Matrix and Tensor Models for Spatiotemporal Traffic Data Imputation and Forecasting

Ph.D. Defense

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4. **Low-Rank Autoregressive Tensor Completion**
5. **Laplacian Convolutional Representation**
6. **Hankel Tensor Factorization**
7. **Experiments**
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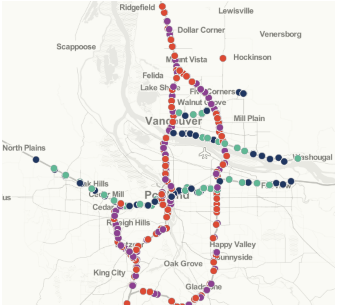
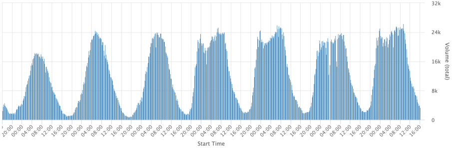
Multivariate Traffic Time Series

Many spatiotemporal traffic time series data are in the form of **matrix**.

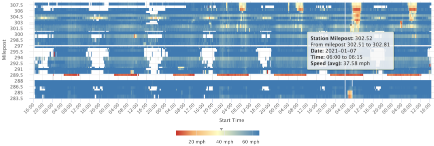
- **Example**: Portland highway traffic data[1].



Highway network & sensor locations



Traffic volume



Traffic speed

- $\boldsymbol{X} \in \mathbb{R}^{N \times T}$ with N spatial locations $\times$ T time steps
- Traffic volume/speed shows strong spatial/temporal dependencies

[1] https://portal.its.pdx.edu/home

Multiple Data Behaviors

Spatiotemporal traffic data are time series, but they involve multiple data behaviors.

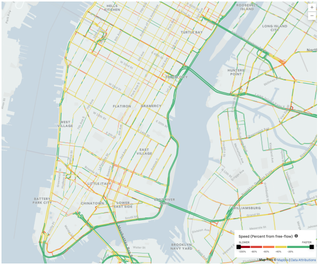
- Incompleteness & sparsity
- High-dimensionality
- Multidimensionality
- Noises & outliers
- Nonstationarity
- ......

In addition, spatiotemporal correlations are also very important.

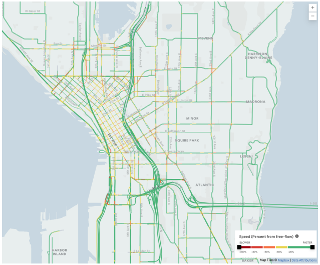
Multiple Data Behaviors

Sparsity & high-dimensionality

- Uber (hourly) movement speed data



NYC movement



Seattle movement

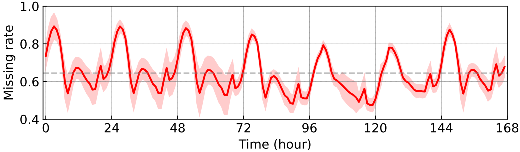
- {(road segment, time slot (hour)), average speed}
- Computing hourly speed: Road segments have 5+ unique trips.

Issue: Insufficient sampling of ridesharing vehicles on the road network!

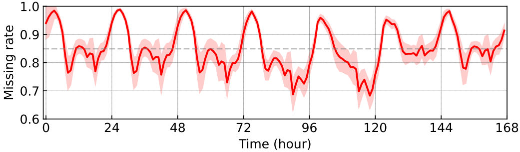
Multiple Data Behaviors

Sparsity & high-dimensionality

- **NYC** movement speed data (2019)
  - **98,210** road segments & 8,760 time steps (hours)
  - Overall missing rate: **64.43%**

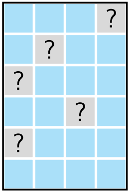


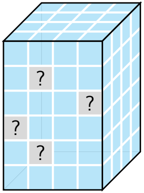
- **Seattle** movement speed data (2019)
  - **63,490** road segments & 8,760 time steps (hours)
  - Overall missing rate: **84.95%**



Problem Formulation

- **Objective A:** Given a multivariate time series data like $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ or a multidimensional time series data like $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{M \times N \times T}$ with the observed index set Ω, impute the missing values of the data.



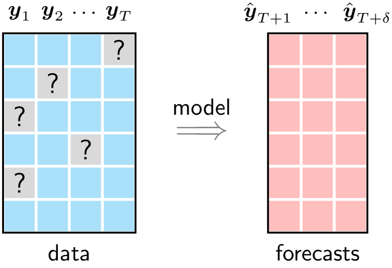


[Q]

- How to reconstruct missing values from observed data?
  - Matrix completion: From $\mathcal{P}_{\Omega}(\boldsymbol{Y})$ (observed) to $\mathcal{P}_{\Omega}^{\perp}(\boldsymbol{Y})$ (unobserved)
  - Tensor completion: From $\mathcal{P}_{\Omega}(\boldsymbol{\mathcal{Y}})$ (observed) to $\mathcal{P}_{\Omega}^{\perp}(\boldsymbol{\mathcal{Y}})$ (unobserved)
- How to make use of spatiotemporal correlations?
- How to make use of traffic time series dynamics?

Problem Formulation

- **Objective B:** Given a partially observed data $\boldsymbol{Y}\in\mathbb{R}^{N\times T}$ consisting of time series $\boldsymbol{y}_1,\ldots,\boldsymbol{y}_T\in\mathbb{R}^N$, forecast data points $\boldsymbol{y}_{T+\delta},\delta\in\mathbb{N}^+$.

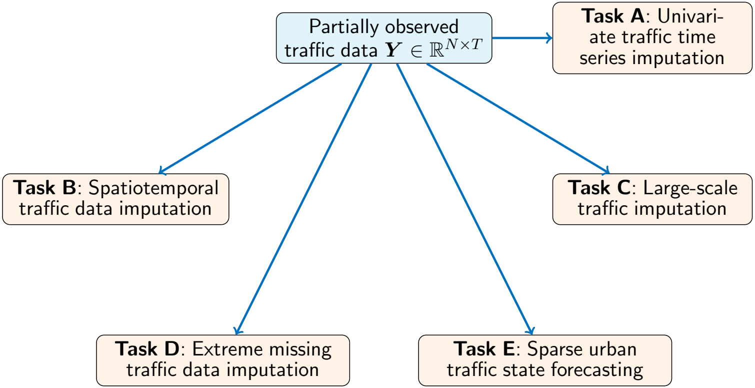


[Q]

- How to learn from *high-dimensional* and *sparse* data?
- How to model *nonstationarity* in time series?
- How to perform forecasting on these time series?

Whole Picture

We are working on **spatiotemporal traffic data imputation and forecasting**.



Imputation & Forecasting

Traffic data imputation

- Time series autoregression (Schafer'97, Chen & Shao'00)
- Principal component analysis (Qu et al.'09, Li et al.'13)
- Matrix factorization (Asif et al.'13, Asif et al.'16)
- Tensor factorization (Tan et al.'13, Chen et al.'19)
- Low-rank tensor completion (Ran et al.'16, Chen et al.'20)
- Temporal matrix/tensor factorization (Chen & Sun'22)

Time series forecasting on sparse data

- Autoregression predictor (Anava et al.'15)
- Prediction on the imputed data (e.g., Che et al.'18)
- Dynamic tensor completion (Tan et al.'16)
- Temporal matrix factorization (Yu et al.'16, Chen & Sun'22)
- Online matrix factorization (Gultekin & Paisley'18)

Tensor Factorization

- Revisit tensor factorization

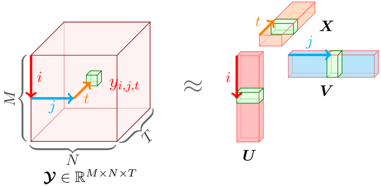
1927	1960s	1970	2009	2011
Higher-Order SVD	Tucker Decomposition	CP Decomposition	Tensor Decompositions and Applications	Tensor-Train Decomposition
Frank Lauren Hitchcock	Ledyard R. Tucker	J. Douglas Carroll Jih-Jie Chang Richard A. Harshman	Tamara G. Kolda	Ivan Oseledets

Frank Lauren Hitchcock

Tamara G. Kolda

Ivan Oseledets

- **CP tensor factorization**: Factorize $\boldsymbol{\mathcal{Y}}$ into the combination of three rank-R factor matrices (i.e., low-dimensional latent factors).



$$\begin{cases} y_{i,j,t} \approx \sum_{r=1}^{R} u_{i,r} v_{j,r} x_{t,r} \\ \boldsymbol{\mathcal{Y}} \approx \sum_{r=1}^{R} \boldsymbol{u}_r \otimes \boldsymbol{v}_r \otimes \boldsymbol{x}_r \end{cases}$$

Matrix/Tensor Completion

Matrix nuclear norm ($\|\boldsymbol{X}\|_*$) minimization
Candès & Recht'09

Singular value thresholding
Cai et al.'10

Truncated nuclear norm ($\|\boldsymbol{X}\|_{r,*}, r \in \mathbb{N}^+$) minimization
Zhang et al.'12
Hu et al.'12

Tensor nuclear norm ($\|\boldsymbol{\mathcal{X}}\|_*$) minimization
Liu et al.'13

Generalized nonconvex nonsmooth low-rank minimization
Lu et al.'14

Tensor nuclear norm minimization with linear transform
Lu et al.'19

Low-rank Hankel matrix/tensor ($\mathcal{H}_\tau(\cdot)$) completion
Yokota et al.'18
Sedighin et al.'20
Cai et al.'21
Yamamoto et al.'22

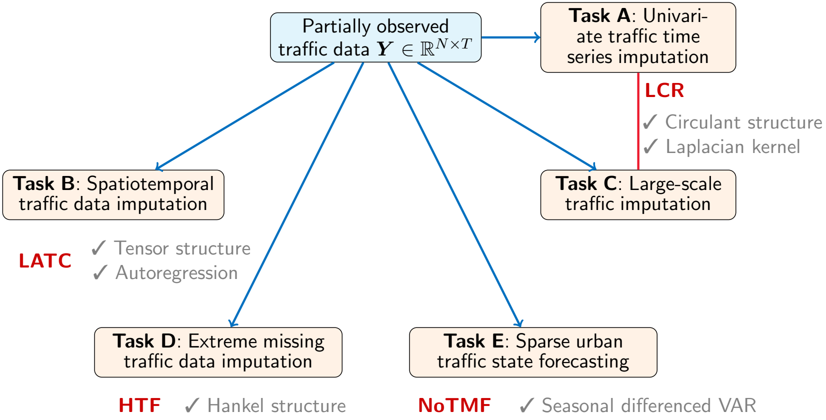
Circulant/Convolution nuclear norm ($\|\mathcal{C}(\boldsymbol{X})\|_*$ or $\|\mathcal{C}_{\tilde{\tau}}(\boldsymbol{X})\|_*$) minimization
Liu'22
Liu & Zhang'23

This research

- Integrate temporal modeling techniques (e.g., temporal smoothing and time series autoregression) into low-rank matrix and tensor methods
- Implement spatiotemporal traffic data imputation and forecasting on partially observed data

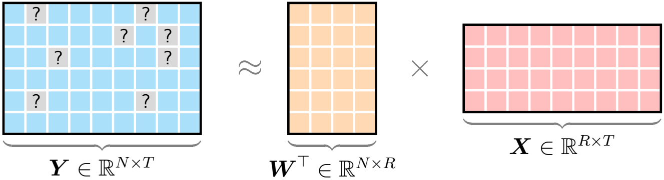
Overview

We are working on **spatiotemporal traffic data imputation and forecasting**.



Matrix Factorization

A simple approach to reconstruct missing values.



MF (Koren et al.'09)

Estimating low-dimensional $\boldsymbol{W}, \boldsymbol{X}$:

$$\min_{\boldsymbol{W}, \boldsymbol{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\boldsymbol{Y} - \boldsymbol{W}^\top \boldsymbol{X})\|_F^2$$

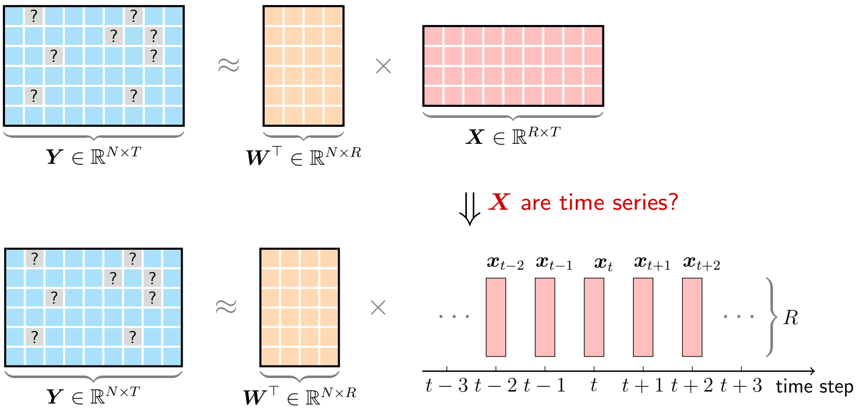
on data $\boldsymbol{Y}$ w/ observed index set Ω.

- ✓ Learn from sparse data
- ✗ Temporal correlations
- ✗ Time series forecasting

How to build temporal correlations on MF?

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.



Why? Temporal factor matrix $\boldsymbol{X} \in \mathbb{R}^{R\times T}$ is the low-dimensional representation of time series dynamics of $\boldsymbol{Y} \in \mathbb{R}^{N\times T}$.

Temporal Matrix Factorization

MF (Koren et al.'09)

Estimating low-dimensional $\boldsymbol{W},\boldsymbol{X}$:

$$\min_{\boldsymbol{W},\boldsymbol{X}} \frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2$$

on data $\boldsymbol{Y}$ w/ observed index set Ω.

$+$

dth-order VAR

$$\boldsymbol{x}_t=\sum_{k=1}^{d}\boldsymbol{A}_k\boldsymbol{x}_{t-k}+\underbrace{\boldsymbol{\epsilon}_t}_{\mathcal{N}(\boldsymbol{0},\boldsymbol{I})}$$

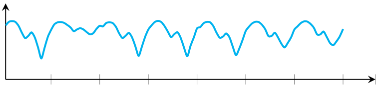
w/ coefficients $\{\boldsymbol{A}_k\}$.

$\Downarrow$ Yu et al.'16, Chen & Sun'21

$$\boxed{\min_{\boldsymbol{W},\boldsymbol{X},\{\boldsymbol{A}_k\}_{k=1}^{d}} \frac{1}{2}\underbrace{\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2}_{\text{MF on data } \boldsymbol{Y}}+\frac{\gamma}{2}\underbrace{\sum_{t=d+1}^{T}\left\|\boldsymbol{x}_t-\sum_{k=1}^{d}\boldsymbol{A}_k\boldsymbol{x}_{t-k}\right\|_2^2}_{\text{VAR on temporal factors } \boldsymbol{X}}}$$

Nonstationary Temporal Matrix Factorization

Traffic data are nonstationary due to daily patterns of traffic flow.



- Season-m differencing ($m \in \mathbb{N}^+$, e.g., daily/weekly):

$$\boldsymbol{x}_t \approx \sum_{k=1}^{d} \boldsymbol{A}_k \boldsymbol{x}_{t-k} \quad \Rightarrow \quad \boldsymbol{x}_t - \boldsymbol{x}_{t-m} \approx \sum_{k=1}^{d} \boldsymbol{A}_k (\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k})$$

- **(Ours)** Optimization problem:

$$\min_{\boldsymbol{W}, \boldsymbol{X}, \{\boldsymbol{A}_k\}_{k=1}^{d}} \frac{1}{2} \underbrace{\|\mathcal{P}_{\Omega}(\boldsymbol{Y} - \boldsymbol{W}^\top \boldsymbol{X})\|_F^2}_{\text{MF on data } \boldsymbol{Y}} + \frac{\rho}{2} \underbrace{(\|\boldsymbol{W}\|_F^2 + \|\boldsymbol{X}\|_F^2)}_{\text{Regularization}}$$
$$+ \frac{\gamma}{2} \underbrace{\sum_{t=d+m+1}^{T} \left\| (\boldsymbol{x}_t - \boldsymbol{x}_{t-m}) - \sum_{k=1}^{d} \boldsymbol{A}_k (\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k}) \right\|_2^2}_{\text{VAR on seasonal differenced temporal factors}}$$

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF

- Optimization problem:

$$\min_{\boldsymbol{W},\boldsymbol{X},\boldsymbol{A}} \ \frac{1}{2}\underbrace{\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^{\top}\boldsymbol{X})\|_F^2}_{\text{MF on data } \boldsymbol{Y}}+\frac{\rho}{2}\underbrace{(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)}_{\text{Regularization}}$$
$$+\frac{\gamma}{2}\underbrace{\|\boldsymbol{X}\boldsymbol{\Psi}_0^{\top}-\boldsymbol{A}(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^{\top}\|_F^2}_{\text{Temporal modeling on } \boldsymbol{X}}$$

where $\boldsymbol{\Psi}_0\in\mathbb{R}^{(T-d-m)\times T}$ and $\boldsymbol{\Psi}\in\mathbb{R}^{(T-d-m)\times(dT)}$ are temporal operators.

- Alternating minimization (let f be the obj.):

$$\begin{cases} \text{Spatial factors} & \boldsymbol{W}:=\{\boldsymbol{W}\mid\frac{\partial f}{\partial \boldsymbol{W}}=\boldsymbol{0}\} & \text{(least squares)} \\ \text{Temporal factors} & \boldsymbol{X}:=\{\boldsymbol{X}\mid\frac{\partial f}{\partial \boldsymbol{X}}=\boldsymbol{0}\} & \text{(conjugate gradient)} \\ \text{VAR coefficients} & \boldsymbol{A}:=\{\boldsymbol{A}\mid\frac{\partial f}{\partial \boldsymbol{A}}=\boldsymbol{0}\} & \text{(least squares)} \end{cases}$$

Nonstationary Temporal Matrix Factorization

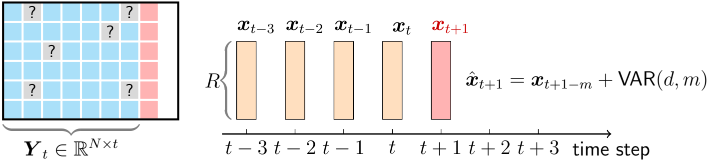
NoTMF forecasting on streaming data?

Implementation

- Estimate $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{A}$
- Forecast $\hat{\boldsymbol{x}}_{t+1}$ with VAR
- Return $\hat{\boldsymbol{y}}_{t+1} = \boldsymbol{W}^\top \hat{\boldsymbol{x}}_{t+1}$

✓ Sparse input $\boldsymbol{Y}_t$

✓ Forecast in latent spaces



Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

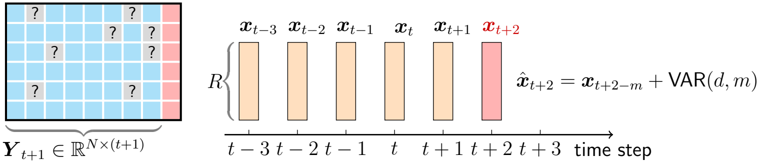
- Online forecasting (Gultekin & Paisley'18):
  - Fix the spatial factor matrix $\boldsymbol{W}$
  - Use input data $\boldsymbol{Y}_{t+1}$ to update the temporal factor matrix $\boldsymbol{X}$ and the coefficient matrix $\boldsymbol{A}$

Implementation

- Estimate $\boldsymbol{X}, \boldsymbol{A}$
- Forecast $\hat{\boldsymbol{x}}_{t+2}$ with VAR
- Return $\hat{\boldsymbol{y}}_{t+2} = \boldsymbol{W}^\top \hat{\boldsymbol{x}}_{t+2}$

✓ Sparse input $\boldsymbol{Y}_{t+1}$

✓ Forecast in latent spaces



Matrix/Tensor Completion

Cornerstone: Nuclear norm minimization in matrix/tensor completion

LRMC (Candès & Recht'09)

Estimating the matrix $\boldsymbol{X}$:

$$\min_{\boldsymbol{X}} \ \|\boldsymbol{X}\|_* \quad \text{s.t. } \mathcal{P}_\Omega(\boldsymbol{X}) = \mathcal{P}_\Omega(\boldsymbol{Y})$$

on data $\boldsymbol{Y}$ w/ observed index set Ω.

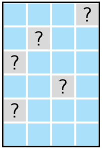
vs.

LRTC (Liu et al.'13)

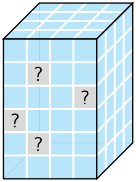
Estimating the tensor $\boldsymbol{\mathcal{X}}$:

$$\min_{\boldsymbol{\mathcal{X}}} \ \|\boldsymbol{\mathcal{X}}\|_* \quad \text{s.t. } \mathcal{P}_\Omega(\boldsymbol{\mathcal{X}}) = \mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}})$$

on data $\boldsymbol{\mathcal{Y}}$ w/ observed index set Ω.



$\mathcal{P}_\Omega(\boldsymbol{Y}) \in \mathbb{R}^{N\times T}$

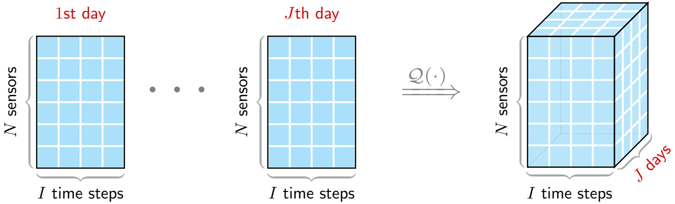


$\mathcal{P}_\Omega(\boldsymbol{\mathcal{Y}}) \in \mathbb{R}^{N\times I\times J}$

- **Limitation**: Only cover global consistency

Low-Rank Autoregressive Tensor Completion

- Introduce traffic tensors with day dimension[2] (Tan et al.'13, Chen et al.'19, ...)



[2] There are $T = IJ$ time steps in total.

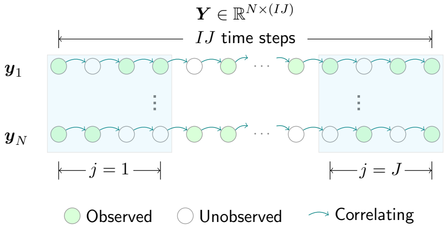
Low-Rank Autoregressive Tensor Completion

- Build temporal correlations with autoregression

On the time series $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$:

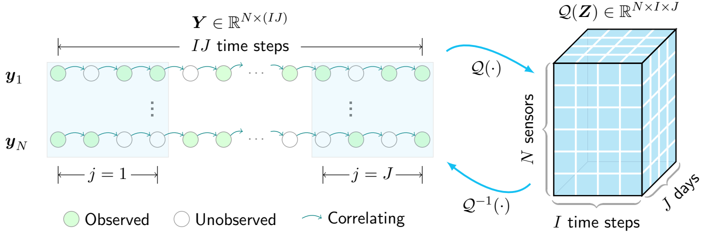
$$\boxed{\|\boldsymbol{Y}\|_{\boldsymbol{A},\mathcal{H}} \triangleq \sum_{n,t}\Big(y_{n,t} - \sum_{k} a_{n,k} y_{n,t-h_k}\Big)^2}$$

with the time lag set $\mathcal{H} = \{h_1, \ldots, h_d\}$ and the coefficient matrix $\boldsymbol{A} \in \mathbb{R}^{N \times d}$.



Low-Rank Autoregressive Tensor Completion

Local consistency w/ autoregression **Global consistency** w/ tensor structure



LATC

Optimization problem:

$$\min_{\boldsymbol{Z},\boldsymbol{A}} \; \|\mathcal{Q}(\boldsymbol{Z})\|_{r,*} + \frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}}$$
$$\text{s.t. } \mathcal{P}_\Omega(\boldsymbol{Z}) = \mathcal{P}_\Omega(\boldsymbol{Y})$$

on data $\boldsymbol{Y}$ w/ observed index set Ω.

$\Rightarrow$

Two subproblems

$$\begin{cases} \boldsymbol{Z} := \underset{\mathcal{P}_\Omega(\boldsymbol{Z})=\mathcal{P}_\Omega(\boldsymbol{Y})}{\arg\min} \; \|\mathcal{Q}(\boldsymbol{Z})\|_{r,*} + \frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}} \\ \boldsymbol{A} := \frac{1}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}} \qquad \text{(Least squares)} \end{cases}$$

Low-Rank Autoregressive Tensor Completion

$\boldsymbol{Z}$-subproblem:

$$\boldsymbol{Z} := \underset{\mathcal{P}_{\Omega}(\boldsymbol{Z})=\mathcal{P}_{\Omega}(\boldsymbol{Y})}{\arg\min} \|\mathcal{Q}(\boldsymbol{Z})\|_{r,*} + \frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}}$$

- Augmented Lagrangian function:[3]

$$\mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{Z},\boldsymbol{\mathcal{W}}) = \|\boldsymbol{\mathcal{X}}\|_{r,*}+\frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}}+\frac{\lambda}{2}\|\boldsymbol{\mathcal{X}}-\mathcal{Q}(\boldsymbol{Z})\|_F^2+\langle\boldsymbol{\mathcal{W}},\boldsymbol{\mathcal{X}}-\mathcal{Q}(\boldsymbol{Z})\rangle+\pi(\boldsymbol{Z})$$

Implementation

Repeat
- Compute $\boldsymbol{Z}$
- Compute $\boldsymbol{A}$

$\Longrightarrow$

Implementation

Repeat
- Repeat
  # Alternating Direction Method of Multipliers (ADMM)
  - Compute $\boldsymbol{\mathcal{X}}$
  - Compute $\boldsymbol{Z}$
  - Compute $\boldsymbol{\mathcal{W}}$
- Compute $\boldsymbol{A}$

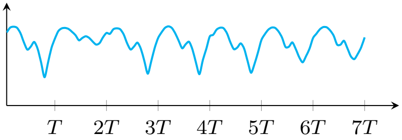
[3] $\boldsymbol{\mathcal{W}} \in \mathbb{R}^{N\times I\times J}$ (Lagrange multiplier); $\langle\cdot,\cdot\rangle$ (inner product). The indicator function:

$$\pi(\boldsymbol{Z}) = \begin{cases} 0, & \text{if } \mathcal{P}_{\Omega}(\boldsymbol{Z}) = \mathcal{P}_{\Omega}(\boldsymbol{Y}), \\ +\infty, & \text{otherwise.} \end{cases}$$

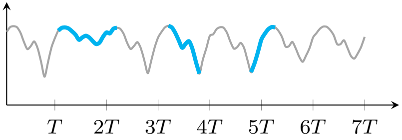
Laplacian Convolutional Representation

Motivation: Time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



- Local trends (e.g., short-term time series trends):

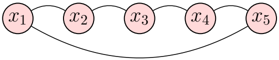


- **[Question]** How to characterize both global and local trends in sparse time series data?

Laplacian Convolutional Representation

Local trend modeling

- Intuition of (circulant) Laplacian matrix



Undirected and circulant graph

$$\xrightarrow{\text{Modeling}} \quad \boldsymbol{L}=\begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (2,-1,0,0,-1)^\top$$

$$\Downarrow$$

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1,\cdots,-1}_{\tau}, 0,\cdots,0, \underbrace{-1,\cdots,-1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series $\boldsymbol{x}=(x_1,\ldots,x_T)^\top \in \mathbb{R}^T$.

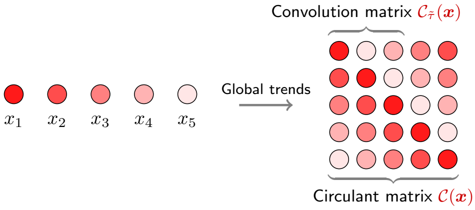
- (Laplacian) Temporal regularization:

$$\mathcal{R}_\tau(\boldsymbol{x})=\frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2=\frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2$$

Reformulate temporal regularization with circular convolution

Laplacian Convolutional Representation

Global trend modeling: Circulant matrix $\mathcal{C}(\boldsymbol{x})$ vs. convolution matrix $\mathcal{C}_{\tilde{\tau}}(\boldsymbol{x})$



- Circulant/Convolution nuclear norm minimization
  - A balance between global and local trends modeling?

CircNNM (Liu'22, Liu & Zhang'23)

Estimating $\boldsymbol{x}$:

$$\begin{aligned}\min_{\boldsymbol{x}}\ & \|\mathcal{C}(\boldsymbol{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon\end{aligned}$$

on data $\boldsymbol{y}$ w/ observed index set Ω.

ConvNNM (Liu'22, Liu & Zhang'23)

Estimating $\boldsymbol{x}$:

$$\begin{aligned}\min_{\boldsymbol{x}}\ & \|\mathcal{C}_{\tilde{\tau}}(\boldsymbol{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon\end{aligned}$$

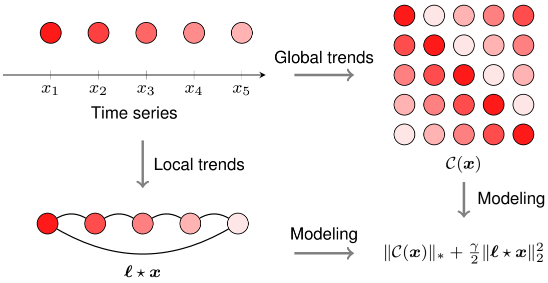
on data $\boldsymbol{y}$ w/ observed index set Ω.

Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\boldsymbol{y} \in \mathbb{R}^T$ with observed index set Ω, LCR utilizes circulant matrix and Laplacian kernel to characterize **global and local trends** in time series, respectively, i.e.,

$$\begin{aligned} \min_{\boldsymbol{x}} \ & \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 \\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$



Laplacian Convolutional Representation

- Augmented Lagrangian function:[4]

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w})=\|\mathcal{C}(\boldsymbol{x})\|_*+\frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2+\frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2+\langle\boldsymbol{w},\boldsymbol{x}-\boldsymbol{z}\rangle+\frac{\eta}{2}\|\mathcal{P}_\Omega(\boldsymbol{z}-\boldsymbol{y})\|_2^2$$

- The ADMM scheme:

$$\begin{cases}\boldsymbol{x}:=\arg\min\limits_{\boldsymbol{x}}\ \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) & \text{(Nuclear norm minimization)}\\ \boldsymbol{z}:=\arg\min\limits_{\boldsymbol{z}}\ \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) & \text{(Closed-form solution)}\\ \boldsymbol{w}:=\boldsymbol{w}+\lambda(\boldsymbol{x}-\boldsymbol{z}) & \text{(Standard update)}\end{cases}$$

- Optimize $\boldsymbol{x}$?

$$\|\mathcal{C}(\boldsymbol{x})\|_*=\|\mathcal{F}(\boldsymbol{x})\|_1\qquad\&\qquad\frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2=\frac{1}{2T}\|\mathcal{F}(\boldsymbol{\ell})\circ\mathcal{F}(\boldsymbol{x})\|_2^2$$

Nuclear norm minimization $\Rightarrow$ ℓ_1**-norm minimization with FFT** in $\mathcal{O}(T\log T)$ time.

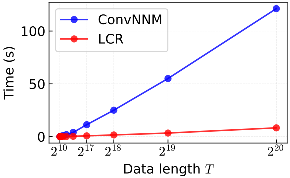
[4] $\boldsymbol{w}\in\mathbb{R}^T$ (Lagrange multiplier); $\langle\cdot,\cdot\rangle$ (inner product).

Laplacian Convolutional Representation

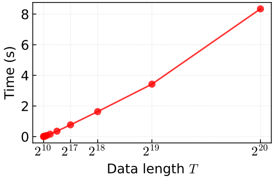
Empirical time complexity

On the synthetic data $\boldsymbol{y}\in\mathbb{R}^{T}$ with $T\in\{2^{10},2^{11},\dots,2^{20}\}$

- Ours: **LCR**
  - An FFT implementation in $\mathcal{O}(T\log T)$
  - The logarithmic factor $\log T$ makes the FFT highly efficient
- Baseline: **ConvNNM**[5] (Liu'22, Liu & Zhang'23)
  - Convolution matrix $\mathcal{C}_{\tilde{\tau}}(\boldsymbol{y})\in\mathbb{R}^{T\times\tilde{\tau}}$ with kernel size $\tilde{\tau}=2^4$
  - Singular value thresholding in $\mathcal{O}(\tilde{\tau}^2T)$



ConvNNM vs. LCR



LCR

[5] Convolution nuclear norm minimization.

Laplacian Convolutional Representation

LCR

On time series $\boldsymbol{y} \in \mathbb{R}^{T}$,

$$\begin{aligned}\min_{\boldsymbol{x}}\ & \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 \\ \text{s.t. } & \|\mathcal{P}_{\Omega}(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon\end{aligned}$$

w/ observed index set Ω.

$\Longrightarrow$

LCR-2D

On time series $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$,

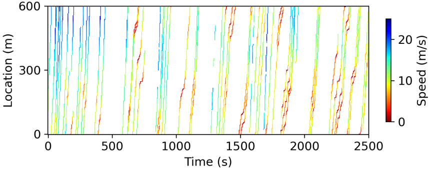
$$\begin{aligned}\min_{\boldsymbol{X}}\ & \|\mathcal{C}(\boldsymbol{X})\|_* + \frac{\gamma}{2}\|(\boldsymbol{\ell}_s\boldsymbol{\ell}^\top) \star \boldsymbol{X}\|_F^2 \\ \text{s.t. } & \|\mathcal{P}_{\Omega}(\boldsymbol{X}-\boldsymbol{Y})\|_F \leq \epsilon\end{aligned}$$

w/ observed index set Ω.

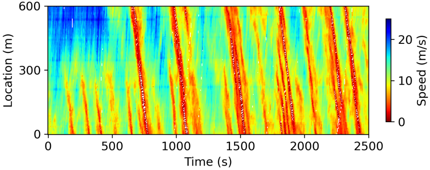
Hankel Tensor Factorization

Motivation: Spatiotemporal data reconstruction

- Speed field reconstruction problem in vehicular traffic flow.



200-by-500 matrix (NGSIM) $\Downarrow$ Reconstruct speed field from 5% sparse trajectories?



- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal dependencies?

Hankel Tensor Factorization

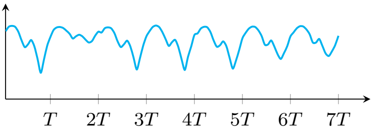
- Hankel matrix
  - Given $\boldsymbol{x} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\boldsymbol{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4\times 2}$$

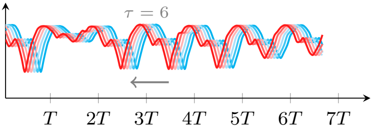
Hankel matrix (Source: Twitter)

Hankel Tensor Factorization

- Hankel matrix
  - Automatic temporal modeling



Traffic time series



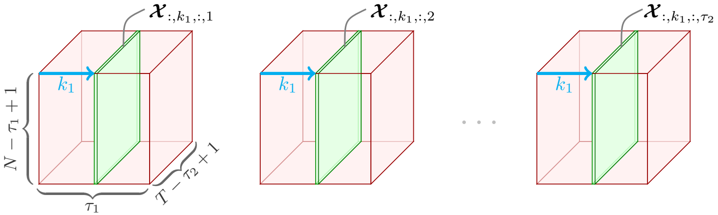
Hankel matrix

Hankel Tensor Factorization

- Hankel tensor: Given any matrix $\boldsymbol{X} \in \mathbb{R}^{N \times T}$, we have

$$\boldsymbol{\mathcal{X}} \triangleq \mathcal{H}_{\tau_1, \tau_2}(\boldsymbol{X})$$

  - Window lengths: $\tau_1, \tau_2 \in \mathbb{N}^+$;
  - Tensor size: $(N-\tau_1+1) \times \tau_1 \times (T-\tau_2+1) \times \tau_2$;



(Figure) 4th order Hankel tensor: A sequence of third-order tensors.

  - Slice: $\boldsymbol{\mathcal{X}}_{:,k_1,:,k_2}, \forall k_1, k_2$;
  - Slice size: $(N-\tau_1+1) \times (T-\tau_2+1)$.

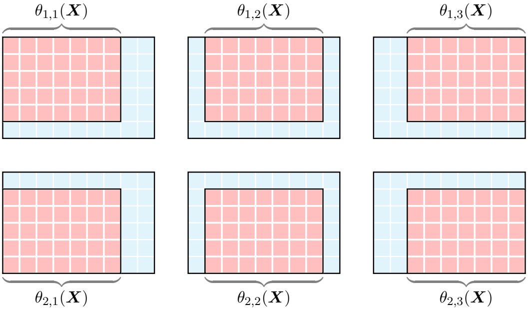
Hankel Tensor Factorization

Hankel indexing:

- Sampling function for the Hankelization:

$$\theta_{k_1,k_2}(\boldsymbol{X}) \triangleq [\mathcal{H}_{\tau_1,\tau_2}(\boldsymbol{X})]_{:,k_1,:,k_2},$$

  referring to the tensor slice with $k_1 \in \{1,\ldots,\tau_1\}$, $k_2 \in \{1,\ldots,\tau_2\}$.
- **[Importance]** Developing memory-efficient algorithms.



- Tensor slices $\theta_{k_1,k_2}(\boldsymbol{X})$ vs. data matrix $\boldsymbol{X}$

Hankel Tensor Factorization

Ours:

- Convolutional tensor decomposition (circular convolution $\star_{\text{row}}$):

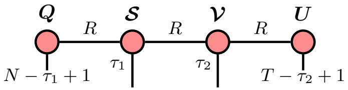
$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q} \star_{\text{row}} \boldsymbol{s}_{k_1}^\top)(\boldsymbol{U} \star_{\text{row}} \boldsymbol{v}_{k_2}^\top)^\top$$

Baselines:

- Tensor-train decomposition:

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q}\boldsymbol{S}_{k_1})(\boldsymbol{U}\boldsymbol{V}_{k_2})^\top$$

  - $\{\boldsymbol{S}_{k_1}, \boldsymbol{V}_{k_2}\}$ are **circulant matrices** $\Rightarrow$ convolutional decomposition
  - $\{\boldsymbol{S}_{k_1}, \boldsymbol{V}_{k_2}\}$ are **diagonal matrices** $\Rightarrow$ CP decomposition



- CP tensor decomposition (Khatri-Rao product $\odot$):

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q} \odot \boldsymbol{s}_{k_1}^\top)(\boldsymbol{U} \odot \boldsymbol{v}_{k_2}^\top)^\top$$

Hankel Tensor Factorization

HTF (convolutional decomposition)

- Optimization problem:

$$\min_{\boldsymbol{Q},\boldsymbol{S},\boldsymbol{U},\boldsymbol{V}}\ \frac{1}{2}\underbrace{\sum_{k_1,k_2}\left\|\mathcal{P}_{\Omega_{k_1,k_2}}(\theta_{k_1,k_2}(\boldsymbol{Y})-(\boldsymbol{Q}\star_{\text{row}}\boldsymbol{s}_{k_1})(\boldsymbol{U}\star_{\text{row}}\boldsymbol{v}_{k_2})^\top)\right\|_F^2}_{\text{Tensor decomposition on Hankel tensor slices}}$$

$$+\frac{\rho}{2}(\|\boldsymbol{Q}\|_F^2+\|\boldsymbol{S}\|_F^2+\|\boldsymbol{U}\|_F^2+\|\boldsymbol{V}\|_F^2)$$

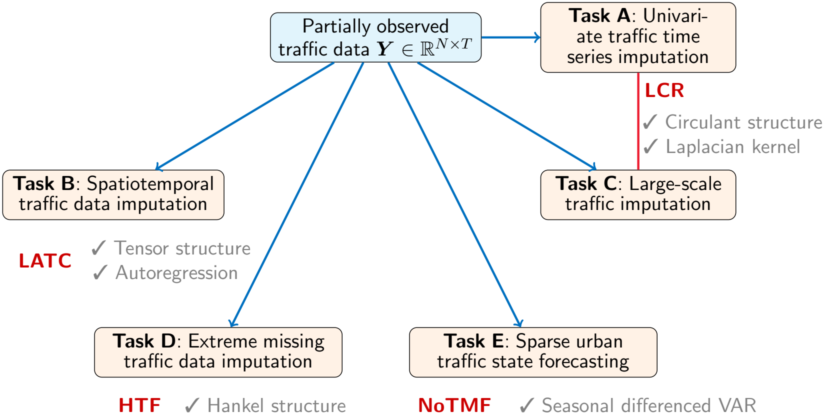
- Alternating minimization (let f be the obj.):

$$\begin{cases}\boldsymbol{Q}:=\{\boldsymbol{Q}\mid\frac{\partial f}{\partial\boldsymbol{Q}}=\boldsymbol{0}\} & \text{(conjugate gradient)}\\ \boldsymbol{s}_{k_1}:=\{\boldsymbol{s}_{k_1}\mid\frac{\partial f}{\partial\boldsymbol{s}_{k_1}}=\boldsymbol{0}\},\ \forall k_1 & \text{(conjugate gradient)}\\ \boldsymbol{U}:=\{\boldsymbol{U}\mid\frac{\partial f}{\partial\boldsymbol{U}}=\boldsymbol{0}\} & \text{(conjugate gradient)}\\ \boldsymbol{v}_{k_2}:=\{\boldsymbol{v}_{k_2}\mid\frac{\partial f}{\partial\boldsymbol{v}_{k_2}}=\boldsymbol{0}\},\ \forall k_2 & \text{(conjugate gradient)}\end{cases}$$

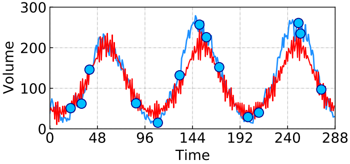
- Memory-efficient but still computationally costly!

Overview

We are working on **spatiotemporal traffic data imputation and forecasting**.



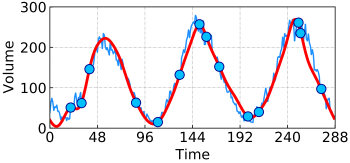
Univariate Traffic Time Series Imputation



CircNNM:

$$\min_{\boldsymbol{x}} \; \|\mathcal{C}(\boldsymbol{x})\|_* \quad \text{s.t.} \; \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

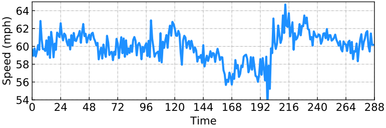
$\Downarrow$ Plus temporal regularization



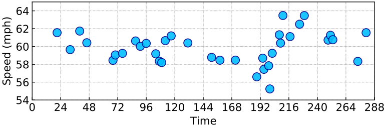
LCR:

$$\min_{\boldsymbol{x}} \; \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 \quad \text{s.t.} \; \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

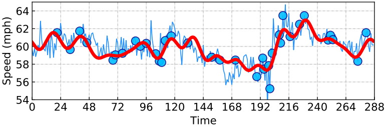
Univariate Traffic Time Series Imputation



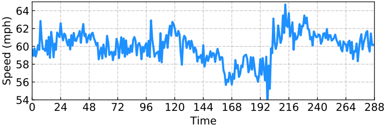
Mask **90%** observations



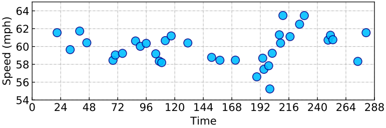
Reconstruct time series



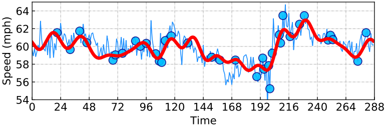
Univariate Traffic Time Series Imputation

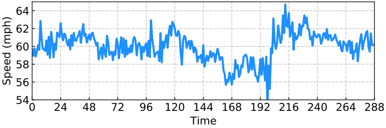


Mask **90%** observations

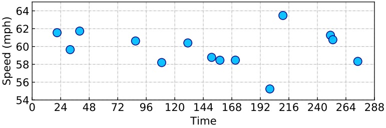


Reconstruct time series

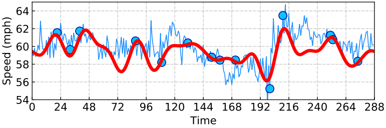




Mask **95%** observations



Reconstruct time series



Spatiotemporal Traffic Data Imputation

LATC vs. baseline models (in MAPE/RMSE)

- On the Seattle freeway traffic speed dataset ($\boldsymbol{Y} \in \mathbb{R}^{323\times 8064}$)

Missing rate	LATC	LAMC	LRTC-TNN	BTMF	SPC
30%, Random Missing	**4.90**/**3.16**	5.98/3.73	4.99/3.20	5.91/3.72	5.92/3.62
70%, Random Missing	**5.96**/**3.71**	8.02/4.70	6.10/3.77	6.47/3.98	7.38/4.30
90%, Random Missing	**7.46**/**4.50**	10.56/5.91	8.08/4.80	8.17/4.81	9.75/5.31
30%, Nonrandom Missing	7.10/4.33	6.99/4.25	**6.85**/**4.21**	9.26/5.36	8.87/4.99
70%, Nonrandom Missing	9.40/5.40	9.75/5.60	**9.23**/**5.35**	10.47/6.15	11.32/5.92
30%, Block-out Missing	**9.43**/**5.36**	27.05/13.66	9.52/5.41	14.33/13.60	11.30/5.84

- On the Portland highway traffic volume dataset ($\boldsymbol{Y} \in \mathbb{R}^{1156\times 2976}$)

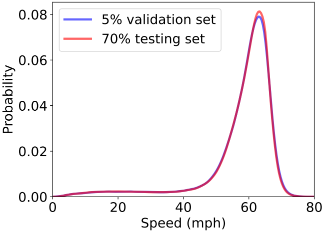
Missing rate	LATC	LAMC	LRTC-TNN	BTMF	SPC
30%, Random Missing	**16.95**/**15.99**	17.93/16.03	17.27/16.08	18.22/19.14	21.29/56.73
70%, Random Missing	**19.59**/**18.70**	21.26/19.37	19.99/18.73	19.96/22.21	24.35/43.32
90%, Random Missing	23.15/22.83	25.64/23.75	**22.90**/**22.68**	23.90/25.71	28.45/39.65
30%, Nonrandom Missing	**19.48**/19.14	19.93/19.69	19.59/**18.91**	19.55/20.38	26.96/60.33
70%, Nonrandom Missing	27.67/45.03	25.75/28.25	30.26/60.85	**23.86**/**26.74**	33.42/47.34
30%, Block-out Missing	**24.01**/**23.50**	29.21/27.60	31.74/74.42	27.85/25.68	31.01/60.33

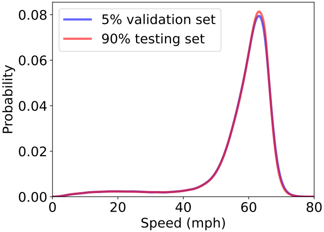
- LATC vs. LAMC: The significance of tensor representation
- LATC vs. LRTC-TNN: The significance of temporal autoregression

Spatiotemporal Traffic Data Imputation

Parameter tuning process: Training set, validation set, and testing set?

- Random missing on the Seattle freeway traffic speed dataset





- Imputation performance (e.g., 70% missing rate)

On the validation set (5% data)

γ/λ	Truncation parameter				
	$r=5$	$r=10$	$r=15$	$r=20$	$r=25$
1/10	7.84/4.52	7.20/4.25	6.82/4.08	6.60/3.98	6.41/3.92
1/5	7.84/4.52	7.20/4.25	6.82/4.08	6.59/3.97	6.41/3.92
1	7.81/4.51	7.18/4.24	6.80/4.07	6.58/3.97	6.39/3.91
5	7.70/4.45	7.09/4.20	6.72/4.04	6.49/3.93	6.29/3.87
10	7.59/4.39	7.00/4.16	6.64/4.00	6.41/3.89	**6.22/3.83**

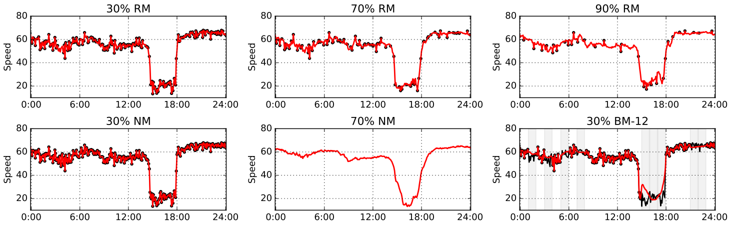
On the testing set (70% data)

γ/λ	Truncation parameter				
	$r=5$	$r=10$	$r=15$	$r=20$	$r=25$
1/10	7.83/4.53	7.18/4.27	6.80/4.09	6.58/3.99	6.41/3.92
1/5	7.83/4.53	7.18/4.26	6.80/4.09	6.57/3.98	6.40/3.92
1	7.80/4.52	7.16/4.25	6.78/4.08	6.55/3.98	6.40/3.92
5	7.70/4.47	7.08/4.21	6.70/4.04	6.46/3.94	6.29/3.87
10	7.58/4.41	6.99/4.17	6.62/4.01	6.39/3.90	**6.21/3.84**

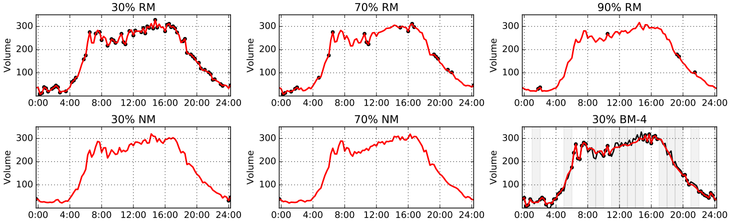
Spatiotemporal Traffic Data Imputation

LATC imputation

- Seattle freeway traffic speed data



- Portland highway traffic volume data



Large-Scale Traffic Data Imputation

LCR vs. baseline models (in MAPE/RMSE)

- PeMS-4W: California freeway traffic speed dataset ($\boldsymbol{Y} \in \mathbb{R}^{11160 \times 8064}$)

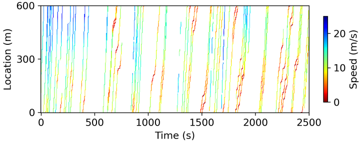
Model	Missing rate			
	30%	50%	70%	90%
LCR-2D	1.50/**1.49**	1.76/**1.69**	**2.07**/**2.06**	**3.19**/**3.05**
LCR_N	**1.48**/1.50	**1.73**/1.73	**2.07**/2.12	3.24/3.22
LCR	1.50/**1.49**	1.76/**1.69**	2.08/2.07	3.21/3.06
CTNNM	2.26/1.84	2.67/2.14	3.40/2.66	5.22/3.90
CircNNM	2.26/1.84	2.69/2.15	3.43/2.67	5.34/3.96
LRMC	2.04/1.80	2.43/2.12	3.08/2.66	6.05/4.43
HaLRTC	1.98/1.73	2.22/1.98	2.84/2.49	4.39/3.66
LRTC-TNN	1.68/1.55	1.93/1.77	2.33/2.14	3.40/3.10
NoTMF	2.95/2.65	3.05/2.73	3.33/2.97	5.22/4.71

Results

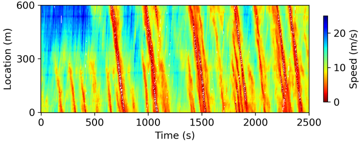
- LCR-2D $>$ CTNNM: The importance of temporal regularization.
- CTNNM $\geq$ CircNNM: Ciculant tensor is superior to circulant matrix.
- LCR $>$ LRMC/LRTC: The importance of global/local modeling.

$$\mathcal{O}(NT\log(NT)) \quad \text{(FFT)} \quad \text{vs.} \quad \mathcal{O}(\min\{N^2T, NT^2\}) \quad \text{(SVD)}$$

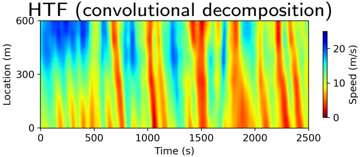
Extreme Missing Traffic Data Imputation



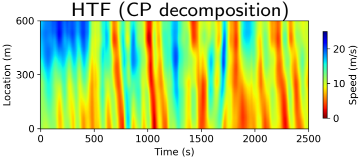
Sparse speed field



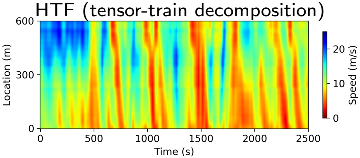
Ground truth speed field



MAPE = **51.92%**



MAPE = 53.93%



MAPE = 56.48%

Extreme Missing Traffic Data Imputation

HTF vs. baseline models (in MAPE/RMSE)

- On the Seattle freeway traffic speed dataset ($\boldsymbol{Y} \in \mathbb{R}^{323 \times 8064}$)

Model	Random missing rate			
	80%	85%	90%	95%
HTF (`convolution`)	**6.21/3.88**	**6.51/4.06**	**6.98/4.30**	**8.02/4.84**
HTF (`tensor-train`)	8.75/5.16	9.86/5.76	9.24/5.36	9.89/5.70
HTF (`CP`)	7.25/4.33	7.93/4.66	8.61/4.96	9.25/5.20
LATC	6.50/4.00	6.90/4.21	7.47/4.51	8.75/5.05
LRTC-TNN	6.97/4.24	7.43/4.43	8.19/4.81	9.60/5.55
LCR	6.75/4.15	7.31/4.38	7.96/4.71	9.78/5.39
BTMF	6.85/4.17	7.36/4.42	8.13/4.79	9.63/5.48

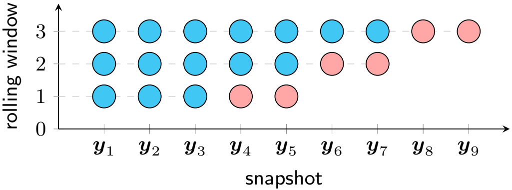
Results

- Convolutional tensor decomposition outperforms both tensor-train and CP tensor decomposition.
- Our HTF model performs better than state-of-the-art baseline models.

Sparse Urban Traffic State Forecasting

NoTMF forecasting

- NYC movement speed dataset:
  - Ten-week data of size 98210×1680
  - Contain 66.56% missing values
- Rolling forecasting setup:
  - Training set: 8-week data
  - Validation set: 1-week data
  - Testing set: 1-week data
  - Time horizon: $\delta = 1, 2, 3, 6$
- Rolling forecasting illustration ($\delta = 2$):



Sparse Urban Traffic State Forecasting

Sparse Urban Traffic State Forecasting

Conclusion

- Matrix nuclear norm ($\|\boldsymbol{X}\|_*$) minimization — Candès & Recht'09
- Singular value thresholding — Cai et al.'10
- Truncated nuclear norm ($\|\boldsymbol{X}\|_{r,*}, r \in \mathbb{N}^+$) minimization — Zhang et al.'12, Hu et al.'12
- Tensor nuclear norm ($\|\boldsymbol{\mathcal{X}}\|_*$) minimization — Liu et al.'13
- Generalized nonconvex nonsmooth low-rank minimization — Lu et al.'14
- Tensor nuclear norm minimization with linear transform — Lu et al.'19
- Low-rank Hankel matrix/tensor ($\mathcal{H}_\tau(\cdot)$) completion — Yokota et al.'18, Sedighin et al.'20, Cai et al.'21, Yamamoto et al.'22
- Circulant/Convolution nuclear norm ($\|\mathcal{C}(\boldsymbol{X})\|_*$ or $\|\mathcal{C}_{\tilde{\tau}}(\boldsymbol{X})\|_*$) minimization — Liu'22, Liu & Zhang'23

Conclusion

- Data (large-scale, high-dimensional, city-wide, sparse)
- Modeling (meaningfulness and importance of temporal correlations)

Conclusion

Low-Rank Autoregressive Tensor Completion (LATC):

- (Highlight) Global & local consistency
  - ✓ Tensor structure $\|\mathcal{Q}(\boldsymbol{\mathcal{Z}})\|_{r,*}$ ✓ Autoregression $\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}}$

Laplacian Convolutional Representation (LCR):

- (Solution) Global & local time series trends
  - Global trend modeling: $\|\mathcal{C}(\boldsymbol{x})\|_*$
  - Local trend modeling: $\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2$
- (Highlight) A unified framework with the FFT implementation.

Hankel Tensor Factorization (HTF):

- (Highlight) Memory-efficient Hankel indexing & convolutional parameterization.

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Thanks for your attention!

Any Questions?

https://xinychen.github.io/papers/thesis.pdf

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Nonstationary Temporal Matrix Factorization

Rewrite VAR in the form of matrix

Temporal operators

For any multivariate time series $\boldsymbol{X} \in \mathbb{R}^{R \times T}$ with $m, d \in \mathbb{N}^{+}$, if we define temporal operators as

$$\begin{aligned} \boldsymbol{\Psi}_k \triangleq & \begin{bmatrix} \mathbf{0}_{(T-d-m)\times(d-k)} & -\boldsymbol{I}_{T-d-m} & \mathbf{0}_{(T-d-m)\times(k+m)} \end{bmatrix} \\ & + \begin{bmatrix} \mathbf{0}_{(T-d-m)\times(d+m-k)} & \boldsymbol{I}_{T-d-m} & \mathbf{0}_{(T-d-m)\times k} \end{bmatrix} \\ & \in \mathbb{R}^{(T-d-m)\times T},\ k = 0, 1, \ldots, d \end{aligned}$$

then

$$\sum_{t=d+m+1}^{T} \|(\boldsymbol{x}_t - \boldsymbol{x}_{t-m}) - \sum_{k=1}^{d} \boldsymbol{A}_k(\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k})\|_2^2$$

$$\equiv \|\boldsymbol{X}\boldsymbol{\Psi}_0^\top - \sum_{k=1}^{d} \boldsymbol{A}_k \boldsymbol{X} \boldsymbol{\Psi}_k^\top\|_F^2 \triangleq \|\boldsymbol{X}\boldsymbol{\Psi}_0^\top - \boldsymbol{A}(\boldsymbol{I}_d \otimes \boldsymbol{X})\boldsymbol{\Psi}^\top\|_F^2$$

where $\boldsymbol{A} \triangleq \begin{bmatrix} \boldsymbol{A}_1 & \cdots & \boldsymbol{A}_d \end{bmatrix}$ and $\boldsymbol{\Psi} \triangleq \begin{bmatrix} \boldsymbol{\Psi}_1 & \cdots & \boldsymbol{\Psi}_d \end{bmatrix}$.

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF:

$$\begin{aligned}\min_{\boldsymbol{W},\boldsymbol{X},\boldsymbol{A}}\ & \frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2+\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)\\ & +\frac{\gamma}{2}\|\boldsymbol{X}\boldsymbol{\Psi}_0^\top-\boldsymbol{A}(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^\top\|_F^2\end{aligned}$$

Alternating minimization method:

- w.r.t. $\boldsymbol{W}$:

$$\frac{\partial f}{\partial \boldsymbol{W}}=-\boldsymbol{X}\mathcal{P}_{\Omega}^\top(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})+\rho\boldsymbol{W}=\boldsymbol{0}\qquad\text{(Least squares)}$$

- w.r.t. $\boldsymbol{X}$:

$$\frac{\partial f}{\partial \boldsymbol{X}}=-\boldsymbol{W}\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})+\rho\boldsymbol{X}+\gamma\sum_{k=0}^{d}\boldsymbol{A}_k^\top\left(\sum_{h=0}^{d}\boldsymbol{A}_h\boldsymbol{X}\boldsymbol{\Psi}_h^\top\right)\boldsymbol{\Psi}_k=\boldsymbol{0}$$

  This generalized Sylvester equation can be solved by **conjugate gradient**.

- w.r.t. $\boldsymbol{A}$:

$$\boldsymbol{A}=\boldsymbol{X}\boldsymbol{\Psi}_0^\top[(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^\top]^\dagger\qquad\text{(Least squares)}$$

Low-Rank Autoregressive Tensor Completion

- Augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{Z},\boldsymbol{\mathcal{W}})=\|\boldsymbol{\mathcal{X}}\|_{r,*}+\frac{\gamma}{2}\|\boldsymbol{Z}\|_{\boldsymbol{A},\mathcal{H}}+\frac{\lambda}{2}\|\boldsymbol{\mathcal{X}}-\mathcal{Q}(\boldsymbol{Z})\|_F^2+\langle\boldsymbol{\mathcal{W}},\boldsymbol{\mathcal{X}}-\mathcal{Q}(\boldsymbol{Z})\rangle+\pi(\boldsymbol{Z})$$

- The ADMM[6] scheme:

$$\begin{cases}\boldsymbol{\mathcal{X}}:=\arg\min\limits_{\boldsymbol{\mathcal{X}}}\ \mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{Z},\boldsymbol{\mathcal{W}}) & \text{(Truncated nuclear norm minimization)}\\ \boldsymbol{Z}:=\arg\min\limits_{\boldsymbol{Z}}\ \mathcal{L}(\boldsymbol{\mathcal{X}},\boldsymbol{Z},\boldsymbol{\mathcal{W}}) & \text{(Generalized Sylvester equation)}\\ \boldsymbol{\mathcal{W}}:=\boldsymbol{\mathcal{W}}+\lambda(\boldsymbol{\mathcal{X}}-\mathcal{Q}(\boldsymbol{Z})) & \text{(Standard update)}\end{cases}$$

✓ Solution to $\boldsymbol{\mathcal{X}}$: singular value thresholding

✓ Solution to $\boldsymbol{Z}$: conjugate gradient

[6] Alternating Direction Method of Multipliers.

Laplacian Convolutional Representation

- Optimize $\boldsymbol{x}$ via FFT (in $\mathcal{O}(T\log T)$ time):

$$\boldsymbol{x} := \arg\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_{*} + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{w}/\lambda\|_2^2$$

$$\Longrightarrow \hat{\boldsymbol{x}} := \arg\min_{\hat{\boldsymbol{x}}} \ \|\hat{\boldsymbol{x}}\|_1 + \frac{\gamma}{2T}\|\hat{\boldsymbol{\ell}}\circ\hat{\boldsymbol{x}}\|_2^2 + \frac{\lambda}{2T}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{z}}+\hat{\boldsymbol{w}}/\lambda\|_2^2$$

where we introduce $\{\hat{\boldsymbol{\ell}},\hat{\boldsymbol{x}},\hat{\boldsymbol{z}},\hat{\boldsymbol{w}}\} \triangleq \mathcal{F}\{\boldsymbol{\ell},\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}\}$ (i.e., FFT).

ℓ_1-norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of ℓ_1-norm minimization in complex space:

$$\min_{\hat{\boldsymbol{x}}} \ \|\hat{\boldsymbol{x}}\|_1 + \frac{\omega}{2}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{h}}\|_2^2$$

with complex-valued vectors $\hat{\boldsymbol{x}},\hat{\boldsymbol{h}}\in\mathbb{C}^T$ and weight parameter ω, element-wise, the solution is given by

$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|}\cdot\max\{0,|\hat{h}_t|-1/\omega\}, t=1,\dots,T.$$

Flipping Operation in LCR

Results on speed fields

Tuning Hyperparameters