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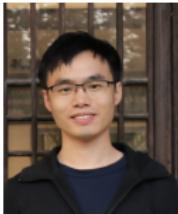


Low-Rank Matrix and Tensor Methods for Spatiotemporal Traffic Data Modeling

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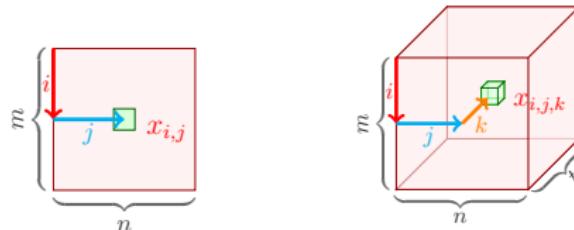
Co-supervisor
Prof. Lijun Sun

Outline

- **Spatiotemporal Traffic Data**
- **Spatiotemporal Traffic Data Imputation**
 - Laplacian convolutional representation
 - Hankel tensor factorization
- **Sparse Traffic Forecasting**
- **Dynamic Pattern Discovery**
- **Conclusion**

Matrix & Tensor

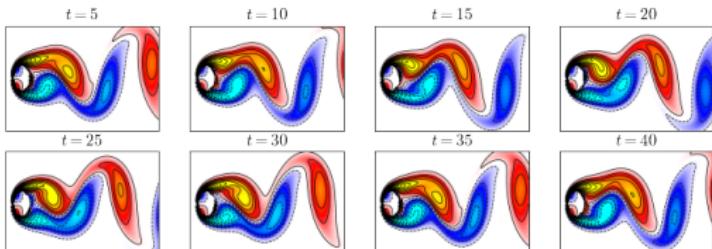
- What is tensor? $\mathbf{X} \in \mathbb{R}^{m \times n}$ vs. $\mathcal{X} \in \mathbb{R}^{m \times n \times t}$



- Tensors are everywhere!



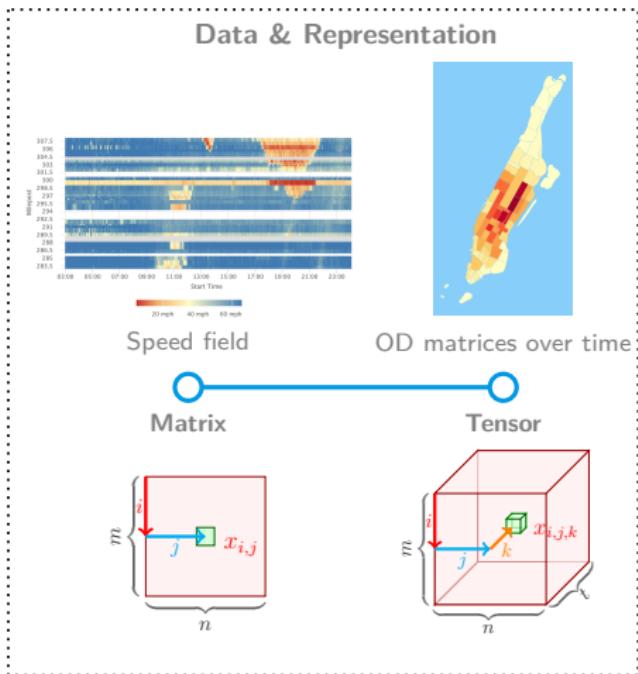
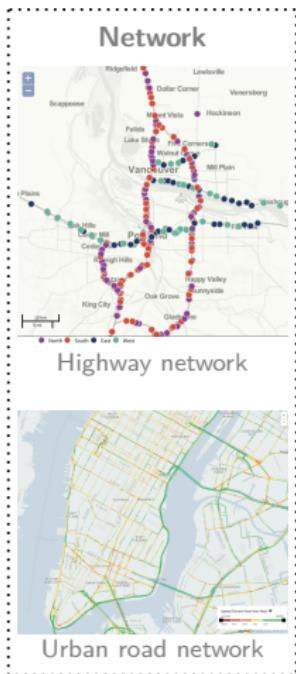
Color image with
RGB channels



Dynamical system (fluid flow)

Spatiotemporal Traffic Data

- Spatiotemporal traffic data are indeed matrices or tensors.



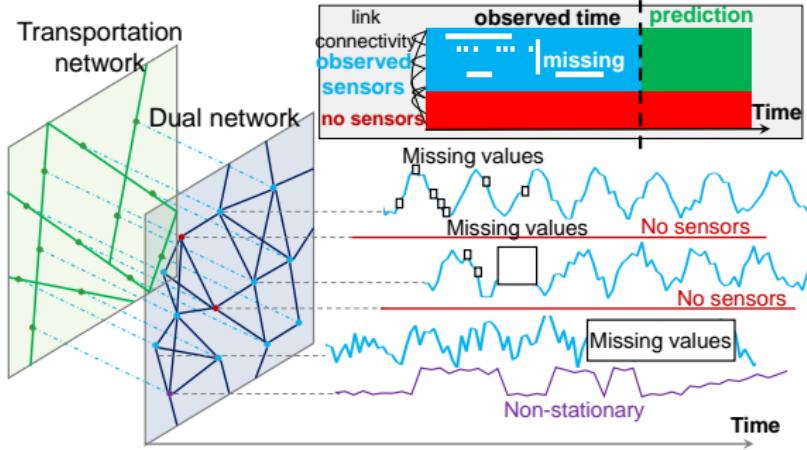
Spatiotemporal Traffic Data Imputation

- ① X. Chen, Z. Cheng, N. Saunier, L. Sun (2022). Laplacian convolutional representation for traffic time series imputation. arXiv preprint arXiv:2212.01529.
(Under 1st review at IEEE Transactions on Signal Processing)
- ② X. Chen Z. Cheng, L. Sun, N. Saunier (2023). Memory-efficient Hankel tensor factorization for extreme missing traffic data imputation. (coming soon)

GitHub repositories:

- **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (960+ stars & 270+ forks)
<https://github.com/xinchen/transdim>
- **awesome-latex-drawing**: Academic drawing examples in LaTeX. (1,000+ stars & 140+ forks)
<https://github.com/xinchen/awesome-latex-drawing>

Laplacian Convolutional Representation



Motivation:

- How to characterize both global and local trends in sparse traffic data?

Laplacian Convolutional Representation

Reformulate Laplacian regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.

Undirected and circulant graph

Modeling

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Laplacian kernel: $\ell = (2, -1, 0, 0, -1)^\top$.
- Define Laplacian kernel:

$$\ell \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \dots, -1}_{\tau}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$.

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|L\mathbf{x}\|_2^2 = \frac{1}{2} \|\ell \star \mathbf{x}\|_2^2 = \frac{1}{2T} \|\mathcal{F}(\ell) \circ \mathcal{F}(\mathbf{x})\|_2^2$$

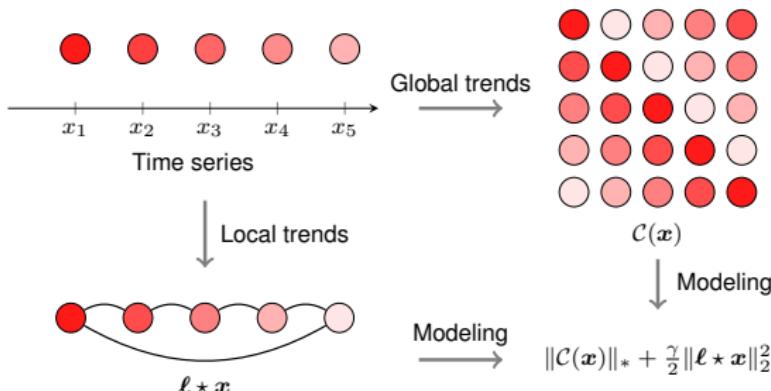
Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\mathbf{y} \in \mathbb{R}^T$ with observed index set Ω , LCR utilizes circulant matrix and Laplacian kernel to characterize **global and local trends** in time series, respectively, i.e.,

$$\begin{aligned} & \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{x}) \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

where $\mathcal{C} : \mathbb{R}^T \rightarrow \mathbb{R}^{T \times T}$ denotes the circulant operator. $\|\cdot\|_*$ denotes the nuclear norm of matrix, namely, the sum of singular values.



Laplacian Convolutional Representation

- LCR model:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{x}) \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

- Augmented Lagrangian function:

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = & \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell \star \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 \\ & + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2 \end{aligned}$$

where $\mathbf{w} \in \mathbb{R}^T$ is the Lagrange multiplier, and $\langle \cdot, \cdot \rangle$ denotes the inner product.

- The ADMM scheme:

$$\left\{ \begin{array}{l} \mathbf{x} := \arg \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) \\ \mathbf{z} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) \\ = \frac{1}{\lambda + \eta} \mathcal{P}_\Omega(\lambda \mathbf{x} + \mathbf{w} + \eta \mathbf{y}) + \frac{1}{\lambda} \mathcal{P}_\Omega^\perp(\lambda \mathbf{x} + \mathbf{w}) \\ \mathbf{w} := \mathbf{w} + \lambda(\mathbf{x} - \mathbf{z}) \end{array} \right.$$

Laplacian Convolutional Representation

- Optimize \mathbf{x} via fast Fourier transform (FFT in $\mathcal{O}(T \log T)$ time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell \star \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2 \\ &= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - (\lambda \hat{\mathbf{z}} - \hat{\mathbf{w}}) \oslash (\lambda \hat{\ell}^* \circ \hat{\ell} + \lambda \mathbb{I}_T)\|_2^2\end{aligned}$$

where $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\}$ refers to $\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$ in the frequency domain.

ℓ_1 -norm Minimization in Complex Space (Liu & Zhang'22)

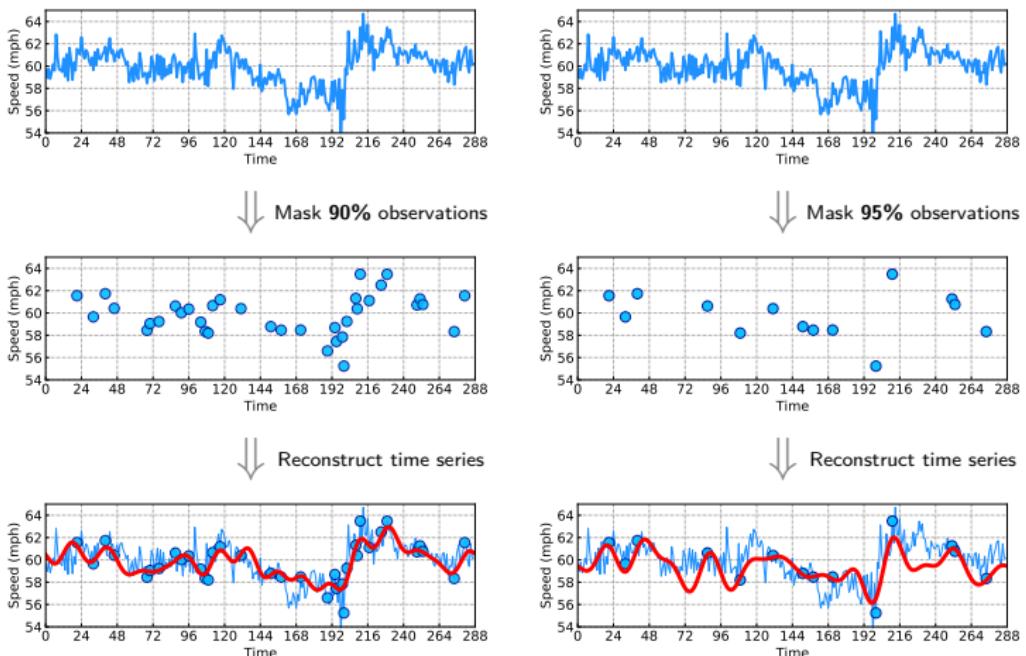
For any ℓ_1 -norm minimization problem in complex space:

$$\min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{h}}\|_2^2$$

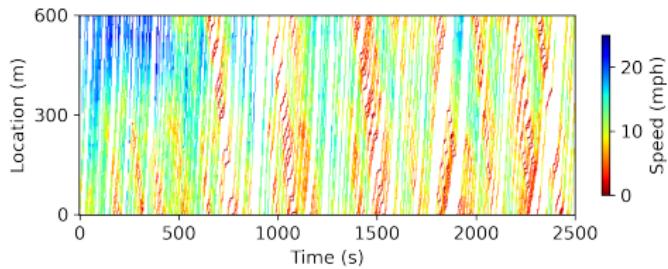
with complex-valued $\hat{\mathbf{x}}, \hat{\mathbf{h}} \in \mathbb{C}^T$, element-wise, the solution is given by

$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|} \cdot \max\{0, |\hat{h}_t| - T/\lambda\}, t = 1, \dots, T.$$

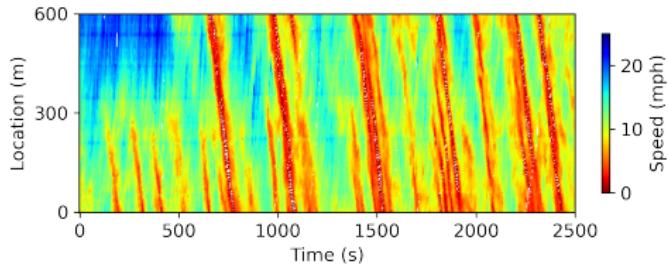
Laplacian Convolutional Representation



Hankel Tensor Factorization



200-by-500 matrix
(NGSIM) \Downarrow Reconstruct speed field from
20% sparse trajectories?



Motivation:

- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal local dependencies?

Hankel Tensor Factorization

- Hankel matrix

- Given $\mathbf{y} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\mathbf{y}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4 \times 2}$$

- On time series $\mathbf{y} = (y_1, y_2, \dots, y_5)^\top$ with $\tau = 2$:

$$\mathcal{H}_\tau(\mathbf{y}) = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \\ y_3 & y_4 \\ y_4 & y_5 \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\implies \hat{\mathbf{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \\ \hat{y}_5 \end{bmatrix} = \mathcal{H}_\tau^{-1} \left(\begin{bmatrix} v_1 x_1 & v_1 x_2 \\ v_2 x_1 & v_2 x_2 \\ v_3 x_1 & v_3 x_2 \\ v_4 x_1 & v_4 x_2 \end{bmatrix} \right) = \begin{bmatrix} v_1 x_1 \\ (v_1 x_2 + v_2 x_1)/2 \\ (v_2 x_2 + v_3 x_1)/2 \\ (v_3 x_2 + v_4 x_1)/2 \\ v_4 x_2 \end{bmatrix}$$

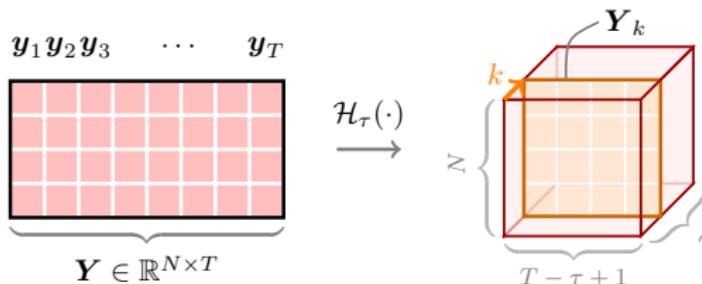
- Automatic **temporal** modeling

Hankel Tensor Factorization

Hankel Tensor Factorization

- Hankelization from $\mathbf{Y} \in \mathbb{R}^{N \times T}$ to Hankel tensor $\mathcal{H}_\tau(\mathbf{Y})$.

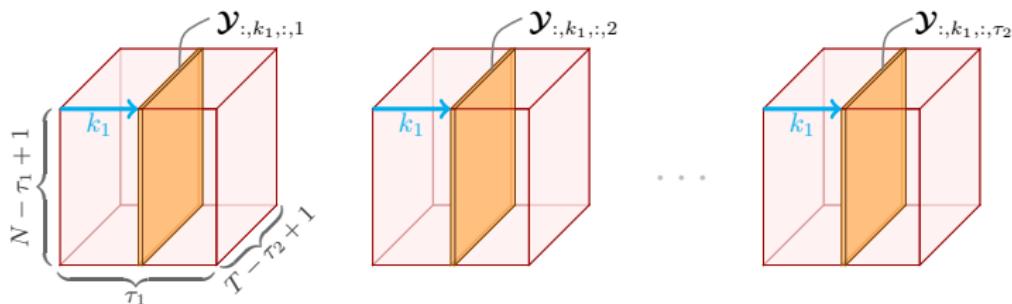
- Tensor size: $N \times (T - \tau + 1) \times \tau$;
- Slices: $\mathbf{Y}_k = \begin{bmatrix} | & | & | \\ \mathbf{y}_k & \mathbf{y}_{k+1} & \cdots & \mathbf{y}_{T-\tau+k} \\ | & | & | \end{bmatrix}, k = 1, 2, \dots, \tau$;
- Slice size: $N \times (T - \tau + 1)$.



- Automatic **temporal** modeling

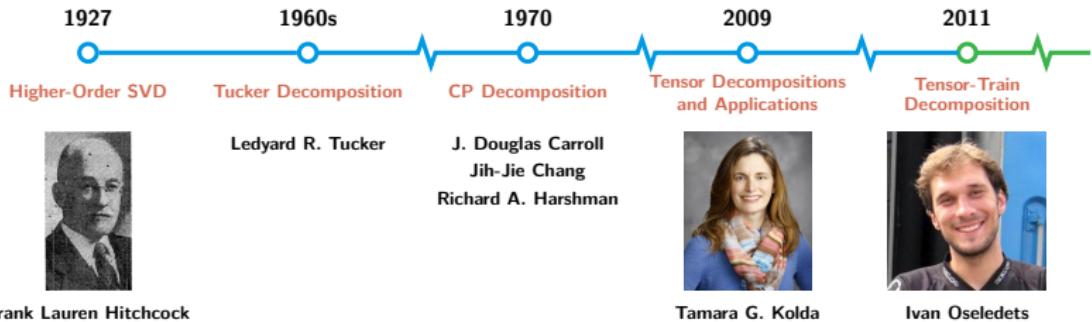
Hankel Tensor Factorization

- Hankelization from $\mathbf{Y} \in \mathbb{R}^{N \times T}$ to $\mathcal{Y} \triangleq \mathcal{H}_{\tau_1, \tau_2}(\mathbf{Y})$ (Hankel tensor).
 - Tensor size: $(N - \tau_1 + 1) \times \tau_1 \times (T - \tau_2 + 1) \times \tau_2$;
 - Slice: $\mathcal{Y}_{:, k_1, :, k_2}, \forall k_1, k_2$;
 - Slice size: $(N - \tau_1 + 1) \times (T - \tau_2 + 1)$.

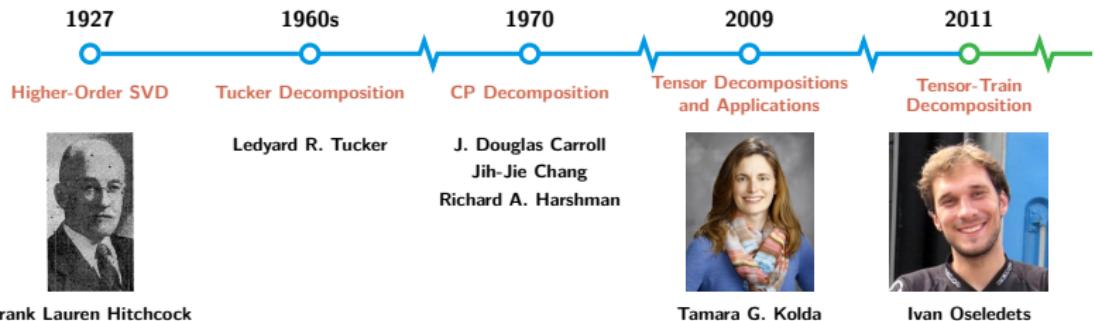


- Automatic **spatial** and **temporal** modeling

- Revisit tensor factorization (TF)



- Revisit tensor factorization (TF)



Frank Lauren Hitchcock



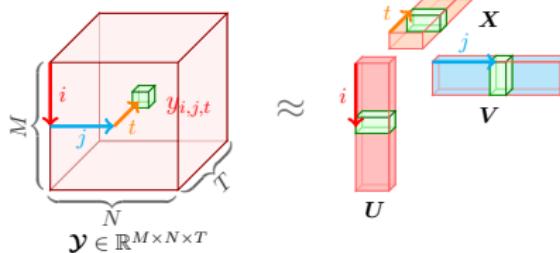
Tamara G. Kolda



Ivan Oseledets

- **CP decomposition:** Factorize \mathcal{Y} into the combination of rank- R factor

$$\text{matrices, i.e., } \mathcal{Y} \approx \sum_{r=1}^R \mathbf{u}_r \otimes \mathbf{v}_r \otimes \mathbf{x}_r.$$



Hankel Tensor Factorization

- Hankel tensor factorization:

$$\min_{\mathbf{Q}, \mathbf{S}, \mathbf{U}, \mathbf{V}} \frac{1}{2} \left\| \mathcal{P}_{\dot{\Omega}} \left(\mathcal{H}_{\tau_1, \tau_2}(\mathbf{Y}) - \sum_{r=1}^R \mathbf{q}_r \otimes \mathbf{s}_r \otimes \mathbf{u}_r \otimes \mathbf{v}_r \right) \right\|_F^2$$

- Advantage/disadvantage:

- ✓ Automatic spatial and temporal modeling
- ✗ High memory consumption

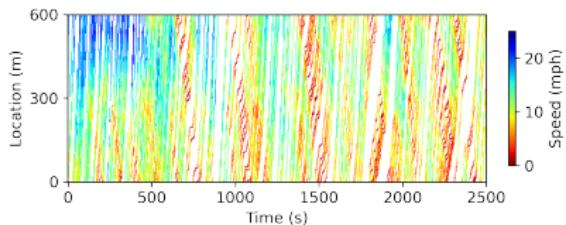
- Space complexity:

$$\mathcal{O}(\tau_1 \tau_2 (N - \tau_1 + 1)(T - \tau_2 + 1))$$

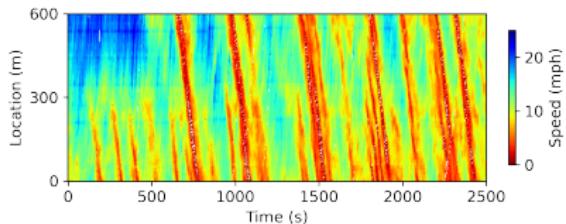
- **(Contribution)** Reduce the space complexity to $\mathcal{O}(NT)$.

Hankel Tensor Factorization

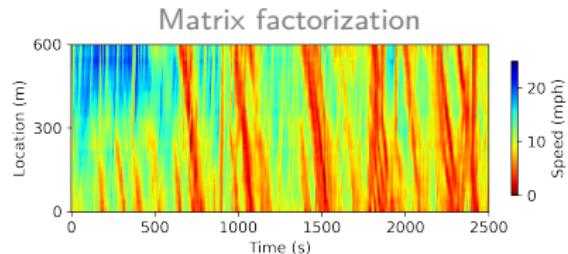
Which Model Is Better?



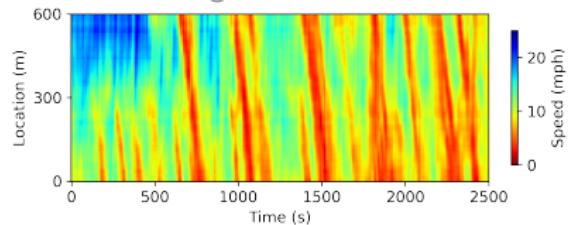
Sparse speed field



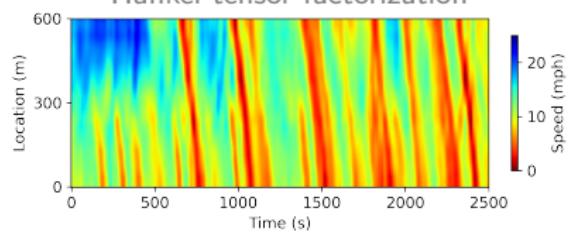
Ground truth speed field



Matrix factorization



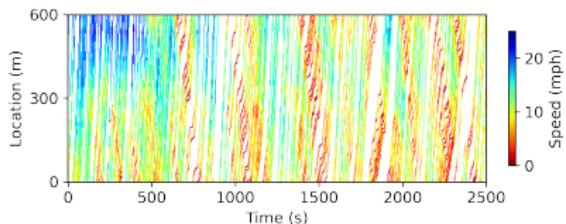
Smoothing matrix factorization



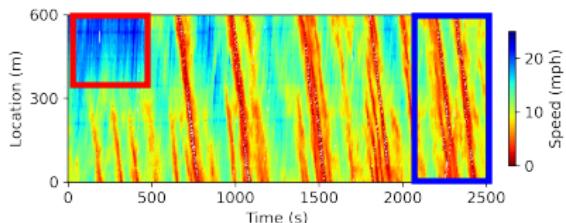
Hankel tensor factorization

Hankel Tensor Factorization

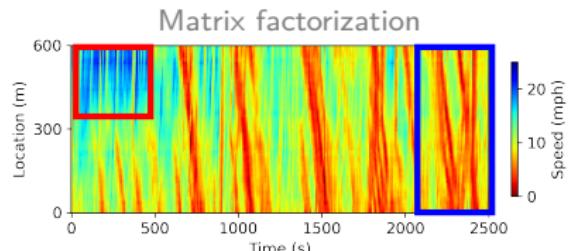
Which Model Is Better?



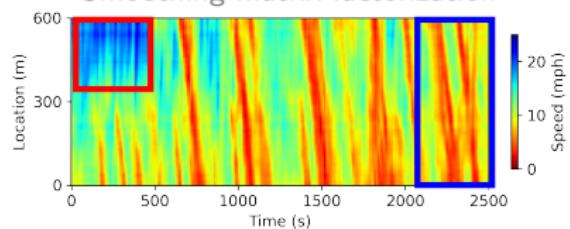
Sparse speed field



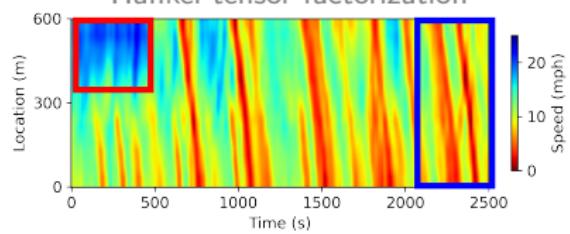
Ground truth speed field



Matrix factorization



Smoothing matrix factorization



Hankel tensor factorization

Sparse Traffic Forecasting

- ③ X. Chen, L. Sun (2022). Bayesian temporal factorization for multidimensional time series prediction. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44 (9): 4659–4673.
 - 100+ citations on Google Scholar
 - ESI highly cited paper (top 1%)
 - ESI hot paper (top 0.1%)
- ④ X. Chen, C. Zhang, X.-L. Zhao, N. Saunier, L. Sun (2022). Nonstationary temporal matrix factorization for sparse traffic time series forecasting. *arXiv preprint arXiv:2203.10651*.
(Under 2nd review at *Transportation Research Part C: Emerging Technologies*)

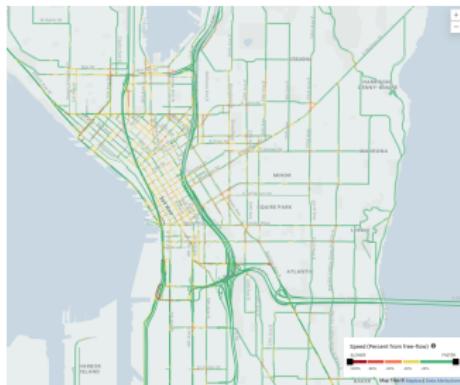
Sparse Traffic Forecasting

Motivation:

- Uber (hourly) movement speed data¹



NYC movement



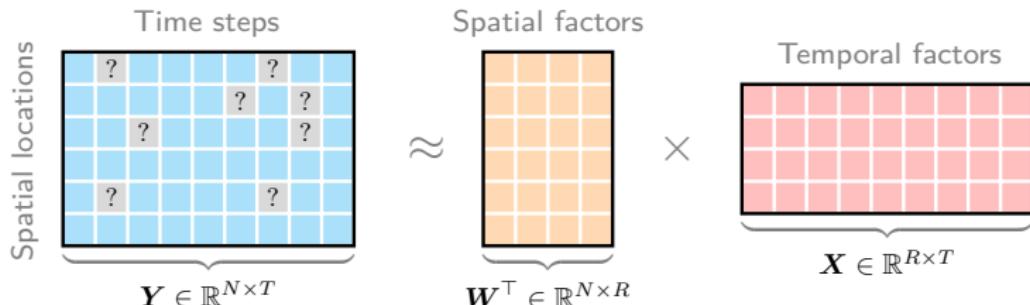
Seattle movement

- The average speed on a given road segment for each hour of each day.
- Hourly speeds are computed when road segments have 5+ unique trips.
- **Issue:** insufficient sampling of ridesharing vehicles on the road network.

¹<https://movement.uber.com/>

Sparse Traffic Forecasting

- Spatiotemporal data can be reconstructed by low-dimensional latent factors!



- MF optimization problem

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \left\| \mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) \right\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)$$

with factor matrices \mathbf{W} and \mathbf{X} .

- Objective function $f(\mathbf{W}, \mathbf{X})$ or f ;
- Rank $R \in \mathbb{N}^+$ ($R < \min\{N, T\}$).

Sparse Traffic Forecasting

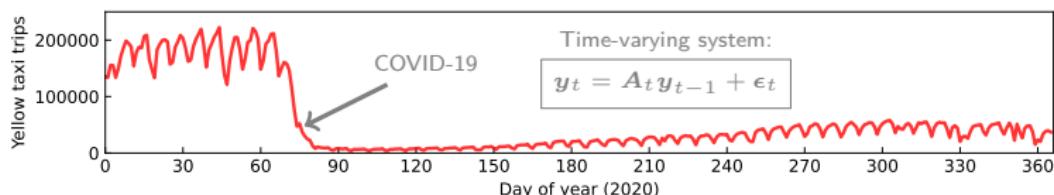
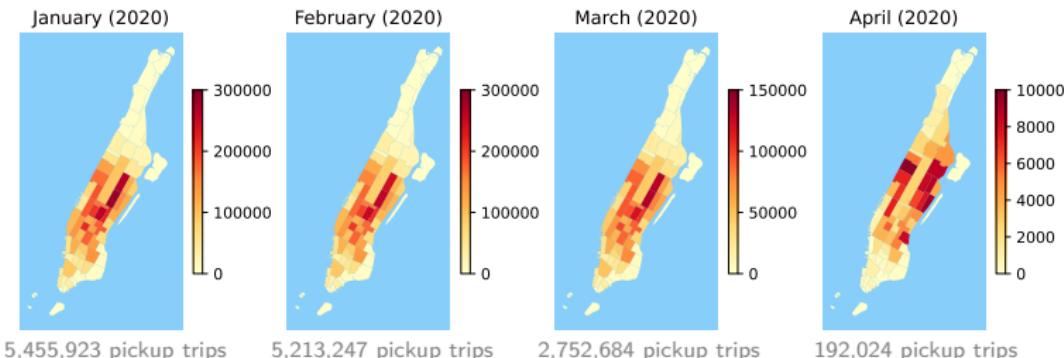
Dynamic Pattern Discovery

- ⑤ X. Chen, C. Zhang, X. Chen, N. Saunier, L. Sun (2022). Discovering dynamic patterns from spatiotemporal data with time-varying low-rank autoregression. arXiv preprint arXiv:2211.15482.
(Under 2nd review at IEEE Transactions on Knowledge and Data Engineering)

Dynamic Pattern Discovery

Motivation:

- NYC (yellow) taxi data²



- How to characterize the dynamic patterns?

²<https://www1.nyc.gov/site/tlc/about/tlc-trip-record-data.page>

Dynamic Pattern Discovery

- Given a sequence of spatiotemporal measurements
 $\mathbf{y}_t \in \mathbb{R}^N$, $t = 1, 2, \dots, T$

$$\min_{\{\mathbf{A}_t\}} \underbrace{\frac{1}{2} \sum_t \|\mathbf{y}_t - \mathbf{A}_t \mathbf{y}_{t-1}\|_2^2}_{\text{Time-varying autoregression}}$$

[Over-parameterization] $\mathcal{O}(N^2(T-1))$ parameters vs. $\mathcal{O}(NT)$ data.

- (Ours)** Parameterize coefficients via TF:

$$\min_{\mathbf{W}, \mathbf{G}, \mathbf{V}, \mathbf{x}} \underbrace{\frac{1}{2} \sum_t \left\| \mathbf{y}_t - \mathbf{W} \mathbf{G} (\mathbf{x}_t^\top \otimes \mathbf{V})^\top \mathbf{y}_{t-1} \right\|_2^2}_{\text{Let } \mathbf{A}_t = \mathbf{G} \times_1 \mathbf{W} \times_2 \mathbf{V} \times_3 \mathbf{x}_t^\top \text{ be the TF}}$$

- Alternating minimization (Let f be the obj.)

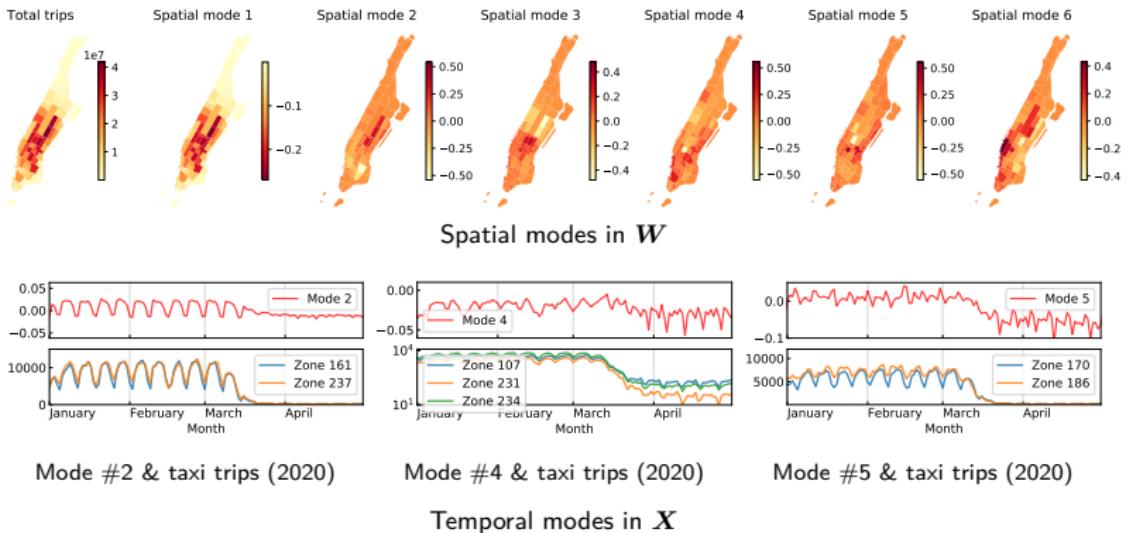
$$\begin{aligned} \mathbf{W} &:= \{\mathbf{W} \mid \frac{\partial f}{\partial \mathbf{W}} = \mathbf{0}\} & \mathbf{G} &:= \{\mathbf{G} \mid \frac{\partial f}{\partial \mathbf{G}} = \mathbf{0}\} \\ \mathbf{V} &:= \{\mathbf{V} \mid \frac{\partial f}{\partial \mathbf{V}} = \mathbf{0}\} & \mathbf{x}_t &:= \{\mathbf{x}_t \mid \frac{\partial f}{\partial \mathbf{x}_t} = \mathbf{0}\} \end{aligned}$$

- Solve each subproblem by **conjugate gradient** or **least squares**.

- Time-varying autoregression with TF

$$\min_{\mathbf{W}, \mathbf{G}, \mathbf{V}, \mathbf{x}} \frac{1}{2} \sum_t \left\| \mathbf{y}_t - \mathbf{W} \mathbf{G} (\mathbf{x}_t^\top \otimes \mathbf{V})^\top \mathbf{y}_{t-1} \right\|_2^2$$

- NYC taxi dataset (pickup)



- Produce interpretable patterns and identify the changing point of system (mainly due to COVID-19).

Conclusion

Prior Works

- ⑥ X. Chen, M. Lei, N. Saunier, L. Sun (2022). Low-rank autoregressive tensor completion for spatiotemporal traffic data imputation. *IEEE Transactions on Intelligent Transportation Systems*, 23 (8): 12301–12310.
- ⑦ X. Chen, Y. Chen, N. Saunier, L. Sun (2021). Scalable low-rank tensor learning for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*, 129: 103226.
- ⑧ X. Chen, J. Yang, L. Sun (2020). A nonconvex low-rank tensor completion model for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*, 117: 102673.
- ⑨ X. Chen, Z. He, Y. Chen, Y. Lu, J. Wang (2019). Missing traffic data imputation and pattern discovery with a Bayesian augmented tensor factorization model. *Transportation Research Part C: Emerging Technologies*, 104: 66-77.
- ⑩ X. Chen, Z. He, L. Sun (2019). A Bayesian tensor decomposition approach for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*, 98: 73-84.

References

A short list:

- [Liu & Zhang'22]



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Thanks for your attention!

Any Questions?

Slides: https://xinychen.github.io/slides/traffic_data_modeling.pdf

About me:

- 🏠 Homepage: <https://xinychen.github.io>
- ✉️ Google Scholar: [user=mCrW04wAAAAJhl](#) (690 citations)
- ⌚ GitHub: <https://github.com/xinychen> (3.2k+ stars)
- 💻 Blog: <https://medium.com/@xinyu.chen> (70k+ views)
- ✉️ How to reach me: chenxy346@gmail.com