

Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 5: Modeling and Errors

Xinyu Chen

Assistant Professor

University of Central Florida

Quizzes Now!

- **Today's participation** (ungraded survey): Please check out
“**Class Participation Quiz 11**”
Time slot: **2:30PM – 3:00PM**
on Canvas.
- Online engagement (graded quizzes)
“**Quiz 11**”
Deadline: **11:59PM, February 9, 2026**
on Canvas.

Truncation Errors & Taylor Series

Learning objectives:

- Define truncation errors in numerical methods
- Understand the role of Taylor series in function approximation
- Derive finite difference approximations using Taylor expansions
- Apply Taylor series to estimate derivatives numerically

Truncation Errors

What is a truncation error?

- Error from using an approximation instead of an exact mathematical procedure
 - Example: Approximating a derivative

$$\frac{dv}{dt} \approx \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

Intuition:

numerator → difference in velocity Δv
denominator → difference in time Δt

The difference equation is an approximation → introduces truncation error

Truncation Errors

Why study truncation errors?

- Understand accuracy of numerical methods
- Choose appropriate approximations for given problems
- Estimate error bounds for computations
- Improve algorithms by reducing error terms

Taylor Series

Taylor theorem:

- Any **smooth function** can be approximated by a **polynomial**
- If f and its first $n + 1$ derivatives are continuous on $[a, x]$, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

Recall that

- $f'(a)$: first-order derivative
- $f''(a)$: second-order derivative
- ...
- $f^{(n)}(a)$: n th-order derivative

with

- $n!$: factorial of integer n [Python: `np.prod(np.arange(1, n+1))`]

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Taylor Series

Intuition: We predict $f(x_{i+1})$ using information at x_i :

- **Zeroth-order** (constant) approximation:

$$f(x_{i+1}) \approx f(x_i)$$

Only works if f is **constant** between x_i and x_{i+1}

Taylor Series

Intuition: We predict $f(x_{i+1})$ using information at x_i :

- **Zeroth-order** (constant) approximation:

$$f(x_{i+1}) \approx f(x_i)$$

Only works if f is **constant** between x_i and x_{i+1}

- **First-order** Taylor approximation:

Add **slope** information:

$$f(x_{i+1}) \approx f(x_i) + \cancel{f'(x_i)} \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

- Represents a **straight line** (linear approximation)
 - Exact if f is linear

Taylor Series

Intuition: We predict $f(x_{i+1})$ using information at x_i :

- **Zeroth-order** (constant) approximation:

$$f(x_{i+1}) \approx f(x_i)$$

Only works if f is **constant** between x_i and x_{i+1}

- **First-order** Taylor approximation:

Add slope information:

$$f(x_{i+1}) \approx f(x_i) + \cancel{f'(x_i)} \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

- Represents a **straight line** (linear approximation)
 - Exact if f is linear

- **Euler's formula:**

$$\underbrace{y_{i+1}}_{\text{next value}} = \underbrace{y_i}_{\text{current value}} + \underbrace{\Delta x}_{\text{step size}} \cdot \underbrace{f(x_i, y_i)}_{\text{slope}} \quad x_{i+1} = x_i + \underbrace{\Delta x}_{\text{step size}}$$

Taylor Series

Intuition: We predict $f(x_{i+1})$ using information at x_i :

- **Second-order** Taylor approximation:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

- Captures quadratic behavior
- Better accuracy for smooth functions

Taylor Series

- General Taylor polynomial

$$\begin{aligned}f(x_{i+1}) &\approx f(x_i) \\&\quad + \color{red}{f'(x_i)}(x_{i+1} - x_i) \\&\quad + \frac{\color{red}{f''(x_i)}}{2!}(x_{i+1} - x_i)^2 \\&\quad + \dots \\&\quad + \frac{\color{red}{f^{(n)}(x_i)}}{n!}(x_{i+1} - x_i)^n \\&= \sum_{k=0}^n \frac{\color{red}{f^{(k)}(x_i)}}{k!}(x_{i+1} - x_i)^k\end{aligned}$$

Higher $n \rightarrow$ better approximation (if function is smooth)

Taylor Series Approximation of a Polynomial

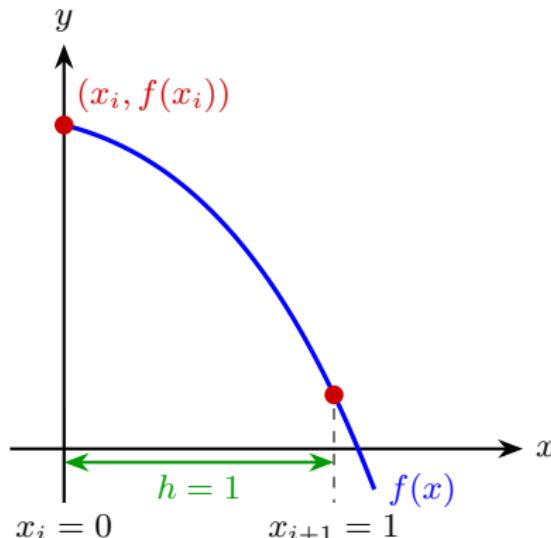
Problem statement:

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

at $x_{i+1} = 1$ starting from $x_i = 0$ with step size $h = x_{i+1} - x_i = 1$.

- **Goal:** Predict $f(1)$ using Taylor approximations of increasing order.



Taylor Series Approximation of a Polynomial

Problem statement:

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

- **Goal:** Predict $f(1)$ using Taylor approximations of increasing order.

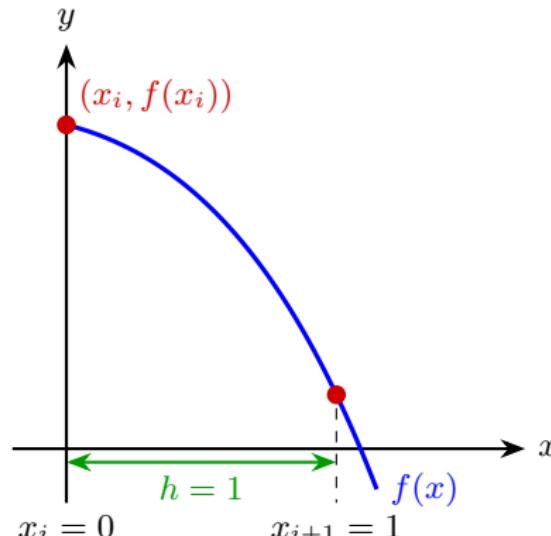
- Function $f(x)$:

$$f(0) = 1.2$$

$$\begin{aligned} f(1) &= -0.1 - 0.15 - 0.5 \\ &\quad - 0.25 + 1.2 = 0.2 \end{aligned}$$

- True value to predict:

$$f(1) = 0.2$$



Taylor Series Approximation of a Polynomial

First-order approximation for $f(1)$:

- Need first-order derivative at $x_i = 0$:

$$\begin{aligned}f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow f'(0) &= \textcolor{blue}{-0.25}\end{aligned}$$

- First-order Taylor series:

$$\begin{aligned}f(x_{i+1}) &\approx f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i) \\ \Rightarrow f(1) &\approx 1.2 + (\textcolor{blue}{-0.25}) \times 1 = \textcolor{red}{0.95}\end{aligned}$$

Taylor Series Approximation of a Polynomial

Second-order approximation for $f(1)$:

- Need second-order derivative at $x_i = 0$:

$$\begin{aligned}f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow f''(x) &= -1.2x^2 - 0.9x - 1 \\ \Rightarrow f''(0) &= \textcolor{blue}{-1}\end{aligned}$$

- Second-order Taylor series:

$$\begin{aligned}f(x_{i+1}) &\approx f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i) + \frac{f''(x_i)}{2!} (x_{i+1} - x_i)^2 \\ \Rightarrow f(1) &\approx 1.2 - 0.25 \times 1 + \left(\frac{\textcolor{blue}{-1}}{2}\right) \times 1^2 = \textcolor{red}{0.45}\end{aligned}$$

Taylor Series Approximation of a Polynomial

Third-order approximation for $f(1)$:

- Need third-order derivative at $x_i = 0$:

$$\begin{aligned}f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow f''(x) &= -1.2x^2 - 0.9x - 1 \\ \Rightarrow f'''(x) &= -2.4x - 0.9 \\ \Rightarrow f'''(0) &= \textcolor{blue}{-0.9}\end{aligned}$$

- Third-order Taylor series:

$$f(1) \approx 1.2 - 0.25 \times 1 - 0.5 \times 1^2 + \left(\frac{\textcolor{blue}{-0.9}}{3!} \right) \times 1^3 = \textcolor{red}{0.3}$$

Taylor Series Approximation of a Polynomial

Fourth-order approximation for $f(1)$:

- Need fourth-order derivative at $x_i = 0$:

$$\begin{aligned}f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow f''(x) &= -1.2x^2 - 0.9x - 1 \\ \Rightarrow f'''(x) &= -2.4x - 0.9 \\ \Rightarrow f^{(4)}(x) &= -2.4 \\ \Rightarrow f^{(4)}(0) &= \textcolor{blue}{-2.4}\end{aligned}$$

- Fourth-order Taylor series:

$$f(1) \approx 1.2 - 0.25 \times 1 - 0.5 \times 1^2 - 0.15 \times 1^3 - \left(\frac{\textcolor{blue}{-2.4}}{4!} \right) \times 1^4 = \textcolor{red}{0.2}$$

Taylor Series Approximation of a Polynomial

- Error:

$$\varepsilon = |f(x_{i+1}) - \hat{f}(x_{i+1})|$$

where the approximation is

$$\hat{f}(x_{i+1}) = \sum_{k=0}^n \frac{f^{(k)}(x_i)}{k!} (x_{i+1} - x_i)^k$$

- Summary of results:

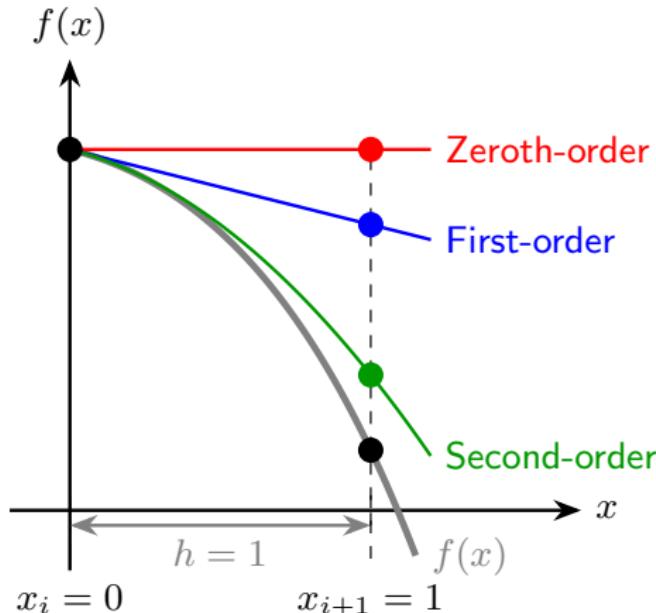
Order (n)	Approximation $\hat{f}(1)$	Error ε
1	0.95	$ 0.2 - 0.95 = 0.75$
2	0.45	$ 0.2 - 0.45 = 0.25$
3	0.3	$ 0.2 - 0.3 = 0.1$
4	0.2	$ 0.2 - 0.2 = 0$

- Error decreases as order increases
- With $n = 4$, approximation is exact because $f(x)$ is a 4th-degree polynomial

Taylor Series Approximation of a Polynomial

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$



Taylor Series Approximation of a Polynomial

Concluding remarks:

- Taylor series approximates functions using derivatives at a point
- Higher-order terms improve accuracy by capturing curvature
- For an m -degree polynomial, a Taylor series of order m gives the exact function
- Truncation error quantifies the approximation error
- This example illustrates the power of Taylor series for function approximation in numerical methods

Taylor Series Approximation of $\sin(x)$

Maclaurin series:

- A Taylor series expansion of a function about 0
- Taylor series approximation:

$$f(x_{i+1}) \approx \sum_{k=0}^n \frac{f^{(k)}(x_i)}{k!} (x_{i+1} - x_i)^k$$

- Set $x_i = 0$ and $x = x_{i+1}$, then

$$f(x) \approx \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

- Derivative of $f(x) = \sin(x)$:

$$f'(x) = \underbrace{\cos(x)}_{\cos(0)=1}, \quad f''(x) = -\underbrace{\sin(x)}_{\sin(0)=0}, \quad f'''(x) = -\underbrace{\cos(x)}_{\cos(0)=1}, \quad f^{(4)}(x) = \underbrace{\sin(x)}_{\sin(0)=0}$$

- Formula:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \dots$$

Quick Summary

Monday's Class:

- Definition of truncation error
- Taylor series of any smooth function (approximated by polynomial)
- Approximation with zeroth-, first-, second-, and higher-order information
- Example: Taylor series approximation of a polynomial function
- Revisit Taylor series approximation of $\sin(x)$