Memory-Efficient Hankel Tensor Factorization for Extreme Missing Traffic Data Imputation

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Introduction

- Motivation: Spatiotemporal traffic flow reconstruction with sparse data.
  - Example: Speed field reconstruction.

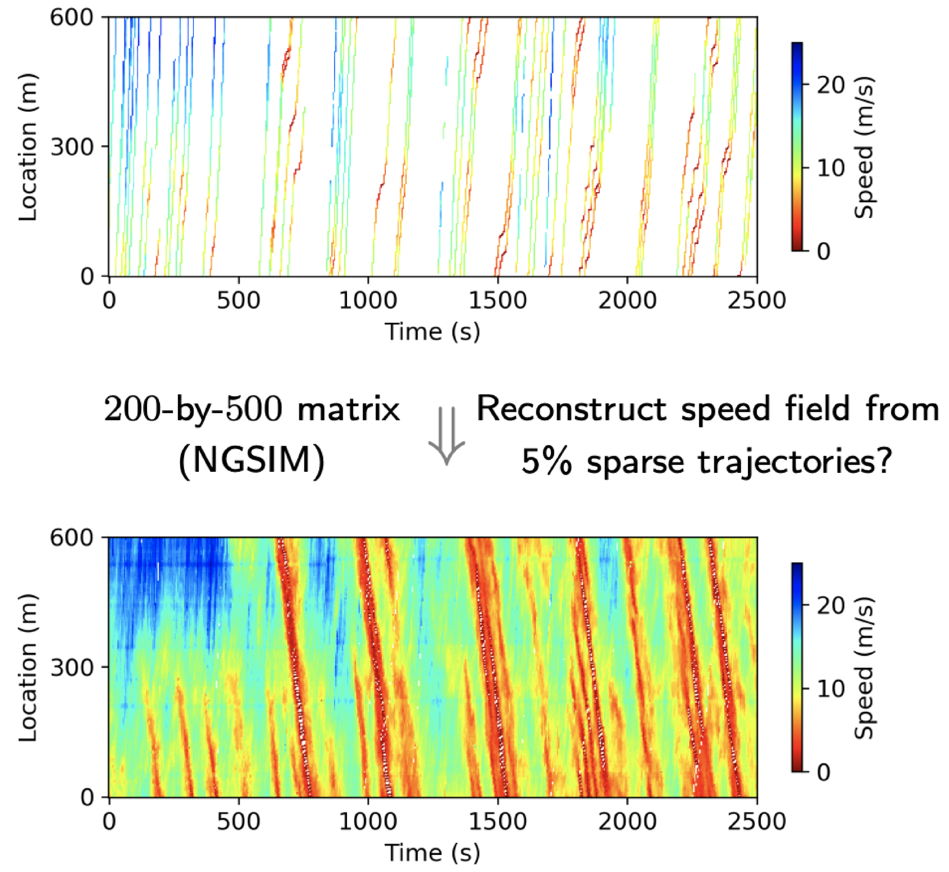


Fig. 1: Speed field reconstruction probem in vehicular traffic flow.

- Challenges: How to characterize both spatial and temporal dependencies on sparse traffic data?
- Tensor factorization: Factorizing the tensor $\boldsymbol{\mathcal{Y}}\in\mathbb{R}^{M\times N\times T}$ into the combination of three rank-R factor matrices (i.e., low-dimensional latent factors), formally,

$$\boldsymbol{\mathcal{Y}}\approx\sum_{r=1}^{R}\boldsymbol{u}_r\otimes\boldsymbol{v}_r\otimes\boldsymbol{x}_r\quad\text{or}\quad y_{i,j,t}\approx\sum_{r=1}^{R}u_{i,r}v_{j,r}x_{t,r}\tag{1}$$

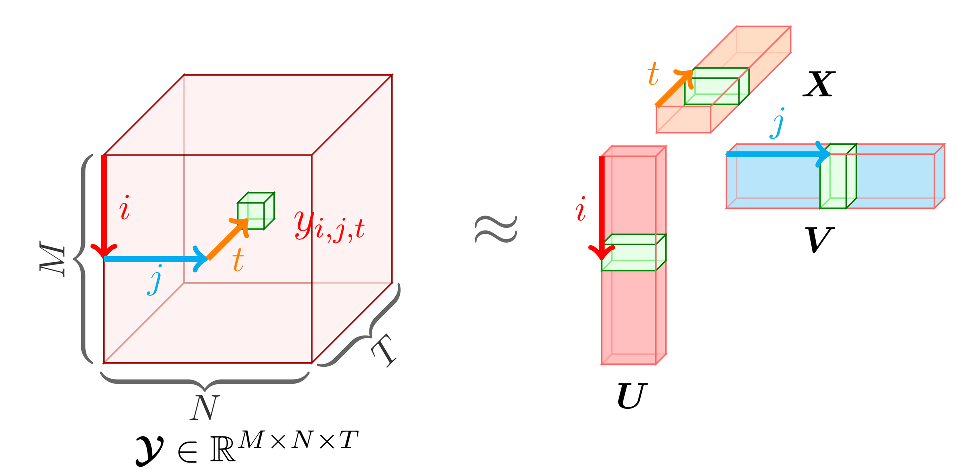


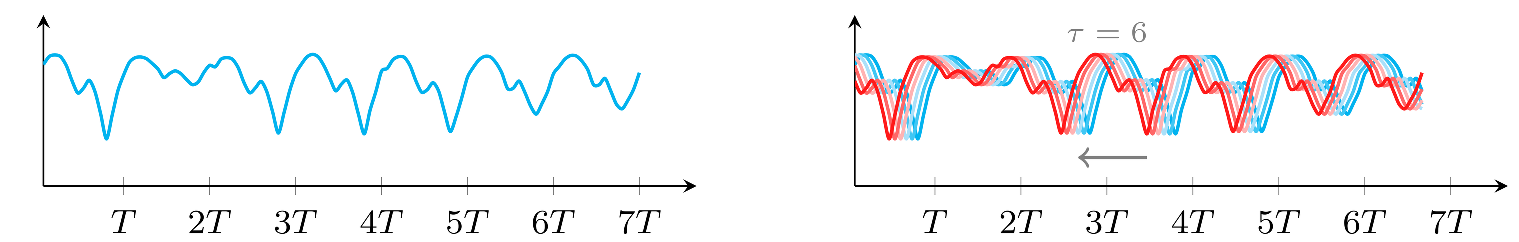
Fig. 2: Illustration of CP tensor factorization.

Summary

- This study advances a memory-efficient Hankel tensor factorization (HTF) model specially designed to handle the extreme missing traffic data imputation.
- We introduce a novel Hankel indexing mechanism that eliminates the need for explicit construction of a Hankel tensor yet preserves the properties of the Hankel structure.
- The proposed HTF employs a convolutional matrix factorization formula for each slice of the Hankel tensor, in which the formula stems from the tensor-train decomposition.
- We validate the HTF model's effectiveness on both speed field reconstruction and extreme missing traffic data imputation tasks. The results underline the effectiveness and significance of convolutional parameterization in HTF, outperforming several state-of-the-art (SOTA) baseline models.

Methodology

- Main assumptions: (1) Traffic data show strong spatial and temporal dependencies; (2) Factorization on the Hankel matrix/tensor achieves automatic spatiotemporal modeling.
- Hankel matrix: For any univariate traffic time series (e.g., traffic speed):



(a) Traffic time series. (b) Hankel matrix.

Fig. 3: Building Hankel matrix on the traffic time series data with the window length $\tau\in\mathbb{N}^{+}$.

- Hankel tensor: On spatiotemporal traffic data $\boldsymbol{X}\in\mathbb{R}^{N\times T}$, we have

$$\boldsymbol{\mathcal{X}}\triangleq\mathcal{H}_{\tau_1,\tau_2}(\boldsymbol{X})\tag{2}$$

of size $(N-\tau_1+1)\times\tau_1\times(T-\tau_2+1)\times\tau_2$, see Fig. 4 for illustration.

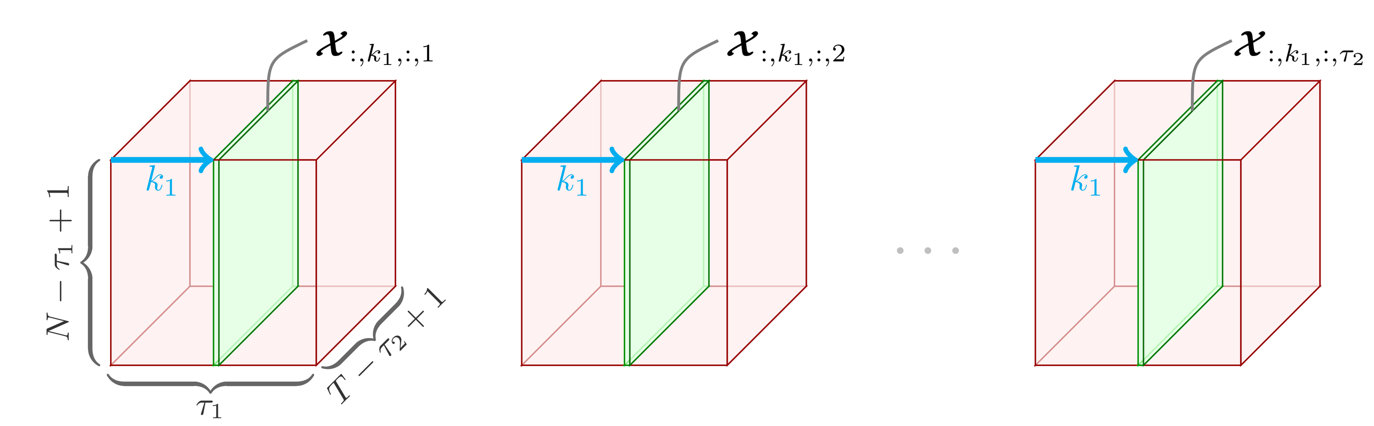


Fig. 4: Fourth-order Hankel tensor built on $\boldsymbol{X}$ with the window lengths $\tau_1,\tau_2\in\mathbb{N}^{+}$.

- Hankel indexing: Sampling function for the Hankel tensor:

$$\theta_{k_1,k_2}(\boldsymbol{X})\triangleq[\mathcal{H}_{\tau_1,\tau_2}(\boldsymbol{X})]_{:,k_1,:,k_2}\tag{3}$$

referring to as the tensor slice with $k_1\in\{1,\ldots,\tau_1\}$, $k_2\in\{1,\ldots,\tau_2\}$. This is important for developing memory-efficient algorithms.

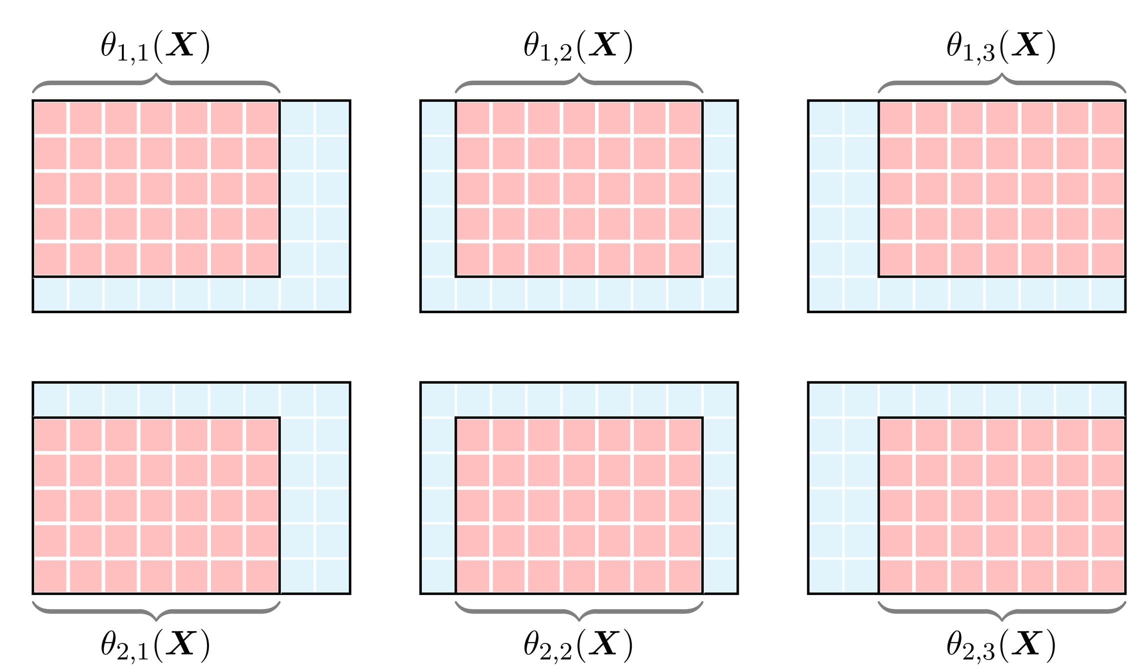


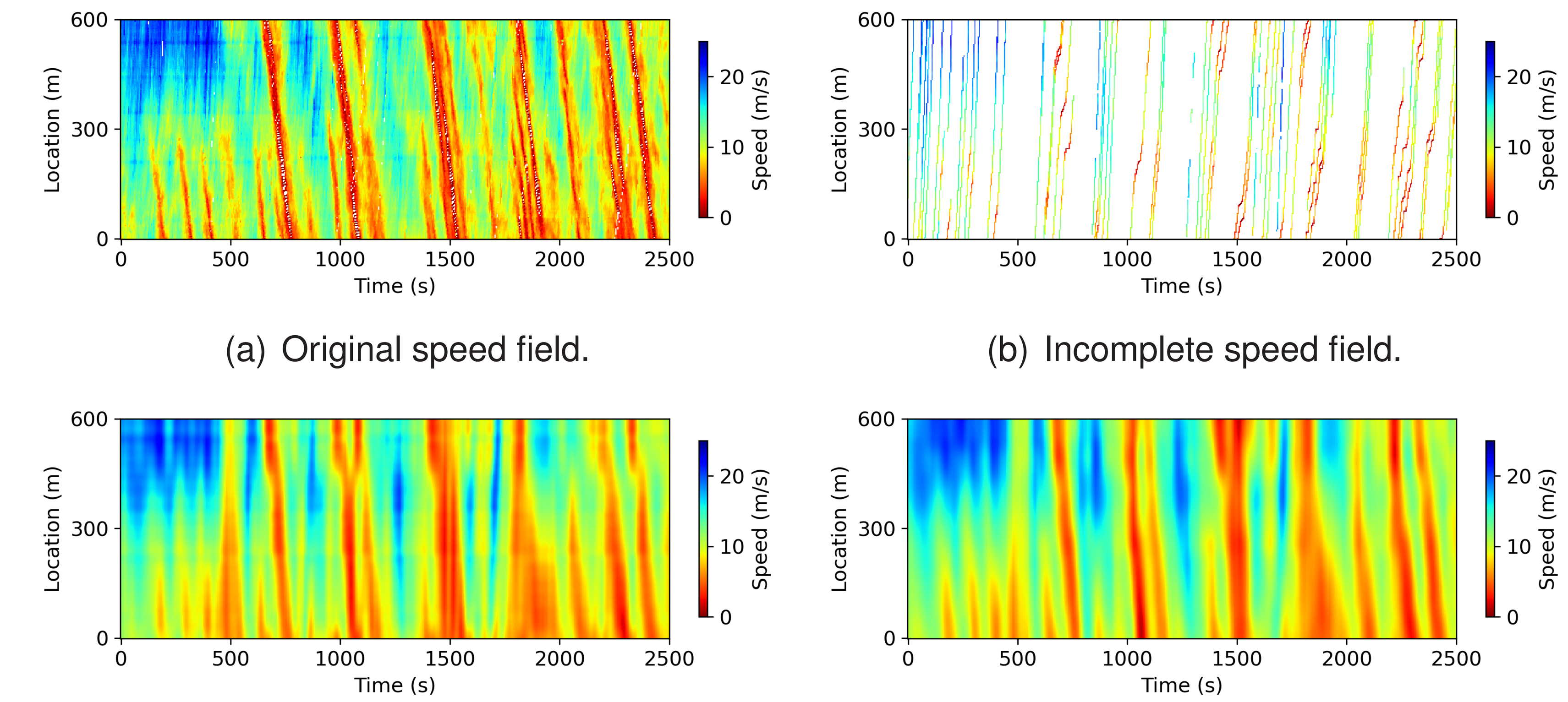
Fig. 5: Hankel tensor built on $\boldsymbol{X}$ with the window lengths $\tau_1,\tau_2\in\mathbb{N}^{+}$.

- Hankel tensor factorization (HTF): (circular convolution $\star_{\text{row}}$)

$$\begin{aligned}\min_{\boldsymbol{Q},\boldsymbol{S},\boldsymbol{U},\boldsymbol{V}}\ &\frac{1}{2}\sum_{k_1,k_2}\left\|\mathcal{P}_{\Omega_{k_1,k_2}}\left(\theta_{k_1,k_2}(\boldsymbol{Y})-(\boldsymbol{Q}\star_{\text{row}}\boldsymbol{s}_{k_1}^{\top})(\boldsymbol{U}\star_{\text{row}}\boldsymbol{v}_{k_2}^{\top})^{\top}\right)\right\|_F^2\\&+\frac{\rho}{2}\left(\|\boldsymbol{Q}\|_F^2+\|\boldsymbol{S}\|_F^2+\|\boldsymbol{U}\|_F^2+\|\boldsymbol{V}\|_F^2\right)\end{aligned}\tag{4}$$

Experiments & Results

- NGSIM speed field reconstruction $\boldsymbol{Y}\in\mathbb{R}^{N\times T}$:
  - $N=200$ (i.e., 3-meter spatial resolution).
  - $T=500$ (i.e., 5-second time resolution).
  - Reconstruction on the 95% masked trajectories.



(a) Original speed field. (b) Incomplete speed field.

(c) HTF with tensor-train (RMSE = 2.51 m/s). (d) HTF with convolution (RMSE = 2.42 m/s).

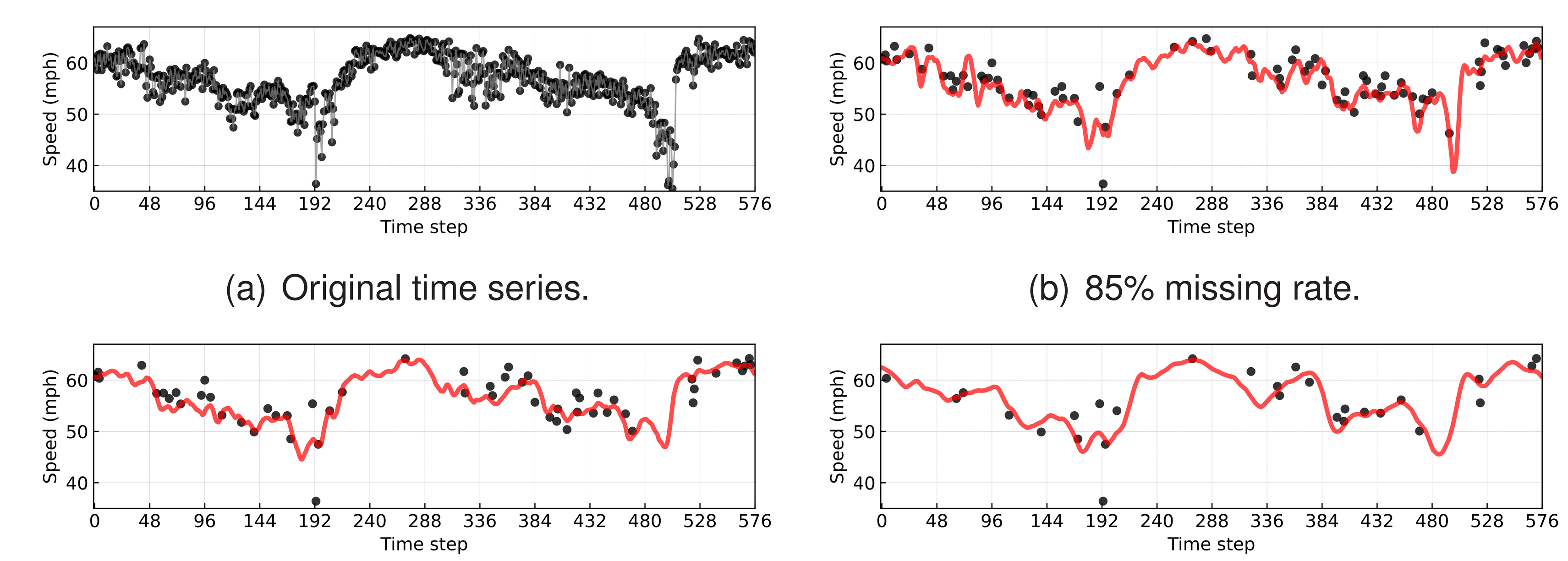
Fig. 6: Speed field reconstruction with HTF models.

- Seattle freeway traffic speed imputation:
  - 323 detectors & 8,064 time steps (i.e., 5-minute resolution & 4-week data).

Table 1: Performance comparison (in MAPE/RMSE) for imputation tasks on the Seattle freeway traffic speed dataset.

Missing rate	HTF (convolution)	HTF (tensor-train)	LATC	LRTC-TNN	LCR-2D	BTMF
80%	**6.21/3.88**	8.75/5.16	6.50/4.00	6.97/4.24	6.75/4.15	6.85/4.17
85%	**6.51/4.06**	9.86/5.76	6.90/4.21	7.43/4.43	7.31/4.38	7.36/4.42
90%	**6.98/4.30**	9.24/5.36	7.47/4.51	8.19/4.81	7.96/4.71	8.13/4.79
95%	**8.02/4.84**	9.89/5.70	8.75/5.05	9.60/5.55	9.78/5.39	9.63/5.48

- HTF with convolutional decomposition performs better than tensor-train decomposition.
- HTF outperforms some SOTA methods such as LATC, LCR-2D, and BTMF.



(a) Original time series. (b) 85% missing rate.

(c) 85% missing rate. (d) 85% missing rate.

Fig. 7: Reconstructed time series achieved by HTF on the Seattle freeway traffic speed data. This example corresponds to the detector #1 and 5th-6th days. Black dot: observations; Red curves: imputed values.