The writings in Chapter 3 can be confusing to some readers due to omission of certain phrases and/or sentences.

QUESTION 1

For instance, on page 73 of your thesis, you wrote:

> At any time $t = 1, \ldots, T$, the insurer's liability in a VA contract is the present value of all payments, net of the fee income. For example, suppose that the per-period risk-free rate is r, then the insurer's time-t liability for a GMMB contract is $V_t = e^{-r(T-t)} \cdot (G_T - F_T)^+ - \sum_{s=t+1}^{T} e^{-r(T-s)} F_s \eta_n$. Also, the insurer's time-t liability for a GMWB contract is $V_t = \sum_{s=t+1}^{T} e^{-r(T-s)}[(I_s - F_s)^+ - \eta_n F_s]$.

Let's pick the GMWB case (so as not to be repetitive). The insurer's liability in this contract is the PV of withdrawal benefits paid following depletion of sub-account F_t , which is offset by PV of net management fees viewed as "income". As a result, the PV of unhedged liability of GMWB contract at *t* is given by V_t as written down in the last part of the above cited paragraph. *Is this what you mean?*

Given the dynamic of $X_t = (S_t, F_t, G_t)$, this GMWB liability is path dependent in the sense that its value depends on the entire path of $\{X_{t+h}\}_{h=0}^{T}$.

Then in sub-section 3.2.3 of your thesis, you discuss Dynamic Hedging for Variable Annuities.

The question is: how does a dynamic delta hedging apply to, say, the GMWB contract?

The insurer constructs and maintains (by periodic rebalancing) a hedge portfolio at time $t = 0, \ldots, T-1$ to offset the delta of the GMWB's *future* liabilities. Similar to delta hedging the GMMB, the time t hedge portfolio consists of Δ_t units of underlying stock S_t, where Δ_t captures sensitivity of the GMWB's future liability beyond time t with respect to the time t stock price S_t.

This takes us back to sub-section 3.2.2 on page 73 again to your equation (3.2) (so, the sequence of presentation in the thesis alone can cause some confusion to readers).

In this part, you state in your thesis:

Given the time t state, $X_t = (S_t, F_t, G_t)$, following Cathcart et al. (2015) the sensitivity of Vt with respect to S_t can be estimated by a pathwise estimator (Glasserman, 2004), which is equation (3.2).

There are several clarifications that can be made:

(1) I think that you mean to say:

Given an outer scenario at t, $X_t = (S_t, F_t, G_t)$, the pathwise estimator of Δ_t can be estimated from a *single* inner simulation path as:

(2) I think that you also mean to write: ..... $\left[1\{\tilde{I}_{t,s} > \tilde{F}_{t,s}\}.\left(\frac{\partial \tilde{I}_{t,s}}{\partial \tilde{S}_t} - \frac{\partial \tilde{F}_{t,s}}{\partial \tilde{S}_t}\right) - \eta_n \frac{\partial \tilde{F}_{t,s}}{\partial \tilde{S}_t}\right]$

where $S_t = (S_{t,t+1}, \dots, S_{t,T})$, $F_t = (F_{t,t+1}, \dots, F_{t,T})$, and $G_t = (G_{t,t+1}, \dots, G_{t,T})$ are the *inner* simulation sample paths for the stock price, sub-account value, and guarantee value, respectively.

(3) At each time step, $s = t+1, \dots, T$, in the inner sample paths, sensitivities can be calculated recursively as presented on page 73 just before subsection 3.2.3. A question may arise as to how the initial condition of the recursion gets to be determined. Here they are:

1. At *each* inner simulation, the sample path at *t* is initialized as:

$$(\tilde{X}_{t,t}) = (\tilde{S}_{t,t}, \tilde{F}_{t,t}, \tilde{G}_{t,t}) = (\tilde{S}_t, \tilde{F}_t, \tilde{G}_t) = (\tilde{X}_t)$$

2. *Before* withdrawal at *t*, $\tilde{F}_t$ is set to be *proportional* to $\tilde{S}_t$, such that:

$$\frac{\partial \tilde{F}_{t,t}}{\partial \tilde{S}_t} = \frac{\partial \tilde{F}_t}{\partial \tilde{S}_t} = \frac{\tilde{F}_t}{\tilde{S}_t}$$

3. *Before* withdrawal at t, $\tilde{G}_t$ and $\tilde{I}_t$ are *fixed as constants,* such that:

$$\frac{\partial \tilde{G}_{t,t}}{\partial \tilde{S}_t} = \frac{\partial \tilde{G}_t}{\partial \tilde{S}_t} = 0 \ and \ \frac{\partial \tilde{I}_{t,t}}{\partial \tilde{S}_t} = \frac{\partial \tilde{I}_t}{\partial \tilde{S}_t} = 0$$

This is summarized by your second statement at the bottom of page 73 just before subsection 3.2.3.

Here is an important point to make. No calculation is needed for $F_t \leq I_t$. In this case, the sub-account is depleted at/before t, so GMWB's future liabilities beyond t no longer depend on the stock. So, no hedging is necessary. We can skip the recursive calculations and set $\Delta_t = 0$ without a need for simulation (or set inner simulation output to 0).

Let's move on to your presentation in subsection 3.2.3 on Dynamic Hedging for Variable Annuities in your thesis following the scheme presented in Figure 3.1. You start with the following paragraph:

> Consider a VA contract which delta hedge portfolio at any time t, $t = 0, 1, \dots, T-1$, consists of Δ_t units in the underlying stock and B_t amount of a risk-free zero-coupon bond maturing at time T. The value of the hedge portfolio at time $(t-1)$ is:
>
> $$H_{t-1} = \Delta_{t-1} S_{t-1} + B_{t-1},$$

How do you get to this equation? Perhaps, a little bit of information about the set-up is useful for the purpose of readability. You assume that the time of expiration of VA is T months and VA's sub-account is invested in a stock index, and the hedging portfolio is rebalanced every month. Then at $t \leq T$, a dynamic equation can be specified for the sub-account value F_t, in the absence of withdrawal, as:

$$F_t = F_{t-1} \cdot \frac{S_t}{S_{t-1}} \cdot (1 - \eta_g)$$

where $\eta_g > \eta_n$ is the gross rate of management fee that is deducted from the fund value at each period.

If we assume that the delta hedge for the VA embedded option consists of Δ_t units in the underlying stock and an amount of B_t in a risk-free zero-coupon bond maturing at T. Then the value of the delta hedge portfolio at *T*-1 is given by the equation cited from your thesis. At *t*, the value of this hedge changes to the equation stated on page 74 in your thesis:

$$H_t^{bf} = \Delta_{t-1} S_t + B_{t-1} e^r.$$

where $e^{-rt} = \frac{D_{t-1}}{D_t}$ and D_t is the PV at t=0 of $1 payable at *t*, discounted at the risk-free rate. The equation as stated is incorrect as it should be e^{-rt}.

This is the hedge brought forward at *t* (assuming no rebalancing between *t*-1 and *t*). Therefore, the hedging error, which is the cash flow incurred by the insurer, is simply the difference between the cost of the hedge at *t* and the value of the hedge brought forward.

Part of the P&L of VA contract includes the cost to set up the initial hedge portfolio, the periodic hedging gains and losses due to rebalancing at each *t*=1,...,*T*, the unwinding of hedge, the payment of guaranteed benefit, and the management fee income. The PV of these cash follows, discounted at risk-free rate, is the insurer's overall gain/loss from VA and this is the loss random variable to which a risk measure is applied, giving rise to the equation (3.3) stated in your thesis:

$$HE_t = H_t - H_t^{bf}, \quad t = 1, \ldots, T-1. \tag{3.3}$$

At *t*=0, the insurer's overall gain/loss L of VA contract and its dynamic hedging program can be stated as:

$$L = H_0 + \sum_{t=1}^{t-1} HE_t e^{-rt} + H_T^{bf} e^{-rT} + V_0$$

$$= H_0 + \sum_{t=1}^{t-1} e^{-rt}(H_t + H_t^{bf}) - H_T^{bf} e^{-rT} + V_0$$

$$= B_0 + S_0 V_0 + \sum_{t=1}^{t-1} e^{-rt}\left(B_t + S_t\Delta_t - B_{t-1}\frac{e^{-r(t-1)}}{e^{-rt}} - S_t\Delta_{t-1}\right) - e^{-rT} S_T \Delta_{T-1} + V_0$$

which leads to equation (3.4) stated in your thesis:

$$L = H_0 + \sum_{t=1}^{T-1} e^{-rt} HE_t - e^{-rT} H_T^{bf} + V_0 = \sum_{t=0}^{T-1} \Delta_t (e^{-rt} S_t - e^{-r(t+1)} S_{t+1}) + V_0, \qquad (3.4)$$

where the second equality holds by a telescopic sum simplification of $e^{-rt}B_t$, $t = 0, \ldots, T-1$.

Note that V_0 is a discounted payoff of VA contract at t=0, net of fee income in the absence of hedging. The terms $H_0 + \sum_{t=1}^{T-1} HE_t e^{-rt} + H_T^{bf} e^{-rT}$ is the gain/loss from hedging, H_0 is the initial cost of setting up the hedge, $\sum_{t=1}^{T-1} HE_t e^{-rt}$ is the gain/loss from all rebalancing trades in dynamic hedging, and $H_T^{bf} e^{-rT}$ is the proceeds from unwinding hedging.

Equation (3.4) shows that in bond holdings of hedging portfolio, all intermediate bond transactions cancel out since the interest rate at which bond value accumulates is the same as rate at which profit and loss from bond transactions are discounted. In the stock holdings of the hedging portfolio, the gain/loss arises from the initial set-up of stock future holdings and the profit and loss from each stock future trading.

In (3.4), Δ_t and V_0 are determined by using a risk-neutral measure $\mathbb{Q}$, while the distribution of L, which is the tail risk measure of the loss random variable, is determined under the real-world probability measure $\mathbb{P}$. If Δ_t and V_0 cannot be calculated analytically, a nested simulation, where the outer scenarios are generated under the $\mathbb{P}$ measures to estimate the loss distribution, and the inner sample paths are generated under the $\mathbb{Q}$ measure.

QUESTION 2

Estimating tail risk measures for portfolios of complex VAs usually requires nested simulation. In the nested simulation, the outer simulation stage involves projecting scenarios of key risk factors under a P measure, while the inner simulations are used to value pay-offs under guarantees of varying complexity, under a Q measure.

For insurers, large-scale nested simulations for assessing VA losses are generally time intensive; i.e., a large number of outer level simulations are needed to estimate extreme tails for risk-measure calculations, and a large number of inner simulations are also needed at each time step, because embedded options are out of money. In addition, the required calculations need to be repeated for each VA policy, or cluster of policies.

Consequently, insurers are interested in techniques for nested simulation models that produce accurate results within the confine of a finite computational budget.

There are two approaches that can be used to improve the efficiency of inner simulations. The first one uses a so-called proxy model to replace the inner simulation step, and the second uses a dynamic, non-uniform allocation of the inner simulation budget. Some work uses a combination of the two approaches.

Proxy models are tractable analytic functions or flexible empirical functions that are used to replace the inner simulation stage of a nested simulation. Empirical proxies may be intrinsic or extrinsic to the simulation process. Extrinsic proxy functions are selected to be close to the inner simulation values and therefore require detailed information about the payoffs that are evaluated in the inner simulation step. Empirical proxies are constructed by using an initial pilot simulation to develop factors or functionals that can subsequently be used in place of the inner simulation. A prominent empirical proxy method discussed in research is the least squares Monte Carlo (LSMC) method.

Chapter 3 sets out to introduce a proxy model to reduce computational requirement of inner simulations by replacing the inner simulation step of a nested simulation. This proxy model is chosen to be a flexible empirical function, which is drawn from ML arsenals. This raises the following questions.

1. How does an ML algorithm (say, LSTM), serving as a proxy model, actually “replace” the inner simulation stage of a nested simulation? The answer to this question requires that you explicitly map the inner simulation stage of a nested simulation to the LSTM serving as a proxy model by describing the inner structure of the LSTM. This has an additional benefit of “demystifying” the blackboxness of ML algorithms (Recall that in STAT 974, which you took with me as a master student, we discussed how a GARCH(1,1) model is mapped into an LSTM architecture. Similar steps can be undertaken in your thesis to show in what “precise” ways LSTM, as a proxy model, replaces the inner simulation stage of a nested simulation).

2. Do the accuracy in the performance of proxy models via ML algorithms stay the same or do they deteriorate over time? If the latter is the case, do you need to backtest them regularly and will this backtest option always be available to you regardless of the length of contracts? This question btw leads to the possibility of entertaining the incorporation of transfer learning to LSTM or RL.

I believe that “tiny” additions of this sort to your thesis can go a few distances in elevating some (may not be all) confusion to some readers.