POLYTECHNIQUE MONTRÉAL
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IVADO

Matrix and Tensor Models for Spatiotemporal Traffic Data Imputation and Forecasting

Ph.D. Defense

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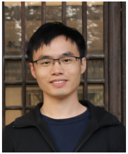

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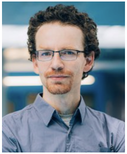

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Outline

- **Background**
- **Literature Review**
  - Tensor Factorization
  - Tensor Factorization (TF)
- **Nonstationary Temporal Matrix Factorization**
- **Low-Rank Autoregressive Tensor Completion**
- **Laplacian Convolutional Representation**
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  - Reformulate Laplacian Regularization
  - Traffic Time Series Imputation
- **Hankel Tensor Factorization**
  - Motivation
  - Hankel Structure
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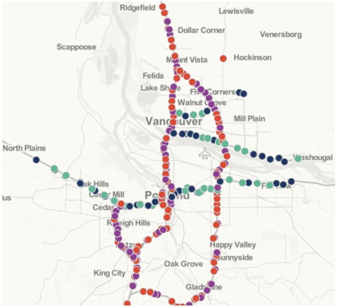
- **Conclusion**

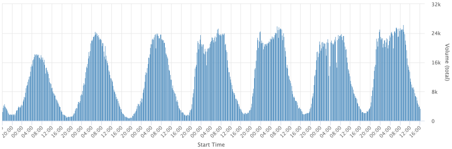
Multivariate Traffic Time Series

Many spatiotemporal traffic time series data are in the form of **matrix**.

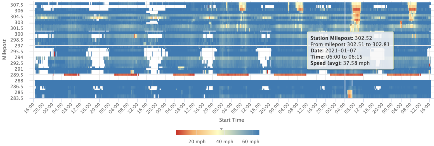
- **Example**: Portland highway traffic data[1].



Highway network & sensor locations



Traffic volume



Traffic speed

- $\boldsymbol{X} \in \mathbb{R}^{N \times T}$ with N spatial locations $\times$ T time steps
- Traffic volume/speed shows strong spatial/temporal dependencies

[1] https://portal.its.pdx.edu/home

Multiple Data Behaviors

Spatiotemporal traffic data are time series, but they involve multiple data behaviors.

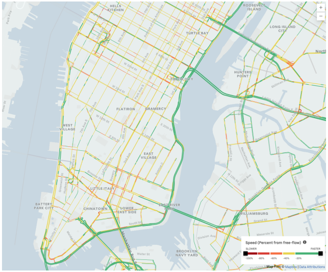
- Incompleteness & sparsity
- High-dimensionality
- Multidimensionality
- Noises & outliers
- Time-varying behavior
- Nonstationarity
- ......

In addition, spatiotemporal correlations are also very important.

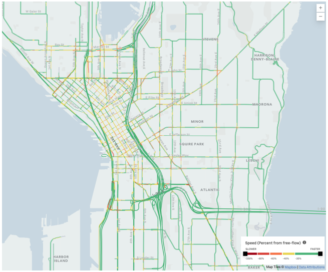
Multiple Data Behaviors

Sparsity & high-dimensionality

- Uber (hourly) movement speed data[2]



NYC movement



Seattle movement

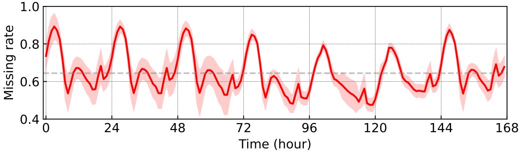
- The average speed on a given road segment for each hour of each day.
- Hourly speeds are computed when road segments have 5+ unique trips.
- **Issue**: insufficient sampling of ridesharing vehicles on the road network.

[2] https://movement.uber.com/

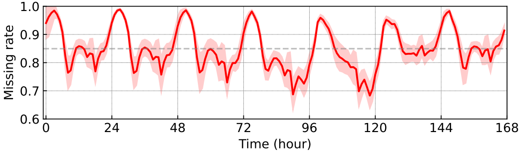
Multiple Data Behaviors

Sparsity & high-dimensionality

- **NYC** movement speed data (2019)
  - **98,210** road segments & 8,760 time steps (hours)
  - Overall missing rate: **64.43%**

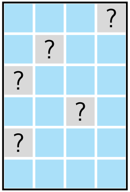


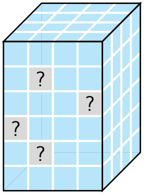
- **Seattle** movement speed data (2019)
  - **63,490** road segments & 8,760 time steps (hours)
  - Overall missing rate: **84.95%**



Problem Formulation

- **Objective A:** Given a multivariate time series data like $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ or a multidimensional time series data like $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{M \times N \times T}$, impute the missing values of the data.



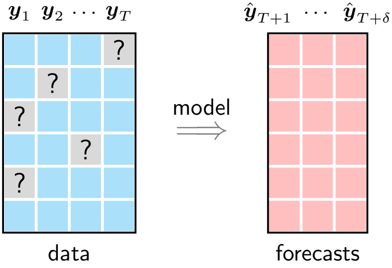


[Q]

- How to reconstruct missing values from observed data?
- How to make use of spatiotemporal correlations?
- How to make use of traffic time series dynamics?

Problem Formulation

- **Objective C-1:** Given a partially observed data $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ consisting of time series $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_T \in \mathbb{R}^N$, forecast data points $\boldsymbol{y}_{T+\delta}, \delta \in \mathbb{N}^+$.

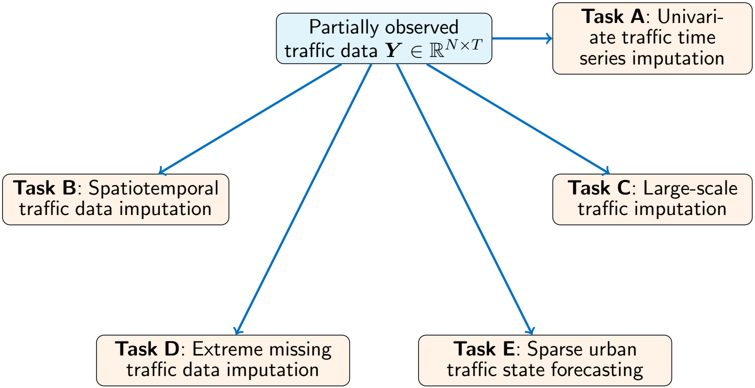


[Q]

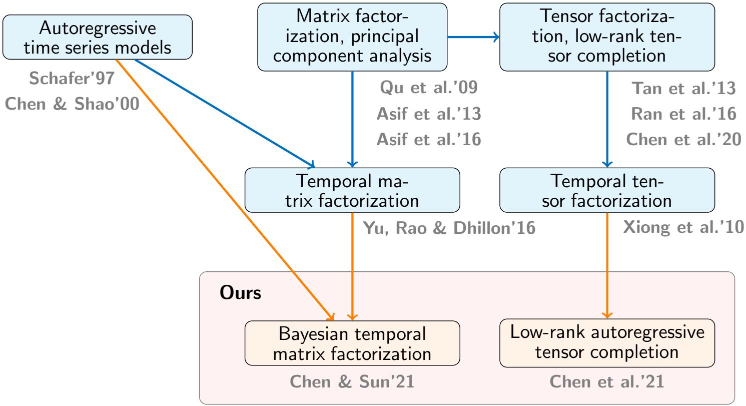
- How to learn from *high-dimensional* and *sparse* data?
- How to model *nonstationarity* in time series?
- How to perform forecasting on these time series?

Whole Picture

We are working on **spatiotemporal traffic data modeling**.



Spatiotemporal Traffic Data Imputation



Spatiotemporal Traffic Forecasting

Large-scale traffic data, e.g., PeMS traffic speed data and Uber movement speed data

High-dimensional forecasting

- Transfer learning — Mallick et al.'21
- Temporal matrix factorization — Chen & Sun'21
- High-dimensional vector autoregressive model — Wang et al.'21

Multidimensional forecasting

- Matrix autoregressive model — Chen, Xiao & Yang'21; Hsu, Huang & Tsay'21
- Autoregressive tensor factorization — Takeuchi, Kashima & Ueda'17

Forecasting on incomplete data

- AR predictor — Anava, Hazan & Zeevi'15
- Temporal matrix factorization — Yu, Rao & Dhillon'16; Chen & Sun'21
- Temporal tensor factorization — Xiong et al.'10; Chen & Sun'21

Low-Rank Tensor Models

- Low-rank matrix/tensor completion

Candès & Recht'09: Convex nuclear norm minimization for matrix completion.

$$\min_{\boldsymbol{X}} \; \|\boldsymbol{X}\|_*$$
$$\text{s.t. } \mathcal{P}_\Omega(\boldsymbol{X}) = \mathcal{P}_\Omega(\boldsymbol{Y})$$

Cai, Candès & Shen'10: Singular value thresholding algorithm.

$$\begin{cases} \boldsymbol{X}^\ell = \mathcal{D}_\tau\left(\boldsymbol{Z}^{\ell-1}\right) \\ \boldsymbol{Z}^\ell = \boldsymbol{Z}^{\ell-1} + \delta_\ell \mathcal{P}_\Omega\left(\boldsymbol{Y} - \boldsymbol{X}^\ell\right) \end{cases}$$

Zhang et al.'12: Nonconvex truncated nuclear norm minimization.

Liu et al.'13: Convex nuclear norm minimization for tensor completion.

$$\min_{\boldsymbol{\mathcal{X}}} \; \|\boldsymbol{\mathcal{X}}\|_*$$
$$\text{s.t. } \mathcal{P}_\Omega(\boldsymbol{\mathcal{X}}) = \mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}})$$

Lu, Peng & Wei'19: Tensor nuclear norm induced by linear transform.

Tensor Factorization

- Revisit tensor factorization (TF)

1927 — Higher-Order SVD

Frank Lauren Hitchcock

1960s — Tucker Decomposition

Ledyard R. Tucker

1970 — CP Decomposition

J. Douglas Carroll
Jih-Jie Chang
Richard A. Harshman

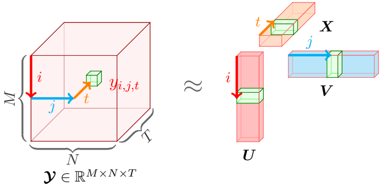
2009 — Tensor Decompositions and Applications

Tamara G. Kolda

2011 — Tensor-Train Decomposition

Ivan Oseledets

- **CP tensor factorization**: Factorize $\boldsymbol{\mathcal{Y}}$ into the combination of three rank-R factor matrices (i.e., low-dimensional latent factors).



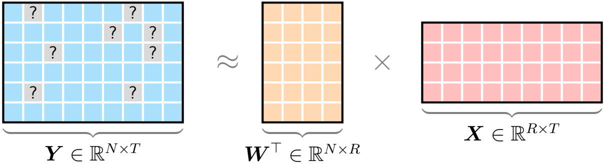
$$\begin{cases} y_{i,j,t} \approx \sum_{r=1}^{R} u_{i,r} v_{j,r} x_{t,r} \\ \boldsymbol{\mathcal{Y}} \approx \sum_{r=1}^{R} \boldsymbol{u}_r \otimes \boldsymbol{v}_r \otimes \boldsymbol{x}_r \end{cases}$$

Matrix Factorization

Low-Rank Matrix Factorization (Koren et al.'09)

Given any partially observed data matrix $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ with observed index set Ω, then the optimization problem of matrix factorization with rank R can formulated as

$$\min_{\boldsymbol{W},\boldsymbol{X}} \frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2+\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)$$



✗ Cannot capture temporal correlations.

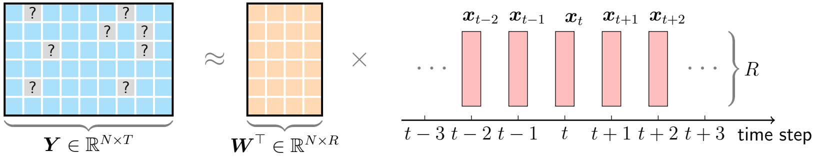
✗ Cannot perform time series forecasting.

Temporal Matrix Factorization

Temporal Matrix Factorization (Yu et al.'16; Chen & Sun'21)

Given any partially observed time series data $\boldsymbol{Y}\in\mathbb{R}^{N\times T}$ with observed index set Ω, then temporal matrix factorization assumes a dth-order vector autoregressive (VAR) process on the temporal factor matrix:

$$\min_{\boldsymbol{W},\boldsymbol{X},\{\boldsymbol{A}_k\}_{k=1}^{d}} \frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2+\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)$$
$$+\frac{\lambda}{2}\sum_{t=d+1}^{T}\left\|\boldsymbol{x}_t-\sum_{k=1}^{d}\boldsymbol{A}_k\boldsymbol{x}_{t-k}\right\|_2^2$$



✗ VAR is usually built on stationary time series (temporal factors).

Nonstationary Temporal Matrix Factorization

Rewrite VAR in the form of matrix

Temporal operators

For any multivariate time series $\boldsymbol{X} \in \mathbb{R}^{R \times T}$ with $m, d \in \mathbb{N}^{+}$, if we define temporal operators as

$$\begin{aligned}\boldsymbol{\Psi}_k \triangleq& \begin{bmatrix}\mathbf{0}_{(T-d-m)\times(d-k)} & -\boldsymbol{I}_{T-d-m} & \mathbf{0}_{(T-d-m)\times(k+m)}\end{bmatrix} \\ &+ \begin{bmatrix}\mathbf{0}_{(T-d-m)\times(d+m-k)} & \boldsymbol{I}_{T-d-m} & \mathbf{0}_{(T-d-m)\times k}\end{bmatrix} \\ &\in \mathbb{R}^{(T-d-m)\times T},\ k = 0, 1, \ldots, d\end{aligned}$$

then

$$\sum_{t=d+m+1}^{T} \|(\boldsymbol{x}_t - \boldsymbol{x}_{t-m}) - \sum_{k=1}^{d} \boldsymbol{A}_k(\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k})\|_2^2$$

$$\equiv \|\boldsymbol{X}\boldsymbol{\Psi}_0^\top - \sum_{k=1}^{d} \boldsymbol{A}_k \boldsymbol{X} \boldsymbol{\Psi}_k^\top\|_F^2 \triangleq \|\boldsymbol{X}\boldsymbol{\Psi}_0^\top - \boldsymbol{A}(\boldsymbol{I}_d \otimes \boldsymbol{X})\boldsymbol{\Psi}^\top\|_F^2$$

where $\boldsymbol{A} \triangleq \begin{bmatrix}\boldsymbol{A}_1 & \cdots & \boldsymbol{A}_d\end{bmatrix}$ and $\boldsymbol{\Psi} \triangleq \begin{bmatrix}\boldsymbol{\Psi}_1 & \cdots & \boldsymbol{\Psi}_d\end{bmatrix}$.

Nonstationary Temporal Matrix Factorization

Nonstationary temporal matrix factorization (NoTMF)

Given any partially observed time series data $\boldsymbol{Y}\in\mathbb{R}^{N\times T}$ with observed index set Ω, then we assume a season-m differencing on the latent temporal factors:

$$\begin{aligned}\min_{\boldsymbol{W},\boldsymbol{X},\{\boldsymbol{A}_k\}_{k=1}^{d}}\ &\frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2+\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)\\ &+\frac{\lambda}{2}\sum_{t=d+m+1}^{T}\left\|(\boldsymbol{x}_t-\boldsymbol{x}_{t-m})-\sum_{k=1}^{d}\boldsymbol{A}_k(\boldsymbol{x}_{t-k}-\boldsymbol{x}_{t-m-k})\right\|_2^2\end{aligned}$$

- First-order differencing $\boldsymbol{x}_t'=\boldsymbol{x}_t-\boldsymbol{x}_{t-1}$.
- Second-order differencing $\boldsymbol{x}_t''=(\boldsymbol{x}_t-\boldsymbol{x}_{t-1})-(\boldsymbol{x}_{t-1}-\boldsymbol{x}_{t-2})$.
- Twice-differenced series $\boldsymbol{x}_t''=(\boldsymbol{x}_t-\boldsymbol{x}_{t-m})-(\boldsymbol{x}_{t-1}-\boldsymbol{x}_{t-m-1})$.

✓ Stationarizing a time series with differencing can improve the prediction.[3]

[3] Stationarity and differencing: https://otexts.com/fpp2/stationarity.html

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF:

$$\begin{aligned}\min_{\boldsymbol{W},\boldsymbol{X},\boldsymbol{A}}\ &\frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2+\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)\\&+\frac{\lambda}{2}\|\boldsymbol{X}\boldsymbol{\Psi}_0^\top-\boldsymbol{A}(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^\top\|_F^2\end{aligned}$$

Alternating minimization method:

- w.r.t. $\boldsymbol{W}$:

$$\frac{\partial f}{\partial\boldsymbol{W}}=-\boldsymbol{X}\mathcal{P}_{\Omega}^\top(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})+\rho\boldsymbol{W}=\boldsymbol{0}$$

- w.r.t. $\boldsymbol{X}$:

$$\frac{\partial f}{\partial\boldsymbol{X}}=-\boldsymbol{W}\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})+\rho\boldsymbol{X}+\lambda\sum_{k=0}^{d}\boldsymbol{A}_k^\top\left(\sum_{h=0}^{d}\boldsymbol{A}_h\boldsymbol{X}\boldsymbol{\Psi}_h^\top\right)\boldsymbol{\Psi}_k=\boldsymbol{0}$$

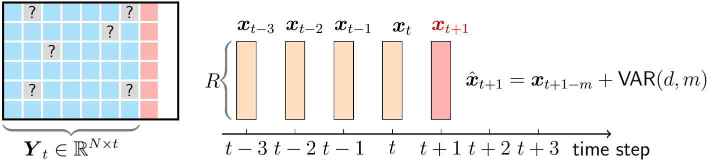
- w.r.t. $\boldsymbol{A}$:

$$\boldsymbol{A}=\boldsymbol{X}\boldsymbol{\Psi}_0^\top[(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^\top]^\dagger$$

Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

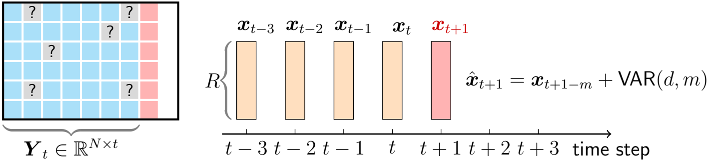
- NoTMF: Use $\boldsymbol{Y}_t$ to estimate $\{\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{A}\}$.



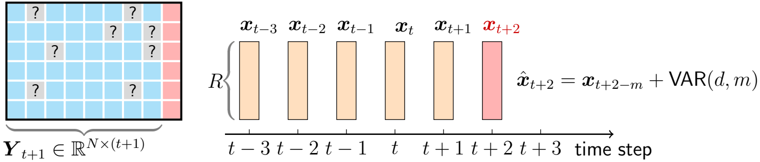
Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

- NoTMF: Use $\boldsymbol{Y}_t$ to estimate $\{\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{A}\}$.



- Online forecasting (Gultekin & Paisley'18): Fix $\boldsymbol{W}$ and use $\boldsymbol{Y}_{t+1}$ to update $\{\boldsymbol{X}, \boldsymbol{A}\}$.



Low-Rank Autoregressive Tensor Completion

Low-rank matrix completion (Candès & Recht'09)

For any partially observed data matrix $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ with observed index set Ω, then low-rank matrix completion takes the form of

$$\begin{aligned} \min_{\boldsymbol{X}} \ & \|\boldsymbol{X}\|_* \\ \text{s.t. } & \mathcal{P}_\Omega(\boldsymbol{X}) = \mathcal{P}_\Omega(\boldsymbol{Y}). \end{aligned}$$

Low-rank tensor completion (Liu et al.'13)

For any partially observed data matrix $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{M \times N \times T}$ with observed index set Ω, then low-rank matrix completion takes the form of

$$\begin{aligned} \min_{\boldsymbol{\mathcal{X}}} \ & \|\boldsymbol{\mathcal{X}}\|_* \\ \text{s.t. } & \mathcal{P}_\Omega(\boldsymbol{\mathcal{X}}) = \mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}}). \end{aligned}$$

- **Limitation:** Only cover the global consistency.
- **Comment:** For modeling spatiotemporal traffic data, local consistency (e.g., temporal correlations) is also important.

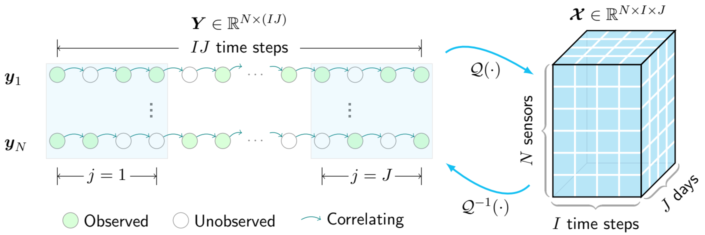
Low-Rank Autoregressive Tensor Completion

Optimization problem:

$$\min_{\boldsymbol{\mathcal{X}}} \ \|\boldsymbol{\mathcal{X}}\|_* + \frac{\lambda}{2}\sum_{n=1}^{N}\sum_{t=d+1}^{T}\left(z_{n,t} - \sum_{k=1}^{d} a_{n,k} z_{n,t-k}\right)^2$$

$$\text{s.t.} \ \begin{cases} \boldsymbol{\mathcal{X}} = \mathcal{Q}(\boldsymbol{Z}), \\ \mathcal{P}_{\Omega}(\boldsymbol{Z}) = \mathcal{P}_{\Omega}(\boldsymbol{Y}). \end{cases}$$

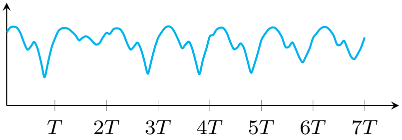
- **Advantage:** Global consistency + local consistency.



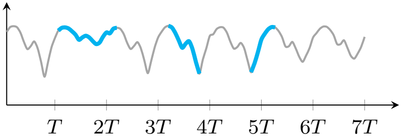
Laplacian Convolutional Representation

Motivation: Time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



- Local trends (e.g., short-term time series trends):

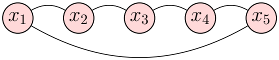


- **[Question]** How to characterize both global and local trends in sparse time series data?

Laplacian Convolutional Representation

[Local trend modeling] Reformulate temporal regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.



Undirected and circulant graph

$\xrightarrow{\text{Modeling}}$

$$\boldsymbol{L}=\begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Define Laplacian kernel:

$$\boldsymbol{\ell}\triangleq(2,-1,0,0,-1)^\top$$

$$\Downarrow$$

$$\boldsymbol{\ell}\triangleq(\underbrace{2\tau}_{\text{degree}},\underbrace{-1,\cdots,-1}_{\tau},0,\cdots,0,\underbrace{-1,\cdots,-1}_{\tau})^\top\in\mathbb{R}^T$$

for any time series $\boldsymbol{x}=(x_1,\ldots,x_T)^\top\in\mathbb{R}^T$.

- (Laplacian) Temporal regularization:

$$\mathcal{R}_\tau(\boldsymbol{x})=\frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2=\frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2$$

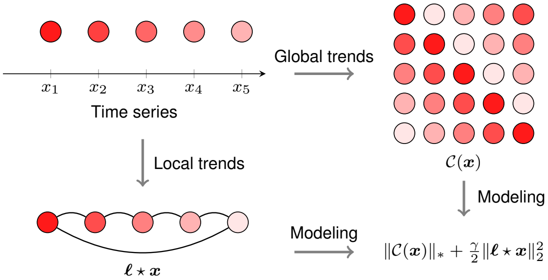
Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\boldsymbol{y} \in \mathbb{R}^T$ with observed index set Ω, LCR utilizes circulant matrix and Laplacian kernel to characterize **global and local trends** in time series, respectively, i.e.,

$$\begin{aligned} \min_{\boldsymbol{x}} \ & \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 \\ \text{s.t.} \ & \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

where $\mathcal{C}: \mathbb{R}^T \to \mathbb{R}^{T\times T}$ denotes the circulant operator. $\|\cdot\|_*$ denotes the nuclear norm of matrix, namely, the sum of singular values.



Laplacian Convolutional Representation

- Augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) = \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2 + \langle\boldsymbol{w},\boldsymbol{x}-\boldsymbol{z}\rangle + \frac{\eta}{2}\|\mathcal{P}_\Omega(\boldsymbol{z}-\boldsymbol{y})\|_2^2$$

where $\boldsymbol{w}\in\mathbb{R}^T$ is the Lagrange multiplier, and $\langle\cdot,\cdot\rangle$ denotes the inner product.

- The ADMM scheme:

$$\begin{cases}\boldsymbol{x} := \arg\min_{\boldsymbol{x}}\ \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) & \text{(Nuclear norm minimization)}\\ \boldsymbol{z} := \arg\min_{\boldsymbol{z}}\ \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) & \text{(Closed-form solution)}\\ \quad = \frac{1}{\lambda+\eta}\mathcal{P}_\Omega(\lambda\boldsymbol{x}+\boldsymbol{w}+\eta\boldsymbol{y}) + \frac{1}{\lambda}\mathcal{P}_\Omega^\perp(\lambda\boldsymbol{x}+\boldsymbol{w}) & \\ \boldsymbol{w} := \boldsymbol{w}+\lambda(\boldsymbol{x}-\boldsymbol{z}) & \text{(Standard update)}\end{cases}$$

- Optimize $\boldsymbol{x}$?

$$\|\mathcal{C}(\boldsymbol{x})\|_* = \|\mathcal{F}(\boldsymbol{x})\|_1 \qquad \& \qquad \frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 = \frac{1}{2T}\|\mathcal{F}(\boldsymbol{\ell})\circ\mathcal{F}(\boldsymbol{x})\|_2^2$$

Nuclear norm minimization $\Rightarrow$ ℓ_1**-norm minimization with FFT** (in $\mathcal{O}(T\log T)$ time).

Laplacian Convolutional Representation

- Optimize $\boldsymbol{x}$ via FFT (in $\mathcal{O}(T\log T)$ time):

$$\boldsymbol{x} := \arg\min_{\boldsymbol{x}} \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{w}/\lambda\|_2^2$$

$$\Longrightarrow \hat{\boldsymbol{x}} := \arg\min_{\hat{\boldsymbol{x}}} \|\hat{\boldsymbol{x}}\|_1 + \frac{\gamma}{2T}\|\hat{\boldsymbol{\ell}}\circ\hat{\boldsymbol{x}}\|_2^2 + \frac{\lambda}{2T}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{z}}+\hat{\boldsymbol{w}}/\lambda\|_2^2$$

where we introduce $\{\hat{\boldsymbol{\ell}},\hat{\boldsymbol{x}},\hat{\boldsymbol{z}},\hat{\boldsymbol{w}}\}\triangleq\mathcal{F}\{\boldsymbol{\ell},\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}\}$ (i.e., FFT).

ℓ_1-norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of ℓ_1-norm minimization in complex space:

$$\min_{\hat{\boldsymbol{x}}} \|\hat{\boldsymbol{x}}\|_1 + \frac{\omega}{2}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{h}}\|_2^2$$

with complex-valued vectors $\hat{\boldsymbol{x}},\hat{\boldsymbol{h}}\in\mathbb{C}^T$ and weight parameter ω, element-wise, the solution is given by

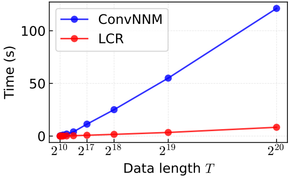
$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|}\cdot\max\{0,|\hat{h}_t|-1/\omega\}, t=1,\ldots,T.$$

Laplacian Convolutional Representation

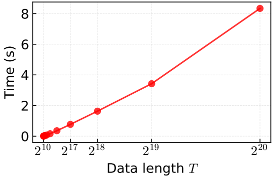
Empirical time complexity

On the synthetic data $\boldsymbol{y} \in \mathbb{R}^T$ with $T \in \{2^{10}, 2^{11}, \dots, 2^{20}\}$

- Ours: **LCR**
  - An FFT implementation in $\mathcal{O}(T \log T)$
  - The logarithmic factor $\log T$ makes the FFT highly efficient
- Baseline: **ConvNNM** (Convolution nuclear norm minimization, Liu & Zhang'23)
  - Convolution matrix $\mathcal{C}_{\tilde{\tau}}(\boldsymbol{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$ with kernel size $\tilde{\tau} \in \mathbb{N}^+$
  - Singular value thresholding in $\mathcal{O}(\tilde{\tau}^2 T)$



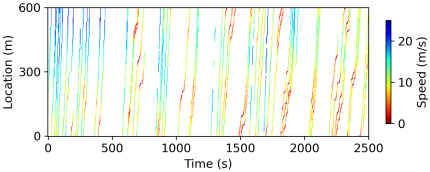
ConvNNM vs. LCR



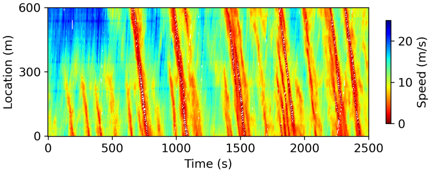
LCR

Motivation: Spatiotemporal data reconstruction

- Speed field reconstruction problem in vehicular traffic flow.



200-by-500 matrix (NGSIM) $\Downarrow$ Reconstruct speed field from 5% sparse trajectories?



- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal dependencies?

Hankel Tensor Factorization

- Hankel matrix
  - Given $\boldsymbol{x} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\boldsymbol{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4\times 2}$$

Hankel matrix (Source: Twitter)

Hankel Tensor Factorization

- Hankel matrix
  - On time series $\boldsymbol{y}=(y_1,y_2,\ldots,y_5)^\top$ with $\tau=2$:

$$\mathcal{H}_\tau(\boldsymbol{y})=\begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \\ y_3 & y_4 \\ y_4 & y_5 \end{bmatrix}\approx\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}\otimes\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Longrightarrow\quad\hat{\boldsymbol{y}}=\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \\ \hat{y}_5 \end{bmatrix}=\mathcal{H}_\tau^{-1}\left(\begin{bmatrix} v_1x_1 & v_1x_2 \\ v_2x_1 & v_2x_2 \\ v_3x_1 & v_3x_2 \\ v_4x_1 & v_4x_2 \end{bmatrix}\right)=\begin{bmatrix} v_1x_1 \\ (v_1x_2+v_2x_1)/2 \\ (v_2x_2+v_3x_1)/2 \\ (v_3x_2+v_4x_1)/2 \\ v_4x_2 \end{bmatrix}$$

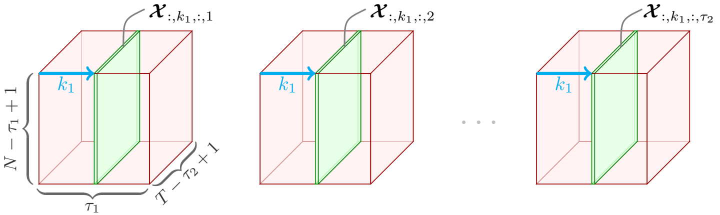
  - Automatic temporal modeling.

Hankel Tensor Factorization

- Hankel tensor: Given any matrix $\boldsymbol{X} \in \mathbb{R}^{N \times T}$, we have

$$\boldsymbol{\mathcal{X}} \triangleq \mathcal{H}_{\tau_1,\tau_2}(\boldsymbol{X})$$

  - Window lengths: $\tau_1, \tau_2 \in \mathbb{N}^+$;
  - Tensor size: $(N-\tau_1+1) \times \tau_1 \times (T-\tau_2+1) \times \tau_2$;



(Figure) 4th order Hankel tensor: A sequence of third-order tensors.

  - Slice: $\boldsymbol{\mathcal{X}}_{:,k_1,:,k_2}, \forall k_1, k_2$;
  - Slice size: $(N-\tau_1+1) \times (T-\tau_2+1)$.

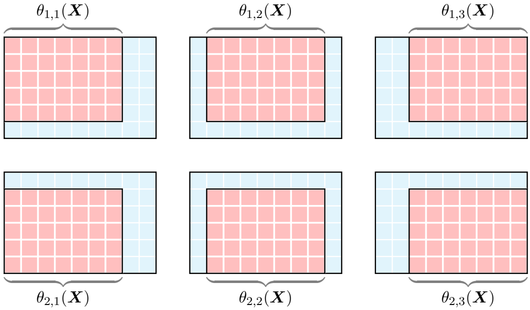
Hankel Tensor Factorization

Hankel indexing:

- Sampling function for the Hankelization:

$$\theta_{k_1,k_2}(\boldsymbol{X}) \triangleq [\mathcal{H}_{\tau_1,\tau_2}(\boldsymbol{X})]_{:,k_1,:,k_2},$$

  referring to the tensor slice with $k_1 \in \{1,\ldots,\tau_1\}$, $k_2 \in \{1,\ldots,\tau_2\}$.
- **[Importance]** Developing memory-efficient algorithms.



- Tensor slices $\theta_{k_1,k_2}(\boldsymbol{X})$ vs. data matrix $\boldsymbol{X}$

Hankel Tensor Factorization

Ours:

- Convolutional tensor decomposition (circular convolution $\star_{\text{row}}$):

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q} \star_{\text{row}} \boldsymbol{s}_{k_1}^\top)(\boldsymbol{U} \star_{\text{row}} \boldsymbol{v}_{k_2}^\top)^\top$$

Baselines:

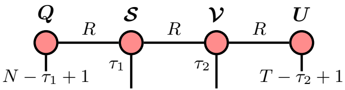
- CP tensor decomposition (Khatri-Rao product $\odot$):

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q} \odot \boldsymbol{s}_{k_1}^\top)(\boldsymbol{U} \odot \boldsymbol{v}_{k_2}^\top)^\top$$

- Tensor-train decomposition:

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q}\boldsymbol{S}_{k_1})(\boldsymbol{U}\boldsymbol{V}_{k_2})^\top$$

  - $\{\boldsymbol{S}_{k_1}, \boldsymbol{V}_{k_2}\}$ are **circulant matrices** $\Rightarrow$ convolutional decomposition
  - $\{\boldsymbol{S}_{k_1}, \boldsymbol{V}_{k_2}\}$ are **diagonal matrices** $\Rightarrow$ CP decomposition



Hankel Tensor Factorization

HTF (convolutional decomposition)

- Optimization problem:

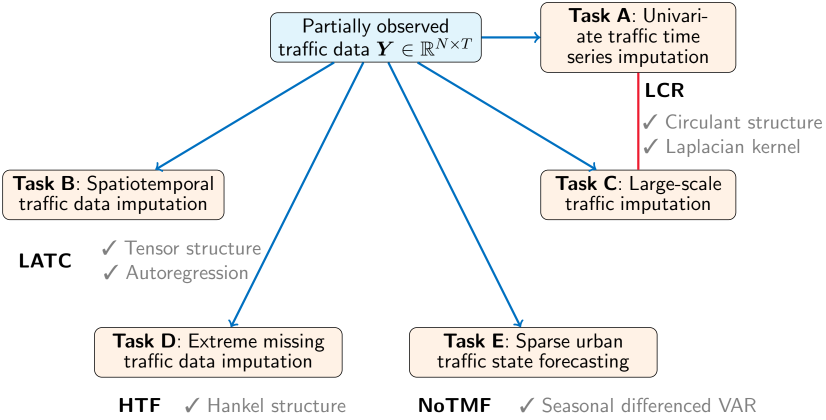
$$\min_{\boldsymbol{Q},\boldsymbol{S},\boldsymbol{U},\boldsymbol{V}}\ \frac{1}{2}\sum_{k_1,k_2}\left\|\mathcal{P}_{\Omega_{k_1,k_2}}(\theta_{k_1,k_2}(\boldsymbol{Y})-(\boldsymbol{Q}\star_{\text{row}}\boldsymbol{s}_{k_1})(\boldsymbol{U}\star_{\text{row}}\boldsymbol{v}_{k_2})^\top)\right\|_F^2 + \frac{\rho}{2}(\|\boldsymbol{Q}\|_F^2+\|\boldsymbol{S}\|_F^2+\|\boldsymbol{U}\|_F^2+\|\boldsymbol{V}\|_F^2)$$

- Alternating minimization:

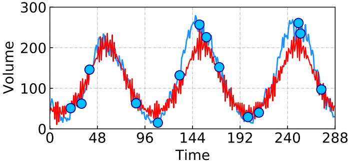
$$\begin{cases} \boldsymbol{Q} \coloneqq \{\boldsymbol{Q}\mid \dfrac{\partial f}{\partial \boldsymbol{Q}}=\boldsymbol{0}\} \\ \boldsymbol{s}_{k_1} \coloneqq \{\boldsymbol{s}_{k_1}\mid \dfrac{\partial f}{\partial \boldsymbol{s}_{k_1}}=\boldsymbol{0}\},\ k_1\in\{1,2,\ldots,\tau_1\} \\ \boldsymbol{U} \coloneqq \{\boldsymbol{U}\mid \dfrac{\partial f}{\partial \boldsymbol{U}}=\boldsymbol{0}\} \\ \boldsymbol{v}_{k_2} \coloneqq \{\boldsymbol{v}_{k_2}\mid \dfrac{\partial f}{\partial \boldsymbol{v}_{k_2}}=\boldsymbol{0}\},\ k_2\in\{1,2,\ldots,\tau_2\} \end{cases}$$

Whole Picture

We are working on **spatiotemporal traffic data modeling**.



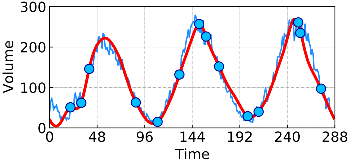
Univariate Traffic Time Series Imputation



CircNNM:

$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_*$$
$$\text{s.t.} \ \|\mathcal{P}_{\Omega}(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

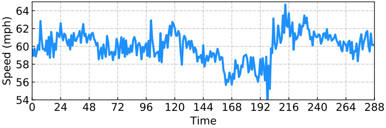
$\Downarrow$ Plus temporal regularization (TR)



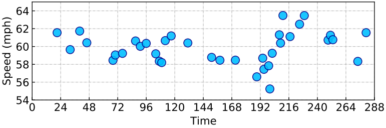
LCR:

$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2$$
$$\text{s.t.} \ \|\mathcal{P}_{\Omega}(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

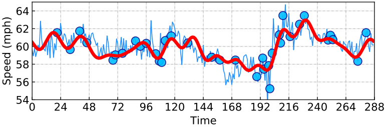
Univariate Traffic Time Series Imputation



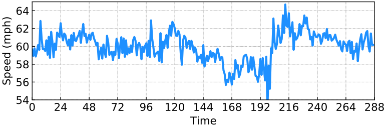
Mask **90%** observations



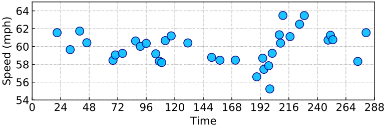
Reconstruct time series



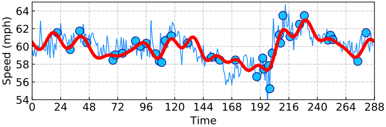
Univariate Traffic Time Series Imputation

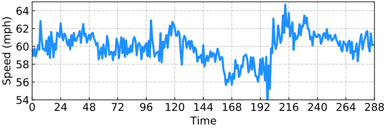


Mask **90%** observations

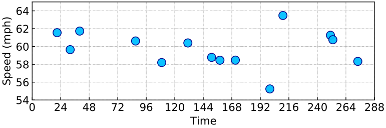


Reconstruct time series

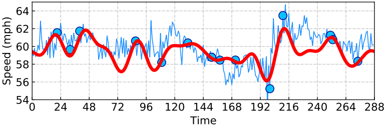




Mask **95%** observations



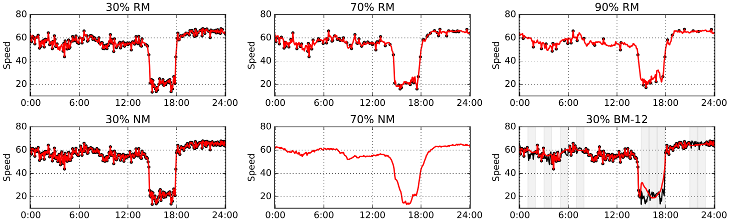
Reconstruct time series



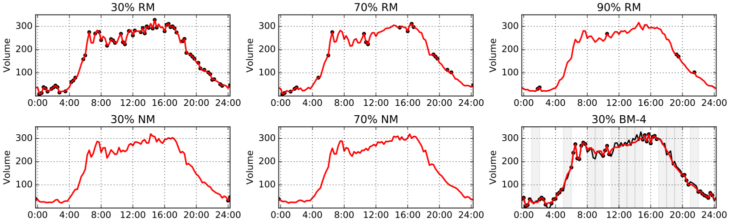
Spatiotemporal Traffic Data Imputation

LATC imputation

- Seattle freeway traffic speed data



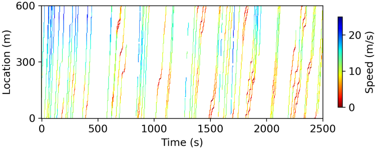
- Portland highway traffic volume data



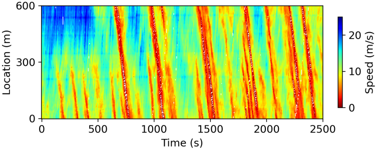
Large-Scale Traffic Data Imputation

LCR

Extreme Missing Traffic Data Imputation

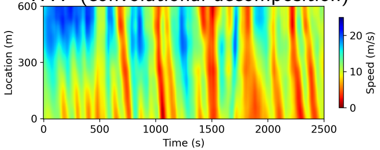


Sparse speed field



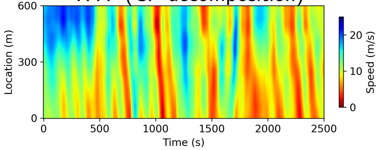
Ground truth speed field

HTF (convolutional decomposition)



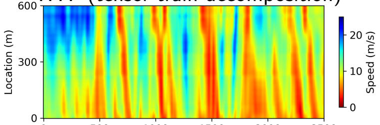
MAPE = **51.92%**

HTF (CP decomposition)



MAPE = 53.93%

HTF (tensor-train decomposition)



Sparse Urban Traffic State Forecasting

NoTMF

Conclusion

Time-Varying Autoregression:

- (Highlight) Interpretable model with tensor factorization.
  ✓ Parameter compression ✓ Pattern discovery

Laplacian Convolutional Representation:

- (Solution) Time series trend modeling in the low-rank framework?
  - Global time series trend modeling (low-rank model):

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* \\ \text{s. t.} \quad & \|\mathcal{P}_{\Omega}(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

  - Local time series trend modeling (temporal regularization):

$$\mathcal{R}_{\tau}(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2$$

- (Highlight) A unified framework with the FFT implementation.

Hankel Tensor Factorization:

- (Highlight) Memory-efficient Hankel indexing & convolutional parameterization.

Our studies:

❷ X. Chen, Z. Cheng, N. Saunier, L. Sun (2022). Laplacian convolutional representation for traffic time series imputation. arXiv preprint arXiv:2212.01529.

❸ X. Chen, L. Sun (2022). Bayesian temporal factorization for multidimensional time series prediction. IEEE Transactions on Pattern Analysis and Machine Intelligence. 44 (9): 4659-4673.

GitHub repository:

- **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (1,000 stars & 270 forks on GitHub) https://github.com/xinychen/transdim

Thank you for your attention!

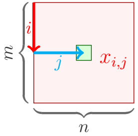
Any Questions?

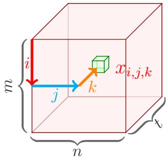
About me:

- Homepage: https://xinychen.github.io
- GitHub: https://github.com/xinychen
- How to reach me: chenxy346@gmail.com

What Is Tensors?

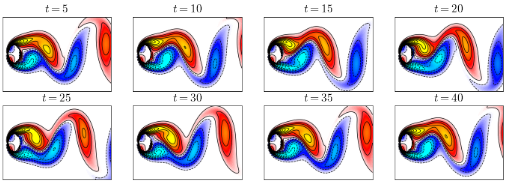
- What is tensor? $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ vs. $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{m \times n \times t}$





- Tensors are everywhere!

Color image with RGB channels



Dynamical system (fluid flow)

Appendix