

# The Relevance of $t$ -Statistics for Small Sample Sizes

An Introductory Class to Higher Statistics

**Xinyu Chen**

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# Outline

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Answering a lot questions, e.g.,

- ❶ How was  $t$ -statistic developed?
- ❷ Normal distribution vs. student  $t$ -distribution?
- ❸ What is  $t$ -statistic?
- ❹ How to calculate a  $t$ -test?
- ❺ What are the hypotheses and the assumptions?
- ❻ How to interpret results?

Huge real-world applications

# Development

The problems of **small sample sizes**

1876

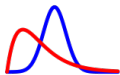
*t*-distribution as  
a posterior distribution



F. R. Helmert



J. Lüroth



(Source: Wiki)

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

1895

*t*-distribution as Pearson  
type IV distribution



K. Pearson



(Source: Wiki)

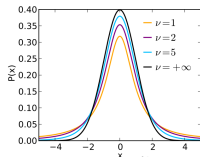
Diagram of the Pearson system

1908

Student *t*-distribution  
"The Probable Error of a Mean" (Biometrika)



W. S. Gosset

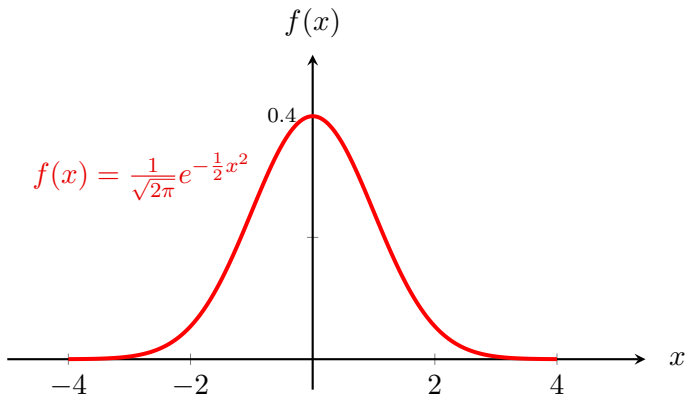


(Source: Wiki)

Probability density function

## Revisiting Normal Distribution

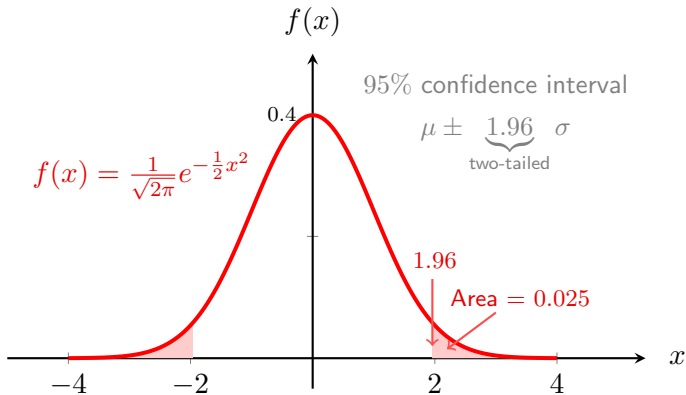
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Probability density function of the standard normal distribution

## Revisiting Normal Distribution

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Probability density function of the standard normal distribution

# Implementing $z$ -Test

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## Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh with a population standard deviation of 5 kWh. A random sample of 40 households has an average daily energy consumption of 32 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

# Implementing $z$ -Test

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## Steps:

### ❶ Formulate Hypotheses

- Null Hypothesis ( $H_0$ ): The population mean is  $\mu = 30$  kWh.
- Alternative Hypothesis ( $H_a$ ): The population mean is not  $\mu = 30$  kWh ( $\mu \neq 30$ ).

### ❷ Use the $z$ -test formula since the population standard deviation is known:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{32 - 30}{5/\sqrt{40}} = \frac{2}{5/6.32} = \frac{2}{0.79} \approx 2.53$$

- $\bar{x} = 32$  (sample mean)
- $\mu = 30$  (population mean)
- $n = 40$  (sample size)
- $\sigma = 5$  (population standard deviation)

# Implementing $z$ -Test

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- ③ Decision rule at a 95% confidence interval

- Reject  $H_0$  if  $|z| > 1.96$ .
- Otherwise, fail to reject  $H_0$ .

- ④ Interpretation

- The test statistic  $|z| = 2.53 > 1.96$  (exceeding the critical value).
- Thus, we reject the null hypothesis.
- The sample provides sufficient evidence to conclude that the average daily energy consumption is not 30 kWh.



## Student $t$ -Distribution

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- Probability density function (w/ random variable  $x$ ):

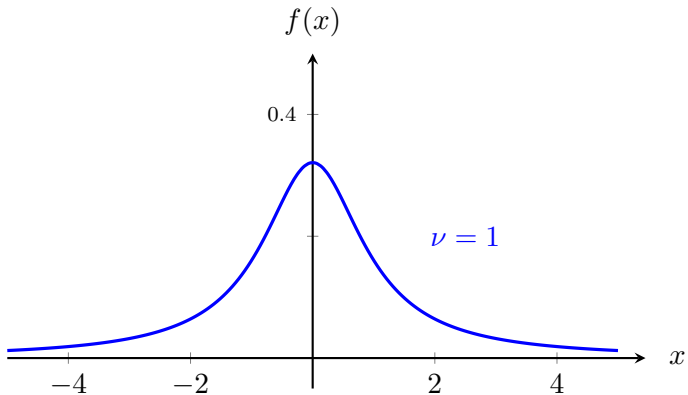
$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

- $\nu \in \mathbb{Z}^+$ : Degrees of freedom
  - $\Gamma(\cdot)$ : The Gamma function
- Example of  $\nu = 5$  degrees of freedom:

$$f(x) = \frac{\Gamma(3)}{\sqrt{5\pi}\Gamma(2.5)} \left(1 + \frac{x^2}{5}\right)^{-3}$$

# Student $t$ -Distribution

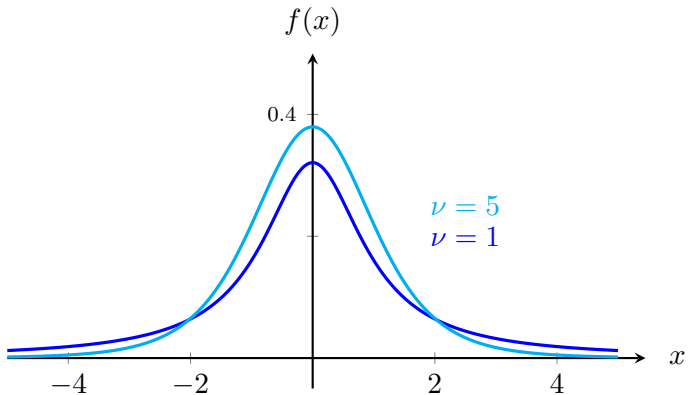
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Student  $t$ -distribution

# Student $t$ -Distribution

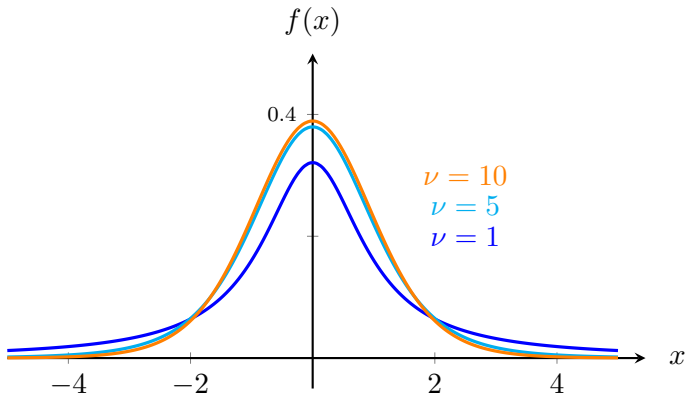
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Student  $t$ -distribution

# Student $t$ -Distribution

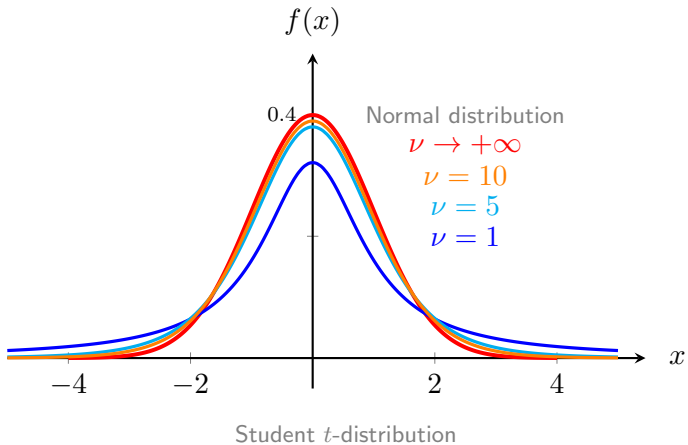
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Student  $t$ -distribution

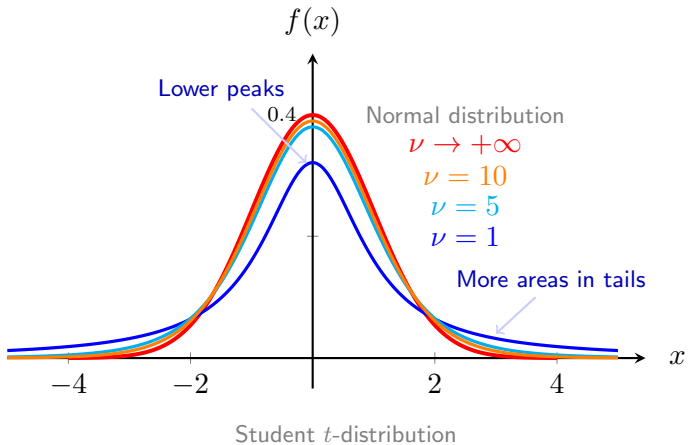
# Student $t$ -Distribution

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# Student $t$ -Distribution

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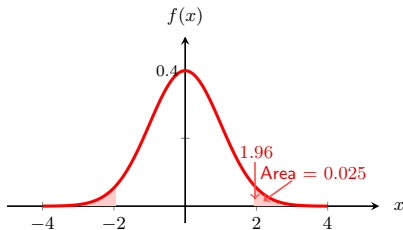
## 95% Confidence Interval

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For the population mean  $\mu$  (given) and standard deviation  $\sigma$  (given or not?)

- If **population standard deviation  $\sigma$**  is known

$$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$



Standard normal distribution

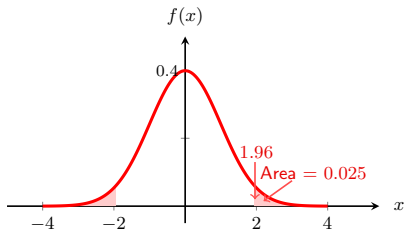
## 95% Confidence Interval

For the population mean  $\mu$  (given) and standard deviation  $\sigma$  (given or not?)

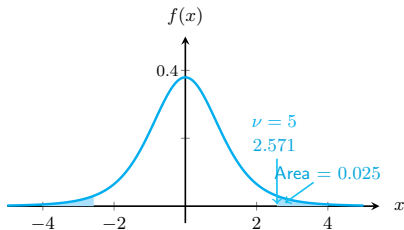
- If **population standard deviation  $\sigma$**  is known
- If  $\sigma$  is unknown, using **sample standard deviation  $s$**  instead

$$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

$$\bar{x} \pm ? \times \frac{s}{\sqrt{n}}$$



Standard normal distribution



Student  $t$ -distribution

- Heavier tails in student  $t$ -distribution ( $\nu = n - 1$  degrees of freedom) is important for small sample size  $n$



- Formula of  $t$ -statistic:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

- $\bar{x}$  sample mean
  - $\mu$  population mean
  - $s$  sample standard deviation
  - $n$  sample size (usually small  $n$ )
- The  $t$ -statistic quantifies the difference relative to variability in the data.
- (Interpretation) A high absolute value of  $t$  (larger than the critical value from the  $t$ -table) suggests a statistically significant difference.
- The problem of small sample size!

# Implementing $t$ -Test

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## Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh with a population standard deviation of 5 kWh. A random sample of 6 households has an average daily energy consumption of 32 kWh, with a sample standard deviation of 6 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

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### ❷ Use the $t$ -test formula since the population standard deviation is not known:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{32 - 30}{6/\sqrt{6}} = \frac{2}{6/2.449} \approx 0.816$$

- $\bar{x} = 32$  (sample mean)
- $s = 6$  (sample standard deviation)
- $n = 6$  (sample size)
- $\sigma = 30$  (population mean)

## *t*-Table

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### Small sample sizes

- Degrees of freedom for a *t*-test:

$$\nu = \underbrace{n}_{\text{sample size}} - 1 = 6 - 1 = 5$$

- t*-distributions with  $\nu$  degrees of freedom at a 95% confidence interval (two-tailed)

$\nu = 1$	$\nu = 5$	$\nu = 10$	$\nu \rightarrow +\infty$
12.706	2.571	2.228	1.960

- The critical *t*-value

$$t_{\nu, (1-0.95)/2} = t_{5, 0.025} = 2.571$$

# Implementing $t$ -Test

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### Steps:

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$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{32 - 30}{6/\sqrt{6}} = \frac{2}{6/2.449} \approx 0.816$$

- ③ Decision rule at a 95% confidence interval

- Reject  $H_0$  if  $|t| > 2.571$ .
- Otherwise, fail to reject  $H_0$ .

- ④ Interpretation

- The test statistic  $|t| = 0.816 < 2.571$ .
- Thus, we fail to reject the null hypothesis.
- There is not enough evidence to conclude that the average daily energy consumption differs from the company's claim of 30 kWh.

Summary...

## Teaching Concept

# Method

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- use math
- use figures
- use examples
- use data
- use codes
- use latex to create all examples



# Thanks for your attention!

## Any Questions?

### About me:

 Homepage: <https://xinychen.github.io>

 How to reach me: [chenxy346@gmail.com](mailto:chenxy346@gmail.com)