

Definition, Properties, and Derivatives of Matrix Traces

A Class for Undergraduate Students

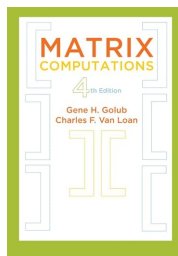
Xinyu Chen

March 31, 2024

Class Targets

Throughout this class, you will:

- Understand some basic concepts (e.g., norms, traces, and derivatives)
- Connect them with linear algebra and machine learning
- Use matrix norms and traces in matrix computations (very useful!)



Vector & Matrix

Notation:

- On the vector $\boldsymbol{x} \in \mathbb{R}^n$ of length n

$$\boldsymbol{x} = (x_1, x_2, \dots, x_n)^\top \quad \text{or} \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

whose i -th entry is x_i , $i \in [n]$.¹

¹The set of integers, $\{1, 2, \dots, n\}$, is represented by $[n]$, $n \in \mathbb{Z}^+$.

Vector & Matrix

Notation:

- On the vector $\mathbf{x} \in \mathbb{R}^n$ of length n

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

whose i -th entry is x_i , $i \in [n]$.¹

- On the matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$ with m rows and n columns

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

whose (i, j) -th entry is x_{ij} , $i \in [m], j \in [n]$.

¹The set of integers, $\{1, 2, \dots, n\}$, is represented by $[n]$, $n \in \mathbb{Z}^+$.

Vector Norms

A number of concepts to mention, e.g., ℓ_0 -norm, ℓ_1 -norm, and ℓ_2 -norm.

- **Definition.** For any vector $\mathbf{x} \in \mathbb{R}^n$, the ℓ_2 -norm of \mathbf{x} is given by

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^n x_i^2}$$

where $x_i, \forall i \in [n]$ is the i -th entry of \mathbf{x} .

Vector Norms

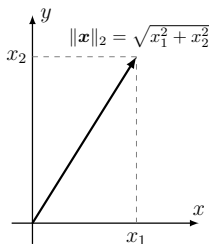
A number of concepts to mention, e.g., ℓ_0 -norm, ℓ_1 -norm, and ℓ_2 -norm.

- **Definition.** For any vector $\mathbf{x} \in \mathbb{R}^n$, the ℓ_2 -norm of \mathbf{x} is given by

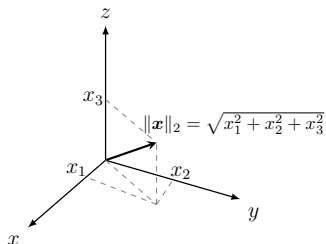
$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^n x_i^2}$$

where $x_i, \forall i \in [n]$ is the i -th entry of \mathbf{x} .

- Intuitive examples:



On $\mathbf{x} = (x_1, x_2)^\top$

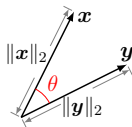


On $\mathbf{x} = (x_1, x_2, x_3)^\top$

Inner Product

- Basics: For any $\mathbf{x} = (x_1, x_2)^\top$ and $\mathbf{y} = (y_1, y_2)^\top$, the angle θ can be computed by

$$\cos \theta = \frac{x_1 y_1 + x_2 y_2}{\sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}}$$



- in which
 - ℓ_2 -norm:

$$\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2} \quad \|\mathbf{y}\|_2 = \sqrt{y_1^2 + y_2^2}$$

- inner product:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = x_1 y_1 + x_2 y_2$$

- It leads to

$$\cos \theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2}$$

Inner Product

- **Definition.** For any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, the inner product is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^n x_i y_i$$

Example. Given $\mathbf{x} = (1, 2, 3, 4)^\top$ and $\mathbf{y} = (2, -1, 3, 0)^\top$, write down the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$.

In this case,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = 1 \times 2 + 2 \times (-1) + 3 \times 3 + 4 \times 0 = 9$$

Inner Product

- **Definition.** For any matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{m \times n}$, the inner product is

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum_{i=1}^m \sum_{j=1}^n x_{ij} y_{ij}$$

Example. Given $\mathbf{X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, write down the inner product $\langle \mathbf{X}, \mathbf{Y} \rangle$.

In this case,

$$\langle \mathbf{X}, \mathbf{Y} \rangle = 2 \times 2 + 1 \times (-1) + 1 \times (-1) + 2 \times 2 + 1 \times (-1) + 3 \times 2 = 11$$

Frobenius Norm

- **Definition.** For any matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$, the Frobenius norm of \mathbf{X} is given by

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n x_{ij}^2}$$

where x_{ij} , $\forall i \in [m], j \in [n]$ is the (i, j) -th entry of \mathbf{X} .

Example. Given $\mathbf{X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, write down the Frobenius norm of \mathbf{X} .

$$\|\mathbf{X}\|_F = \sqrt{2^2 + 1^2 + 1^2 + 1^2 + 2^2 + 1^2 + 3^2} = \sqrt{21}$$

Frobenius Norm

- Connection with ℓ_2 -norm:

$$\|\mathbf{X}\|_F = \sqrt{\sum_{j=1}^n \sum_{i=1}^m x_{ij}^2} = \sqrt{\sum_{j=1}^n \|\mathbf{x}_j\|_2^2}$$

with the column vectors $\mathbf{x}_j \in \mathbb{R}^m$, $j \in [n]$ such that

$$\mathbf{X} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ | & | & & | \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Definition of Matrix Trace

- **Definition.** For any **square matrix** $\mathbf{X} \in \mathbb{R}^{n \times n}$, the matrix trace (denoted by $\text{tr}(\cdot)$) is the sum of diagonal entries, i.e.,

$$\text{tr}(\mathbf{X}) = \sum_{i=1}^n \underbrace{x_{ii}}_{\text{diagonal}}$$

where $x_{ii}, \forall i \in [n]$ is the (i, i) -th entry of \mathbf{X} . Thus, $\text{tr}(\mathbf{X}) = \text{tr}(\mathbf{X}^\top)$.

Example. Given $\mathbf{X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, write down the matrix trace of \mathbf{X} .

$$\text{tr}(\mathbf{X}) = 2 + 2 + 3 = 7$$

Property: $\text{tr}(\mathbf{X} + \mathbf{Y}) = \text{tr}(\mathbf{X}) + \text{tr}(\mathbf{Y})$

- **Property.** For any square matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n \times n}$, it always holds that

$$\text{tr}(\mathbf{X} + \mathbf{Y}) = \text{tr}(\mathbf{X}) + \text{tr}(\mathbf{Y})$$

Example. Given $\mathbf{X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, write down $\text{tr}(\mathbf{X} + \mathbf{Y})$.

In this case,

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 2+2 & 1-1 & 1+0 \\ 1-1 & 2+2 & 1-1 \\ 0+0 & 0-1 & 3+2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & -1 & 5 \end{bmatrix}$$

Thus, $\text{tr}(\mathbf{X} + \mathbf{Y}) = 4 + 4 + 5 = 13$. Note that $\text{tr}(\mathbf{X}) = 7$ and $\text{tr}(\mathbf{Y}) = 6$, it shows that $\text{tr}(\mathbf{X} + \mathbf{Y}) = \text{tr}(\mathbf{X}) + \text{tr}(\mathbf{Y}) = 13$.

- **Variant.** For any $\alpha, \beta \in \mathbb{R}$, we have

$$\text{tr}(\alpha \mathbf{X} + \beta \mathbf{Y}) = \alpha \text{tr}(\mathbf{X}) + \beta \text{tr}(\mathbf{Y})$$

Property: $\text{tr}(\mathbf{XY}) = \text{tr}(\mathbf{YX})$

- **Property.** For any matrices $\mathbf{X} \in \mathbb{R}^{m \times n}$ and $\mathbf{Y} \in \mathbb{R}^{n \times m}$, it always holds that

$$\text{tr}(\mathbf{XY}) = \text{tr}(\mathbf{YX})$$

- **Proof.**

$$\begin{aligned} \text{tr}(\mathbf{XY}) &= [\mathbf{XY}]_{11} + [\mathbf{XY}]_{22} + \cdots + [\mathbf{XY}]_{mm} \\ &= \underbrace{x_{11}y_{11} + x_{12}y_{21} + \cdots + x_{1n}y_{n1}}_{\text{the first row of } \mathbf{X} \text{ times the first column of } \mathbf{Y}} \\ &\quad + x_{21}y_{12} + x_{22}y_{22} + \cdots + x_{2n}y_{n2} \\ &\quad + \cdots + x_{m1}y_{1m} + x_{m2}y_{2m} + \cdots + x_{mn}y_{nm} \end{aligned}$$

Property: $\text{tr}(\mathbf{XY}) = \text{tr}(\mathbf{YX})$

- **Property.** For any matrices $\mathbf{X} \in \mathbb{R}^{m \times n}$ and $\mathbf{Y} \in \mathbb{R}^{n \times m}$, it always holds that

$$\text{tr}(\mathbf{XY}) = \text{tr}(\mathbf{YX})$$

- **Proof.**

$$\begin{aligned} \text{tr}(\mathbf{XY}) &= [\mathbf{XY}]_{11} + [\mathbf{XY}]_{22} + \cdots + [\mathbf{XY}]_{mm} \\ &= \underbrace{x_{11}y_{11} + x_{12}y_{21} + \cdots + x_{1n}y_{n1}}_{\text{the first row of } \mathbf{X} \text{ times the first column of } \mathbf{Y}} \\ &\quad + x_{21}y_{12} + x_{22}y_{22} + \cdots + x_{2n}y_{n2} \\ &\quad + \cdots + x_{m1}y_{1m} + x_{m2}y_{2m} + \cdots + x_{mn}y_{nm} \\ &= x_{11}y_{11} + x_{12}y_{21} + \cdots + x_{1n}y_{n1} \\ &\quad + x_{21}y_{12} + x_{22}y_{22} + \cdots + x_{2n}y_{n2} \\ &\quad + \cdots + x_{m1}y_{1m} + x_{m2}y_{2m} + \cdots + x_{mn}y_{nm} \end{aligned}$$

Property: $\text{tr}(\mathbf{XY}) = \text{tr}(\mathbf{YX})$

- **Property.** For any matrices $\mathbf{X} \in \mathbb{R}^{m \times n}$ and $\mathbf{Y} \in \mathbb{R}^{n \times m}$, it always holds that

$$\text{tr}(\mathbf{XY}) = \text{tr}(\mathbf{YX})$$

- **Proof.**

$$\begin{aligned}
 \text{tr}(\mathbf{XY}) &= [\mathbf{XY}]_{11} + [\mathbf{XY}]_{22} + \cdots + [\mathbf{XY}]_{mm} \\
 &= \underbrace{x_{11}y_{11} + x_{12}y_{21} + \cdots + x_{1n}y_{n1}}_{\text{the first row of } \mathbf{X} \text{ times the first column of } \mathbf{Y}} \\
 &\quad + x_{21}y_{12} + x_{22}y_{22} + \cdots + x_{2n}y_{n2} \\
 &\quad + \cdots + x_{m1}y_{1m} + x_{m2}y_{2m} + \cdots + x_{mn}y_{nm} \\
 &= x_{11}y_{11} + x_{12}y_{21} + \cdots + x_{1n}y_{n1} \\
 &\quad + x_{21}y_{12} + x_{22}y_{22} + \cdots + x_{2n}y_{n2} \\
 &\quad + \cdots + x_{m1}y_{1m} + x_{m2}y_{2m} + \cdots + x_{mn}y_{nm} \\
 &= \underbrace{y_{11}x_{11} + y_{12}x_{21} + \cdots + y_{1m}x_{m1}}_{\text{the first row of } \mathbf{Y} \text{ times the first column of } \mathbf{X}} \\
 &\quad + y_{21}x_{12} + y_{22}x_{22} + \cdots + y_{2m}x_{m2} \\
 &\quad + \cdots + y_{n1}x_{1n} + \cdots + y_{n2}x_{2n} + \cdots + y_{nm}x_{mn} \\
 &= [\mathbf{YX}]_{11} + [\mathbf{YX}]_{22} + \cdots + [\mathbf{YX}]_{nn} = \text{tr}(\mathbf{YX})
 \end{aligned}$$

Property: $\text{tr}(\mathbf{XY}) = \text{tr}(\mathbf{YX})$

Example. Given $\mathbf{X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, write down $\text{tr}(\mathbf{XY})$ and $\text{tr}(\mathbf{YX})$, respectively.

In this case,

$$\mathbf{XY} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & 6 \end{bmatrix} \quad \mathbf{YX} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & -2 \\ -1 & -2 & 5 \end{bmatrix}$$

Thus,

$$\text{tr}(\mathbf{XY}) = 3 + 2 + 6 = 11 \quad \text{tr}(\mathbf{YX}) = 3 + 3 + 5 = 11$$

Property: $\|X\|_F^2 = \text{tr}(X^\top X)$

- **Property.** For any matrix $X \in \mathbb{R}^{m \times n}$, it always holds that

$$\|X\|_F^2 = \text{tr}(X^\top X)$$

- **Proof.**

$$\begin{aligned} \text{tr}(X^\top X) &= [X^\top X]_{11} + [X^\top X]_{22} + \cdots + [X^\top X]_{nn} \\ &= x_{11}^2 + x_{21}^2 + \cdots + x_{m1}^2 \\ &\quad + x_{12}^2 + x_{22}^2 + \cdots + x_{m2}^2 \\ &\quad + \cdots + x_{1n}^2 + x_{2n}^2 + \cdots + x_{mn}^2 \\ &= \sum_{i=1}^m x_{i1}^2 + \sum_{i=1}^m x_{i2}^2 + \cdots + \sum_{i=1}^m x_{in}^2 \\ &= \sum_{i=1}^m \sum_{j=1}^n x_{ij}^2 \\ &= \|X\|_F^2 \end{aligned}$$

Property: $\langle X, Y \rangle = \text{tr}(X^\top Y)$

- **Property.** For any matrices $X, Y \in \mathbb{R}^{m \times n}$, it always holds that

$$\langle X, Y \rangle = \text{tr}(X^\top Y)$$

- **Proof.**

$$\begin{aligned} \text{tr}(X^\top Y) &= [X^\top Y]_{11} + [X^\top Y]_{22} + \cdots + [X^\top Y]_{nn} \\ &= x_{11}y_{11} + x_{21}y_{21} + \cdots + x_{m1}y_{m1} \\ &\quad + x_{12}y_{12} + x_{22}y_{22} + \cdots + x_{m2}y_{m2} \\ &\quad + \cdots + x_{1n}y_{1n} + x_{2n}y_{2n} + \cdots + x_{mn}y_{mn} \\ &= \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle + \cdots + \langle x_n, y_n \rangle \\ &= \langle X, Y \rangle \end{aligned}$$

where $x_i, y_i \in \mathbb{R}^m$, $\forall i \in [n]$ are the i -th column vectors of X and Y , respectively.

Property: $\langle \mathbf{X}, \mathbf{Y} \rangle = \text{tr}(\mathbf{X}^\top \mathbf{Y})$

Example. Given $\mathbf{X} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, write down $\langle \mathbf{X}, \mathbf{Y} \rangle$ and $\text{tr}(\mathbf{X}^\top \mathbf{Y})$, respectively.

Recall that

$$\langle \mathbf{X}, \mathbf{Y} \rangle = 2 \times 2 + 1 \times (-1) + 1 \times (-1) + 2 \times 2 + 1 \times (-1) + 3 \times 2 = 11$$

For the matrix,

$$\mathbf{X}^\top \mathbf{Y} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -2 \\ 1 & -2 & 5 \end{bmatrix}$$

we have $\text{tr}(\mathbf{X}^\top \mathbf{Y}) = 3 + 3 + 5 = 11$.

Derivatives

A quick revisit!

- **Derivative.** Given a scalar function $f(x)$ of the single variable x , the derivative is defined by

$$\frac{d f(x)}{d x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$$

Derivatives

A quick revisit!

- **Derivative.** Given a scalar function $f(x)$ of the single variable x , the derivative is defined by

$$\frac{d f(x)}{d x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$$

- **Partial derivatives.** Given a scalar function $f(x, y)$ of two variables x, y , the partial derivatives are defined by

$$\begin{cases} \frac{\partial f(x, y)}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\ \frac{\partial f(x, y)}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \end{cases}$$

Derivatives

Example. Given $f(\mathbf{x}) = \|\mathbf{x}\|_2^2$, write down the derivative $\frac{d f(\mathbf{x})}{d \mathbf{x}}$.

First, notice that the function $f(\mathbf{x})$ can be written as

$$f(\mathbf{x}) = f(x_1, x_2, \dots, x_n) = x_1^2 + x_2^2 + \dots + x_n^2$$

Hence, the partial derivatives of $f(x_1, x_2, \dots, x_n)$ with respect to x_1, x_2, \dots, x_n are

$$\begin{aligned} \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_1} &= 2x_1 \\ \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_2} &= 2x_2 \\ &\vdots \\ \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_n} &= 2x_n \end{aligned} \quad \Rightarrow \quad \frac{d f(\mathbf{x})}{d \mathbf{x}} = \begin{bmatrix} \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_1} \\ \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_2} \\ \vdots \\ \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{bmatrix} = 2\mathbf{x}$$

Derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{X})$

- **Function.** For any square matrix $\mathbf{X} \in \mathbb{R}^{n \times n}$, what is the derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{X})$?
- **Derivative.** Since $f(\mathbf{X}) = \sum_{i=1}^n x_{ii}$, we have

$$\begin{aligned} \frac{d f(\mathbf{X})}{d \mathbf{X}} &= \begin{bmatrix} \frac{\partial f(\mathbf{X})}{\partial x_{11}} & \frac{\partial f(\mathbf{X})}{\partial x_{12}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{1n}} \\ \frac{\partial f(\mathbf{X})}{\partial x_{21}} & \frac{\partial f(\mathbf{X})}{\partial x_{22}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(\mathbf{X})}{\partial x_{n1}} & \frac{\partial f(\mathbf{X})}{\partial x_{n2}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{nn}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \mathbf{I}_n \end{aligned}$$

Derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{A}\mathbf{X})$

- **Function.** For any matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{X} \in \mathbb{R}^{n \times m}$, what is the derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{A}\mathbf{X})$?
- **Derivative.** Since

$$f(\mathbf{X}) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} x_{ji}$$

Hence, the partial derivative of $f(\mathbf{X})$ with respect to the entry x_{ji} is given by

$$\frac{\partial f(\mathbf{X})}{\partial x_{ji}} = a_{ij}$$

As a result, we have

$$\frac{d f(\mathbf{X})}{d \mathbf{X}} = \begin{bmatrix} \frac{\partial f(\mathbf{X})}{\partial x_{11}} & \frac{\partial f(\mathbf{X})}{\partial x_{12}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{1m}} \\ \frac{\partial f(\mathbf{X})}{\partial x_{21}} & \frac{\partial f(\mathbf{X})}{\partial x_{22}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(\mathbf{X})}{\partial x_{n1}} & \frac{\partial f(\mathbf{X})}{\partial x_{n2}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{nm}} \end{bmatrix} = \mathbf{A}^\top$$

- By the way, what is the derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{X}\mathbf{A})$? How about $f(\mathbf{X}) = \text{tr}(\mathbf{A}^\top \mathbf{X}^\top)$?

Derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{AXB})$

- **Function.** For any matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{X} \in \mathbb{R}^{n \times d}$, and $\mathbf{B} \in \mathbb{R}^{d \times m}$, what is the derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{AXB})$?
- **Derivative.** Since

$$f(\mathbf{X}) = \sum_{i=1}^m [\mathbf{AXB}]_{i,i} = \sum_{i=1}^m \sum_{j=1}^n a_{ij} [\mathbf{XB}]_{j,i} = \sum_{i=1}^m \sum_{j=1}^n a_{ij} \sum_{k=1}^d x_{jk} b_{ki}$$

Hence, the partial derivative of $f(\mathbf{X})$ with respect to the entry x_{jk} is given by

$$\frac{\partial f(\mathbf{X})}{\partial x_{jk}} = \sum_{i=1}^m a_{ij} b_{ki} = [\mathbf{A}^\top \mathbf{B}^\top]_{j,k}$$

As a result, we have

$$\frac{d f(\mathbf{X})}{d \mathbf{X}} = \begin{bmatrix} \frac{\partial f(\mathbf{X})}{\partial x_{11}} & \frac{\partial f(\mathbf{X})}{\partial x_{12}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{1d}} \\ \frac{\partial f(\mathbf{X})}{\partial x_{21}} & \frac{\partial f(\mathbf{X})}{\partial x_{22}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{2d}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(\mathbf{X})}{\partial x_{n1}} & \frac{\partial f(\mathbf{X})}{\partial x_{n2}} & \cdots & \frac{\partial f(\mathbf{X})}{\partial x_{nd}} \end{bmatrix} = \mathbf{A}^\top \mathbf{B}^\top$$

Derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{X}^\top \mathbf{A} \mathbf{X})$

- **Function.** For any matrices $\mathbf{A}, \mathbf{X} \in \mathbb{R}^{n \times n}$, what is the derivative of $f(\mathbf{X}) = \text{tr}(\mathbf{X}^\top \mathbf{A} \mathbf{X})$?
- **Derivative.**

$$\begin{aligned}
 \frac{d f(\mathbf{X})}{d \mathbf{X}} &= \frac{d \text{tr}(\mathbf{X}^\top \mathbf{B})}{d \mathbf{X}} + \frac{d \text{tr}(\mathbf{C} \mathbf{X})}{d \mathbf{X}} \\
 &= \frac{d \text{tr}(\mathbf{B}^\top \mathbf{X})}{d \mathbf{X}} + \frac{d \text{tr}(\mathbf{C} \mathbf{X})}{d \mathbf{X}} \\
 &= \mathbf{B} + \mathbf{C}^\top \\
 &= \mathbf{A} \mathbf{X} + \mathbf{A}^\top \mathbf{X}
 \end{aligned}$$

where $\mathbf{B} \triangleq \mathbf{A} \mathbf{X}$ and $\mathbf{C} \triangleq \mathbf{X}^\top \mathbf{A}$.

Derivative of $f(X) = \text{tr}(AXBXC)$

- **Function.** For any matrices $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^{n \times d}$, $B \in \mathbb{R}^{d \times n}$, and $C \in \mathbb{R}^{d \times m}$, what is the derivative of $f(X) = \text{tr}(AXBXC)$?
- **Derivative.**

$$\begin{aligned} \frac{d f(X)}{d X} &= \frac{d \text{tr}(A X D)}{d X} + \frac{d \text{tr}(E X C)}{d X} \\ &= A^{\top} D^{\top} + E^{\top} C^{\top} \\ &= \mathbf{A^{\top} C^{\top} X^{\top} B^{\top} + B^{\top} X^{\top} A^{\top} C^{\top}} \end{aligned}$$

where $D \triangleq BXC$ and $E \triangleq AXB$.

Derivative of $f(\mathbf{X}) = \|\mathbf{A}\mathbf{X}\|_F^2$

- **Function.** For any matrices $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{X} \in \mathbb{R}^{n \times d}$, what is the derivative of $f(\mathbf{X}) = \|\mathbf{A}\mathbf{X}\|_F^2$?
- **Derivative.** Since

$$f(\mathbf{X}) = \text{tr}(\mathbf{X}^\top \mathbf{A}^\top \mathbf{A} \mathbf{X})$$

Hence, we have

$$\begin{aligned} \frac{d f(\mathbf{X})}{d \mathbf{X}} &= \frac{d \text{tr}(\mathbf{B} \mathbf{X})}{d \mathbf{X}} + \frac{d \text{tr}(\mathbf{X}^\top \mathbf{B}^\top)}{d \mathbf{X}} \\ &= \frac{d \text{tr}(\mathbf{B} \mathbf{X})}{d \mathbf{X}} + \frac{d \text{tr}(\mathbf{B} \mathbf{X})}{d \mathbf{X}} \\ &= 2\mathbf{B}^\top \\ &= 2\mathbf{A}^\top \mathbf{A} \mathbf{X} \end{aligned}$$

where $\mathbf{B} \triangleq \mathbf{X}^\top \mathbf{A}^\top \mathbf{A}$.

Orthogonal Procrustes Problem (Optional)

- **Orthogonal Procrustes problem:**

For any $Q \in \mathbb{R}^{m \times r}$, $m \geq r$, the solution to

$$\begin{aligned} \min_F \quad & \|F - Q\|_F^2 \\ \text{s. t.} \quad & \underbrace{F^\top F = I_r}_{\text{orthogonal}} \end{aligned}$$

is

$$F := UV^\top$$

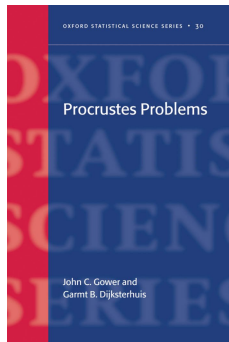
where

$$\underbrace{Q = U\Sigma V^\top}_{\text{singular value decomposition}}$$

- Equivalent form:

$$\|F - Q\|_F^2 = \text{tr}(\underbrace{F^\top F}_{=I_r} - F^\top Q - Q^\top F + \underbrace{Q^\top Q}_{\text{const.}}) = -2 \text{tr}(F^\top Q) + \text{const.}$$

$$\implies F =: \arg \min_{F^\top F = I_r} \|F - Q\|_F^2 = \arg \max_{F^\top F = I_r} \text{tr}(F^\top Q)$$



A Quick Look

Content:

- Vector structure, ℓ_2 -norm
- Matrix structure, Frobenius norm
- Inner product
- Definition, properties, and derivatives of matrix trace (including a lot of examples)

For your need!

- Slides: https://xinychen.github.io/slides/matrix_trace.pdf
- E-book:
https://xinychen.github.io/books/spatiotemporal_low_rank_models.pdf

Reference material:

- The matrix cookbook:
<https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf>

Special thanks to

- Zhanhong Cheng (McGill University)
- Benzhenh Li (UESTC)
- Fuqiang Liu (McGill University)
- Sheng Liu (UESTC)

for providing feedback and suggestions.

Thanks for your attention!

Any Questions?

About me:

- Homepage: <https://xinychen.github.io>
- How to reach me: chenxy346@gmail.com