Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 2: Mathematical Modeling & Engineering Problem Solving

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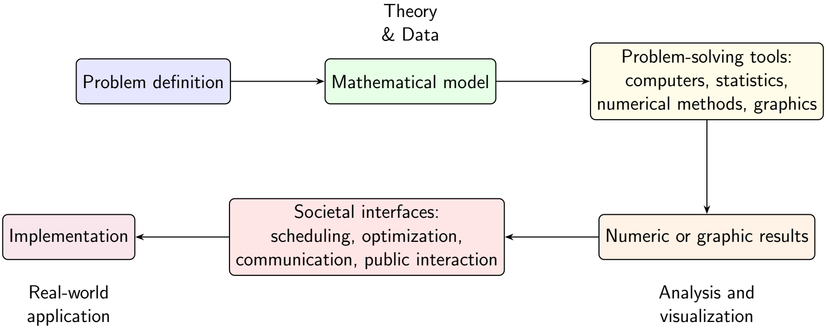
How to understand

Applied Numerical Methods for Civil Engineering?

Numerical methods are techniques by which **mathematical problems** are formulated so that they can be solved with **arithmetic operations**.

Engineering Problem Solving Process



Bungee Jumping

Engineering Task.

- A bungee jumping company needs to **predict velocity vs. time** during free fall to design safe bungee cords.
- **Key Questions**:
  - What is the **maximum velocity** reached?
    (Safe limit: **45 m/s**)
  - How long until maximum velocity?
  - What cord length is needed?

Physical Forces F_g and F_a

Two Main Forces: Physical Forces Acting on Jumper

$$\boxed{F = F_g - F_a = m \cdot g - c_d \cdot v^2}$$

- Gravity (Downward)

$$F_g = m \cdot g$$

with

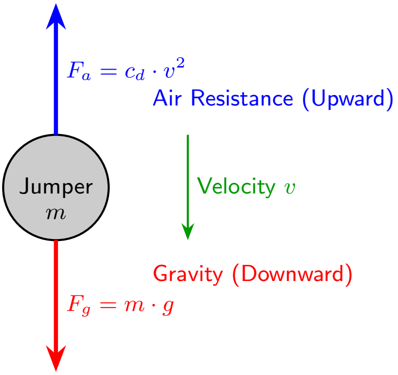
  - $m =$ mass (kg)
  - $g = 9.81\,\mathrm{m/s^2}$, gravitational acceleration

- Air Resistance (Upward)

$$F_a = c_d \cdot v^2$$

with

  - $c_d =$ drag coefficient (kg/m)
  - $v =$ velocity



Newton's Second Law

Mathematical Model - Newton's Second Law

- From $F = m \cdot a$:

$$F = \boxed{m\frac{\mathrm{d}v}{\mathrm{d}t} = m \cdot g - c_d \cdot v^2}$$

- Divide by m:

$$\underbrace{\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{c_d}{m}v^2}_{\text{Ordinary Differential Equation!!!}}$$

  in terms of the differential rate of change in velocity.

- Initial condition:

$$v(0) = 0 \quad \text{(starts from rest)}$$

- **Problem definition**: Solve the velocity of the jumper in free fall as a function of time.
- **Why Numerical Methods?**
  - Real engineering problems often **do not have simple analytical solutions**!

Euler's Method (Numerical)

Euler's Method - The Simplest Numerical Approach

- Essential idea:

  Approximate continuous change with a small discrete time step size Δt.

- Rewrite the formula of bungee jumper velocity:

$$\begin{aligned}\underbrace{v_{t+\Delta t}}_{\text{new}} &= v_t + \Delta t \cdot \frac{\mathrm{d}v_t}{\mathrm{d}t} \\ &= \underbrace{v_t}_{\text{old}} + \underbrace{\Delta t}_{\text{time step size}} \cdot \underbrace{\left(g - \frac{c_d}{m} v_t^2\right)}_{\text{acceleration}}\end{aligned}$$

  from the ordinary differential equation:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{c_d}{m} v^2$$

Euler's Method (Numerical)

Euler's Method - The Simplest Numerical Approach

- Formula of bungee jumper velocity:

$$\underbrace{v_{t+\Delta t}}_{\text{new}} = \underbrace{v_t}_{\text{old}} + \underbrace{\Delta t}_{\text{time step size}} \cdot \underbrace{\left(g - \frac{c_d}{m} v_t^2\right)}_{\text{acceleration}}$$

- Computing **bungee jumper velocity** (step-by-step):
  - Start at $t = 0$ and $v = 0$
  - Repeat across different time steps:
    - Compute **acceleration**:

$$a = g - \frac{c_d}{m} v_t^2$$

    - Update **velocity**:

$$v_{t+\Delta t} = v_t + \Delta t \cdot a$$

    - Increment time step: $t = t + \Delta t$

A Real Case

Input. Mass $m = 50\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, drag coefficient $c_d = 0.25\,\text{kg/m}$, and initial velocity $v_0 = 0$. (Given time step size $\Delta t = 1\,\text{s}$)
Output. Bungee jumper velocity v_t.

- At time $t = 1$:

$$a = g - \frac{c_d}{m} v_0^2 = 9.81 - 0.005 \times 0^2 = 9.81$$
$$v_1 = v_0 + \Delta t \cdot a = 0 + 9.81 = \mathbf{9.81}$$

- At time $t = 2$:

$$a = g - \frac{c_d}{m} v_1^2 = 9.81 - 0.005 \times 9.81^2 = 9.33$$
$$v_2 = v_1 + \Delta t \cdot a = 9.81 + 1 \times 9.33 = \mathbf{19.14}$$

- At time $t = 3$

A Real Case

Input. Mass $m = 50\,\text{kg}$, $g = 9.81\,\text{m/s}^2$, drag coefficient $c_d = 0.25\,\text{kg/m}$, and initial velocity $v_0 = 0$. (Given time step size $\Delta t = 1\,\text{s}$)
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- At time $t = 2$:

$$a = g - \frac{c_d}{m} v_1^2 = 9.81 - 0.005 \times 9.81^2 = 9.33$$
$$v_2 = v_1 + \Delta t \cdot a = 9.81 + 1 \times 9.33 = \mathbf{19.14}$$

- At time $t = 3$

$$a = g - \frac{c_d}{m} v_2^2 = 9.81 - 0.005 \times 19.14^2 = 7.98$$
$$v_3 = v_2 + \Delta t \cdot a = 19.14 + 1 \times 7.98 = \mathbf{27.12}$$

- ...

The Basic Syntax of a `for` Loop in Python

Description.

- A `for` loop in Python is a control flow statement used to iterate over items of any sequence (such as a list, tuple, string, set, or dictionary) in the order that they appear.
- It is primarily used when you need to execute a block of code a specific, predetermined number of times or for each item in a collection.

The Basic Syntax of a `for` Loop in Python

Fibonacci Sequence.

- Definition: Given $f(1) = f(2) = 1$, the Fibonacci sequence takes the form of

$$f(n) = f(n-1) + f(n-2),\ n > 2$$

- Write down $f(3)$, $f(4)$, $f(5)$, $f(6)$, $f(7)$, $\cdots$ by yourself?

The Basic Syntax of a `for` Loop in Python

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- Write down $f(3)$, $f(4)$, $f(5)$, $f(6)$, $f(7)$, $\cdots$ by yourself?

$$\begin{aligned}
f(3) &= f(2) + f(1) = 2 \\
f(4) &= f(3) + f(2) = 3 \\
f(5) &= f(4) + f(3) = 5 \\
f(6) &= f(5) + f(4) = 8 \\
f(7) &= f(6) + f(5) = 13
\end{aligned}$$

The Basic Syntax of a `for` Loop in Python

Fibonacci Sequence.

- Definition: Given $f(1) = f(2) = 1$, the Fibonacci sequence takes the form of

$$f(n) = f(n-1) + f(n-2),\ n > 2$$

- Python programming

```
import numpy as np

def fib(n):          # Input n>2
    f = np.zeros(n)
    f[0] = 1
    f[1] = 1
    for i in range(2, n):
        f[i] = f[i - 1] + f[i - 2]
    return f[n - 1]
```

Python Programming for Euler's Method

- **Python programming example**. Computing **bungee jumper velocity**:
  - Start at $t = 0$ and $v = 0$
  - Repeat across different time steps:
    - Compute **acceleration**:

$$a = g - \frac{c_d}{m} v_t^2$$

    - Update **velocity**:

$$v_{t+\Delta t} = v_t + \Delta t \cdot a$$

    - Increment time step: $t = t + \Delta t$

```python
import numpy as np

def euler(m, g, cd, v0, delta_t, time_steps):
    v = np.zeros(time_steps)                # Velocity
    v[0] = v0                               # Initial velocity
    for i in range(time_steps - 1):         # Repeat
        a = g - cd / m * (v[i] ** 2)        # Acceleration
        v[i + 1] = v[i] + delta_t * a       # Velocity
    return v
```

A Real Case

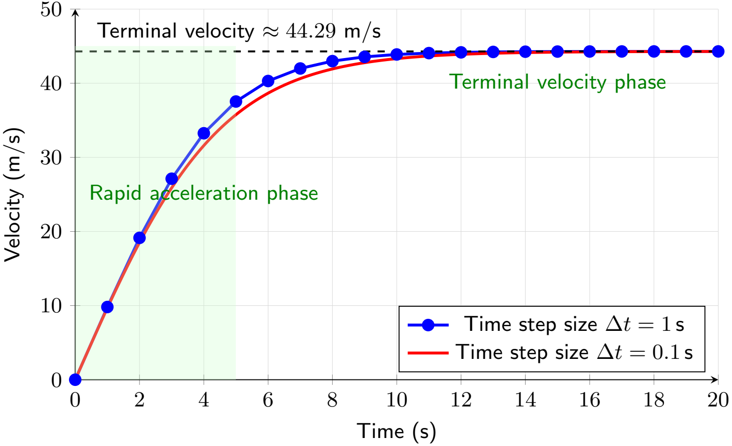
- Mass: $m = 50\,\text{kg}$
- Gravitational acceleration: $g = 9.81\,\text{m/s}^2$
- Drag coefficient: $c_d = 0.25\,\text{kg/m}$

```python
import numpy as np

# Parameters
m = 50                  # Mass (kg)
g = 9.81                # Gravitational acceleration (m/s^2)
cd = 0.25               # Drag coefficient
v0 = 0                  # Initial velocity

# Time setup
delta_t = 1             # Time step size
t_end = 20              # Total time
time_steps = int(t_end / delta_t) + 1

# Euler's method
t = np.linspace(0, t_end, time_steps)
v = euler(m, g, cd, v0, delta_t, time_steps)
```

Velocity vs. Time

Bungee jumper **velocity vs. time** (w/ air resistance)

- Comparison between $\Delta t = 1\,\mathrm{s}$ and $\Delta t = 0.1\,\mathrm{s}$
- Input: $m = 50\,\mathrm{kg}$, $g = 9.81\,\mathrm{m/s^2}$, and $c_d = 0.25\,\mathrm{kg/m}$



Velocity vs. Time

Terminal velocity (solving a simple quadratic equation):

$$\underbrace{a = g - \frac{c_d}{m} v^2 = 0}_{\text{acceleration } = 0} \quad \Rightarrow \quad v = \sqrt{\frac{mg}{c_d}}$$

In this case:

$$v = \sqrt{\frac{mg}{c_d}} = \sqrt{\frac{50 \times 9.81}{0.25}} = 44.29\,\text{m/s}$$

Numerical method insight.

- Demonstrates importance of time step selection in simulations
- Fine time steps give more accurate results
- Coarse time steps are faster to compute but less accurate

Numerical vs. Analytical Solution

Going back to the ordinary differential equation:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - \frac{c_d}{m} v^2$$

which has solution:

$$v_t = \sqrt{\frac{mg}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right) \qquad \text{tangent: } \tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$$

```python
import numpy as np

def analytical_solution(m, g, cd, t):
    v_term = np.sqrt(m * g / cd)
    v_analytical = v_term * np.tanh(np.sqrt(g * cd / m) * t)
    return v_analytical
```

```python
delta_t = 1         # Time step size
t_end = 20          # Total time
time_steps = int(t_end / delta_t) + 1

# Computing the analytical solution
t = np.linspace(0, t_end, time_steps)
v_analytical = analytical_solution(m, g, cd, t)
```

Numerical Error Analysis

How to analyze errors?

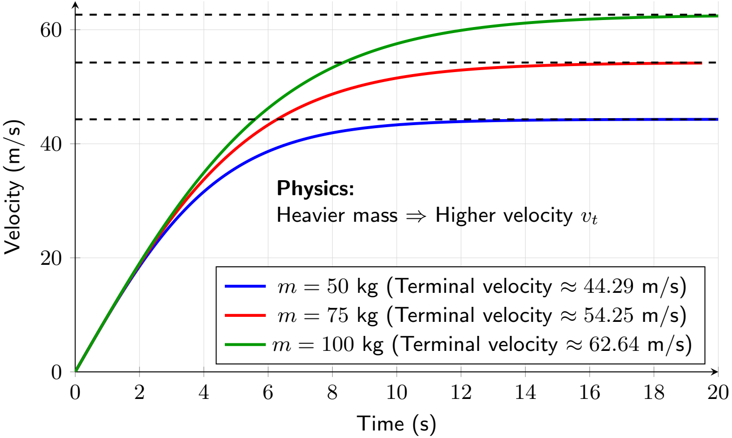
```
error = v - v_analytical
plt.plot(t, error, 'red')
plt.xlabel('Time (s)')
plt.ylabel('Error (m/s)')
plt.show()
```

- Why errors?
  - Euler method assumes constant acceleration over Δt.
  - Smaller $\Delta t \rightarrow$ Smaller error, but more computation.
- Time step comparison:
  - Time step size $\Delta t = 1\,\mathrm{s}$: Error $\approx 1.96\,\mathrm{m/s}$
  - Time step size $\Delta t = 0.1\,\mathrm{s}$: Error $\approx 0.18\,\mathrm{m/s}$
  - Time step size $\Delta t = 0.01\,\mathrm{s}$: Error $\approx 0.02\,\mathrm{m/s}$
- Engineering trade-off: Accuracy vs. Computational cost

Velocity vs. Time (Different Mass)

Bungee jumper **velocity vs. time** (w/ air resistance)

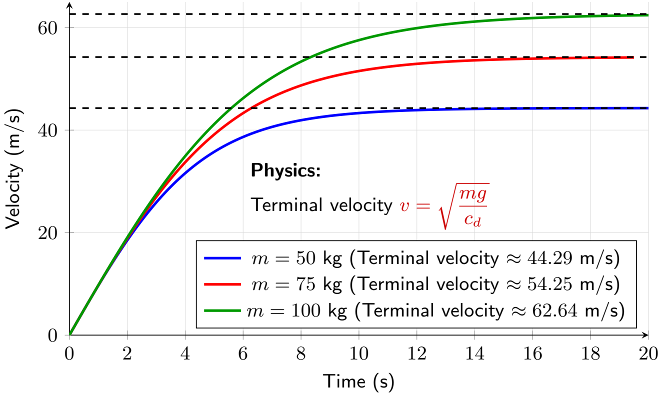
- Comparison among mass $m = 50\,\text{kg}$, $75\,\text{kg}$, $100\,\text{kg}$
- Input: $g = 9.81\,\text{m/s}^2$, and $c_d = 0.25\,\text{kg/m}$



Engineering Safety Analysis

Safe limit: Typically **45 m/s** (160 km/h) for bungee jumping

- Input: $g = 9.81\,\text{m/s}^2$, and $c_d = 0.25\,\text{kg/m}$



- Terminal velocity exceeds safe limit? Increase drag coefficient (baggy clothing); Deploy parachute earlier; Use heavier cord for more drag.

Parameter Sensitivity

How do mass and drag affect terminal velocity?

```python
mass = [75, 100]
drag = [0.15, 0.25, 0.5]

for m in mass:
    for cd in drag:
        v_term = np.sqrt(m * g / cd)
        print('Mass: {}'.format(m))
        print('Drag coefficient: {}'.format(cd))
        print('Terminal velocity: {}'.format(v_term))
        print()
```

Results:

- Lighter jumpers → Lower terminal velocity
- Higher drag coefficient → Lower terminal velocity
- **Design implication**: Need different cords for different jumper weights!

Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "`Class Participation Quiz 3`"

  Time slot: **3:00PM – 3:30PM**

  on Canvas.

- Online engagement (graded quizzes)

  "`Quiz 3`" (14 questions)

  Deadline: **11:59PM, January 21, 2026**

  on Canvas.

Quick Summary

Wednesday's Class:

- Bungee jumping velocity vs. time
  - Newton's second law $F = F_g - F_a = mg - c_d \cdot v^2 = m \cdot a$
  - Ordinary differential equation (the differential rate of change in velocity $\rightarrow$ acceleration)

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \underbrace{g - \frac{c_d}{m} v^2}_{\text{acceleration}}$$

  - Euler's method for numerical computing

$$\underbrace{v_{t+\Delta t}}_{\text{new}} = \underbrace{v_t}_{\text{old}} + \underbrace{\Delta t}_{\text{time step size}} \cdot \underbrace{\left(g - \frac{c_d}{m} v_t^2\right)}_{\text{acceleration}}$$

  - Numerical error analysis
  - Sensitivity across different parameters
- Python programming
  - Fibonacci sequence
  - Numerical computing

Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "`Class Participation Quiz 4`"

  Time slot: **2:30PM – 3:00PM**

  on Canvas.

- Online engagement (graded quizzes)

  "`Quiz 4`" (15 questions)

  Deadline: **11:59PM, January 23, 2026**

  on Canvas.

Euler's Method

Euler's Method is the **simplest numerical technique** for solving **Ordinary Differential Equations (ODEs)**.

- It approximates continuous change using small, discrete steps.
- When to use it?
    - When you know the rate of change $\frac{\mathrm{d}y}{\mathrm{d}x}$
    - When you need a quick, approximate solution
    - When other methods are too complex

Euler's Method

Euler's Method is the **simplest numerical technique** for solving **Ordinary Differential Equations (ODEs)**.

- It approximates continuous change using small, discrete steps.
- When to use it?
  - When you know the rate of change $\frac{dy}{dx}$
  - When you need a quick, approximate solution
  - When other methods are too complex

Bungee jumping velocity vs. time?

- We know the rate of change in velocity:

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

- We need an approximate solution:

$$\underbrace{v_{t+\Delta t}}_{\text{new velocity}} = \underbrace{v_t}_{\text{old velocity}} + \underbrace{\Delta t}_{\text{time step size}} \cdot \underbrace{\frac{dv}{dt}}_{\text{acceleration}}$$

Mathematical Formulation

- **Example**. Given an ODE:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

  with initial condition $y(x_0) = y_0$

- **Euler's formula**:

$$\underbrace{y_{i+1}}_{\text{next value}} = \underbrace{y_i}_{\text{current value}} + \underbrace{\Delta x}_{\text{step size}} \cdot \underbrace{f(x_i, y_i)}_{\text{slope}}$$

$$x_{i+1} = x_i + \underbrace{\Delta x}_{\text{step size}}$$

- **Interpretation**:
  - $f(x_i, y_i) =$ slope at current point
  - $\Delta x =$ step size (small values!)
  - step size $\times$ slope $=$ predicted change in y
  - Add to current y to get next y

Simple Example

- **Toy example**: Solve
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$$
with $y(0) = 1$, find $y(1)$ using step size $\Delta x = 0.5$.
- ❶ Initialize $x_0 = 0$ and $y_0 = 1$
- ❷ First step $(0 \to \Delta x)$

$$f(x_0, y_0) = x_0 + y_0 = 1 \quad y_1 = y_0 + \Delta x \cdot f(x_0, y_0) = 1.5 \quad x_1 = x_0 + \Delta x = 0.5$$

- ❸ Second step $(\Delta x \to 2\Delta x)$

$$f(x_1, y_1) = x_1 + y_1 = 2 \quad y_2 = y_1 + \Delta x \cdot f(x_1, y_1) = 2.5 \quad x_2 = x_1 + \Delta x = 1$$

So we have $y(1) \approx y(x_2) = 2.5$.

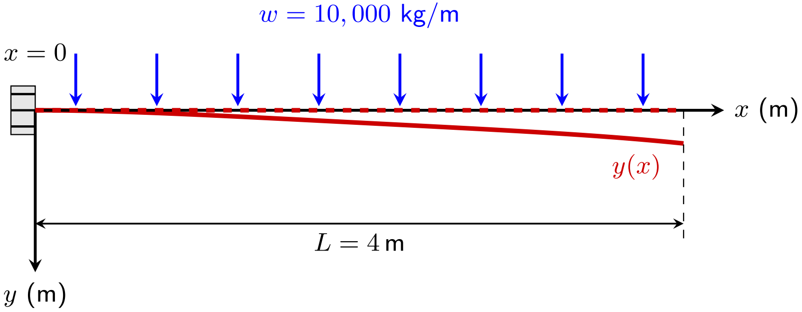
- Hint (Keep in mind!):

$$\underbrace{y_{i+1}}_{\text{next value}} = \underbrace{y_i}_{\text{current value}} + \underbrace{\Delta x}_{\text{step size}} \cdot \underbrace{f(x_i, y_i)}_{\text{sloop}} \qquad x_{i+1} = x_i + \Delta x$$

Cantilever Beam Deflection

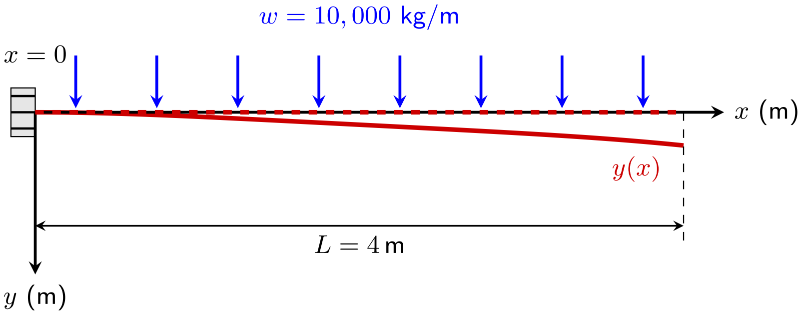
Engineering Task.

- Calculate the **deflection of a cantilever beam** under uniform load.
- Needed for: **Building codes**, **safety checks**, **material selection**.



Cantilever Beam Deflection

- Use **Euler's method** to find deflection $y(x)$ from $x = 0$ to $x = L$.
- $y(x)$ is downward deflection at point x (x is distance from fixed end).
- **Given parameters**:
  - Uniform load: $w = 10,000\,\text{kg/m}$
  - Beam length: $L = 4\,\text{m}$
  - Modulus: $E = 2 \times 10^{11}$ Pa (steel)
  - Moment of inertia: $I = 3.25 \times 10^{-4}\,\text{m}^4$



Cantilever Beam Deflection

- Use **Euler's method** to find deflection $y(x)$ from $x = 0$ to $x = L$.

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \underbrace{\frac{w}{24 \cdot E \cdot I}}_{\text{constant}}(4x^3 - 12Lx^2 + 12L^2x)}$$

- x is distance from fixed end.
- $y(x)$ is downward deflection at point x.
- **Given parameters**:
  - Uniform load: $w = 10,000\,\text{kg/m}$
  - Beam length: $L = 4\,\text{m}$
  - Modulus: $E = 2 \times 10^{11}$ Pa (steel)
  - Moment of inertia: $I = 3.25 \times 10^{-4}\,\text{m}^4$
- Compute the **constant factor**:

$$c = \frac{w}{24 \cdot E \cdot I} = \frac{10^4}{24 \times (2 \times 10^{11}) \times (3.25 \times 10^{-4})} = 6.41 \times 10^{-6}$$

Euler's Method (Numerical)

- **Idea**: Given the step size $\Delta x = 0.125\,\text{m}$, we start from $y(0) = 0$ and **update the deflection** by

$$\underbrace{y_{i+1}}_{\text{next deflection}} = \underbrace{y_i}_{\text{current deflection}} + \underbrace{\Delta x}_{\text{step size}} \cdot \underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{sloop}}$$

update the position by

$$\underbrace{x_{i+1}}_{\text{next position}} = \underbrace{x_i}_{\text{current position}} + \underbrace{\Delta x}_{\text{step size}}$$

where the sloop is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = c(4x^3 - 12Lx^2 + 12L^2x)$$

- Number of steps (repeat `for` loop)

$$\frac{L}{\Delta x} = \frac{4}{0.125} = 32\,\text{steps}$$

- Compute the **constant factor**:

$$c = \frac{w}{24 \cdot E \cdot I} = \frac{10^4}{24 \times (2 \times 10^{11}) \times (3.25 \times 10^{-4})} = 6.41 \times 10^{-6}$$

Python Programming

Compute the constant factor with Python programming.

- Given **parameters**: uniform load $w = 10,000\,\text{kg/m}$, modulus $E = 2 \times 10^{11}$ Pa, and moment of inertia $I = 3.25 \times 10^{-4}\,\text{m}^4$.
- Compute the **constant factor**:

$$c = \frac{w}{24 \cdot E \cdot I} = \frac{10^4}{24 \times (2 \times 10^{11}) \times (3.25 \times 10^{-4})} = 6.41 \times 10^{-6}$$

```python
import numpy as np

def const(w, E, I):
    return w / (24 * E * I)

w = 10 ** 4              # uniform load
E = 2 * 10 ** 11         # modulus
I = 3.25 * 10 ** (-4) # moment of inertia
c = const(w, E, I)      # constant factor
print(c)
```

Python Programming

- Update deflection and position:

$$\underbrace{y_{i+1}}_{\text{next deflection}} = \underbrace{y_i}_{\text{current deflection}} + \underbrace{\Delta x}_{\text{step size}} \cdot \underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}}_{\text{sloop}}$$

$$\underbrace{x_{i+1}}_{\text{next position}} = \underbrace{x_i}_{\text{current position}} + \underbrace{\Delta x}_{\text{step size}}$$

with

$$\frac{\mathrm{d}y}{\mathrm{d}x} = c(4x^3 - 12Lx^2 + 12L^2x)$$

```python
import numpy as np

def sloop(c, L, x):
    return c * (4*x**3 - 12*L*x**2 + 12*L**2*x)

def euler_deflection(c, L, x, y, delta_x):
    y_plus = y + delta_x * sloop(c, L, x) # deflection
    x_plus = x + delta_x                  # position
    return x_plus, y_plus
```

Python Programming

```python
import numpy as np

delta_x = 0.125              # step size
L = 4                        # beam length
n = int(L / delta_x) + 1     # number of steps
x = np.linspace(0, L, n)
y = np.zeros(n)
for i in range(n - 1):
    y[i + 1] = y[i] + delta_x * sloop(c, L, x[i])
```

Cantilever Beam Deflection

Quick Summary

Friday's Class:

- Apply Euler's method to engineering problem
- Compute numerical vs. analytical solutions
- Understand error accumulation
- Implement numerical methods with Python