



**POLYTECHNIQUE
MONTRÉAL**

UNIVERSITÉ
D'INGÉNIERIE



Laplacian Convolutional Representation for Traffic Time Series Imputation

Xinyu Chen

Postdoctoral Associate, MIT

Ph.D., University of Montreal

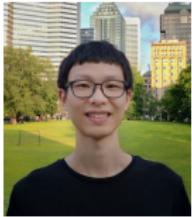
July 8, 2024



Université
de Montréal



Xinyu Chen
PolyMtl



Zhanhong Cheng
McGill



HanQin Cai
UCF



Nicolas Saunier
PolyMtl



Lijun Sun
McGill

Publication:

- X. Chen, Z. Cheng, H.Q. Cai, N. Saunier, L. Sun (2024). Laplacian convolutional representation for traffic time series imputation. *IEEE Transactions on Knowledge and Data Engineering*. Early Access.
<https://doi.org/10.1109/TKDE.2024.3419698>

Open source:

- Spatiotemporal data modeling initiative:
<https://spatiotemporal-data.github.io> (coming soon)

Outline

- **Motivation**

- Data-Driven ITS
- Time Series Imputation
- Speed Field Reconstruction
- Marginal Idea Works

- **Preliminaries**

- Revisit Laplacian Matrix
- Revisit Circular Convolution
- Reformulate Laplacian regularization

- **Global Trend Modeling**

- **Laplacian Convolutional Representation**

- Model Description
- Solution Algorithm
- Empirical Time Complexity

- **Experiments**

- Traffic Volume & Speed Imputation
- Speed Field Reconstruction

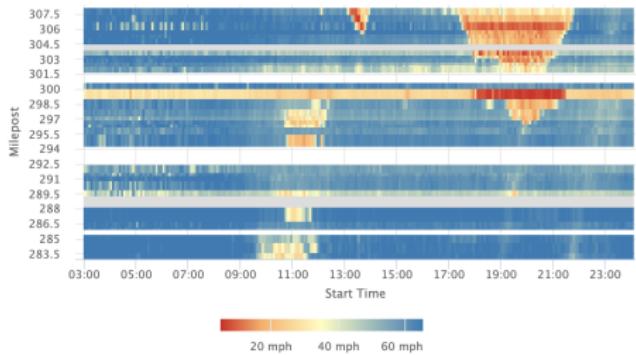
- **Conclusion**

Motivation

- Portland highway traffic flow data¹



Highway network & sensor locations

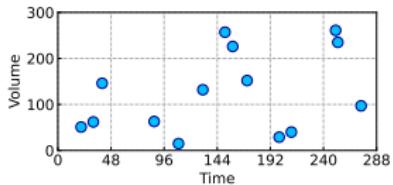


Traffic speed field

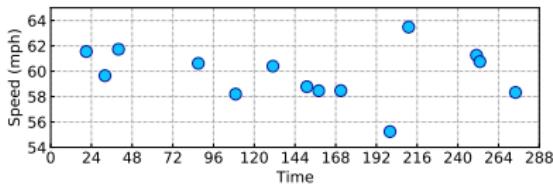
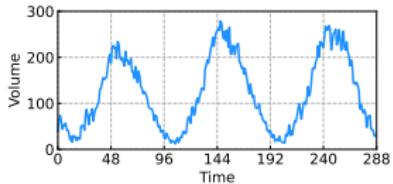
- Speed field $\mathbf{X} \in \mathbb{R}^{N \times T}$ (N locations & T time steps)
- Speed field shows strong spatial/temporal dependencies

¹<https://portal.its.pdx.edu/home>

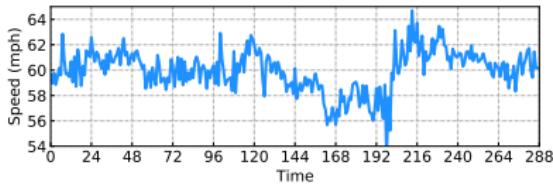
Motivation



↓ Reconstruct
traffic volume?

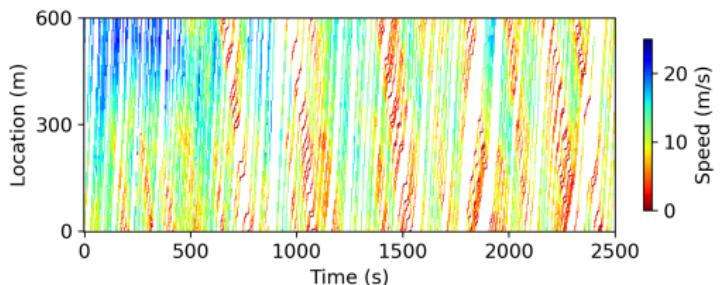


↓ Reconstruct
traffic speed?



- How to utilize the global trends of traffic time series?
- How to produce local consistency of traffic data?

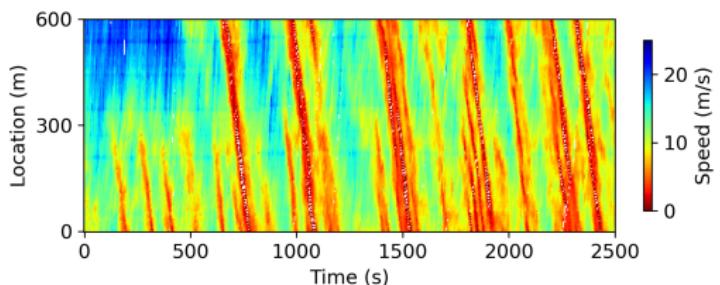
Motivation



200-by-500 matrix
(NGSIM)



Reconstruct speed field from
20% sparse trajectories?



- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal local dependencies?

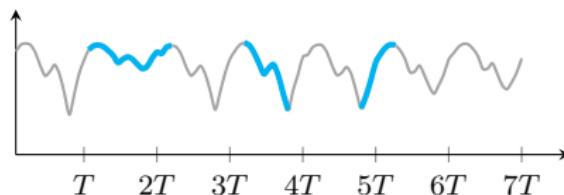
Motivation

Time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



- Local trends (e.g., short-term time series trends):

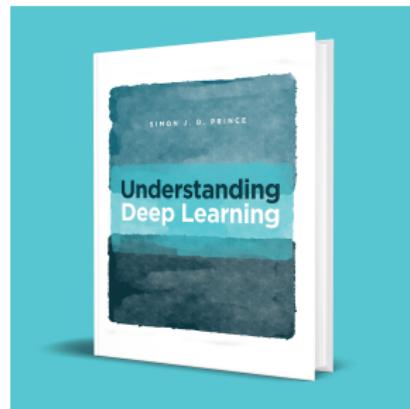


How to characterize both global and local trends in sparse time series?

Revisit Laplacian Matrix

- The graph Laplacian matrix is

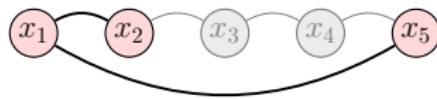
$$L = \underbrace{D}_{\text{degree matrix}} - \underbrace{A}_{\text{adjacency matrix}}$$



<https://udlbook.github.io/udlbook/>

Revisit Laplacian Matrix

- Intuition of (circulant) Laplacian matrix.



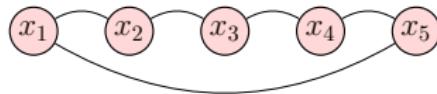
Modeling

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

Revisit Laplacian Matrix

- Intuition of (circulant) Laplacian matrix.



Modeling

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

Revisit Circular Convolution

Reformulate Laplacian regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.

Undirected and circulant graph

$$\xrightarrow{\text{Modeling}} \mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Laplacian kernel: $\ell = (2, -1, 0, 0, -1)^\top$.

$$\mathbf{L}\mathbf{x} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \star \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \ell \star \mathbf{x}$$

where \star denotes the circular convolution.

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\ell \star \mathbf{x}\|_2^2$$

“... The circulant graph has an adjacency matrix that is a circulant matrix.”

— “*Circulant graph*” on Wikipedia

Reformulate Laplacian Regularization

Reformulate Laplacian regularization with circular convolution.

- Define Laplacian kernel:

$$\ell \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \dots, -1}_{\tau}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$.

²It refers to the Convolution theorem.

Reformulate Laplacian Regularization

Reformulate Laplacian regularization with circular convolution.

- Define Laplacian kernel:

$$\ell \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \dots, -1}_{\tau}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$.

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\ell * \mathbf{x}\|_2^2$$

- Property with discrete Fourier transform (denoted by $\mathcal{F}(\cdot)$)²:

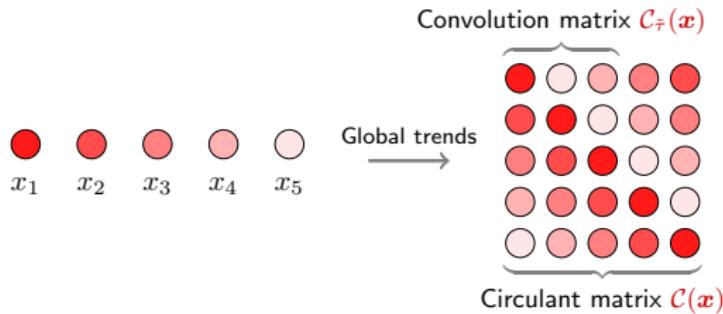
$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\ell * \mathbf{x}\|_2^2 = \underbrace{\frac{1}{2T} \|\mathcal{F}(\ell) \circ \mathcal{F}(\mathbf{x})\|_2^2}_{\text{w/ FFT in } \mathcal{O}(T \log T) \text{ time}}$$

- FFT: Fast implementation of discrete Fourier transform.

²It refers to the Convolution theorem.

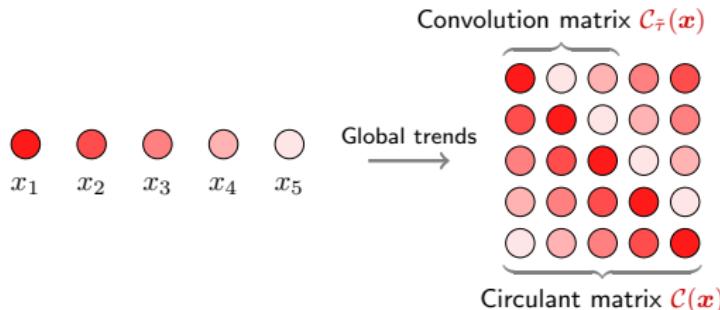
Global Trend Modeling

Circulant matrix $\mathcal{C}(\mathbf{x})$ vs. convolution matrix $\mathcal{C}_{\tilde{\tau}}(\mathbf{x})$



Global Trend Modeling

Circulant matrix $\mathcal{C}(\mathbf{x})$ vs. convolution matrix $\mathcal{C}_{\tilde{\tau}}(\mathbf{x})$



- Circulant/Convolution nuclear norm minimization (w/ $\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1$)
 - A balance between global and local trends modeling?

CircNNM (Liu'22, Liu & Zhang'23)

Estimating \mathbf{x} :

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

on data \mathbf{y} w/ observed index set Ω .

ConvNNM (Liu'22, Liu & Zhang'23)

Estimating \mathbf{x} :

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}_{\tilde{\tau}}(\mathbf{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

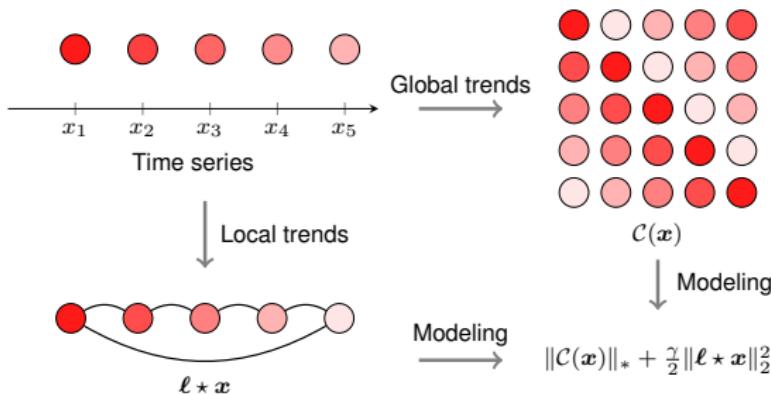
on data \mathbf{y} w/ observed index set Ω .

Global + Local Trends?

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\mathbf{y} \in \mathbb{R}^T$ with observed index set Ω , LCR utilizes **circulant matrix** and **Laplacian kernel** to characterize global and local trends in time series, respectively, i.e.,

$$\begin{aligned} \min_{\mathbf{x}} \quad & \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\ell * \mathbf{x}\|_2^2}_{\text{local}} \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$



Laplacian Convolutional Representation

- LCR model:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{x}) \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

- Augmented Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2$$

where $\mathbf{w} \in \mathbb{R}^T$ is the Lagrange multiplier, and $\langle \cdot, \cdot \rangle$ denotes the inner product.

Laplacian Convolutional Representation

- LCR model:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{x}) \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

- Augmented Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2$$

where $\mathbf{w} \in \mathbb{R}^T$ is the Lagrange multiplier, and $\langle \cdot, \cdot \rangle$ denotes the inner product.

- The ADMM scheme:

$$\left\{ \begin{array}{ll} \mathbf{x} := \arg \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Nuclear norm minimization)} \\ \mathbf{z} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Closed-form solution)} \\ = \frac{1}{\lambda + \eta} \mathcal{P}_\Omega(\lambda \mathbf{x} + \mathbf{w} + \eta \mathbf{y}) + \frac{1}{\lambda} \mathcal{P}_\Omega^\perp(\lambda \mathbf{x} + \mathbf{w}) & \\ \mathbf{w} := \mathbf{w} + \lambda(\mathbf{x} - \mathbf{z}) & \text{(Standard update)} \end{array} \right.$$

- Optimize \mathbf{x} ?

$$\underbrace{\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1}_{\text{property of circulant matrix}} \quad \& \quad \underbrace{\frac{1}{2} \|\ell * \mathbf{x}\|_2^2 = \frac{1}{2T} \|\mathcal{F}(\ell) \circ \mathcal{F}(\mathbf{x})\|_2^2}_{\text{property of circular convolution}}$$

Laplacian Convolutional Representation

- Optimize \mathbf{x} via FFT (in $\mathcal{O}(T \log T)$ time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell \star \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2\end{aligned}$$

where we introduce $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$ (i.e., FFT).

Laplacian Convolutional Representation

- Optimize \mathbf{x} via FFT (in $\mathcal{O}(T \log T)$ time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell \star \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2\end{aligned}$$

where we introduce $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$ (i.e., FFT).

ℓ_1 -norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of ℓ_1 -norm minimization in complex space:

$$\min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\delta}{2} \|\hat{\mathbf{x}} - \hat{\mathbf{h}}\|_2^2$$

with complex-valued $\hat{\mathbf{x}}, \hat{\mathbf{h}} \in \mathbb{C}^T$ and weight parameter δ , element-wise, the solution is given by

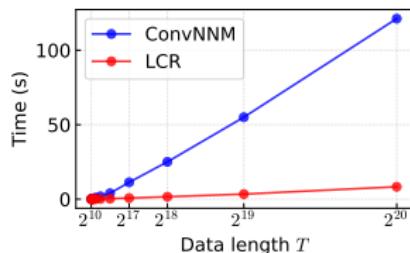
$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|} \cdot \max\{0, |\hat{h}_t| - 1/\delta\}, t = 1, \dots, T.$$

Laplacian Convolutional Representation

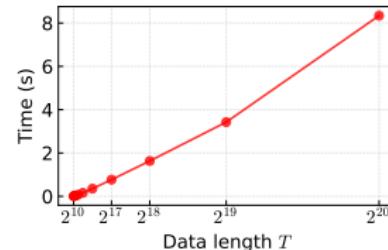
Empirical time complexity

On the synthetic data $\mathbf{y} \in \mathbb{R}^T$ with $T \in \{2^{10}, 2^{11}, \dots, 2^{20}\}$

- Ours: **LCR**
 - An FFT implementation in $\mathcal{O}(T \log T)$
 - The logarithmic factor $\log T$ makes the FFT highly efficient
- Baseline: **ConvNNM** (Liu'22, Liu & Zhang'23)
 - Convolution matrix $\mathcal{C}_{\tilde{\tau}}(\mathbf{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$ with kernel size $\tilde{\tau} = 2^4$
 - Singular value thresholding in $\mathcal{O}(\tilde{\tau}^2 T)$

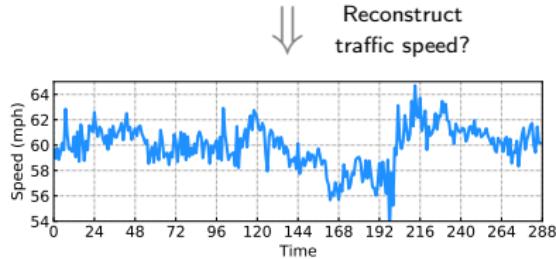
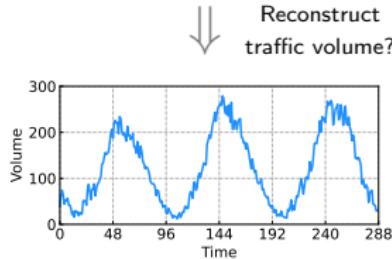
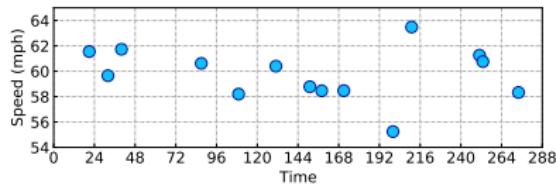
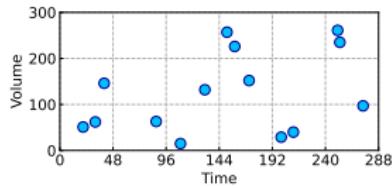


ConvNNM vs. LCR



LCR

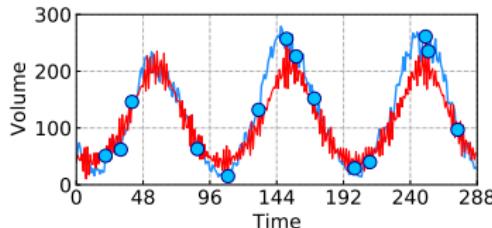
Experiments



- How to utilize the global trends of traffic time series?
- How to produce local consistency of traffic data?

Experiments

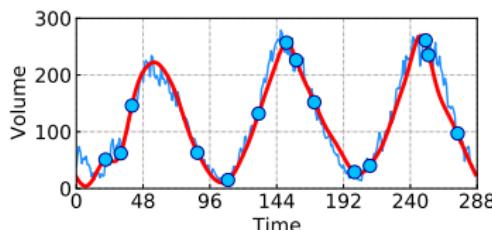
- Substantial performance gains?



CircNNM:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* \\ \text{s. t. } \quad & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

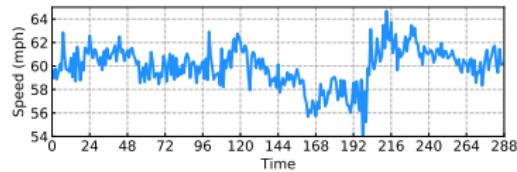
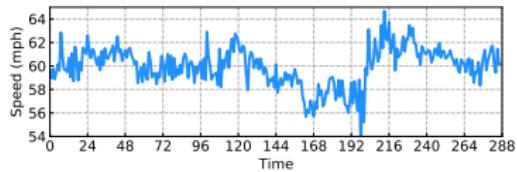
↓ Plus **local** time series trends



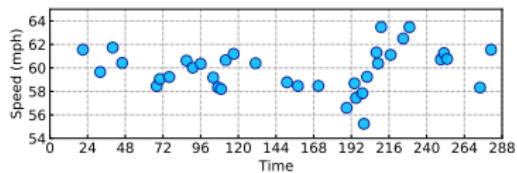
LCR:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2} \|\boldsymbol{\ell} * \boldsymbol{x}\|_2^2 \\ \text{s. t. } \quad & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

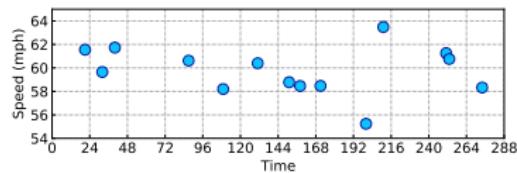
Experiments



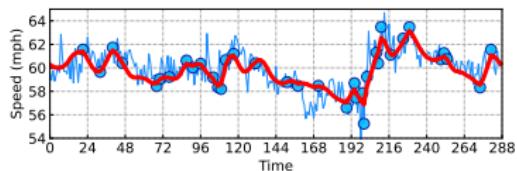
↓ Mask 90% observations



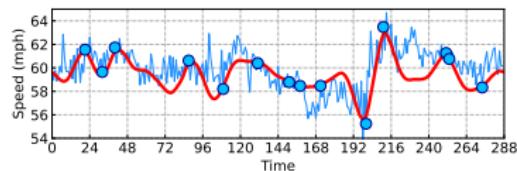
↓ Mask 95% observations



↓ Reconstruct time series



↓ Reconstruct time series



Experiments

- The start data points and end data points are connected?

Undirected and circulant graph

Modeling →

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Flipping operation on $\mathbf{x} \in \mathbb{R}^5$:

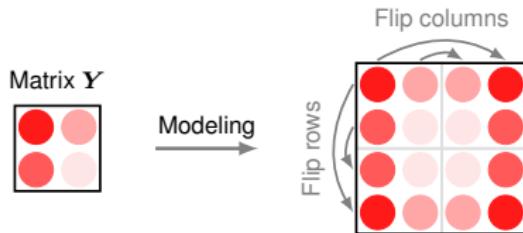
$$\mathbf{x}_{\text{new}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{J}\mathbf{x} \end{bmatrix} = (\underbrace{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5}_{\text{original time series}}, \underbrace{\mathbf{x}_5, \mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1}_{\text{flipped time series}})^{\top} \in \mathbb{R}^{10}$$

where $\mathbf{J} \in \mathbb{R}^{5 \times 5}$ is the exchange matrix.

Experiments

Speed field reconstruction³

- Flipping operation on a matrix:

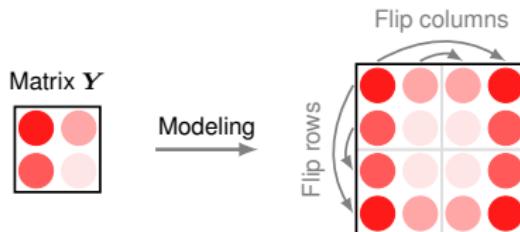


³Highway Drone (HighD) dataset at <https://www.hightd-dataset.com/>

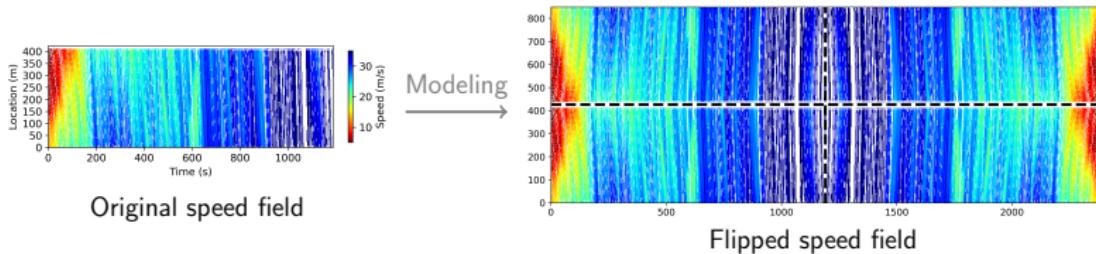
Experiments

Speed field reconstruction³

- Flipping operation on a matrix:



- Flipping operation on a speed field of vehicular traffic flow:



³Highway Drone (HighD) dataset at <https://www.hightd-dataset.com/>

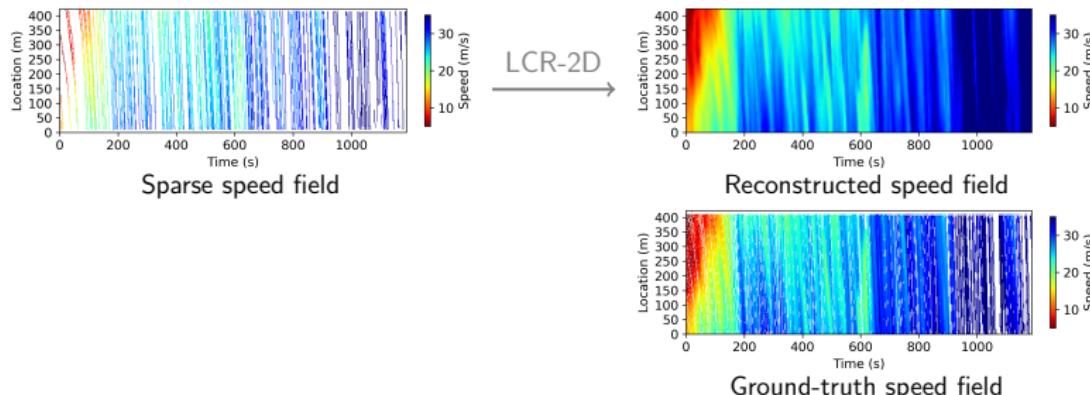
Experiments

Speed field reconstruction⁴

- Scenario: Mask trajectories of 70% vehicles
- LCR-2D on partially observed $\mathbf{Y} \in \mathbb{R}^{N \times T}$:

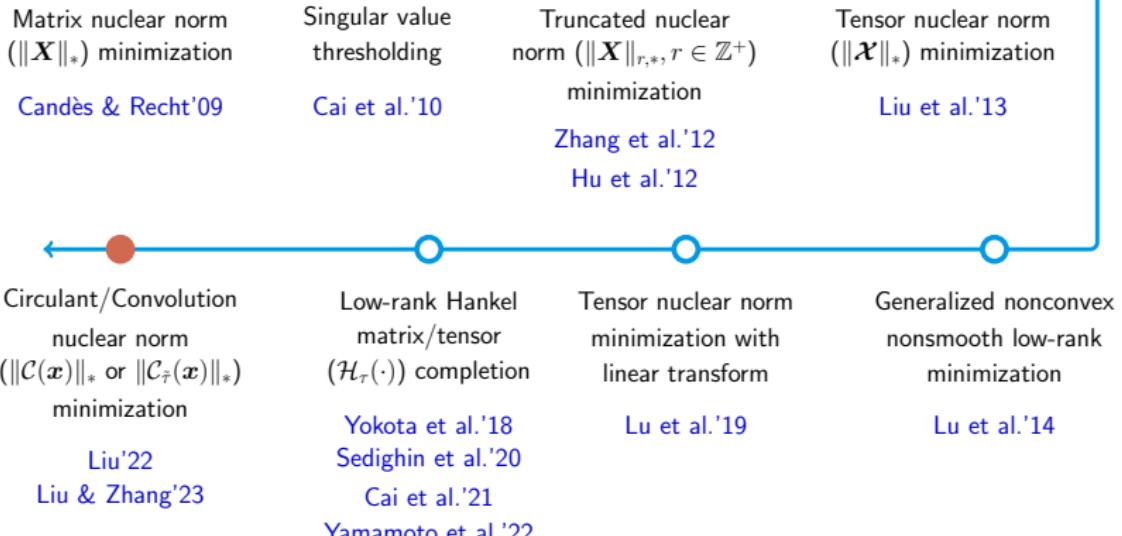
$$\min_{\mathbf{X}} \underbrace{\|\mathcal{C}(\mathbf{X})\|_*}_{\text{global trend}} + \frac{\gamma}{2} \underbrace{\|(\ell_s \ell^\top) * \mathbf{X}\|_F^2}_{\text{local trend}}$$

s.t. $\|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \epsilon$



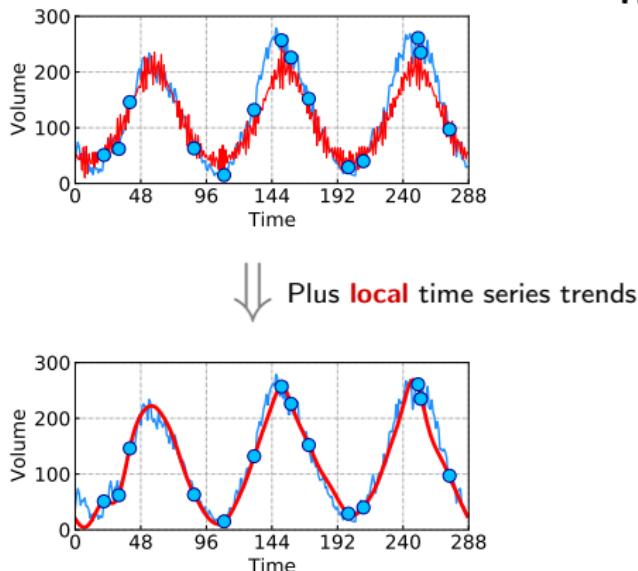
⁴Highway Drone (HighD) dataset at <https://www.hightd-dataset.com/>

Contributions



(Ours) LCR:

- ✓ Local trend modeling
- ✓ An FFT implementation



Highlights:

- Rethinking the importance of local trend modeling in traffic data imputation tasks.
- Finding a unified **global and local trend** modeling framework whose optimization can be efficiently solved by **FFT**:

$$\min_{\mathbf{x}} \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\ell * \mathbf{x}\|_2^2}_{\text{local}}$$

$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$

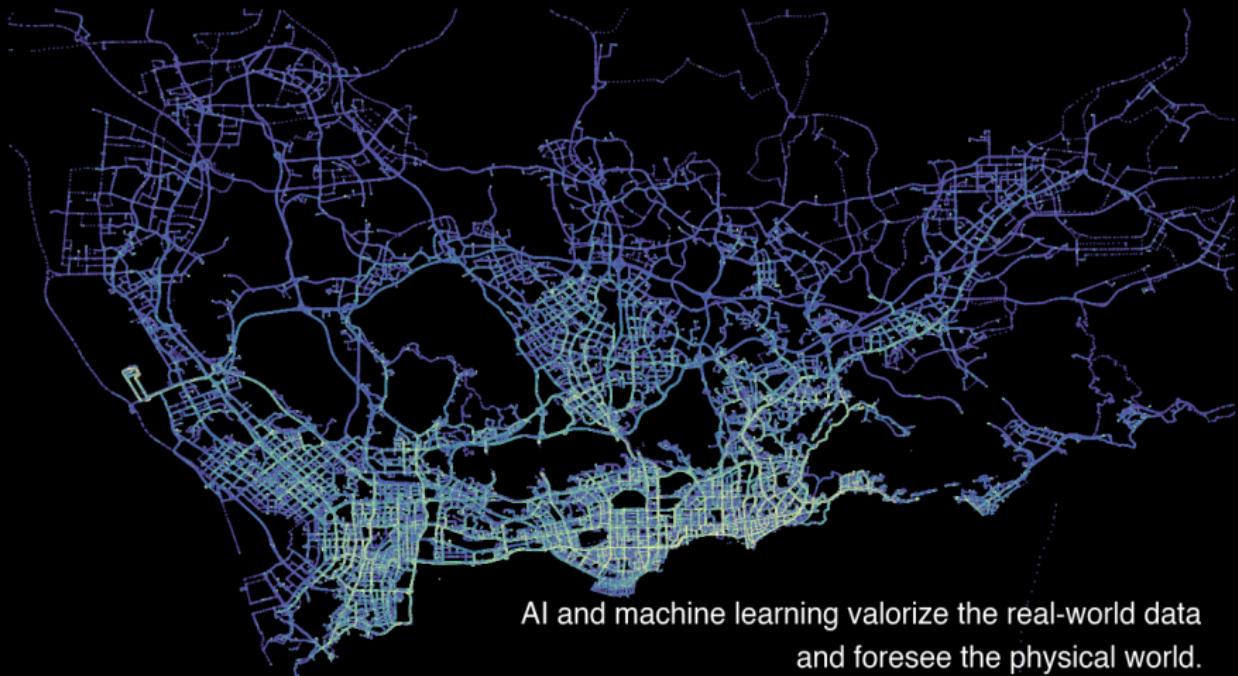
Concluding Remarks

- **(Starting point)** How to impute traffic time series?
 - ✓ Low-rank models ✓ Temporal regularization
- **(Solution)** Time series trend modeling in the low-rank framework?
 - Global time series trend modeling (low-rank model):
$$\min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_*$$
$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$
 - Local time series trend modeling (temporal regularization):
$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\boldsymbol{\ell} \star \mathbf{x}\|_2^2$$
- **(Highlight)** A unified framework with the **FFT** implementation.

References

A short list:

- [Liu'22] G. Liu (2022). Time series forecasting via learning convolutionally low-rank models. *IEEE Transactions on Information Theory*, 68(5): 3362–3380.
- [Liu & Zhang'23] G. Liu and W. Zhang (2023). Recovery of future data via convolution nuclear norm minimization. *IEEE Transactions on Information Theory*, 69(1): 650–665.



AI and machine learning valorize the real-world data
and foresee the physical world.

Source: <https://spatiotemporal-data.github.io>



POLYTECHNIQUE
MONTRÉAL

UNIVERSITÉ
D'INGÉNIERIE



Thanks for your attention!

Any Questions?

Slides: <https://xinychen.github.io/slides/LCR24.pdf>

About me:

- 🏠 Homepage: <https://xinychen.github.io>
- /github/ GitHub: <https://github.com/xinychen>
- ✉️ How to reach me: chenxy346@gmail.com