The Relevance of t-Statistics for Small Sample Sizes

An Introductory Class to Higher Statistics

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Outline

Answering a lot questions, e.g.,

❶ How was t-statistic developed?

❷ Standard normal distribution vs. student t-distribution?

❸ What is t-statistic?

❹ What types of t-tests there are?

❺ What are the hypotheses and the assumptions?

❻ How a t-test is calculated?

❼ How do you interpret results?

❽ Huge real-world applications

Development

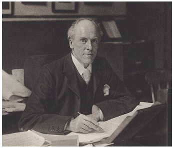
The problems of small samples

1876 — t-**distribution as a posterior distribution**

1895 — t-**distribution as Pearson type IV distribution**

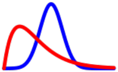
1908 — **Student** t-**distribution**
"The Probable Error of a Mean" (Biometrika)


F. R. Helmert **J. Lüroth**



K. Pearson

W. S. Gosset

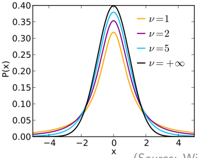


(Source: Wiki)

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

(Source: Wiki)

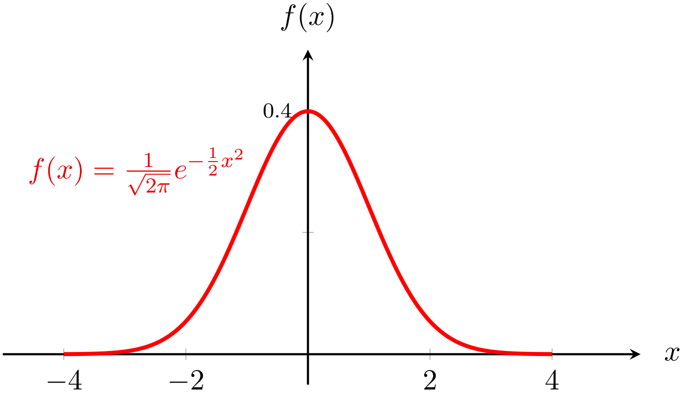
Diagram of the Pearson system



(Source: Wiki)

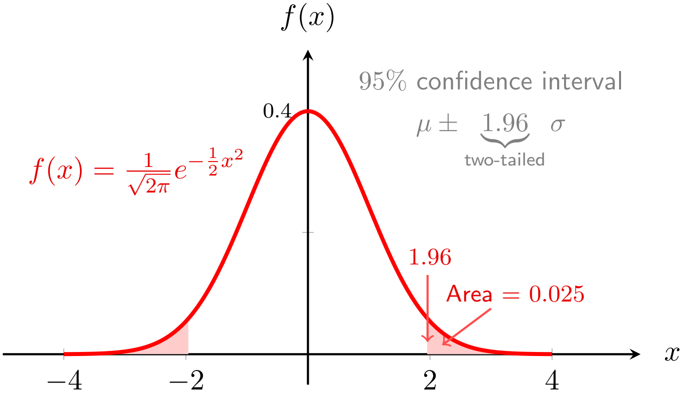
Probability density function

Revisiting Normal Distribution



Probability density function of the standard normal distribution

Revisiting Normal Distribution



Probability density function of the standard normal distribution

Implementing z-Test

Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh with a population standard deviation of 5 kWh. A random sample of 40 households has an average daily energy consumption of 32 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

Implementing z-Test

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Steps:

1. Formulate Hypotheses
   - Null Hypothesis (H_0): The population mean is $\mu = 30\,$kWh.
   - Alternative Hypothesis (H_a): The population mean is not $\mu = 30\,$kWh ($\mu \neq 30$).
2. Use the z-test formula since the population standard deviation is known:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{32 - 30}{5/\sqrt{40}} = \frac{2}{5/6.32} = \frac{2}{0.79} \approx 2.53$$

- $\bar{x} = 32$ (sample mean)
- $\mu = 30$ (population mean)
- $n = 40$ (sample size)
- $\sigma = 5$ (population standard deviation)

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❸ Decision rule at a 95% confidence interval

- Reject H_0 if $|z| > 1.96$.
- Otherwise, fail to reject H_0.

❹ Interpretation

- The test statistic $z = 2.53$ exceeds the critical value of 1.96.
- Thus, we reject the null hypothesis.
- The sample provides sufficient evidence to conclude that the **average daily energy consumption is not 30 kWh**.

Student t-Distribution

- Probability density function (w/ random variable x):

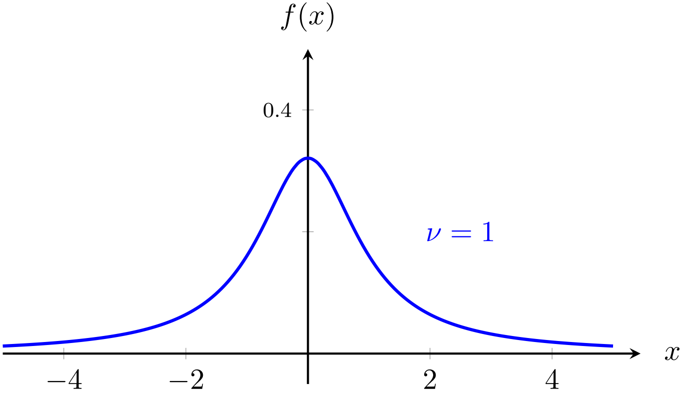
$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\Big(1 + \frac{x^2}{\nu}\Big)^{-\frac{\nu+1}{2}}$$

  - $\nu \in \mathbb{Z}^+$: Degrees of freedom
  - $\Gamma(\cdot)$: The Gamma function

- Example of $\nu = 5$ degrees of freedom:

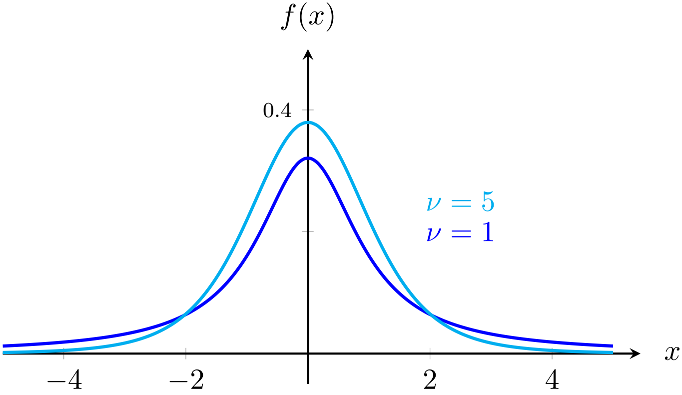
$$f(x) = \frac{\Gamma(3)}{\sqrt{5\pi}\,\Gamma(2.5)}\Big(1 + \frac{x^2}{5}\Big)^{-3}$$

Student t-Distribution



Student t-distribution

Student t-Distribution



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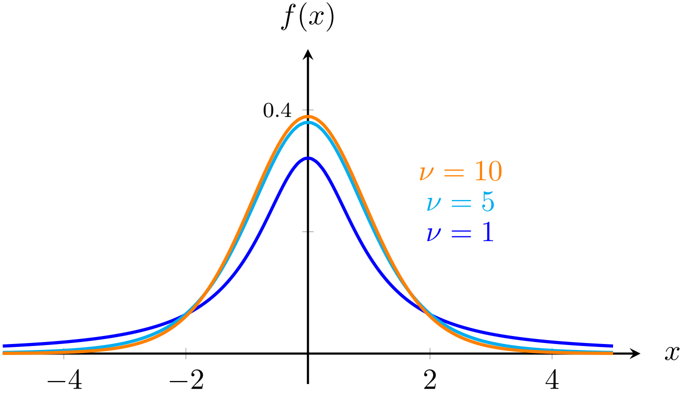

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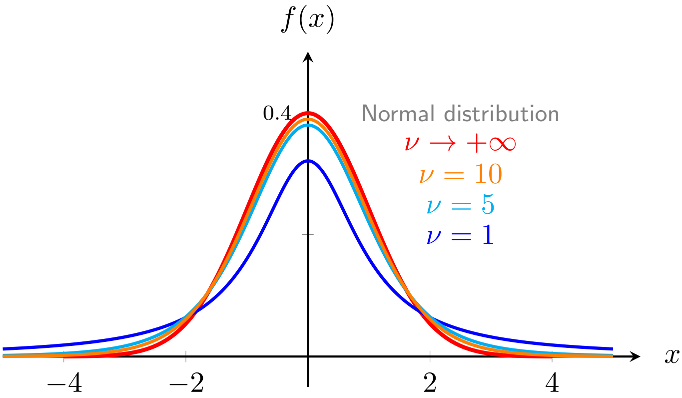

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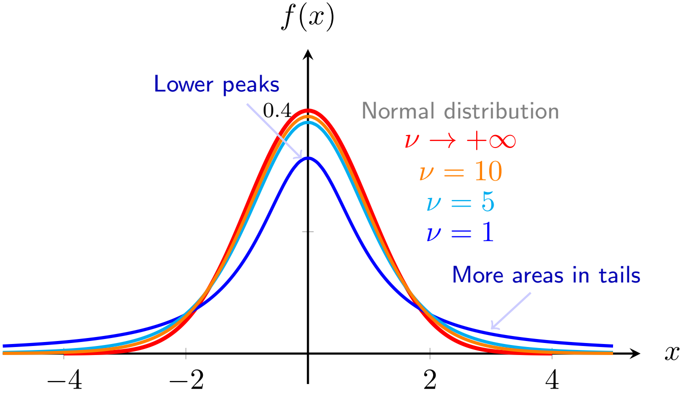


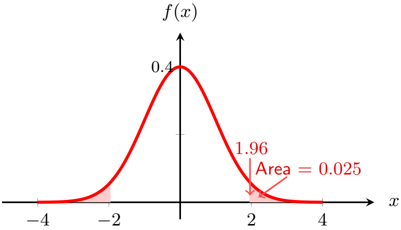
Student t-distribution

95% Confidence Interval

For the population mean μ (given) and standard deviation σ (given or not?)

- If population standard deviation σ is known

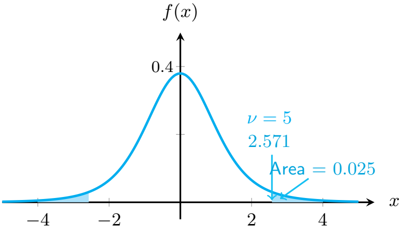
$$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$



Standard normal distribution

- If σ is unknown, using sample standard deviation s instead

$$\bar{x} \pm ? \times \frac{s}{\sqrt{n}}$$



Student t-distribution

Development

- The t-statistic depends on the type of test, but for a one-sample t-test:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

  - $\bar{x}$ sample mean
  - μ population mean
  - s sample standard deviation
  - n sample size
- The t-statistic quantifies the difference relative to variability in the data.
- (Interpretation) A high absolute value of t (larger than the critical value from the t-table) suggests a statistically significant difference.
- The problem of small sample size!

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2. Use the t-test formula since the population standard deviation is not known:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{32 - 30}{6/\sqrt{6}} = \frac{2}{6/2.449} \approx 0.816$$

- $\bar{x} = 32$ (sample mean)
- $s = 6$ (sample standard deviation)
- $n = 6$ (sample size)
- $\sigma = 30$ (population mean)

Small sample sizes

- Degrees of freedom for a t-test:

$$\nu = \underbrace{n}_{\text{sample size}} - 1 = 6 - 1 = 5$$

- t-distributions with ν degrees of freedom at a 95% confidence interval (two-tailed)

$\nu = 1$	$\nu = 5$	$\nu = 10$	$\nu \to +\infty$
12.706	2.571	2.228	1.960

- The critical t-value

$$t_{\nu,(1-0.95)/2} = t_{5,0.025} = 2.571$$

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❸ Decision rule at a 95% confidence interval

- Reject H_0 if $|t| > 2.571$.
- Otherwise, fail to reject H_0.

❹ Interpretation

- The test statistic $|t| = 0.816 < 2.571$.
- Thus, we fail to reject the null hypothesis.
- There is not enough evidence to conclude that the **average daily energy consumption differs from the company's claim of 30 kWh.**

Teaching Concept

Method

use math
use figures
use examples
use data
use codes
use latex to create all examples

Thanks for your attention!

Any Questions?

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