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Discrete Mathematics 152 (1996) 299–302

DISCRETE MATHEMATICS

Communication

A Hajós-like theorem for list coloring

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Received 22 November 1995
Communicated by C. Benzaken

Abstract

We give an analogue of Hajós's theorem for the list-chromatic number, emphasizing the role of complete bipartite graphs.

We consider only finite, undirected graphs, without loops. Given a graph $G = (V, E)$, a k-coloring of the vertices of G is a mapping $c: V \to \{1, 2, \dots, k\}$ such that for every edge xy of G we have $c(x) \neq c(y)$. The graph G is called k-colorable if it admits a k-coloring, and the chromatic number of G is the smallest integer k such that G is k-colorable. Vizing [4], as well as Erdős et al. [1] introduced a variant of the coloring problem as follows. Suppose that each vertex v is assigned a list $L(v)$ of possible colors; we then want to find a vertex-coloring c such that $c(v) \in L(v)$ for all $v \in V$. In the case where such a c exists we will say that the graph G is L-colorable; we may also say that c is an L-coloring of G. Given an integer k, the graph G is called *k-choosable* if it is L-colorable for every assignment L that satisfies $|L(v)| \geqslant k$ for all $v \in V$. Finally, the *choice number or list-chromatic number* $Ch(G)$ of G is the smallest k such that G is k-choosable. Hajós [2] proved a classic theorem on the chromatic number of a graph. Here we give the analogue for the choice number. Hajós proposed a transformation of graphs based on three operations, with which all graphs that are not q-colorable can be built from the complete graph on $q + 1$ vertices. Call $\mathscr{G}_q$ the class of all graphs that are not q-colorable. This class $\mathscr{G}_q$ is closed under the three following operations, defined by Hajós.

(1) Add vertices or edges.

(2) Let G_1, G_2 be two vertex-disjoint graphs, and a_1b_1 and a_2b_2 be edges in G_1 and G_2, respectively. Make a graph G from $G_1 \cup G_2$ by deleting the edge a_ib_i from G_i (for

SSDI 0012-365X(95)00350-9

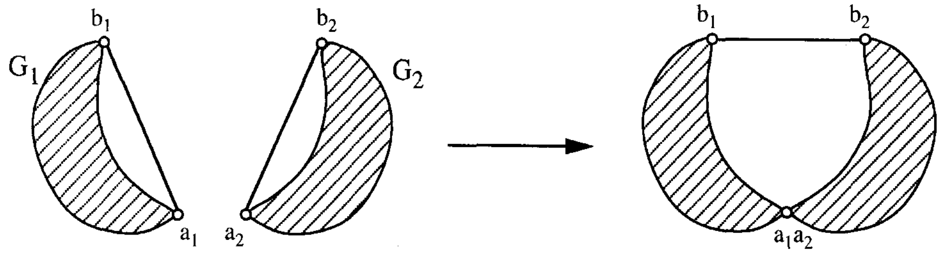

Fig. 1

$i = 1, 2$), identifying a_1 and a_2 (the resulting vertex is called a_1a_2), and adding the edge b_1b_2 (see Fig. 1).

(3) Identify two non-adjacent vertices.

Theorem 1 (Hajós [2]). *Every graph in $\mathcal{G}_q$ can be obtained by Operations* (1)–(3) *from the complete graph K_{q+1}.*

Now we call $\mathcal{G}'_q$ the class of graphs that are not q-choosable. Clearly, $\mathcal{G}'_q$ is closed under Operation (1). Lemma 1 shows that the same holds for Operation (2). However, the same does not hold for Operation (3); for example, observe that the complete bipartite graph $K_{2,4}$ is not 2-choosable while identifying any two non-adjacent vertices in it yields a 2-choosable graph. This leads us to a revision of Operation (3) as follows:

(3′) If G is not L-colorable for some L with $|L(x)| \geqslant q$ for each $x \in V(G)$, then identify two non-adjacent vertices u and v such that $L(u) = L(v)$.

Clearly, $\mathcal{G}'_q$ is closed under Operation (3′).

It is not hard to see that K_{q,q^q} is not q-choosable. Thus, for every integer q there exist integers α, β such that the complete bipartite graph $K_{\alpha,\beta}$ is not q-choosable. For a fixed integer q, let α, β be any such pair.

Theorem 2. *Every graph in $\mathcal{G}'_q$ can be obtained by Operations* (1), (2) *and* (3′) *from any complete bipartite graph $K_{\alpha,\beta}$ in $\mathcal{G}'_q$.*

Lemma 1. *$\mathcal{G}'_q$ is closed under Operation* (2).

Proof. We use the same notation as in the description of Operation (2). For $i = 1, 2$, since G_i is not q-choosable, there exists an assignment L_i with $|L_i(v)| = q$ for all $v \in V(G_i)$ and such that G_i is not L_i-colorable. We may assume that $L_1(a_1) = L_2(a_2)$ by a suitable permutation of the colors. Now we make an assignment L on $V(G)$ by setting $L(v) = L_i(v)$ in $v \in G_i$. We claim that G is not L-colorable. Indeed, suppose that there is an L-coloring c of G. Then, since $c(b_1) \neq c(b_2)$ and $c(a_1 a_2) = c(a_1) = c(a_2)$, we have either $c(b_1) \neq c(a_1)$ or $c(b_2) \neq c(a_2)$ and so c is either an L_1-coloring of G_1 or an L_2-coloring of G_2, a contradiction. Since $|L(v)| = q$ for all $v \in V(G)$, this shows that G is not q-choosable. $\square$

In order to prove Theorem 2, it suffices to prove the following two lemmas.

Lemma 2. *Every graph is $\mathscr{G}'_q$ can be obtained by Operations* (1), (2) *and* (3′) *from a family of complete multipartite graphs in $\mathscr{G}'_q$.*

Lemma 3. *Every complete multipartite graph G in $\mathscr{G}'_q$ can be obtained by Operations* (1) *and* (3′) *from any $K_{\alpha,\beta}$ in $\mathscr{G}'_q$.*

Proof of Lemma 2. Assume that there exists a counterexample $G = (V, E)$ to Lemma 2, that is, G is not q-choosable but cannot be obtained by Operations (1), (2) and (3′) from complete multipartite graphs. We may assume that E is maximal for this property, or else add edges as long as this still yields a counterexample. Since G is not q-choosable, there exists an assignment L with $|L(v)| = q$ for all $v \in V(G)$ such that G is not L-colorable. Since G is not complete multipartite, the relation of non-adjacency is not transitive, i.e., G contains three vertices x, y, z such that xy, $yz \notin E$ and $xz \in E$. Let G'_1, G'_2 be two copies of G (where the images of any vertex v will be denoted v_1 and v_2 respectively), and define $G_1 = G'_1 + x_1y_1$ and $G_2 = G'_2 + y_2z_2$. Clearly, G_1, G_2 are in $\mathscr{G}'_q$; actually each of G_1, G_2 is not L-colorable. By the maximality of E, both G_1, G_2 can be obtained from complete multipartite graphs by Operations (1), (2) and (3′). We show how to obtain G from G_1, G_2 by Operations (2) and (3′); this contradiction will complete the proof. First, make a graph G_0 by applying Operation (2) to the graphs G_1, G_2 and the edges x_1y_1 and z_2y_2 where y_1 and y_2 are the vertices to be identified. Define an assignment L_0 on $V(G_0)$ by setting $L_0(v_i) = L(v)$ for each $v \in V(G)$ and each $i = 1, 2$. As in the proof of Lemma 1, G_0 is not L_0-colorable. Finally, since $L(v_1) = L(v_2)$ for each $v \in V(G) - \{y\}$, we can identify the two vertices of every such pair; this yields G by Operation (3′). □

Proof of Lemma 3. Let $G = K_{r_1, \ldots, r_t}$ be a complete multipartite graph in $\mathscr{G}'_q$, with stable sets $R_1, \ldots, R_t$. We may assume $q \geqslant 2$ and $t \geqslant 2$. Let L be an assignment on the vertices of G such that $|L(v)| \geqslant q$ for each $v \in V(G)$ and G is not L-colorable. Let $K_{\alpha,\beta}$ be any complete bipartite graph in $\mathscr{G}'_q$. First, apply Operation (1) by adding vertices and edges to $K_{\alpha,\beta}$ so as to obtain a graph $G' = K_{s_1, \ldots, s_t}$ with $s_i \geqslant r_i$ for $i = 1, \ldots, t$, calling S_i the stable set of size s_i. We may consider G as an induced subgraph of G', with $R_i \subseteq S_i$ for $i = 1, \ldots, t$. Now, extend the assignment L to an assignment L' on the vertices of G' by picking an arbitrary v_i in each R_i and setting $L'(v) = L(v_i)$ for each $v \in S_i - R_i$ and for each i. Clearly, G' is not L'-colorable. Then we can apply Operation (3′) by identifying all vertices of $S_i - R_i$ with v_i. This yields G. □

Lemma 3 shows in particular that all complete bipartite graphs in $\mathscr{G}'_q$ are 'equivalent' under Operations (1), (2) and (3′). Let $\mathscr{B}'_q$ be the set of complete bipartite graphs in $\mathscr{G}'_q$. Note that the class $\mathscr{B}'_q$ ordered by subgraph inclusion may contain several minimal elements; for example, $K_{3,3}$ and $K_{2,4}$ both are minimal elements in $\mathscr{B}'_2$ (and they are the only ones).

Theorem 2 shows that, for fixed q, the minimal elements in the class $\mathscr{B}'_q$ form the basis for the non-q-choosability. However, determining the choice number of a complete bipartite graph seems to be a difficult problem. In fact, even the question of 3-choosability for $K_{p,q}$ is not completely solved. In [3] this problem was tackled and answered for most values of p, q. The remaining unanswered cases are: is $K_{5,q}$ 3-choosable for $9 \leqslant q \leqslant 14$; is $K_{6,q}$ 3-choosable for $6 \leqslant q \leqslant 10$?

Acknowledgements

The author is grateful to Frédéric Maffray for helpful comments.

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