Intuitive Understanding of Newton-Raphson Method

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Abstract

Newton-Raphson method is a classical and powerful algorithm for numerically determining the root of some equations. This post will introduce some simple and intuitive examples for showing how does the Newton-Raphson method work for finding the root of equations.

1 Start with a Quadratic Equation

Quadratic equation is one of the most classical equation in math, which takes the form of $ax^2 + bx + c = 0$. If we have a quadratic equation $x^2 - 4 = 0$, then as we know, the positive solution is $x = 2$.

For graphical convenience, we define a function as $f(x) = x^2 - 4$. The derivative of the function is $f'(x) = 2x$.

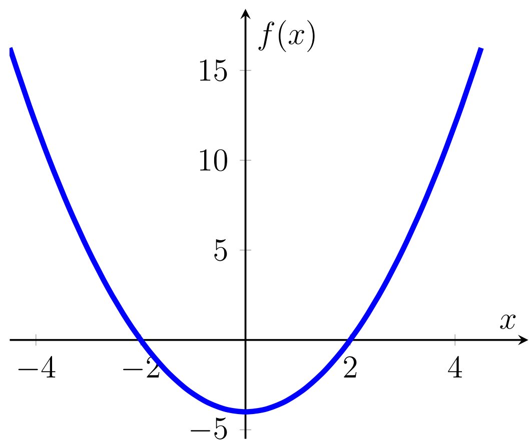

Figure 1: $f(x) = x^2 - 4$ (blue curve).

If we use $x_0 = 4$ as starting point, then $(4, 12)$ is the point of intersection.

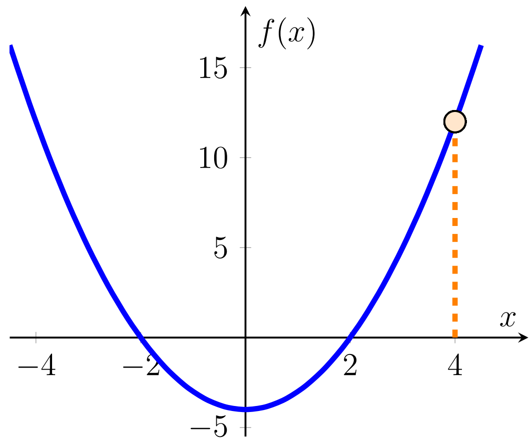

Figure 2: $f(x) = x^2 - 4$ (blue curve); $x = 4$ (dashed orange line); $(4, 12)$ (orange dot).

For the point $(4, 12)$, the tangent line is $f(x) = 8x - 20$.

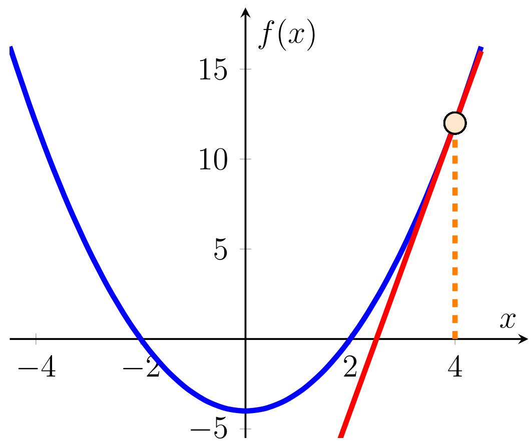

Figure 3: $f(x) = x^2 - 4$ (blue curve); $x = 4$ (dashed orange line); $(4, 12)$ (orange dot); $y = 8x - 20$ (red line).

The point of intersection of tangent line and x axis is $(\frac{5}{2}, 0)$. If we use $x_1 = \frac{5}{2}$ as a new point, then $(\frac{5}{2}, \frac{9}{4})$ is the point of intersection.

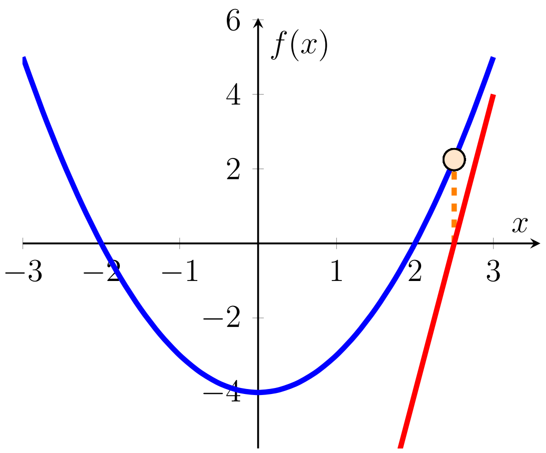

Figure 4: $f(x) = x^2 - 4$ (blue curve); $x = \frac{5}{2}$ (dashed orange line); $(\frac{5}{2}, \frac{9}{4})$ (orange dot); $y = 8x - 20$ (red line).

For the point $(\frac{5}{2}, \frac{9}{4})$, the tangent line is $f(x) = 5x - \frac{41}{4}$.

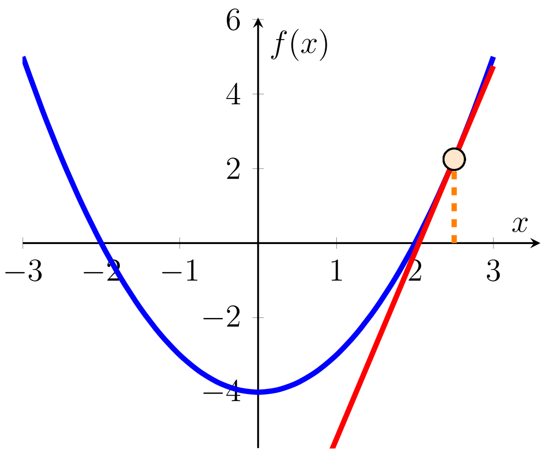

Figure 5: $f(x) = x^2 - 4$ (blue curve); $x = \frac{5}{2}$ (dashed orange line); $(\frac{5}{2}, \frac{9}{4})$ (orange dot); $y = 5x - \frac{41}{4}$ (red line).

The point of intersection of tangent line and x axis is $(\frac{41}{20}, 0)$. As can be seen, the x coordinate $\frac{41}{20} = 2.05$ is very close to the solution $x = 2$.

2 Revisit Newton's Motivation

In the rather early stage, Newton tried to motivate the idea using equation $x^3 - 2x - 5 = 0$. As we know, the solution is very close to 2. Perhaps, we can try to use $x_0 = 2$ as starting point.

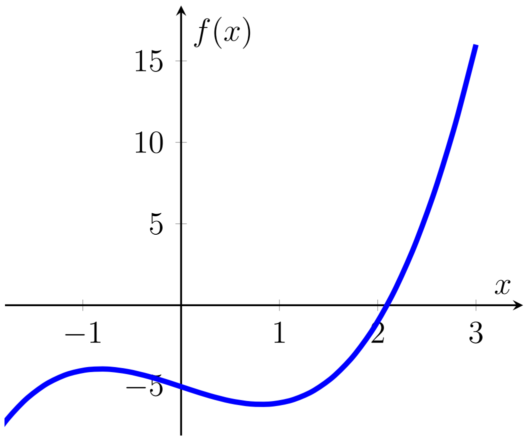

Figure 6: $f(x) = x^3 - 2x - 5$ (blue curve).

Recall that $f'(x) = 3x^2 - 2$.

1. Use $x_0 = 2$, then $(x_0, f(x_0)) = (2, -1)$ and $f'(x_0) = 10$.
2. Tangent line is $f(x) = 10x - 21$, and the point of intersection of tangent line and x axis is $(2.1, 0)$.
3. Use $x_1 = 2.1$, then $(x_1, f(x_1)) = (2.1, 0.061)$ and $f'(x_1) = 11.23$.
4. Tangent line is $f(x) = 11.23x - 23.522$, and the point of intersection of tangent line and x axis is $(2.094568, 0)$.

Note that $x_2 = 2.094568$ is very close to the final estimate achieved by Newton, i.e., $x = 2.09455148$.

3 Explain Newton-Raphson Iteration

For any equation $f(x) = 0$, the Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Suppose $x^3 - 2x^2 - 11x + 12 = 0$, how to use the Newton-Raphson iteration to obtain the solution with the starting point $x_0 = 2$?

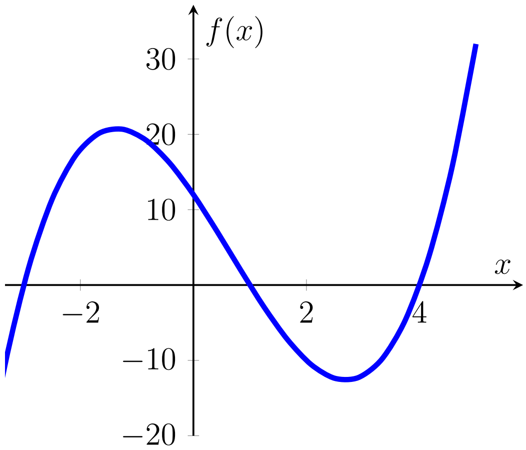

Figure 7: $f(x) = x^3 - 2x^2 - 11x + 12$ (blue curve).

Note that $f'(x) = 3x^2 - 4x - 11$.
If $x_0 = 2$, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 0.5714285714285714$$

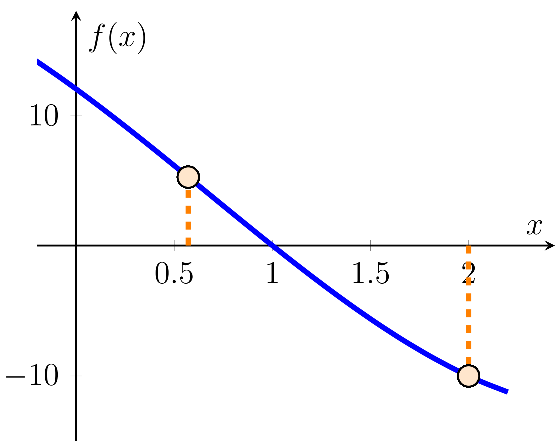

Figure 8: $f(x) = x^3 - 2x^2 - 11x + 12$ (blue curve); $x_0 = 2, x_1 = 0.57143$ (orange line).

As $x_1 = 0.5714285714285714$, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.9978678038379531$$

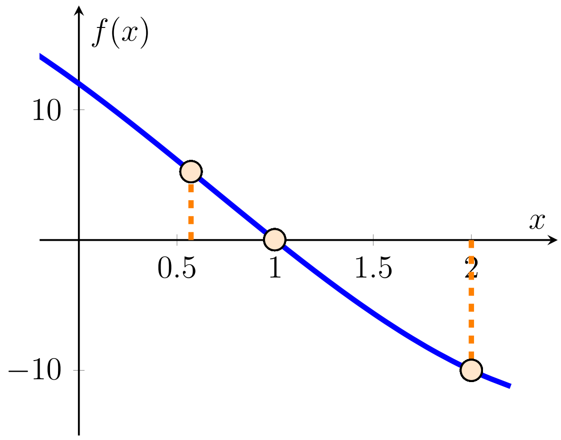

Figure 9: $f(x) = x^3 - 2x^2 - 11x + 12$ (blue curve); $x_0 = 2, x_1 = 0.57143, x_2 = 0.99787$ (orange line).