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Matrix and Tensor Models for Spatiotemporal Traffic Data Imputation and Forecasting

Ph.D. Defense

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December 11, 2023



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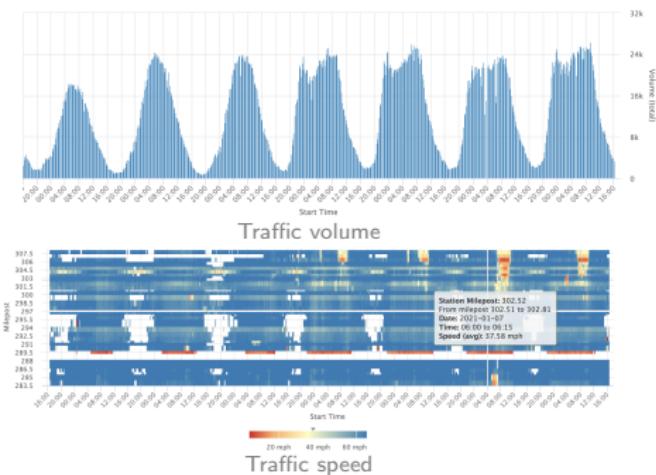
Outline

1. Background
2. Literature Review
3. Nonstationary Temporal Matrix Factorization
4. Low-Rank Autoregressive Tensor Completion
5. Laplacian Convolutional Representation
6. Hankel Tensor Factorization
7. Experiments
8. Conclusion

Traffic Flow Data

Many spatiotemporal traffic time series data are in the form of **matrix**.

- Portland highway traffic data¹



- $X \in \mathbb{R}^{N \times T}$ with N spatial locations \times T time steps
- Traffic volume/speed shows strong spatial/temporal dependencies

¹<https://portal.its.pdx.edu/home>

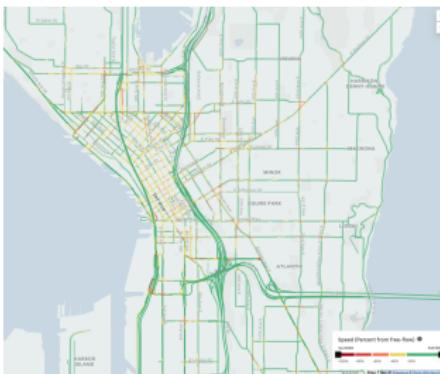
Urban Movement Data

High-dimensional & sparse

- Uber (hourly) movement speed data



NYC movement

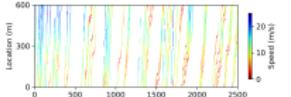
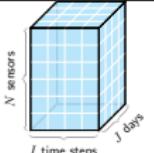
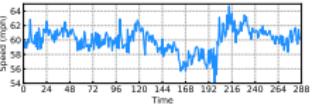
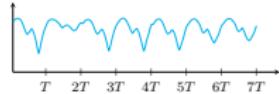


Seattle movement

- $\{(\text{road segment, time slot (hour)}, \text{average speed})\}$
- Computing hourly speed: Road segments have 5+ unique trips.

Issue: Insufficient sampling of ridesharing vehicles on the road network!

Spatiotemporal Traffic Data

Sparse	
High-dimensional	
Multidimensional	
Noises & outliers	
Nonstationary	

Traffic data show strong spatiotemporal patterns and correlations.

Problem Formulation

Objective A: Impute missing values in the data matrix $\mathbf{Y} \in \mathbb{R}^{N \times T}$ (or tensor $\mathcal{Y} \in \mathbb{R}^{M \times N \times T}$).



- Matrix completion (Observed index set Ω)

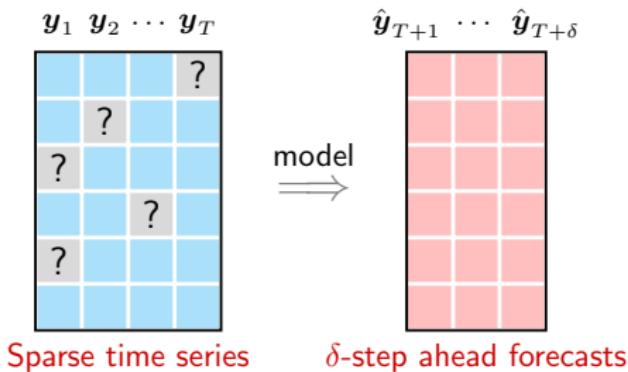
$$\underbrace{\mathcal{P}_\Omega(\mathbf{Y})}_{\text{Partially observed}} \xrightarrow{\text{Estimate}} \underbrace{\mathcal{P}_\Omega^\perp(\mathbf{Y})}_{\text{Unobserved}}$$

Modeling process:

- How to make use of spatiotemporal traffic patterns?
- How to make use of traffic time series dynamics?

Problem Formulation

Objective B: Given a partially observed data $\mathbf{Y} \in \mathbb{R}^{N \times T}$ consisting of time series $\mathbf{y}_1, \dots, \mathbf{y}_T \in \mathbb{R}^N$, forecast data points $\hat{\mathbf{y}}_{T+\delta}, \delta \in \mathbb{N}^+$.

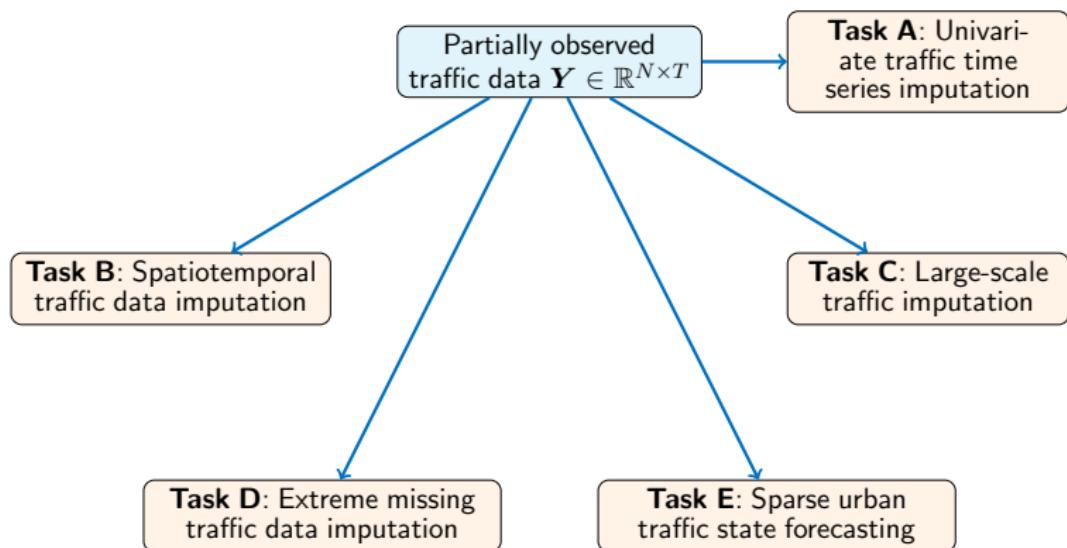


Modeling process:

- How to characterize time series dynamics in high-dimensional and sparse traffic data?

Tasks

We are working on **spatiotemporal traffic data imputation and forecasting**.



Imputation & Forecasting

Traffic data imputation

- Time series autoregression
(Schafer'97, Chen & Shao'00)
- Principal component analysis
(Qu et al.'09, Li et al.'13)
- Matrix factorization
(Asif et al.'13, Asif et al.'16)
- Tensor factorization
(Tan et al.'13, Chen et al.'19)
- Low-rank tensor completion
(Ran et al.'16, Chen et al.'20)
- Temporal matrix/tensor factorization
(Chen & Sun'22)

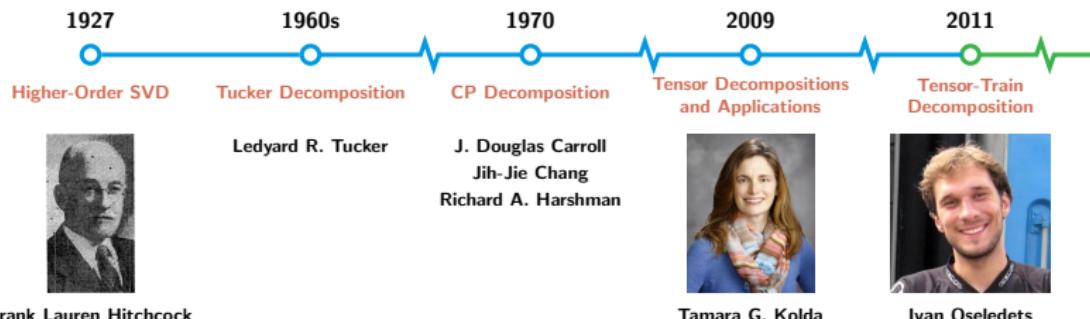
Time series forecasting on sparse data

- Autoregression predictor
(Anava et al.'15)
- Prediction on the imputed data
(e.g., Che et al.'18)
- Dynamic tensor completion
(Tan et al.'16)
- Temporal matrix factorization
(Yu et al.'16, Chen & Sun'22)
- Online matrix factorization
(Gultekin & Paisley'18)

Autoregression, matrix/tensor factorization/completion, ...

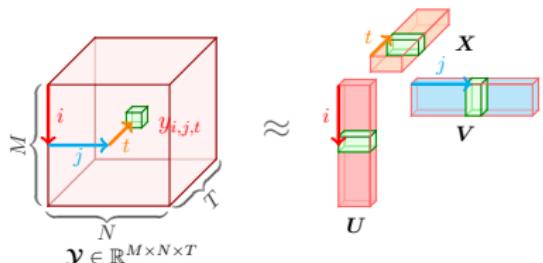
Tensor Factorization

- Revisit tensor factorization



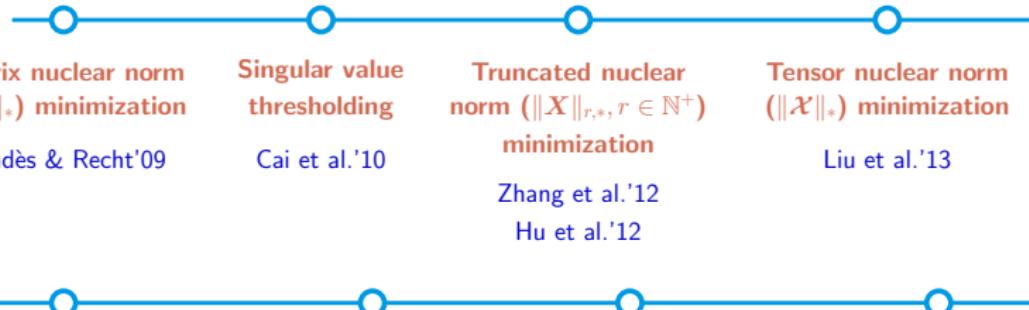
Frank Lauren Hitchcock

- **CP tensor factorization:** Factorize \mathcal{Y} into the combination of three rank- R factor matrices (i.e., low-dimensional latent factors).



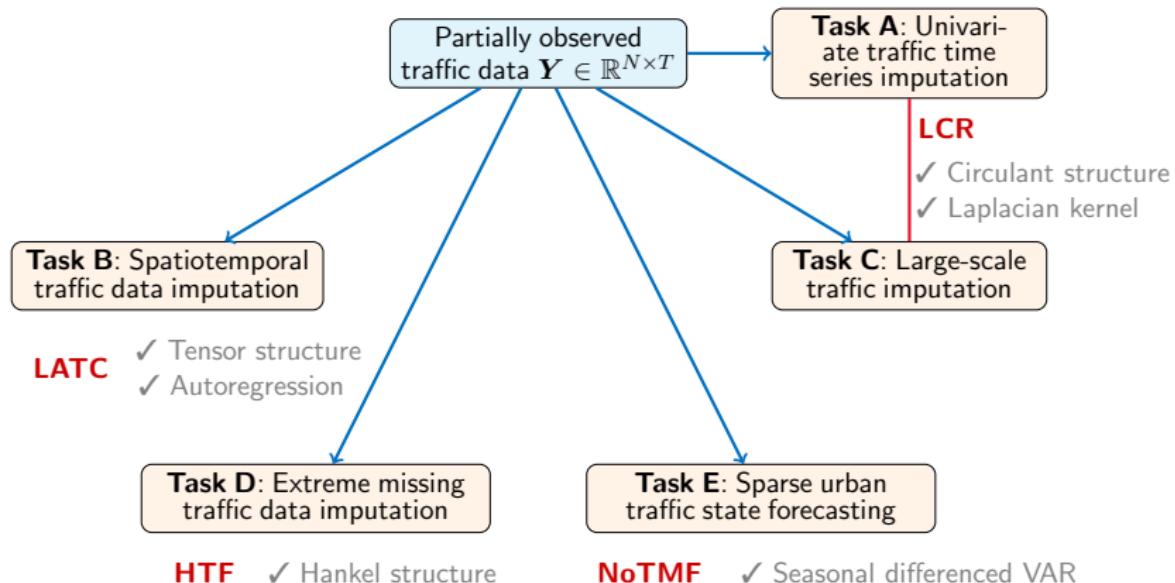
$$\begin{cases} y_{i,j,t} \approx \sum_{r=1}^R u_{i,r} v_{j,r} x_{t,r} \\ \mathcal{Y} \approx \sum_{r=1}^R \mathbf{u}_r \otimes \mathbf{v}_r \otimes \mathbf{x}_r \end{cases}$$

Matrix/Tensor Completion



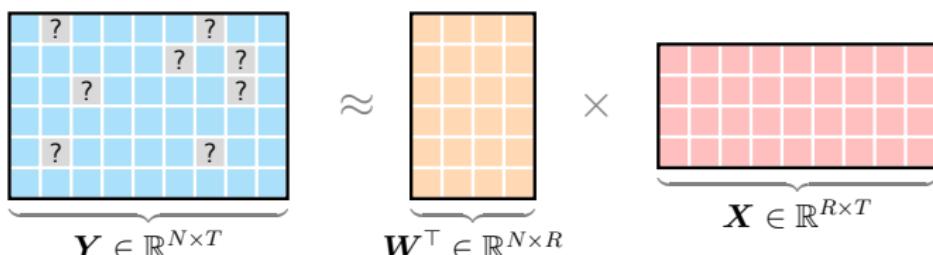
Overview

Matrix/Tensor methods + temporal modeling (e.g., smoothing & autoregression)



Matrix Factorization

A simple approach to reconstruct missing values.



MF (Koren et al.'09)

Estimating low-dimensional \mathbf{W}, \mathbf{X} :

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2$$

on data \mathbf{Y} w/ observed index set Ω .

- ✓ Learn from sparse data
- ✓ Spatial factor matrix \mathbf{W}
- ✓ Temporal factor matrix \mathbf{X}

How to build temporal correlations on MF?

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

$$\begin{matrix} \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \end{matrix} \underbrace{\quad}_{\mathbf{Y} \in \mathbb{R}^{N \times T}}$$

 \approx

$$\underbrace{\begin{matrix} \text{?} & \text{?} & \dots & \text{?} \\ \text{?} & \text{?} & \dots & \text{?} \\ \vdots & \vdots & \ddots & \vdots \\ \text{?} & \text{?} & \dots & \text{?} \end{matrix}}_{\mathbf{W}^T \in \mathbb{R}^{N \times R}}$$

 \times

$$\underbrace{\begin{matrix} \text{?} & \text{?} & \dots & \text{?} \\ \text{?} & \text{?} & \dots & \text{?} \\ \vdots & \vdots & \ddots & \vdots \\ \text{?} & \text{?} & \dots & \text{?} \end{matrix}}_{\mathbf{X} \in \mathbb{R}^{R \times T}}$$

$$\begin{matrix} \text{?} & \text{?} & \dots & \text{?} \\ \text{?} & \text{?} & \dots & \text{?} \\ \vdots & \vdots & \ddots & \vdots \\ \text{?} & \text{?} & \dots & \text{?} \end{matrix} \underbrace{\quad}_{\mathbf{Y} \in \mathbb{R}^{N \times T}}$$

 \approx

$$\underbrace{\begin{matrix} \text{?} & \text{?} & \dots & \text{?} \\ \text{?} & \text{?} & \dots & \text{?} \\ \vdots & \vdots & \ddots & \vdots \\ \text{?} & \text{?} & \dots & \text{?} \end{matrix}}_{\mathbf{W}^T \in \mathbb{R}^{N \times R}}$$

 \times

$$\downarrow \text{ } \mathbf{X} \text{ is time series?}$$

$$\dots \quad \underbrace{\begin{matrix} \mathbf{x}_{t-2} \\ \mathbf{x}_{t-1} \\ \mathbf{x}_t \\ \mathbf{x}_{t+1} \\ \mathbf{x}_{t+2} \end{matrix}}_{\begin{matrix} t-3 \\ t-2 \\ t-1 \\ t \\ t+1 \\ t+2 \\ t+3 \end{matrix}} \dots \Biggr\}_R$$

time step

Why? Temporal factor matrix $\mathbf{X} \in \mathbb{R}^{R \times T}$ is the low-dimensional representation of time series dynamics of $\mathbf{Y} \in \mathbb{R}^{N \times T}$.

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

MF (Koren et al.'09)

Estimating low-dimensional \mathbf{W}, \mathbf{X} :

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2$$

on data \mathbf{Y} w/ observed index set Ω .

dth-order VAR

$$\mathbf{x}_t = \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} + \underbrace{\boldsymbol{\epsilon}_t}_{\mathcal{N}(\mathbf{0}, \mathbf{I})}$$

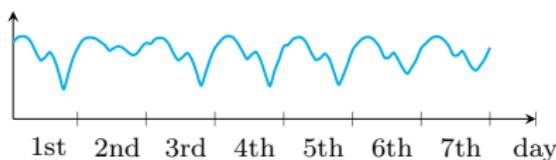
w/ coefficients $\{\mathbf{A}_k\}$.

↓ Yu et al.'16
Chen & Sun'21

$$\min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} \frac{1}{2} \underbrace{\|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2}_{\text{MF on data } \mathbf{Y}} + \frac{\gamma}{2} \underbrace{\sum_{t=d+1}^T \left\| \mathbf{x}_t - \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} \right\|_2^2}_{\text{VAR on temporal factors } \mathbf{X}}$$

Nonstationary Temporal Matrix Factorization

Traffic data are nonstationary due to daily patterns of traffic flow.



- Season- m differencing ($m \in \mathbb{N}^+$, e.g., daily/weekly):

$$\mathbf{x}_t \approx \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} \quad \Rightarrow \quad \mathbf{x}_t - \mathbf{x}_{t-m} \approx \sum_{k=1}^d \mathbf{A}_k (\mathbf{x}_{t-k} - \mathbf{x}_{t-m-k})$$

- (Ours) Optimization problem:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} & \frac{1}{2} \underbrace{\|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2}_{\text{MF on data } \mathbf{Y}} + \frac{\rho}{2} \underbrace{(\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)}_{\text{Regularization}} \\ & + \frac{\gamma}{2} \underbrace{\sum_{t=d+m+1}^T \left\| (\mathbf{x}_t - \mathbf{x}_{t-m}) - \sum_{k=1}^d \mathbf{A}_k (\mathbf{x}_{t-k} - \mathbf{x}_{t-m-k}) \right\|_2^2}_{\text{VAR on seasonal differenced temporal factors}} \end{aligned}$$

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF

- Optimization problem:

$$\min_{\mathbf{W}, \mathbf{X}, \mathbf{A}} \frac{1}{2} \underbrace{\|\mathcal{P}_{\Omega}(\mathbf{Y} - \mathbf{W}^T \mathbf{X})\|_F^2}_{\text{MF on data } \mathbf{Y}} + \frac{\rho}{2} \underbrace{(\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)}_{\text{Regularization}} + \frac{\gamma}{2} \underbrace{\|\mathbf{X} \Psi_0^T - \mathbf{A} (\mathbf{I}_d \otimes \mathbf{X}) \Psi^T\|_F^2}_{\text{Temporal modeling on } \mathbf{X}}$$

where $\Psi_0 \in \mathbb{R}^{(T-d-m) \times T}$ and $\Psi \in \mathbb{R}^{(T-d-m) \times (dT)}$ are temporal operators.

- Alternating minimization (let f be the obj.):

$$\left\{ \begin{array}{ll} \text{Spatial factors} & \mathbf{W} := \{\mathbf{W} \mid \frac{\partial f}{\partial \mathbf{W}} = \mathbf{0}\} \quad (\text{least squares}) \\ \text{Temporal factors} & \mathbf{X} := \{\mathbf{X} \mid \frac{\partial f}{\partial \mathbf{X}} = \mathbf{0}\} \quad (\text{conjugate gradient}) \\ \text{VAR coefficients} & \mathbf{A} := \{\mathbf{A} \mid \frac{\partial f}{\partial \mathbf{A}} = \mathbf{0}\} \quad (\text{least squares}) \end{array} \right.$$

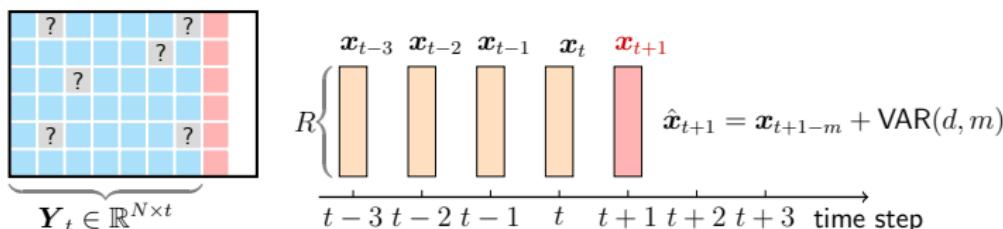
Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

Implementation

- Estimate $\mathbf{W}, \mathbf{X}, \mathbf{A}$
- Forecast $\hat{\mathbf{x}}_{t+1}$ with VAR
- Return $\hat{\mathbf{y}}_{t+1} = \mathbf{W}^\top \hat{\mathbf{x}}_{t+1}$

- ✓ Sparse input \mathbf{Y}_t
- ✓ Forecast in latent spaces



Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

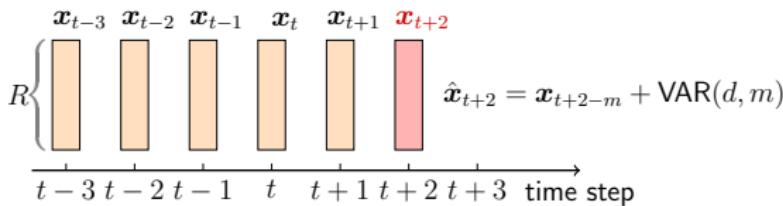
- Online forecasting (Gultekin & Paisley'18):
 - Fix the spatial factor matrix \mathbf{W}
 - Use input data \mathbf{Y}_{t+1} to update the temporal factor matrix \mathbf{X} and the coefficient matrix \mathbf{A}

Implementation

- Estimate \mathbf{X}, \mathbf{A}
- Forecast $\hat{\mathbf{x}}_{t+2}$ with VAR
- Return $\hat{\mathbf{y}}_{t+2} = \mathbf{W}^\top \hat{\mathbf{x}}_{t+2}$

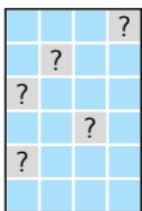
- ✓ Sparse input \mathbf{Y}_{t+1}
- ✓ Forecast in latent spaces

$$\mathbf{Y}_{t+1} \in \mathbb{R}^{N \times (t+1)}$$

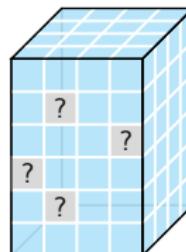


Matrix/Tensor Completion

Problem? Impute missing values in matrices/tensors.



$$\mathcal{P}_\Omega(\mathbf{Y}) \in \mathbb{R}^{N \times T}$$



$$\mathcal{P}_\Omega(\mathbf{Y}) \in \mathbb{R}^{N \times I \times J}$$

Cornerstone: Nuclear norm minimization

LRMC (Candès & Recht'09)

Estimating the matrix \mathbf{X} :

$$\min_{\mathbf{X}} \|\mathbf{X}\|_*$$

$$\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{Y})$$

on data \mathbf{Y} w/ observed index set Ω .

vs.

LRTC (Liu et al.'13)

Estimating the tensor \mathcal{X} :

$$\min_{\mathcal{X}} \|\mathcal{X}\|_*$$

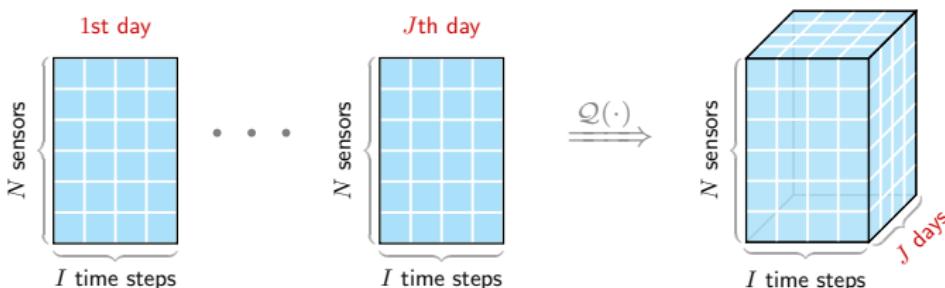
$$\text{s.t. } \mathcal{P}_\Omega(\mathcal{X}) = \mathcal{P}_\Omega(\mathbf{Y})$$

on data \mathbf{Y} w/ observed index set Ω .

Limitation: Nuclear norm minimization only covers global consistency.

Low-Rank Autoregressive Tensor Completion

- Introduce traffic tensors with day dimension² (Tan et al.'13, Chen et al.'19, ...).



- Build temporal correlations with univariate autoregression.

On the time series $\mathbf{Y} \in \mathbb{R}^{N \times T}$:

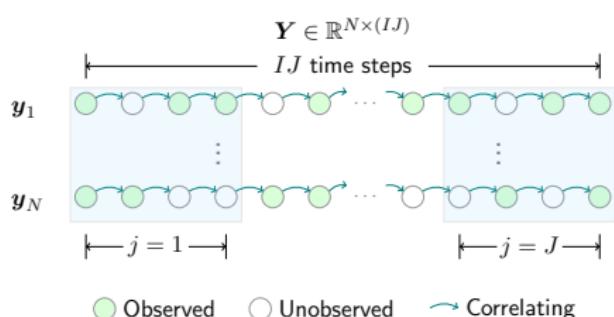
$$\|\mathbf{Y}\|_{\mathbf{A}, \mathcal{H}} \triangleq \sum_{n,t} \left(y_{n,t} - \sum_k \mathbf{a}_{n,k} y_{n,t-h_k} \right)^2$$

w/ the time lag set $\mathcal{H} = \{h_1, \dots, h_d\}$ and the coefficient matrix $\mathbf{A} \in \mathbb{R}^{N \times d}$.

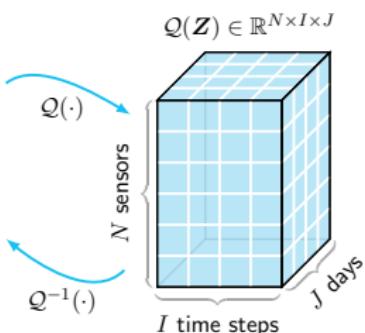
²There are $T = IJ$ time steps in total.

Low-Rank Autoregressive Tensor Completion

Local consistency w/ autoregression



Global consistency w/ tensor structure



LATC

Optimization problem:

$$\min_{\mathbf{Z}, \mathbf{A}} \underbrace{\|Q(\mathbf{Z})\|_{r,*}}_{\text{Global}} + \frac{\gamma}{2} \underbrace{\|\mathbf{Z}\|_{\mathbf{A}, \mathcal{H}}}_{\text{Local}}$$

s.t. $\mathcal{P}_\Omega(\mathbf{Z}) = \mathcal{P}_\Omega(\mathbf{Y})$

on data \mathbf{Y} w/ observed index set Ω .

Two subproblems

$$\Rightarrow \begin{cases} \mathbf{Z} := \underset{\mathcal{P}_\Omega(\mathbf{Z}) = \mathcal{P}_\Omega(\mathbf{Y})}{\arg \min} \|Q(\mathbf{Z})\|_{r,*} + \frac{\gamma}{2} \|\mathbf{Z}\|_{\mathbf{A}, \mathcal{H}} \\ \mathbf{A} := \frac{1}{2} \|\mathbf{Z}\|_{\mathbf{A}, \mathcal{H}} \end{cases} \quad (\text{Least squares})$$

Low-Rank Autoregressive Tensor Completion

Z -subproblem:

$$Z := \arg \min_{\mathcal{P}_\Omega(Z) = \mathcal{P}_\Omega(Y)} \|Q(Z)\|_{r,*} + \frac{\gamma}{2} \|Z\|_{A,\mathcal{H}}$$

- Augmented Lagrangian function:³

$$\mathcal{L}(X, Z, W) = \|X\|_{r,*} + \frac{\gamma}{2} \|Z\|_{A,\mathcal{H}} + \frac{\lambda}{2} \|X - Q(Z)\|_F^2 + \langle W, X - Q(Z) \rangle + \pi(Z)$$

Implementation

Repeat

- Compute Z
- Compute A



Implementation

Repeat

- Repeat
 - # Alternating Direction Method of Multipliers (ADMM)
 - Compute X
 - Compute Z
 - Compute W
- Compute A

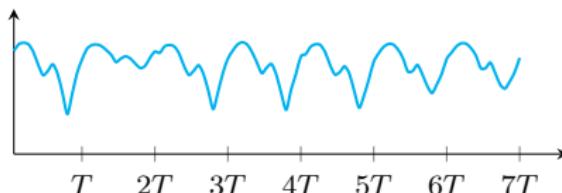
³ $W \in \mathbb{R}^{N \times I \times J}$ (Lagrange multiplier); $\langle \cdot, \cdot \rangle$ (inner product). The indicator function:

$$\pi(Z) = \begin{cases} 0, & \text{if } \mathcal{P}_\Omega(Z) = \mathcal{P}_\Omega(Y), \\ +\infty, & \text{otherwise.} \end{cases}$$

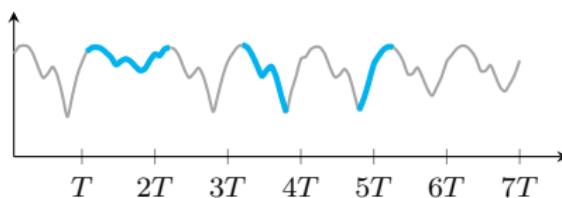
Laplacian Convolutional Representation

Motivation: Time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



- Local trends (e.g., short-term time series trends):

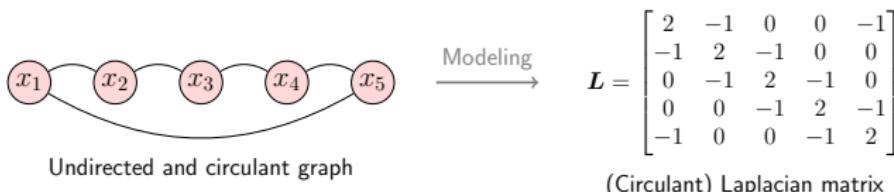


How to characterize both global and local trends in sparse time series?

Laplacian Convolutional Representation

Local trend modeling

- Intuition of (circulant) Laplacian matrix



- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (2, -1, 0, 0, -1)^\top$$

↓

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \dots, -1}_{\tau}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$.

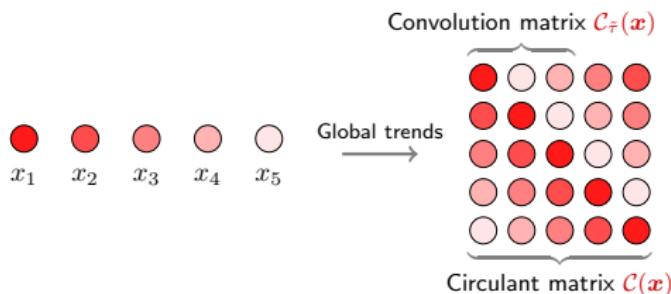
- (Laplacian) Temporal regularization:

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2$$

Reformulate temporal regularization with circular convolution.

Laplacian Convolutional Representation

Global trend modeling: Circulant matrix $\mathcal{C}(\mathbf{x})$ vs. convolution matrix $\mathcal{C}_{\tilde{\tau}}(\mathbf{x})$



- Circulant/Convolution nuclear norm minimization
 - A balance between global and local trends modeling?

CircNNM (Liu'22, Liu & Zhang'23)

Estimating \mathbf{x} :

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

on data \mathbf{y} w/ observed index set Ω .

ConvNNM (Liu'22, Liu & Zhang'23)

Estimating \mathbf{x} :

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}_{\tilde{\tau}}(\mathbf{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

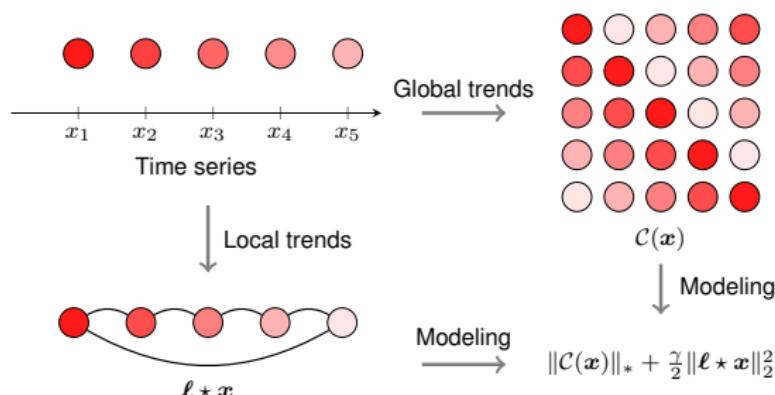
on data \mathbf{y} w/ observed index set Ω .

Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\mathbf{y} \in \mathbb{R}^T$ with observed index set Ω , LCR utilizes **circulant matrix** and **Laplacian kernel** to characterize **global and local trends** in time series, respectively, i.e.,

$$\begin{aligned} & \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 \\ & \text{s.t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$



Laplacian Convolutional Representation

- Augmented Lagrangian function:⁴

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2$$

- The ADMM scheme:

$$\begin{cases} \mathbf{x} := \arg \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Nuclear norm minimization)} \\ \mathbf{z} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Closed-form solution)} \\ \mathbf{w} := \mathbf{w} + \lambda(\mathbf{x} - \mathbf{z}) & \text{(Standard update)} \end{cases}$$

- Optimize \mathbf{x} ?

$$\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1 \quad \& \quad \frac{1}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2 = \frac{1}{2T} \|\mathcal{F}(\boldsymbol{\ell}) \circ \mathcal{F}(\mathbf{x})\|_2^2$$

Nuclear norm minimization \Rightarrow **ℓ_1 -norm minimization with FFT** in $\mathcal{O}(T \log T)$ time.

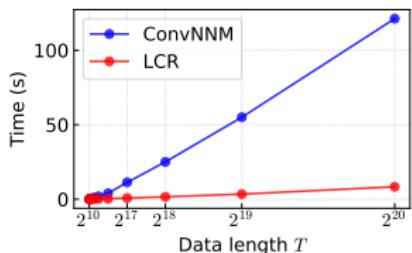
⁴ $\mathbf{w} \in \mathbb{R}^T$ (Lagrange multiplier); $\langle \cdot, \cdot \rangle$ (inner product).

Laplacian Convolutional Representation

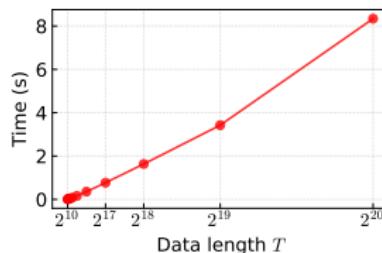
Empirical time complexity

On the synthetic data $\mathbf{y} \in \mathbb{R}^T$ with $T \in \{2^{10}, 2^{11}, \dots, 2^{20}\}$

- Ours: **LCR**
 - An FFT implementation in $\mathcal{O}(T \log T)$
 - The logarithmic factor $\log T$ makes the FFT highly efficient
- Baseline: **ConvNNM**⁵ ([Liu'22](#), [Liu & Zhang'23](#))
 - Convolution matrix $C_{\tilde{\tau}}(\mathbf{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$ with kernel size $\tilde{\tau} = 2^4$
 - Singular value thresholding in $\mathcal{O}(\tilde{\tau}^2 T)$



ConvNNM vs. LCR



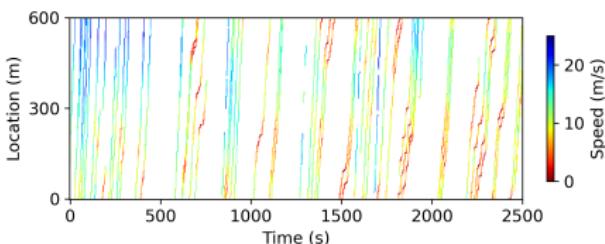
LCR

⁵Convolution nuclear norm minimization.

Hankel Tensor Factorization

Motivation: Spatiotemporal data reconstruction

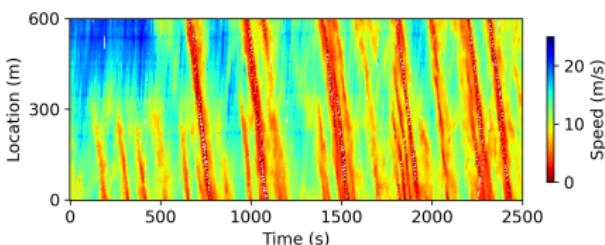
- Sparse speed field reconstruction problem in vehicular traffic flow.



200-by-500 matrix
(NGSIM)



Reconstruct speed field from
5% sparse trajectories?



How to characterize both spatial and temporal dependencies?

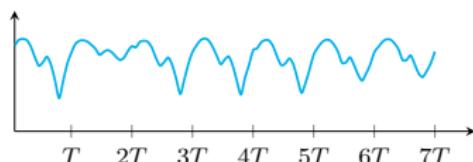
Hankel Tensor Factorization

- Hankel matrix

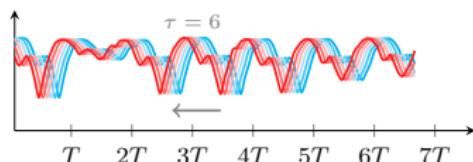
- Given $\mathbf{x} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4 \times 2}$$

- Automatic temporal modeling



Traffic time series



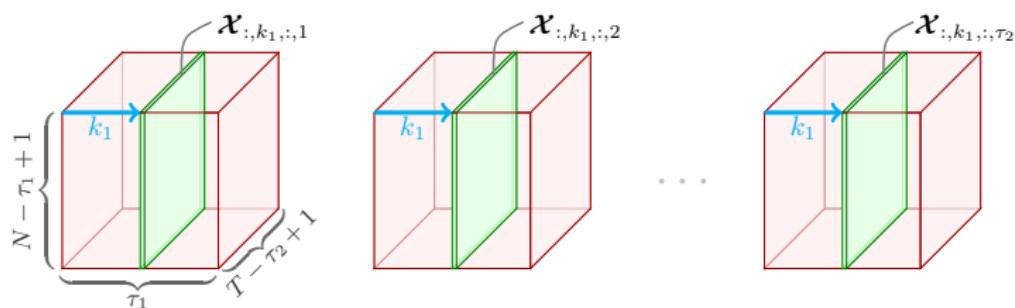
Hankel matrix

Hankel Tensor Factorization

- Hankel tensor: Given any matrix $\mathbf{X} \in \mathbb{R}^{N \times T}$, we have

$$\mathcal{X} \triangleq \mathcal{H}_{\tau_1, \tau_2}(\mathbf{X})$$

- Window lengths: $\tau_1, \tau_2 \in \mathbb{N}^+$;
- Tensor size: $(N - \tau_1 + 1) \times \tau_1 \times (T - \tau_2 + 1) \times \tau_2$;



(Figure) 4th order Hankel tensor: A sequence of third-order tensors.

- Slice: $\mathcal{X}_{:,k_1,:,:,\tau_2}, \forall k_1, k_2$;
- Slice size: $(N - \tau_1 + 1) \times (T - \tau_2 + 1)$.

Hankel Tensor Factorization

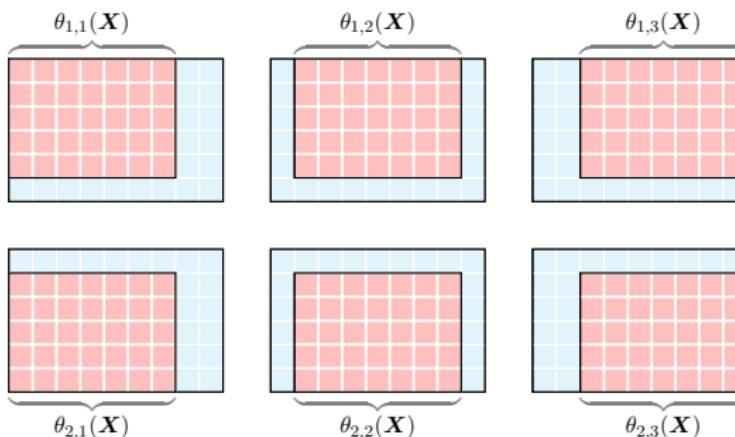
Hankel indexing

- Sampling function for the Hankelization:

$$\theta_{k_1, k_2}(\mathbf{X}) \triangleq [\mathcal{H}_{\tau_1, \tau_2}(\mathbf{X})]_{:, k_1, :, k_2},$$

referring to as the tensor slice with $k_1 \in \{1, \dots, \tau_1\}$, $k_2 \in \{1, \dots, \tau_2\}$.

- [Importance] Developing memory-efficient algorithms.



- Tensor slices $\theta_{k_1, k_2}(\mathbf{X})$ vs. data matrix \mathbf{X}

Hankel Tensor Factorization

Ours:

- Convolutional tensor decomposition (circular convolution \star_{row}):

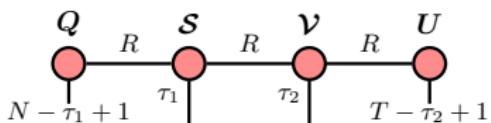
$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \star_{\text{row}} \mathbf{s}_{k_1}^{\top})(\mathbf{U} \star_{\text{row}} \mathbf{v}_{k_2}^{\top})^{\top}$$

Baselines:

- Tensor-train decomposition:

$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \mathbf{S}_{k_1})(\mathbf{U} \mathbf{V}_{k_2})^{\top}$$

- $\{\mathbf{S}_{k_1}, \mathbf{V}_{k_2}\}$ are **circulant matrices** \Rightarrow convolutional decomposition
- $\{\mathbf{S}_{k_1}, \mathbf{V}_{k_2}\}$ are **diagonal matrices** \Rightarrow CP decomposition



- CP tensor decomposition (Khatri-Rao product \odot):

$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \odot \mathbf{s}_{k_1}^{\top})(\mathbf{U} \odot \mathbf{v}_{k_2}^{\top})^{\top}$$

Hankel Tensor Factorization

HTF (convolutional decomposition)

- Optimization problem:

$$\min_{Q, S, U, V} \frac{1}{2} \underbrace{\sum_{k_1, k_2} \left\| \mathcal{P}_{\Omega_{k_1, k_2}} (\theta_{k_1, k_2}(Y) - (Q \star_{\text{row}} s_{k_1})(U \star_{\text{row}} v_{k_2})^T) \right\|_F^2}_{\text{Tensor decomposition on Hankel tensor slices}} + \frac{\rho}{2} (\|Q\|_F^2 + \|S\|_F^2 + \|U\|_F^2 + \|V\|_F^2)$$

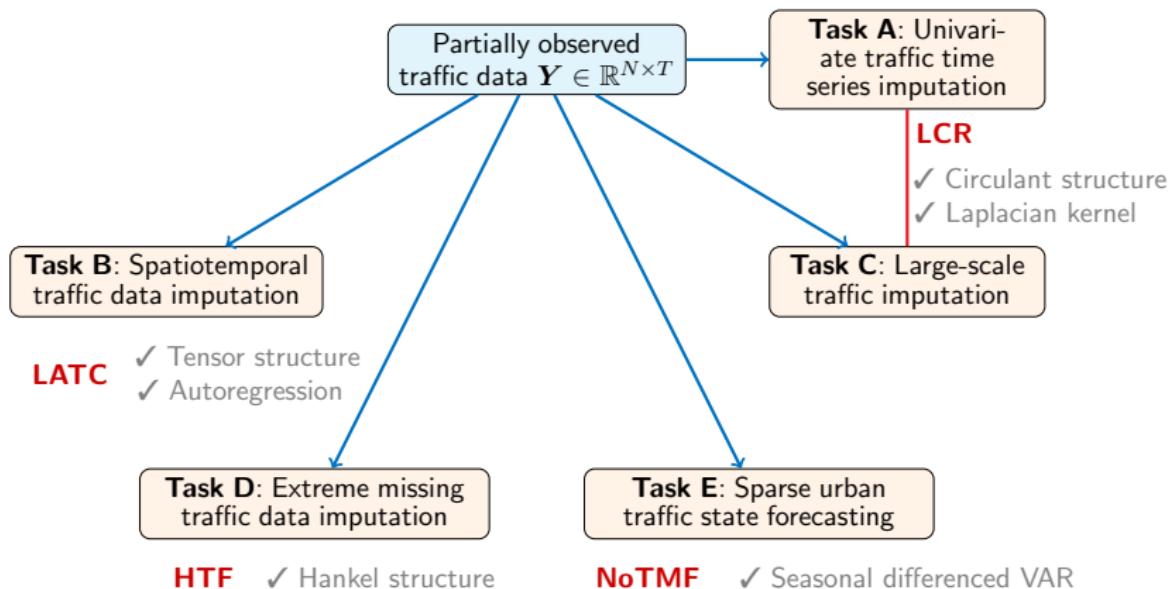
- Alternating minimization (let f be the obj.):

$$\begin{cases} Q := \{Q \mid \frac{\partial f}{\partial Q} = \mathbf{0}\} & \text{(conjugate gradient)} \\ s_{k_1} := \{s_{k_1} \mid \frac{\partial f}{\partial s_{k_1}} = \mathbf{0}\}, \forall k_1 & \text{(conjugate gradient)} \\ U := \{U \mid \frac{\partial f}{\partial U} = \mathbf{0}\} & \text{(conjugate gradient)} \\ v_{k_2} := \{v_{k_2} \mid \frac{\partial f}{\partial v_{k_2}} = \mathbf{0}\}, \forall k_2 & \text{(conjugate gradient)} \end{cases}$$

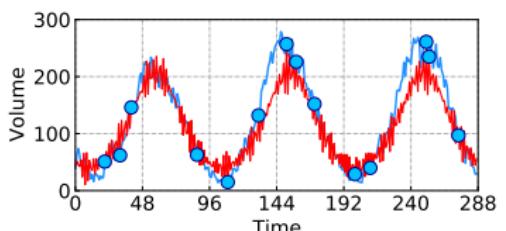
- Memory-efficient but still computationally costly!

Overview

We are working on **spatiotemporal traffic data imputation and forecasting**.



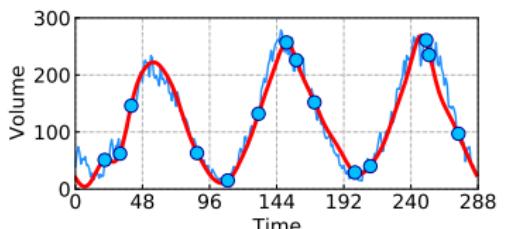
Task A: Univariate Traffic Time Series Imputation



CircNNM:

$$\begin{aligned} & \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* \\ \text{s. t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

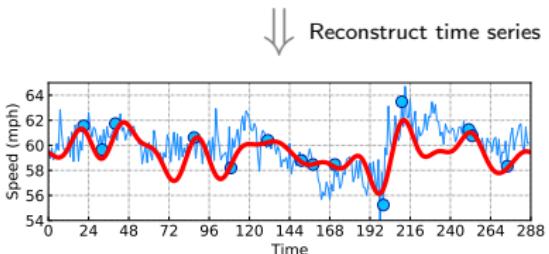
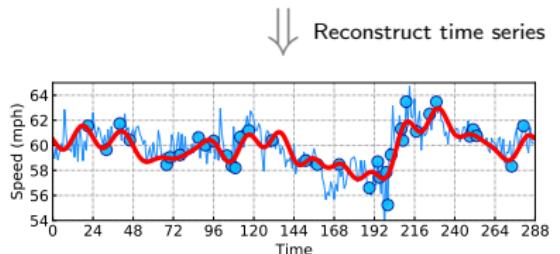
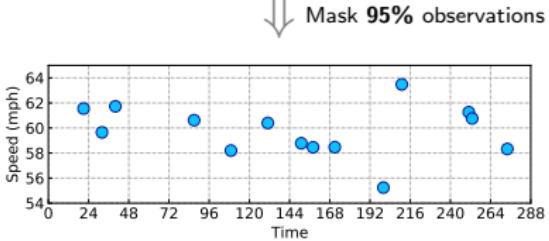
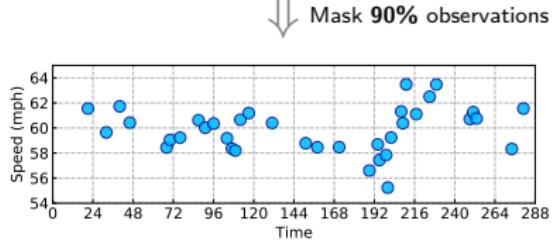
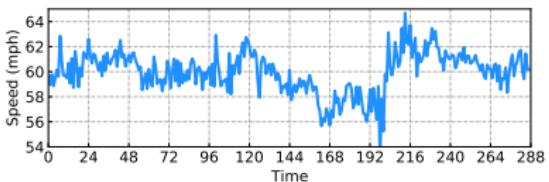
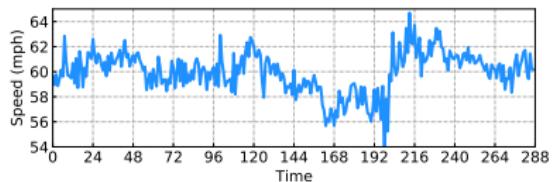
↓ Plus temporal regularization



LCR:

$$\begin{aligned} & \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 \\ \text{s. t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

Task A: Univariate Traffic Time Series Imputation



Task B: Spatiotemporal Traffic Data Imputation

LATC vs. baseline models (in MAPE/RMSE)

- On the Seattle freeway traffic speed dataset ($\mathbf{Y} \in \mathbb{R}^{323 \times 8064}$)

Missing rate	LATC	LAMC	LRTC-TNN	BTMF	SPC
30%, Random Missing	4.90/3.16	5.98/3.73	4.99/3.20	5.91/3.72	5.92/3.62
70%, Random Missing	5.96/3.71	8.02/4.70	6.10/3.77	6.47/3.98	7.38/4.30
90%, Random Missing	7.46/4.50	10.56/5.91	8.08/4.80	8.17/4.81	9.75/5.31
30%, Nonrandom Missing	7.10/4.33	6.99/4.25	6.85/4.21	9.26/5.36	8.87/4.99
70%, Nonrandom Missing	9.40/5.40	9.75/5.60	9.23/5.35	10.47/6.15	11.32/5.92
30%, Block-out Missing	9.43/5.36	27.05/13.66	9.52/5.41	14.33/13.60	11.30/5.84

- On the Portland highway traffic volume dataset ($\mathbf{Y} \in \mathbb{R}^{1156 \times 2976}$)

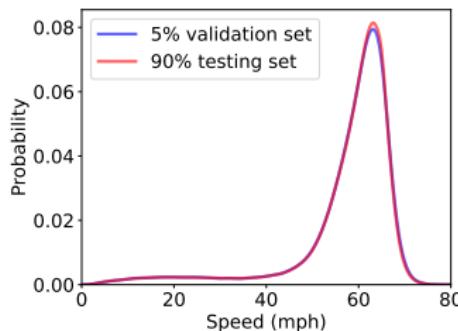
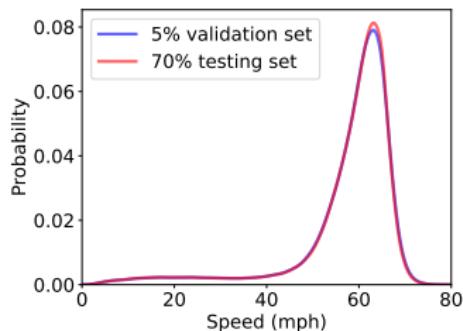
Missing rate	LATC	LAMC	LRTC-TNN	BTMF	SPC
30%, Random Missing	16.95/15.99	17.93/16.03	17.27/16.08	18.22/19.14	21.29/56.73
70%, Random Missing	19.59/18.70	21.26/19.37	19.99/18.73	19.96/22.21	24.35/43.32
90%, Random Missing	23.15/22.83	25.64/23.75	22.90/22.68	23.90/25.71	28.45/39.65
30%, Nonrandom Missing	19.48/19.14	19.93/19.69	19.59/ 18.91	19.55/20.38	26.96/60.33
70%, Nonrandom Missing	27.67/45.03	25.75/28.25	30.26/60.85	23.86/26.74	33.42/47.34
30%, Block-out Missing	24.01/23.50	29.21/27.60	31.74/74.42	27.85/25.68	31.01/60.33

- LATC vs. LAMC: The significance of tensor representation
- LATC vs. LRTC-TNN: The significance of temporal autoregression

Task B: Spatiotemporal Traffic Data Imputation

Parameter tuning process: Training set, validation set, and testing set?

- Random missing on the Seattle freeway traffic speed dataset



- Imputation performance (e.g., 70% missing rate)

On the validation set (5% data)

γ/λ	Truncation parameter				
	$r = 5$	$r = 10$	$r = 15$	$r = 20$	$r = 25$
1/10	7.84/4.52	7.20/4.25	6.82/4.08	6.60/3.98	6.41/3.92
1/5	7.84/4.52	7.20/4.25	6.82/4.08	6.59/3.97	6.41/3.92
1	7.81/4.51	7.18/4.24	6.80/4.07	6.58/3.97	6.39/3.91
5	7.70/4.45	7.09/4.20	6.72/4.04	6.49/3.93	6.29/3.87
10	7.59/4.39	7.00/4.16	6.64/4.00	6.41/3.89	6.22/3.83

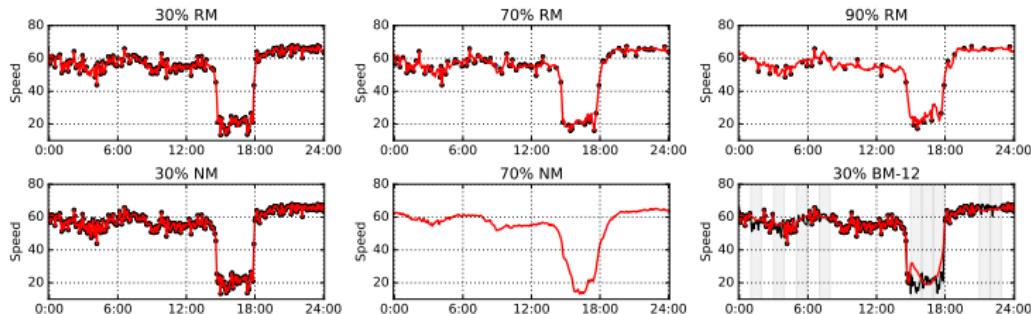
On the testing set (70% data)

γ/λ	Truncation parameter				
	$r = 5$	$r = 10$	$r = 15$	$r = 20$	$r = 25$
1/10	7.83/4.53	7.18/4.27	6.80/4.09	6.58/3.99	6.41/3.92
1/5	7.83/4.53	7.18/4.26	6.80/4.09	6.57/3.98	6.40/3.92
1	7.80/4.52	7.16/4.25	6.78/4.08	6.55/3.98	6.40/3.92
5	7.70/4.47	7.08/4.21	6.70/4.04	6.46/3.94	6.29/3.87
10	7.58/4.41	6.99/4.17	6.62/4.01	6.39/3.90	6.21/3.84

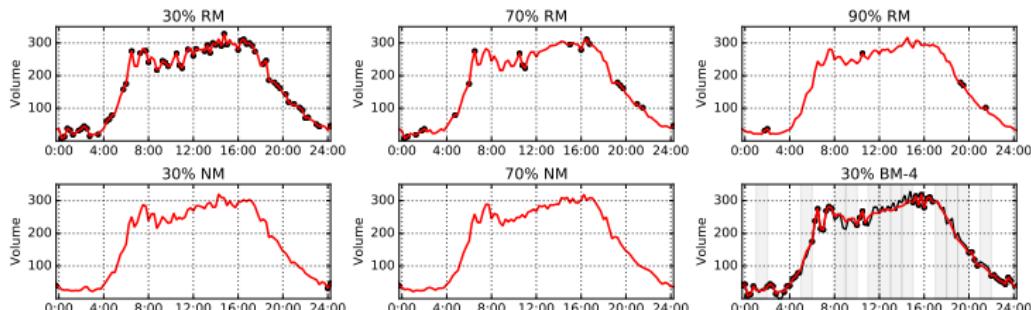
Task B: Spatiotemporal Traffic Data Imputation

LATC imputation

- Seattle freeway traffic speed data



- Portland highway traffic volume data



Task C: Large-Scale Traffic Data Imputation

LCR vs. baseline models (in MAPE/RMSE)

- PeMS-4W: California freeway traffic speed dataset ($Y \in \mathbb{R}^{11160 \times 8064}$)

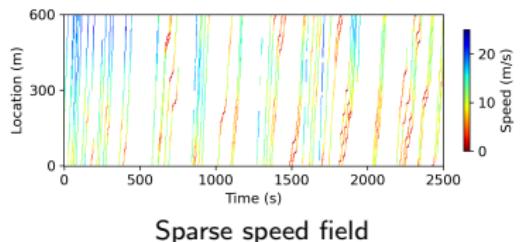
Model	Missing rate			
	30%	50%	70%	90%
LCR-2D	1.50/1.49	1.76/1.69	2.07/2.06	3.19/3.05
LCR_N	1.48/1.50	1.73/1.73	2.07/2.12	3.24/3.22
LCR	1.50/1.49	1.76/1.69	2.08/2.07	3.21/3.06
CTNNM	2.26/1.84	2.67/2.14	3.40/2.66	5.22/3.90
CircNNM	2.26/1.84	2.69/2.15	3.43/2.67	5.34/3.96
LRMC	2.04/1.80	2.43/2.12	3.08/2.66	6.05/4.43
HaLRTC	1.98/1.73	2.22/1.98	2.84/2.49	4.39/3.66
LRTC-TNN	1.68/1.55	1.93/1.77	2.33/2.14	3.40/3.10
NoTMF	2.95/2.65	3.05/2.73	3.33/2.97	5.22/4.71

Results

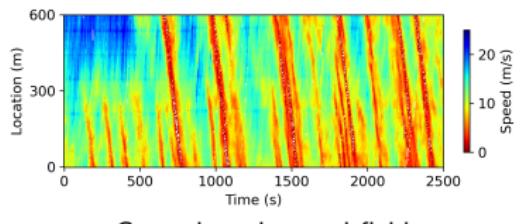
- LCR-2D > CTNNM: The importance of temporal regularization.
- CTNNM \geq CircNNM: Cyclic tensor is superior to circulant matrix.
- LCR > LRMC/LRTC: The importance of global/local modeling.

$$\mathcal{O}(NT \log(NT)) \text{ (FFT)} \quad \text{vs.} \quad \mathcal{O}(\min\{N^2T, NT^2\}) \text{ (SVD)}$$

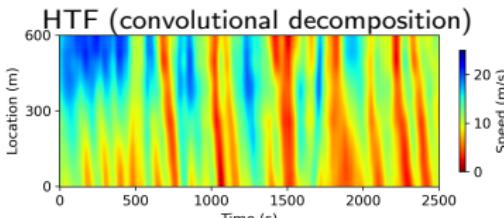
Task D: Extreme Missing Traffic Data Imputation



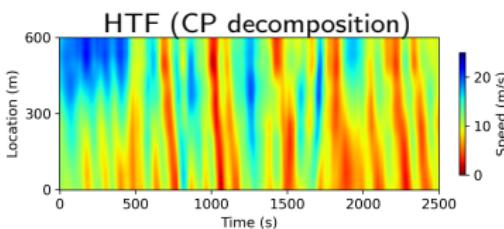
Sparse speed field



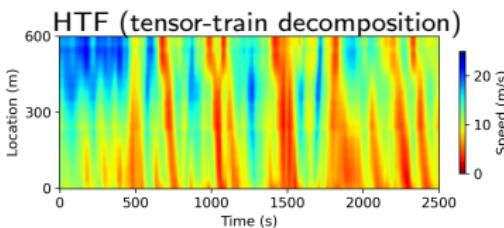
Ground truth speed field



MAPE = 51.92%



MAPE = 53.93%



MAPE = 56.48%

Task D: Extreme Missing Traffic Data Imputation

HTF vs. baseline models (in MAPE/RMSE)

- On the Seattle freeway traffic speed dataset ($Y \in \mathbb{R}^{323 \times 8064}$)

Model	Random missing rate			
	80%	85%	90%	95%
HTF (convolution)	6.21/3.88	6.51/4.06	6.98/4.30	8.02/4.84
HTF (tensor-train)	8.75/5.16	9.86/5.76	9.24/5.36	9.89/5.70
HTF (CP)	7.25/4.33	7.93/4.66	8.61/4.96	9.25/5.20
LATC	6.50/4.00	6.90/4.21	7.47/4.51	8.75/5.05
LRTC-TNN	6.97/4.24	7.43/4.43	8.19/4.81	9.60/5.55
LCR	6.75/4.15	7.31/4.38	7.96/4.71	9.78/5.39
BTMF	6.85/4.17	7.36/4.42	8.13/4.79	9.63/5.48

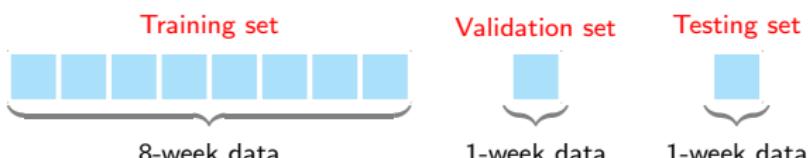
Results

- Convolutional tensor decomposition outperforms both tensor-train and CP tensor decomposition.
- Our HTF model performs better than state-of-the-art baseline models.

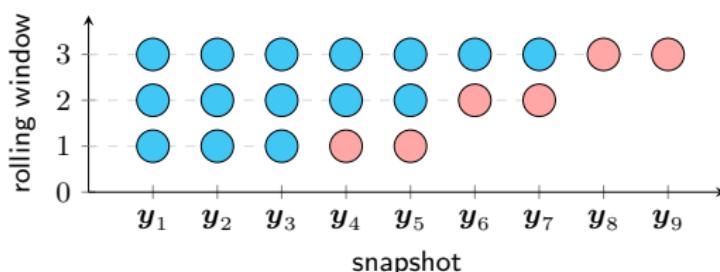
Task E: Sparse Urban Traffic State Forecasting

NoTMF forecasting

- NYC movement speed dataset:
 - **10-week data** of size 98210×1680
 - Contain **66.56%** missing values
- Rolling forecasting setup (Time horizon $\delta = 1, 2, 3, 6$):



- Rolling forecasting illustration ($\delta = 2$):



Task E: Sparse Urban Traffic State Forecasting

NoTMF vs. baseline models (in MAPE/RMSE)

- On the NYC Uber movement speed dataset

δ	d	NoTMF ($m = 24$)	NoTMF ($m = 168$)	NoTMF-1st ($m = 168$)	TRMF	BTMF	BTRMF
1	1	13.63/2.88	13.53/2.86	13.45/2.85	14.50/3.12	14.94/3.13	15.93/3.33
	2	13.47/2.84	13.41/2.84	13.42/2.84	14.14/3.05	15.70/3.41	15.90/3.35
	3	13.46/2.84	13.39/2.83	13.43/2.84	13.87/2.96	15.80/3.34	16.08/3.43
	6	13.41/2.83	13.39/2.83	13.41/2.83	14.00/2.98	15.45/3.27	16.26/3.48
2	1	13.91/2.96	13.76/2.94	13.70/2.92	15.85/3.43	15.33/3.21	16.85/3.56
	2	13.77/2.92	13.63/2.89	13.72/2.92	15.04/3.31	15.87/3.32	17.27/3.71
	3	13.72/2.91	13.61/2.89	13.73/2.92	15.25/3.36	15.69/3.33	17.24/3.74
	6	13.59/2.87	13.57/2.88	13.68/2.91	14.92/3.24	15.91/3.39	18.18/3.97
3	1	14.30/3.05	14.06/3.02	14.02/3.00	17.52/3.83	15.86/3.32	18.61/3.91
	2	14.01/2.98	13.84/2.94	13.96/2.98	17.32/4.00	16.30/3.40	18.90/4.10
	3	13.95/2.97	13.79/2.93	13.98/2.98	16.91/3.71	16.56/3.49	18.68/4.05
	6	13.78/2.92	13.73/2.92	13.91/2.96	16.72/3.65	15.49/3.27	20.45/4.66
6	1	14.61/3.11	14.67/3.20	14.98/3.32	21.20/4.70	15.99/3.32	22.40/4.69
	2	14.30/3.03	14.33/3.09	14.90/3.28	20.87/5.01	16.04/3.33	23.56/5.63
	3	14.26/3.03	14.28/3.09	14.86/3.26	20.08/4.65	15.67/3.28	24.27/5.72
	6	14.06/2.97	14.16/3.06	14.80/3.23	20.40/4.35	16.38/3.50	26.34/6.60

NoTMF performs better than other TMF models (e.g., TRMF, BTMF & BTRMF).

Task E: Sparse Urban Traffic State Forecasting

NoTMF vs. baseline models (in MAPE/RMSE)

- On the Seattle Uber movement speed dataset

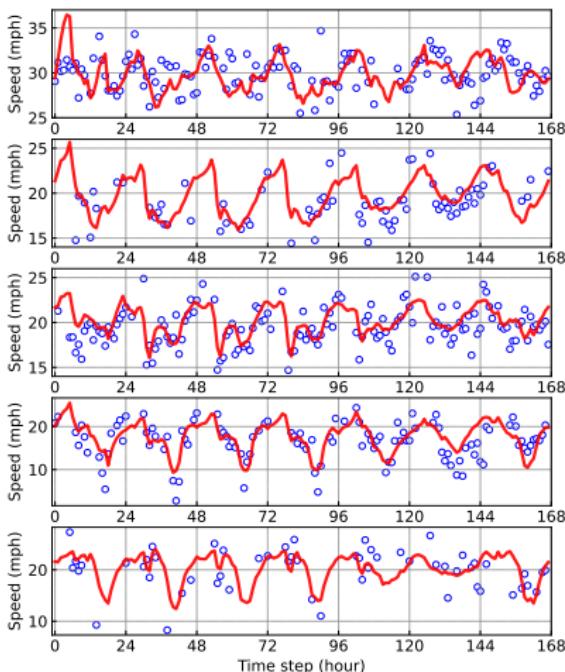
δ	d	NoTMF ($m = 24$)	NoTMF ($m = 168$)	NoTMF-1st ($m = 168$)	TRMF	BTMF	BTRMF
1	1	10.45/3.32	10.26 /3.22	10.26 / 3.21	11.58/3.79	12.23/3.89	12.52/4.01
	2	10.53/3.34	10.29/3.23	10.23 / 3.21	10.92/3.51	12.95/4.18	13.16/4.31
	3	10.42/3.30	10.30/3.22	10.25 / 3.21	10.86/3.47	12.96/4.22	13.89/4.64
	6	10.50/3.32	10.21 / 3.21	10.27/3.22	10.99/3.51	12.91/4.18	13.90/4.67
2	1	10.90/3.55	10.32/3.25	10.25 / 3.23	12.07/4.02	12.74/4.06	13.31/4.32
	2	10.90/3.52	10.31/3.24	10.25 / 3.23	12.59/4.24	13.68/4.45	13.44/4.43
	3	10.81/3.49	10.31/3.24	10.27 / 3.23	12.01/3.96	13.55/4.46	13.66/4.56
	6	10.57/3.38	10.25 / 3.23	10.27/ 3.23	12.18/3.98	13.56/4.42	14.67/4.92
3	1	11.27/3.71	10.41 / 3.29	10.41 / 3.29	13.47/4.62	13.16/4.15	14.01/4.52
	2	11.26/3.71	10.30 / 3.27	10.34/ 3.27	14.48/5.19	13.63/4.37	14.39/4.76
	3	11.11/3.62	10.35 / 3.28	10.38/ 3.28	14.04/4.83	13.76/4.42	14.67/4.84
	6	10.96/3.55	10.30 / 3.26	10.30 / 3.26	13.32/4.51	13.28/4.29	15.64/5.31
6	1	11.88/3.97	10.63/3.43	10.60 / 3.42	15.59/5.32	13.63/4.30	16.39/5.28
	2	11.58/3.83	10.55 / 3.40	10.56/ 3.40	18.66/7.20	13.27/4.19	16.77/5.58
	3	11.54/3.81	10.57/3.39	10.53 / 3.38	17.94/6.32	13.88/4.36	17.35/5.70
	6	11.27/3.70	10.53/ 3.35	10.50 / 3.35	15.12/5.24	13.30/4.24	16.63/5.62

NoTMF performs better than other TMF models (e.g., TRMF, BTMF & BTRMF).

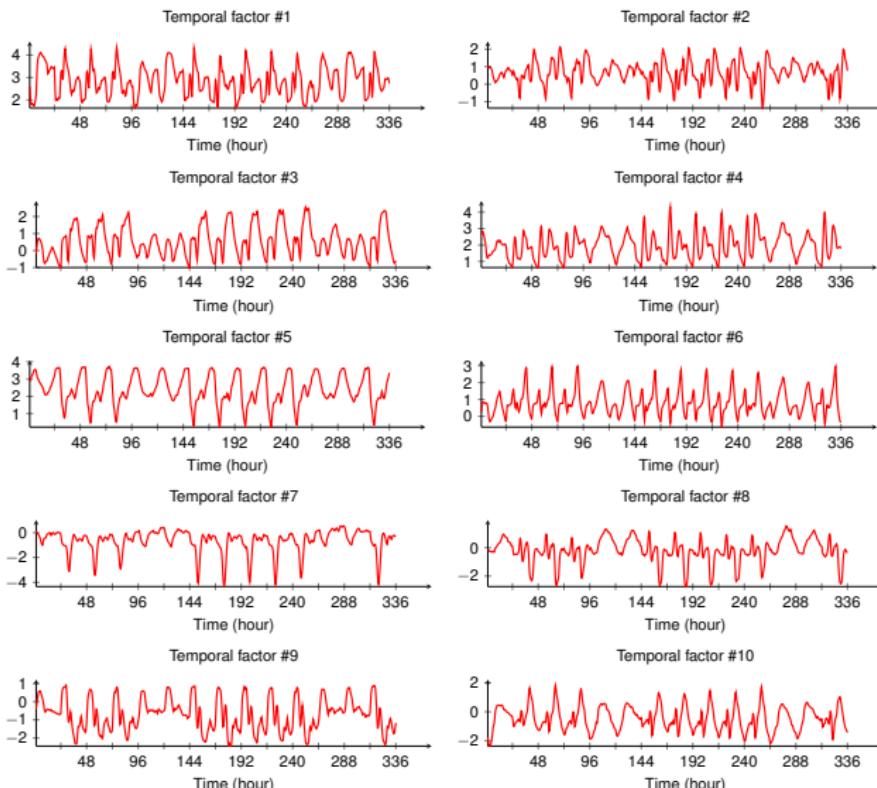
Task E: Sparse Urban Traffic State Forecasting

NoTMF forecasting ($\delta = 6$)

- On the NYC Uber movement speed dataset



Task E: Sparse Urban Traffic State Forecasting



Conclusion

Sparse	
High-dimensional	
Multidimensional	
Noises & outliers	
Nonstationary	

Low-rank framework:

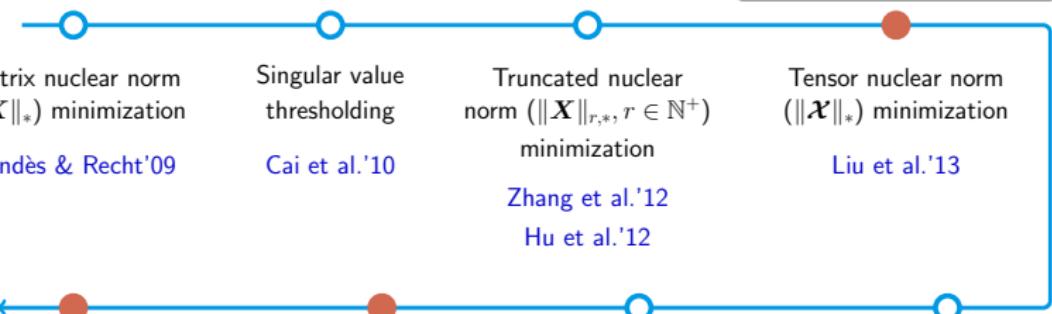
- NoTMF: matrix factorization
- LATC: low-rank tensor completion
- LCR: circulant matrix nuclear norm minimization
- HTF: tensor factorization

⇒ Temporal modeling:

- NoTMF: seasonal differenced vector autoregression
- LATC: univariate autoregression
- LCR: temporal smoothing
- HTF: automatic temporal modeling with Hankel tensor

Highlights & Contributions

(Ours) LATC:
 ✓ Temporal autoregression



Circulant/Convolution nuclear norm ($\ \mathcal{C}(\mathbf{X})\ _*$ or $\ \mathcal{C}_{\tilde{\tau}}(\mathbf{X})\ _*$) minimization Liu'22 Liu & Zhang'23	Low-rank Hankel matrix/tensor ($\mathcal{H}_\tau(\cdot)$) completion Yokota et al.'18 Sedighin et al.'20 Cai et al.'21 Yamamoto et al.'22	Tensor nuclear norm minimization with linear transform Lu et al.'19	Generalized nonconvex nonsmooth low-rank minimization Lu et al.'14
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(Ours) LCR:
 ✓ Local trend modeling
 ✓ An FFT implementation

(Ours) HTF:
 ✓ Memory-efficient
 ✓ Conv. para.

Collaborators



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Thanks for your attention!

Any Questions?

<https://xinychen.github.io/papers/thesis.pdf>

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Supplementary Material

- Model settings:
https://xinychen.github.io/papers/model_settings_v1.pdf

Nonstationary Temporal Matrix Factorization

Rewrite VAR in the form of matrix

Temporal operators

For any multivariate time series $\mathbf{X} \in \mathbb{R}^{R \times T}$ with $m, d \in \mathbb{N}^+$, if we define temporal operators as

$$\begin{aligned}\Psi_k &\triangleq \begin{bmatrix} \mathbf{0}_{(T-d-m) \times (d-k)} & -\mathbf{I}_{T-d-m} & \mathbf{0}_{(T-d-m) \times (k+m)} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{0}_{(T-d-m) \times (d+m-k)} & \mathbf{I}_{T-d-m} & \mathbf{0}_{(T-d-m) \times k} \end{bmatrix} \\ &\in \mathbb{R}^{(T-d-m) \times T}, \quad k = 0, 1, \dots, d\end{aligned}$$

then

$$\begin{aligned}&\sum_{t=d+m+1}^T \|(\mathbf{x}_t - \mathbf{x}_{t-m}) - \sum_{k=1}^d \mathbf{A}_k (\mathbf{x}_{t-k} - \mathbf{x}_{t-m-k})\|_2^2 \\ &\equiv \|\mathbf{X} \Psi_0^\top - \sum_{k=1}^d \mathbf{A}_k \mathbf{X} \Psi_k^\top\|_F^2 \triangleq \|\mathbf{X} \Psi_0^\top - \mathbf{A} (\mathbf{I}_d \otimes \mathbf{X}) \Psi^\top\|_F^2\end{aligned}$$

where $\mathbf{A} \triangleq [\mathbf{A}_1 \quad \cdots \quad \mathbf{A}_d]$ and $\Psi \triangleq [\Psi_1 \quad \cdots \quad \Psi_d]$.

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{X}, \mathbf{A}} \quad & \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2) \\ & + \frac{\gamma}{2} \|\mathbf{X} \Psi_0^\top - \mathbf{A} (\mathbf{I}_d \otimes \mathbf{X}) \Psi^\top\|_F^2 \end{aligned}$$

Alternating minimization method:

- w.r.t. \mathbf{W} :

$$\frac{\partial f}{\partial \mathbf{W}} = -\mathbf{X} \mathcal{P}_\Omega^\top (\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) + \rho \mathbf{W} = \mathbf{0} \quad (\text{Least squares})$$

- w.r.t. \mathbf{X} :

$$\frac{\partial f}{\partial \mathbf{X}} = -\mathbf{W} \mathcal{P}_\Omega (\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) + \rho \mathbf{X} + \gamma \sum_{k=0}^d \mathbf{A}_k^\top \left(\sum_{h=0}^d \mathbf{A}_h \mathbf{X} \Psi_h^\top \right) \Psi_k = \mathbf{0}$$

This generalized Sylvester equation can be solved by **conjugate gradient**.

- w.r.t. \mathbf{A} :

$$\mathbf{A} = \mathbf{X} \Psi_0^\top [(\mathbf{I}_d \otimes \mathbf{X}) \Psi^\top]^\dagger \quad (\text{Least squares})$$

Low-Rank Autoregressive Tensor Completion

- Augmented Lagrangian function:

$$\mathcal{L}(\mathbf{X}, \mathbf{Z}, \mathbf{W}) = \|\mathbf{X}\|_{r,*} + \frac{\gamma}{2} \|\mathbf{Z}\|_{A,\mathcal{H}} + \frac{\lambda}{2} \|\mathbf{X} - \mathcal{Q}(\mathbf{Z})\|_F^2 + \langle \mathbf{W}, \mathbf{X} - \mathcal{Q}(\mathbf{Z}) \rangle + \pi(\mathbf{Z})$$

- The ADMM⁶ scheme:

$$\begin{cases} \mathbf{X} := \arg \min_{\mathbf{X}} \mathcal{L}(\mathbf{X}, \mathbf{Z}, \mathbf{W}) & \text{(Truncated nuclear norm minimization)} \\ \mathbf{Z} := \arg \min_{\mathbf{Z}} \mathcal{L}(\mathbf{X}, \mathbf{Z}, \mathbf{W}) & \text{(Generalized Sylvester equation)} \\ \mathbf{W} := \mathbf{W} + \lambda(\mathbf{X} - \mathcal{Q}(\mathbf{Z})) & \text{(Standard update)} \end{cases}$$

⁶Alternating Direction Method of Multipliers.

Laplacian Convolutional Representation

- Optimize \mathbf{x} via FFT (in $\mathcal{O}(T \log T)$ time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2\end{aligned}$$

where we introduce $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$ (i.e., FFT).

ℓ_1 -norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of ℓ_1 -norm minimization in complex space:

$$\min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\omega}{2} \|\hat{\mathbf{x}} - \hat{\mathbf{h}}\|_2^2$$

with complex-valued vectors $\hat{\mathbf{x}}, \hat{\mathbf{h}} \in \mathbb{C}^T$ and weight parameter ω , element-wise, the solution is given by

$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|} \cdot \max\{0, |\hat{h}_t| - 1/\omega\}, t = 1, \dots, T.$$

Laplacian Convolutional Representation

LCR

On time series $\mathbf{y} \in \mathbb{R}^T$,

$$\min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2$$

$$\text{s.t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$

w/ observed index set Ω .

LCR-2D

On time series $\mathbf{Y} \in \mathbb{R}^{N \times T}$,

$$\Rightarrow \min_{\mathbf{X}} \|\mathcal{C}(\mathbf{X})\|_* + \frac{\gamma}{2} \|(\boldsymbol{\ell}_s \boldsymbol{\ell}^\top) * \mathbf{X}\|_F^2$$

$$\text{s.t. } \|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \epsilon$$

w/ observed index set Ω .

Flipping Operation in LCR

Results on speed fields

Tuning Hyperparameters