Problem Set 7

Xinxin Zhang

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Economic Questions

I want to use dynamic model to study consumption and portfolio choices for individuals.

Model description

Individual has an initial wealth, A. The individual receives income, y, and makes consumption, c, every period. He can invest in two types of assets every period. One is risk-free asset with R_f as interest rate and another one is risky asset with R_m as interest rate. The rate of time preference is β. The individual wants to maximize his utility for infinite horizon.
Individual only know his current wealth (A),current income (y) and historical interest rate (R_{-1}) at the start of each period. He needs to predict current interest rate R based on historical information and makes decisions on consumption (c) and how much he invests in risky asset (w_m) for this period.

Objective function

The objective function for the individual is

$$max_{c_t,w_{m,t}} \sum_{t=0}^{\infty} E_t(\beta^t u(c_t)) \tag{1}$$

Bellman equation

The Bellman equation is

$$\begin{aligned} V(A, y, R_{-1}) = max_{c,w_m} u(c) + \beta E_{y',R|y,R_{-1}}(V(A', y', R)) \\ w_m >= 0 \\ w_m <= 1 \\ c > 0 \end{aligned} \tag{2}$$

State variable is (A,y,R_{-1}), where A and y are wealth and income at the start of each period, R_{-1} is a vector of past returns by assets.
Control variable is (c,w_m), where c is consumption and w_m is the percentage of wealth invested in risky asset during this period.
Transition equation is $A' = (A + y - c)(R_f + w_m(R_m - R_f)) = S * R_{t-1}$, where

S is saving and R_{t-1} is weighted average return of portfolio during this period.

Solve Model

There are two first order conditions. First FOC is

$$\begin{aligned}\frac{\partial V}{\partial c} &= u_c'(c) + \beta E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)\frac{\partial A'}{\partial c}) \\ &= u_c'(c) - \beta E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)(R_f + w_m(R_m - R_f)) \\ &= u_c'(c) - \beta R_f E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)) - \beta w_m E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)(R_m - R_f))) \\ &= 0\end{aligned} \tag{3}$$

$$u_c'(c) = \beta R_f E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)) + \beta w_m E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)(R_m - R_f))) \tag{4}$$

Second FOC is

$$\begin{aligned}\frac{\partial V}{\partial w_m} &= \beta E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)\frac{\partial A'}{\partial w_m}) \\ &= \beta E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)(A+y-c)(R_m - R_f)) \\ &= \beta(A+y-c)E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)(R_m - R_f)) \\ &= 0\end{aligned} \tag{5}$$

$$E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)(R_m - R_f)) = 0 \tag{6}$$

Applying equation (6) to equation (4), we can get

$$u_c'(c) = \beta R_f E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)) \tag{7}$$

From Euler equation

$$\frac{\partial V}{\partial A} = u_c'(c)\frac{\partial c}{\partial A} + \beta E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)\frac{dA'}{dA}) \tag{8}$$

$$\begin{aligned}\frac{dA'}{dA} =& (\frac{\partial A'}{\partial A} + \frac{\partial A'}{\partial w_m} * \frac{\partial w_m}{\partial A} + \frac{\partial A'}{\partial c} * \frac{\partial c}{\partial A}) \\ &= R_{t-1} + (A+y-c)(R_m - R_f)\frac{\partial w_m}{\partial A} - R_{t-1} * \frac{\partial c}{\partial A} \\ &= (A+y-c)(R_m - R_f)\frac{\partial w_m}{\partial A} + R_{t-1} * (1 - \frac{\partial c}{\partial A})\end{aligned} \tag{9}$$

Applying equation (9) to equation (8), we can derive

$$\begin{aligned}\frac{\partial V}{\partial A} &= u_c'(c)\frac{\partial c}{\partial A} \\ &+ \beta(A+y-c)\frac{\partial w_m}{\partial A} * E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)(R_m - R_f)) \\ &+ \beta E_{y',R|y,R_{-1}}(V_{A'}'(A',y',R)R_{t-1})(1 - \frac{\partial c}{\partial A})\end{aligned} \tag{10}$$

applying equation (6) and (7) to equation (10)

$$\begin{aligned}\frac{\partial V}{\partial A} &= u'_c(c)\frac{\partial c}{\partial A} + u'_c(c)(1 - \frac{\partial c}{\partial A}) \\ &= u'_c(c)\end{aligned} \tag{11}$$

Using two FOCs, equation (6) and equation (11) we can find solution for consumption (c) and portfolio choice (w_m).

Derive CAPM

Manipulate equation (6)

$$E_{y',R|y,R_{-1}}(V'_{A'}(A',y',R)(R_m - R_f)) = E_{y',R|y,R_{-1}}(V'_{A'}(A',y',R)R_m) - R_f E_{y',R|y,R_{-1}}(V'_{A'}(A',y',R)) = 0 \tag{12}$$

$$E_{y',R|y,R_{-1}}(V'_{A'}(A',y',R)R_m) = R_f E_{y',R|y,R_{-1}}(V'_{A'}(A',y',R)) \tag{13}$$

$$E_{y',R|y,R_{-1}}(V'_{A'}(A',y',R)R_m) = E_{y',R|y,R_{-1}}(V'_{A'}(A',y',R)) * E_{R|R_{-1}}(R_m) + cov(V'_{A'}(A',y',R), R_m) \tag{14}$$

Applying equation (11) to equation (13)

$$R_f(u'_{c'}(c')) = E_{y',R|y,R_{-1}}(u'_{c'}) * E_{R|R_{-1}} R_m + cov(u'_{c'}(c'), R_m) \tag{15}$$

We can derive CAPM from equation (14)

$$E_{R|R_{-1}}(R_m) = R_f - \frac{cov(u'_{c'}(c'), R_m)}{E_{y',R|y,R_{-1}}(u'_{c'}(c')} \tag{16}$$

Summary

To summarize, I want to study the dynamic consumption and portfolio choices for individuals. Using two FOCs, equation (6) and (11), we can find solution for consumption (c) and percentage of investment in risky asset (w_m) every period. We can drive CAPM from this dynamic model.