



# **Modeling Urban Traffic Data with Matrix and Tensor Approaches**

● 2024 INFORMS Annual Meeting

**Xinyu Chen**

Postdoctoral Associate, MIT

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# Urban Traffic Data

- Transport & mobility application scenarios



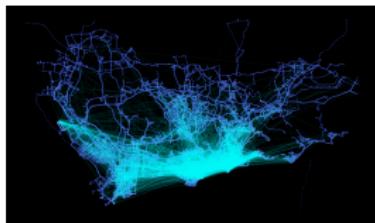
Highway (Portland)



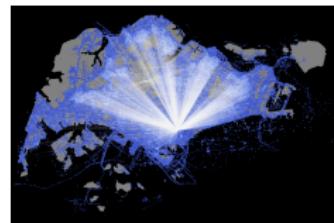
Uber movement (NYC)



Uber movement (Seattle)

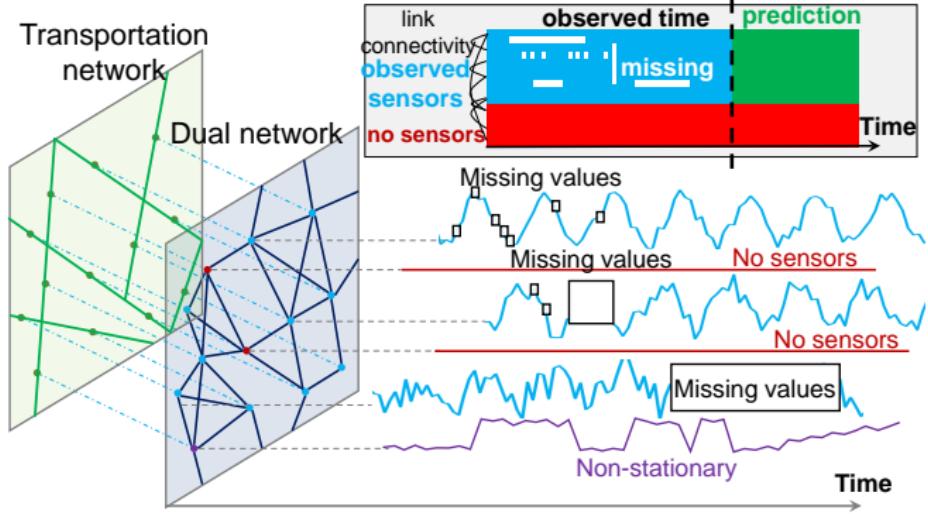


Taxi trajectory (Shenzhen)



Passenger flow (Singapore)

- Challenges: Sparsity, high-dimensionality, and multi-dimensionality, time-varying system



## Papers:

- X. Chen, Z. Cheng, H.Q. Cai, N. Saunier, L. Sun (2024). "Laplacian Convolutional Representation for Traffic Time Series Imputation". IEEE Transactions on Knowledge and Data Engineering, 36 (11): 6490–6502.
- X. Chen, L. Sun (2022). "Bayesian Temporal Factorization for Multidimensional Time Series Prediction". IEEE Transactions on Pattern Analysis and Machine Intelligence, 44 (9): 4659–4673.
- X. Chen, X.L. Zhao, C. Cheng (2024). "Forecasting Urban Traffic States with Sparse Data Using Hankel Temporal Matrix Factorization". INFORMS Journal on Computing. Early access.
- X. Chen, C. Zhang, X. Chen, N. Saunier, L. Sun (2024). "Discovering Dynamic Patterns from Spatiotemporal Data with Time-Varying Low-Rank Autoregression". IEEE Transactions on Knowledge and Data Engineering, 36 (2): 504–517.

ML  $\Rightarrow$  Imputation & Prediction & Pattern Discovery

# Laplacian Convolutional Representation for Traffic Time Series Imputation

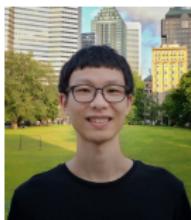
IEEE Transactions on Knowledge and Data Engineering, 2024

Code: <https://github.com/xinychen/transdim>

Blog: [https://spatiotemporal-data.github.io/posts/ts\\_conv](https://spatiotemporal-data.github.io/posts/ts_conv)



Xinyu Chen  
UdeM → MIT



Zhanhong Cheng  
McGill → UF



HanQin Cai  
UCF



Nicolas Saunier  
PolyMtl



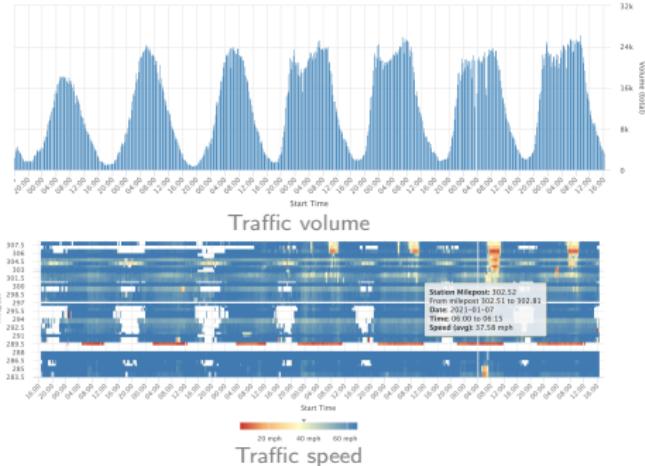
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McGill

# Traffic Flow Data

- Portland highway traffic data<sup>1</sup>



Highway network & sensor locations



- $X \in \mathbb{R}^{N \times T}$  with  $N$  spatial locations  $\times T$  time steps
- Traffic volume/speed shows strong spatial/temporal dependencies

<sup>1</sup><https://portal.its.pdx.edu/home>

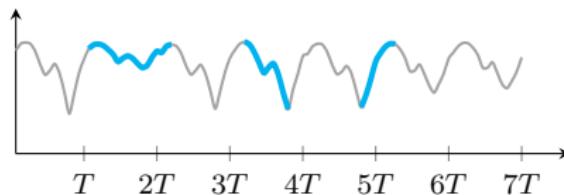
# Time Series Imputation

## Motivation: Traffic imputation

- Global trends (e.g., daily/weekly periodicity):



- Local trends (e.g., short-term time series trends):



How to characterize both global and local trends in sparse data?

# Local Trend Modeling

- Intuition of (circulant) Laplacian matrix

The diagram illustrates the intuition behind the (circulant) Laplacian matrix. On the left, an undirected and circulant graph is shown with five nodes labeled  $x_1$ ,  $x_2$ ,  $x_3$ ,  $x_4$ , and  $x_5$ . The nodes are arranged horizontally, and each node is connected to its immediate neighbors ( $x_{i-1}$  and  $x_{i+1}$ ). Below this graph is the label "Undirected and circulant graph". An arrow labeled "Modeling" points from the graph to the right, where the (Circulant) Laplacian matrix  $\mathbf{L}$  is given as a 5x5 matrix:

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

Below the matrix is the label "(Circulant) Laplacian matrix".

- Laplacian kernel:

$$\boldsymbol{\ell} \triangleq \underbrace{(2, -1, 0, 0, -1)}_{\text{first column of } \mathbf{L}}^\top$$

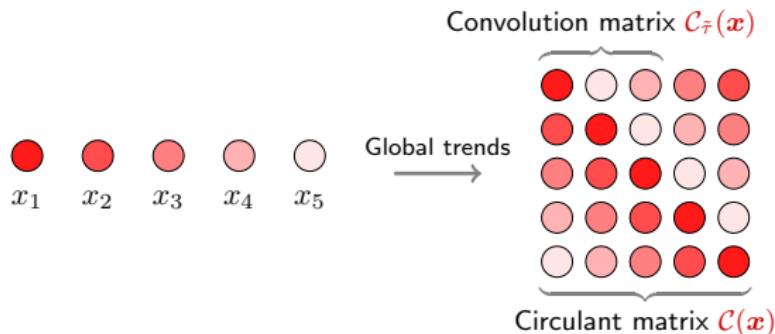
extending to the degree  $2\tau$  for  $\mathbf{x} \in \mathbb{R}^T$ .

- Temporal regularization:

$$\mathcal{R}(\mathbf{x}) = \underbrace{\frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2}_{\text{mat-vec mul.}} = \underbrace{\frac{1}{2} \|\boldsymbol{\ell} \star \mathbf{x}\|_2^2}_{\text{conv. } \star}$$

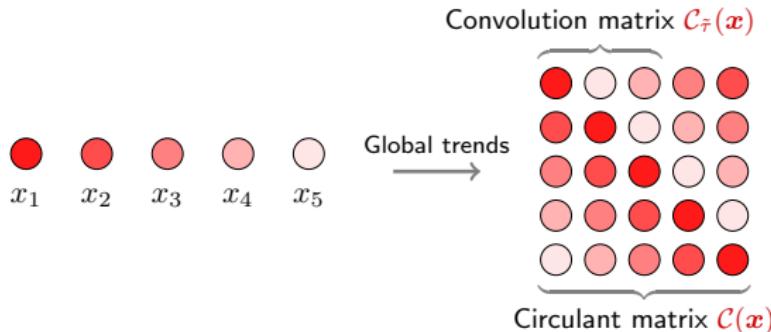
# Global Trend Modeling

Circulant matrix  $\mathcal{C}(\mathbf{x})$  vs. convolution matrix  $\mathcal{C}_{\tilde{\tau}}(\mathbf{x})$



# Global Trend Modeling

Circulant matrix  $\mathcal{C}(\mathbf{x})$  vs. convolution matrix  $\mathcal{C}_{\tilde{\tau}}(\mathbf{x})$



- Circulant/Convolution nuclear norm minimization
  - A balance between global and local trends modeling?

CircNNM (Liu'22, Liu & Zhang'23)

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

on data  $\mathbf{y}$  w/ observed index set  $\Omega$ .

ConvNNM (Liu'22, Liu & Zhang'23)

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}_{\tilde{\tau}}(\mathbf{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

on data  $\mathbf{y}$  w/ observed index set  $\Omega$ .

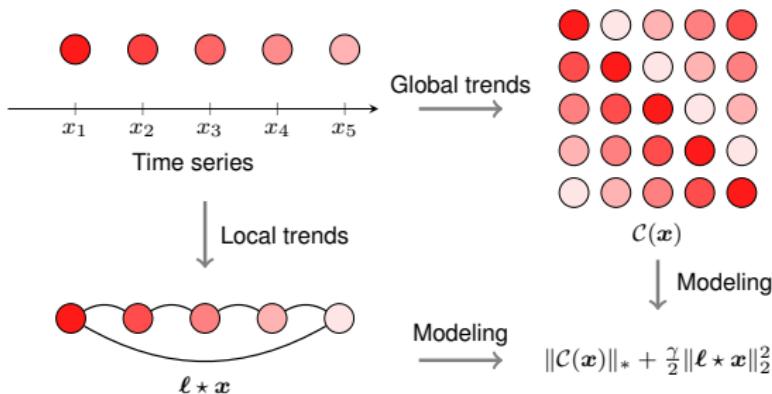
# Global + Local Trends?

## Laplacian Convolutional Representation (LCR)

For any partially observed time series  $\mathbf{y} \in \mathbb{R}^T$  with observed index set  $\Omega$ , LCR utilizes **circulant matrix** and **Laplacian kernel** to characterize global/local time series trends:

$$\min_{\mathbf{x}} \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\ell * \mathbf{x}\|_2^2}_{\text{local}}$$

s.t.  $\|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$



# Laplacian Convolutional Representation

- Augmented Lagrangian function:<sup>2</sup>

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = \underbrace{\|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2}_{\text{global} + \text{local}} + \underbrace{\frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle}_{\text{Lagrangian multiplier } \mathbf{w}} + \underbrace{\frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2}_{\text{observations } \mathbf{y}}$$

- The ADMM scheme:

$$\begin{cases} \mathbf{x} := \arg \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & (\text{Nuclear norm minimization}) \\ \mathbf{z} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & (\text{Closed-form solution}) \\ \mathbf{w} := \mathbf{w} + \lambda(\mathbf{x} - \mathbf{z}) & (\text{Standard update}) \end{cases}$$

- Optimize  $\mathbf{x}$ ?

$$\underbrace{\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1}_{\text{property of circulant matrix}} \quad \& \quad \underbrace{\frac{1}{2} \|\ell * \mathbf{x}\|_2^2 = \frac{1}{2T} \|\mathcal{F}(\ell) \circ \mathcal{F}(\mathbf{x})\|_2^2}_{\text{property of circular convolution}}$$

Nuclear norm minimization  $\Rightarrow$   **$\ell_1$ -norm minimization with FFT** in  $\mathcal{O}(T \log T)$  time.

<sup>2</sup> $\mathbf{w} \in \mathbb{R}^T$  (Lagrange multiplier);  $\langle \cdot, \cdot \rangle$  (inner product).

# Laplacian Convolutional Representation

- Optimize  $\mathbf{x}$  via FFT (in  $\mathcal{O}(T \log T)$  time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2\end{aligned}$$

where  $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$  (i.e., FFT).

## $\ell_1$ -norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of  $\ell_1$ -norm minimization in complex space:

$$\min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\delta}{2} \|\hat{\mathbf{x}} - \hat{\mathbf{h}}\|_2^2$$

with complex-valued  $\hat{\mathbf{x}}, \hat{\mathbf{h}} \in \mathbb{C}^T$  and weight parameter  $\delta$ , element-wise, the solution is

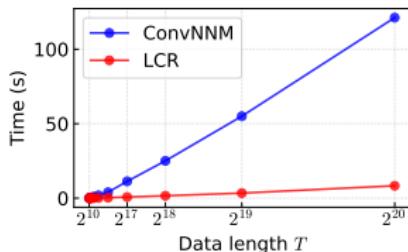
$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|} \cdot \underbrace{\max\{0, |\hat{h}_t| - 1/\delta\}}_{\text{shrinkage (e.g., ReLU)}} , t \in [T].$$

# Laplacian Convolutional Representation

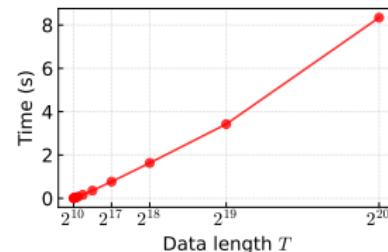
## Empirical time complexity

On the synthetic data  $\mathbf{y} \in \mathbb{R}^T$  with  $T \in \{2^{10}, 2^{11}, \dots, 2^{20}\}$

- Ours: **LCR**
  - An FFT implementation in  $\mathcal{O}(T \log T)$
  - The logarithmic factor  $\log T$  makes the FFT highly efficient
- Baseline: **ConvNNM** (Liu'22, Liu & Zhang'23)
  - Convolution matrix  $\mathcal{C}_{\tilde{\tau}}(\mathbf{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$  with kernel size  $\tilde{\tau} = 2^4$
  - Singular value thresholding in  $\mathcal{O}(\tilde{\tau}^2 T)$



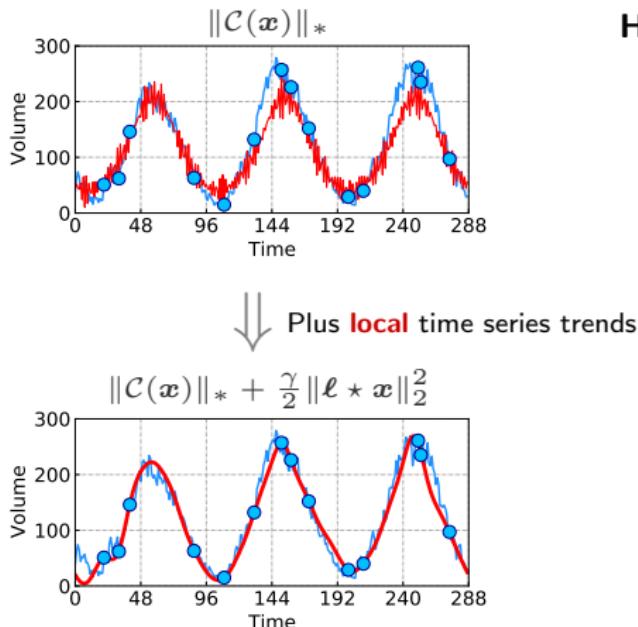
ConvNNM vs. LCR



LCR

# Experiments

- Traffic speed imputation<sup>3</sup>



## Highlights:

- Rethinking the importance of local trend modeling in traffic data imputation tasks.
- Finding a unified **global and local trend** modeling framework whose optimization can be efficiently solved by **FFT**:

$$\min_{\mathbf{x}} \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\ell * \mathbf{x}\|_2^2}_{\text{local}}$$

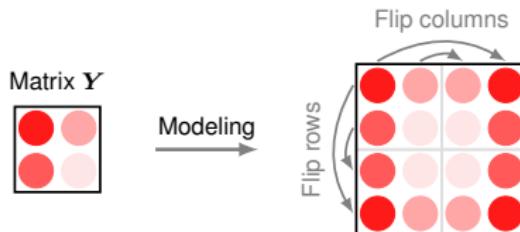
$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$

<sup>3</sup>Blue dot: partial obs.; red line: imputation.

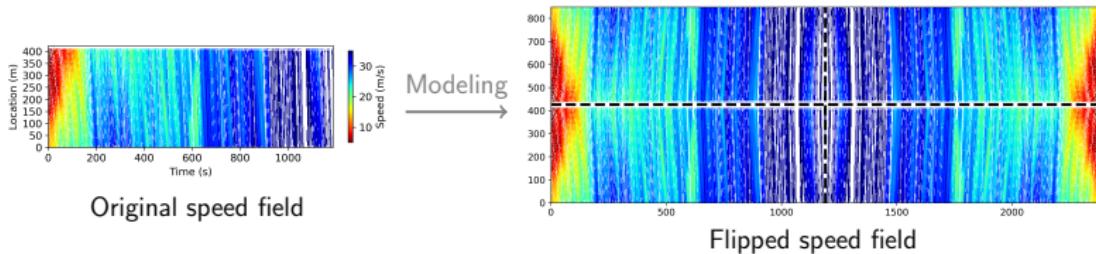
# Experiments

## Speed field reconstruction<sup>4</sup>

- Flipping operation on a matrix:



- Flipping operation on a speed field of vehicular traffic flow:



<sup>4</sup>Highway Drone (HighD) dataset at <https://www.hightd-dataset.com/>

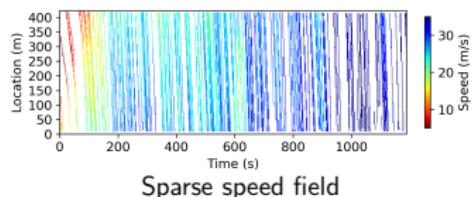
# Experiments

## Speed field reconstruction

- Scenario: Mask trajectories of 70% vehicles
- LCR-2D on partially observed  $\mathbf{Y} \in \mathbb{R}^{N \times T}$ :

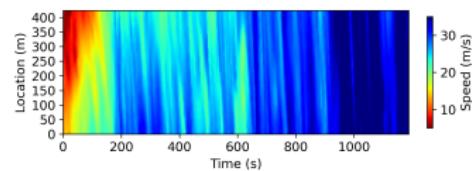
$$\min_{\mathbf{X}} \underbrace{\|\mathcal{C}(\mathbf{X})\|_*}_{\text{global trend}} + \frac{\gamma}{2} \underbrace{\|(\ell_s \ell^\top) \star \mathbf{X}\|_F^2}_{\text{local trend}}$$

s.t.  $\|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \epsilon$

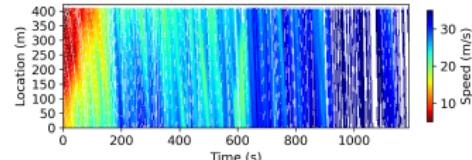


Sparse speed field

LCR-2D



Reconstructed speed field



Ground-truth speed field

# Bayesian Temporal Factorization for Multidimensional Time Series Prediction

IEEE Transactions on Pattern Analysis and Machine Intelligence, 2022

Code: <https://github.com/xinyuchen/transdim>



Xinyu Chen  
UdeM → MIT



Lijun Sun  
McGill

# Forecasting Urban Traffic States with Sparse Data Using Hankel Temporal Matrix Factorization

INFORMS Journal on Computing, 2024

Code: <https://github.com/xinyuchen/tracebase>



Xinyu Chen  
UdeM → MIT



Xi-Le Zhao  
UESTC



Chun Cheng  
DUT

# Revisit Traffic Prediction

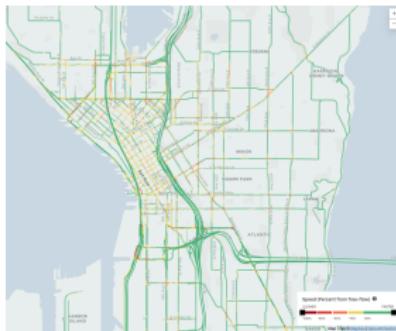
A classical problem w/ new ideas?

- Uber (hourly) movement speed data<sup>5</sup>



NYC movement

**64.43%** (missing)



Seattle movement

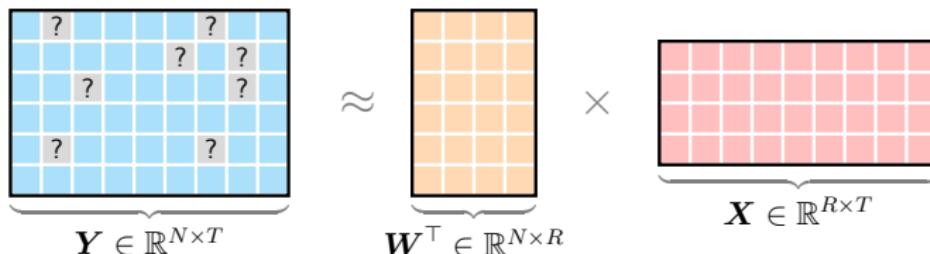
**84.95%** (missing)

- {road segment, time step (hour), average speed}
- Computing hourly speed: Road segments have 5+ unique trips.
- Challenge: Forecasting network-wide traffic states with sparse data.

<sup>5</sup><https://movement.uber.com/> (not available now)

# Matrix Factorization

A simple approach to reconstruct missing values.



## MF (Koren et al.'09)

Estimating low-dimensional  $\mathbf{W}, \mathbf{X}$ :

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2$$

on data  $\mathbf{Y}$  w/ observed index set  $\Omega$ .

- ✓ Learn from sparse data
- ✓ Spatial factor matrix  $\mathbf{W}$
- ✓ Temporal factor matrix  $\mathbf{X}$
- ✗ Temporal correlations?

# Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

$$\begin{array}{c} \text{?} & & \text{?} \\ & \text{?} & \text{?} \\ \text{?} & & \text{?} \\ \text{?} & & \text{?} \end{array} \underbrace{\quad}_{\mathbf{Y} \in \mathbb{R}^{N \times T}} \approx \underbrace{\quad}_{\mathbf{W}^\top \in \mathbb{R}^{N \times R}} \times \underbrace{\quad}_{\mathbf{X} \in \mathbb{R}^{R \times T}}$$

↓  **$\mathbf{X}$  is time series?**

$$\begin{array}{c} \text{?} & & \text{?} \\ & \text{?} & \text{?} \\ \text{?} & & \text{?} \\ \text{?} & & \text{?} \end{array} \underbrace{\quad}_{\mathbf{Y} \in \mathbb{R}^{N \times T}} \approx \underbrace{\quad}_{\mathbf{W}^\top \in \mathbb{R}^{N \times R}} \times \underbrace{\quad}_{\begin{array}{ccccccccc} \mathbf{x}_{t-2} & \mathbf{x}_{t-1} & \mathbf{x}_t & \mathbf{x}_{t+1} & \mathbf{x}_{t+2} \\ \dots & & & & \\ t-3 & t-2 & t-1 & t & t+1 & t+2 & t+3 \end{array} \xrightarrow{\text{time step}}} \dots \Big\} R$$

**Why?**  $\mathbf{X} \in \mathbb{R}^{R \times T}$  is the low-dimensional representation of time series dynamics of  $\mathbf{Y} \in \mathbb{R}^{N \times T}$ .

# Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

MF (Koren et al.'09)

Estimating low-dimensional  $\mathbf{W}, \mathbf{X}$ :

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2$$

on data  $\mathbf{Y}$  w/ observed index set  $\Omega$ .

dth-order VAR

$$+ \quad \mathbf{x}_t = \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} + \underbrace{\boldsymbol{\epsilon}_t}_{\mathcal{N}(\mathbf{0}, \mathbf{I})}$$

w/ coefficients  $\{\mathbf{A}_k\}$ .

↓  
Yu et al.'16  
Chen & Sun'22

$$\min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} \frac{1}{2} \underbrace{\|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2}_{\text{MF on data } \mathbf{Y}} + \frac{\gamma}{2} \underbrace{\sum_{t=d+1}^T \left\| \mathbf{x}_t - \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} \right\|_2^2}_{\text{VAR on temporal factors } \mathbf{X}}$$

# Bayesian Temporal Matrix Factorization

- Bayesian network (Chen & Sun'22)

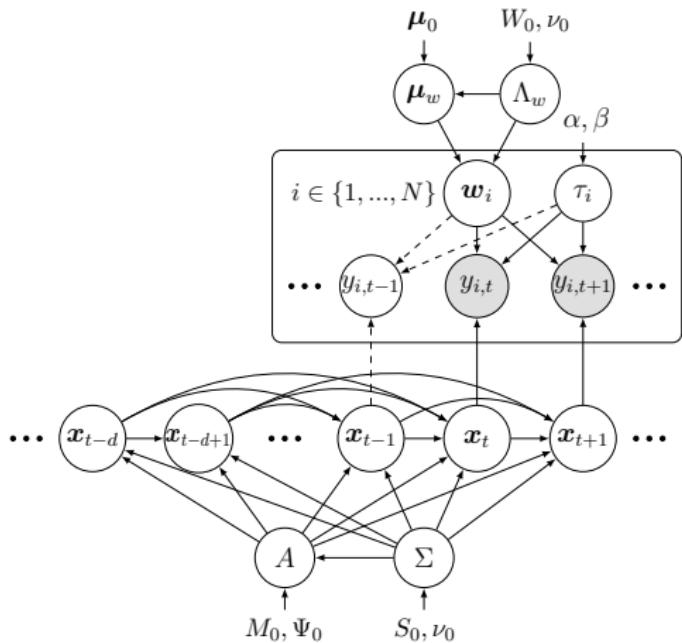
- Observations  $(i, t) \in \Omega$ :

$$y_{i,t} \sim \mathcal{N}(\underbrace{\boldsymbol{w}_i^\top \boldsymbol{x}_t}_{\text{MF}}, \tau_i^{-1})$$

- Prior of parameters:

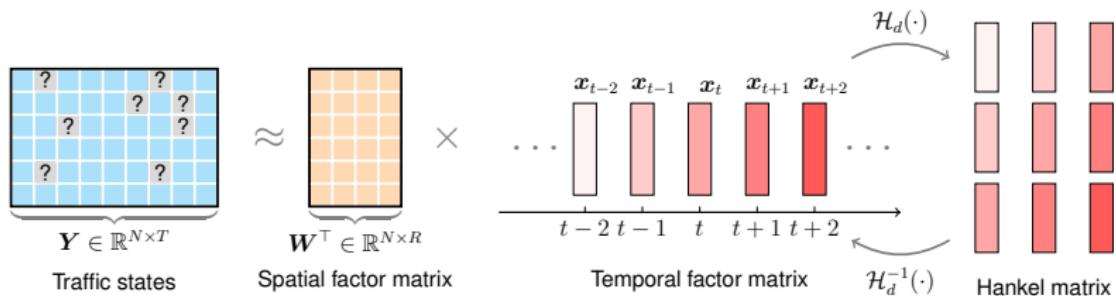
$$\begin{cases} \boldsymbol{w}_i \sim \mathcal{N}(\boldsymbol{\mu}_w, \Lambda_w^{-1}) \\ \boldsymbol{x}_t \sim \mathcal{N}(\underbrace{\boldsymbol{A}\boldsymbol{x}_{t-1}}_{\text{VAR}}, \boldsymbol{\Sigma}) \\ \tau_i \sim \text{Gamma}(\alpha, \beta) \end{cases}$$

- Conjugate prior of hyperparameters.



# Hankel Temporal Matrix Factorization

- HTMF (Chen, Zhao, & Cheng'24)
  - Automatic temporal modeling via Hankel matrices.



- Optimization problem

$$\min_{\mathbf{W}, \mathbf{X}, \mathbf{F}} \underbrace{\frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2}_{\text{MF on observations } \mathbf{Y}} + \underbrace{\frac{\gamma}{2} \|\mathbf{F} - \mathbf{X}\|_F^2}_{\text{bias mitigation}}$$

s.t.  $\underbrace{\text{rank}(\mathcal{H}_d(\mathbf{F})) = R}_{\text{rank-}R \text{ Hankel matrix}}$

# Discovering Dynamic Patterns from Spatiotemporal Data with Time-Varying Low-Rank Autoregression

IEEE Transactions on Knowledge and Data Engineering, 2024

Code: <https://github.com/xinychen/vars>

Blog: [https://spatiotemporal-data.github.io/posts/time\\_varying\\_model](https://spatiotemporal-data.github.io/posts/time_varying_model)



Xinyu Chen  
UdeM → MIT



Chengyuan Zhang\*  
McGill



Xiaoxu Chen  
McGill



Nicolas Saunier  
PolyMtl

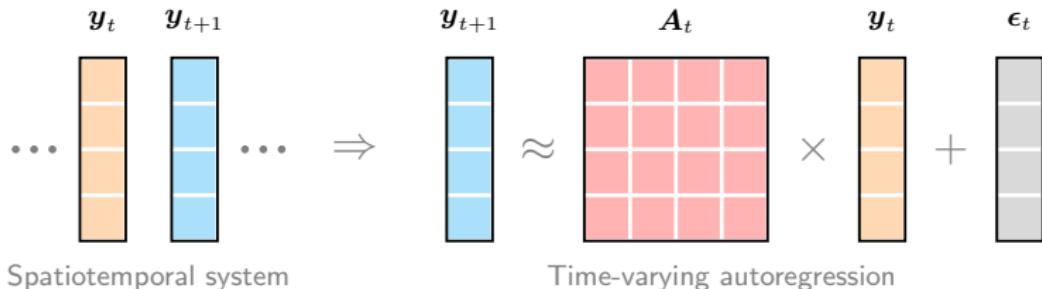


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# Autoregression

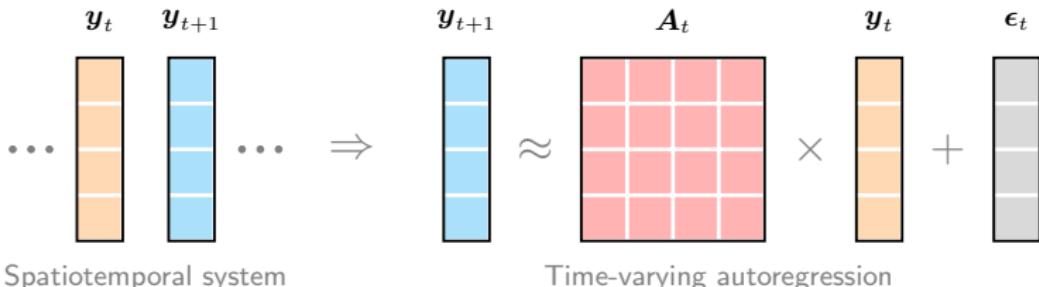
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- How to characterize dynamical systems?



# Autoregression

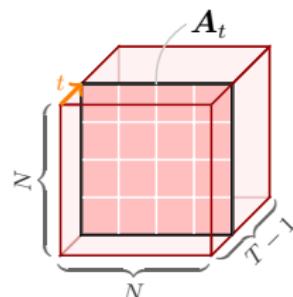
- How to characterize dynamical systems?

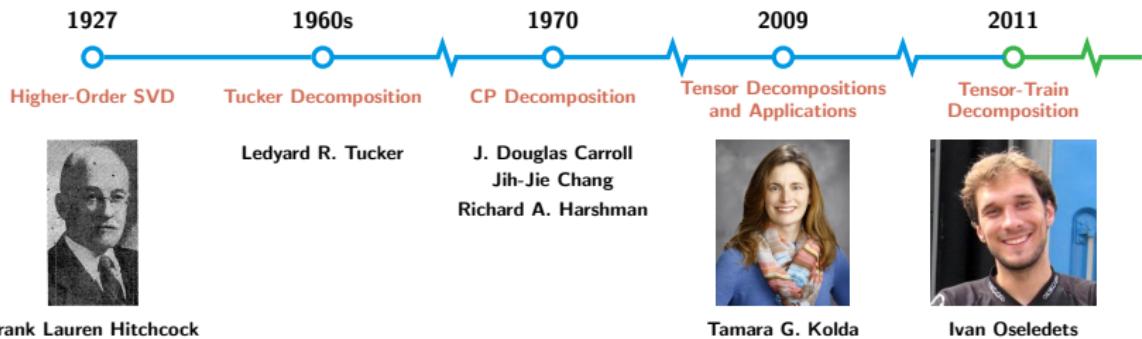


- On spatiotemporal systems  $\mathbf{Y} \in \mathbb{R}^{N \times T}$ :

$$\underbrace{y_{t+1} = \mathbf{A}y_t + \epsilon_t}_{\text{time-invariant (e.g., DMD)}} \quad \text{v.s.} \quad \underbrace{y_{t+1} = \mathbf{A}_t y_t + \epsilon_t}_{\text{time-varying, but over-param.}}$$

- How to discover spatial/temporal modes (patterns) from the tensor  $\mathcal{A} \triangleq \{\mathbf{A}_t\}_{t \in [T-1]}$ ?

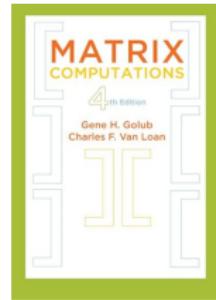




# Time-Varying Autoregression

- Tensor factorization<sup>6</sup>:

$$\mathcal{A} = \underbrace{\mathcal{G} \times_1 \mathbf{W} \times_2 \mathbf{V} \times_3 \mathbf{X}}_{\text{Tucker decomposition}}$$
$$A_t = \mathcal{G} \times_1 \underbrace{\mathbf{W}}_{\text{spatial modes}} \times_2 \mathbf{V} \times_3 \underbrace{\mathbf{x}_t^\top}_{\text{temporal modes}}$$



- (Ours) Time-varying low-rank autoregression:

$$\min_{\mathcal{G}, \mathbf{W}, \mathbf{V}, \mathbf{X}} \frac{1}{2} \sum_{t \in [T-1]} \left\| \mathbf{y}_{t+1} - \underbrace{(\mathcal{G} \times_1 \mathbf{W} \times_2 \mathbf{V} \times_3 \mathbf{x}_t^\top)}_{\text{Tucker decomposition}} \mathbf{y}_t \right\|_2^2$$

- Dominant spatial/temporal patterns in e.g.,  $\{\mathbf{W}, \mathbf{X}\}$ .

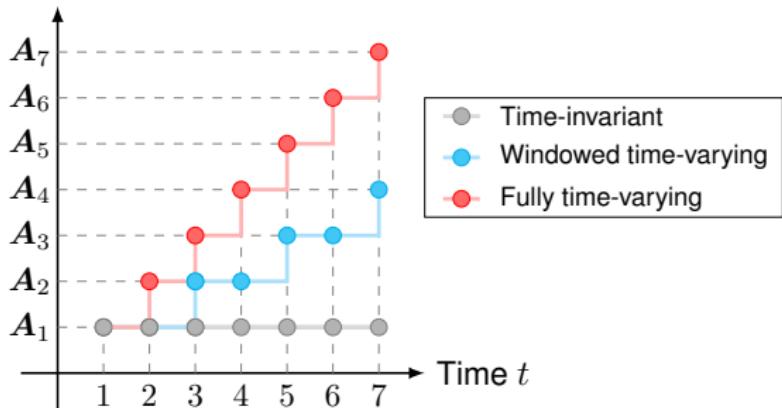
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<sup>6</sup> $\times_k$ ,  $\forall k$  is the mode- $k$  product between tensor and matrix/vector.

- On the data  $\mathbf{Y} \in \mathbb{R}^{N \times T}$ :

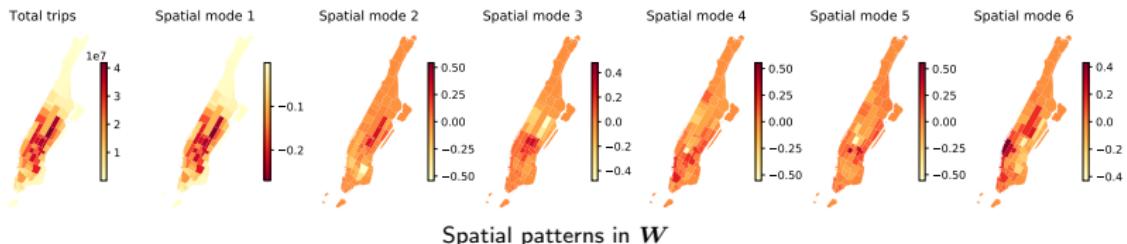
$$\underbrace{\mathbf{y}_{t+1} = \mathbf{A}\mathbf{y}_t + \epsilon_t}_{\text{time-invariant (e.g., DMD)}} \quad \text{v.s.} \quad \underbrace{\mathbf{y}_{t+1} = \mathbf{A}_t \mathbf{y}_t + \epsilon_t}_{\text{fully time-varying (ours)}}$$

Coefficients

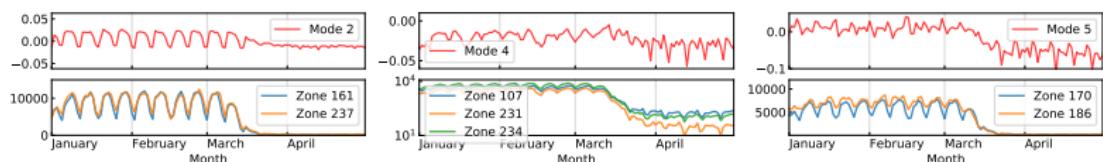


# NYC Taxi Data

- NYC taxi dataset (pickup)



Spatial patterns in  $W$

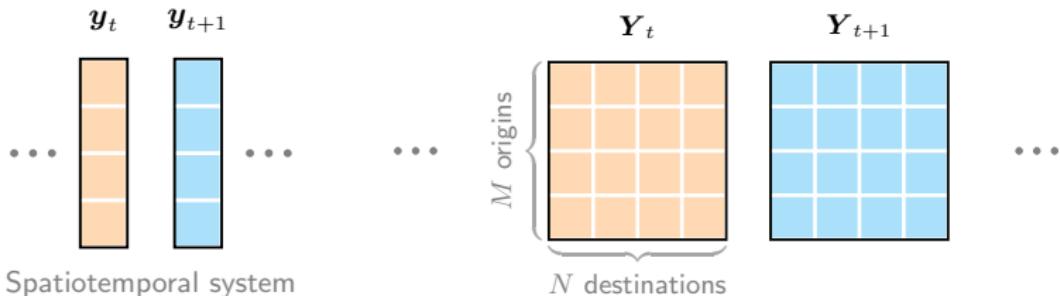


Pattern #2 & taxi trips (2020)

Pattern #4 & taxi trips (2020)

Pattern #5 & taxi trips (2020)

- Discovering **spatial/temporal patterns** from 2D and 3D spatiotemporal systems with unsupervised learning:
  - Time-varying autoregression **on the data**
  - Tensor factorization **on the coefficients**





# Thanks for your attention!

Any Questions?

Slides: <https://xinychen.github.io/slides/informs24.pdf>

## About me:

- 🏠 Homepage: <https://xinychen.github.io>
- ✉️ How to reach me: [chenxy346@gmail.com](mailto:chenxy346@gmail.com)
- ✉️ Or send to: [xinychen@mit.edu](mailto:xinychen@mit.edu)