

# Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

## Week 5: Modeling and Errors

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## Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

**"Class Participation Quiz 11"**

Time slot: **2:30PM – 3:00PM**

on Canvas.

- **Online engagement** (graded quizzes)

**"Quiz 11"**

Deadline: **11:59PM, February 9, 2026**

on Canvas.

## Truncation Errors & Taylor Series

### Learning objectives:

- Define truncation errors in numerical methods
- Understand the role of Taylor series in function approximation
- Derive finite difference approximations using Taylor expansions
- Apply Taylor series to estimate derivatives numerically

## Truncation Errors

## What is a truncation error?

- Error from using an approximation instead of an exact mathematical procedure
- Example: Approximating a derivative

$$\frac{dv}{dt} \approx \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

Intuition:  $\frac{\text{numerator} \rightarrow \text{difference in velocity } \Delta v}{\text{denominator} \rightarrow \text{difference in time } \Delta t}$

The difference equation is an approximation  $\rightarrow$  introduces truncation error

## Truncation Errors

## Why study truncation errors?

- Understand accuracy of numerical methods
- Choose appropriate approximations for given problems
- Estimate error bounds for computations
- Improve algorithms by reducing error terms

## Taylor Series

## Taylor theorem:

- Any **smooth function** can be approximated by a **polynomial**
- If  $f$  and its first  $n + 1$  derivatives are continuous on  $[a, x]$ , then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

Recall that

- $f'(a)$ : first-order derivative
- $f''(a)$ : second-order derivative
- $\dots$
- $f^{(n)}(a)$ :  $n$ th-order derivative

with

- $n!$ : factorial of integer  $n$  [Python: `np.prod(np.arange(1, n+1))`]

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1$$

## Taylor Series

**Intuition:** We predict  $f(x_{i+1})$  using information at  $x_i$ :

- **Zeroth-order** (constant) approximation:

$$f(x_{i+1}) \approx f(x_i)$$

Only works if  $f$  is **constant** between  $x_i$  and  $x_{i+1}$

## Taylor Series

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Only works if  $f$  is **constant** between  $x_i$  and  $x_{i+1}$

- **First-order** Taylor approximation:

Add **slope** information:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i) \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

- Represents a **straight line** (linear approximation)
- Exact if  $f$  is linear



## Taylor Series

**Intuition:** We predict  $f(x_{i+1})$  using information at  $x_i$ :

- **Zeroth-order** (constant) approximation:

$$f(x_{i+1}) \approx f(x_i)$$

Only works if  $f$  is **constant** between  $x_i$  and  $x_{i+1}$

- **First-order** Taylor approximation:

Add **slope** information:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i) \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

- Represents a **straight line** (linear approximation)
- Exact if  $f$  is linear

- **Euler's formula:**

$$\underbrace{y_{i+1}}_{\text{next value}} = \underbrace{y_i}_{\text{current value}} + \underbrace{\Delta x}_{\text{step size}} \cdot \underbrace{f(x_i, y_i)}_{\text{slope}} \quad x_{i+1} = x_i + \underbrace{\Delta x}_{\text{step size}}$$

## Taylor Series

**Intuition:** We predict  $f(x_{i+1})$  using information at  $x_i$ :

- **Second-order** Taylor approximation:

$$f(x_{i+1}) \approx f(x_i) + \textcolor{red}{f}'(x_i)(x_{i+1} - x_i) + \frac{\textcolor{red}{f}''(x_i)}{2!}(x_{i+1} - x_i)^2$$

- Captures quadratic behavior
- Better accuracy for smooth functions

## Taylor Series

- General Taylor polynomial

$$\begin{aligned}f(x_{i+1}) &\approx f(x_i) \\&+ f'(x_i)(x_{i+1} - x_i) \\&+ \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 \\&+ \dots \\&+ \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n \\&= \sum_{k=0}^n \frac{f^{(k)}(x_i)}{k!}(x_{i+1} - x_i)^k\end{aligned}$$

Higher  $n \rightarrow$  better approximation (if function is smooth)

## Taylor Series Approximation of a Polynomial

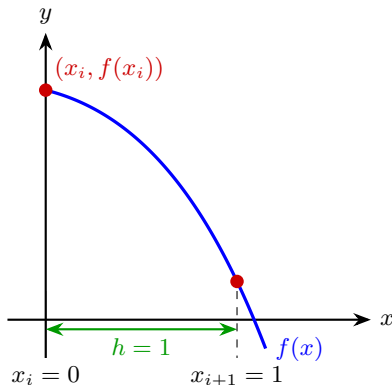
### Problem statement:

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

at  $x_{i+1} = 1$  starting from  $x_i = 0$  with step size  $h = x_{i+1} - x_i = 1$ .

- Goal:** Predict  $f(1)$  using Taylor approximations of increasing order.



## Taylor Series Approximation of a Polynomial

### Problem statement:

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

- Goal:** Predict  $f(1)$  using Taylor approximations of increasing order.

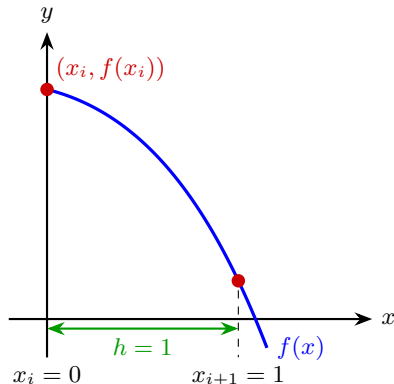
- Function  $f(x)$ :

$$f(0) = 1.2$$

$$\begin{aligned} f(1) &= -0.1 - 0.15 - 0.5 \\ &\quad - 0.25 + 1.2 = 0.2 \end{aligned}$$

- True value to predict:

$$f(1) = 0.2$$



## Taylor Series Approximation of a Polynomial

**First-order approximation** for  $f(1)$ :

- Need first-order derivative at  $x_i = 0$ :

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$\Rightarrow f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$$\Rightarrow f'(0) = -0.25$$

- First-order Taylor series:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i)$$

$$\Rightarrow f(1) \approx 1.2 + (-0.25) \times 1 = 0.95$$

## Taylor Series Approximation of a Polynomial

**Second-order approximation** for  $f(1)$ :

- Need second-order derivative at  $x_i = 0$ :

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$\Rightarrow f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$$\Rightarrow f''(x) = -1.2x^2 - 0.9x - 1$$

$$\Rightarrow f''(0) = -1$$

- Second-order Taylor series:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i) + \frac{f''(x_i)}{2!} (x_{i+1} - x_i)^2$$

$$\Rightarrow f(1) \approx 1.2 - 0.25 \times 1 + \left(\frac{-1}{2}\right) \times 1^2 = 0.45$$

## Taylor Series Approximation of a Polynomial

**Third-order approximation** for  $f(1)$ :

- Need third-order derivative at  $x_i = 0$ :

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$\Rightarrow f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$$\Rightarrow f''(x) = -1.2x^2 - 0.9x - 1$$

$$\Rightarrow f'''(x) = -2.4x - 0.9$$

$$\Rightarrow f'''(0) = -0.9$$

- Third-order Taylor series:

$$f(1) \approx 1.2 - 0.25 \times 1 - 0.5 \times 1^2 + \left( \frac{-0.9}{3!} \right) \times 1^3 = 0.3$$



## Taylor Series Approximation of a Polynomial

**Fourth-order approximation** for  $f(1)$ :

- Need fourth-order derivative at  $x_i = 0$ :

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$\Rightarrow f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

$$\Rightarrow f''(x) = -1.2x^2 - 0.9x - 1$$

$$\Rightarrow f'''(x) = -2.4x - 0.9$$

$$\Rightarrow f^{(4)}(x) = -2.4$$

$$\Rightarrow f^{(4)}(0) = -2.4$$

- Fourth-order Taylor series:

$$f(1) \approx 1.2 - 0.25 \times 1 - 0.5 \times 1^2 - 0.15 \times 1^3 - \left( \frac{-2.4}{4!} \right) \times 1^4 = 0.2$$

## Taylor Series Approximation of a Polynomial

- Absolute error:

$$\varepsilon = |f(x_{i+1}) - \hat{f}(x_{i+1})|$$

where the approximation is

$$\hat{f}(x_{i+1}) = \sum_{k=0}^n \frac{f^{(k)}(x_i)}{k!} (x_{i+1} - x_i)^k$$

- Summary of results:

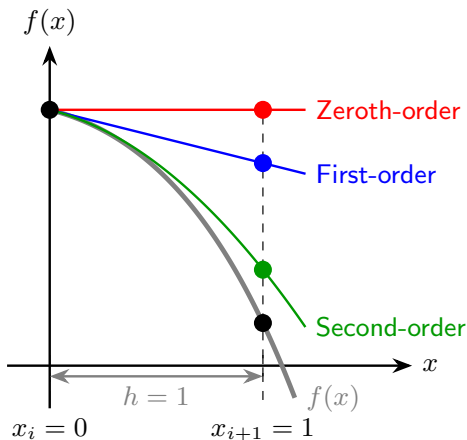
Order ( $n$ )	Approximation $\hat{f}(1)$	Absolute error $\varepsilon$
1	0.95	$ 0.2 - 0.95  = 0.75$
2	0.45	$ 0.2 - 0.45  = 0.25$
3	0.3	$ 0.2 - 0.3  = 0.1$
4	0.2	$ 0.2 - 0.2  = 0$

- Error decreases as order increases
- With  $n = 4$ , approximation is exact because  $f(x)$  is a 4th-degree polynomial

## Taylor Series Approximation of a Polynomial

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$



## Taylor Series Approximation of a Polynomial

### Concluding remarks:

- **Taylor series** approximates functions using derivatives at a point
- **Higher-order terms** improve accuracy by capturing curvature
- For an  $m$ -degree polynomial, a Taylor series of order  $m$  gives the **exact** function
- **Truncation error** quantifies the approximation error
- This example illustrates the **power of Taylor series** for function approximation in numerical methods

## Taylor Series Approximation of $\sin(x)$

### Maclaurin series:

- A Taylor series expansion of a function about 0
- Taylor series approximation:

$$f(x_{i+1}) \approx \sum_{k=0}^n \frac{f^{(k)}(x_i)}{k!} (x_{i+1} - x_i)^k$$

- Set  $x_i = 0$  and  $x = x_{i+1}$ , then

$$f(x) \approx \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

## Taylor Series Approximation of $\sin(x)$

### Maclaurin series:

- A Taylor series expansion of a function about 0
- Taylor series approximation:

$$f(x_{i+1}) \approx \sum_{k=0}^n \frac{f^{(k)}(x_i)}{k!} (x_{i+1} - x_i)^k$$

- Set  $x_i = 0$  and  $x = x_{i+1}$ , then

$$f(x) \approx \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

- Derivatives of  $f(x) = \sin(x)$ :

$$f'(x) = \underbrace{\cos(x)}_{\cos(0)=1}, \quad f''(x) = -\underbrace{\sin(x)}_{\sin(0)=0}, \quad f'''(x) = -\underbrace{\cos(x)}_{\cos(0)=1}, \quad f^{(4)}(x) = \underbrace{\sin(x)}_{\sin(0)=0}$$

- Formula:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \dots$$

## Quick Summary

### Monday's Class:

- Definition of truncation error
- Taylor series of any smooth function (approximated by polynomial)
- Approximation with zeroth-, first-, second-, and higher-order information
- Example: Taylor series approximation of a polynomial function
- Revisit Taylor series approximation of  $\sin(x)$

## Function $f(x) = x^m$

The effect of **nonlinearity** and **step size** on the Taylor series approximation

- Function

$$f(x) = x^m$$

for  $m = 1, 2, 3, 4$  over the range  
from  $x = 1$  to 2.



## Function $f(x) = x^m$

The effect of **nonlinearity** and **step size** on the Taylor series approximation

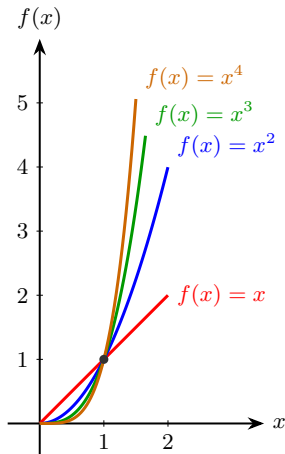
- Function

$$f(x) = x^m$$

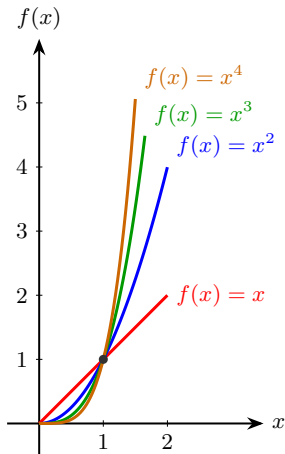
for  $m = 1, 2, 3, 4$  over the range from  $x = 1$  to 2.

- First-order** Taylor series expansion:

$$\begin{aligned}\hat{f}(x_{i+1}) &= f(x_i) + f'(x_i)(x_{i+1} - x_i) \\ &= f(x_i) + mx_i^{m-1}(x_{i+1} - x_i)\end{aligned}$$



## Function $f(x) = x^m$ : Nonlinearity



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + mx_i^{m-1}(x_{i+1} - x_i)$$

- For  $m = 1$ , the function is  $f(x) = x$  and derivative is  $f'(x) = 1$ :

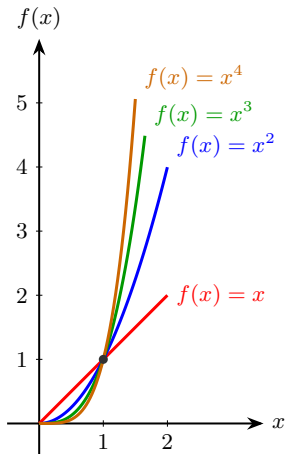
- True value  $f(2) = 2$
- Approximation:

$$\hat{f}(2) = f(1) + 1 \times (2 - 1) = 2$$

- Absolute error:

$$|\hat{f}(2) - f(2)| = 0$$

## Function $f(x) = x^m$ : Nonlinearity



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + mx_i^{m-1}(x_{i+1} - x_i)$$

- For  $m = 2$ , the function is  $f(x) = x^2$  and derivative is  $f'(x) = 2x$ :

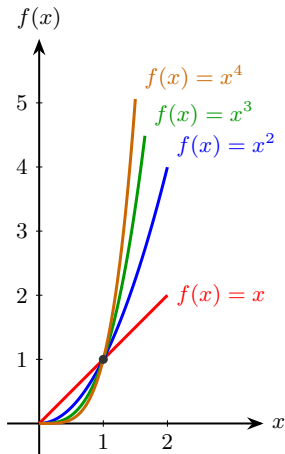
- True value  $f(2) = 4$
- Approximation:

$$\hat{f}(2) = f(1) + 2 \times (2 - 1) = 3$$

- Absolute error:

$$|\hat{f}(2) - f(2)| = 1$$

## Function $f(x) = x^m$ : Nonlinearity



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + mx_i^{m-1}(x_{i+1} - x_i)$$

- For  $m = 3$ , the function is  $f(x) = x^3$  and derivative is  $f'(x) = 3x^2$ :

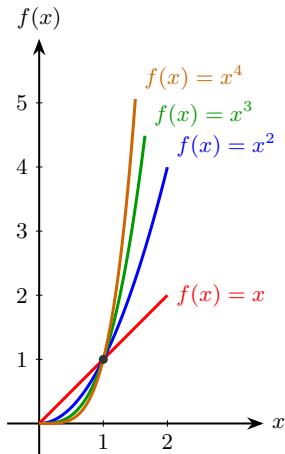
- True value  $f(2) = 8$
- Approximation:

$$\hat{f}(2) = f(1) + 3 \times (2 - 1) = 4$$

- Absolute error:

$$|\hat{f}(2) - f(2)| = 4$$

## Function $f(x) = x^m$ : Nonlinearity



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + mx_i^{m-1}(x_{i+1} - x_i)$$

- For  $m = 4$ , the function is  $f(x) = x^4$  and derivative is  $f'(x) = 4x^3$ :

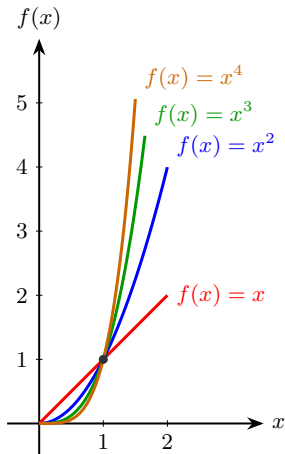
- True value  $f(2) = 16$
- Approximation:

$$\hat{f}(2) = f(1) + 4 \times (2 - 1) = 5$$

- Absolute error:

$$|\hat{f}(2) - f(2)| = 11$$

## Function $f(x) = x^m$ : Nonlinearity



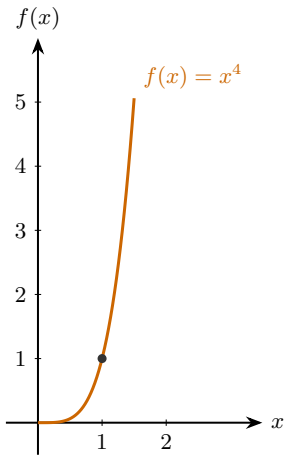
First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + mx_i^{m-1}(x_{i+1} - x_i)$$

**[Effect of nonlinearity]** Error increases as the function becomes more nonlinear.

Exp. $m$	$f(2)$	$\hat{f}(2)$	Absolute error
1	2	2	0
2	4	3	1
3	8	4	4
4	16	5	11

## Function $f(x) = x^4$ : Step Size $h$



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + 4x_i^3 \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

Evaluate  $x + h$  where  $x = 1$ :

- For  $h = 1$ :

- True value  $f(2) = 16$
- Approximation:

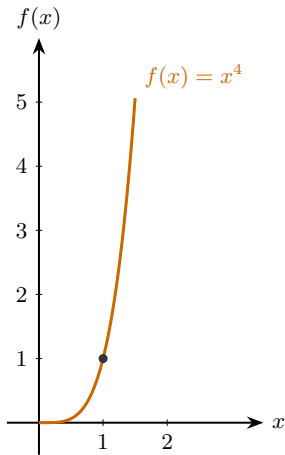
$$\hat{f}(2) = f(1) + 4 \times (2 - 1) = 5$$

- For  $h = 0.5$ :

- True value  $f(1.5) = 5.0625$
- Approximation:

$$\hat{f}(1.5) = f(1) + 4 \times (1.5 - 1) = 3$$

## Function $f(x) = x^4$ : Step Size $h$



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + 4x_i^3 \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

Evaluate  $x + h$  where  $x = 1$ :

- For  $h = 0.25$ :
  - True value  $f(1.25) = 2.441406$
  - Approximation:

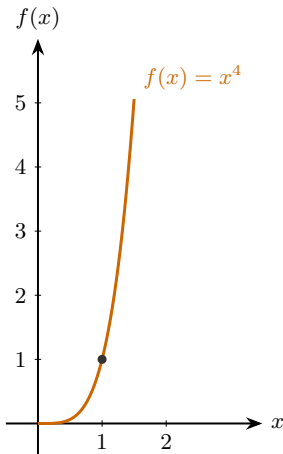
$$\hat{f}(1.25) = f(1) + 4 \times (1.25 - 1) = 2$$

- For  $h = 0.125$ :
  - True value  $f(1.125) = 1.601807$
  - Approximation:

$$\hat{f}(1.125) = f(1) + 4 \times (1.125 - 1) = 1.5$$



## Function $f(x) = x^4$ : Step Size $h$



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + \underbrace{4x_i^3 (x_{i+1} - x_i)}_{\text{step size } h}$$

Evaluate  $x + h$  where  $x = 1$ :

$h$	$f(x + h)$	$\hat{f}(x + h)$	Absolute error
1	16	5	11
0.5	5.0625	3	2.0625
0.25	2.441406	2	0.441406
0.125	1.601807	1.5	0.101807

**[Effect of step size]** Discrepancy will decrease as  $h$  is reduced.

## Taylor Series Approximation

### Maclaurin series:

- A Taylor series expansion of a function about 0
- Taylor series approximation:

$$f(x_{i+1}) \approx \sum_{k=0}^n \frac{f^{(k)}(x_i)}{k!} (x_{i+1} - x_i)^k$$

- Set  $x_i = 0$  and  $x = x_{i+1}$ , then

$$f(x) \approx \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

## Taylor Series Approximation of $\exp(x)$

### Maclaurin series:

- A Taylor series expansion of a function about 0

$$f(x) \approx \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

- Derivatives of  $f(x) = e^x$ :

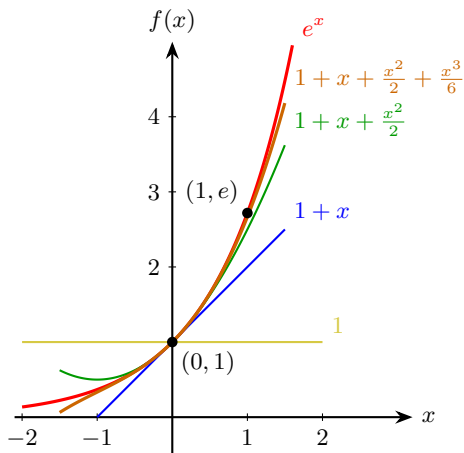
$$f'(x) = f''(x) = \dots = f^{(n)}(x) = e^x \quad \Rightarrow \quad f^{(n)}(0) = 1 \quad \text{for all } n$$

- Formula:

$$\begin{aligned} f(x) &\approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} \\ &= \sum_{k=0}^n \frac{x^k}{k!} \end{aligned}$$

## Taylor Series Approximation of $\exp(x)$

$$f(x) = e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$



## Taylor Series Approximation of $\exp(x)$ for $x = 1$

**Problem statement:**

TBD

TBD

TBD