

# The Relevance of $t$ -Statistics for Small Sample Sizes

An Introductory Class to Higher Statistics

**Xinyu Chen**

December 6, 2024



# Outline

---

Answering a lot questions, e.g.,

- ❶ How was  $t$ -statistic developed?
- ❷ Normal distribution vs. student  $t$ -distribution?
- ❸ What is  $t$ -statistic?
- ❹ How to calculate a  $t$ -test?
- ❺ What are the hypotheses and the assumptions?
- ❻ How to interpret results?

# Development

The problems of **small sample sizes**

1876

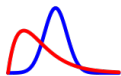
*t*-distribution as  
a posterior distribution



F. R. Helmert



J. Lüroth



(Source: Wiki)

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

1895

*t*-distribution as Pearson  
type IV distribution



K. Pearson



(Source: Wiki)

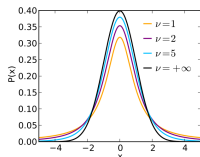
Diagram of the Pearson system

1908

Student *t*-distribution  
"The Probable Error of a Mean" (Biometrika)



W. S. Gosset

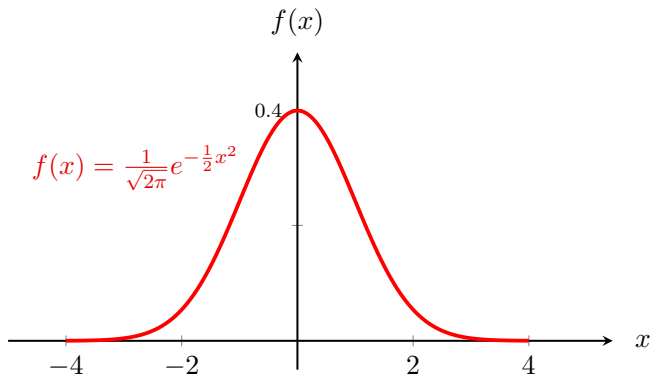


(Source: Wiki)

Probability density function

## Revisiting Normal Distribution

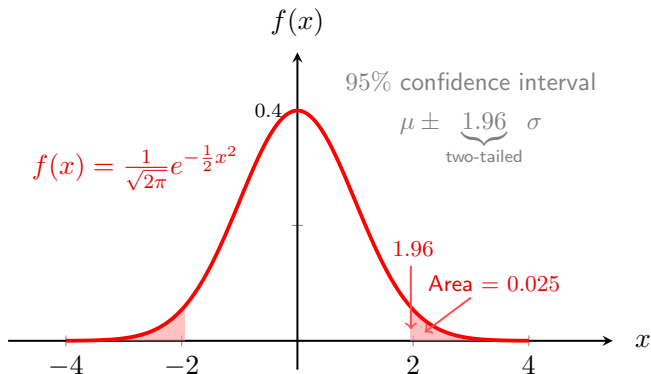
---



Probability density function of the standard normal distribution

## Revisiting Normal Distribution

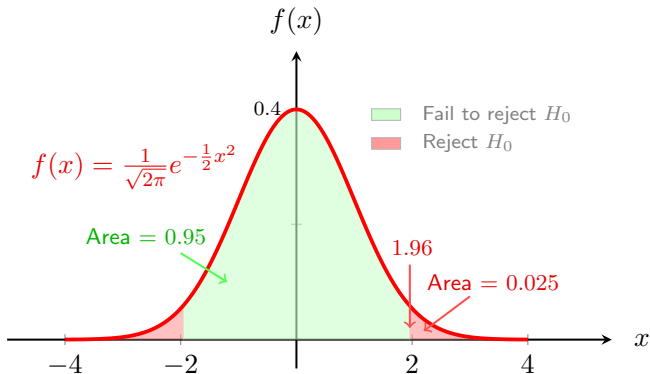
---



Probability density function of the standard normal distribution

# Connecting with Hypothesis Test

- Hypothesis test
  - Population: mean  $\mu$ , standard deviation  $\sigma$
  - Sample: mean  $\bar{x}$ , sample size  $n$
  - Null hypothesis ( $H_0$ ): The population mean is  $\mu$
  - $z$ -test:  $z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$  ( $z \uparrow$  implies statistically significant difference)
- 95% confidence interval



# Implementing $z$ -Test

---

## Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh with a population standard deviation of 5 kWh. A random sample of 40 households has an average daily energy consumption of 32 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

# Implementing $z$ -Test

## Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh with a population standard deviation of 5 kWh. A random sample of 40 households has an average daily energy consumption of 32 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

## Steps:

### ❶ Formulate Hypotheses

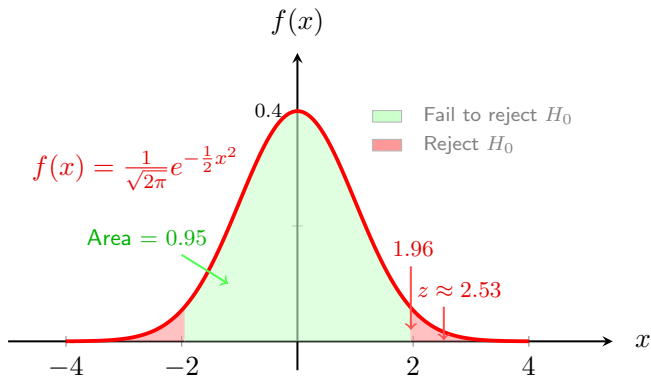
- Null Hypothesis ( $H_0$ ): The population mean is  $\mu = 30$  kWh.
- Alternative Hypothesis ( $H_a$ ): The population mean is not  $\mu = 30$  kWh ( $\mu \neq 30$ ).

### ❷ Use the $z$ -test formula since the population standard deviation is known:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{32 - 30}{5/\sqrt{40}} = \frac{2}{5/6.32} = \frac{2}{0.79} \approx 2.53$$

- $\bar{x} = 32$  (sample mean)
- $\mu = 30$  (population mean)
- $n = 40$  (sample size)
- $\sigma = 5$  (population standard deviation)





Test statistic  $|z| > 1.96 \Rightarrow$  Reject the null hypothesis

# Implementing $z$ -Test

## Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh with a population standard deviation of 5 kWh. A random sample of 40 households has an average daily energy consumption of 32 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

### Steps:

- ② Use the  $z$ -test formula since the population standard deviation is known:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{32 - 30}{5/\sqrt{40}} = \frac{2}{5/6.32} = \frac{2}{0.79} \approx 2.53$$

- ③ Decision rule at a 95% confidence interval

- Reject  $H_0$  if  $|z| > 1.96$ .
- Otherwise, fail to reject  $H_0$ .

- ④ Interpretation

- The test statistic  $|z| = 2.53 > 1.96$  (exceeding the critical value).
- Thus, we reject the null hypothesis.
- The sample provides sufficient evidence to conclude that the average daily energy consumption is not 30 kWh.

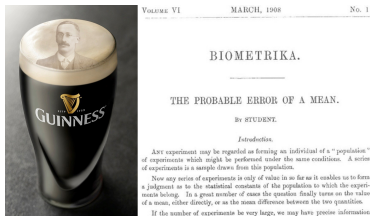
# Student $t$ -Distribution

In the case of **small sample sizes**?

- Switch to student  $t$ -distribution and  $t$ -test
- Probability density function:

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

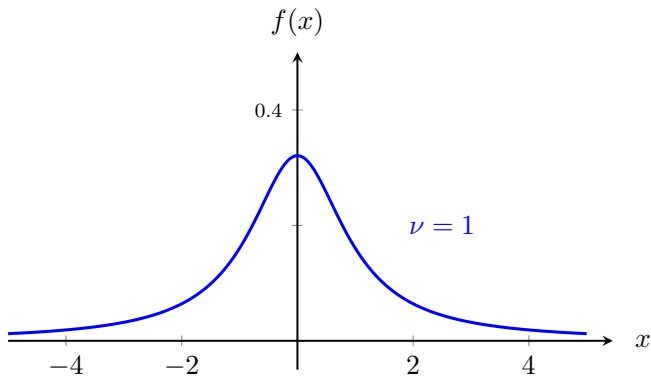
- $x \in \mathbb{R}$ : random variable
- $\nu \in \mathbb{Z}^+$ : degrees of freedom
- $\Gamma(\cdot)$ : Gamma function



Gosset'1908 (known as "student")

# Student $t$ -Distribution

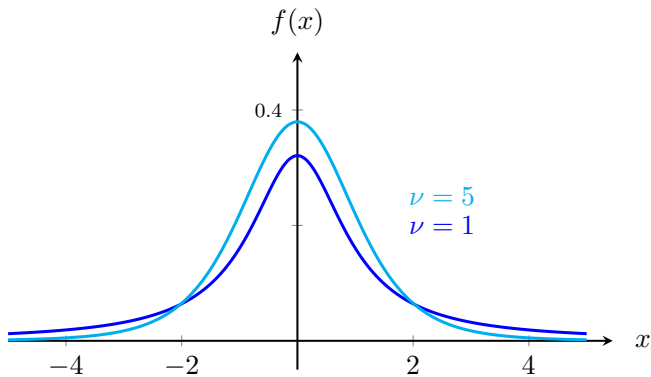
---



Student  $t$ -distribution of  $\nu$  degrees of freedom

# Student $t$ -Distribution

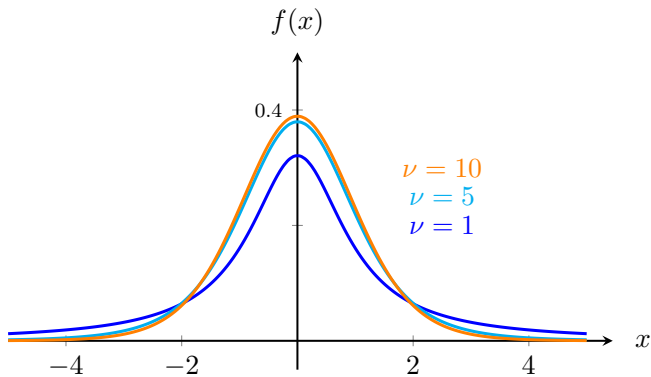
---



Student  $t$ -distribution of  $\nu$  degrees of freedom

# Student $t$ -Distribution

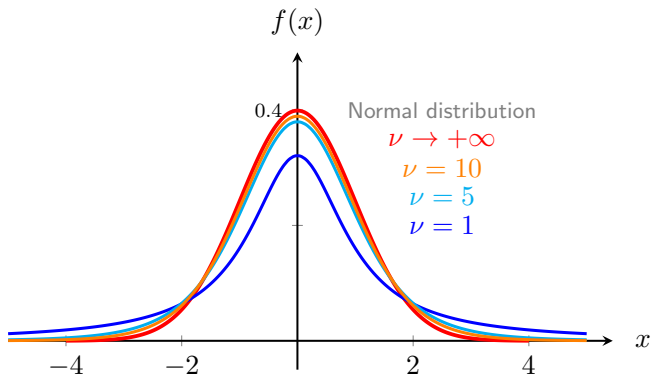
---



Student  $t$ -distribution of  $\nu$  degrees of freedom

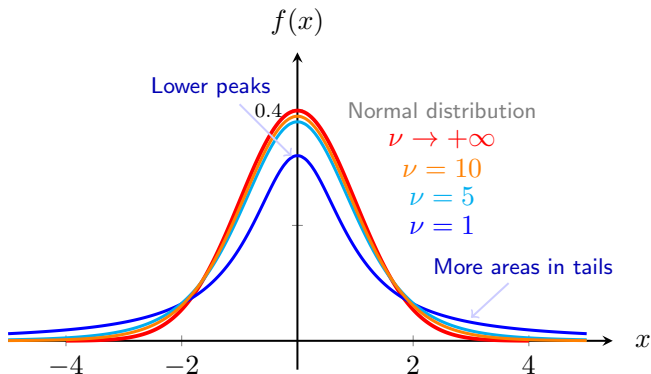
# Student $t$ -Distribution

---



Student  $t$ -distribution of  $\nu$  degrees of freedom

# Student $t$ -Distribution



Student  $t$ -distribution of  $\nu$  degrees of freedom



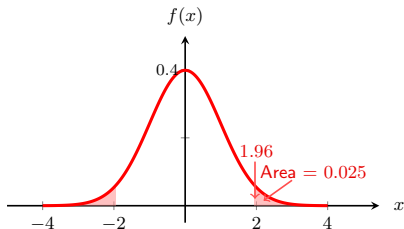
## 95% Confidence Interval

For the population mean  $\mu$  (✓) and standard deviation  $\sigma$  (✓/X)

- If population standard deviation  $\sigma$  is known

$$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

- Use  $z$ -test



Standard normal distribution

## 95% Confidence Interval

For the population mean  $\mu$  (✓) and standard deviation  $\sigma$  (✓/X)

- If **population standard deviation  $\sigma$**  is known

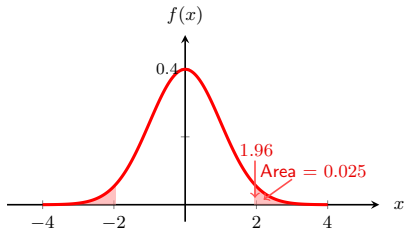
$$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

- Use **z-test**

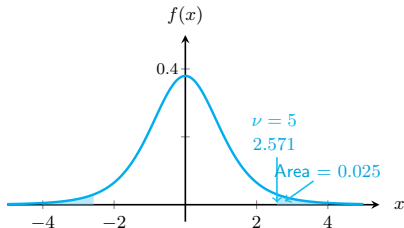
- If  $\sigma$  is unknown, using **sample standard deviation  $s$**  instead

$$\bar{x} \pm ? \times \frac{s}{\sqrt{n}}$$

- Use **t-test**



Standard normal distribution



Student  $t$ -distribution

- **Heavier tail** in student  $t$ -distribution ( $\nu = n - 1$  degrees of freedom) is important for small sample size  $n$

## Definition of $t$ -Statistic

---

- Formula of  $t$ -statistic for small sample sizes

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

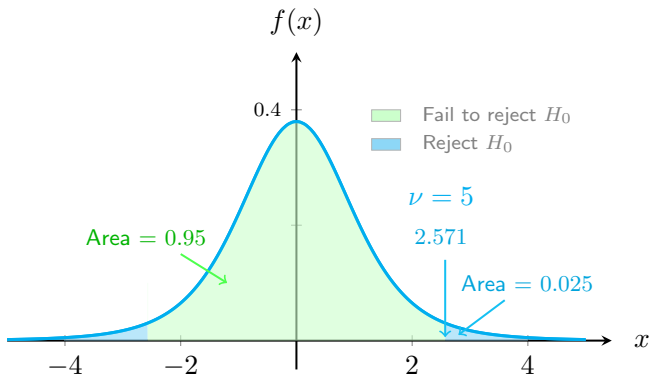
- Population: mean  $\mu$
- Sample: mean  $\bar{x}$ , standard deviation  $s$ , sample size  $n$  (small value)

## Definition of $t$ -Statistic

- Formula of  $t$ -statistic for small sample sizes

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

- Population: mean  $\mu$
- Sample: mean  $\bar{x}$ , standard deviation  $s$ , sample size  $n$  (small value)
- A high absolute value of  $t$  suggests a statistically significant difference.



## Implementing $t$ -Test for Small Sample Size

---

### Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh. A random sample of 6 households has an average daily energy consumption of 32 kWh, with a sample standard deviation of 6 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

# Implementing $t$ -Test for Small Sample Size

## Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh. A random sample of 6 households has an average daily energy consumption of 32 kWh, with a sample standard deviation of 6 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

## Steps:

### ❶ Formulate Hypotheses

- Null Hypothesis ( $H_0$ ): The population mean is  $\mu = 30$  kWh.
- Alternative Hypothesis ( $H_a$ ): The population mean is not  $\mu = 30$  kWh ( $\mu \neq 30$ ).

### ❷ Use the $t$ -test formula since the population standard deviation is not known:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{32 - 30}{6/\sqrt{6}} = \frac{2}{6/2.449} \approx 0.816$$

- $\bar{x} = 32$  (sample mean)
- $s = 6$  (sample standard deviation)
- $n = 6$  (sample size)
- $\sigma = 30$  (population mean)

## Critical Values in $t$ -Table

---

### Small sample sizes

- Degrees of freedom for a  $t$ -test:

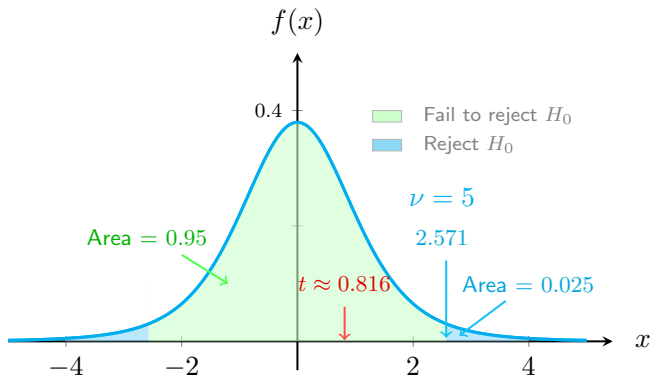
$$\nu = \underbrace{n}_{\text{sample size}} - 1 = 6 - 1 = 5$$

- $t$ -distributions with  $\nu$  degrees of freedom at a 95% confidence interval (two-tailed)

| $\nu = 1$ | $\nu = 5$ | $\nu = 10$ | $\nu \rightarrow +\infty$ |
|-----------|-----------|------------|---------------------------|
| 12.706    | 2.571     | 2.228      | 1.960                     |

- The critical  $t$ -value

$$t_{\nu, (1-0.95)/2} = t_{5, 0.025} = 2.571$$



Test statistic  $|t| < 2.571 \Rightarrow$  Fail to reject the null hypothesis



# Implementing $t$ -Test for Small Sample Size

## Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh. A random sample of 6 households has an average daily energy consumption of 32 kWh, with a sample standard deviation of 6 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

## Steps:

- ② Use the  $t$ -test formula:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{32 - 30}{6/\sqrt{6}} = \frac{2}{6/2.449} \approx 0.816$$

- ③ Decision rule at a 95% confidence interval

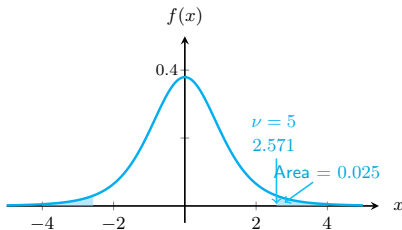
- Reject  $H_0$  if  $|t| > 2.571$ .
- Otherwise, fail to reject  $H_0$ .

- ④ Interpretation

- The test statistic  $|t| = 0.816 < 2.571$ .
- Thus, we fail to reject the null hypothesis.
- There is not enough evidence to conclude that the average daily energy consumption differs from the company's claim of 30 kWh.

# Summary

- Student  $t$ -distribution of  $\nu$  degrees of freedom



Student  $t$ -distribution

- Population: mean  $\mu$  (✓), standard deviation  $\sigma$  (X)
- Sample: mean  $\bar{x}$ , standard deviation  $s$ , and small sample size  $n$

- $t$ -statistic:  $t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \Rightarrow t\text{-test}$

- 95% confidence interval:  $\bar{x} \pm \underbrace{t_{\nu, 0.025}}_{\nu = n - 1} \times \frac{s}{\sqrt{n}}$



W. S. Gosset in Guinness

## **Teaching Concept**

# Method

---

- use math
- use figures
- use examples
- use data
- use codes
- use latex to create all examples

# Thanks for your attention!

## Any Questions?

### About me:

 Homepage: <https://xinychen.github.io>

 How to reach me: [chenxy346@gmail.com](mailto:chenxy346@gmail.com)