



**POLYTECHNIQUE
MONTRÉAL**

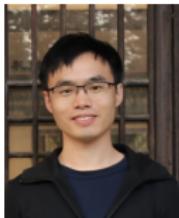
UNIVERSITÉ
D'INGÉNIERIE



Low-Rank Matrix and Tensor Factorization for Speed Field Reconstruction

Xinyu Chen

March 9, 2023



Ph.D. candidate
Xinyu Chen
Polytechnique Montréal



Supervisor
Prof. Nicolas Saunier
Polytechnique Montréal



Co-supervisor
Prof. Lijun Sun
McGill University

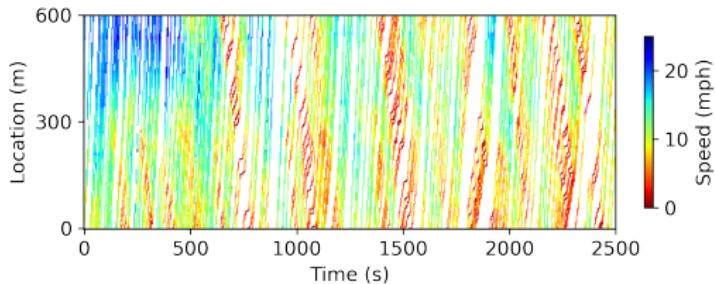
Source

- ① **Slides:** https://xinychen.github.io/slides/MF_TF_SFR.pdf
- ② **Jupyter Notebook:** https://github.com/xinychen/transdim/blob/master/toy-examples/MF_TF_SFR.ipynb

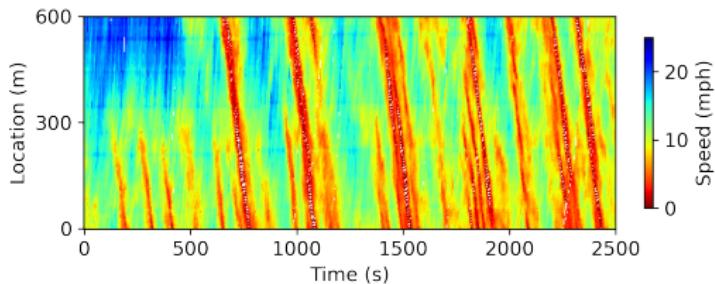
Outline

- **Motivation**
- **Matrix Factorization**
 - Optimization Problem
 - GD vs. SGD vs. ALS
- **Smoothing Matrix Factorization**
 - Spatial/Temporal Smoothing
 - Alternating Minimization
- **Temporal Matrix Factorization**
- **Tensor Factorization**
 - Basic Idea
 - CP Tensor Factorization
 - Hankel Tensor and Its Factorization

Motivation

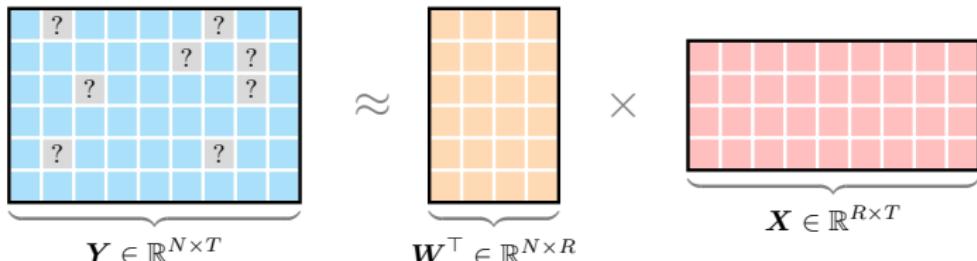


↓ Reconstruct speed field
from sparse trajectories?



Matrix Factorization

- Spatiotemporal data can be reconstructed by low-dimensional latent factors!



- MF optimization problem

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \left\| \mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) \right\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)$$

with factor matrices \mathbf{W} and \mathbf{X} .

Matrix Factorization

- MF optimization problem

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \left\| \mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) \right\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)$$

- Partial derivatives

$$\begin{cases} \frac{\partial f}{\partial \mathbf{W}} = -\mathbf{X} \mathcal{P}_\Omega^\top (\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) + \rho \mathbf{W} \\ \frac{\partial f}{\partial \mathbf{X}} = -\mathbf{W} \mathcal{P}_\Omega (\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) + \rho \mathbf{X} \end{cases}$$

- Gradient descent (**GD**) vs. Steepest gradient descent (**SGD**)

$$\begin{cases} \mathbf{W} := \mathbf{W} - \alpha \frac{\partial f}{\partial \mathbf{W}} \\ \mathbf{X} := \mathbf{X} - \alpha \frac{\partial f}{\partial \mathbf{X}} \end{cases} \quad \text{vs.} \quad \begin{cases} \alpha := \arg \min_\alpha f(\mathbf{W} - \alpha \frac{\partial f}{\partial \mathbf{W}}, \mathbf{X}) \\ \mathbf{W} := \mathbf{W} - \alpha \frac{\partial f}{\partial \mathbf{W}} \\ \beta := \arg \min_\beta f(\mathbf{W}, \mathbf{X} - \beta \frac{\partial f}{\partial \mathbf{X}}) \\ \mathbf{X} := \mathbf{X} - \beta \frac{\partial f}{\partial \mathbf{X}} \end{cases}$$

Matrix Factorization

- MF optimization problem

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \left\| \mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) \right\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)$$

- Partial derivatives

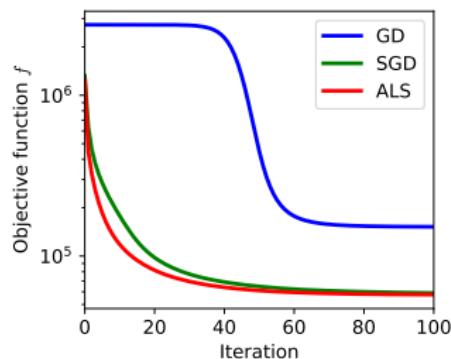
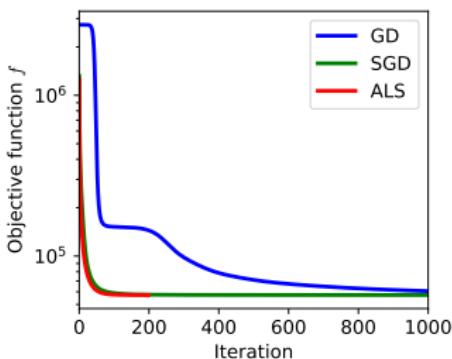
$$\begin{cases} \frac{\partial f}{\partial \mathbf{W}} = -\mathbf{X} \mathcal{P}_\Omega^\top (\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) + \rho \mathbf{W} \\ \frac{\partial f}{\partial \mathbf{X}} = -\mathbf{W} \mathcal{P}_\Omega (\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) + \rho \mathbf{X} \end{cases}$$

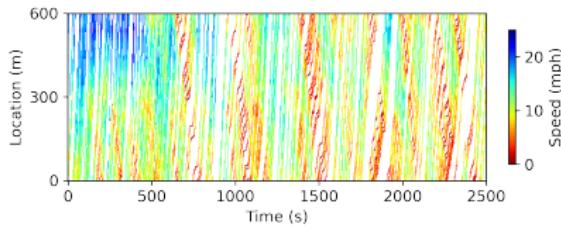
- Alternating least squares (**ALS**)

$$\begin{cases} \mathbf{w}_i := \left(\sum_{t:(i,t) \in \Omega} \mathbf{x}_t \mathbf{x}_t^\top + \rho \mathbf{I}_R \right)^{-1} \sum_{t:(i,t) \in \Omega} \mathbf{x}_t y_{i,t} \\ \mathbf{x}_t := \left(\sum_{i:(i,t) \in \Omega} \mathbf{w}_i \mathbf{w}_i^\top + \rho \mathbf{I}_R \right)^{-1} \sum_{i:(i,t) \in \Omega} \mathbf{w}_i y_{i,t} \end{cases}$$

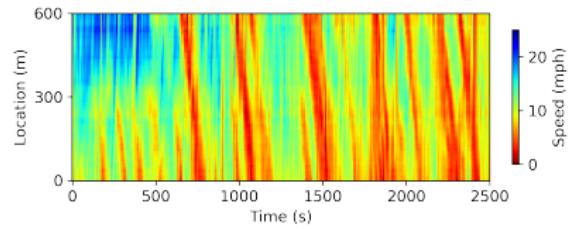
Matrix Factorization

- Objective function f vs. iteration
 - Rank $R = 10$, weight parameter $\rho = 10$;
 - GD step size $\alpha = 10^{-4}$.

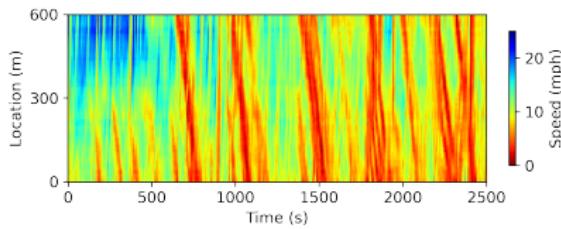




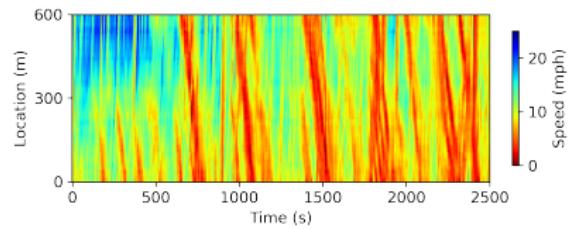
Sparse speed field



MF with GD



MF with SGD



MF with ALS

- Reconstruction errors

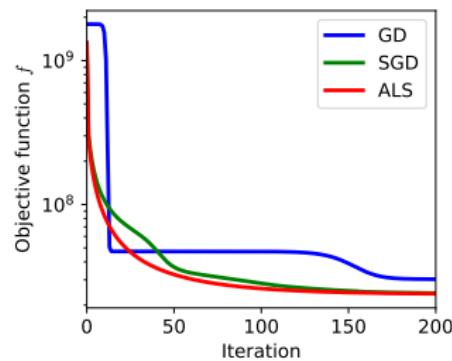
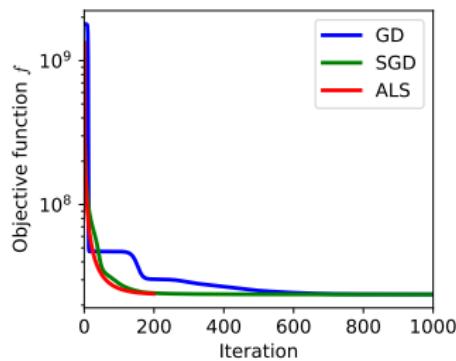
$$\text{MAPE} = \begin{cases} 50.66\% & (\text{GD}) \\ 45.13\% & (\text{SGD}) \\ 45.84\% & (\text{ALS}) \end{cases}$$

$$\text{RMSE} = \begin{cases} 2.33 & (\text{GD}) \\ 2.79 & (\text{SGD}) \quad (\text{mph}) \\ 2.80 & (\text{ALS}) \end{cases}$$

Matrix Factorization

Seattle freeway speed dataset (randomly generate 60% missing values)

- Objective function f vs. iteration
 - Rank $R = 10$, weight parameter $\rho = 10^2$;
 - GD step size $\alpha = 2 \times 10^{-5}$.



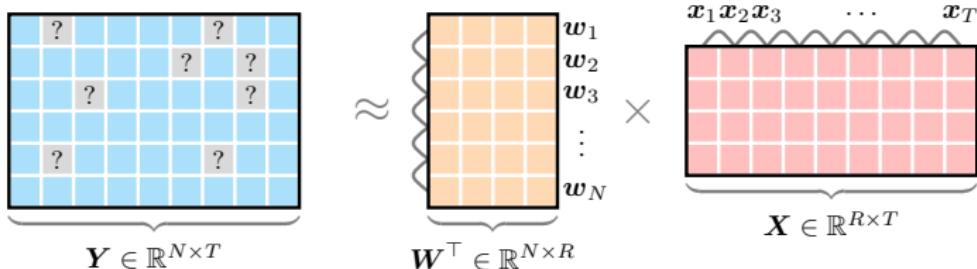
- Reconstruction errors

$$\text{MAPE} = \begin{cases} 9.14\% & (\text{GD}) \\ 9.12\% & (\text{SGD}) \\ 9.13\% & (\text{ALS}) \end{cases}$$

$$\text{RMSE} = \begin{cases} 5.24 & (\text{GD}) \\ 5.24 & (\text{SGD}) \quad (\text{mph}) \\ 5.24 & (\text{ALS}) \end{cases}$$

Smoothing Matrix Factorization

- Spatial/temporal local dependencies are also important!



- Spatial/temporal smoothing

$$\Psi = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \Rightarrow \begin{cases} \mathbf{W}\Psi_1^\top & \text{with } \Psi_1 \in \mathbb{R}^{(N-1) \times N} \\ \mathbf{X}\Psi_2^\top & \text{with } \Psi_2 \in \mathbb{R}^{(T-1) \times T} \end{cases}$$

Smoothing Matrix Factorization

- Spatial/temporal smoothing

$$\begin{aligned}\mathbf{W}\Psi_1^\top &= \begin{bmatrix} | & & & & | \\ \mathbf{w}_2 - \mathbf{w}_1 & \cdots & \mathbf{w}_N - \mathbf{w}_{N-1} \\ | & & & & | \end{bmatrix} \\ \mathbf{X}\Psi_2^\top &= \begin{bmatrix} | & & & & | \\ \mathbf{x}_2 - \mathbf{x}_1 & \cdots & \mathbf{x}_T - \mathbf{x}_{T-1} \\ | & & & & | \end{bmatrix}\end{aligned}$$

- SMF optimization problem

$$\begin{aligned}\min_{\mathbf{W}, \mathbf{X}} \quad & \frac{1}{2} \left\| \mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) \right\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2) \\ & + \frac{\lambda}{2} (\|\mathbf{W}\Psi_1^\top\|_F^2 + \|\mathbf{X}\Psi_2^\top\|_F^2)\end{aligned}$$

- Alternating minimization

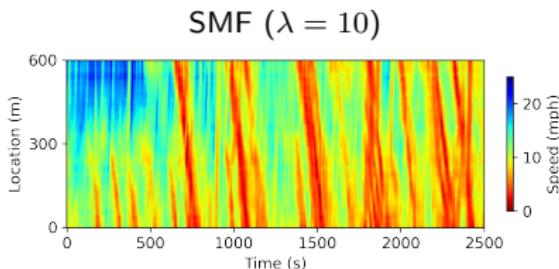
$$\mathbf{W} := \{ \mathbf{W} \mid \frac{\partial f}{\partial \mathbf{W}} = \mathbf{0} \} \quad \mathbf{X} := \{ \mathbf{X} \mid \frac{\partial f}{\partial \mathbf{X}} = \mathbf{0} \}$$

- Solving each matrix equation by the conjugate gradient method.

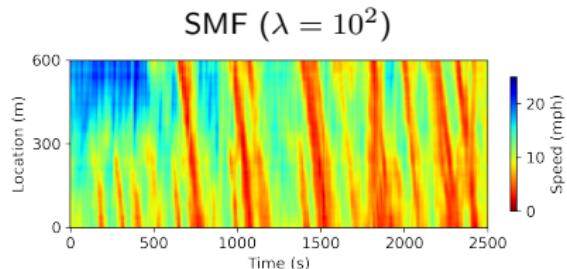
Smoothing Matrix Factorization

- Speed field reconstruction
 - Rank $R = 10$, weight parameter $\rho = 10$.
 - Recall that the reconstruction errors of MF:

$$\text{MAPE} = \begin{cases} 50.66\% & (\text{GD}) \\ 45.13\% & (\text{SGD}) \\ 45.84\% & (\text{ALS}) \end{cases} \quad \text{RMSE} = \begin{cases} 2.33 & (\text{GD}) \\ 2.79 & (\text{SGD}) \quad (\text{mph}) \\ 2.80 & (\text{ALS}) \end{cases}$$



MAPE = 44.06%, RMSE = 2.16mph

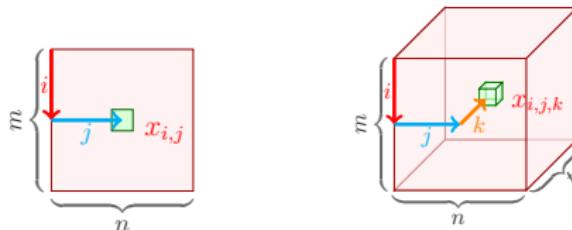


MAPE = 48.00%, RMSE = 1.60mph

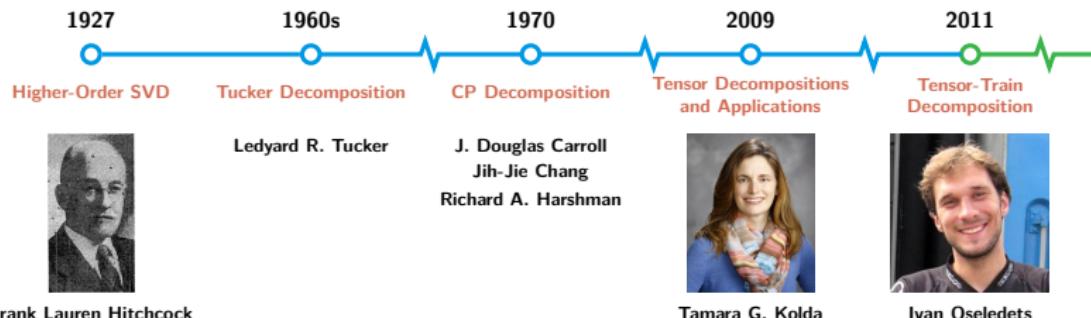
Temporal Matrix Factorization

Tensor Factorization

- What is tensor? $\mathbf{X} \in \mathbb{R}^{m \times n}$ vs. $\mathcal{X} \in \mathbb{R}^{m \times n \times t}$

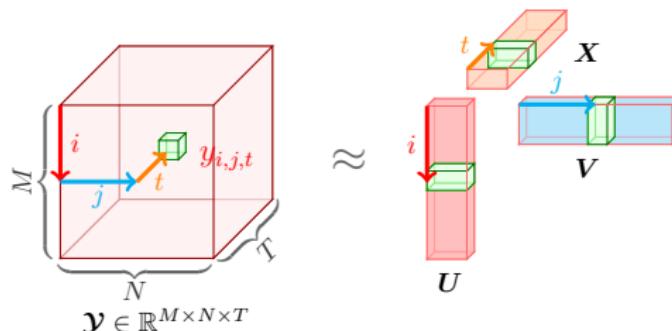


- Revisit tensor factorization



CP Tensor Factorization

- Factorize \mathcal{Y} into the combination of three rank- R factor matrices (i.e., low-dimensional latent factors).



- Understanding CP factorization¹:

$$\left\{ \begin{array}{l} y_{i,j,t} \approx \sum_{r=1}^R u_{i,r} v_{j,r} x_{t,r} \quad (\text{sum of latent factors}) \\ \mathcal{Y} \approx \sum_{r=1}^R \mathbf{u}_r \otimes \mathbf{v}_r \otimes \mathbf{x}_r \quad (\text{sum of rank-one tensors}) \end{array} \right.$$

¹The symbol \otimes denotes the outer product.

Hankel Tensor and Its Factorization

- Hankel matrix

- Given $\mathbf{y} = (1, 2, 3, 4, 5, 6)^\top$ and window length $\tau = 3$, we have

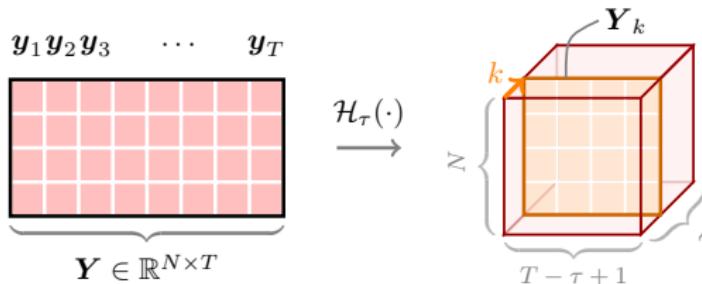
$$\mathcal{H}_\tau(\mathbf{y}) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{4 \times 3}$$

- (Hankelization) Hankel tensor $\mathcal{H}_\tau(\mathbf{Y})$

- Tensor size: $N \times (T - \tau + 1) \times \tau$;

- Slices: $\mathbf{Y}_k = \begin{bmatrix} | & | & & | \\ \mathbf{y}_k & \mathbf{y}_{k+1} & \cdots & \mathbf{y}_{T-\tau+k} \\ | & | & & | \end{bmatrix}, k = 1, 2, \dots, \tau$;

- Slice size: $N \times (T - \tau + 1)$.



Hankel Tensor and Its Factorization

- HTF optimization problem

$$\begin{aligned} \min_{\mathbf{U}, \mathbf{V}, \mathbf{X}} \quad & \frac{1}{2} \left\| \mathcal{P}_\Omega \left(\mathcal{H}_\tau(\mathbf{Y}) - \sum_{r=1}^R \mathbf{u}_r \otimes \mathbf{v}_r \otimes \mathbf{x}_r \right) \right\|_F^2 \\ & + \frac{\rho}{2} (\|\mathbf{U}\|_F^2 + \|\mathbf{V}\|_F^2 + \|\mathbf{X}\|_F^2) \end{aligned}$$

- HTF's advantage/disadvantage over MF

- Advantage: automatic temporal modeling
 - Disadvantage: high memory consumption

- Speed field reconstruction

- Rank $R = 10$, weight parameter $\rho = 10$.



POLYTECHNIQUE
MONTRÉAL

UNIVERSITÉ
D'INGÉNIERIE



Thanks for your attention!

Any Questions?

About me:

-  Homepage: <https://xinychen.github.io>
-  GitHub: <https://github.com/xinychen> (3K+ stars)
-  Blog: <https://medium.com/@xinyu.chen> (60K+ views)
-  How to reach me: chenxy346@gmail.com