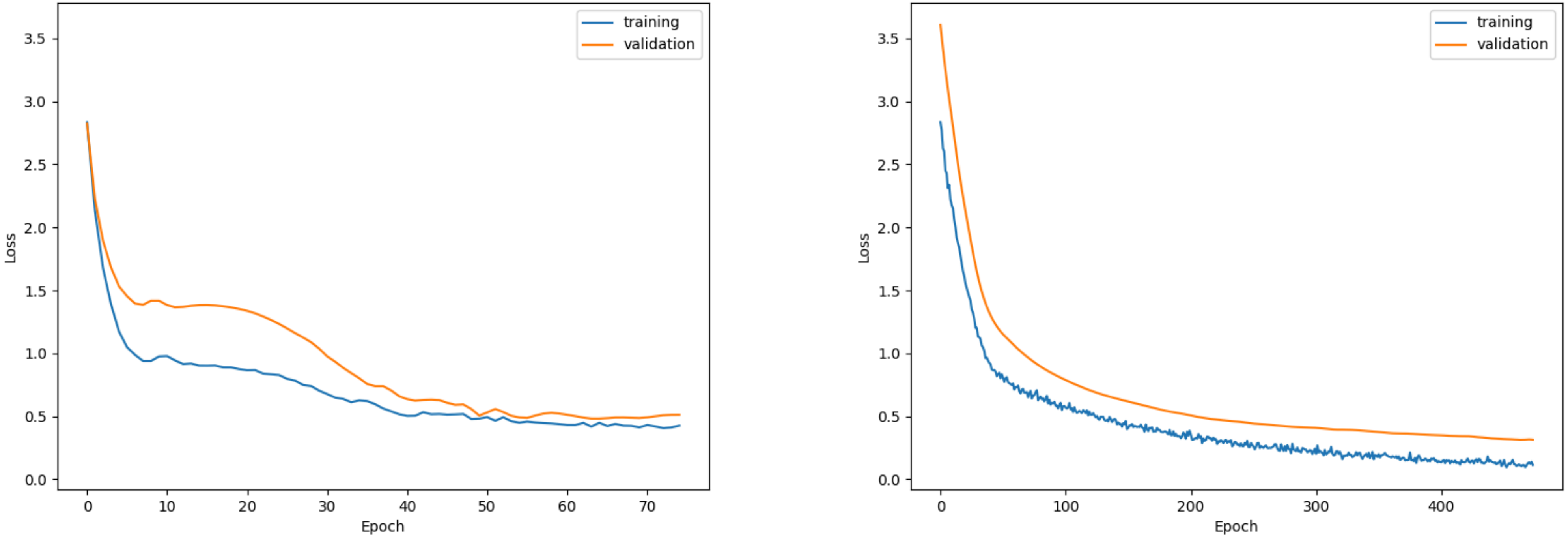



*Figure 2: The effect of transfer learning on limited training samples*

- *Left: fine-tuning; right: layer freezing*

## Experimental Setup

- **Source domain**: VA contract with specific features/parameters
- **Target domain**: New VA contract or same contract with different features
- **Architecture**: LSTM-based metamodel
- **Transfer approach**: Pre-trained weights from source model

# Learning Dynamic Lapse

## The Effect of Similarity between Source and Target

# Learning Curves: Learning Dynamic Lapse

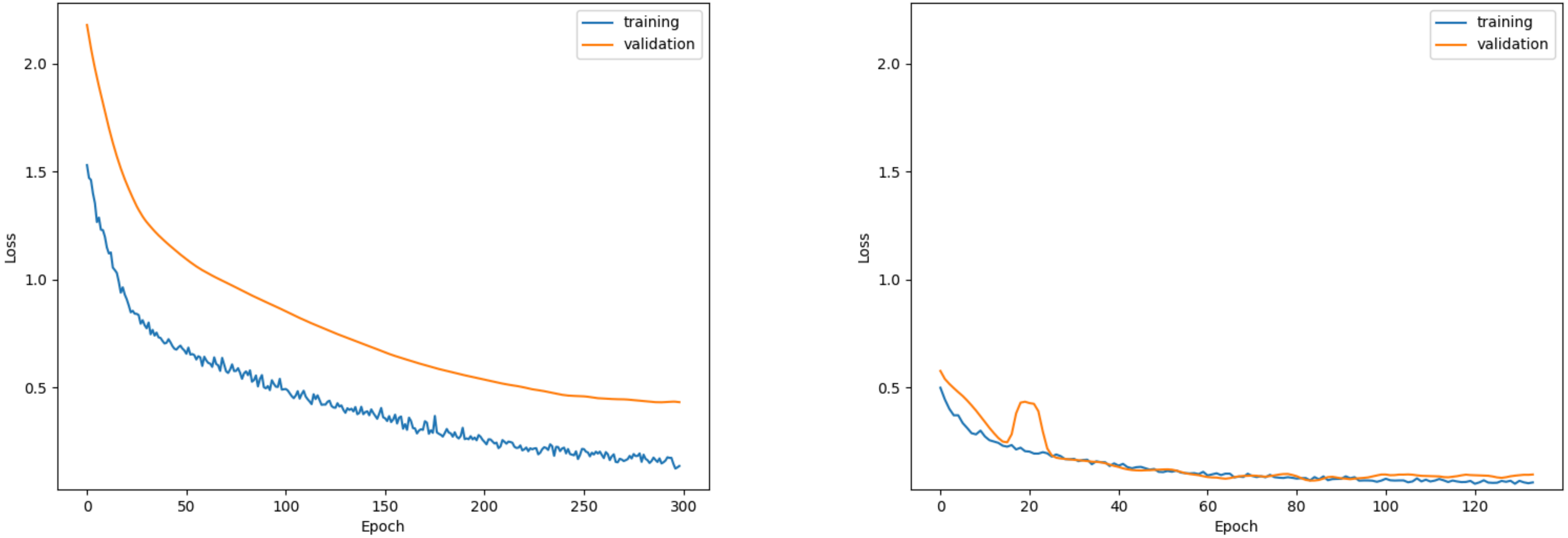



*Figure 3: The effect of similarity between source and target on the convergence speed*

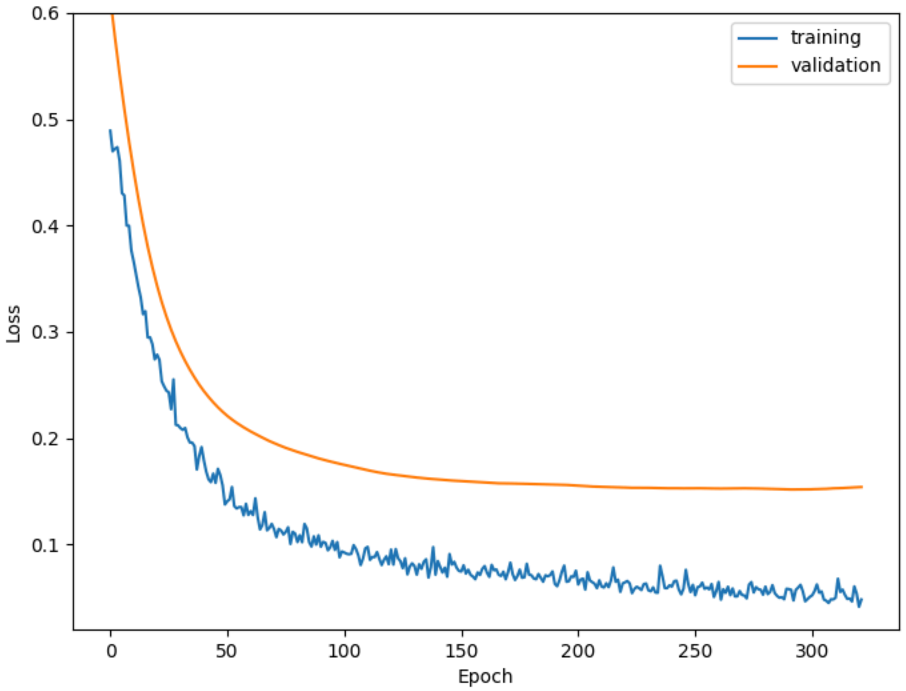
- *Left: from No Lapse; right: from Static Lapse*

## Results: Accuracy Comparison

| Lapse Type | Extensive | Fine-tuning | Layer Freezing | Without TL |
|---|---|---|---|---|
| No Lapse | N/A | 0.4894 | **0.3361** | N/A |
| Static Lapse | N/A | 0.0794 | 0.0763 | N/A |
| Dynamic Lapse | **0.0587** | N/A | N/A | **0.2950** |

*Table 1: Comparison of different TL methods on GMMB contracts (best MSE values)*

# Learning Curves: Effect of Similarity



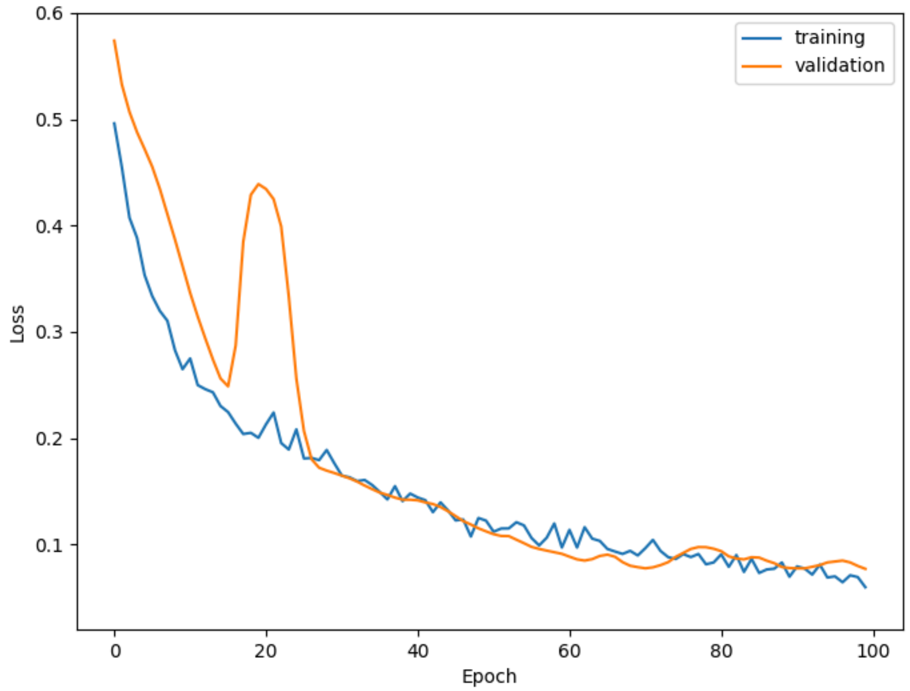



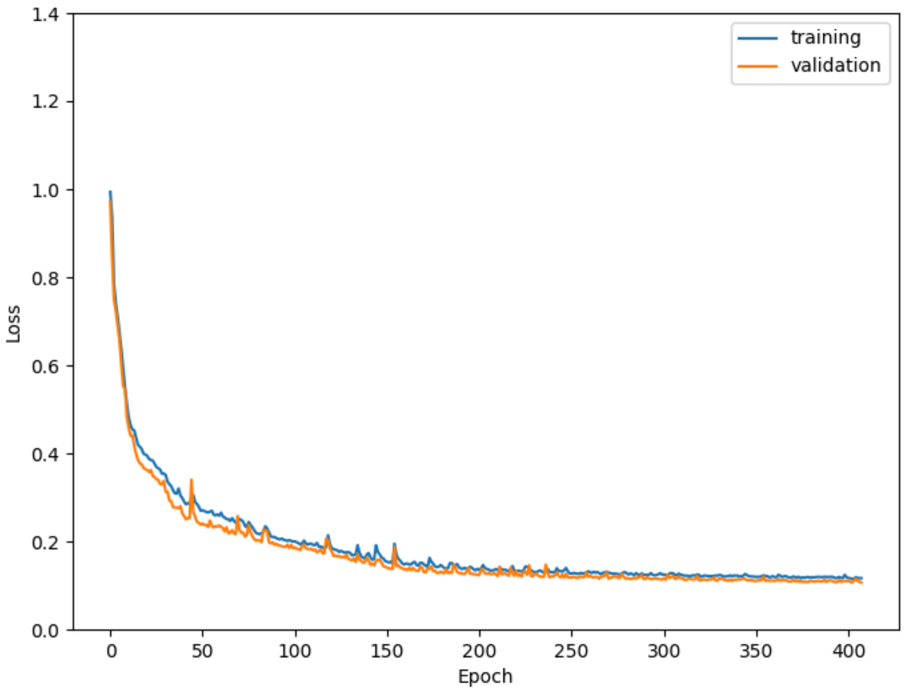

*Figure 4: The effect of similarity between source and target on the convergence speed*

- *Left: from No Lapse; right: from Static Lapse*

# Transfer Knowledge to other Contract Types

## From GMMB to GMWB Contract

# Learning Curves: Transfer Knowledge to other Contract Types



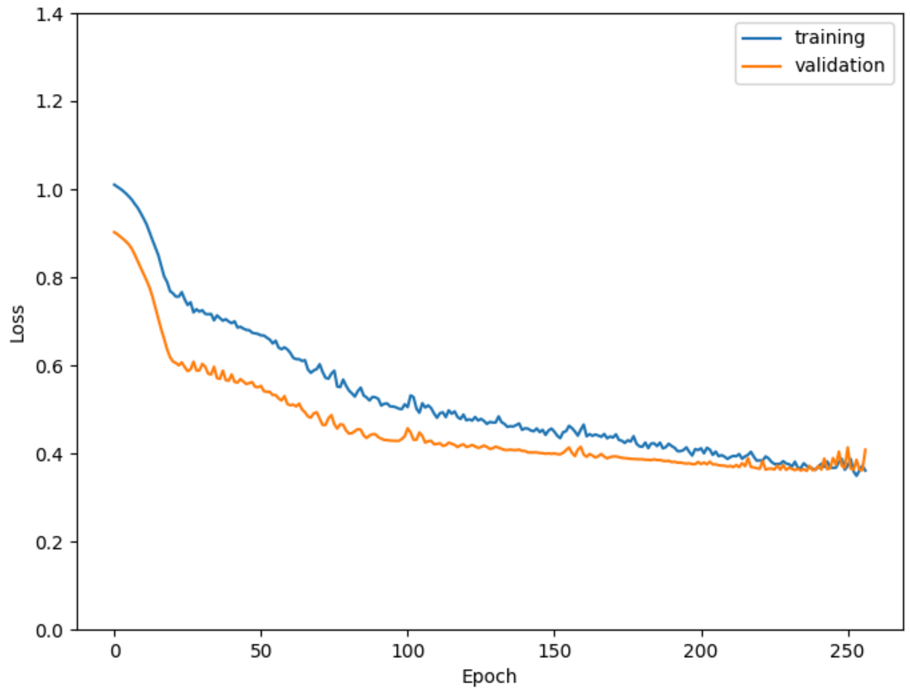




*Figure 5: Learning curves of GMWB contracts*

- *Left: 50000 samples; right: 2000 samples*

# Learning Curves: Effect of Similarity

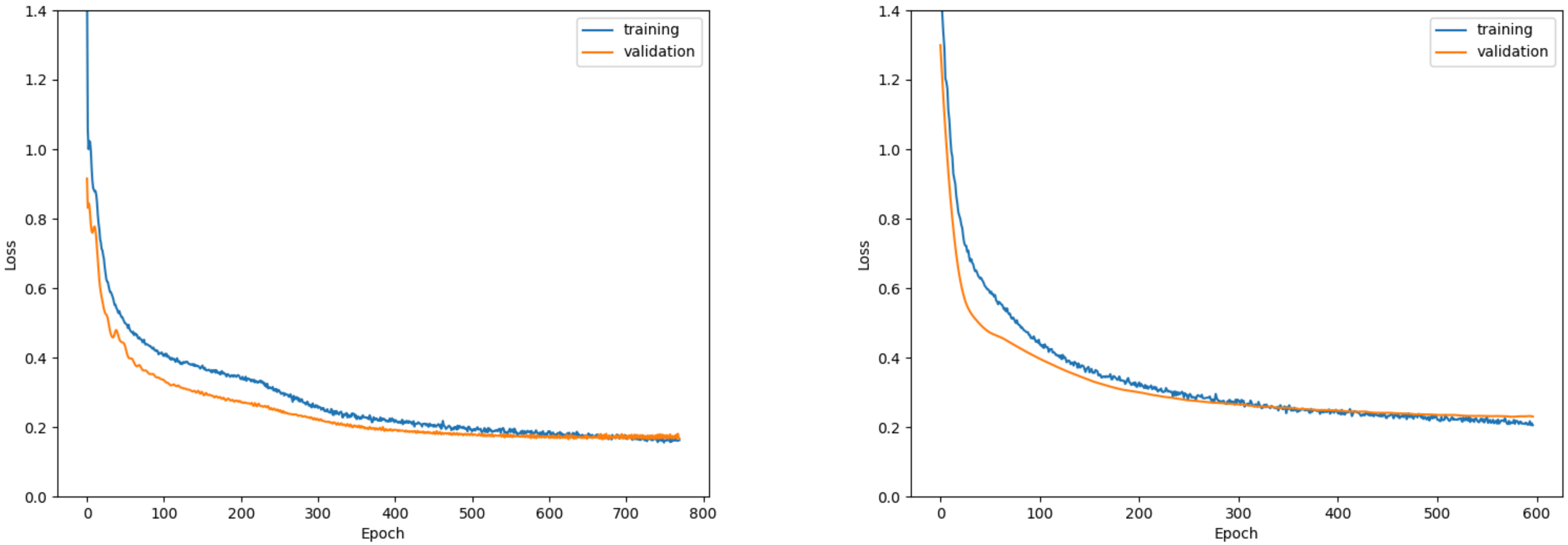



*Figure 6: Comparison of different TL methods on GMWB contracts*

- *Left: fine-tuning; right: layer freezing*

## Performance Comparison (GMMB → GMWB)

| Model | Training MSE | True MSE |
| --- | --- | --- |
| Without TL | 0.3588 | 0.4188 |
| Fine Tuning | 0.1690 | 0.1780 |
| Layer Freezing | 0.1828 | 0.2295 |
| Extensive Training | 0.0853 | 0.0726 |

- Fine-tuning outperforms layer freezing for dissimilar tasks
- Both TL methods better than training from scratch
- Extensive training still superior with abundant data

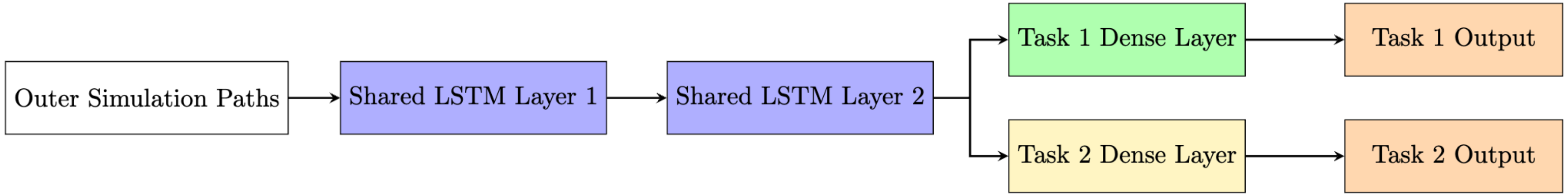
## Practical Implementation

- **Framework**: TensorFlow/PyTorch implementation
- **Workflow**:
    i. Train base model on source contract
    ii. Freeze early layers
    iii. Fine-tune later layers on target contract
    iv. Deploy for production use

**PyTorch** is recommended.

- It is more flexible and easier to customize
- It is more intuitive and easier to understand
- It is more efficient and easier to debug

# Multi-task Learning




- LSTM layers shared across multiple tasks
- Task-specific fully connected layers
- Objective: Minimize sum of loss functions across all tasks

# Multi-task Learning Framework

**Input**: Set of $K$ tasks $\{\mathcal{T}_k\}_{k=1}^{K}$ with datasets $\mathcal{D}_k = \{(X_k^{(i)}, L_k^{(i)})\}_{i=1}^{M_k}$, shared parameters $\theta_0$, and task-specific parameters $\theta_k$ for each task $k$

**Algorithm**:

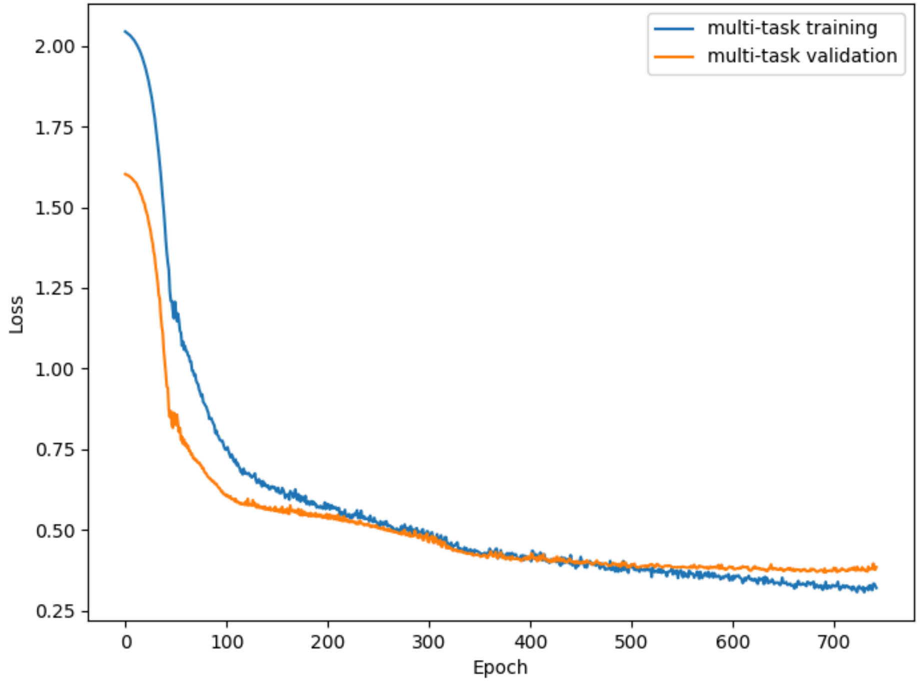
1. Train the multi-head LSTM metamodel on all $K$ tasks simultaneously by minimizing the multi-task loss function:

$$\min_{\theta_0, \{\theta_k\}_{k=1}^{K}} \sum_{k=1}^{K} \frac{1}{M_k} \sum_{i=1}^{M_k} \left( f_i(X_k^{(i)}; \theta_0, \theta_k) - L_k^{(i)} \right)^2$$

2. Update both the shared parameters $\theta_0$ and task-specific parameters $\{\theta_k\}_{k=1}^{K}$ simultaneously using backpropagation and gradient descent with learning rate $\alpha$

**Output**: Trained multi-task LSTM metamodel $f(\cdot; \theta_0, \{\theta_k\}_{k=1}^{K})$ for all $K$ tasks

# Multi-task Learning: GMMB and GMWB



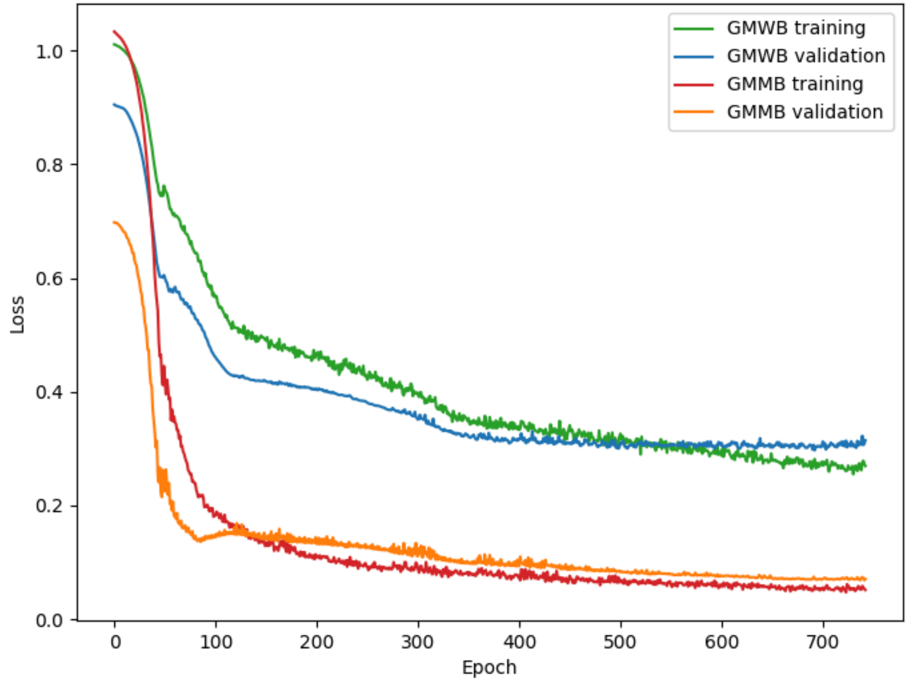




*Figure 5: Multi-task learning of GMMB and GMWB*

- *Left: total MSE; right: individual MSE*

# Multi-task Learning: GMMB and GMWB

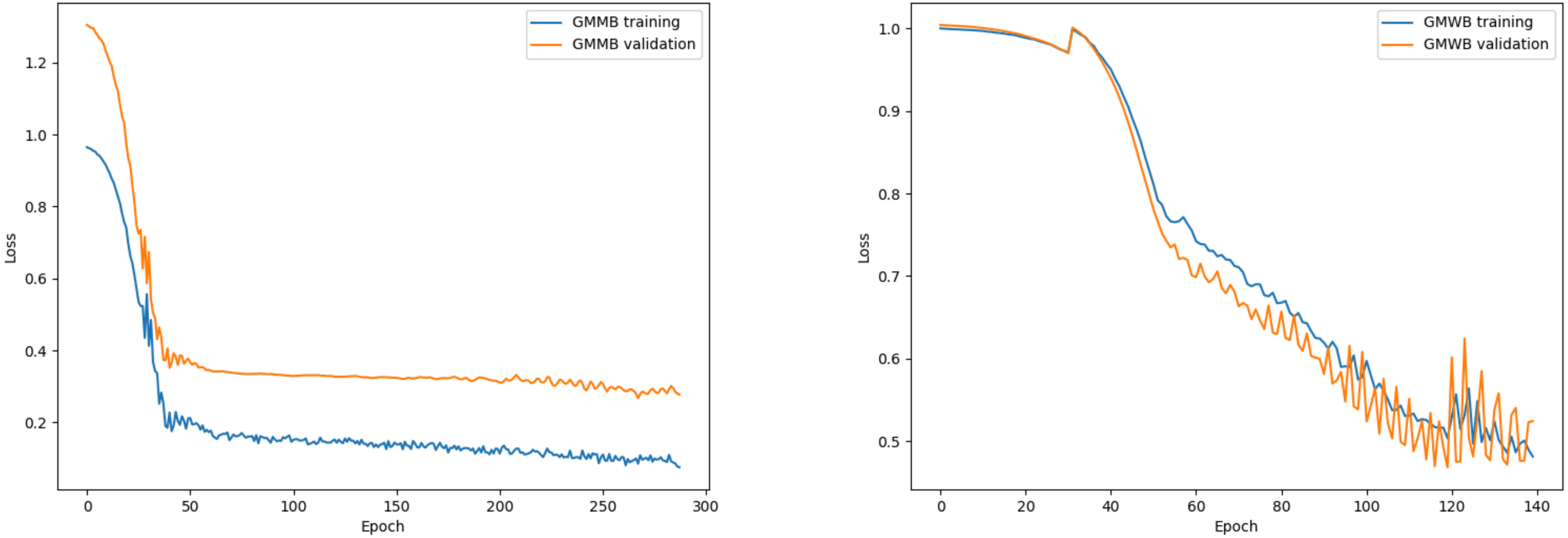



*Figure 6: Training without multi-task learning*

- *Left: GMMB; right: GMWB*

## Challenges & Limitations

- Determining optimal layer freezing strategy
- Handling significantly different contract structures
- Quantifying uncertainty in transfer learning predictions
- Regulatory acceptance of black-box approaches

## Conclusions

- Transfer learning significantly improves metamodeling for VA contracts:
    - Faster training convergence
    - Better prediction accuracy
    - Reduced computational requirements
- Enables more frequent risk assessments and faster decision-making

## Future Directions

- Incorporating domain knowledge into transfer process
- Extension to other insurance and financial products
- Multi-task learning with more than two tasks