Parallel Two-dimensional Fast Fourier Transform

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Sequential 2D Discrete Fourier Transform (DFT)

Suppose the 2D signals are stored in a $N_1 \times N_2$ matrix $\mathbf{x}$. The 2D DFT of $\mathbf{x}$ is defined as

$$X_{r_1,r_2} = \sum_{l_1=0}^{N_1-1} \sum_{l_2=0}^{N_2-1} x_{l_1,l_2} \exp(-\frac{2\pi i}{N_1} r_1 l_1) \exp(-\frac{2\pi i}{N_2} r_2 l_2)$$

$$r_1 = 0, 1, \cdots, N_1 - 1, \text{ and } r_2 = 0, 1, \cdots, N_2 - 1$$

Total computational cost: $O(N_1^2 N_2^2)$.

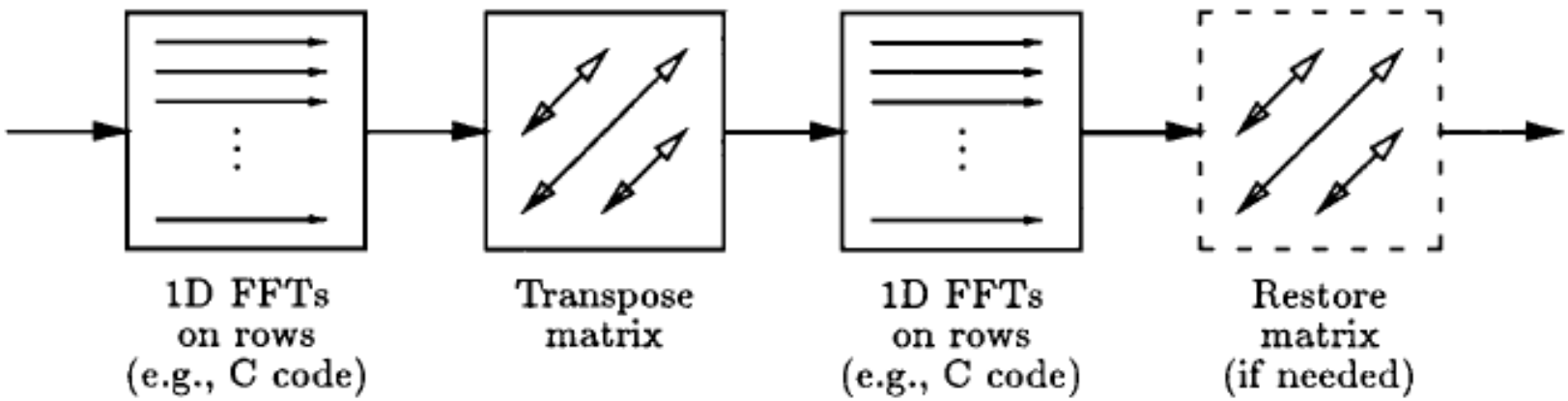
This can be reduced significantly by separating the 2D-DFT into a series of 1D-DFTs, which can each be implemented using a fast 1D-FFT algorithm.

Sequential 2D FFT

$$
\begin{aligned}
X_{r_1,r_2} &= \sum_{l_1=0}^{N_1-1}\sum_{l_2=0}^{N_2-1} x_{l_1,l_2}\exp(-\frac{2\pi i}{N_1}r_1 l_1)\exp(-\frac{2\pi i}{N_2}r_2 l_2) \\
&= \sum_{l_1=0}^{N_1-1}\exp(-\frac{2\pi i}{N_1}r_1 l_1)\left(\sum_{l_2=0}^{N_2-1} x_{l_1,l_2}\exp(-\frac{2\pi i}{N_2}r_2 l_2)\right) \\
&= \sum_{l_1=0}^{N_1-1}\exp(-\frac{2\pi i}{N_1}r_1 l_1)\tilde{X}_{l_1,r_2} \\
&= \sum_{l_1=0}^{N_1-1}\tilde{X}_{l_1,r_2}\exp(-\frac{2\pi i}{N_1}r_1 l_1)
\end{aligned}
$$

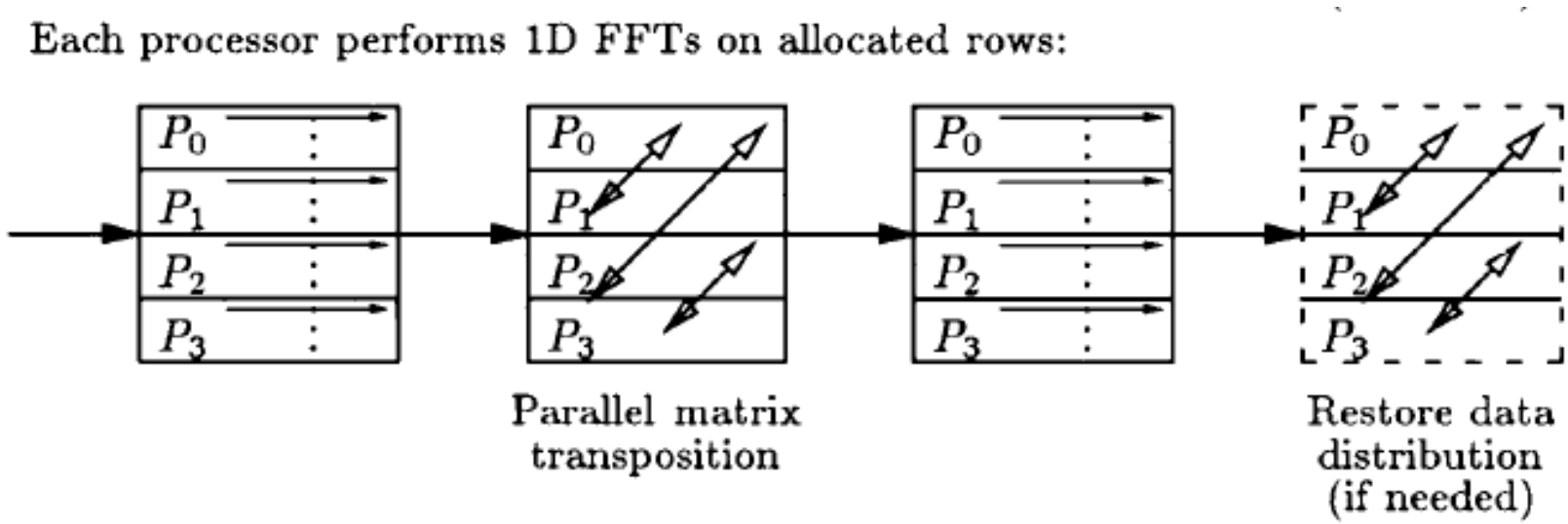
Do 1D FFTs on the N_1 rows (of length N_2), and then do 1D FFT2 on the N_2 columns (of length N_1). Total computational cost is $O(N_1 N_2 log_2(N_1 N_2))$.

- Step 1: 1D FFTs on the rows (locally).
- Step 2: matrix transpose.
- Step 3: 1D FFTs on the rows again (locally).
- Step 4: matix transpose.



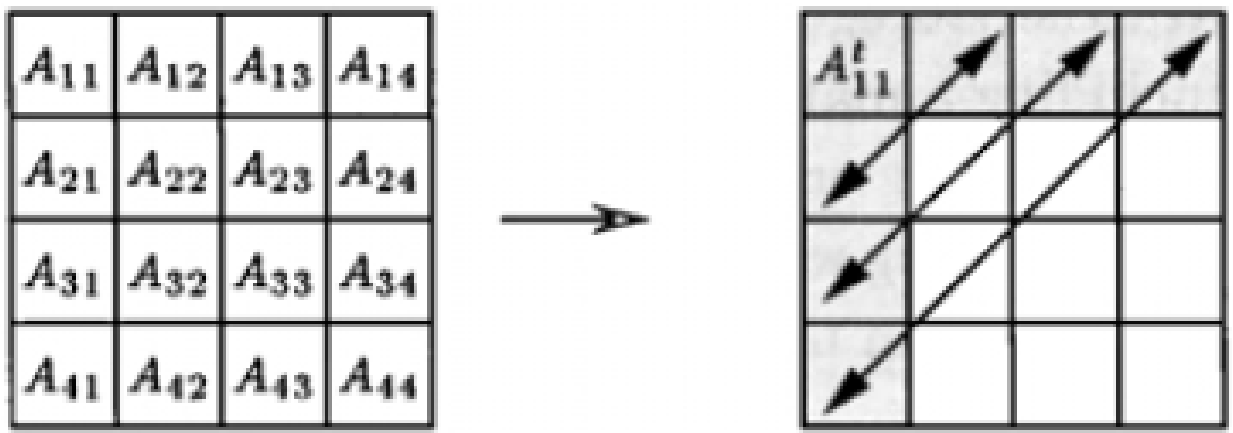
Parallel 2D FFT

1D FFT can be computed locally.
Communication is only needed during the transpose step.



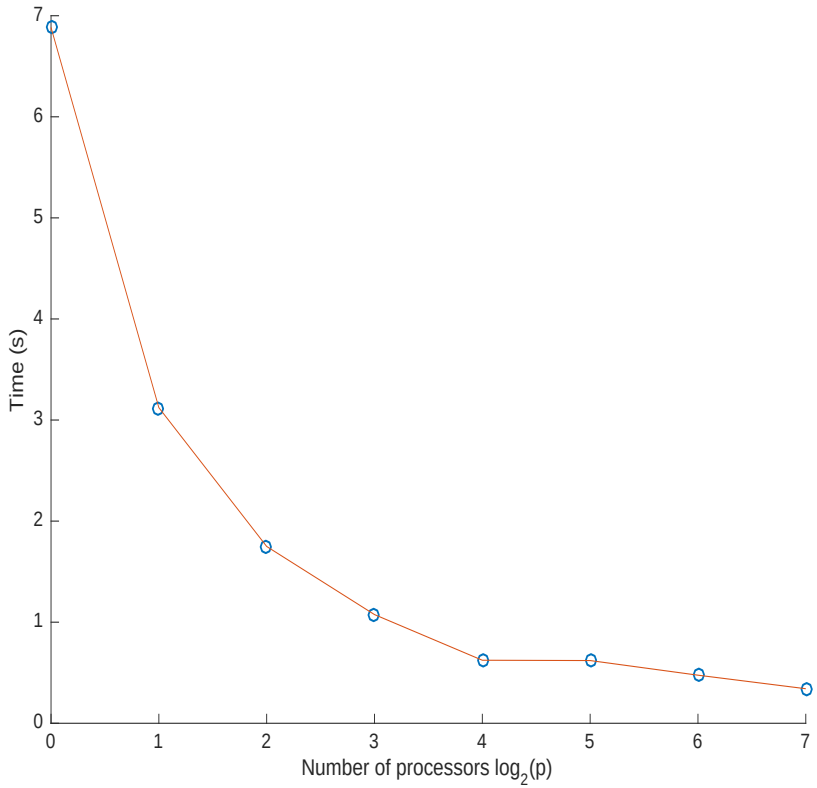
Implementation

- Use FFTW library to do the 1D FFT.
- The hardest part of the multi-dimensional distributed FFT is the transpose.
- Imagine there are blocks in each processor and use MPI Alltoall function to exchange blocks of data with every other processor.
- Then do the transpose for each block of data.

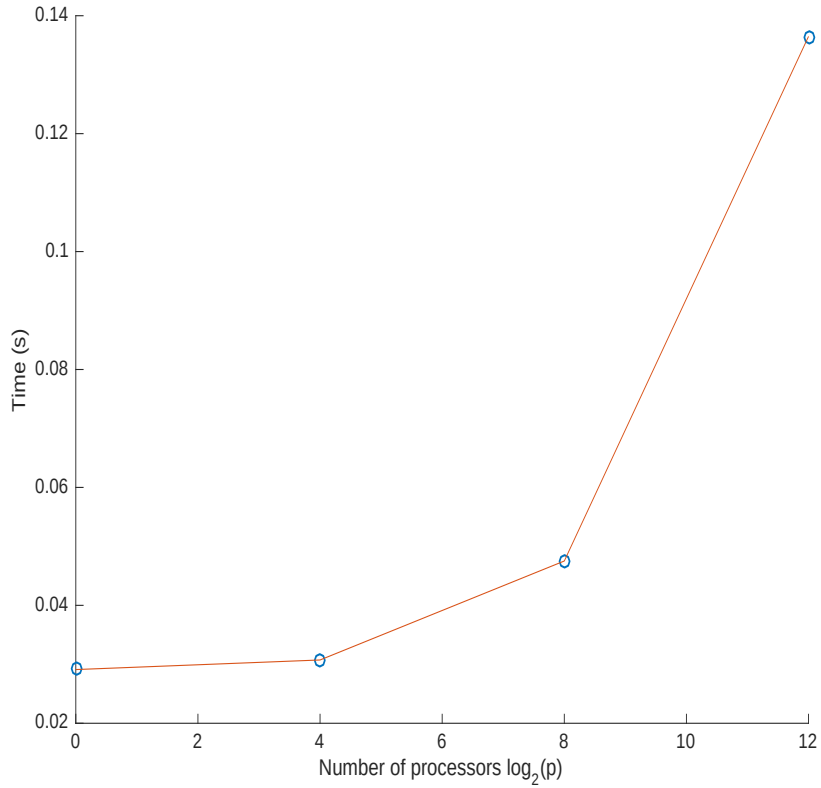


Scalability

- Strong scalability: square matrix N=8192, number of processors: 2^j, $j = 0, \cdots, 10$.
- Weak scalability: keep the number of data in each processor fixed 512×512.



(a) Strong scalability



(b) Weak scalability

- Strong and weak scalability are clear when p is small.
- Communication in the transpose step dominates when p is large.
- When p is too large, running time may increase.
- 2D version FFTW takes 10s to compute this problem when N=8192. This method is faster than 2d_fftw when p=1.
- Efficient transpose algorithm is needed to improve the performance.

For Further Reading I

Chu, Eleanor and Alan George
Inside the FFT black box: serial and parallel fast Fourier transform algorithms..
CRC Press, 1999.

Matteo Frigo and Steven G. Johnson
FFTW Manual