Matrix and Tensor Models for Spatiotemporal Traffic Data Imputation and Forecasting

Xinyu Chen

Ph.D., University of Montreal, Canada

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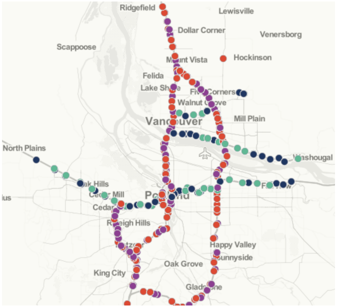
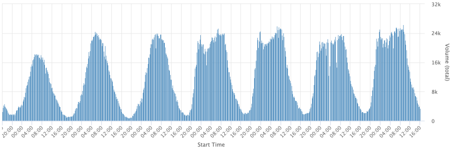
Outline

Spatiotemporal Traffic Data

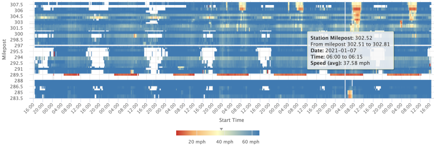
- Many spatiotemporal tra c time series data are in the form of **matrix**.
  - Portland highway tra c data[1]



Highway network & sensor locations



Tra c volume



Tra c speed

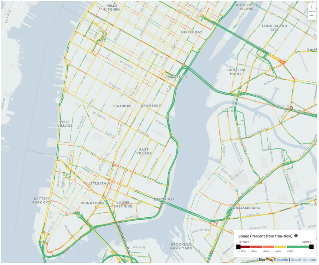
- $\boldsymbol{X} \in \mathbb{R}^{N \times T}$ with N spatial locations T time steps
- Tra c volume/speed shows strong spatial/temporal dependencies

[1] https://portal.its.pdx.edu/home

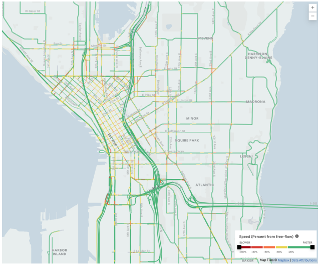
Urban Movement Data

High-dimensional & sparse

Uber (hourly) movement speed data



NYC movement



Seattle movement

*f*road segment, time slot (hour), average speed*g*

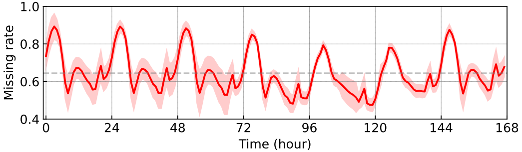
Computing hourly speed: Road segments have 5+ unique trips.

Issue: Insu cient sampling of ridesharing vehicles on the road network!

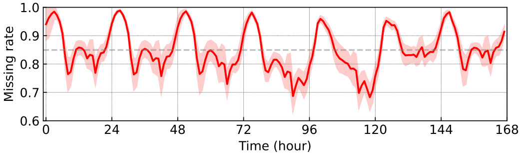
Urban Movement Data

High-dimensional & sparse

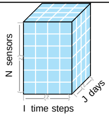
- **NYC** movement speed data (2019)
  - **98,210** road segments & 8,760 time steps (hours)
  - Overall missing rate: **64.43%**



- **Seattle** movement speed data (2019)
  - **63,490** road segments & 8,760 time steps (hours)
  - Overall missing rate: **84.95%**



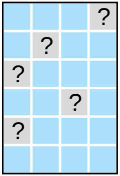
Spatiotemporal Tra c Data

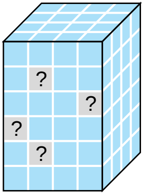
Sparse	
High-dimensional	
Multidimensional	
Noises & outliers	
Nonstationary	

[redacted]

Problem Formulation

Objective A: Impute missing values in the data matrix $Y \in \mathbb{R}^{N\times T}$ (or tensor $\mathcal{Y} \in \mathbb{R}^{M\times N\times T}$).





- Matrix completion (Observed index set Ω)

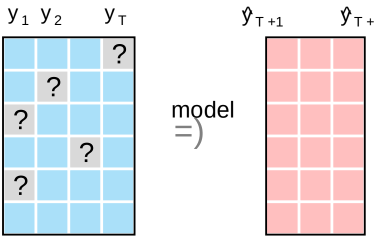
$$\underbrace{\mathcal{P}_{\Omega}(Y)}_{\text{Partially observed}} \xrightarrow{\text{Estimate}} \underbrace{\mathcal{P}_{\Omega^{\perp}}(Y)}_{\text{Unobserved}}$$

Modeling process:

- How to make use of spatiotemporal traffic patterns?
- How to make use of traffic time series dynamics?

Problem Formulation

Objective B: Given a partially observed data $Y \in \mathbb{R}^{N \times T}$ consisting of time series $y_1, \ldots, y_T \in \mathbb{R}^N$, forecast data points $y_{T+\tau}$, $\tau \in \mathbb{N}^+$.



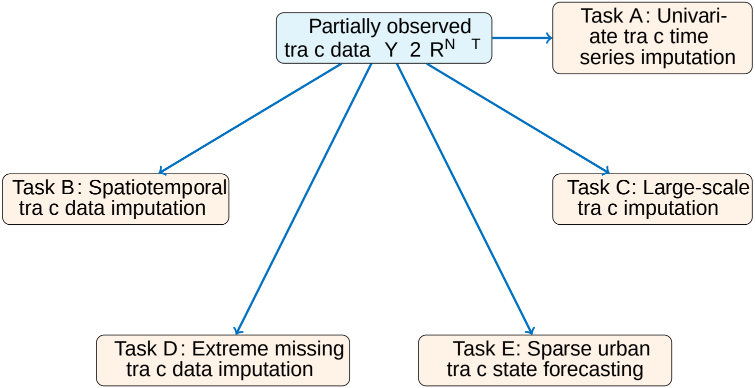
Sparse time series $\quad$ τ-step ahead forecasts

Modeling process:

- How to characterize time series dynamics in high-dimensional and sparse traffic data?

Tasks

We are working on spatiotemporal traffic data imputation and forecasting.



Tensor Factorization

Revisit tensor factorization

1927	1960s	1970	2009	2011
Higher-Order SVD	Tucker Decomposition	CP Decomposition	Tensor Decompositions and Applications	Tensor-Train Decomposition
	Ledyard R. Tucker	J. Douglas Carroll Jih-Jie Chang Richard A. Harshman		
Frank Lauren Hitchcock			Tamara G. Kolda	Ivan Oseledets

CP tensor factorization : Factorize $\mathcal{Y}$ into the combination of three rank-R factor matrices (i.e., low-dimensional latent factors).

$$\begin{cases} y_{i,j,t} \approx \sum_{r=1}^{R} u_{i,r} v_{j,r} x_{t,r} \\ \mathcal{Y} \approx \sum_{r=1}^{R} u_r \circ v_r \circ x_r \end{cases}$$

Matrix/Tensor Completion

Matrix nuclear norm ($\|X\|_*$) minimization
Candes & Recht'09

Singular value thresholding
Cai et al.'10

Truncated nuclear norm ($\|X\|_{r,*}, r \in \mathbb{N}^+$) minimization
Zhang et al.'12
Hu et al.'12

Tensor nuclear norm ($\|\mathcal{X}\|_*$) minimization
Liu et al.'13

Generalized nonconvex nonsmooth low-rank minimization
Lu et al.'14

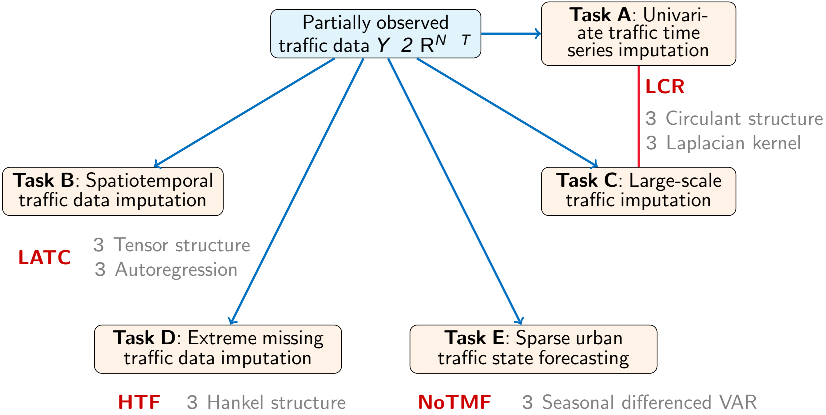
Tensor nuclear norm minimization with linear transform
Lu et al.'19

Low-rank Hankel matrix/tensor ($\mathcal{H}(\cdot)$) completion
Yokota et al.'18
Sedighin et al.'20
Cai et al.'21
Yamamoto et al.'22

Circulant/Convolution nuclear norm ($\|\mathcal{C}(X)\|_*$ or $\|\mathcal{C}_\ell(X)\|_*$) minimization
Liu'22
Liu & Zhang'23

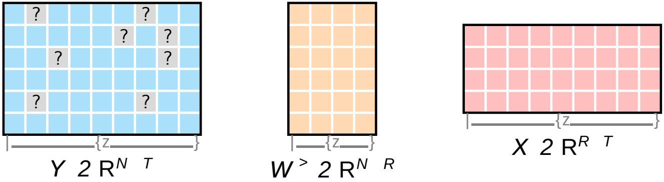
Overview

Matrix/Tensor methods + temporal modeling (e.g., smoothing & autoregression)



Matrix Factorization

A simple approach to reconstruct missing values.



MF (Koren et al.'09)

Estimating low-dimensional $\boldsymbol{W}, \boldsymbol{X}$:

$$\min_{\boldsymbol{W}, \boldsymbol{X}} \frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y} - \boldsymbol{W}^\top \boldsymbol{X})\|_F^2$$

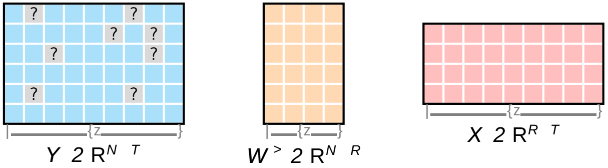
on data $\boldsymbol{Y}$ w/ observed index set Ω.

- ✓ Learn from sparse data
- ✓ Spatial factor matrix $\boldsymbol{W}$
- ✓ Temporal factor matrix $\boldsymbol{X}$

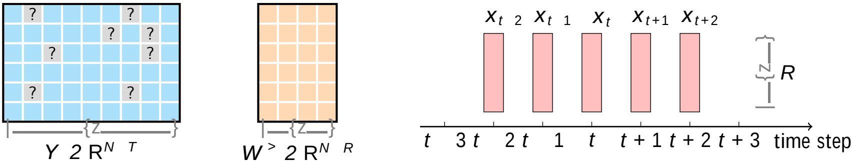
How to build temporal correlations on MF?

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.



$+$ $\boldsymbol{X}$ is time series?



Why? Temporal factor matrix $\boldsymbol{X}\in\mathbb{R}^{R\times T}$ is the low-dimensional representation of time series dynamics of $\boldsymbol{Y}\in\mathbb{R}^{N\times T}$.

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

[redacted]

Estimating low-dimensional W, X:

$$\min_{W,X} \frac{1}{2}\|\mathcal{P}_{\Omega}(Y - W^\top X)\|_F^2$$

on data Y w/ observed index set Ω.

+

[redacted]

$$x_t = \sum_{k=1}^{d} A_k x_{t-k} + \underbrace{\epsilon_t}_{\sim \mathcal{N}(0, I)}$$

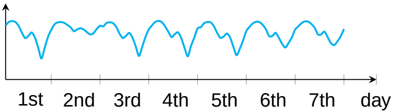
w/ coefficients $\{A_k\}$.

+ Yu et al.'16
Chen & Sun'21

$$\min_{W, X, \{A_k\}_{k=1}^{d}} \frac{1}{2}\underbrace{\|\mathcal{P}_{\Omega}(Y - W^\top X)\|_F^2}_{\text{MF on data } Y} + \frac{\gamma}{2}\underbrace{\sum_{t=d+1}^{T}\Big\|x_t - \sum_{k=1}^{d} A_k x_{t-k}\Big\|_2^2}_{\text{VAR on temporal factors } X}$$

Nonstationary Temporal Matrix Factorization

Traffic data are nonstationary due to daily patterns of traffic flow.



Season-m differencing ($m \in \mathbb{N}^{+}$, e.g., daily/weekly):

$$\boldsymbol{x}_t \approx \sum_{k=1}^{d} \boldsymbol{A}_k \boldsymbol{x}_{t-k} \Rightarrow \boldsymbol{x}_t - \boldsymbol{x}_{t-m} \approx \sum_{k=1}^{d} \boldsymbol{A}_k (\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k})$$

(Ours) Optimization problem:

$$\min_{\boldsymbol{W}, \boldsymbol{X}, \{\boldsymbol{A}_k\}_{k=1}^{d}} \underbrace{\frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y} - \boldsymbol{W}^{\top}\boldsymbol{X})\|_F^2}_{\text{MF on data } \boldsymbol{Y}} + \underbrace{\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2 + \|\boldsymbol{X}\|_F^2)}_{\text{Regularization}}$$
$$+ \underbrace{\frac{\lambda}{2}\sum_{t=d+m+1}^{T} \left\|(\boldsymbol{x}_t - \boldsymbol{x}_{t-m}) - \sum_{k=1}^{d} \boldsymbol{A}_k (\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k})\right\|_2^2}_{\text{VAR on seasonal differenced temporal factors}}$$

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF

Optimization problem:[2]

$$\min_{\boldsymbol{W},\boldsymbol{X},\boldsymbol{A}}\ \frac{1}{2}\underbrace{\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2}_{\text{MF on data }\boldsymbol{Y}}+\frac{\rho}{2}\underbrace{(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)}_{\text{Regularization}}$$

$$+\frac{\lambda}{2}\underbrace{\|\boldsymbol{X}\boldsymbol{\Psi}_0^\top-\boldsymbol{A}(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^\top\|_F^2}_{\text{VAR on }\boldsymbol{X}}$$

where $\boldsymbol{\Psi}_0\in\mathbb{R}^{(T-dm)\times T},\boldsymbol{\Psi}\in\mathbb{R}^{(T-dm)\times(dT)}$ are temporal operators.

Alternating minimization (let f be the obj.):

$$\begin{cases}\text{Spatial factors} & \boldsymbol{W}:=\{\boldsymbol{W}\mid\frac{\partial f}{\partial\boldsymbol{W}}=\boldsymbol{0}\} & \text{(least squares)}\\ \text{Temporal factors} & \boldsymbol{X}:=\{\boldsymbol{X}\mid\frac{\partial f}{\partial\boldsymbol{X}}=\boldsymbol{0}\} & \text{(conjugate gradient)}\\ \text{VAR coefficients} & \boldsymbol{A}:=\{\boldsymbol{A}\mid\frac{\partial f}{\partial\boldsymbol{A}}=\boldsymbol{0}\} & \text{(least squares)}\end{cases}$$

[2] $\boldsymbol{A}\triangleq[\boldsymbol{A}_1,\cdots,\boldsymbol{A}_d]\in\mathbb{R}^{R\times(dR)}$ (coefficient matrix).

Nonstationary Temporal Matrix Factorization

NoTMF forecasting?

Implementation

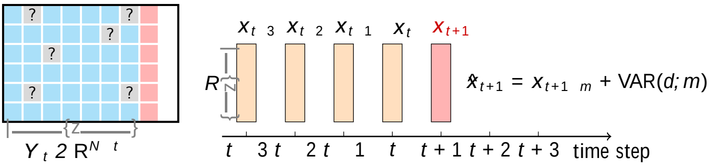
Estimate $\boldsymbol{W}; \boldsymbol{X}; \boldsymbol{A}$

Forecast $\hat{\boldsymbol{x}}_{t+1}$ with VAR

Return $\hat{\boldsymbol{y}}_{t+1} = \boldsymbol{W}^{>} \hat{\boldsymbol{x}}_{t+1}$

3 Sparse input $\boldsymbol{Y}_t$

3 Forecast in latent spaces



Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

Online forecasting (Gultekin & Paisley'18):

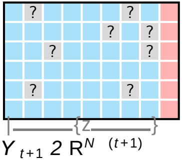
- Fix the spatial factor matrix $\boldsymbol{W}$
- Use input data $\boldsymbol{Y}_{t+1}$ to update the temporal factor matrix $\boldsymbol{X}$ and the coe cient matrix $\boldsymbol{A}$

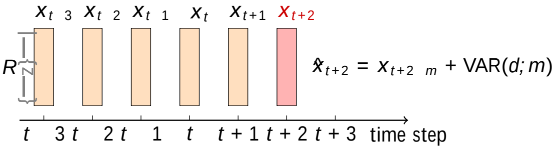
Implementation

- Estimate $\boldsymbol{X}; \boldsymbol{A}$
- Forecast $\hat{\boldsymbol{x}}_{t+2}$ with VAR
- Return $\hat{\boldsymbol{y}}_{t+2} = \boldsymbol{W}^{>} \hat{\boldsymbol{x}}_{t+2}$

3 Sparse input $\boldsymbol{Y}_{t+1}$

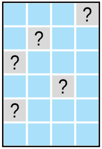
3 Forecast in latent spaces



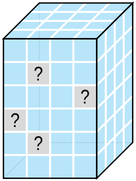


Matrix/Tensor Completion

Problem? Impute missing values in matrices/tensors.



$\mathcal{P}_{\Omega}(\boldsymbol{Y}) \in \mathbb{R}^{N \times T}$



$\mathcal{P}_{\Omega}(\mathcal{Y}) \in \mathbb{R}^{N \times I \times J}$

Cornerstone: Nuclear norm minimization

LRMC (Candes & Recht'09)

Estimating the matrix $\boldsymbol{X}$:

$$\min_{\boldsymbol{X}} \|\boldsymbol{X}\|_{*} \quad \text{s.t. } \mathcal{P}_{\Omega}(\boldsymbol{X}) = \mathcal{P}_{\Omega}(\boldsymbol{Y})$$

on data $\boldsymbol{Y}$ w/ observed index set Ω.

vs.

LRTC (Liu et al.'13)

Estimating the tensor $\mathcal{X}$:

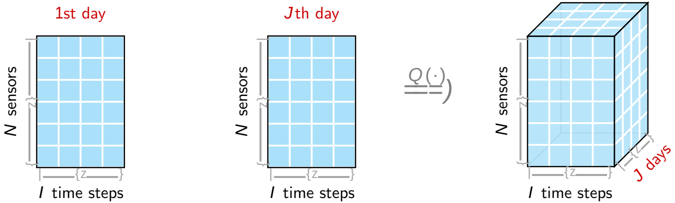
$$\min_{\mathcal{X}} \|\mathcal{X}\|_{*} \quad \text{s.t. } \mathcal{P}_{\Omega}(\mathcal{X}) = \mathcal{P}_{\Omega}(\mathcal{Y})$$

on data $\mathcal{Y}$ w/ observed index set Ω.

Limitation: Nuclear norm minimization only covers global consistency.

Low-Rank Autoregressive Tensor Completion

Introduce traffic tensors with day dimension[3] (Tan et al.'13, Chen et al.'19, ...).



Build temporal correlations with univariate autoregression.

On the time series $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$:

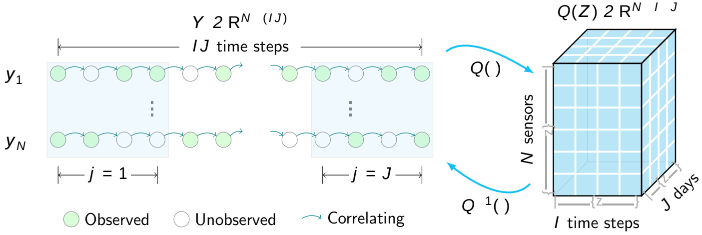
$$\|\boldsymbol{Y}\|_{\boldsymbol{A},\mathcal{H}} \triangleq \sqrt{\sum_{n,t}\left(y_{n,t} - \sum_{k} a_{n,k} y_{n,t-h_k}\right)^2}$$

w/ the time lag set $\mathcal{H} = \{h_1, \ldots, h_d\}$ and the coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{N \times d}$.

[3] There are $T = IJ$ time steps in total.

Low-Rank Autoregressive Tensor Completion

Local consistency w/ autoregression — **Global consistency** w/ tensor structure



LATC

Optimization problem:

$$\min_{\boldsymbol{Z};\boldsymbol{A}} \underbrace{\|Q(\boldsymbol{Z})\|_{r;}}_{\text{global}} + \frac{}{2}\underbrace{\|\boldsymbol{Z}\|_{\boldsymbol{A};H}}_{\text{local}}$$

s.t. $P\ (\boldsymbol{Z}) = P\ (\boldsymbol{Y})$

on data $\boldsymbol{Y}$ w/ observed index set .

$\Longrightarrow$

Two subproblems

$$\begin{cases} \boldsymbol{Z} := \underset{P\ (\boldsymbol{Z}) = P\ (\boldsymbol{Y})}{\arg\min}\ \|Q(\boldsymbol{Z})\|_{r;} + \frac{}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A};H} \\ \boldsymbol{A} := \frac{1}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A};H} \qquad \text{(Least squares)} \end{cases}$$

Low-Rank Autoregressive Tensor Completion

$\boldsymbol{Z}$-subproblem:

$$\boldsymbol{Z} := \underset{P\ (\boldsymbol{Z})=P\ (\boldsymbol{Y})}{\arg\min}\ \|Q(\boldsymbol{Z})\|_{r,} + \frac{}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},H}$$

Augmented Lagrangian function:[4]

$$L(X, \boldsymbol{Z}, W) = \|X\|_{r,} + \frac{}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},H} + \frac{}{2}\|X \quad Q(\boldsymbol{Z})\|_F^2 + \langle W, X \quad Q(\boldsymbol{Z})\rangle + \ (\boldsymbol{Z})$$

Implementation

Repeat

- Compute $\boldsymbol{Z}$
- Compute $\boldsymbol{A}$

$\Rightarrow$

Implementation

Repeat

- Repeat
  - # Alternating Direction Method of Multipliers (ADMM)
  - Compute X
  - Compute $\boldsymbol{Z}$
  - Compute W
- Compute $\boldsymbol{A}$

---

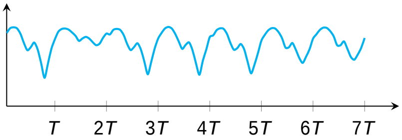
[4] $W \in \mathbb{R}^{N \quad I \quad J}$ (Lagrange multiplier); $\langle\ ,\ \rangle$ (inner product). The indicator function:

$$(\boldsymbol{Z}) = \begin{cases} 0, & \text{if } P\ (\boldsymbol{Z}) = P\ (\boldsymbol{Y}), \\ +\infty, & \text{otherwise.} \end{cases}$$

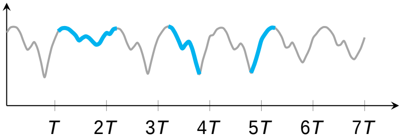
Laplacian Convolutional Representation

Motivation: Time series imputation

Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



Local trends (e.g., short-term time series trends):

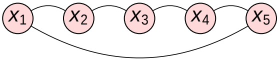


How to characterize both global and local trends in sparse time series?

Laplacian Convolutional Representation

Local trend modeling

Intuition of (circulant) Laplacian matrix



Undirected and circulant graph

$\xrightarrow{\text{Modeling}}$

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (2, -1, 0, 0, -1)^\top$$

$$\Downarrow$$

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \cdots, -1}, 0, \cdots, 0, \underbrace{-1, \cdots, -1})^\top \in \mathbb{R}^T$$

for any time series $\boldsymbol{x} = (x_1, \ldots, x_T)^\top \in \mathbb{R}^T$.

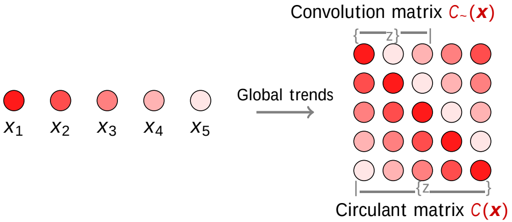
(Laplacian) Temporal regularization:

$$\mathcal{R}_\tau(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2 = \frac{1}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2$$

Reformulate temporal regularization with circular convolution.

Laplacian Convolutional Representation

Global trend modeling: Circulant matrix $C(\boldsymbol{x})$ vs. convolution matrix $C_{\sim}(\boldsymbol{x})$



Circulant/Convolution nuclear norm minimization

A balance between global and local trends modeling?

CircNNM (Liu'22, Liu & Zhang'23)

Estimating $\boldsymbol{x}$:

$$\min_{\boldsymbol{x}}\ \|C(\boldsymbol{x})\|$$
$$\text{s.t. } \|P\ (\boldsymbol{x}\quad \boldsymbol{y})\|_2$$

on data $\boldsymbol{y}$ w/ observed index set .

ConvNNM (Liu'22, Liu & Zhang'23)

Estimating $\boldsymbol{x}$:

$$\min_{\boldsymbol{x}}\ \|C_{\sim}(\boldsymbol{x})\|$$
$$\text{s.t. } \|P\ (\boldsymbol{x}\quad \boldsymbol{y})\|_2$$

on data $\boldsymbol{y}$ w/ observed index set .

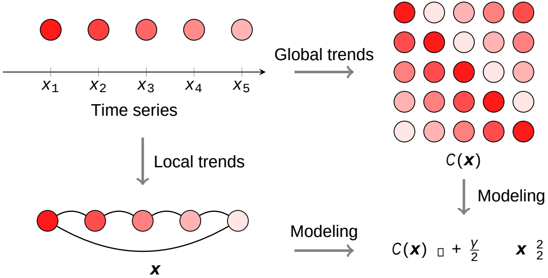
Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\boldsymbol{y} \in \mathbb{R}^T$ with observed index set Ω, LCR utilizes circulant matrix and Laplacian kernel to characterize global and local trends in time series, respectively, i.e.,

$$\min_{\boldsymbol{x}} \underbrace{\|\mathcal{C}(\boldsymbol{x})\|_*}_{\text{global}} + \frac{\gamma}{2}\underbrace{\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2}_{\text{local}}$$

$$\text{s.t. } \|\mathcal{P}_{\Omega}(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon$$



Laplacian Convolutional Representation

Augmented Lagrangian function:[5]

$$L(x;z;w) = kC(x)k \quad + \quad \frac{}{2}k\grave{} \; ? \; x k_2^2 + \frac{}{2}kx \quad z k_2^2 + hw; x \quad z i + \frac{}{2}kP \quad (z \quad y) k_2^2$$

The ADMM scheme:

$$\begin{array}{lll} x := \arg\min_{x} \; L(x;z;w) & & \text{(Nuclear norm minimization)} \\ z := \arg\min_{z} \; L(x;z;w) & & \text{(Closed-form solution)} \\ w := w + \quad (x \quad z) & & \text{(Standard update)} \end{array}$$

Optimize x?

$$kC(x)k \quad = \; kF(x)k_1 \qquad \& \qquad \frac{1}{2}k\grave{} \; ? \; x k_2^2 = \frac{1}{2T}kF(\grave{}) \quad F(x)k_2^2$$

[redacted]

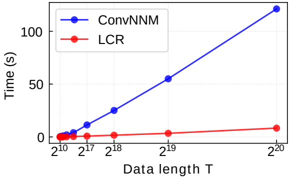
[5] $w \; 2 \; R^T$ (Lagrange multiplier); $h; i$ (inner product).

Laplacian Convolutional Representation

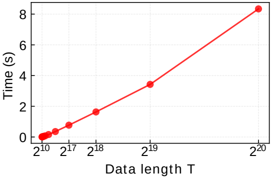
Empirical time complexity

On the synthetic data $\boldsymbol{y} \in \mathbb{R}^T$ with $T \in \{2^{10}, 2^{11}, \ldots, 2^{20}\}$

- Ours: **LCR**
  - An FFT implementation in $O(T \log T)$
  - The logarithmic factor $\log T$ makes the FFT highly efficient
- Baseline: **ConvNNM**[6] (Liu'22, Liu & Zhang'23)
  - Convolution matrix $\mathcal{C}_{\tau}(\boldsymbol{y}) \in \mathbb{R}^{T \times \tau}$ with kernel size $\tau = 2^4$
  - Singular value thresholding in $O(\tau^2 T)$



ConvNNM vs. LCR



LCR

[6] Convolution nuclear norm minimization.

Hankel Tensor Factorization

Motivation: Spatiotemporal data reconstruction

Sparse speed eld reconstruction problem in vehicular tra c ow.

200-by-500 matrix
(NGSIM)

+

Reconstruct speed eld from
5% sparse trajectories?

[redacted]

Hankel Tensor Factorization

Hankel matrix

- Given $\boldsymbol{x} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\textcolor{red}{\mathcal{H}_{\tau}(\boldsymbol{x})} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4 \times 2}$$

- Automatic temporal modeling

Traffic time series

Hankel matrix

Hankel Tensor Factorization

Hankel tensor: Given any matrix $X\ 2\ R^{N\quad T}$, we have

$$X\ ,\ H_{1;\ 2}(X)$$

Window lengths: $_1;\ _2\ 2\ N^+$;
Tensor size: $(N\quad _1 + 1)\quad _1\quad (T\quad _2 + 1)\quad _2$;

(Figure) 4th-order Hankel tensor: A sequence of third-order tensors.

Slice: $X_{:;k_1;:;k_2}$; $8k_1; k_2$;
Slice size: $(N\quad _1 + 1)\quad (T\quad _2 + 1)$.

Hankel Tensor Factorization

Hankel indexing

Sampling function for the Hankel tensor:

$$_{k_1;k_2}(X) \, , \, [H_{\ 1;\ 2}(X)]_{:;k_1;:;k_2};$$

referring to as the tensor slice with k_1 2 f 1; : : : ; $_1$g; k_2 2 f 1; : : : ; $_2$g.

[Importance] Developing memory-e cient algorithms

Tensor slices $_{k_1;k_2}(X)$ vs. data matrix X

Hankel Tensor Factorization

Ours:

Convolutional tensor decomposition (circular convolution $?_{\text{row}}$):

$$_{k_1;k_2}(Y) \quad (Q \, ?_{\text{row}} \, s_{k_1}^{>})(U \, ?_{\text{row}} \, v_{k_2}^{>})^{>}$$

Baselines:

Tensor-train decomposition:

$$_{k_1;k_2}(Y) \quad (QS_{k_1})(UV_{k_2})^{>}$$

- f $S_{k_1}; V_{k_2}$ g are circulant matrices) convolutional decomposition
- f $S_{k_1}; V_{k_2}$ g are diagonal matrices) CP decomposition

CP tensor decomposition (Khatri-Rao product):

$$_{k_1;k_2}(Y) \quad (Q \quad s_{k_1}^{>})(U \quad v_{k_2}^{>})^{>}$$

Hankel Tensor Factorization

HTF (convolutional decomposition)

Optimization problem:

$$\min_{Q,S,U,V} \frac{1}{2} \underbrace{\sum_{k_1,k_2} \left\| \mathcal{P}_{k_1,k_2}(\mathcal{Y}) - (Q \star_{\text{row}} s_{k_1}^\top)(U \star_{\text{row}} v_{k_2}^\top)^\top \right\|_F^2}_{\text{Tensor decomposition on Hankel tensor slices}}$$
$$+ \frac{\lambda}{2}\left(\|Q\|_F^2 + \|S\|_F^2 + \|U\|_F^2 + \|V\|_F^2 \right)$$

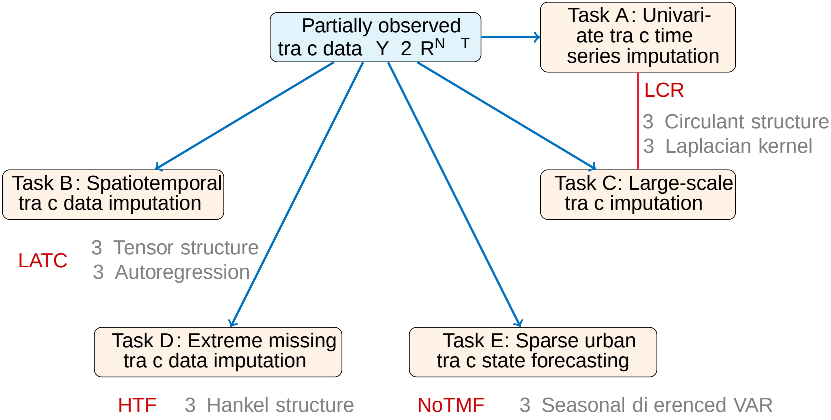
Alternating minimization (let f be the obj.):

$$\begin{cases}
Q := \{Q \mid \frac{\partial f}{\partial Q} = 0\} & \text{(conjugate gradient)} \\
s_{k_1} := \{s_{k_1} \mid \frac{\partial f}{\partial s_{k_1}} = 0\}, \forall k_1 & \text{(conjugate gradient)} \\
U := \{U \mid \frac{\partial f}{\partial U} = 0\} & \text{(conjugate gradient)} \\
v_{k_2} := \{v_{k_2} \mid \frac{\partial f}{\partial v_{k_2}} = 0\}, \forall k_2 & \text{(conjugate gradient)}
\end{cases}$$

Memory-efficient but still computationally costly!

Overview

We are working on spatiotemporal traffic data imputation and forecasting.



Task A: Univariate Traffic Time Series Imputation

CircNNM :

$$\min_{x} \underbrace{\|C(x)\|}_{\text{global}}$$

$$\text{s.t.} \quad \|P(x - y)\|_2$$

Plus temporal regularization

LCR:

$$\min_{x} \underbrace{\|C(x)\|}_{\text{global}} + \frac{\,}{2} \underbrace{\|\ell \star x\|_2^2}_{\text{local}}$$

$$\text{s.t.} \quad \|P(x - y)\|_2$$

Task A: Univariate Tra c Time Series Imputation

Mask 90% observations

Mask 95% observations

Reconstruct time series

Reconstruct time series

[redacted]

Task B: Spatiotemporal Tra c Data Imputation

LATC vs. baseline (in MAPE/RMSE)

On the Seattle freeway tra c speed dataset ($\mathcal{Y} \in \mathbb{R}^{323 \times 8064}$)

Missing rate	LATC	LAMC	LRTC-TNN	BTMF	SPC
30%, Random Missing	4.90/ 3.16	5.98/3.73	4.99/3.20	5.91/3.72	5.92/3.62
70%, Random Missing	5.96/ 3.71	8.02/4.70	6.10/3.77	6.47/3.98	7.38/4.30
90%, Random Missing	7.46/ 4.50	10.56/5.91	8.08/4.80	8.17/4.81	9.75/5.31
30%, Nonrandom Missing	7.10/4.33	6.99/4.25	6.85/ 4.21	9.26/5.36	8.87/4.99
70%, Nonrandom Missing	9.40/5.40	9.75/5.60	9.23/ 5.35	10.47/6.15	11.32/5.92
30%, Block-out Missing	9.43/ 5.36	27.05/13.66	9.52/5.41	14.33/13.60	11.30/5.84

On the Portland highway tra c volume dataset ($\mathcal{Y} \in \mathbb{R}^{1156 \times 2976}$)

Missing rate	LATC	LAMC	LRTC-TNN	BTMF	SPC
30%, Random Missing	16.95/ 15.99	17.93/16.03	17.27/16.08	18.22/19.14	21.29/56.73
70%, Random Missing	19.59/ 18.70	21.26/19.37	19.99/18.73	19.96/22.21	24.35/43.32
90%, Random Missing	23.15/22.83	25.64/23.75	22.90/ 22.68	23.90/25.71	28.45/39.65
30%, Nonrandom Missing	19.48/19.14	19.93/19.69	19.59/ 18.91	19.55/20.38	26.96/60.33
70%, Nonrandom Missing	27.67/45.03	25.75/28.25	30.26/60.85	23.86/ 26.74	33.42/47.34
30%, Block-out Missing	24.01/ 23.50	29.21/27.60	31.74/74.42	27.85/25.68	31.01/60.33

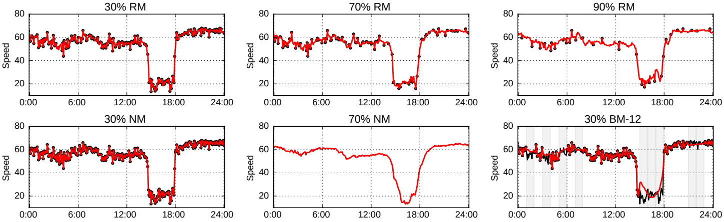
LATC vs. LAMC: The signi cance of tensor representation

LATC vs. LRTC-TNN: The signi cance of temporal autoregression

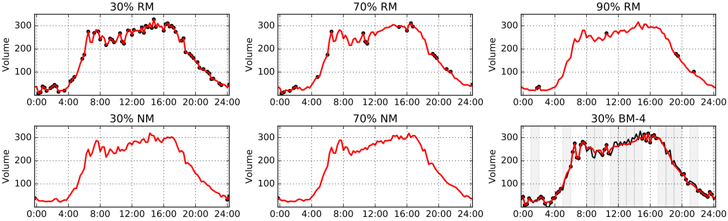
Task B: Spatiotemporal Traffic Data Imputation

LATC imputation

Seattle freeway tra c speed data



Portland highway tra c volume data



Task C: Large-Scale Traffic Data Imputation

LCR vs. baseline (in MAPE/RMSE)

PeMS-4W: California freeway traffic speed dataset ($\boldsymbol{Y} \in \mathbb{R}^{11160 \times 8064}$)

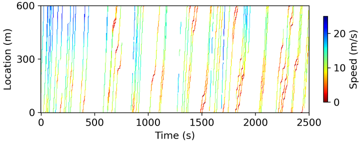
Model	Missing rate			
	30%	50%	70%	90%
LCR-2D	1.50/**1.49**	1.76/**1.69**	**2.07**/**2.06**	**3.19**/**3.05**
LCR_N	**1.48**/1.50	**1.73**/1.73	**2.07**/2.12	3.24/3.22
LCR	1.50/**1.49**	1.76/**1.69**	2.08/2.07	3.21/3.06
CTNNM	2.26/1.84	2.67/2.14	3.40/2.66	5.22/3.90
CircNNM	2.26/1.84	2.69/2.15	3.43/2.67	5.34/3.96
LRMC	2.04/1.80	2.43/2.12	3.08/2.66	6.05/4.43
HaLRTC	1.98/1.73	2.22/1.98	2.84/2.49	4.39/3.66
LRTC-TNN	1.68/1.55	1.93/1.77	2.33/2.14	3.40/3.10
NoTMF	2.95/2.65	3.05/2.73	3.33/2.97	5.22/4.71

Results

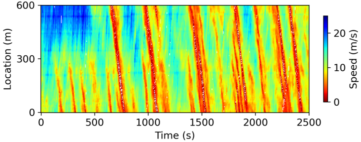
- LCR-2D > CTNNM: The importance of temporal regularization.
- CTNNM CircNNM: Ciculant tensor is superior to circulant matrix.
- LCR > LRMC/LRTC: The importance of global/local modeling.

$$O(NT\log(NT)) \quad \text{(FFT)} \quad \text{vs.} \quad O(\min\{N^2T, NT^2\}) \quad \text{(SVD)}$$

Task D: Extreme Missing Traffic Data Imputation

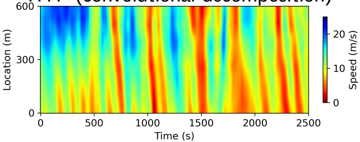


Sparse speed field



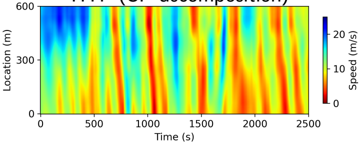
Ground truth speed field

HTF (convolutional decomposition)



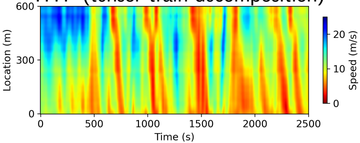
MAPE = **51.92%**

HTF (CP decomposition)



MAPE = 53.93%

HTF (tensor-train decomposition)



MAPE = 56.48%

Task D: Extreme Missing Traffic Data Imputation

HTF vs. baseline (in MAPE/RMSE)

On the Seattle freeway traffic speed dataset ($\boldsymbol{Y} \in \mathbb{R}^{323 \times 8064}$)

Model	Random missing rate			
	80%	85%	90%	95%
HTF (convolution)	**6.21/3.88**	**6.51/4.06**	**6.98/4.30**	**8.02/4.84**
HTF (tensor-train)	8.75/5.16	9.86/5.76	9.24/5.36	9.89/5.70
HTF (CP)	7.25/4.33	7.93/4.66	8.61/4.96	9.25/5.20
LATC	6.50/4.00	6.90/4.21	7.47/4.51	8.75/5.05
LRTC-TNN	6.97/4.24	7.43/4.43	8.19/4.81	9.60/5.55
LCR	6.75/4.15	7.31/4.38	7.96/4.71	9.78/5.39
BTMF	6.85/4.17	7.36/4.42	8.13/4.79	9.63/5.48

Results

- Convolutional tensor decomposition outperforms both tensor-train and CP tensor decomposition.
- Our HTF model performs better than state-of-the-art baseline models.

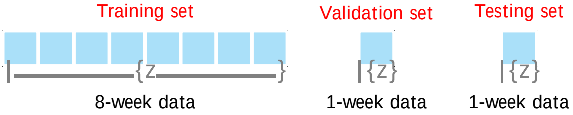
Task E: Sparse Urban Traffic State Forecasting

NoTMF forecasting

NYC Uber movement speed dataset:

10-week data of size 98210 1680; **66.56%** missing values

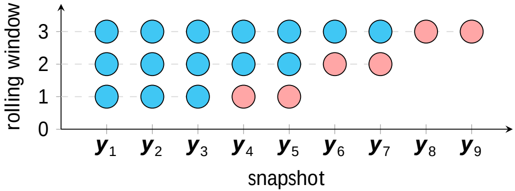
Rolling forecasting setup (Time horizon = 1; 2; 3; 6):



Weight parameter $2\ f10^{\ 2}; 10^{\ 1}; 10^0; 10^1; 10^2 g$

Weight parameter $= f10^{\ 1}\ ; 5\quad 10^{\ 1}\ ;\ \ ; 5\ \ ; 10\ \ g$

Rolling forecasting illustration (= 2):



Task E: Sparse Urban Traffic State Forecasting

NoTMF vs. baseline (in MAPE/RMSE)

On the NYC Uber movement speed dataset

	d	NoTMF (m = 24)	NoTMF (m = 168)	NoTMF-1st (m = 168)	TRMF	BTMF	BTRMF
1	1	13.63/2.88	13.53/2.86	**13.45**/**2.85**	14.50/3.12	14.94/3.13	15.93/3.33
	2	13.47/**2.84**	**13.41**/**2.84**	13.42/**2.84**	14.14/3.05	15.70/3.41	15.90/3.35
	3	13.46/2.84	**13.39**/**2.83**	13.43/2.84	13.87/2.96	15.80/3.34	16.08/3.43
	6	13.41/**2.83**	**13.39**/**2.83**	13.41/**2.83**	14.00/2.98	15.45/3.27	16.26/3.48
2	1	13.91/2.96	13.76/2.94	**13.70**/**2.92**	15.85/3.43	15.33/3.21	16.85/3.56
	2	13.77/2.92	**13.63**/**2.89**	13.72/2.92	15.04/3.31	15.87/3.32	17.27/3.71
	3	13.72/2.91	**13.61**/**2.89**	13.73/2.92	15.25/3.36	15.69/3.33	17.24/3.74
	6	13.59/**2.87**	**13.57**/2.88	13.68/2.91	14.92/3.24	15.91/3.39	18.18/3.97
3	1	14.30/3.05	14.06/3.02	**14.02**/**3.00**	17.52/3.83	15.86/3.32	18.61/3.91
	2	14.01/2.98	**13.84**/**2.94**	13.96/2.98	17.32/4.00	16.30/3.40	18.90/4.10
	3	13.95/2.97	**13.79**/**2.93**	13.98/2.98	16.91/3.71	16.56/3.49	18.68/4.05
	6	13.78/**2.92**	**13.73**/**2.92**	13.91/2.96	16.72/3.65	15.49/3.27	20.45/4.66
6	1	**14.61**/**3.11**	14.67/3.20	14.98/3.32	21.20/4.70	15.99/3.32	22.40/4.69
	2	**14.30**/**3.03**	14.33/3.09	14.90/3.28	20.87/5.01	16.04/3.33	23.56/5.63
	3	**14.26**/**3.03**	14.28/3.09	14.86/3.26	20.08/4.65	15.67/3.28	24.27/5.72
	6	**14.06**/**2.97**	14.16/3.06	14.80/3.23	20.40/4.35	16.38/3.50	26.34/6.60

NoTMF performs better than other TMF models (e.g., TRMF, BTMF & BTRMF).

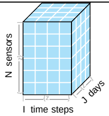
Task E: Sparse Urban Tra c State Forecasting

NoTMF forecasting (= 6)

- On the NYC Uber movement speed dataset

Task E: Sparse Urban Tra c State Forecasting

Conclusion

Sparse	
High-dimensional	
Multidimensional	
Noises & outliers	
Nonstationary	

)

Low-rank framework:

- NoTMF: matrix factorization
- LATC: low-rank tensor completion
- LCR: circulant matrix nuclear norm minimization
- HTF: tensor factorization

Temporal modeling:

- NoTMF: seasonal di erenced vector autoregression
- LATC: univariate autoregression
- LCR: temporal smoothing
- HTF: automatic temporal modeling with Hankel tensor

Highlights & Contributions

(Ours) LATC :
✓ Temporal autoregression

Matrix nuclear norm ($\|\boldsymbol{X}\|_*$) minimization
Candès & Recht'09

Singular value thresholding
Cai et al.'10

Truncated nuclear norm ($\|\boldsymbol{X}\|_{r,*}$; $r \in \mathbb{N}^+$) minimization
Zhang et al.'12
Hu et al.'12

Tensor nuclear norm ($\|\boldsymbol{\mathcal{X}}\|_*$) minimization
Liu et al.'13

Generalized nonconvex nonsmooth low-rank minimization
Lu et al.'14

Tensor nuclear norm minimization with linear transform
Lu et al.'19

Low-rank Hankel matrix/tensor ($\mathcal{H}(\cdot)$) completion
Yokota et al.'18
Sedighin et al.'20
Cai et al.'21
Yamamoto et al.'22

Circulant/Convolution nuclear norm ($\|\mathcal{C}(\boldsymbol{X})\|_*$ or $\|\mathcal{C}_\tau(\boldsymbol{X})\|_*$) minimization
Liu'22
Liu & Zhang'23

(Ours) LCR:
✓ Local trend modeling
✓ An FFT implementation

(Ours) HTF :
✓ Memory-efficient
✓ Conv. para.

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A short list:

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Thanks for your attention!

Any Questions?

Slides: https://xinychen.github.io/slides/sustech23.pdf

About me:

Homepage: https://xinychen.github.io

How to reach me: chenxy346@gmail.com