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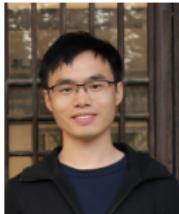


Low-Rank Matrix and Tensor Methods for Spatiotemporal Traffic Data Modeling

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Outline

- **Spatiotemporal Traffic Data**

- Preliminaries: What are matrix & tensor?
 - Traffic data & representation & problems

- **Spatiotemporal Traffic Data Imputation**

- Hankel tensor factorization

- **Sparse Traffic Flow Forecasting**

- Uber movement speed data
 - Temporal matrix factorization

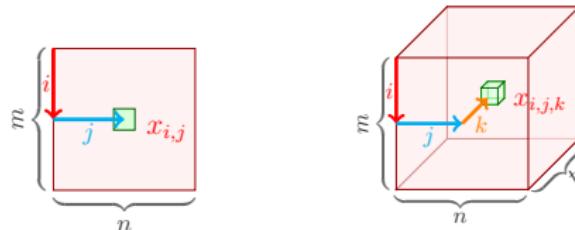
- **Dynamic Pattern Discovery**

- Human mobility changes over COVID-19
 - Time-varying low-rank autoregression

- **Conclusion**

What are Matrix & Tensor?

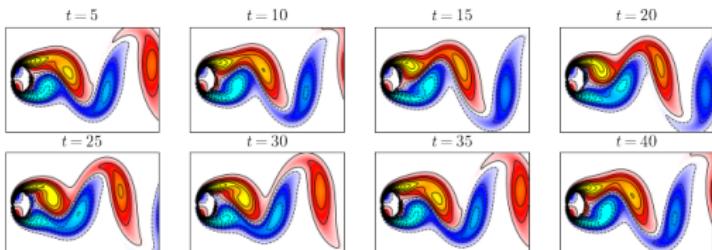
- What is tensor? $\mathbf{X} \in \mathbb{R}^{m \times n}$ vs. $\mathcal{X} \in \mathbb{R}^{m \times n \times t}$



- Tensors are everywhere!



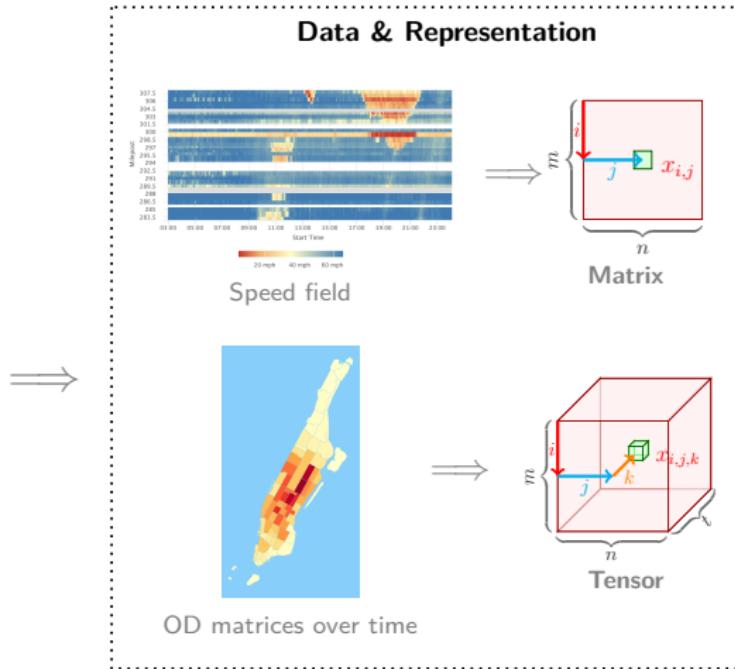
Color image with
RGB channels



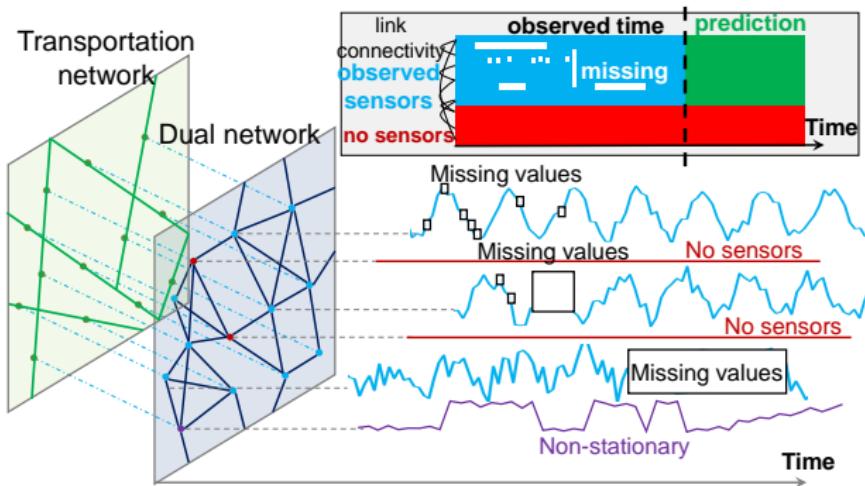
Dynamical system (fluid flow)

Spatiotemporal Traffic Data

- Spatiotemporal traffic data are indeed matrices or tensors.



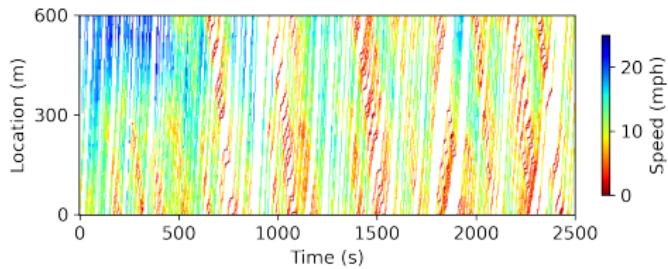
Spatiotemporal Traffic Data



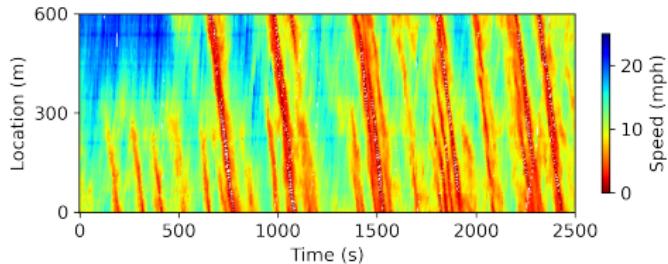
Spatiotemporal Traffic Data Imputation

- ① X. Chen, Z. Cheng, N. Saunier, L. Sun (2022). Laplacian convolutional representation for traffic time series imputation. arXiv preprint arXiv:2212.01529.
(Under 1st review at IEEE Transactions on Signal Processing)
- ② X. Chen Z. Cheng, L. Sun, N. Saunier (2023). Memory-efficient Hankel tensor factorization for extreme missing traffic data imputation. (coming soon)

Hankel Tensor Factorization



200-by-500 matrix
(NGSIM) \Downarrow Reconstruct speed field from
20% sparse trajectories?



Motivation:

- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal local dependencies?

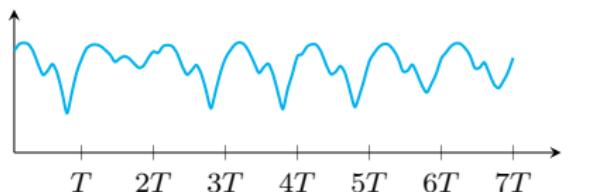
Hankel Tensor Factorization

- Hankel matrix

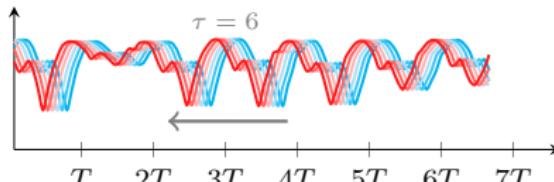
- Given $\mathbf{y} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\mathbf{y}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4 \times 2}$$

- Hankel matrix $\mathcal{H}_\tau(\mathbf{y})$ on time series \mathbf{y} :



↓ Construct Hankel matrix



Hankel Tensor Factorization

- Hankel matrix

- Given $\mathbf{y} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\mathbf{y}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4 \times 2}$$

- On time series $\mathbf{y} = (y_1, y_2, \dots, y_5)^\top$ with $\tau = 2$:

$$\mathcal{H}_\tau(\mathbf{y}) = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \\ y_3 & y_4 \\ y_4 & y_5 \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

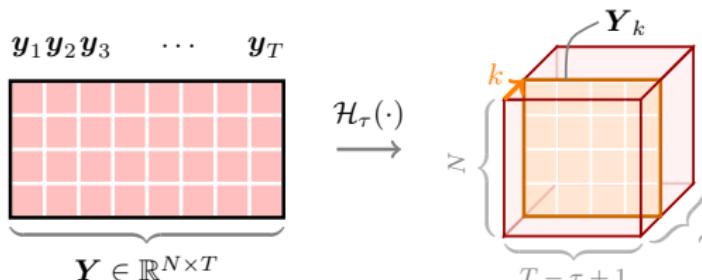
$$\implies \hat{\mathbf{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \\ \hat{y}_5 \end{bmatrix} = \mathcal{H}_\tau^{-1} \left(\begin{bmatrix} v_1 x_1 & v_1 x_2 \\ v_2 x_1 & v_2 x_2 \\ v_3 x_1 & v_3 x_2 \\ v_4 x_1 & v_4 x_2 \end{bmatrix} \right) = \begin{bmatrix} v_1 x_1 \\ (v_1 x_2 + v_2 x_1)/2 \\ (v_2 x_2 + v_3 x_1)/2 \\ (v_3 x_2 + v_4 x_1)/2 \\ v_4 x_2 \end{bmatrix}$$

- Automatic **temporal** modeling

Hankel Tensor Factorization

- Hankelization from $\mathbf{Y} \in \mathbb{R}^{N \times T}$ to Hankel tensor $\mathcal{H}_\tau(\mathbf{Y})$.

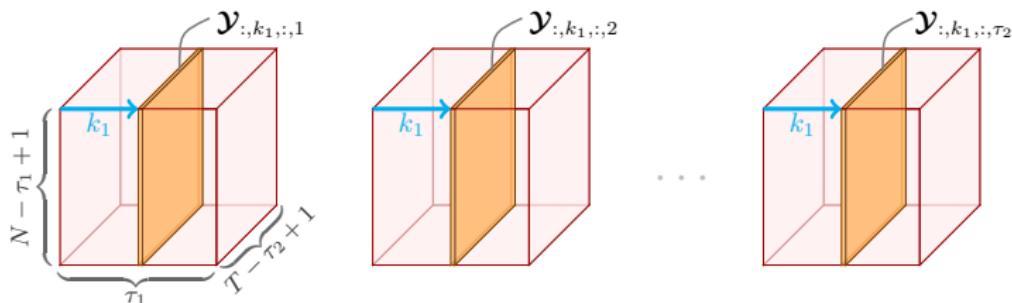
- Tensor size: $N \times (T - \tau + 1) \times \tau$;
- Slices: $\mathbf{Y}_k = \begin{bmatrix} | & | & | \\ \mathbf{y}_k & \mathbf{y}_{k+1} & \cdots & \mathbf{y}_{T-\tau+k} \\ | & | & | \end{bmatrix}, k = 1, 2, \dots, \tau$;
- Slice size: $N \times (T - \tau + 1)$.



- Automatic **temporal** modeling

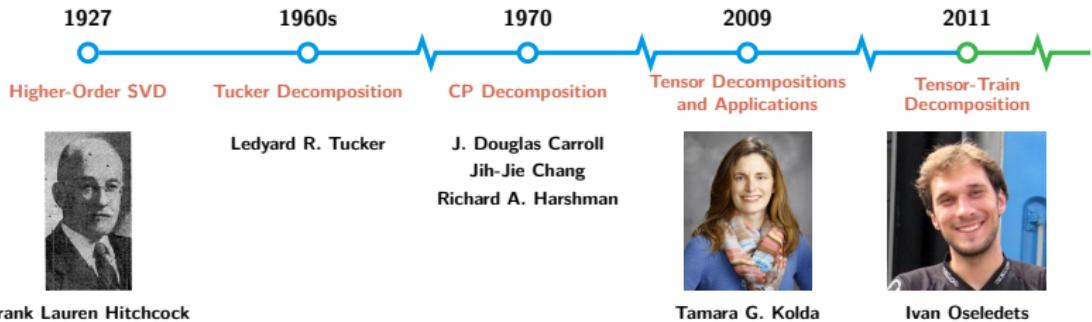
Hankel Tensor Factorization

- Hankelization from $\mathbf{Y} \in \mathbb{R}^{N \times T}$ to $\mathcal{Y} \triangleq \mathcal{H}_{\tau_1, \tau_2}(\mathbf{Y})$ (Hankel tensor).
 - Tensor size: $(N - \tau_1 + 1) \times \tau_1 \times (T - \tau_2 + 1) \times \tau_2$;
 - Slice: $\mathcal{Y}_{:, k_1, :, k_2}, \forall k_1, k_2$;
 - Slice size: $(N - \tau_1 + 1) \times (T - \tau_2 + 1)$.

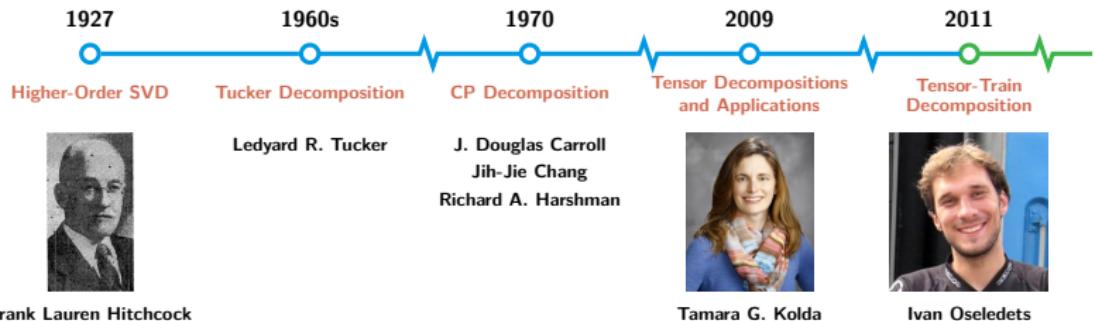


- Automatic **spatial** and **temporal** modeling

- Revisit tensor factorization (TF)



- Revisit tensor factorization (TF)



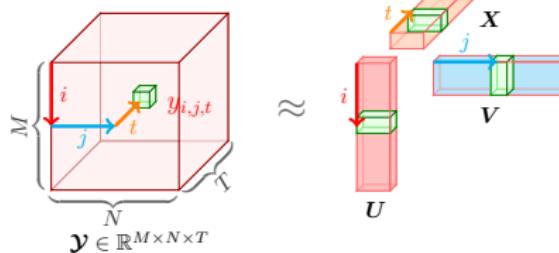
Frank Lauren Hitchcock

Tamara G. Kolda

Ivan Oseledets

- **CP decomposition:** Factorize \mathcal{Y} into the combination of rank- R factor

$$\text{matrices, i.e., } \mathcal{Y} \approx \sum_{r=1}^R \mathbf{u}_r \otimes \mathbf{v}_r \otimes \mathbf{x}_r.$$



Hankel Tensor Factorization

- Hankel tensor factorization:

$$\min_{\mathbf{Q}, \mathbf{S}, \mathbf{U}, \mathbf{V}} \frac{1}{2} \left\| \mathcal{P}_{\dot{\Omega}} \left(\mathcal{H}_{\tau_1, \tau_2}(\mathbf{Y}) - \sum_{r=1}^R \mathbf{q}_r \otimes \mathbf{s}_r \otimes \mathbf{u}_r \otimes \mathbf{v}_r \right) \right\|_F^2$$

- Advantage/disadvantage:

- ✓ Automatic spatial and temporal modeling
- ✗ High memory consumption

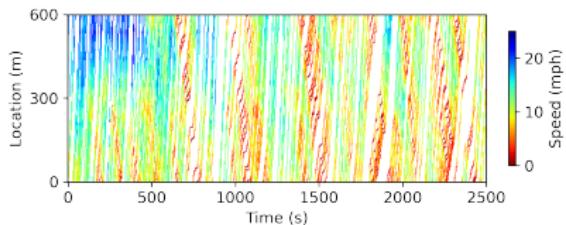
- Space complexity:

$$\mathcal{O}(\tau_1 \tau_2 (N - \tau_1 + 1)(T - \tau_2 + 1))$$

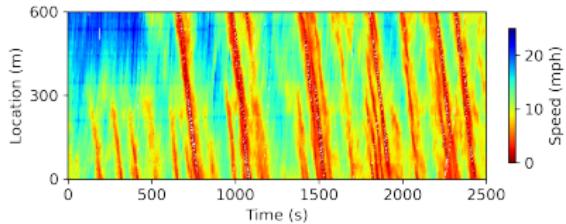
- **(Contribution)** Reduce the space complexity to $\mathcal{O}(NT)$.

Hankel Tensor Factorization

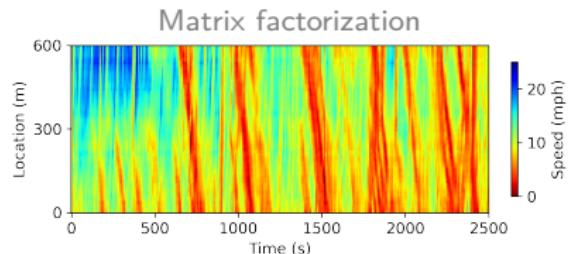
Which Model Is Better?



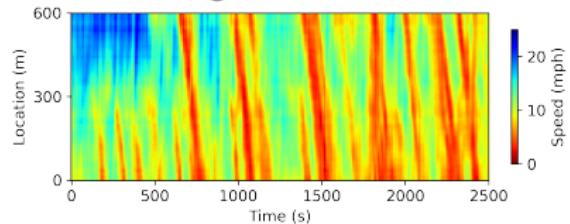
Sparse speed field



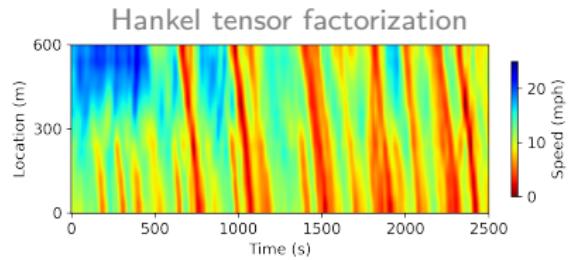
Ground truth speed field



Matrix factorization



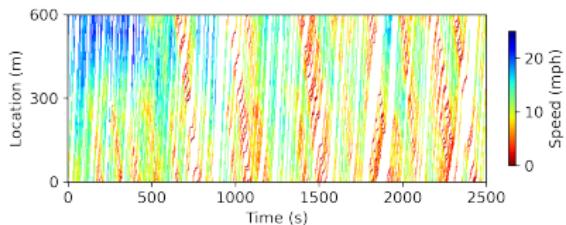
Smoothing matrix factorization



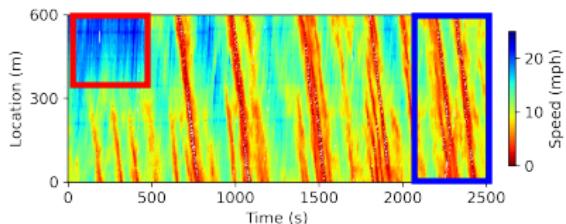
Hankel tensor factorization

Hankel Tensor Factorization

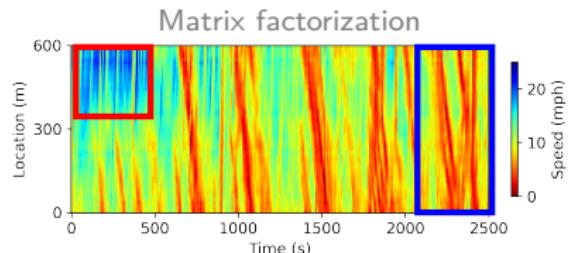
Which Model Is Better?



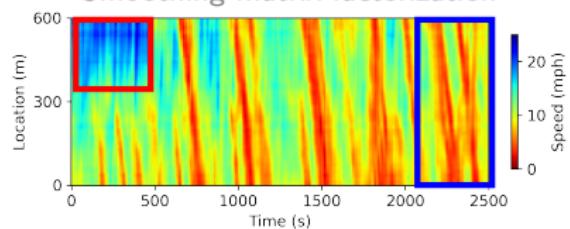
Sparse speed field



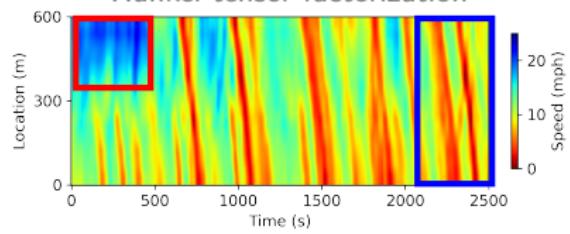
Ground truth speed field



Matrix factorization



Smoothing matrix factorization



Hankel tensor factorization

Sparse Traffic Flow Forecasting

- ③ X. Chen, L. Sun (2022). Bayesian temporal factorization for multidimensional time series prediction. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44 (9): 4659–4673.
 - 100+ citations on Google Scholar
 - ESI highly cited paper (top 1%)
 - ESI hot paper (top 0.1%)
- ④ X. Chen, C. Zhang, X.-L. Zhao, N. Saunier, L. Sun (2022). Nonstationary temporal matrix factorization for sparse traffic time series forecasting. *arXiv preprint arXiv:2203.10651*.
(Under 2nd review at *Transportation Research Part C: Emerging Technologies*)

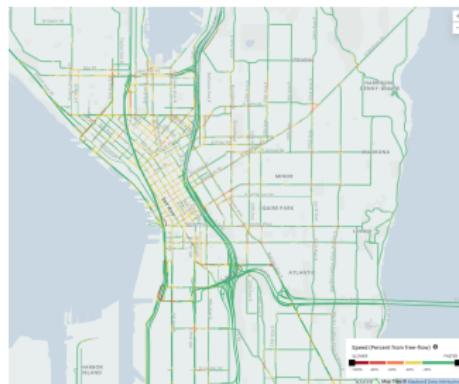
Sparse Traffic Flow Forecasting

Motivation:

- Uber (hourly) movement speed data¹



NYC movement



Seattle movement

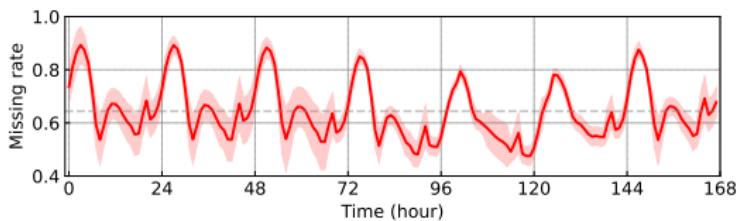
- The average speed on a given road segment for each hour of each day.
- Hourly speeds are computed when road segments have 5+ unique trips.
- **Issue:** insufficient sampling of ridesharing vehicles on the road network.

¹<https://movement.uber.com/>

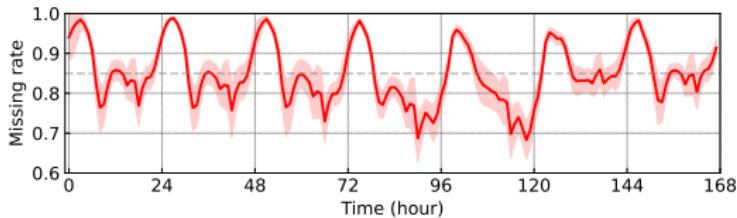
Sparse Traffic Flow Forecasting

High-dimensionality & Sparsity

- **NYC** movement speed data (2019)
 - **98,210** road segments & 8,760 time steps (hours)
 - Whole missing rate: **64.43%**

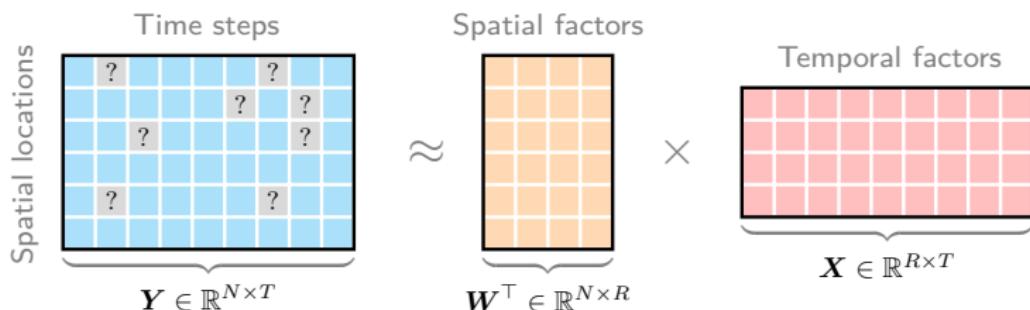


- **Seattle** movement speed data (2019)
 - **63,490** road segments & 8,760 time steps (hours)
 - Whole missing rate: **84.95%**



Sparse Traffic Flow Forecasting

- Spatiotemporal data can be reconstructed by low-dimensional latent factors!



- MF optimization problem

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \left\| \mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) \right\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)$$

with factor matrices \mathbf{W} and \mathbf{X} .

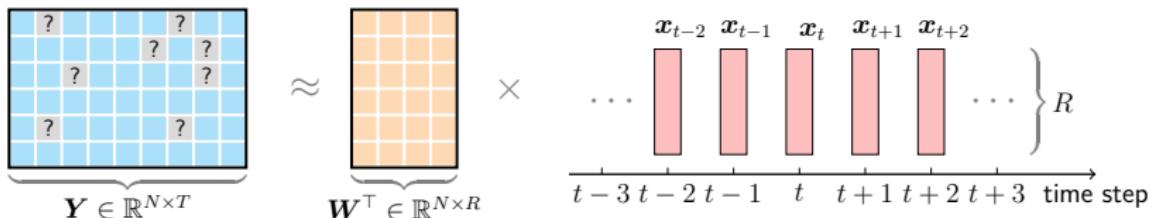
- Disadvantages:
 - Cannot capture temporal correlations.
 - Cannot perform time series forecasting.

Sparse Traffic Flow Forecasting

Temporal matrix factorization (Yu et al.'16; Chen & Sun'22)

Given any partially observed time series data $\mathbf{Y} \in \mathbb{R}^{N \times T}$ with observed index set Ω , then temporal matrix factorization assumes a d th-order vector autoregressive (VAR) process on the temporal factor matrix:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} \quad & \frac{1}{2} \left\| \mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) \right\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2) \\ & + \frac{\lambda}{2} \sum_{t=d+1}^T \left\| \mathbf{x}_t - \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} \right\|_2^2 \end{aligned}$$



GitHub repositories:

- **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (970+ stars & 270+ forks)
<https://github.com/xinychen/transdim>
- **tracebase**: Multivariate time series forecasting on high-dimensional and sparse Uber movement speed data. (30+ stars)
<https://github.com/xinychen/tracebase>
- **awesome-latex-drawing**: Academic drawing examples in LaTeX. (1,000+ stars & 140+ forks)
<https://github.com/xinychen/awesome-latex-drawing>

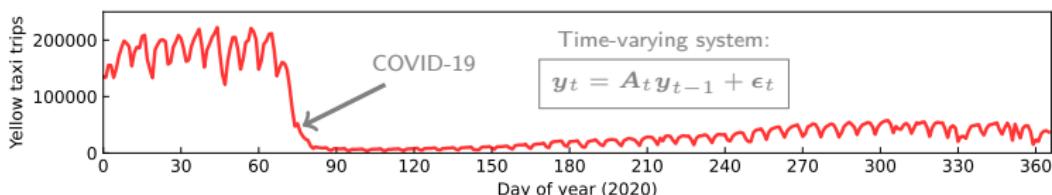
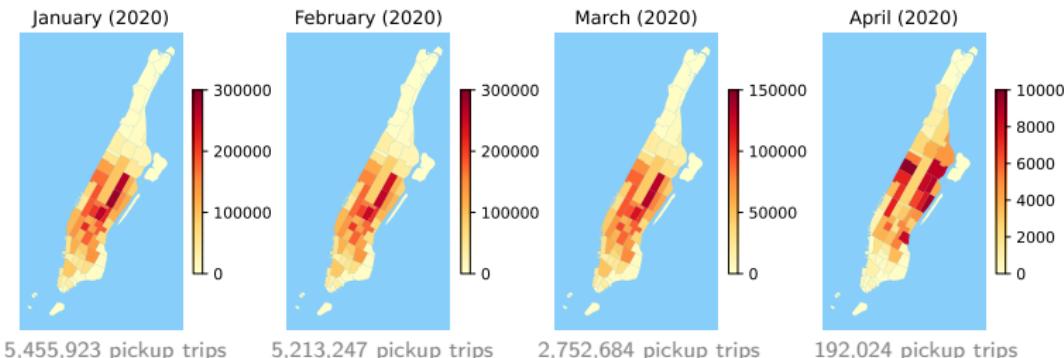
Dynamic Pattern Discovery

- ⑤ X. Chen, C. Zhang[†], X. Chen, N. Saunier, L. Sun (2022). Discovering dynamic patterns from spatiotemporal data with time-varying low-rank autoregression. arXiv preprint arXiv:2211.15482.
(Under 2nd review at IEEE Transactions on Knowledge and Data Engineering)

Dynamic Pattern Discovery

Motivation:

- NYC (yellow) taxi data²



- How to characterize the dynamic patterns?

²<https://www1.nyc.gov/site/tlc/about/tlc-trip-record-data.page>

Dynamic Pattern Discovery

- Given a sequence of spatiotemporal measurements

$$\mathbf{y}_t \in \mathbb{R}^N, t = 1, 2, \dots, T$$

$$\min_{\{\mathbf{A}_t\}} \underbrace{\frac{1}{2} \sum_t \|\mathbf{y}_t - \mathbf{A}_t \mathbf{y}_{t-1}\|_2^2}_{\text{Time-varying autoregression}}$$

[Over-parameterization] $\mathcal{O}(N^2(T-1))$ parameters vs. $\mathcal{O}(NT)$ data.

Dynamic Pattern Discovery

- Given a sequence of spatiotemporal measurements
 $\mathbf{y}_t \in \mathbb{R}^N$, $t = 1, 2, \dots, T$

$$\min_{\{\mathbf{A}_t\}} \underbrace{\frac{1}{2} \sum_t \|\mathbf{y}_t - \mathbf{A}_t \mathbf{y}_{t-1}\|_2^2}_{\text{Time-varying autoregression}}$$

[Over-parameterization] $\mathcal{O}(N^2(T-1))$ parameters vs. $\mathcal{O}(NT)$ data.

- (Ours)** Parameterize coefficients via tensor factorization (TF):

$$\min_{\mathbf{W}, \mathbf{G}, \mathbf{V}, \mathbf{x}} \underbrace{\frac{1}{2} \sum_t \left\| \mathbf{y}_t - \mathbf{W} \mathbf{G} (\mathbf{x}_t^\top \otimes \mathbf{V})^\top \mathbf{y}_{t-1} \right\|_2^2}_{\text{Let } \mathbf{A}_t = \mathbf{G} \times_1 \mathbf{W} \times_2 \mathbf{V} \times_3 \mathbf{x}_t^\top \text{ be the TF}}$$

- Alternating minimization (Let f be the obj.)

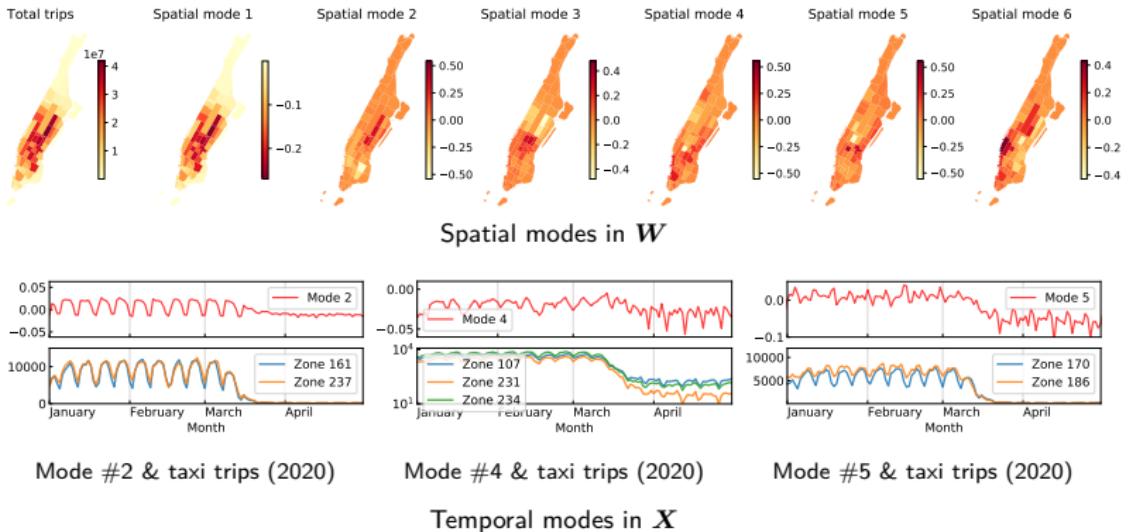
$$\begin{aligned} \mathbf{W} &:= \{\mathbf{W} \mid \frac{\partial f}{\partial \mathbf{W}} = \mathbf{0}\} & \mathbf{G} &:= \{\mathbf{G} \mid \frac{\partial f}{\partial \mathbf{G}} = \mathbf{0}\} \\ \mathbf{V} &:= \{\mathbf{V} \mid \frac{\partial f}{\partial \mathbf{V}} = \mathbf{0}\} & \mathbf{x}_t &:= \{\mathbf{x}_t \mid \frac{\partial f}{\partial \mathbf{x}_t} = \mathbf{0}\} \end{aligned}$$

- Solve each subproblem by **conjugate gradient** and **least squares**.

- Time-varying autoregression with TF

$$\min_{\mathbf{W}, \mathbf{G}, \mathbf{V}, \mathbf{x}} \frac{1}{2} \sum_t \left\| \mathbf{y}_t - \mathbf{W} \mathbf{G} (\mathbf{x}_t^\top \otimes \mathbf{V})^\top \mathbf{y}_{t-1} \right\|_2^2$$

- NYC taxi dataset (pickup)

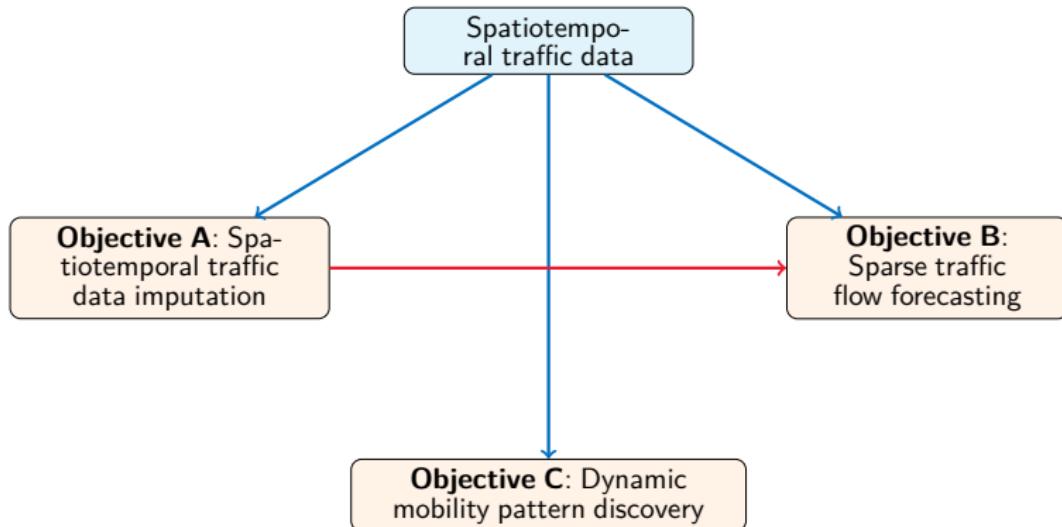


- Produce interpretable patterns and identify the changing point of system (mainly due to COVID-19).

Dynamic Pattern Discovery

Conclusion

We give some studies on **spatiotemporal traffic data modeling**.



Prior Works

Other studies about spatiotemporal data modeling:

- ⑥ X. Chen, M. Lei, N. Saunier, L. Sun (2022). Low-rank autoregressive tensor completion for spatiotemporal traffic data imputation. *IEEE Transactions on Intelligent Transportation Systems*, 23 (8): 12301–12310.
- ⑦ X. Chen, Y. Chen, N. Saunier, L. Sun (2021). Scalable low-rank tensor learning for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*, 129: 103226.
- ⑧ X. Chen, J. Yang, L. Sun (2020). A nonconvex low-rank tensor completion model for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*, 117: 102673.
- ⑨ X. Chen, Z. He, Y. Chen, Y. Lu, J. Wang (2019). Missing traffic data imputation and pattern discovery with a Bayesian augmented tensor factorization model. *Transportation Research Part C: Emerging Technologies*, 104: 66-77.
- ⑩ X. Chen, Z. He, L. Sun (2019). A Bayesian tensor decomposition approach for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*, 98: 73-84.

References

A short list:

- [Liu & Zhang'22] G. Liu and W. Zhang (2022). Recovery of future data via convolution nuclear norm minimization. *IEEE Transactions on Information Theory*, 69(1), 650–665.
- [Yu et al.'16] H.-F. Yu, N. Rao, and I. S. Dhillon (2016). Temporal regularized matrix factorization for high-dimensional time series prediction. *Advances in neural information processing systems (NIPS)*.



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Thanks for your attention!

Any Questions?

Slides: https://xinychen.github.io/slides/traffic_data_modeling_v2.pdf

About me:

- 🏠 Homepage: <https://xinychen.github.io>
- ✉️ Google Scholar: [user=mCrW04wAAAAJhl](#) (690 citations)
- 🐙 GitHub: <https://github.com/xinychen> (3.2k+ stars)
- 💻 Blog: <https://medium.com/@xinyu.chen> (70k+ views)
- ✉️ How to reach me: chenxy346@gmail.com