POLYTECHNIQUE MONTRÉAL

UNIVERSITÉ D'INGÉNIERIE

IVADO

Matrix and Tensor Models for Spatiotemporal Traffic Data Imputation and Forecasting

Ph.D. Defense

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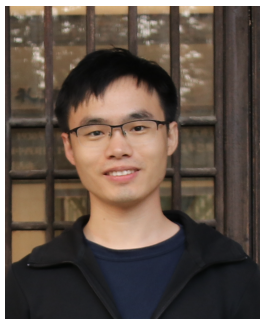
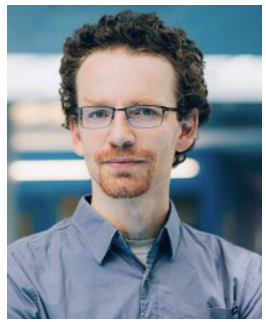


Ph.D. Candidate
Xinyu Chen



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Prof. Nicolas Saunier

Co-supervisor
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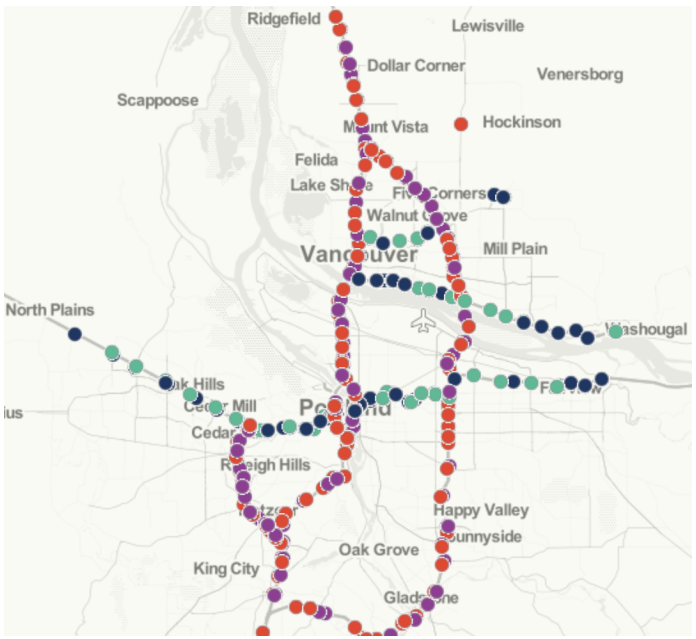
Outline

- **Background**
- **Literature Review**
  - Tensor Factorization
  - Tensor Factorization (TF)
- **Nonstationary Temporal Matrix Factorization**
- **Low-Rank Autoregressive Tensor Completion**
- **Laplacian Convolutional Representation**
  - Motivation
  - Reformulate Laplacian Regularization
  - Traffic Time Series Imputation
- **Hankel Tensor Factorization**
  - Motivation
  - Hankel Structure
- **Experiments**
- **Conclusion**

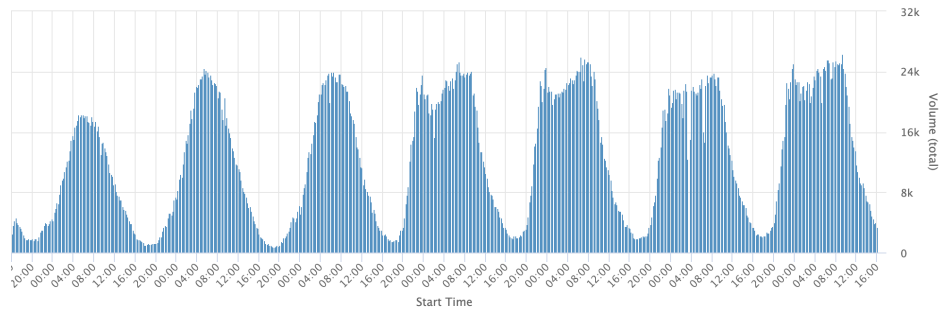
Multivariate Traffic Time Series

Many spatiotemporal traffic time series data are in the form of **matrix**.

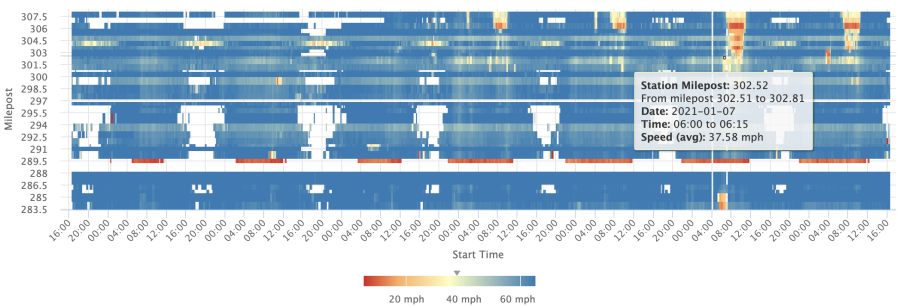
- **Example**: Portland highway traffic data[1].



Highway network & sensor locations



Traffic volume



Traffic speed

- $\boldsymbol{X} \in \mathbb{R}^{N \times T}$ with N spatial locations $\times$ T time steps
- Traffic volume/speed shows strong spatial/temporal dependencies

[1] https://portal.its.pdx.edu/home

Multiple Data Behaviors

Spatiotemporal traffic data are time series, but they involve multiple data behaviors.

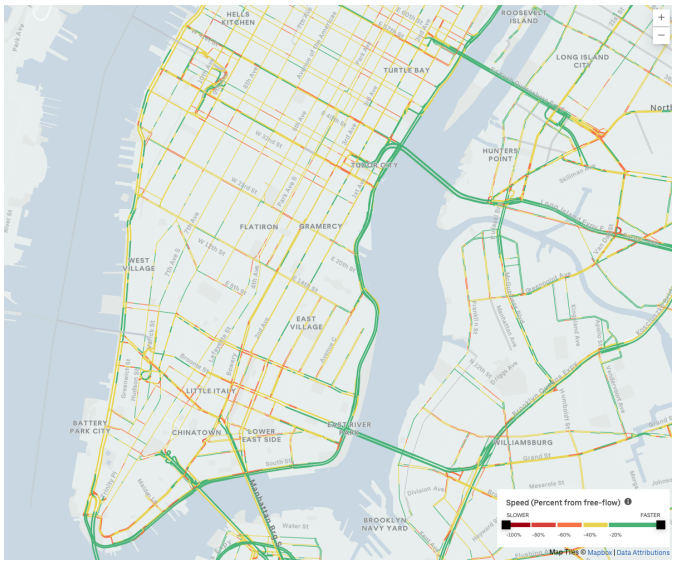
- Incompleteness & sparsity
- High-dimensionality
- Multidimensionality
- Noises & outliers
- Time-varying behavior
- Nonstationarity
- .....

In addition, spatiotemporal correlations are also very important.

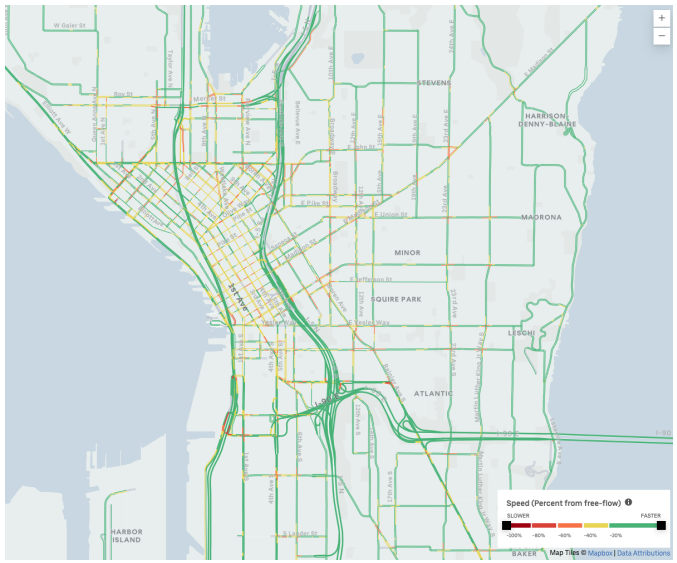
Multiple Data Behaviors

Sparsity & high-dimensionality

- Uber (hourly) movement speed data[2]



NYC movement



Seattle movement

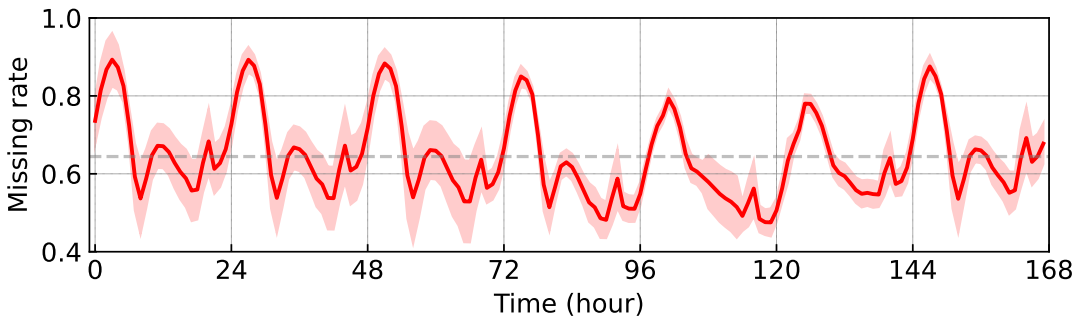
- The average speed on a given road segment for each hour of each day.
- Hourly speeds are computed when road segments have 5+ unique trips.
- **Issue**: insufficient sampling of ridesharing vehicles on the road network.

[2] https://movement.uber.com/

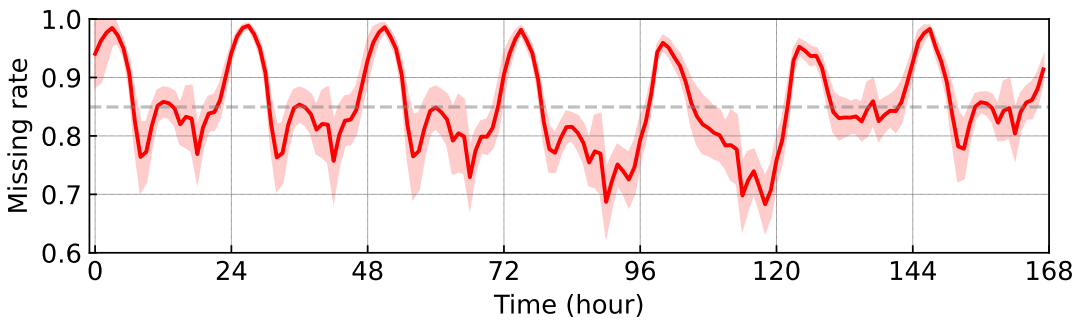
Multiple Data Behaviors

Sparsity & high-dimensionality

- **NYC** movement speed data (2019)
  - **98,210** road segments & 8,760 time steps (hours)
  - Overall missing rate: **64.43%**

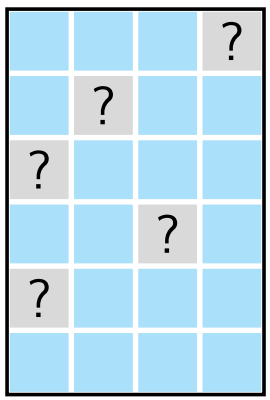


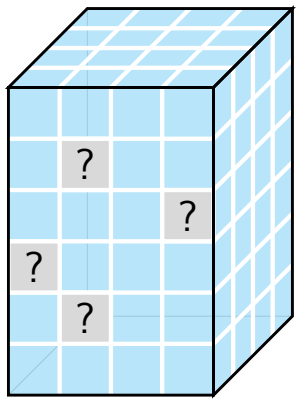
- **Seattle** movement speed data (2019)
  - **63,490** road segments & 8,760 time steps (hours)
  - Overall missing rate: **84.95%**



Problem Formulation

- **Objective A:** Given a multivariate time series data like $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ or a multidimensional time series data like $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{M \times N \times T}$ with the observed index set Ω, impute the missing values of the data.



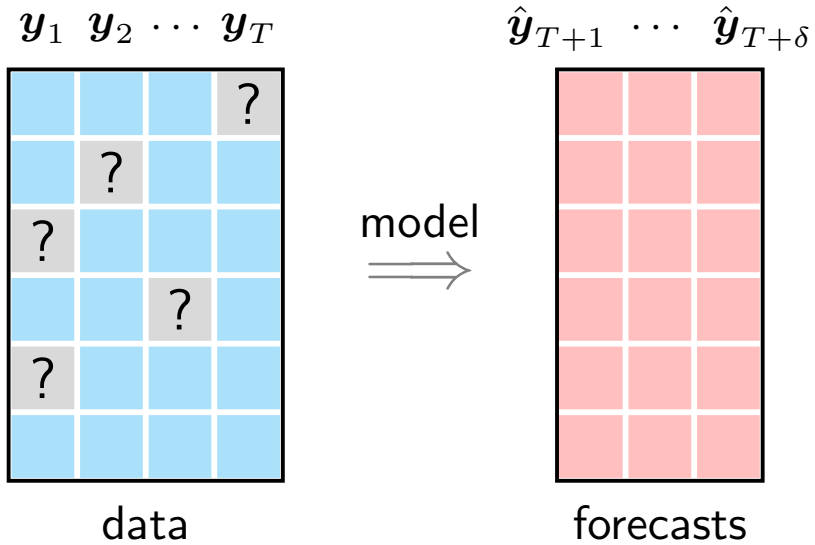


[Q]

- How to reconstruct missing values from observed data?
  - Matrix completion: From $\mathcal{P}_\Omega(\boldsymbol{Y})$ (observed) to $\mathcal{P}_\Omega^\perp(\boldsymbol{Y})$ (unobserved)
  - Tensor completion: From $\mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}})$ (observed) to $\mathcal{P}_\Omega^\perp(\boldsymbol{\mathcal{Y}})$ (unobserved)
- How to make use of spatiotemporal correlations?
- How to make use of traffic time series dynamics?

Problem Formulation

- **Objective B:** Given a partially observed data $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ consisting of time series $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_T \in \mathbb{R}^N$, forecast data points $\boldsymbol{y}_{T+\delta}, \delta \in \mathbb{N}^+$.

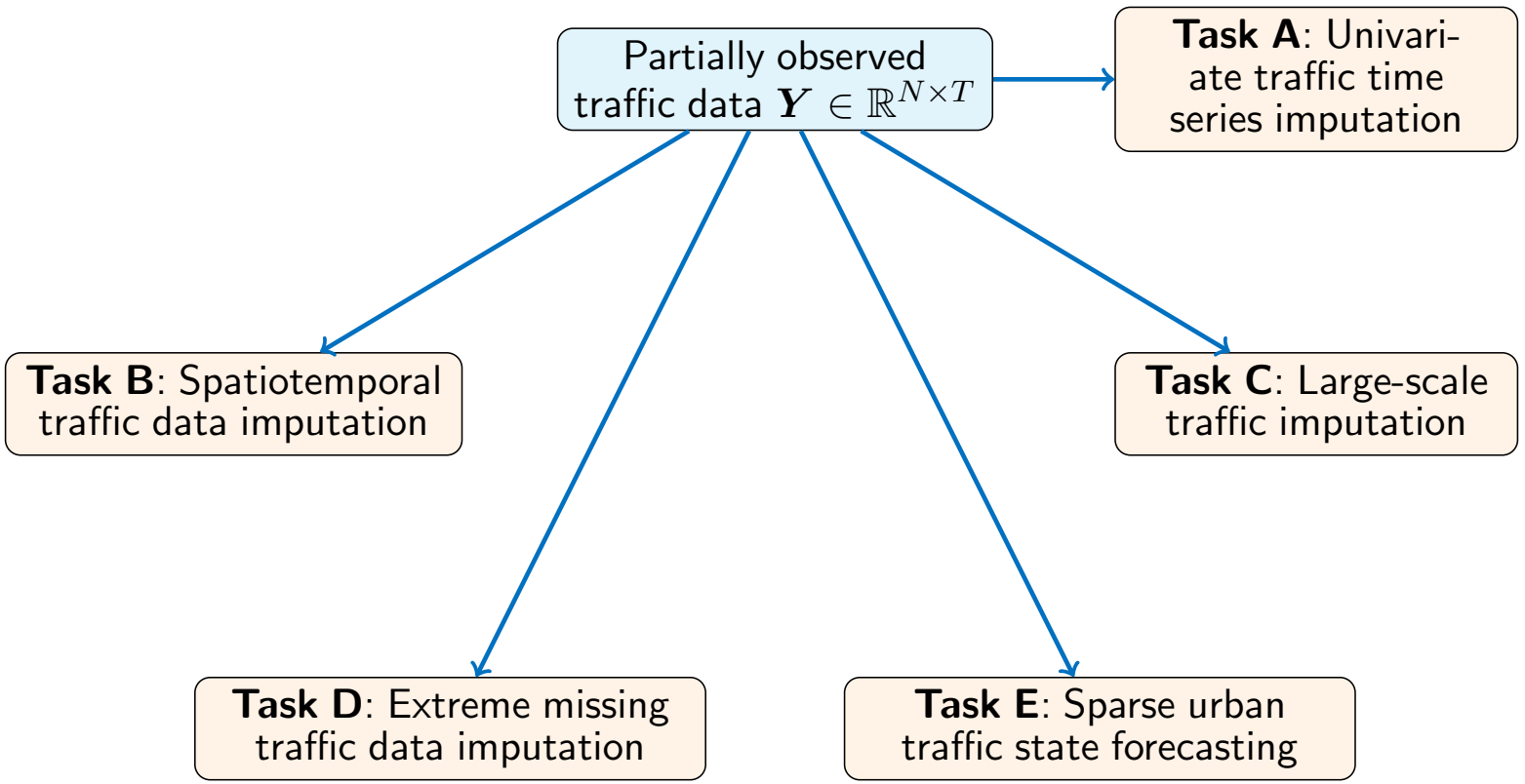


[Q]

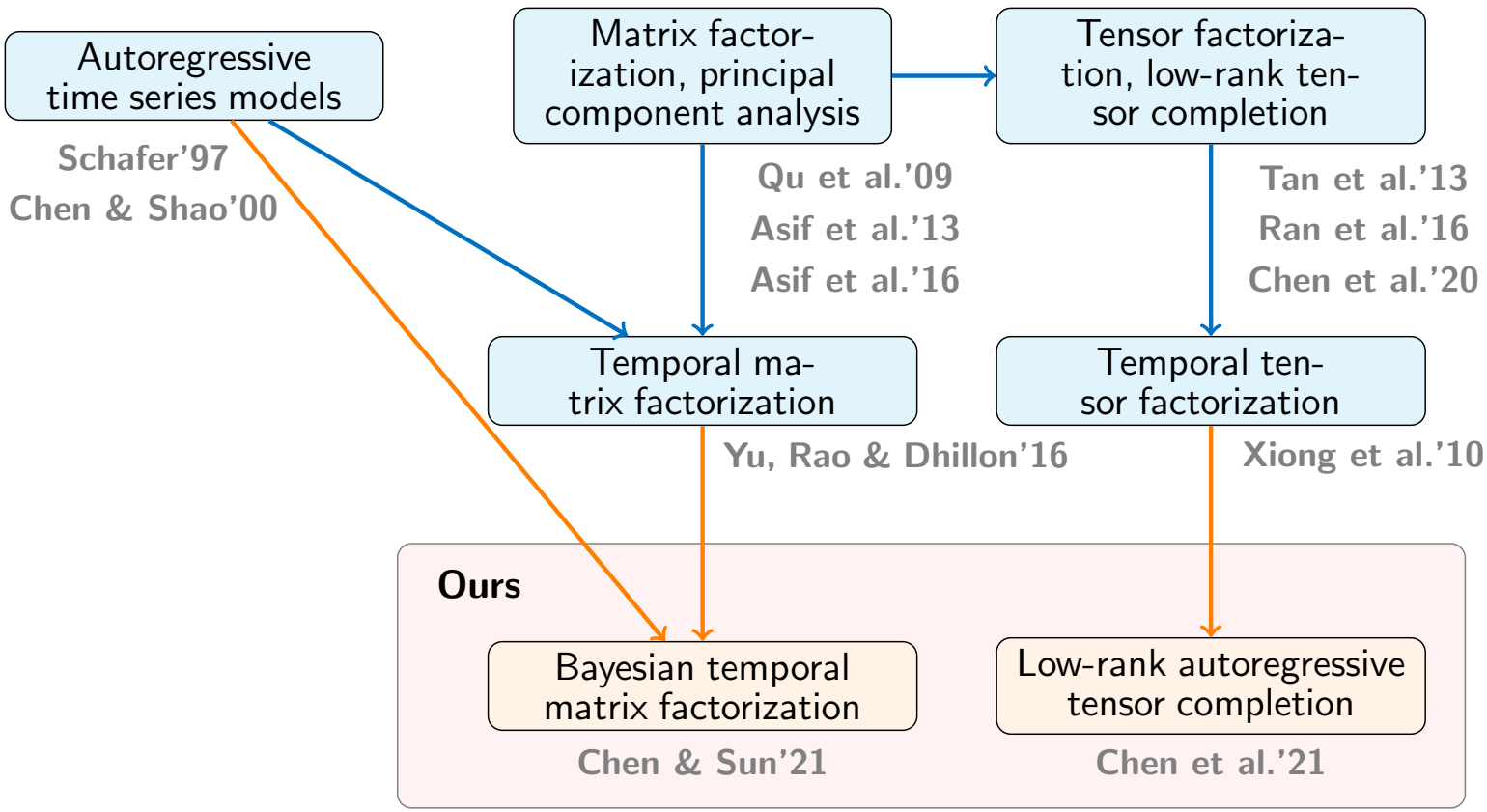
- How to learn from *high-dimensional* and *sparse* data?
- How to model *nonstationarity* in time series?
- How to perform forecasting on these time series?

Whole Picture

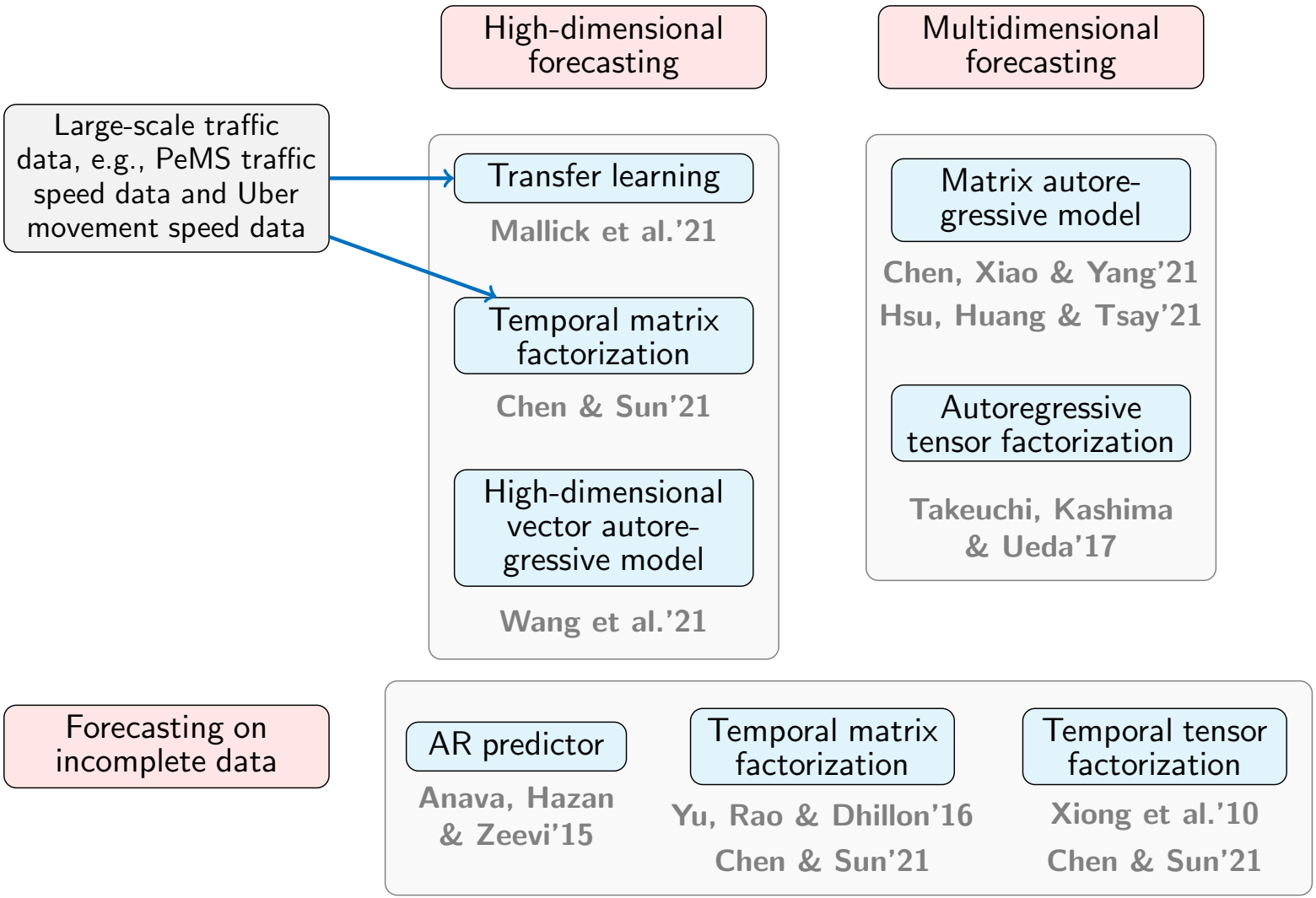
We are working on **spatiotemporal traffic data modeling**.



Spatiotemporal Traffic Data Imputation



Spatiotemporal Traffic Forecasting



Low-Rank Tensor Models

- Low-rank matrix/tensor completion

Candès & Recht'09: Convex nuclear norm minimization for matrix completion.

$$\min_{\boldsymbol{X}} \ \|\boldsymbol{X}\|_* \quad \text{s.t. } \mathcal{P}_\Omega(\boldsymbol{X}) = \mathcal{P}_\Omega(\boldsymbol{Y})$$

Cai, Candès & Shen'10: Singular value thresholding algorithm.

$$\begin{cases} \boldsymbol{X}^\ell = \mathcal{D}_\tau\left(\boldsymbol{Z}^{\ell-1}\right) \\ \boldsymbol{Z}^\ell = \boldsymbol{Z}^{\ell-1} + \delta_\ell \mathcal{P}_\Omega\left(\boldsymbol{Y} - \boldsymbol{X}^\ell\right) \end{cases}$$

Zhang et al.'12: Nonconvex truncated nuclear norm minimization.

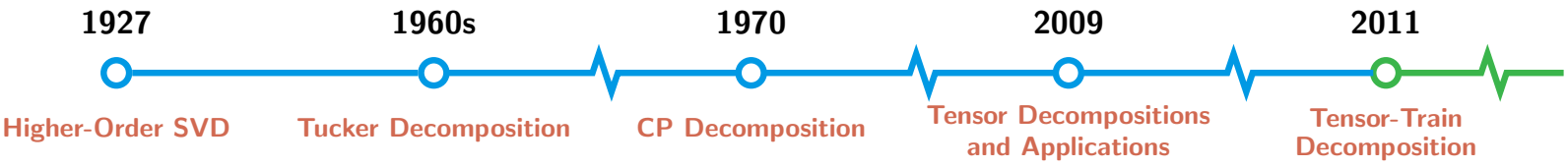
Liu et al.'13: Convex nuclear norm minimization for tensor completion.

$$\min_{\boldsymbol{\mathcal{X}}} \ \|\boldsymbol{\mathcal{X}}\|_* \quad \text{s.t. } \mathcal{P}_\Omega(\boldsymbol{\mathcal{X}}) = \mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}})$$

Lu, Peng & Wei'19: Tensor nuclear norm induced by linear transform.

Tensor Factorization

- Revisit tensor factorization (TF)



Frank Lauren Hitchcock

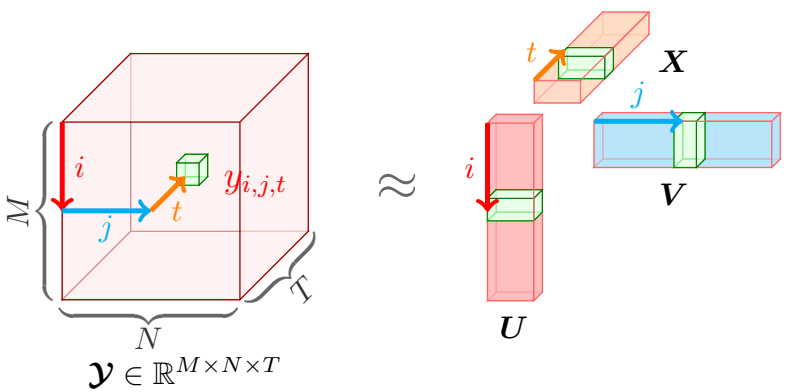
Ledyard R. Tucker

J. Douglas Carroll
Jih-Jie Chang
Richard A. Harshman

Tamara G. Kolda

Ivan Oseledets

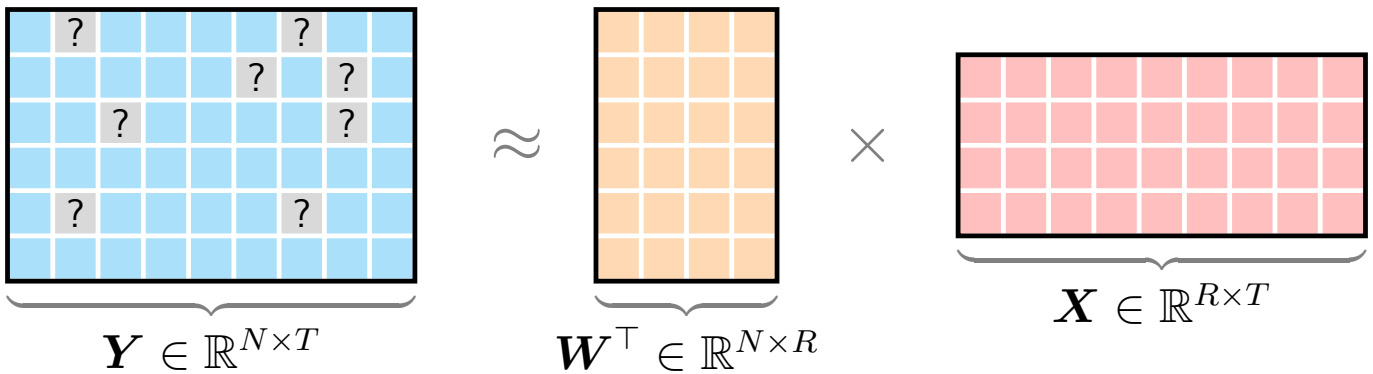
- **CP tensor factorization**: Factorize $\mathcal{Y}$ into the combination of three rank-R factor matrices (i.e., low-dimensional latent factors).



$$\begin{cases} y_{i,j,t} \approx \sum_{r=1}^{R} u_{i,r} v_{j,r} x_{t,r} \\ \mathcal{Y} \approx \sum_{r=1}^{R} \boldsymbol{u}_r \otimes \boldsymbol{v}_r \otimes \boldsymbol{x}_r \end{cases}$$

Matrix Factorization

A simple approach to reconstruct missing values.



MF (Koren et al.'09)

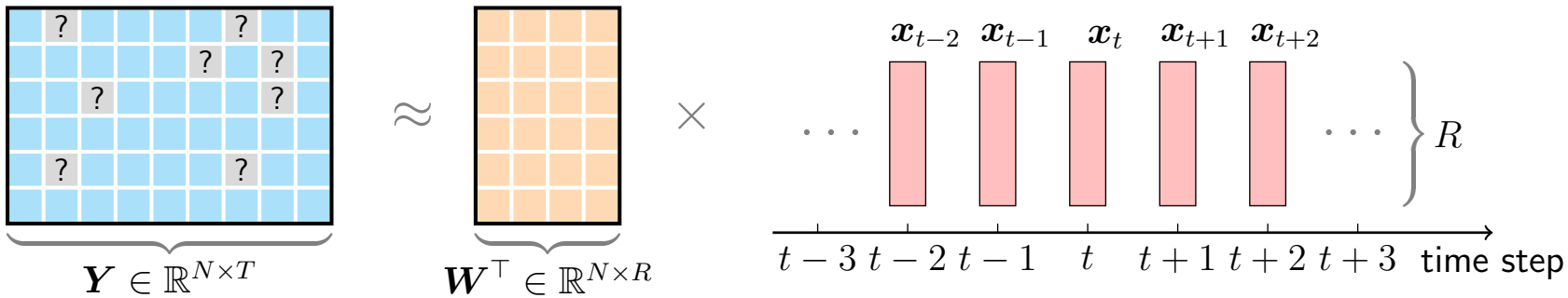
Estimating low-dimensional $\boldsymbol{W}, \boldsymbol{X}$:

$$\min_{\boldsymbol{W},\boldsymbol{X}} \ \frac{1}{2}\|\mathcal{P}_\Omega(\boldsymbol{Y} - \boldsymbol{W}^\top \boldsymbol{X})\|_F^2$$

- ✓ Learn from sparse data
- ✗ Temporal correlations
- ✗ Time series forecasting

Temporal Matrix Factorization

Vector autoregression on the temporal factor matrix.



MF (Koren et al.'09)

Estimating low-dimensional $\boldsymbol{W}, \boldsymbol{X}$:

$$\min_{\boldsymbol{W}, \boldsymbol{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\boldsymbol{Y} - \boldsymbol{W}^\top \boldsymbol{X})\|_F^2$$

$+$

dth-order VAR

$$\boldsymbol{x}_t = \sum_{k=1}^{d} \boldsymbol{A}_k \boldsymbol{x}_{t-k} + \boldsymbol{\epsilon}_t$$

w/ coefficients $\{\boldsymbol{A}_k\}$.

- Temporal matrix factorization (Yu et al.'16; Chen & Sun'21)

$$\min_{\boldsymbol{W}, \boldsymbol{X}, \{\boldsymbol{A}_k\}_{k=1}^d} \frac{1}{2} \|\mathcal{P}_\Omega(\boldsymbol{Y} - \boldsymbol{W}^\top \boldsymbol{X})\|_F^2 + \frac{\gamma}{2} \sum_{t=d+1}^{T} \left\| \boldsymbol{x}_t - \sum_{k=1}^{d} \boldsymbol{A}_k \boldsymbol{x}_{t-k} \right\|_2^2$$

Nonstationary Temporal Matrix Factorization

Nonstationary temporal matrix factorization (NoTMF)

Given any partially observed time series data $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ with observed index set Ω, then we assume a season-m differencing on the latent temporal factors:

$$\begin{aligned}\min_{\boldsymbol{W},\boldsymbol{X},\{\boldsymbol{A}_k\}_{k=1}^{d}} \quad & \frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^{\top}\boldsymbol{X})\|_F^2+\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2) \\ & +\frac{\gamma}{2}\sum_{t=d+m+1}^{T}\left\|(\boldsymbol{x}_t-\boldsymbol{x}_{t-m})-\sum_{k=1}^{d}\boldsymbol{A}_k(\boldsymbol{x}_{t-k}-\boldsymbol{x}_{t-m-k})\right\|_2^2\end{aligned}$$

- First-order differencing $\boldsymbol{x}_t' = \boldsymbol{x}_t - \boldsymbol{x}_{t-1}$.
- Second-order differencing $\boldsymbol{x}_t'' = (\boldsymbol{x}_t - \boldsymbol{x}_{t-1}) - (\boldsymbol{x}_{t-1} - \boldsymbol{x}_{t-2})$.
- Twice-differenced series $\boldsymbol{x}_t'' = (\boldsymbol{x}_t - \boldsymbol{x}_{t-m}) - (\boldsymbol{x}_{t-1} - \boldsymbol{x}_{t-m-1})$.

✓ Stationarizing a time series with differencing can improve the prediction.[3]

[3] Stationarity and differencing: https://otexts.com/fpp2/stationarity.html

Nonstationary Temporal Matrix Factorization

Rewrite VAR in the form of matrix

Temporal operators

For any multivariate time series $\boldsymbol{X} \in \mathbb{R}^{R\times T}$ with $m, d \in \mathbb{N}^{+}$, if we define temporal operators as

$$\begin{aligned}\boldsymbol{\Psi}_k \triangleq & \begin{bmatrix}\boldsymbol{0}_{(T-d-m)\times(d-k)} & -\boldsymbol{I}_{T-d-m} & \boldsymbol{0}_{(T-d-m)\times(k+m)}\end{bmatrix} \\ & + \begin{bmatrix}\boldsymbol{0}_{(T-d-m)\times(d+m-k)} & \boldsymbol{I}_{T-d-m} & \boldsymbol{0}_{(T-d-m)\times k}\end{bmatrix} \\ & \in \mathbb{R}^{(T-d-m)\times T},\ k = 0, 1, \ldots, d\end{aligned}$$

then

$$\sum_{t=d+m+1}^{T} \left\| (\boldsymbol{x}_t - \boldsymbol{x}_{t-m}) - \sum_{k=1}^{d} \boldsymbol{A}_k (\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k}) \right\|_2^2$$

$$\equiv \left\| \boldsymbol{X}\boldsymbol{\Psi}_0^{\top} - \sum_{k=1}^{d} \boldsymbol{A}_k \boldsymbol{X}\boldsymbol{\Psi}_k^{\top} \right\|_F^2 \triangleq \left\| \boldsymbol{X}\boldsymbol{\Psi}_0^{\top} - \boldsymbol{A}(\boldsymbol{I}_d \otimes \boldsymbol{X})\boldsymbol{\Psi}^{\top} \right\|_F^2$$

where $\boldsymbol{A} \triangleq \begin{bmatrix}\boldsymbol{A}_1 & \cdots & \boldsymbol{A}_d\end{bmatrix}$ and $\boldsymbol{\Psi} \triangleq \begin{bmatrix}\boldsymbol{\Psi}_1 & \cdots & \boldsymbol{\Psi}_d\end{bmatrix}$.

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF:

$$\min_{\boldsymbol{W},\boldsymbol{X},\boldsymbol{A}} \frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^{\top}\boldsymbol{X})\|_F^2+\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)+\frac{\gamma}{2}\|\boldsymbol{X}\boldsymbol{\Psi}_0^{\top}-\boldsymbol{A}(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^{\top}\|_F^2$$

Alternating minimization method:

- w.r.t. $\boldsymbol{W}$:

$$\frac{\partial f}{\partial \boldsymbol{W}}=-\boldsymbol{X}\mathcal{P}_{\Omega}^{\top}(\boldsymbol{Y}-\boldsymbol{W}^{\top}\boldsymbol{X})+\rho\boldsymbol{W}=\boldsymbol{0} \qquad \text{(Least squares)}$$

- w.r.t. $\boldsymbol{X}$:

$$\frac{\partial f}{\partial \boldsymbol{X}}=-\boldsymbol{W}\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^{\top}\boldsymbol{X})+\rho\boldsymbol{X}+\gamma\sum_{k=0}^{d}\boldsymbol{A}_k^{\top}\left(\sum_{h=0}^{d}\boldsymbol{A}_h\boldsymbol{X}\boldsymbol{\Psi}_h^{\top}\right)\boldsymbol{\Psi}_k=\boldsymbol{0}$$

  This generalized Sylvester equation can be solved by **conjugate gradient**.

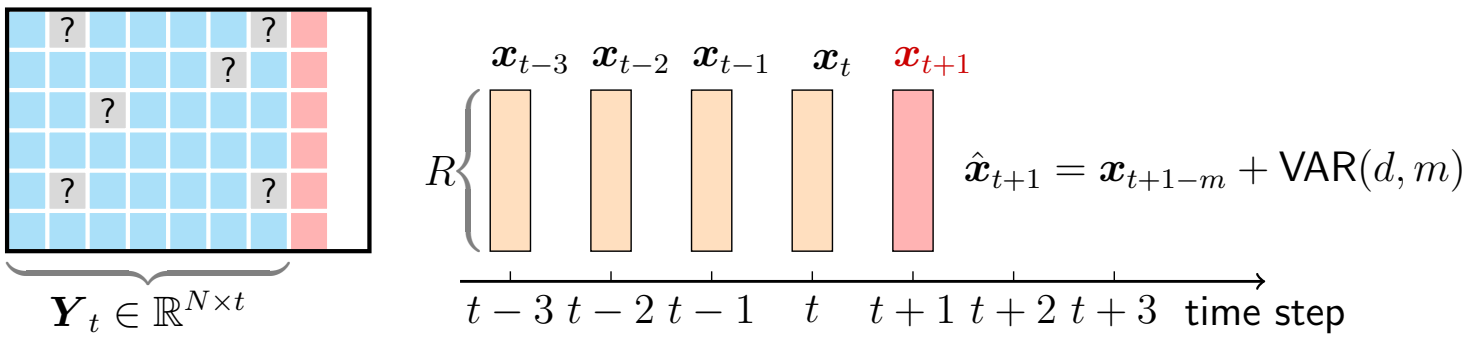
- w.r.t. $\boldsymbol{A}$:

$$\boldsymbol{A}=\boldsymbol{X}\boldsymbol{\Psi}_0^{\top}[(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^{\top}]^{\dagger} \qquad \text{(Least squares)}$$

Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

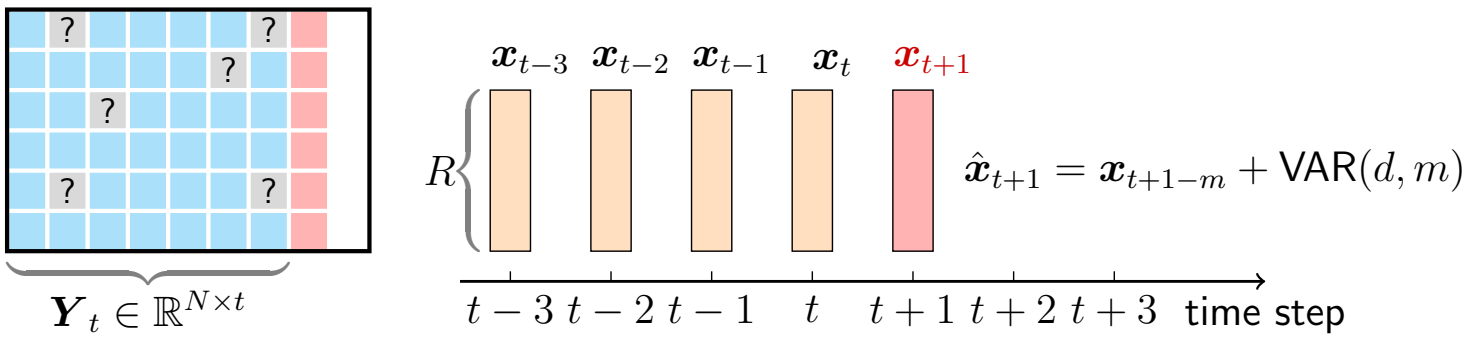
- NoTMF: Use $\boldsymbol{Y}_t$ to estimate $\{\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{A}\}$.



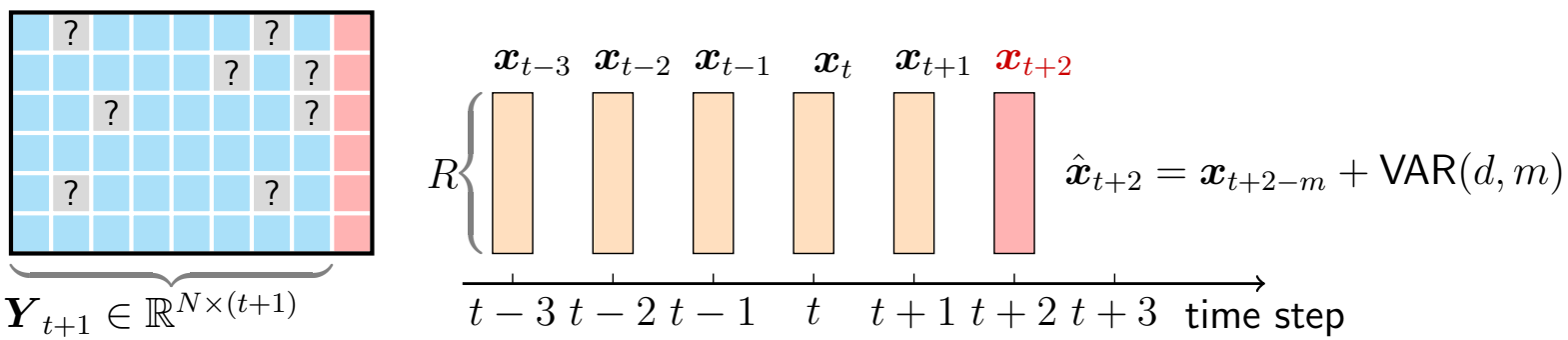
Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

- NoTMF: Use $\boldsymbol{Y}_t$ to estimate $\{\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{A}\}$.



- Online forecasting (Gultekin & Paisley'18): Fix $\boldsymbol{W}$ and use $\boldsymbol{Y}_{t+1}$ to update $\{\boldsymbol{X}, \boldsymbol{A}\}$.



Matrix/Tensor Completion

Cornerstone: Nuclear norm minimization in matrix/tensor completion

LRMC (Candès & Recht'09)

Estimating the matrix $\boldsymbol{X}$:

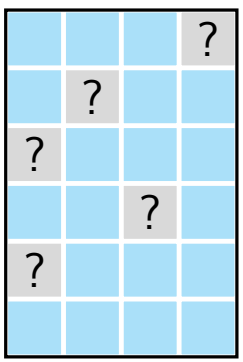
$$\min_{\boldsymbol{X}} \ \|\boldsymbol{X}\|_* \quad \text{s.t. } \mathcal{P}_\Omega(\boldsymbol{X}) = \mathcal{P}_\Omega(\boldsymbol{Y})$$

vs.

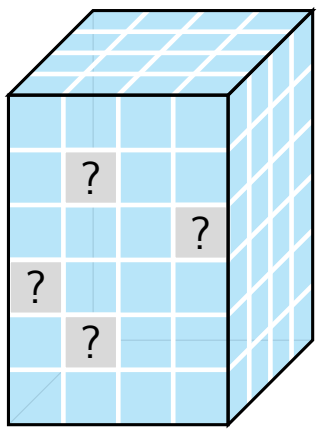
LRTC (Liu et al.'13)

Estimating the tensor $\boldsymbol{\mathcal{X}}$:

$$\min_{\boldsymbol{\mathcal{X}}} \ \|\boldsymbol{\mathcal{X}}\|_* \quad \text{s.t. } \mathcal{P}_\Omega(\boldsymbol{\mathcal{X}}) = \mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}})$$



$\mathcal{P}_\Omega(\boldsymbol{Y}) \in \mathbb{R}^{N \times T}$

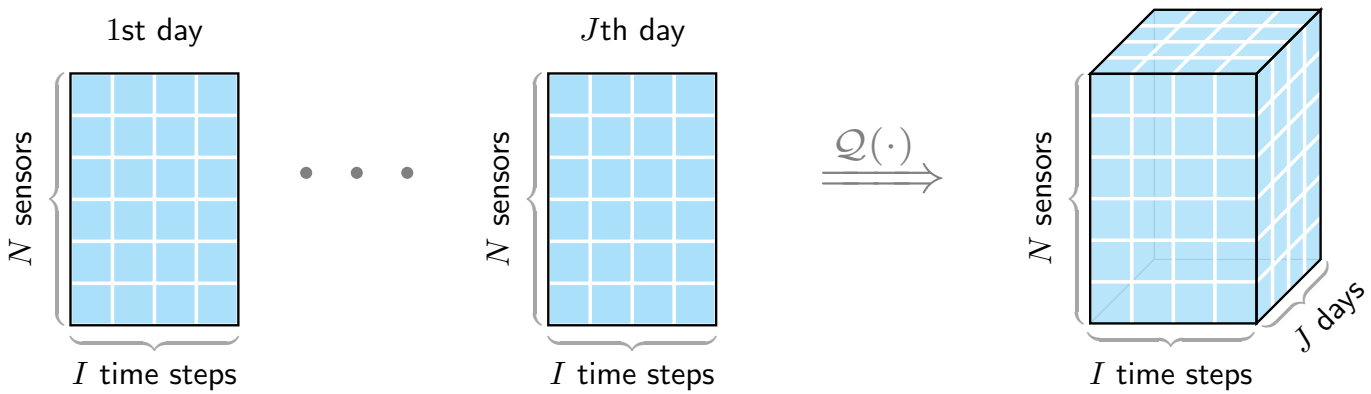


$\mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}}) \in \mathbb{R}^{N \times I \times J}$

- **Limitation**: Only cover global consistency
- **Highlight**: Introduce local consistency (e.g., temporal correlations)

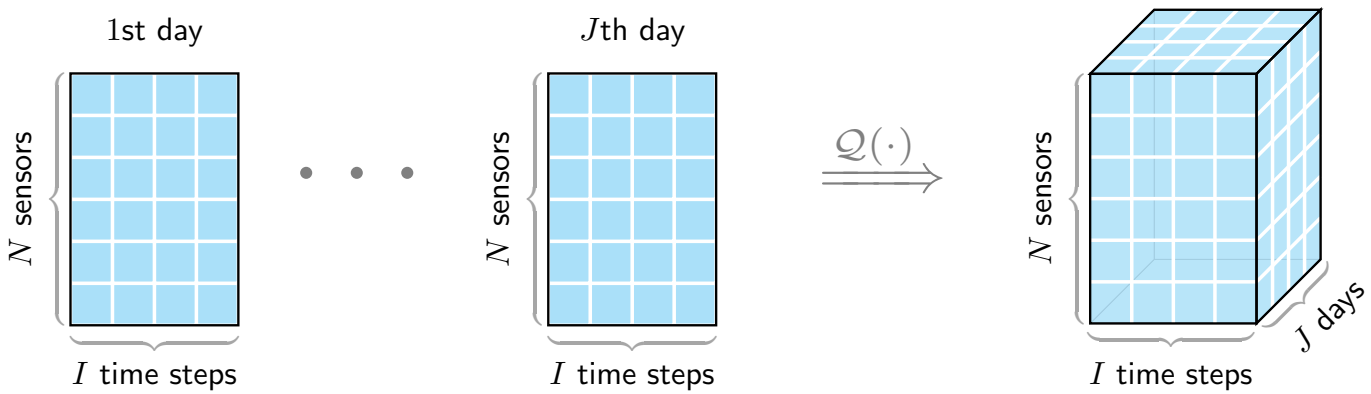
Low-Rank Autoregressive Tensor Completion

- Introduce traffic tensors with day dimension (Tan et al.'13; Chen et al.'19)

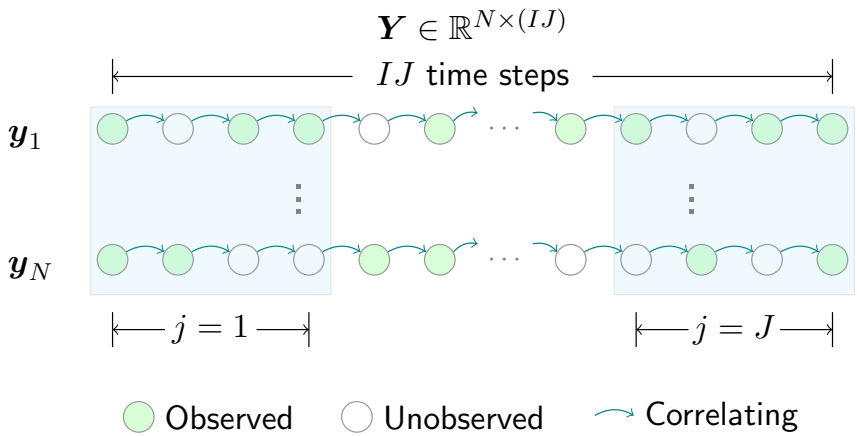


Low-Rank Autoregressive Tensor Completion

- Introduce traffic tensors with day dimension (Tan et al.'13; Chen et al.'19)



- Build temporal correlations with autoregression



Low-Rank Autoregressive Tensor Completion

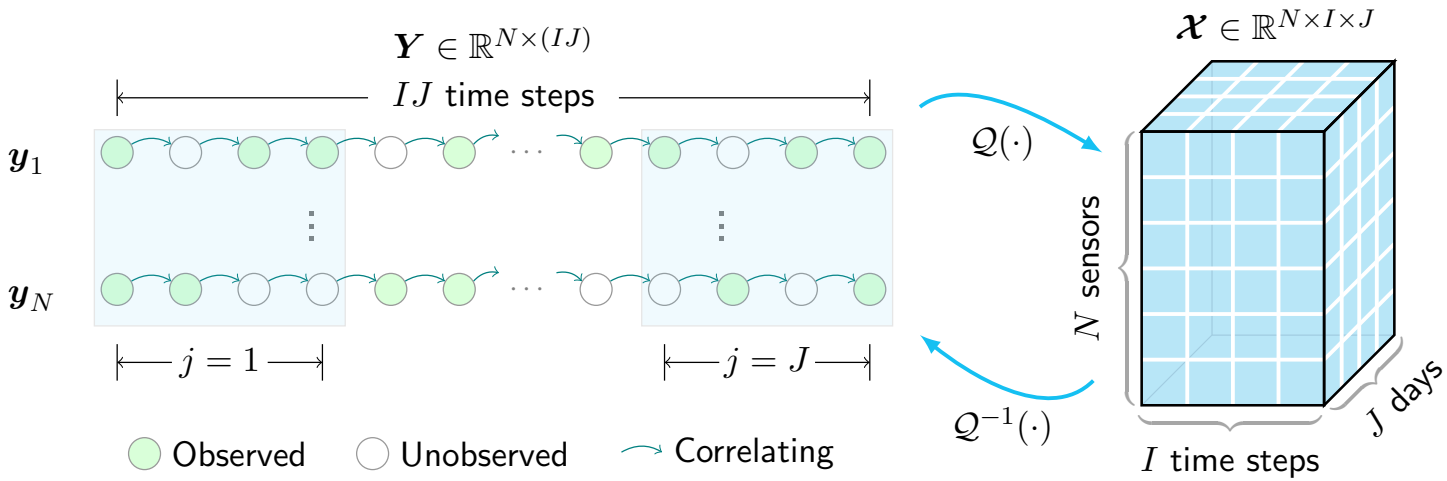
Low-Rank Autoregressive Tensor Completion (LATC)

For any partially observed traffic data $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ with observed index set Ω, LATC takes

$$\min_{\boldsymbol{Z}, \boldsymbol{A}} \ \|\mathcal{Q}(\boldsymbol{Z})\|_{r,*} + \frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}}$$
$$\text{s.t. } \mathcal{P}_{\Omega}(\boldsymbol{Z}) = \mathcal{P}_{\Omega}(\boldsymbol{Y})$$

Advantages:

- ✓ Global consistency (w/ tensor representation)
- ✓ Local consistency (w/ temporal autoregression)



Low-Rank Autoregressive Tensor Completion

- Optimization problem:

$$\min_{\boldsymbol{Z},\boldsymbol{A}} \ \|\mathcal{Q}(\boldsymbol{Z})\|_{r,*} + \frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}}$$
$$\text{s.t. } \mathcal{P}_\Omega(\boldsymbol{Z}) = \mathcal{P}_\Omega(\boldsymbol{Y})$$

- Two subproblems:

$$\begin{cases} \boldsymbol{Z} := \underset{\mathcal{P}_\Omega(\boldsymbol{Z})=\mathcal{P}_\Omega(\boldsymbol{Y})}{\arg\min} \ \|\mathcal{Q}(\boldsymbol{Z})\|_{r,*} + \frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}} \\ \boldsymbol{A} := \frac{1}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}} \qquad \text{(Least squares)} \end{cases}$$

- Optimize $\boldsymbol{Z}$? Write down the augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{Z},\boldsymbol{\mathcal{W}}) = \|\boldsymbol{\mathcal{X}}\|_{r,*} + \frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}} + \frac{\lambda}{2}\|\boldsymbol{\mathcal{X}} - \mathcal{Q}(\boldsymbol{Z})\|_F^2 + \langle \boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{X}} - \mathcal{Q}(\boldsymbol{Z})\rangle + \pi(\boldsymbol{Z})$$

where $\boldsymbol{\mathcal{W}} \in \mathbb{R}^{N\times I\times J}$ is the Lagrange multiplier, and $\langle\cdot,\cdot\rangle$ denotes the inner product.[4]

[4] The indicator function is $\pi(\boldsymbol{Z}) = \begin{cases} 0, & \text{if } \mathcal{P}_\Omega(\boldsymbol{Z}) = \mathcal{P}_\Omega(\boldsymbol{Y}), \\ +\infty, & \text{otherwise.} \end{cases}$

Low-Rank Autoregressive Tensor Completion

- Augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{W}}) = \|\boldsymbol{\mathcal{X}}\|_{r,*} + \frac{\gamma}{2}\|\boldsymbol{\mathcal{Z}}\|_{\boldsymbol{A},\mathcal{H}} + \frac{\lambda}{2}\|\boldsymbol{\mathcal{X}} - \mathcal{Q}(\boldsymbol{\mathcal{Z}})\|_F^2 + \langle \boldsymbol{\mathcal{W}}, \boldsymbol{\mathcal{X}} - \mathcal{Q}(\boldsymbol{\mathcal{Z}})\rangle + \pi(\boldsymbol{\mathcal{Z}})$$

- The ADMM[5] scheme:

$$\begin{cases} \boldsymbol{\mathcal{X}} := \arg\min\limits_{\boldsymbol{\mathcal{X}}} \ \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{W}}) & \text{(Truncated nuclear norm minimization)} \\ \boldsymbol{\mathcal{Z}} := \arg\min\limits_{\boldsymbol{\mathcal{Z}}} \ \mathcal{L}(\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Z}}, \boldsymbol{\mathcal{W}}) & \text{(Generalized Sylvester equation)} \\ \boldsymbol{\mathcal{W}} := \boldsymbol{\mathcal{W}} + \lambda(\boldsymbol{\mathcal{X}} - \mathcal{Q}(\boldsymbol{\mathcal{Z}})) & \text{(Standard update)} \end{cases}$$

✓ Solution to $\boldsymbol{\mathcal{X}}$: singular value thresholding

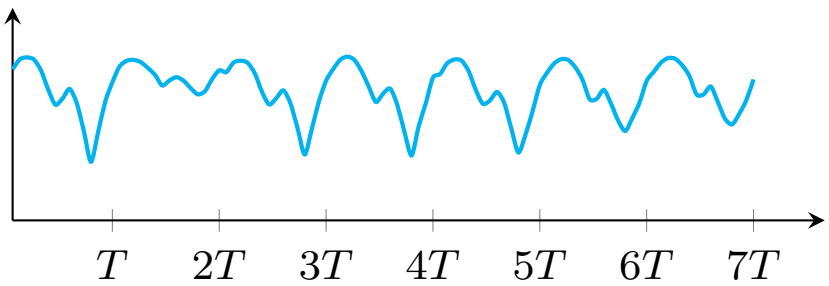
✓ Solution to $\boldsymbol{\mathcal{Z}}$: conjugate gradient

[5] Alternating Direction Method of Multipliers.

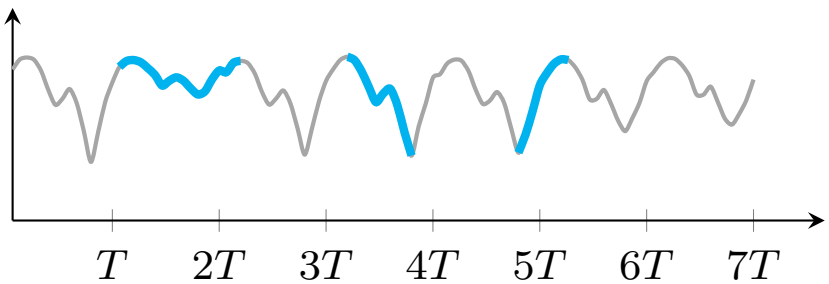
Laplacian Convolutional Representation

Motivation: Time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



- Local trends (e.g., short-term time series trends):

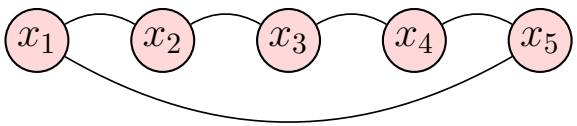


- **[Question]** How to characterize both global and local trends in sparse time series data?

Laplacian Convolutional Representation

[Local trend modeling] Reformulate temporal regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.



Undirected and circulant graph

Modeling $\longrightarrow$

$$\boldsymbol{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (2, -1, 0, 0, -1)^\top$$

$$\Downarrow$$

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \cdots, -1}_{\tau}, 0, \cdots, 0, \underbrace{-1, \cdots, -1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series $\boldsymbol{x} = (x_1, \ldots, x_T)^\top \in \mathbb{R}^T$.

- (Laplacian) Temporal regularization:

$$\mathcal{R}_\tau(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2 = \frac{1}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2$$

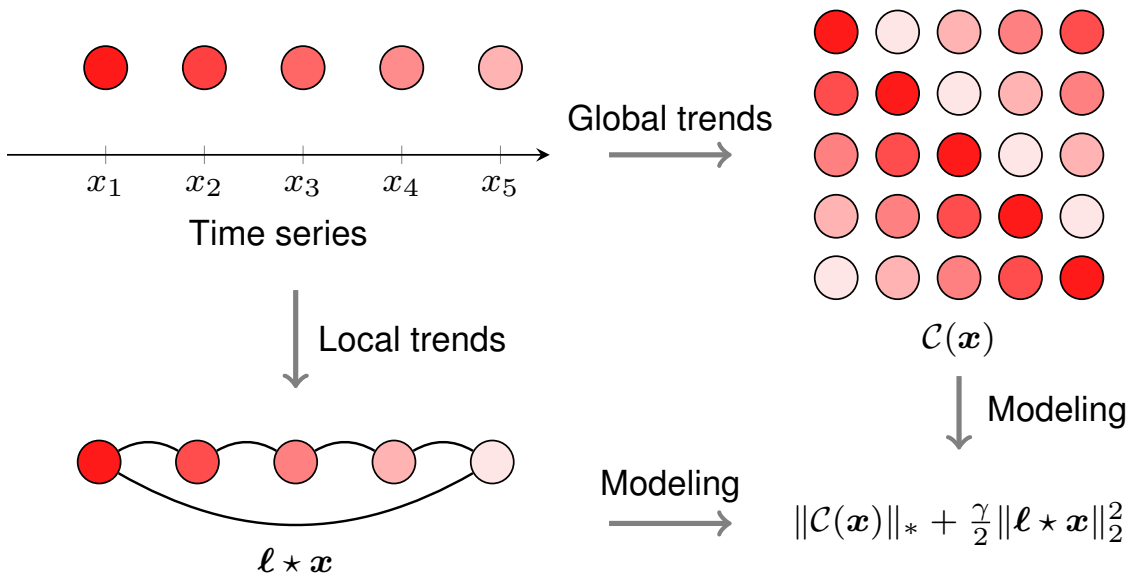
Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\boldsymbol{y} \in \mathbb{R}^T$ with observed index set Ω, LCR utilizes circulant matrix and Laplacian kernel to characterize **global and local trends** in time series, respectively, i.e.,

$$\begin{aligned} \min_{\boldsymbol{x}} \ & \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 \\ \text{s.t.} \ & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

where $\mathcal{C} : \mathbb{R}^T \to \mathbb{R}^{T \times T}$ denotes the circulant operator. $\|\cdot\|_*$ denotes the nuclear norm of matrix, namely, the sum of singular values.



Laplacian Convolutional Representation

- Augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) = \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2 + \langle\boldsymbol{w},\boldsymbol{x}-\boldsymbol{z}\rangle + \frac{\eta}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{z}-\boldsymbol{y})\|_2^2$$

where $\boldsymbol{w}\in\mathbb{R}^T$ is the Lagrange multiplier, and $\langle\cdot,\cdot\rangle$ denotes the inner product.

- The ADMM scheme:

$$\begin{cases} \boldsymbol{x} := \arg\min\limits_{\boldsymbol{x}} \ \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) & \text{(Nuclear norm minimization)} \\ \boldsymbol{z} := \arg\min\limits_{\boldsymbol{z}} \ \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) & \text{(Closed-form solution)} \\ \quad = \dfrac{1}{\lambda+\eta}\mathcal{P}_{\Omega}(\lambda\boldsymbol{x}+\boldsymbol{w}+\eta\boldsymbol{y}) + \dfrac{1}{\lambda}\mathcal{P}_{\Omega}^{\perp}(\lambda\boldsymbol{x}+\boldsymbol{w}) & \\ \boldsymbol{w} := \boldsymbol{w}+\lambda(\boldsymbol{x}-\boldsymbol{z}) & \text{(Standard update)} \end{cases}$$

- Optimize $\boldsymbol{x}$?

$$\|\mathcal{C}(\boldsymbol{x})\|_* = \|\mathcal{F}(\boldsymbol{x})\|_1 \qquad \& \qquad \frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 = \frac{1}{2T}\|\mathcal{F}(\boldsymbol{\ell})\circ\mathcal{F}(\boldsymbol{x})\|_2^2$$

Nuclear norm minimization $\Rightarrow$ ℓ_1-**norm minimization with FFT** (in $\mathcal{O}(T\log T)$ time).

Laplacian Convolutional Representation

- Optimize $\boldsymbol{x}$ via FFT (in $\mathcal{O}(T \log T)$ time):

$$\boldsymbol{x} := \arg\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x} - \boldsymbol{z} + \boldsymbol{w}/\lambda\|_2^2$$

$$\Longrightarrow \hat{\boldsymbol{x}} := \arg\min_{\hat{\boldsymbol{x}}} \ \|\hat{\boldsymbol{x}}\|_1 + \frac{\gamma}{2T}\|\hat{\boldsymbol{\ell}} \circ \hat{\boldsymbol{x}}\|_2^2 + \frac{\lambda}{2T}\|\hat{\boldsymbol{x}} - \hat{\boldsymbol{z}} + \hat{\boldsymbol{w}}/\lambda\|_2^2$$

where we introduce $\{\hat{\boldsymbol{\ell}}, \hat{\boldsymbol{x}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{w}}\} \triangleq \mathcal{F}\{\boldsymbol{\ell}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}\}$ (i.e., FFT).

ℓ_1-norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of ℓ_1-norm minimization in complex space:

$$\min_{\hat{\boldsymbol{x}}} \ \|\hat{\boldsymbol{x}}\|_1 + \frac{\omega}{2}\|\hat{\boldsymbol{x}} - \hat{\boldsymbol{h}}\|_2^2$$

with complex-valued vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{h}} \in \mathbb{C}^T$ and weight parameter ω, element-wise, the solution is given by

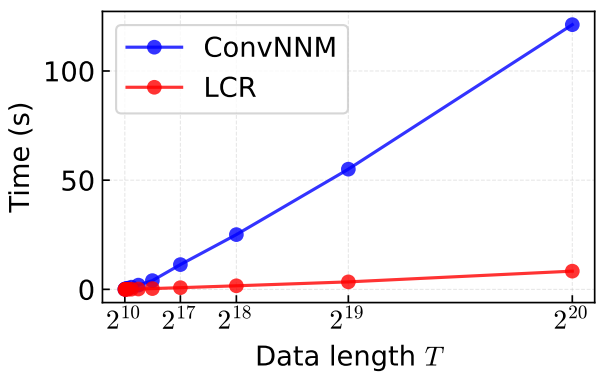
$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|} \cdot \max\{0, |\hat{h}_t| - 1/\omega\}, t = 1, \ldots, T.$$

Laplacian Convolutional Representation

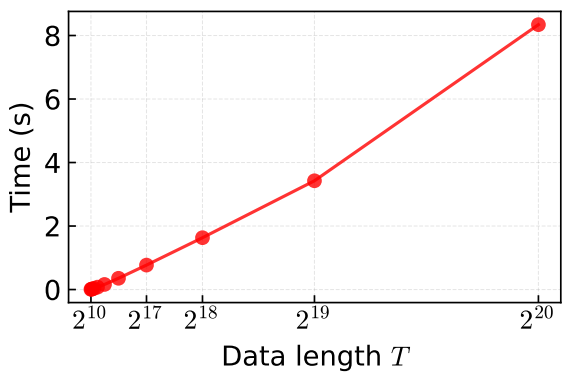
Empirical time complexity

On the synthetic data $\boldsymbol{y} \in \mathbb{R}^T$ with $T \in \{2^{10}, 2^{11}, \ldots, 2^{20}\}$

- Ours: **LCR**
  - An FFT implementation in $\mathcal{O}(T \log T)$
  - The logarithmic factor $\log T$ makes the FFT highly efficient
- Baseline: **ConvNNM**[6] (Liu & Zhang'23)
  - Convolution matrix $\mathcal{C}_{\tilde{\tau}}(\boldsymbol{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$ with kernel size $\tilde{\tau} \in \mathbb{N}^+$
  - Singular value thresholding in $\mathcal{O}(\tilde{\tau}^2 T)$



ConvNNM vs. LCR



LCR

[6] Convolution nuclear norm minimization.

Two-Dimensional LCR (LCR-2D)

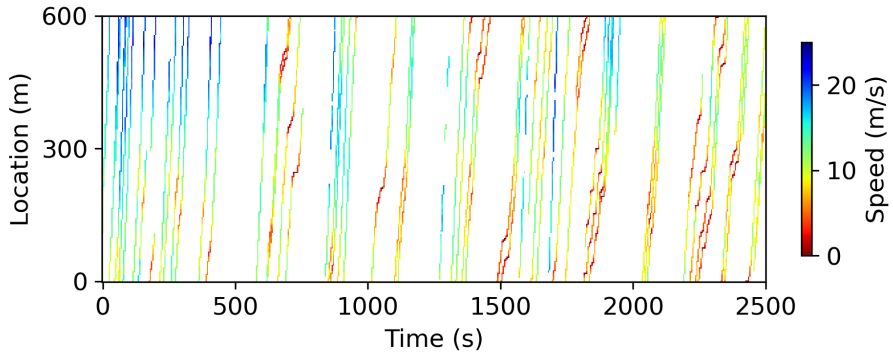
For any partially observed time series $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ with observed index set Ω, LCR can be formulated as follows,

$$\begin{aligned} \min_{\boldsymbol{X}} \ & \|\mathcal{C}(\boldsymbol{X})\|_* + \frac{\gamma}{2}\|(\boldsymbol{\ell}_s \boldsymbol{\ell}^\top) \star \boldsymbol{X}\|_F^2 \\ \text{s.t.} \ & \|\mathcal{P}_\Omega(\boldsymbol{X} - \boldsymbol{Y})\|_F \leq \epsilon \end{aligned}$$

where $\mathcal{C} : \mathbb{R}^{N \times T} \to \mathbb{R}^{N \times N \times T \times T}$ denotes the circulant operator.

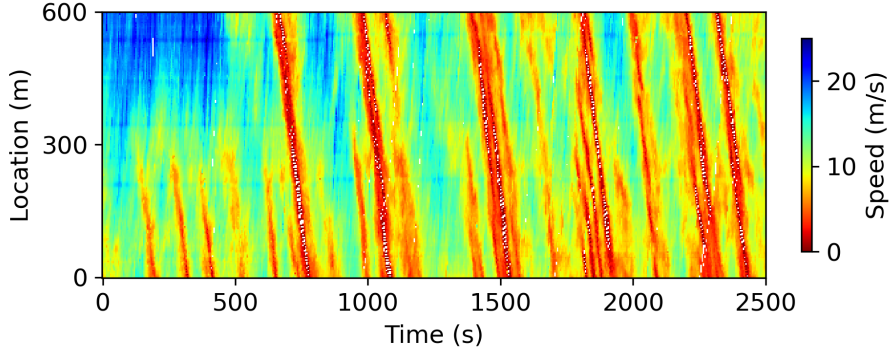
Motivation: Spatiotemporal data reconstruction

- Speed field reconstruction problem in vehicular traffic flow.



200-by-500 matrix (NGSIM)

Reconstruct speed field from 5% sparse trajectories?



- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal dependencies?

Hankel Tensor Factorization

- Hankel matrix
  - Given $\boldsymbol{x} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\boldsymbol{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4\times 2}$$

Hankel matrix (Source: Twitter)

Hankel Tensor Factorization

- Hankel matrix
  - On time series $\boldsymbol{y} = (y_1, y_2, \ldots, y_5)^\top$ with $\tau = 2$:

$$\mathcal{H}_\tau(\boldsymbol{y}) = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \\ y_3 & y_4 \\ y_4 & y_5 \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Longrightarrow \quad \hat{\boldsymbol{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \\ \hat{y}_5 \end{bmatrix} = \mathcal{H}_\tau^{-1} \left(\begin{bmatrix} v_1 x_1 & v_1 x_2 \\ v_2 x_1 & v_2 x_2 \\ v_3 x_1 & v_3 x_2 \\ v_4 x_1 & v_4 x_2 \end{bmatrix} \right) = \begin{bmatrix} v_1 x_1 \\ (v_1 x_2 + v_2 x_1)/2 \\ (v_2 x_2 + v_3 x_1)/2 \\ v_4 x_2 \end{bmatrix}$$

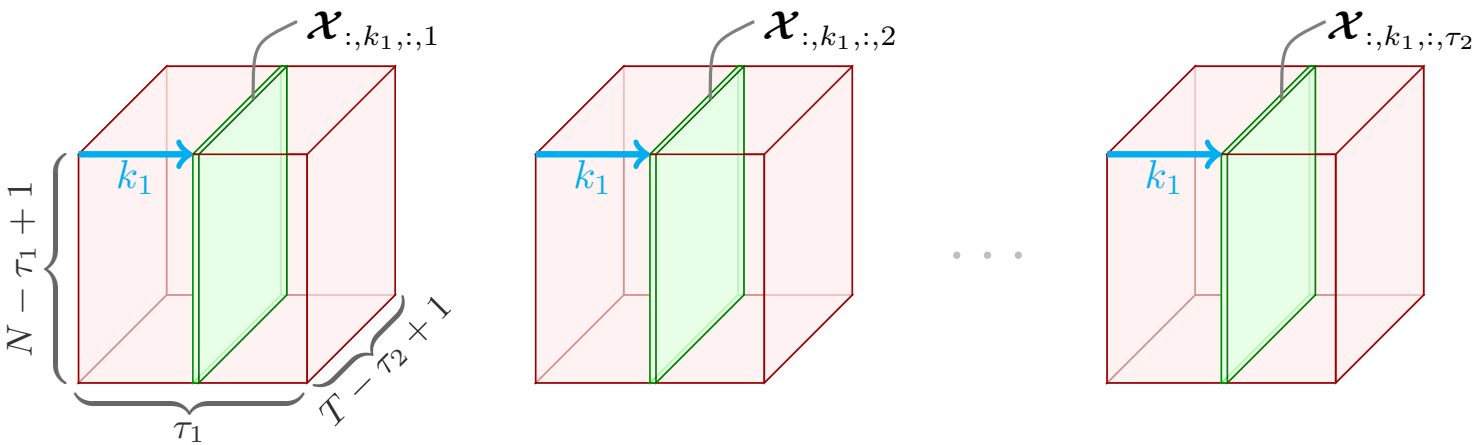
  - Automatic temporal modeling.

Hankel Tensor Factorization

- Hankel tensor: Given any matrix $\boldsymbol{X} \in \mathbb{R}^{N \times T}$, we have

$$\boldsymbol{\mathcal{X}} \triangleq \mathcal{H}_{\tau_1, \tau_2}(\boldsymbol{X})$$

  - Window lengths: $\tau_1, \tau_2 \in \mathbb{N}^+$;
  - Tensor size: $(N - \tau_1 + 1) \times \tau_1 \times (T - \tau_2 + 1) \times \tau_2$;



(Figure) 4th order Hankel tensor: A sequence of third-order tensors.

  - Slice: $\boldsymbol{\mathcal{X}}_{:,k_1,:,k_2}$, $\forall k_1, k_2$;
  - Slice size: $(N - \tau_1 + 1) \times (T - \tau_2 + 1)$.

Hankel Tensor Factorization

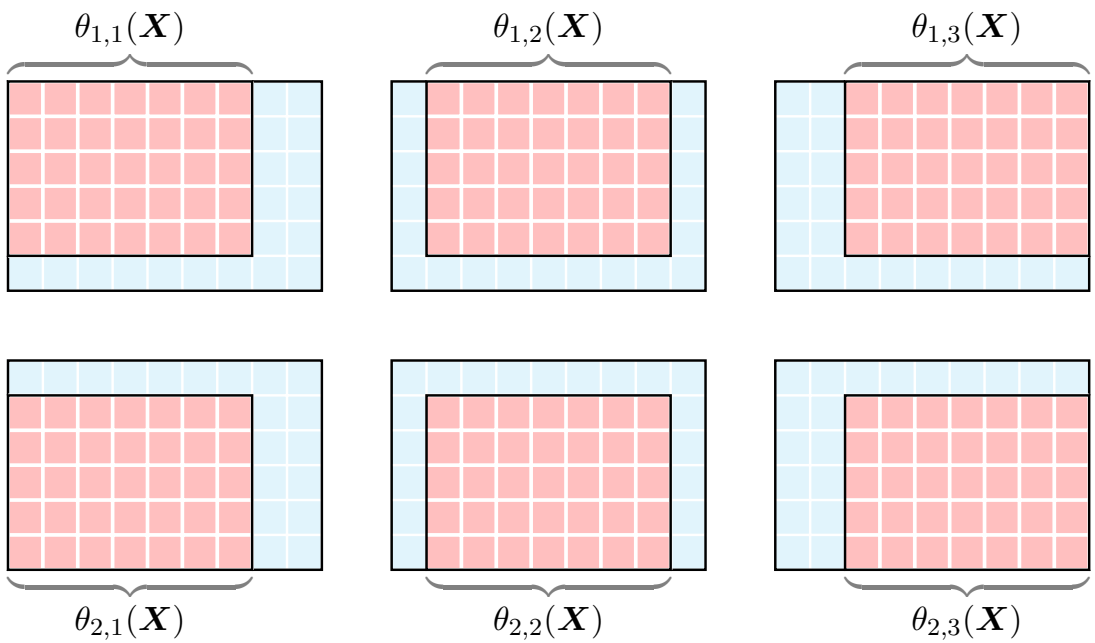
Hankel indexing:

- Sampling function for the Hankelization:

$$\theta_{k_1,k_2}(\boldsymbol{X}) \triangleq [\mathcal{H}_{\tau_1,\tau_2}(\boldsymbol{X})]_{:,k_1,:,k_2},$$

  referring to the tensor slice with $k_1 \in \{1,\ldots,\tau_1\}$, $k_2 \in \{1,\ldots,\tau_2\}$.

- **[Importance]** Developing memory-efficient algorithms.



- Tensor slices $\theta_{k_1,k_2}(\boldsymbol{X})$ vs. data matrix $\boldsymbol{X}$

Hankel Tensor Factorization

Ours:

- Convolutional tensor decomposition (circular convolution $\star_{\text{row}}$):

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q} \star_{\text{row}} \boldsymbol{s}_{k_1}^\top)(\boldsymbol{U} \star_{\text{row}} \boldsymbol{v}_{k_2}^\top)^\top$$

Baselines:

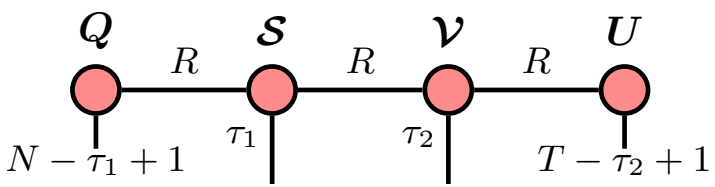
- CP tensor decomposition (Khatri-Rao product $\odot$):

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q} \odot \boldsymbol{s}_{k_1}^\top)(\boldsymbol{U} \odot \boldsymbol{v}_{k_2}^\top)^\top$$

- Tensor-train decomposition:

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q}\boldsymbol{S}_{k_1})(\boldsymbol{U}\boldsymbol{V}_{k_2})^\top$$

  - $\{\boldsymbol{S}_{k_1}, \boldsymbol{V}_{k_2}\}$ are **circulant matrices** $\Rightarrow$ convolutional decomposition
  - $\{\boldsymbol{S}_{k_1}, \boldsymbol{V}_{k_2}\}$ are **diagonal matrices** $\Rightarrow$ CP decomposition



Hankel Tensor Factorization

HTF (convolutional decomposition)

- Optimization problem:

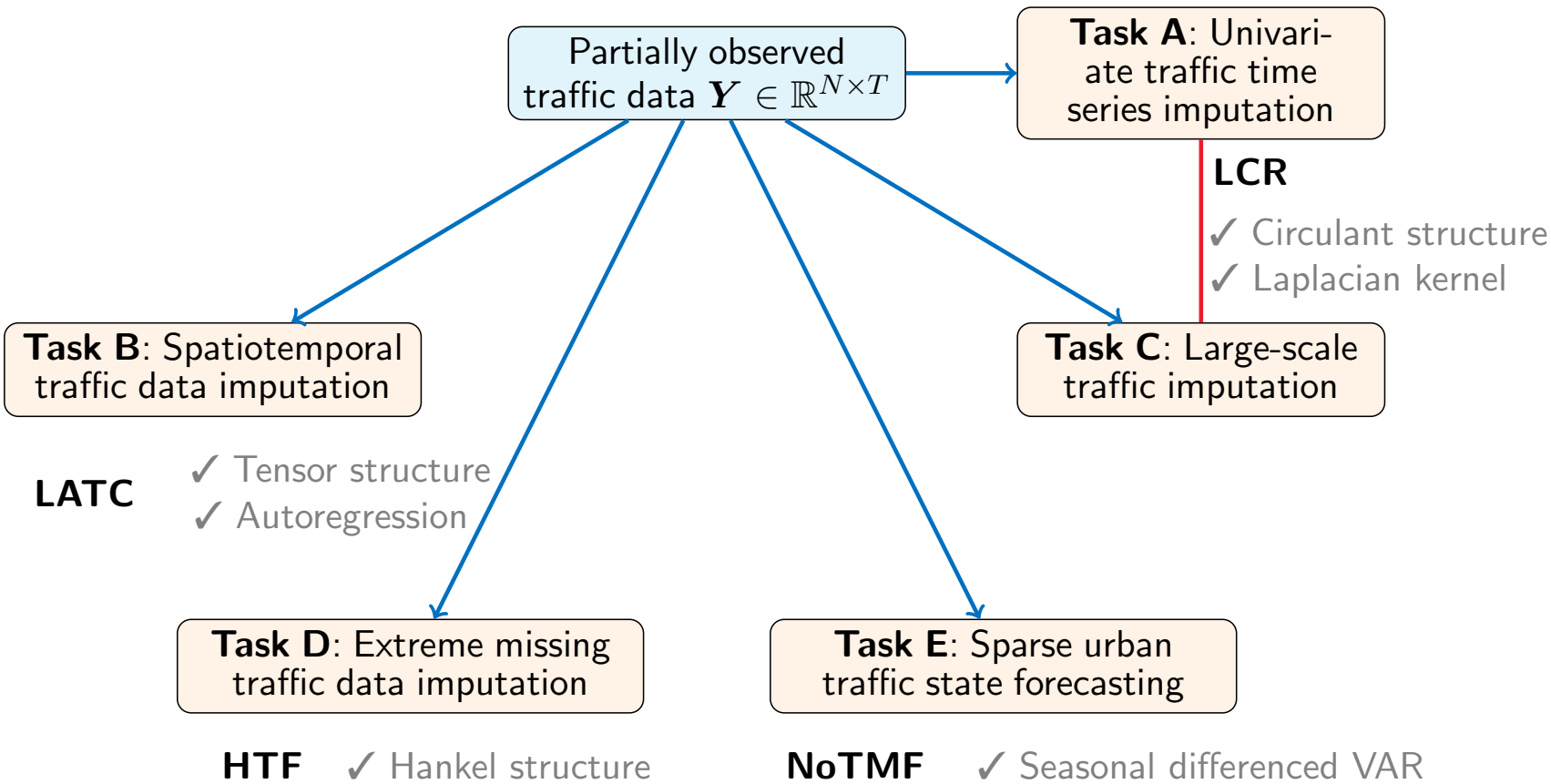
$$\min_{\boldsymbol{Q},\boldsymbol{S},\boldsymbol{U},\boldsymbol{V}} \ \frac{1}{2}\sum_{k_1,k_2}\left\|\mathcal{P}_{\Omega_{k_1,k_2}}\left(\theta_{k_1,k_2}(\boldsymbol{Y})-(\boldsymbol{Q}\star_{\text{row}}\boldsymbol{s}_{k_1})(\boldsymbol{U}\star_{\text{row}}\boldsymbol{v}_{k_2})^\top\right)\right\|_F^2 + \frac{\rho}{2}\left(\|\boldsymbol{Q}\|_F^2+\|\boldsymbol{S}\|_F^2+\|\boldsymbol{U}\|_F^2+\|\boldsymbol{V}\|_F^2\right)$$

- Alternating minimization:

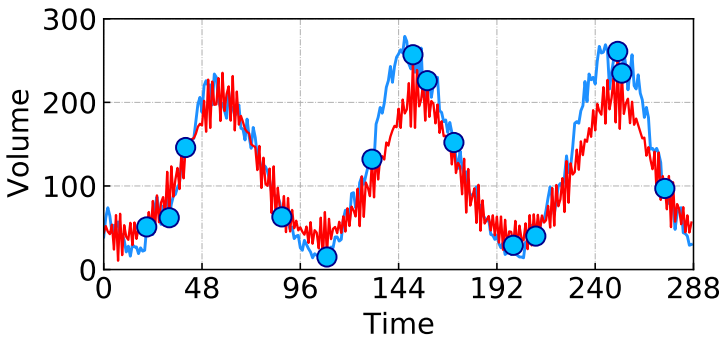
$$\begin{cases} \boldsymbol{Q} \coloneqq \{\boldsymbol{Q} \mid \dfrac{\partial f}{\partial \boldsymbol{Q}} = \boldsymbol{0}\} \\ \boldsymbol{s}_{k_1} \coloneqq \{\boldsymbol{s}_{k_1} \mid \dfrac{\partial f}{\partial \boldsymbol{s}_{k_1}} = \boldsymbol{0}\},\ k_1 \in \{1,2,\ldots,\tau_1\} \\ \boldsymbol{U} \coloneqq \{\boldsymbol{U} \mid \dfrac{\partial f}{\partial \boldsymbol{U}} = \boldsymbol{0}\} \\ \boldsymbol{v}_{k_2} \coloneqq \{\boldsymbol{v}_{k_2} \mid \dfrac{\partial f}{\partial \boldsymbol{v}_{k_2}} = \boldsymbol{0}\},\ k_2 \in \{1,2,\ldots,\tau_2\} \end{cases}$$

Whole Picture

We are working on **spatiotemporal traffic data modeling**.



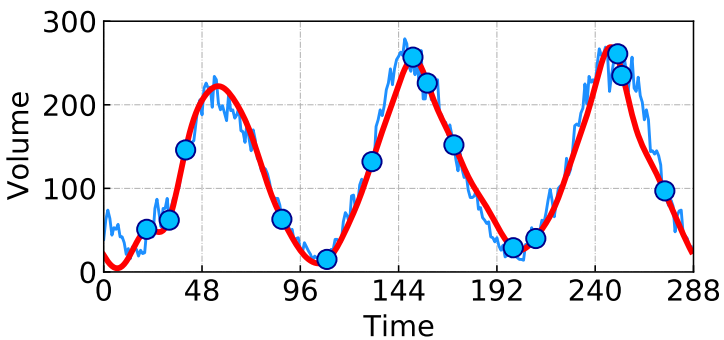
Univariate Traffic Time Series Imputation



CircNNM:

$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_* \quad \text{s.t.} \ \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

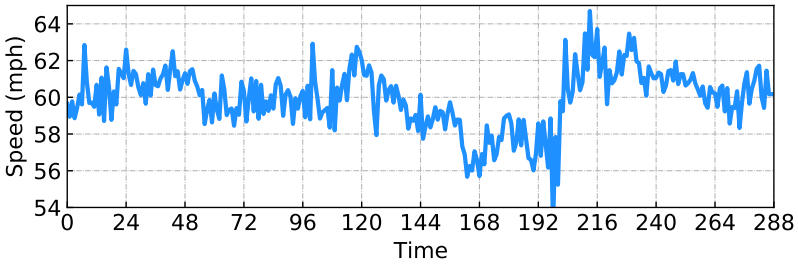
Plus temporal regularization (TR)



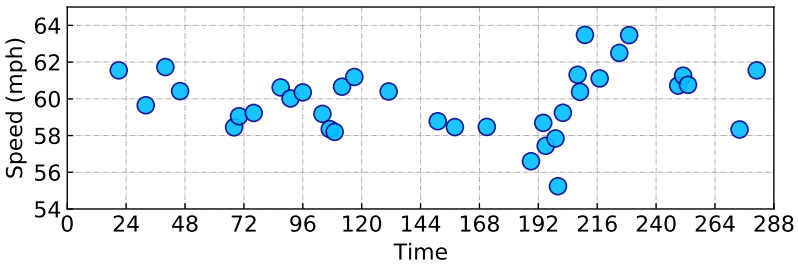
LCR:

$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 \quad \text{s.t.} \ \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

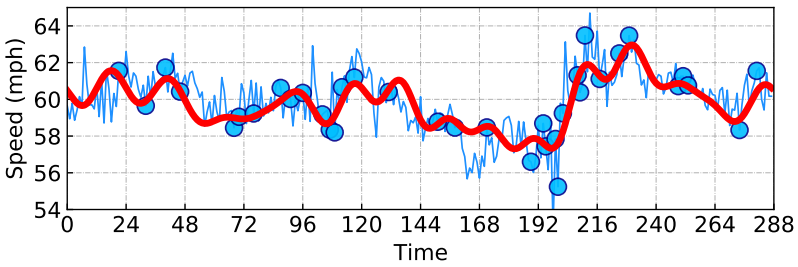
Univariate Traffic Time Series Imputation



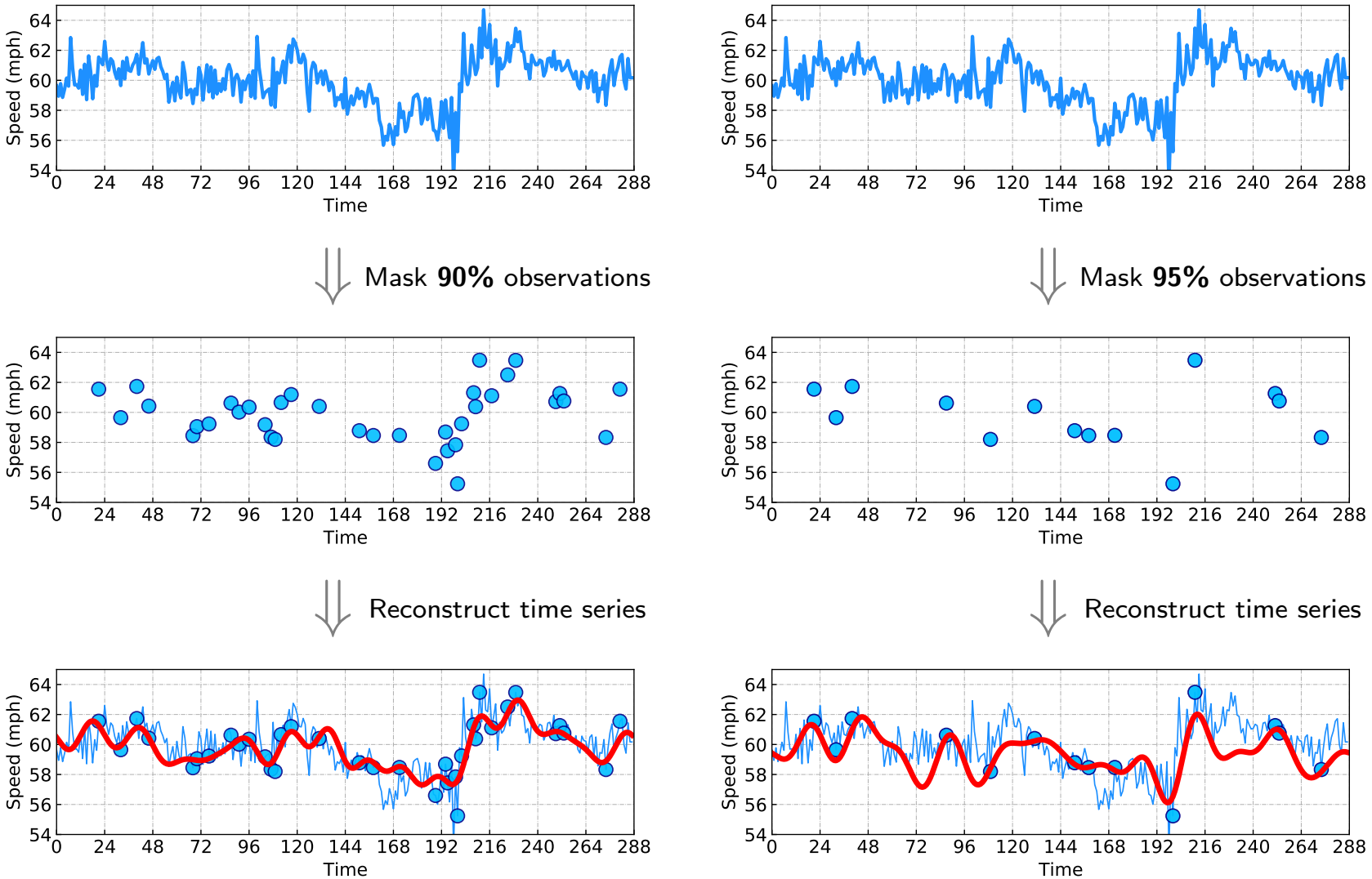
Mask **90%** observations



Reconstruct time series



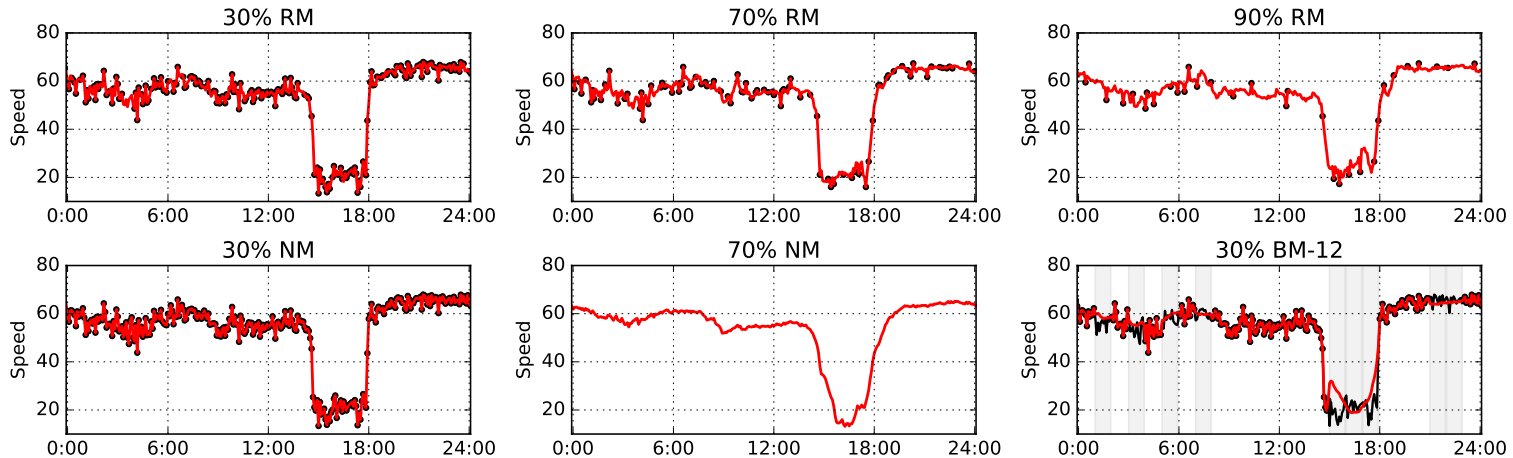
Univariate Traffic Time Series Imputation



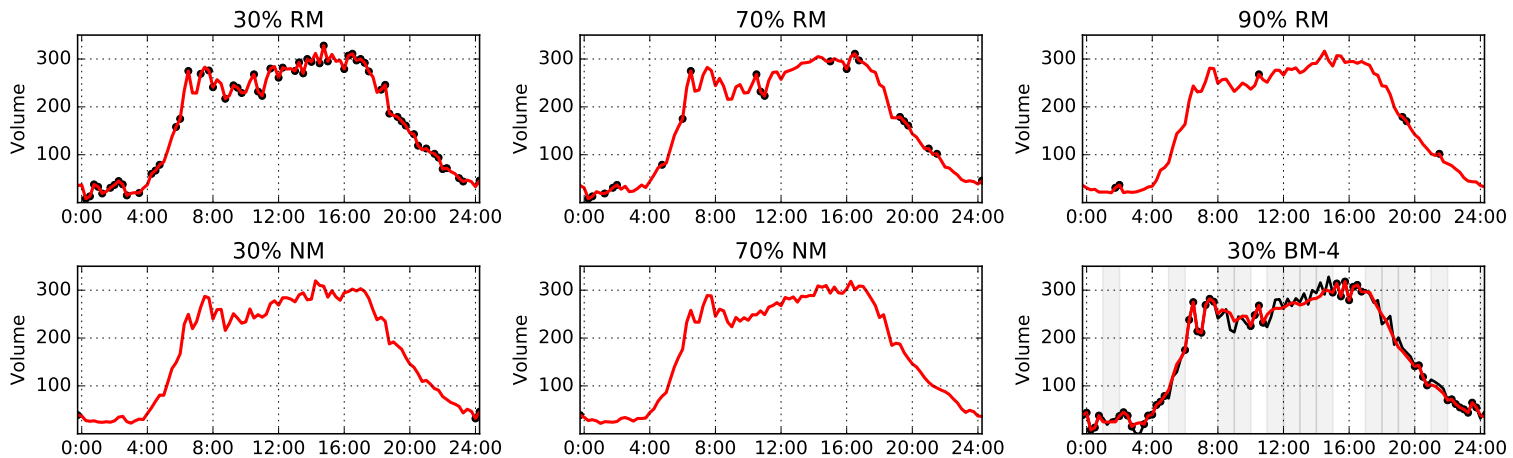
Spatiotemporal Traffic Data Imputation

LATC imputation

- Seattle freeway traffic speed data



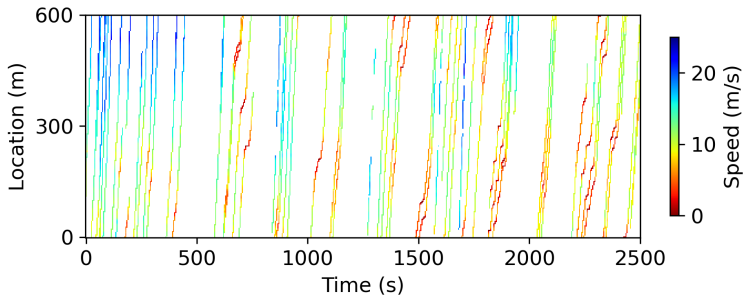
- Portland highway traffic volume data



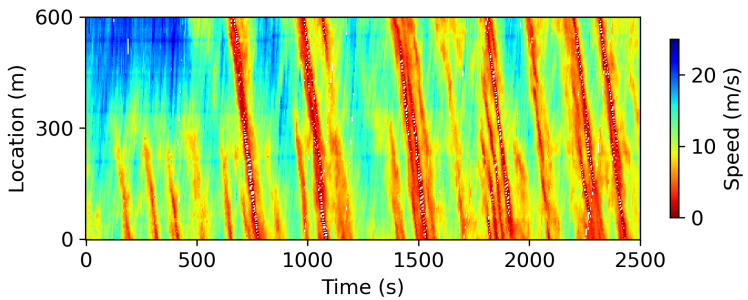
Large-Scale Traffic Data Imputation

LCR

Extreme Missing Traffic Data Imputation

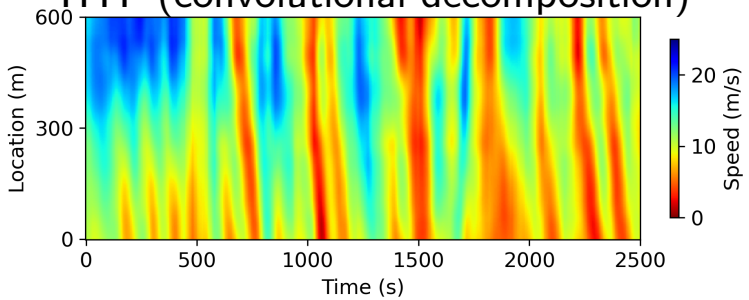


Sparse speed field



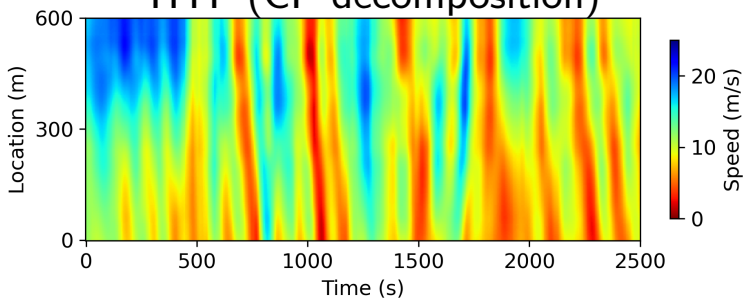
Ground truth speed field

HTF (convolutional decomposition)



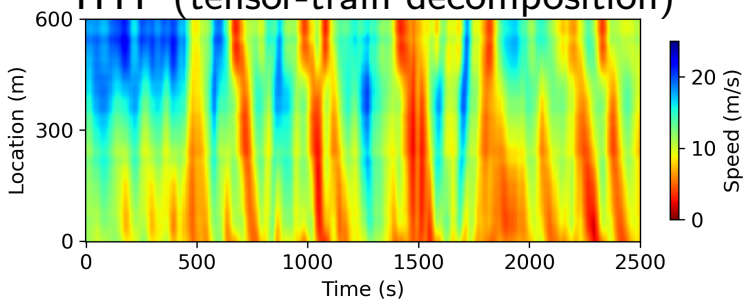
MAPE = **51.92%**

HTF (CP decomposition)



MAPE = 53.93%

HTF (tensor-train decomposition)



MAPE = 56.48%

Sparse Urban Traffic State Forecasting

NoTMF

Conclusion

- Data (large-scale, high-dimensional, city-wide, sparse)
- Modeling (meaningfulness and importance of temporal correlations)

Conclusion

Time-Varying Autoregression:

- (Highlight) Interpretable model with tensor factorization.
  - ✓ Parameter compression ✓ Pattern discovery

Laplacian Convolutional Representation:

- (Solution) Time series trend modeling in the low-rank framework?
  - Global time series trend modeling (low-rank model):

$$\begin{aligned} \min_{\boldsymbol{x}} \ & \|\mathcal{C}(\boldsymbol{x})\|_* \\ \text{s. t.} \ & \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

  - Local time series trend modeling (temporal regularization):

$$\mathcal{R}_\tau(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2$$

- (Highlight) A unified framework with the FFT implementation.

Hankel Tensor Factorization:

- (Highlight) Memory-efficient Hankel indexing & convolutional parameterization.

Our studies:

❷ X. Chen, Z. Cheng, N. Saunier, L. Sun (2022). Laplacian convolutional representation for traffic time series imputation. arXiv preprint arXiv:2212.01529.

❸ X. Chen, L. Sun (2022). Bayesian temporal factorization for multidimensional time series prediction. IEEE Transactions on Pattern Analysis and Machine Intelligence. 44 (9): 4659-4673.

GitHub repository:

- **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (1,000 stars & 270 forks on GitHub)
  https://github.com/xinychen/transdim

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Thanks for your attention!

Any Questions?

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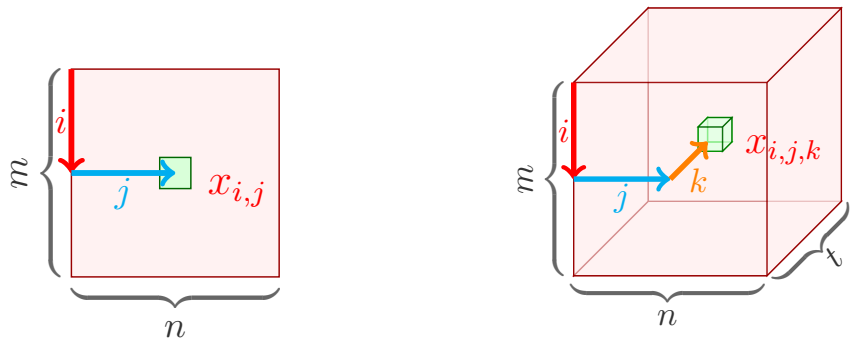
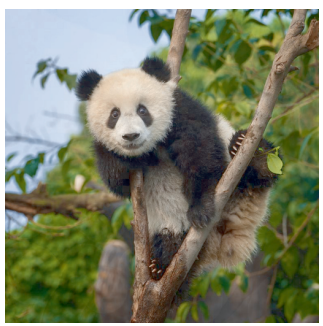
- How to reach me: chenxy346@gmail.com

What Is Tensors?

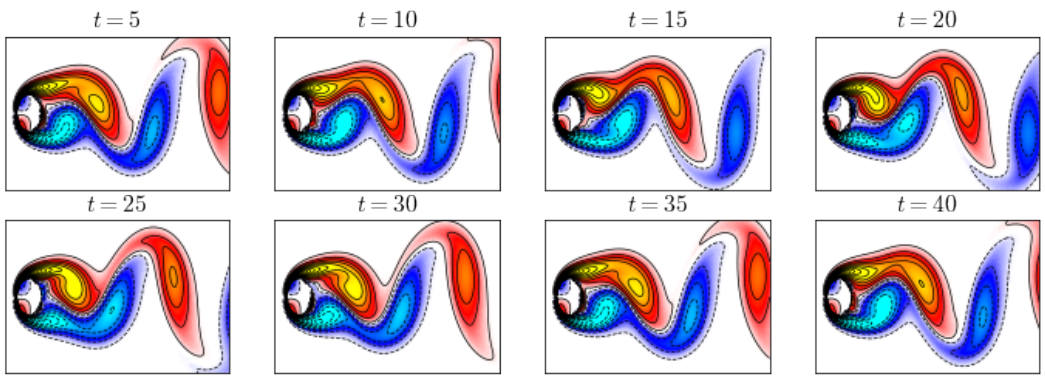
- What is tensor? $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ vs. $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{m \times n \times t}$



- Tensors are everywhere!



Color image with RGB channels



Dynamical system (fluid flow)

Appendix