Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 5: Modeling and Errors

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Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "`Class Participation Quiz 11`"

  Time slot: **2:30PM – 3:00PM**

  on Canvas.

- Online engagement (graded quizzes)

  "`Quiz 11`"

  Deadline: **11:59PM, February 9, 2026**

  on Canvas.

Truncation Errors & Taylor Series

Learning objectives:

- Define truncation errors in numerical methods
- Understand the role of Taylor series in function approximation
- Derive finite difference approximations using Taylor expansions
- Apply Taylor series to estimate derivatives numerically

Truncation Errors

What is a truncation error?

- Error from using an approximation instead of an exact mathematical procedure
- Example: Approximating a derivative

$$\frac{\mathrm{d}\, v}{\mathrm{d}\, t} \approx \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

The difference equation is an approximation $\rightarrow$ introduces truncation error

Truncation Errors

Why study truncation errors?

- Understand accuracy of numerical methods
- Choose appropriate approximations for given problems
- Estimate error bounds for computations
- Improve algorithms by reducing error terms

Taylor Series

Taylor theorem:

- Any smooth function can be approximated by a polynomial
- If f and its first $n+1$ derivatives are continuous on $[a, x]$, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

Recall that

- $f'(a)$: first-order derivative
- $f''(a)$: second-order derivative
- $\cdots$
- $f^{(n)}(a)$: nth-order derivative

with

- $n!$: factorial of integer n [Python: `np.prod(np.arange(1, n+1))`]

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Taylor Series

Intuition: We predict $f(x_{i+1})$ using information at x_i:

- **Zeroth-order** (constant) approximation:

$$f(x_{i+1}) \approx f(x_i)$$

  Only works if f is **constant** between x_i and x_{i+1}

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- **First-order** Taylor approximation:
  Add slope information:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)\underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

  - Represents a **straight line** (linear approximation)
  - Exact if f is linear

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- **Euler's formula**:

$$\underbrace{y_{i+1}}_{\text{next value}} = \underbrace{y_i}_{\text{current value}} + \underbrace{\Delta x}_{\text{step size}} \cdot \underbrace{f(x_i, y_i)}_{\text{slope}} \qquad x_{i+1} = x_i + \underbrace{\Delta x}_{\text{step size}}$$

Taylor Series

Intuition: We predict $f(x_{i+1})$ using information at x_i:

- **Second-order** Taylor approximation:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

  - Captures quadratic behavior
  - Better accuracy for smooth functions

Taylor Series

- **General Taylor polynomial**

$$
\begin{aligned}
f(x_{i+1}) \approx f(x_i) & \\
&+ f'(x_i)(x_{i+1} - x_i) \\
&+ \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 \\
&+ \cdots \\
&+ \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n \\
= & \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!}(x_{i+1} - x_i)^k
\end{aligned}
$$

Higher $n \rightarrow$ better approximation (if function is smooth)

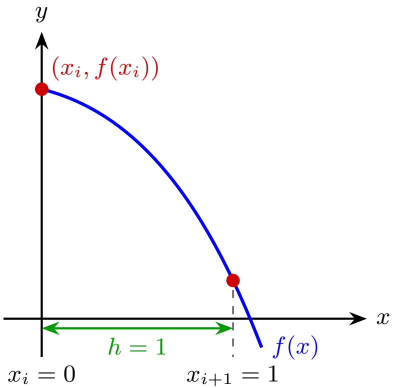
Taylor Series Approximation of a Polynomial

Problem statement:

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

  at $x_{i+1} = 1$ starting from $x_i = 0$ with step size $h = x_{i+1} - x_i = 1$.
- **Goal**: Predict $f(1)$ using Taylor approximations of increasing order.



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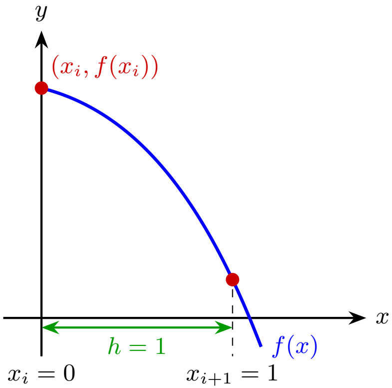
- **Goal**: Predict $f(1)$ using Taylor approximations of increasing order.

  - Function $f(x)$:

$$f(0) = 1.2$$

$$\begin{aligned} f(1) = &-0.1 - 0.15 - 0.5 \\ &-0.25 + 1.2 = 0.2 \end{aligned}$$

  - True value to predict: $f(1) = 0.2$



Taylor Series Approximation of a Polynomial

First-order approximation for $f(1)$:

- Need first-order derivative at $x_i = 0$:

$$\begin{aligned} f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow \quad f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow \quad f'(0) &= -0.25 \end{aligned}$$

- First-order Taylor series:

$$\begin{aligned} f(x_{i+1}) &\approx f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i) \\ \Rightarrow \qquad f(1) &\approx 1.2 + (-0.25) \times 1 = 0.95 \end{aligned}$$

Taylor Series Approximation of a Polynomial

Second-order approximation for $f(1)$:

- Need second-order derivative at $x_i = 0$:

$$
\begin{aligned}
f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\
\Rightarrow \quad f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\
\Rightarrow \quad f''(x) &= -1.2x^2 - 0.9x - 1 \\
\Rightarrow \quad f''(0) &= -1
\end{aligned}
$$

- Second-order Taylor series:

$$
f(x_{i+1}) \approx f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2
$$

$$
\Rightarrow \quad f(1) \approx 1.2 - 0.25 \times 1 + \left(\frac{-1}{2}\right) \times 1^2 = 0.45
$$

Taylor Series Approximation of a Polynomial

Third-order approximation for $f(1)$:

- Need third-order derivative at $x_i = 0$:

$$\begin{aligned} f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow \quad f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow \quad f''(x) &= -1.2x^2 - 0.9x - 1 \\ \Rightarrow \quad f'''(x) &= -2.4x - 0.9 \\ \Rightarrow \quad f'''(0) &= -0.9 \end{aligned}$$

- Third-order Taylor series:

$$f(1) \approx 1.2 - 0.25 \times 1 - 0.5 \times 1^2 + \left(\frac{-0.9}{3!}\right) \times 1^3 = 0.3$$

Taylor Series Approximation of a Polynomial

Fourth-order approximation for $f(1)$:

- Need fourth-order derivative at $x_i = 0$:

$$\begin{aligned} f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow \quad f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow \quad f''(x) &= -1.2x^2 - 0.9x - 1 \\ \Rightarrow \quad f'''(x) &= -2.4x - 0.9 \\ \Rightarrow \quad f^{(4)}(x) &= -2.4 \\ \Rightarrow \quad f^{(4)}(0) &= -2.4 \end{aligned}$$

- Fourth-order Taylor series:

$$f(1) \approx 1.2 - 0.25 \times 1 - 0.5 \times 1^2 - 0.15 \times 1^3 - \left(\frac{-2.4}{4!}\right) \times 1^4 = 0.2$$

Taylor Series Approximation of a Polynomial

- Error:

$$\varepsilon = |f(x_{i+1}) - \hat{f}(x_{i+1})|$$

where the approximation is

$$\hat{f}(x_{i+1}) = \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!}(x_{i+1} - x_i)^k$$

- Summary of results:

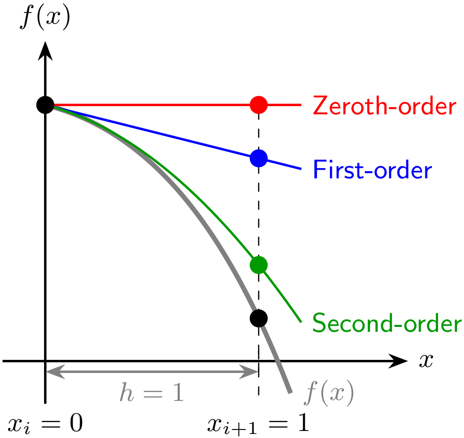
Order (n)	Approximation $\hat{f}(1)$	Error ε
1	0.95	$\lvert 0.2 - 0.95\rvert = 0.75$
2	0.45	$\lvert 0.2 - 0.45\rvert = 0.25$
3	0.3	$\lvert 0.2 - 0.3\rvert = 0.1$
4	0.2	$\lvert 0.2 - 0.2\rvert = 0$

- Error decreases as order increases
- With $n = 4$, approximation is exact because $f(x)$ is a 4th-degree polynomial

Taylor Series Approximation of a Polynomial

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$



Taylor Series Approximation of a Polynomial

Concluding remarks:

- Taylor series approximates functions using derivatives at a point
- Higher-order terms improve accuracy by capturing curvature
- For an m-degree polynomial, a Taylor series of order m gives the exact function
- Truncation error quantifies the approximation error
- This example illustrates the power of Taylor series for function approximation in numerical methods

Taylor Series Approximation of $\sin(x)$

Maclaurin series:

- A Taylor series expansion of a function about 0
- Taylor series approximation:

$$f(x_{i+1}) \approx \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!}(x_{i+1} - x_i)^k$$

- Set $x_i = 0$ and $x = x_{i+1}$, then

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

- Derivative of $f(x) = \sin(x)$:

$$f'(x) = \underbrace{\cos(x)}_{\cos(0)=1}, \quad f''(x) = -\underbrace{\sin(x)}_{\sin(0)=0}, \quad f'''(x) = -\underbrace{\cos(x)}_{\cos(0)=1}, \quad f^{(4)}(x) = \underbrace{\sin(x)}_{\sin(0)=0}$$

- Formula:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \cdots$$

Quick Summary

Monday's Class:

- Definition of truncation error
- Taylor series of any smooth function (approximated by polynomial)
- Approximation with zeroth-, first-, second-, and higher-order information
- Example: Taylor series approximation of a polynomial function
- Revisit Taylor series approximation of $\sin(x)$