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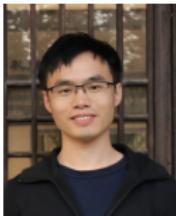


# Spatiotemporal Traffic Data Imputation and Forecasting with Tensor Learning

Ph.D. Research Project

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# Outline

- **Motivation**

- Multivariate traffic time series
- Multidimensional traffic time series
- Multiple data behaviors

- **Literature Review**

- Spatiotemporal traffic data imputation
- Spatiotemporal traffic forecasting
- Low-rank tensor learning

- **Objectives**

- **Methodology**

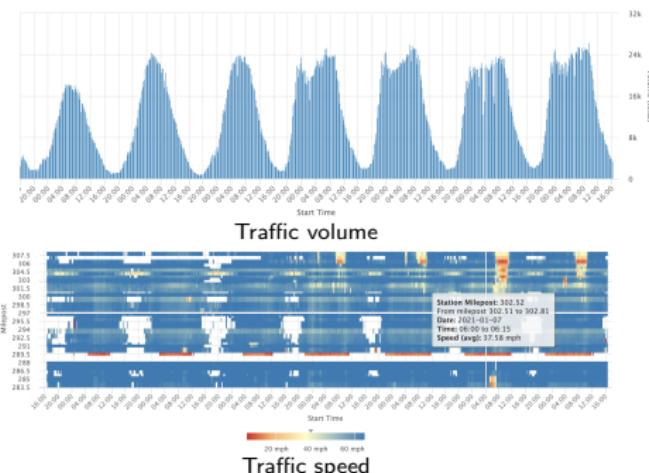
- Spatiotemporal traffic data imputation
- High-dimensional traffic forecasting
- Multidimensional traffic forecasting
- Traffic forecasting on sparse data

- **Conclusion**

# Multivariate Traffic Time Series

Many spatiotemporal traffic time series data are in the form of **matrix**.

- **Example:** Portland highway traffic data<sup>1</sup>.



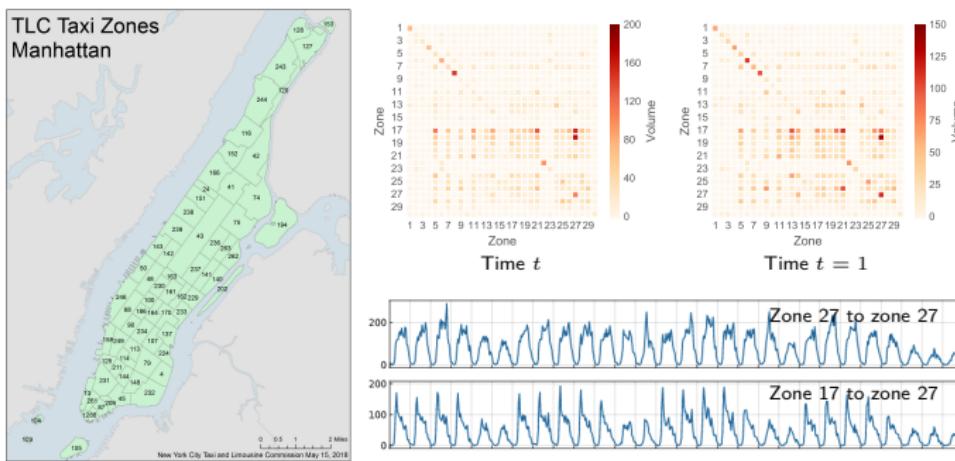
- $X \in \mathbb{R}^{N \times T}$  with  $N$  spatial locations  $\times T$  time steps

<sup>1</sup> <https://portal.its.pdx.edu/home>

# Multidimensional Traffic Time Series

Many spatiotemporal traffic time series data are in the form of **tensor**.

- Example: NYC (hourly) taxi flow data<sup>2</sup>.



- $\mathcal{X} \in \mathbb{R}^{M \times N \times T}$  with  $M$  zones  $\times N$  zones  $\times T$  time steps

<sup>2</sup><https://www1.nyc.gov/site/tlc/about/tlc-trip-record-data.page>

## Multiple Data Behaviors

Spatiotemporal traffic data are time series, but they involve multiple data behaviors.

- Incompleteness & sparsity
  - High-dimensionality
  - Multidimensionality
  - Noises & outliers
  - Nonstationarity
  - ...

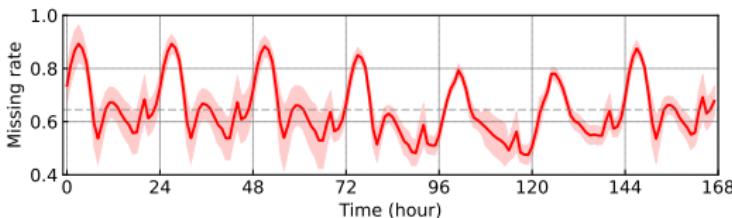
In addition, spatiotemporal correlations are also very important.



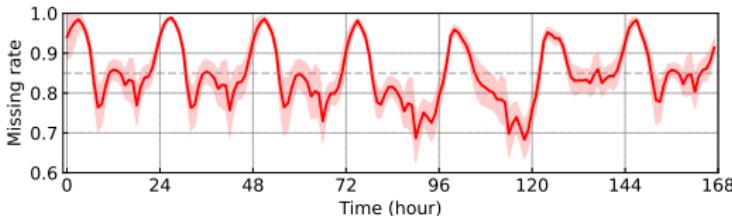
## Multiple Data Behaviors

## Sparsity & high dimensionality

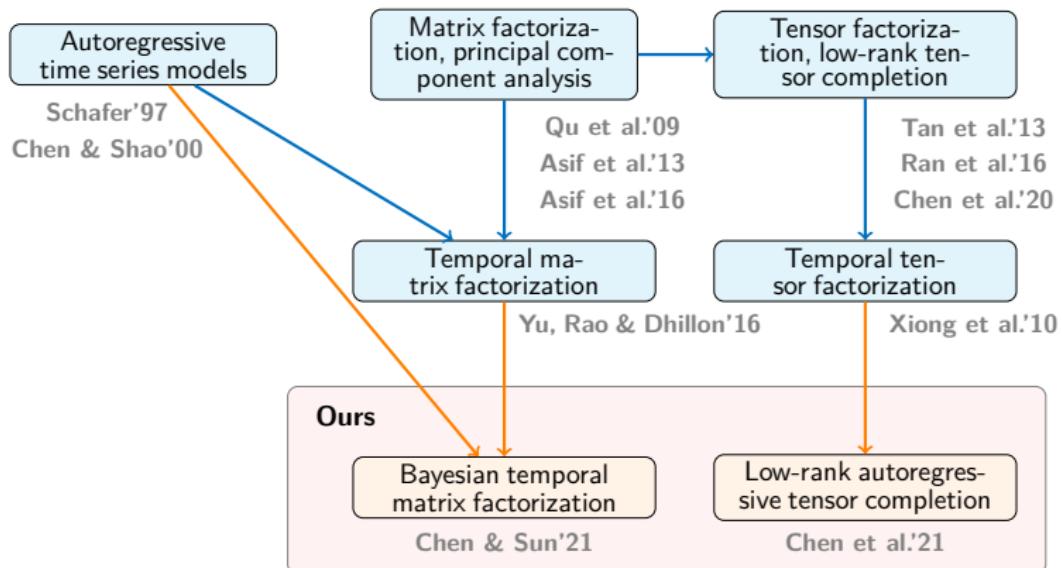
- NYC movement speed data (2019)
    - 98,210 road segments & 8,760 time steps (hours)
    - Overall missing rate: 64.43%



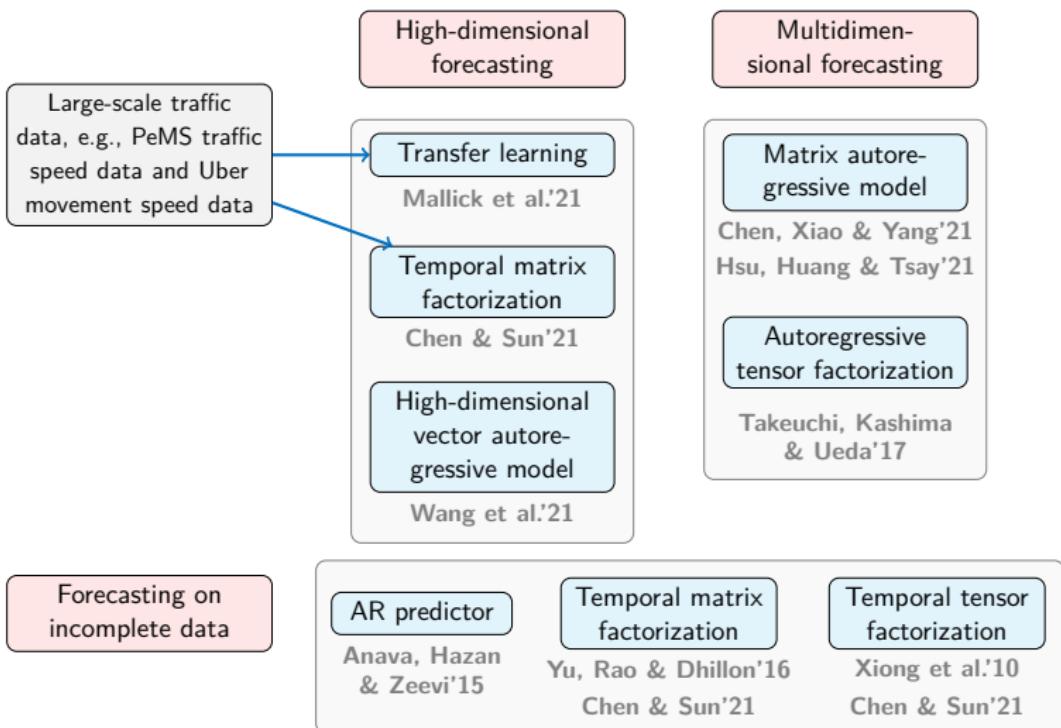
- Seattle movement speed data (2019)
    - 63,490 road segments & 8,760 time steps (hours)
    - Overall missing rate: 84.95%



# Spatiotemporal Traffic Data Imputation



# Spatiotemporal Traffic Forecasting

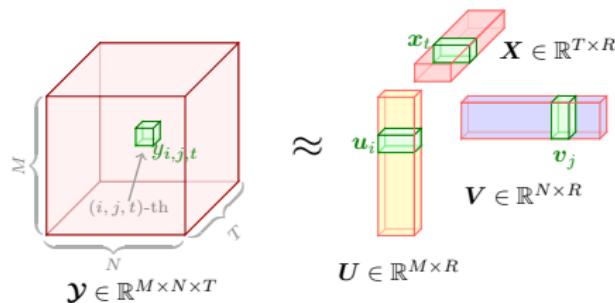


## Low-Rank Tensor Learning

## From data to model

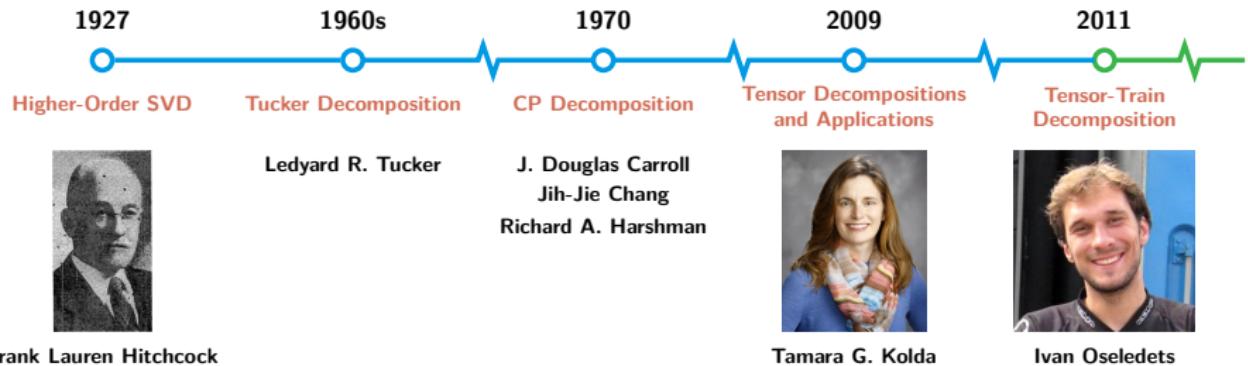
- #### ■ Matrix factorization:

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y}) - \mathcal{P}_\Omega(\mathbf{W}\mathbf{X})\|_F^2 + \frac{\lambda}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)$$



- Tensor factorization:

$$\min_{\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}}) - \mathcal{P}_\Omega\left(\sum_{r=1}^R \boldsymbol{u}_r \circ \boldsymbol{v}_r \circ \boldsymbol{x}_r\right)\|_F^2 + \frac{\lambda}{2} (\|\boldsymbol{U}\|_F^2 + \|\boldsymbol{V}\|_F^2 + \|\boldsymbol{X}\|_F^2)$$



## Low-Rank Tensor Learning

- Low-rank matrix/tensor completion

**Candès & Recht'09:** Convex nuclear norm minimization for matrix completion.

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{X}\|_* \\ \text{s.t. } & \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{Y}) \end{aligned}$$

**Cai, Candès & Shen'10:** Singular value thresholding algorithm.

$$\begin{cases} \mathbf{X}^\ell = \mathcal{D}_\tau(\mathbf{Z}^{\ell-1}) \\ \mathbf{Z}^\ell = \mathbf{Z}^{\ell-1} + \delta_\ell \mathcal{P}_\Omega(\mathbf{Y} - \mathbf{X}^\ell) \end{cases}$$

Zhang et al.'12: Nonconvex truncated nuclear norm minimization.

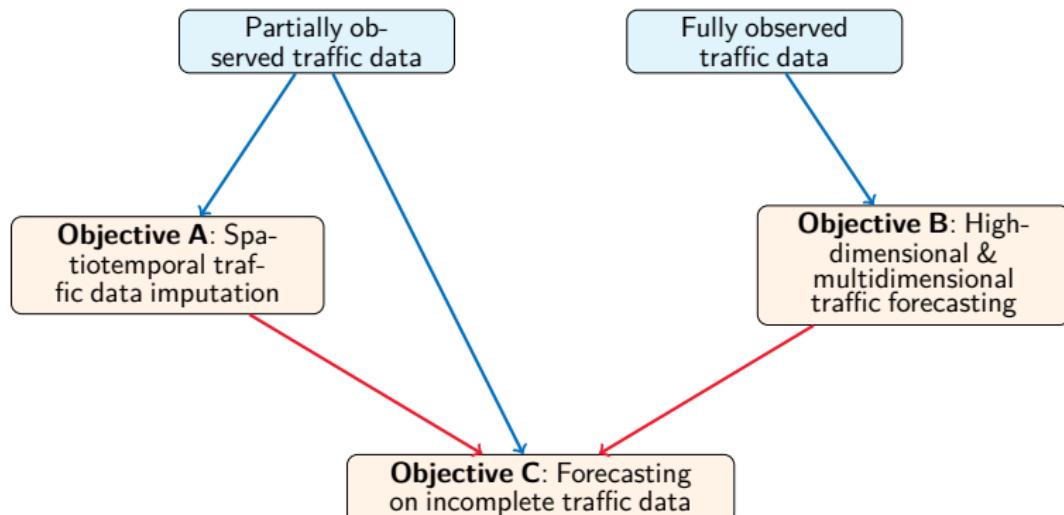
Liu et al.'13: Convex nuclear norm minimization for tensor completion.

$$\begin{aligned} \min_{\boldsymbol{\chi}} \quad & \| \boldsymbol{\chi} \|_* \\ \text{s.t. } & \mathcal{P}_\Omega(\boldsymbol{\chi}) = \mathcal{P}_\Omega(\boldsymbol{\gamma}) \end{aligned}$$

Lu, Peng & Wei'19: Tensor nuclear norm induced by linear transform.

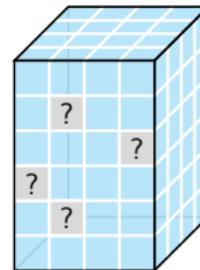
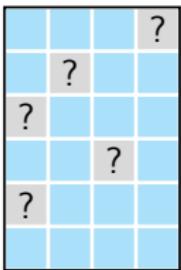
# Objectives: A Whole Structure

We are working on **spatiotemporal traffic data modeling**.



# Spatiotemporal Traffic Data Imputation

- **Objective A:** Given a multivariate time series data like  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  or a multidimensional time series data like  $\mathcal{Y} \in \mathbb{R}^{M \times N \times T}$ , impute the missing values of the data.



[Q]

- How to learn and reconstruct missing values from observed data?
  - How to make use of spatiotemporal correlations?
  - How to make use of traffic time series dynamics?

# Spatiotemporal Traffic Data Imputation

## Low-rank matrix completion (Candès & Recht'09)

For any partially observed data matrix  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  with observed index set  $\Omega$ , then low-rank matrix completion takes the form of

$$\begin{aligned} & \min_{\mathbf{X}} \|\mathbf{X}\|_* \\ & \text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{Y}). \end{aligned} \tag{1}$$

## Low-rank tensor completion (Liu et al.'13)

For any partially observed data matrix  $\mathcal{Y} \in \mathbb{R}^{M \times N \times T}$  with observed index set  $\Omega$ , then low-rank matrix completion takes the form of

$$\begin{aligned} \min_{\boldsymbol{\chi}} \quad & \| \boldsymbol{\chi} \|_* \\ \text{s.t. } & \mathcal{P}_\Omega(\boldsymbol{\chi}) = \mathcal{P}_\Omega(\boldsymbol{y}). \end{aligned} \tag{2}$$

- **Limitation:** Only cover the global consistency.
  - **Comment:** For modeling spatiotemporal traffic data, local consistency (e.g., temporal correlations) is also important.

## Spatiotemporal Traffic Data Imputation

## Low-rank autoregressive matrix completion

Given the multivariate traffic time series data  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  with the observed index set  $\Omega$ , the low-rank matrix completion combines both nuclear norm and univariate autoregressive process:

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathbf{X}\|_* + \frac{\lambda}{2} \sum_{n=1}^N \sum_{t=d+1}^T (z_{n,t} - \sum_{k=1}^d a_{n,k} z_{n,t-k})^2 \\ \text{s.t.} \quad & \begin{cases} \mathbf{X} = \mathbf{Z}, \\ \mathcal{P}_\Omega(\mathbf{Z}) = \mathcal{P}_\Omega(\mathbf{Y}). \end{cases} \end{aligned} \tag{3}$$

## Low-rank autoregressive tensor completion

If  $T$  time steps can be separated into  $I$  time steps per day and  $J$  days, i.e.,  $T = IJ$ , then we can define an operator  $\mathcal{Q}(\cdot)$  to convert multivariate data to third-order tensor:

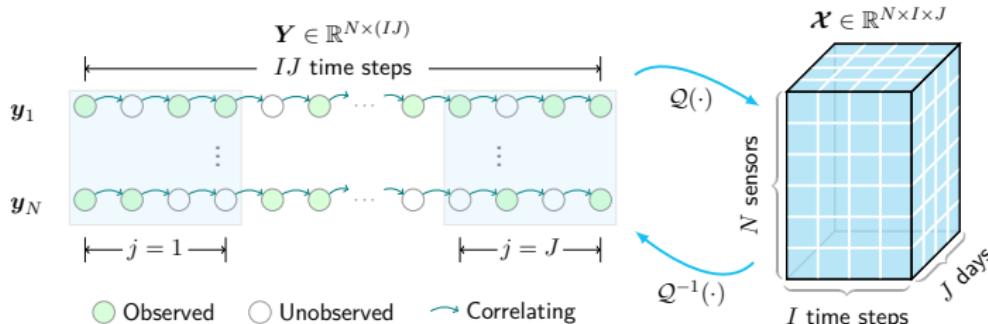
$$\begin{aligned} \min_{\boldsymbol{\mathcal{X}}} \quad & \| \boldsymbol{\mathcal{X}} \|_* + \frac{\lambda}{2} \sum_{n=1}^N \sum_{t=d+1}^T (z_{n,t} - \sum_{k=1}^d a_{n,k} z_{n,t-k})^2 \\ \text{s.t.} \quad & \begin{cases} \boldsymbol{\mathcal{X}} = \mathcal{Q}(\boldsymbol{Z}), \\ \mathcal{P}_{\Omega}(\boldsymbol{Z}) = \mathcal{P}_{\Omega}(\boldsymbol{Y}). \end{cases} \end{aligned} \tag{4}$$

# Spatiotemporal Traffic Data Imputation

Low-rank autoregressive tensor completion

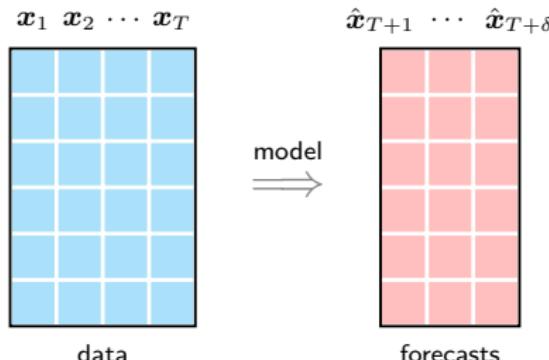
$$\begin{aligned} \min_{\boldsymbol{\mathcal{X}}} \quad & \|\boldsymbol{\mathcal{X}}\|_* + \frac{\lambda}{2} \sum_{n=1}^N \sum_{t=d+1}^T (z_{n,t} - \sum_{k=1}^d a_{n,k} z_{n,t-k})^2 \\ \text{s.t.} \quad & \begin{cases} \boldsymbol{\mathcal{X}} = \mathcal{Q}(\boldsymbol{Z}), \\ \mathcal{P}_\Omega(\boldsymbol{Z}) = \mathcal{P}_\Omega(\boldsymbol{Y}). \end{cases} \end{aligned} \quad (5)$$

- **Advantage:** Global consistency + local consistency.



# High-Dimensional Traffic Forecasting

- **Objective B-1:** Given a multivariate traffic time series  $\mathbf{x}_1, \dots, \mathbf{x}_T \in \mathbb{R}^N$  with  $N \gg T$  ("tall-skinny"), forecast data points  $\mathbf{x}_{T+\delta}, \delta \in \mathbb{N}^+$ .

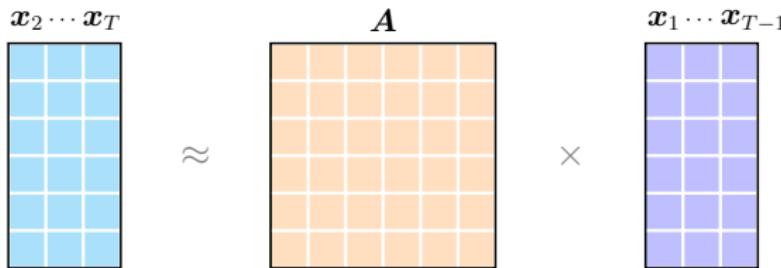


- **Solution:** For time series  $\mathbf{x}_1, \dots, \mathbf{x}_T \in \mathbb{R}^N$ , the  $d$ th-order vector autoregressive (VAR( $d$ )) model:  $\mathbf{x}_t = \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} + \boldsymbol{\epsilon}_t$ .
  - **Advantage:** Co-evolution patterns
  - **Limitation:** Over-parameterization

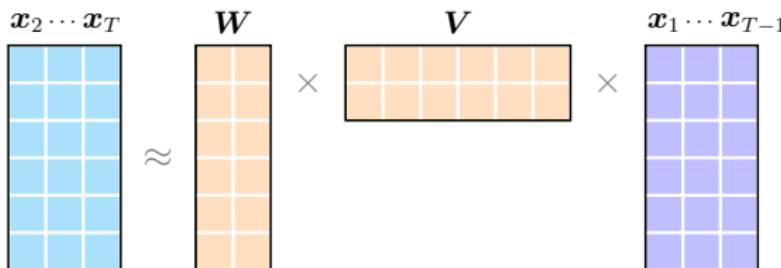
# High-Dimensional Traffic Forecasting

## VAR(1) model

- Over-parameterization in the case of  $N \gg T$ .



- Reduced-rank autoregression:  $A = WV$  with  $W \in \mathbb{R}^{N \times R}$ ,  $V \in \mathbb{R}^{R \times N}$ .



## High-Dimensional Traffic Forecasting

Multivariate reduced-rank regression

- Data:  $Z \in \mathbb{R}^{N \times T}$  (input) &  $Y \in \mathbb{R}^{M \times T}$  (output)
  - Component matrices  $W \in \mathbb{R}^{M \times R}, V \in \mathbb{R}^{R \times N}$  such that  $A = WV$
  - Optimization problem:

$$\mathbf{W}^*, \mathbf{V}^* \triangleq \arg \min_{\mathbf{W}, \mathbf{V}} \frac{1}{2} \|\mathbf{Y} - \mathbf{WVZ}\|_F^2 \quad (6)$$

### Theorem 1 (Izenman'75)

Suppose the optimization problem of multivariate reduced-rank regression trained on the centered data matrices  $Z$  and  $Y$ , then for any positive definite matrix  $\Gamma$ , the solutions are

$$W^* = \Gamma^{-1/2} \xi, V^* = \xi^\top \Gamma^{1/2} \Sigma_{yz} \Sigma_{zz}^{-1}, \quad (7)$$

where  $\xi \in \mathbb{R}^{N \times R}$  consists of the  $R$  eigenvectors corresponding to the  $R$  largest eigenvalues of the matrix  $\Gamma^{1/2} \Sigma_{yz} \Sigma_{zz}^{-1} \Sigma_{zy} \Gamma^{1/2}$ .

## High-Dimensional Traffic Forecasting

## VAR( $d$ ) model

- Recall that  $\mathbf{x}_t = \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} + \boldsymbol{\epsilon}_t$ .
  - Coefficients  $\mathbf{A}_k \in \mathbb{R}^{N \times N}$ ,  $k = 1, \dots, d$  are tensor, e.g.,  $\mathbf{A} \in \mathbb{R}^{N \times N \times d}$ .

## VAR( $d$ ) with Tucker decomposition (Wang et al.'21)

For VAR( $d$ ) on the multivariate time series  $\mathbf{x}_t \in \mathbb{R}^N, t = 1, \dots, T$ , the reduced-rank VAR via Tucker decomposition is given by

$$\min_{\mathcal{G}, U_1, U_2, U_3} \frac{1}{2} \sum_{t=d+1}^T \| \mathbf{x}_t - (\mathcal{G} \times_1 U_1 \times_2 U_2 \times_3 U_3)_{(1)} \mathbf{z}_t \|_2^2 \quad (8)$$

where  $\mathbf{z}_t = (\mathbf{x}_{t-1}^\top, \dots, \mathbf{x}_{t-d}^\top)^\top \in \mathbb{R}^{dN}$ . The multilinear rank is  $(R_1, R_2, R_3)$ .

$\mathcal{G} \in \mathbb{R}^{R_1 \times R_2 \times R_3}$  is the core tensor, while  $\mathbf{U}_1 \in \mathbb{R}^{N \times R_1}$ ,  $\mathbf{U}_2 \in \mathbb{R}^{N \times R_2}$ , and  $\mathbf{U}_3 \in \mathbb{R}^{d \times R_3}$  are the component matrices.

**Advantage:** High compression rate.

**Limitations:** Nonconvex optimization; not scalable to large problems.

# High-Dimensional Traffic Forecasting

## High-dimensional VAR( $d$ )

Given the (high-dimensional) multivariate time series  $\mathbf{x}_t \in \mathbb{R}^N, t = 1, \dots, T$ , for the data  $\mathbf{x}_t \in \mathbb{R}^N, \mathbf{z}_t \in \mathbb{R}^{dN}, t = d+1, \dots, T$ , we have a lower-dimensional problem:

$$\min_{\{\mathbf{U}, \mathbf{W}, \mathbf{V} | \mathbf{U}^\top \mathbf{U} = \mathbf{V}^\top \mathbf{V} = \mathbf{I}_R\}} \frac{1}{2} \sum_{t=d+1}^T \|\mathbf{x}_t - \mathbf{U} \mathbf{W} \mathbf{V}^\top \mathbf{z}_t\|_2^2 \quad (9)$$

or equivalently, we have a canonical form:

$$\min_{\{\mathbf{U}, \mathbf{W}, \mathbf{V} | \mathbf{U}^\top \mathbf{U} = \mathbf{V}^\top \mathbf{V} = \mathbf{I}_R\}} \frac{1}{2} \sum_{t=d+1}^T \|\mathbf{U}^\top \mathbf{x}_t - \mathbf{W} \mathbf{V}^\top \mathbf{z}_t\|_2^2 \quad (10)$$

where  $\mathbf{W} \in \mathbb{R}^{R \times R}$  is the coefficient matrix.

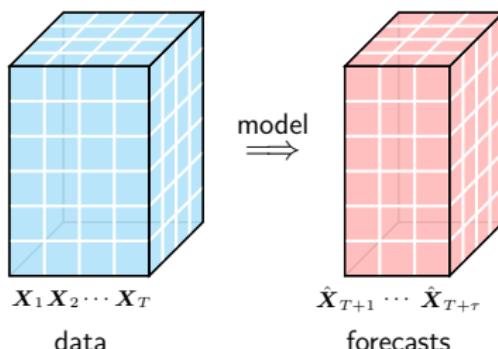
## VARMA( $d, q$ )

$$\mathbf{x}_t = \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} + \boldsymbol{\epsilon}_t + \sum_{p=1}^q \mathbf{B}_p \boldsymbol{\epsilon}_{t-p} \quad (11)$$

Moving average (MA) processes smooth out the data noises.

# Multidimensional Traffic Forecasting

- **Objective B-2:** Given a multidimensional traffic time series  $X_1, \dots, X_T \in \mathbb{R}^{M \times N}$ , forecast data points  $\hat{X}_{T+\tau}, \tau \in \mathbb{N}_+$ .



[Q]

- How to perform forecasting on this kind of data?
- How to preserve the intrinsic tensor representation of data?

# Multidimensional Traffic Forecasting

## Matrix autoregressive model (Chen, Xiao & Yang'21)

Given matrix-variate time series  $\mathbf{X}_t \in \mathbb{R}^{M \times N}$ ,  $t = 1, \dots, T$ , then the  $d$ th-order matrix autoregressive (MAR( $d$ )) model takes the form of

$$\mathbf{X}_t = \sum_{k=1}^d \mathbf{A}_k \mathbf{X}_{t-k} \mathbf{B}_k^\top + \mathbf{E}_t \quad (12)$$

where  $\mathbf{A}_k \in \mathbb{R}^{M \times M}$ ,  $\mathbf{B}_k \in \mathbb{R}^{N \times N}$ ,  $k = 1, \dots, d$  are the coefficient matrices.

### Advantages:

- Preserve the intrinsic tensor representation.
- Reduce parameters in autoregressive models (if  $n = \max\{M, N\}$ ), e.g.,

$$\mathcal{O}(n^4) \text{ in VAR(1)} \quad \text{vs.} \quad \mathcal{O}(n^2) \text{ in MAR(1)}$$

**Limitation:** Not scalable to large problems.

# Multidimensional Traffic Forecasting

Borrowing the idea of Sylvester equation.

## Sylvester-type MAR( $d$ )

Given matrix-variate time series  $\mathbf{X}_t \in \mathbb{R}^{M \times N}, t = 1, \dots, T$ , then we define the MAR( $d$ ) as follows,

$$\mathbf{X}_t = \sum_{k=1}^d \left( \mathbf{A}_k \mathbf{X}_{t-k} + \mathbf{X}_{t-k} \mathbf{B}_k^\top \right) + \mathbf{E}_t \quad (13)$$

where  $\mathbf{A}_k \in \mathbb{R}^{M \times M}, \mathbf{B}_k \in \mathbb{R}^{N \times N}, k = 1, \dots, d$  are the coefficient matrices.

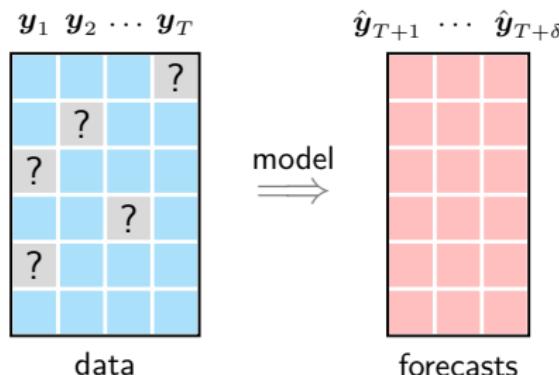
### Advantage:

- Instead of the bilinear form of MAR( $d$ ), solving Sylvester-type MAR( $d$ ) is computationally economic.

# Traffic Forecasting on Sparse Data

## Multivariate traffic time series

- **Objective C-1:** Given a partially observed data  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  consisting of time series  $y_1, \dots, y_T \in \mathbb{R}^N$ , forecast data points  $\hat{y}_{T+\delta}, \delta \in \mathbb{N}^+$ .



[Q]

- How to learn from *high-dimensional* and *sparse* data?
- How to model *nonstationarity* in time series?
- How to perform forecasting on these time series?

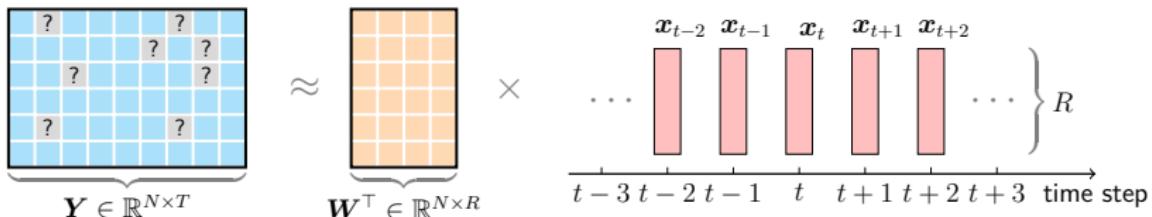
# Traffic Forecasting on Sparse Data

## Multivariate traffic time series

### Temporal matrix factorization (Yu et al.'16; Chen & Sun'21)

Given any partially observed time series data  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  with observed index set  $\Omega$ , then temporal matrix factorization assumes a  $d$ th-order vector autoregressive (VAR) process on the temporal factor matrix:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} \quad & \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2) \\ & + \frac{\lambda}{2} \sum_{t=d+1}^T \|\mathbf{x}_t - \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k}\|_2^2 \end{aligned} \quad (14)$$



VAR is usually built on stationary time series (temporal factors).

# Traffic Forecasting on Sparse Data

## Multivariate traffic time series

### Nonstationary temporal matrix factorization (NoTMF)

Given any partially observed time series data  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  with observed index set  $\Omega$ , then we assume a season- $m$  differencing on the latent temporal factors:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} & \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2) \\ & + \frac{\lambda}{2} \sum_{t=d+m+1}^T \|(\mathbf{x}_t - \mathbf{x}_{t-m}) - \sum_{k=1}^d \mathbf{A}_k (\mathbf{x}_{t-k} - \mathbf{x}_{t-m-k})\|_2^2 \end{aligned} \quad (15)$$

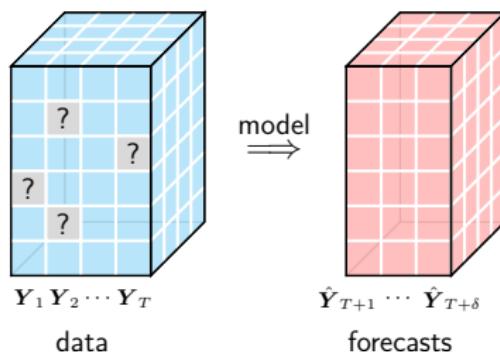
- First-order differencing  $\mathbf{x}'_t = \mathbf{x}_t - \mathbf{x}_{t-1}$ .
  - Second-order differencing  $\mathbf{x}''_t = (\mathbf{x}_t - \mathbf{x}_{t-1}) - (\mathbf{x}_{t-1} - \mathbf{x}_{t-2})$ .
  - Twice-differenced series  $\mathbf{x}'''_t = (\mathbf{x}_t - \mathbf{x}_{t-m}) - (\mathbf{x}_{t-1} - \mathbf{x}_{t-m-1})$ .
- 😊 Stationarizing a time series with differencing can improve the prediction.<sup>4</sup>

<sup>4</sup> Stationarity and differencing: <https://otexts.com/fpp2/stationarity.html>

# Traffic Forecasting on Sparse Data

## Multidimensional traffic time series

- **Objective C-2:** Given a partially observed data  $\mathcal{Y} \in \mathbb{R}^{M \times N \times T}$  consisting of time series  $\mathbf{Y}_1, \dots, \mathbf{Y}_T \in \mathbb{R}^{M \times N}$ , forecast data points  $\hat{\mathbf{Y}}_{T+\delta}, \delta \in \mathbb{N}^+$ .



- **Solution:** Temporal tensor factorization, e.g., CP factorization + VAR (on latent temporal factors).

## Conclusion

## Contributions

- *Objective A: Spatiotemporal traffic data imputation.* Develop a low-rank temporal modeling framework and improve the imputation accuracy, efficiency, and scalability.
  - *Objective B: High-dimensional and multidimensional forecasting.* Fast and accurate forecasting approach for high-dimensional and large-scale data; tensor representation based autoregressive model for multidimensional data.
  - *Objective C: Forecasting on sparse data.* Low-rank temporal modeling framework for traffic time series forecasting in the presence of missing values.

# Research Work during Ph.D. Research

- Publications
  - [J1] X. Chen, M. Lei, N. Saunier, L. Sun (2021). Low-rank autoregressive tensor completion for spatiotemporal traffic data imputation. *IEEE Transactions on Intelligent Transportation Systems*. (Early access)
  - [J2] X. Chen, Y. Chen, N. Saunier, L. Sun (2021). Scalable low-rank tensor learning for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*, 129: 103226.
  - [C1] X. Chen, M. Lei, N. Saunier, L. Sun (2021). Low-rank autoregressive tensor completion for spatiotemporal traffic data imputation. *The 7th SIGKDD Workshop on Mining and Learning from Time Series (MiLeTS)* at KDD 2021.
- Preprint (under review)
  - [P1] X. Chen, C. Zhang, X.L. Zhao, N. Saunier, L. Sun (2022). Nonstationary temporal matrix factorization for multivariate time series forecasting. *IEEE Transactions on Knowledge and Data Engineering*.
- Open-source projects
  - **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (780 stars & 240 forks on GitHub)  
<https://github.com/xinchen/transdim>
  - **tracebase**: Multivariate time series forecasting on high-dimensional and sparse Uber movement speed data. (19 stars & 5 forks on GitHub)  
<https://github.com/xinchen/tracebase>



# Thanks for your attention!

Any Questions?

## About me:

-  Homepage: <https://xinychen.github.io>
-  GitHub: <https://github.com/xinychen> (2.4K+ stars)
-  Blog: <https://medium.com/@xinyu.chen> (30K+ views)
-  How to reach me: [chenxy346@gmail.com](mailto:chenxy346@gmail.com)