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# Matrix and Tensor Models for Spatiotemporal Traffic Data Imputation and Forecasting

Ph.D. Defense

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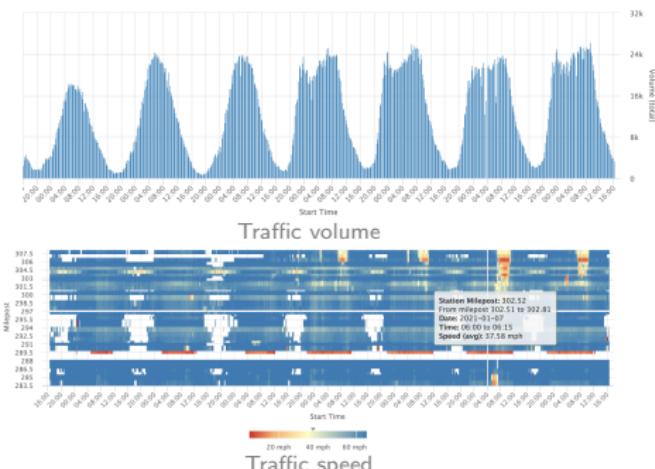
# Outline

- **Background**
- **Literature Review**
- **Nonstationary Temporal Matrix Factorization**
- **Low-Rank Autoregressive Tensor Completion**
- **Laplacian Convolutional Representation**
  - Motivation
  - Reformulate Laplacian Regularization
  - Traffic Time Series Imputation
- **Hankel Tensor Factorization**
  - Motivation
  - Hankel Structure
- **Experiments**
- **Conclusion**

# Multivariate Traffic Time Series

Many spatiotemporal traffic time series data are in the form of **matrix**.

- Example: Portland highway traffic data<sup>1</sup>.



- $X \in \mathbb{R}^{N \times T}$  with  $N$  spatial locations  $\times$   $T$  time steps
- Traffic volume/speed shows strong spatial/temporal dependencies

<sup>1</sup><https://portal.its.pdx.edu/home>

## Multiple Data Behaviors

Spatiotemporal traffic data are time series, but they involve multiple data behaviors.

- Incompleteness & sparsity
- High-dimensionality
- Multidimensionality
- Noises & outliers
- Time-varying behavior
- Nonstationarity
- .....

In addition, spatiotemporal correlations are also very important.

# Multiple Data Behaviors

## Sparsity & high-dimensionality

- Uber (hourly) movement speed data<sup>2</sup>



NYC movement



Seattle movement

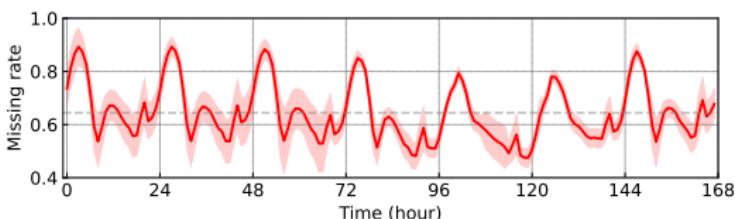
- The average speed on a given road segment for each hour of each day.
- Hourly speeds are computed when road segments have 5+ unique trips.
- **Issue:** insufficient sampling of ridesharing vehicles on the road network.

<sup>2</sup><https://movement.uber.com/>

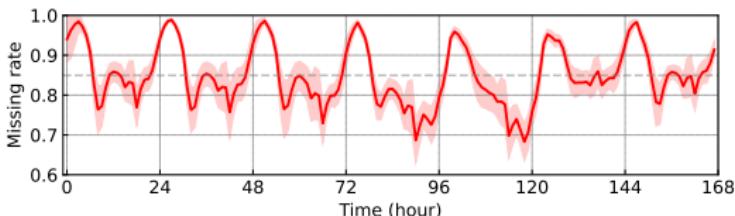
# Multiple Data Behaviors

## Sparsity & high-dimensionality

- **NYC** movement speed data (2019)
  - 98,210 road segments & 8,760 time steps (hours)
  - Overall missing rate: 64.43%

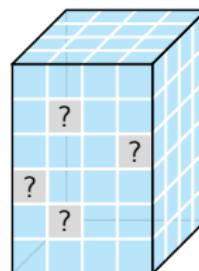
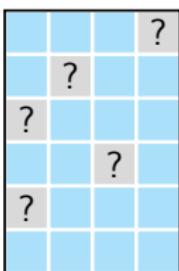


- **Seattle** movement speed data (2019)
  - 63,490 road segments & 8,760 time steps (hours)
  - Overall missing rate: 84.95%



## Problem Formulation

- **Objective A:** Given a multivariate time series data like  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  or a multidimensional time series data like  $\mathcal{Y} \in \mathbb{R}^{M \times N \times T}$  with the observed index set  $\Omega$ , impute the missing values of the data.

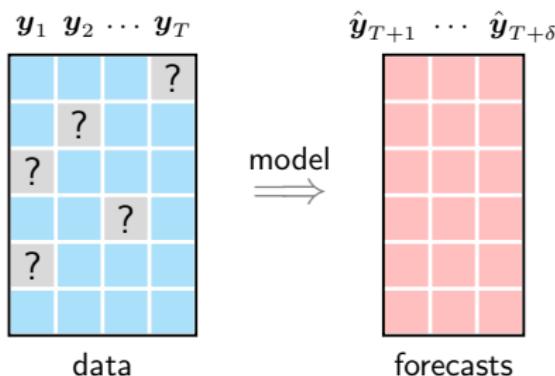


[Q]

- How to reconstruct missing values from observed data?
  - Matrix completion: From  $\mathcal{P}_\Omega(\mathbf{Y})$  (observed) to  $\mathcal{P}_\Omega^\perp(\mathbf{Y})$  (unobserved)
  - Tensor completion: From  $\mathcal{P}_\Omega(\mathcal{Y})$  (observed) to  $\mathcal{P}_\Omega^\perp(\mathcal{Y})$  (unobserved)
- How to make use of spatiotemporal correlations?
- How to make use of traffic time series dynamics?

## Problem Formulation

- **Objective B:** Given a partially observed data  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  consisting of time series  $\mathbf{y}_1, \dots, \mathbf{y}_T \in \mathbb{R}^N$ , forecast data points  $\hat{\mathbf{y}}_{T+\delta}, \delta \in \mathbb{N}^+$ .

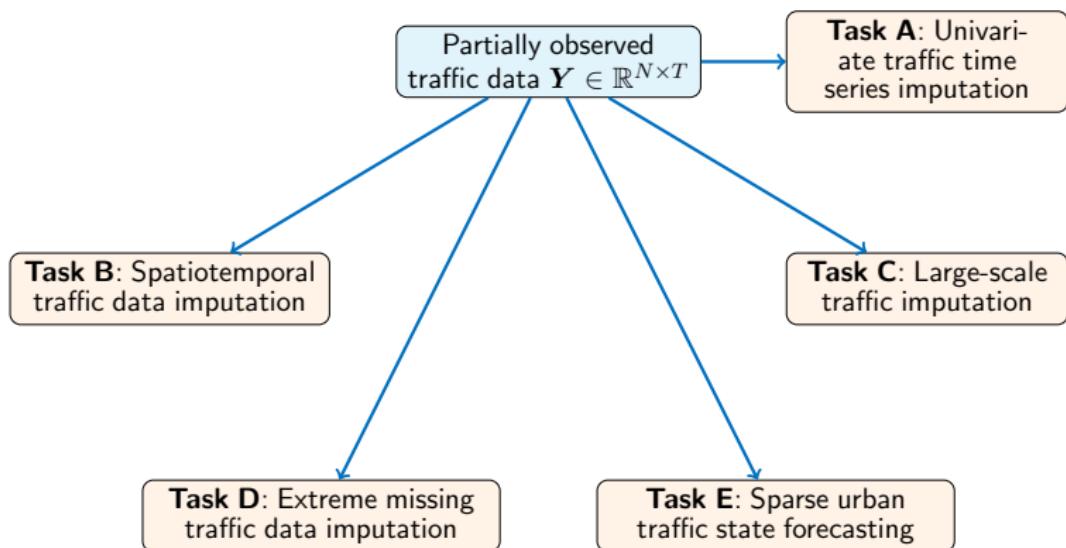


[Q]

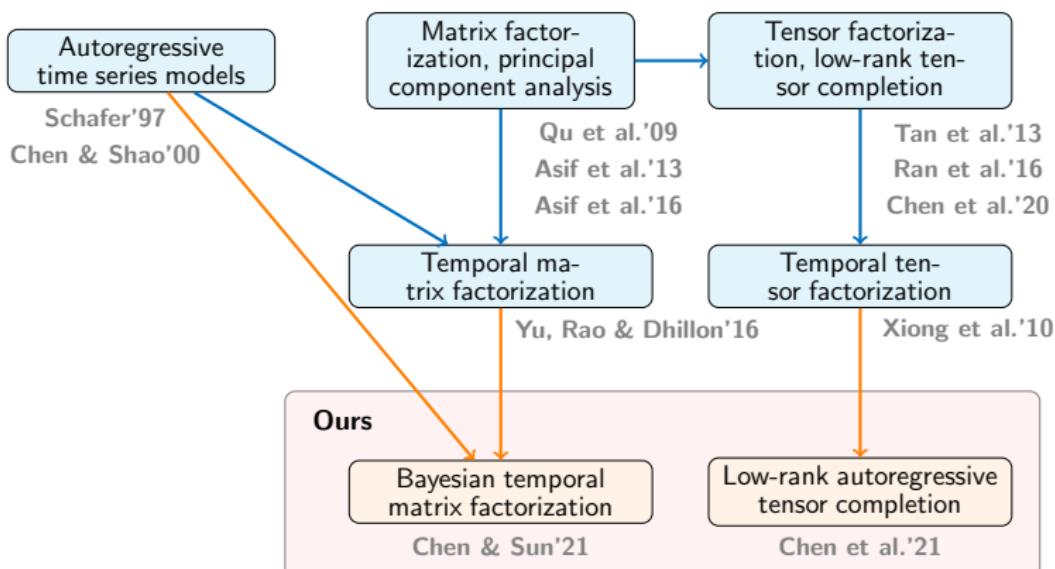
- How to learn from *high-dimensional* and *sparse* data?
- How to model *nonstationarity* in time series?
- How to perform forecasting on these time series?

# Whole Picture

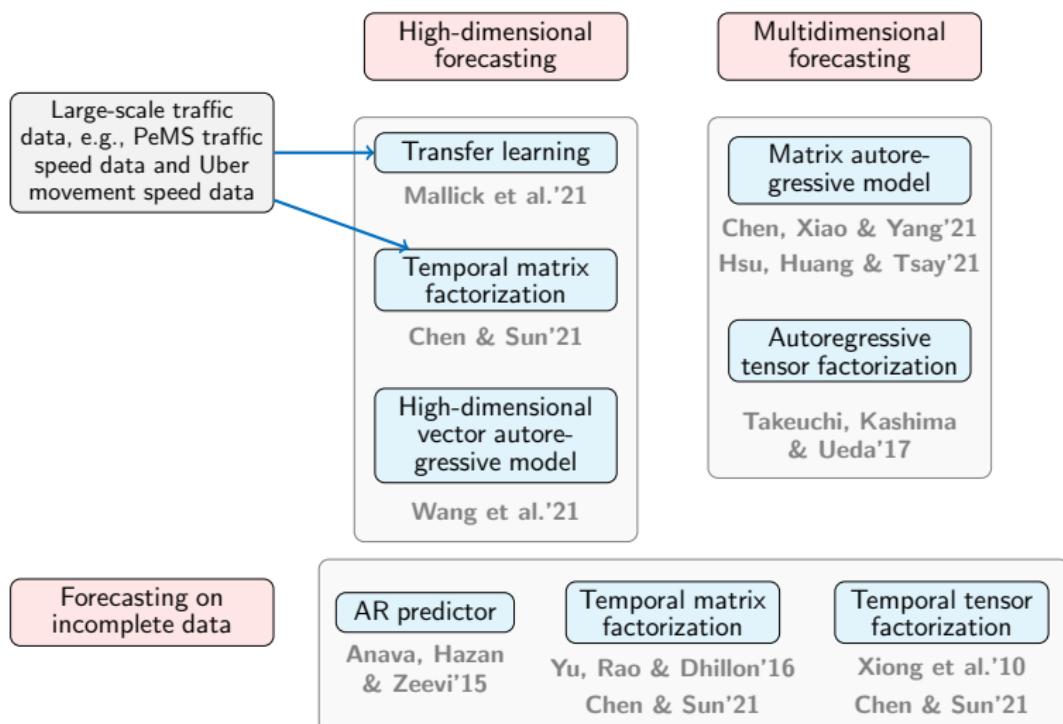
We are working on **spatiotemporal traffic data modeling**.



# Spatiotemporal Traffic Data Imputation

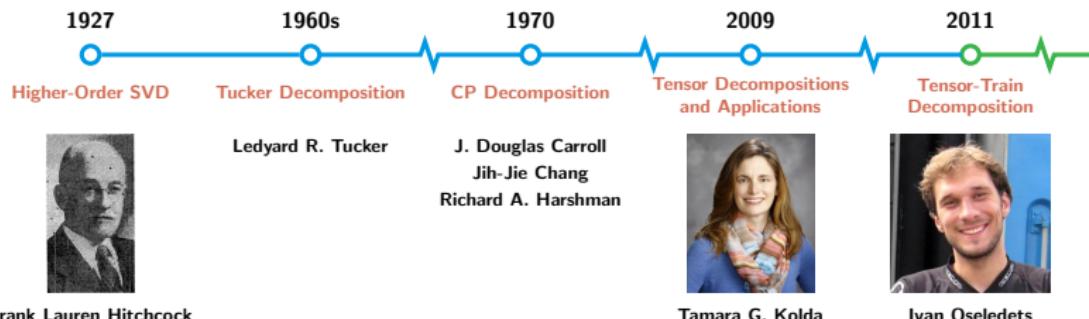


# Spatiotemporal Traffic Forecasting



# Tensor Factorization

- Revisit tensor factorization



Frank Lauren Hitchcock

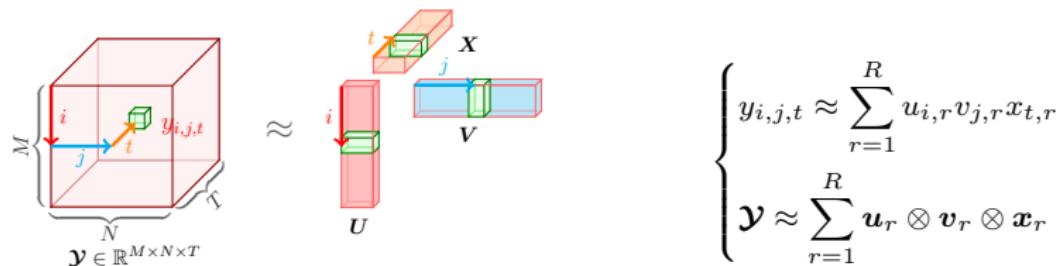
Ledyard R. Tucker

J. Douglas Carroll  
Jih-Jie Chang  
Richard A. Harshman

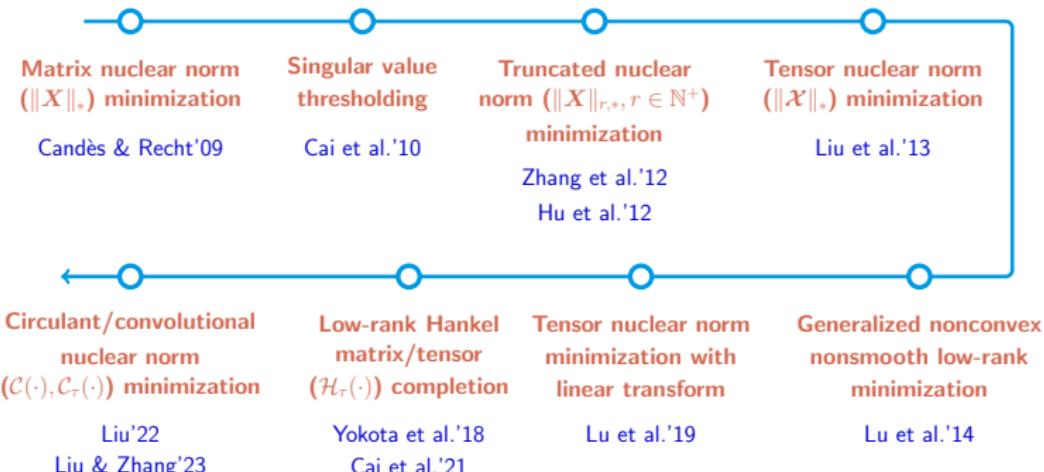
Tamara G. Kolda

Ivan Oseledets

- CP tensor factorization:** Factorize  $\mathcal{Y}$  into the combination of three rank- $R$  factor matrices (i.e., low-dimensional latent factors).



# Matrix/Tensor Completion

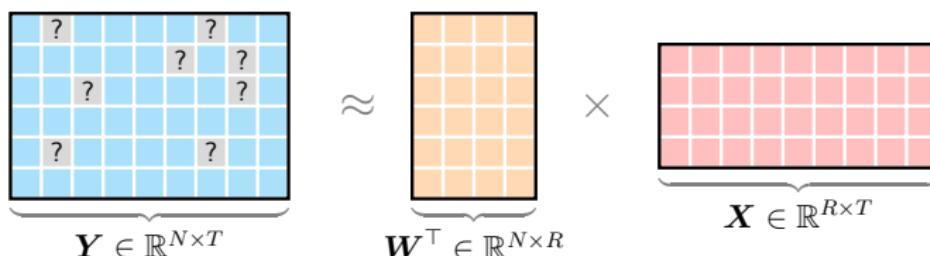


### This research

- Integrate temporal modeling techniques (e.g., temporal smoothing and time series autoregression) into low-rank matrix and tensor methods
- Implement spatiotemporal traffic data imputation and forecasting on partially observed data

# Matrix Factorization

A simple approach to reconstruct missing values.



## MF (Koren et al.'09)

Estimating low-dimensional  $\mathbf{W}, \mathbf{X}$ :

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2$$

on data  $\mathbf{Y}$  w/ observed index set  $\Omega$ .

- ✓ Learn from sparse data
- ✗ Temporal correlations
- ✗ Time series forecasting

# Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

$$\begin{array}{c} \text{?} \quad \text{?} \\ \text{?} \quad \text{?} \quad \text{?} \\ \text{?} \quad \text{?} \end{array} \underbrace{\quad \quad \quad}_{\mathbf{Y} \in \mathbb{R}^{N \times T}} \approx \underbrace{\quad \quad \quad}_{\mathbf{W}^\top \in \mathbb{R}^{N \times R}} \times \underbrace{\quad \quad \quad}_{\mathbf{X} \in \mathbb{R}^{R \times T}}$$

⇓ **X** are time series?

$$\begin{array}{c} \text{?} \quad \text{?} \quad \text{?} \\ \text{?} \quad \text{?} \quad \text{?} \\ \text{?} \quad \text{?} \end{array} \underbrace{\quad \quad \quad}_{\mathbf{Y} \in \mathbb{R}^{N \times T}} \approx \underbrace{\quad \quad \quad}_{\mathbf{W}^\top \in \mathbb{R}^{N \times R}} \times \begin{array}{c} \mathbf{x}_{t-2} \quad \mathbf{x}_{t-1} \quad \mathbf{x}_t \quad \mathbf{x}_{t+1} \quad \mathbf{x}_{t+2} \\ \dots \qquad \qquad \qquad \qquad \qquad \dots \end{array} \left. \right\} R$$

time step

$t-3 \quad t-2 \quad t-1 \quad t \quad t+1 \quad t+2 \quad t+3$

# Temporal Matrix Factorization

## MF (Koren et al.'09)

Estimating low-dimensional  $\mathbf{W}, \mathbf{X}$ :

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2$$

on data  $\mathbf{Y}$  w/ observed index set  $\Omega$ .

## dth-order VAR

$$+ \quad \mathbf{x}_t = \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} + \boldsymbol{\epsilon}_t$$

w/ coefficients  $\{\mathbf{A}_k\}$ .

↓ Yu et al.'16  
Chen & Sun'21

$$\min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2 + \frac{\gamma}{2} \sum_{t=d+1}^T \left\| \mathbf{x}_t - \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} \right\|_2^2$$

# Nonstationary Temporal Matrix Factorization

## Nonstationary temporal matrix factorization (NoTMF)

Given any partially observed time series data  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  with observed index set  $\Omega$ , then we assume a season- $m$  differencing on the latent temporal factors:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} & \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2) \\ & + \frac{\gamma}{2} \sum_{t=d+m+1}^T \left\| (\mathbf{x}_t - \mathbf{x}_{t-m}) - \sum_{k=1}^d \mathbf{A}_k (\mathbf{x}_{t-k} - \mathbf{x}_{t-m-k}) \right\|_2^2 \end{aligned}$$

- First-order differencing  $\mathbf{x}'_t = \mathbf{x}_t - \mathbf{x}_{t-1}$ .
  - Second-order differencing  $\mathbf{x}''_t = (\mathbf{x}_t - \mathbf{x}_{t-1}) - (\mathbf{x}_{t-1} - \mathbf{x}_{t-2})$ .
  - Twice-differenced series  $\mathbf{x}'''_t = (\mathbf{x}_t - \mathbf{x}_{t-m}) - (\mathbf{x}_{t-1} - \mathbf{x}_{t-m-1})$ .
- ✓ Stationarizing a time series with differencing can improve the prediction.<sup>3</sup>

<sup>3</sup>Stationarity and differencing: <https://otexts.com/fpp2/stationarity.html>

# Nonstationary Temporal Matrix Factorization

Rewrite VAR in the form of matrix

## Temporal operators

For any multivariate time series  $\mathbf{X} \in \mathbb{R}^{R \times T}$  with  $m, d \in \mathbb{N}^+$ , if we define temporal operators as

$$\begin{aligned}\Psi_k &\triangleq \begin{bmatrix} \mathbf{0}_{(T-d-m) \times (d-k)} & -\mathbf{I}_{T-d-m} & \mathbf{0}_{(T-d-m) \times (k+m)} \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{0}_{(T-d-m) \times (d+m-k)} & \mathbf{I}_{T-d-m} & \mathbf{0}_{(T-d-m) \times k} \end{bmatrix} \\ &\in \mathbb{R}^{(T-d-m) \times T}, \quad k = 0, 1, \dots, d\end{aligned}$$

then

$$\begin{aligned}&\sum_{t=d+m+1}^T \|(\mathbf{x}_t - \mathbf{x}_{t-m}) - \sum_{k=1}^d \mathbf{A}_k (\mathbf{x}_{t-k} - \mathbf{x}_{t-m-k})\|_2^2 \\ &\equiv \|\mathbf{X} \Psi_0^\top - \sum_{k=1}^d \mathbf{A}_k \mathbf{X} \Psi_k^\top\|_F^2 \triangleq \|\mathbf{X} \Psi_0^\top - \mathbf{A} (\mathbf{I}_d \otimes \mathbf{X}) \Psi^\top\|_F^2\end{aligned}$$

where  $\mathbf{A} \triangleq [\mathbf{A}_1 \quad \cdots \quad \mathbf{A}_d]$  and  $\Psi \triangleq [\Psi_1 \quad \cdots \quad \Psi_d]$ .

# Nonstationary Temporal Matrix Factorization

Rewrite NoTMF:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{X}, \mathbf{A}} \quad & \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2 + \frac{\rho}{2} (\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2) \\ & + \frac{\gamma}{2} \|\mathbf{X} \Psi_0^\top - \mathbf{A} (\mathbf{I}_d \otimes \mathbf{X}) \Psi^\top\|_F^2 \end{aligned}$$

Alternating minimization method:

- w.r.t.  $\mathbf{W}$ :

$$\frac{\partial f}{\partial \mathbf{W}} = -\mathbf{X} \mathcal{P}_\Omega^\top (\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) + \rho \mathbf{W} = \mathbf{0} \quad (\text{Least squares})$$

- w.r.t.  $\mathbf{X}$ :

$$\frac{\partial f}{\partial \mathbf{X}} = -\mathbf{W} \mathcal{P}_\Omega (\mathbf{Y} - \mathbf{W}^\top \mathbf{X}) + \rho \mathbf{X} + \gamma \sum_{k=0}^d \mathbf{A}_k^\top \left( \sum_{h=0}^d \mathbf{A}_h \mathbf{X} \Psi_h^\top \right) \Psi_k = \mathbf{0}$$

This generalized Sylvester equation can be solved by **conjugate gradient**.

- w.r.t.  $\mathbf{A}$ :

$$\mathbf{A} = \mathbf{X} \Psi_0^\top [(\mathbf{I}_d \otimes \mathbf{X}) \Psi^\top]^\dagger \quad (\text{Least squares})$$

# Nonstationary Temporal Matrix Factorization

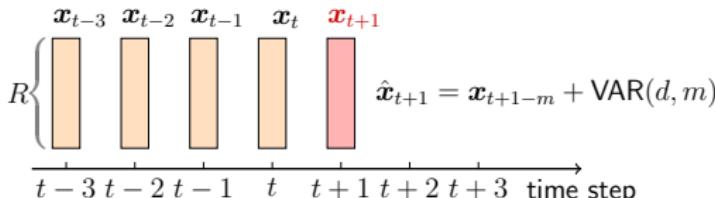
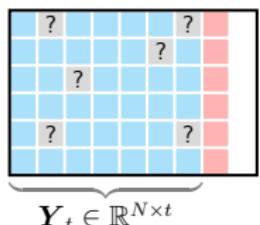
## NoTMF forecasting on streaming data?

- NoTMF: Use  $\mathbf{Y}_t$  to estimate  $\mathbf{W}, \mathbf{X}, \mathbf{A}$ .

### Implementation

- Estimate  $\mathbf{W}, \mathbf{X}, \mathbf{A}$
- Forecast  $\hat{\mathbf{x}}_{t+1}$  with VAR
- Return  $\hat{\mathbf{y}}_{t+1} = \mathbf{W}^\top \hat{\mathbf{x}}_{t+1}$

- ✓ Sparse input  $\mathbf{Y}_t$
- ✓ Low-dimensional temporal factors
- ✓ Forecast in latent spaces



# Nonstationary Temporal Matrix Factorization

## NoTMF forecasting on streaming data?

- Online forecasting (Gultekin & Paisley'18): Fix  $\mathbf{W}$  and use  $\mathbf{Y}_{t+1}$  to update  $\mathbf{X}, \mathbf{A}$ .

### Implementation

- Estimate  $\mathbf{X}, \mathbf{A}$
  - Forecast  $\hat{\mathbf{x}}_{t+2}$  with VAR
  - Return  $\hat{\mathbf{y}}_{t+2} = \mathbf{W}^\top \hat{\mathbf{x}}_{t+2}$
- ✓ Sparse input  $\mathbf{Y}_{t+1}$
  - ✓ Fixed spatial factors  $\mathbf{W}$
  - ✓ Forecast in latent spaces

$$\mathbf{Y}_{t+1} \in \mathbb{R}^{N \times (t+1)}$$

$$R \left\{ \begin{array}{c} \mathbf{x}_{t-3} \quad \mathbf{x}_{t-2} \quad \mathbf{x}_{t-1} \quad \mathbf{x}_t \quad \mathbf{x}_{t+1} \quad \mathbf{x}_{t+2} \\ t-3 \quad t-2 \quad t-1 \quad t \quad t+1 \quad t+2 \quad t+3 \end{array} \right. \text{ time step}$$
$$\hat{\mathbf{x}}_{t+2} = \mathbf{x}_{t+2-m} + \text{VAR}(d, m)$$

# Matrix/Tensor Completion

**Cornerstone:** Nuclear norm minimization in matrix/tensor completion

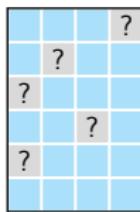
LRMC (Candès & Recht'09)

Estimating the matrix  $\mathbf{X}$ :

$$\min_{\mathbf{X}} \|\mathbf{X}\|_*$$

$$\text{s.t. } \mathcal{P}_\Omega(\mathbf{X}) = \mathcal{P}_\Omega(\mathbf{Y})$$

on data  $\mathbf{Y}$  w/ observed index set  $\Omega$ .



$$\mathcal{P}_\Omega(\mathbf{Y}) \in \mathbb{R}^{N \times T}$$

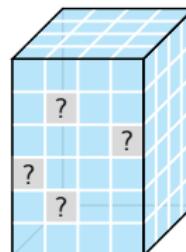
LRTC (Liu et al.'13)

Estimating the tensor  $\mathcal{X}$ :

$$\min_{\mathcal{X}} \|\mathcal{X}\|_*$$

$$\text{s.t. } \mathcal{P}_\Omega(\mathcal{X}) = \mathcal{P}_\Omega(\mathcal{Y})$$

on data  $\mathcal{Y}$  w/ observed index set  $\Omega$ .

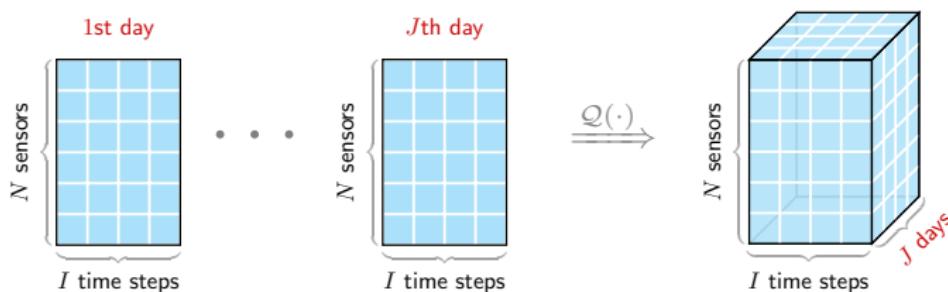


$$\mathcal{P}_\Omega(\mathcal{Y}) \in \mathbb{R}^{N \times I \times J}$$

- **Limitation:** Only cover global consistency

# Low-Rank Autoregressive Tensor Completion

- Introduce traffic tensors with day dimension<sup>4</sup> (Tan et al.'13, Chen et al.'19, ...)



<sup>4</sup>There are  $T = IJ$  time steps in total.

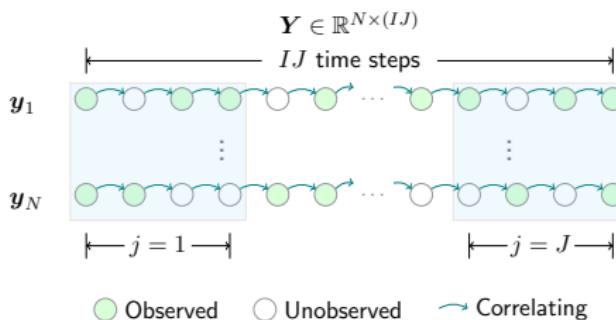
# Low-Rank Autoregressive Tensor Completion

- Build temporal correlations with autoregression

On the time series  $\mathbf{Y} \in \mathbb{R}^{N \times T}$ :

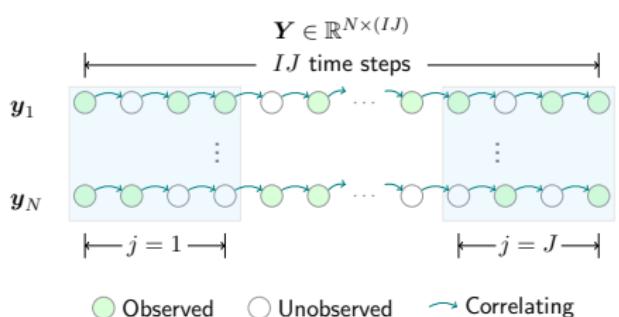
$$\|\mathbf{Y}\|_{\mathbf{A}, \mathcal{H}} \triangleq \sum_{n,t} \left( y_{n,t} - \sum_k \mathbf{a}_{n,k} y_{n,t-h_k} \right)^2$$

with the time lag set  $\mathcal{H} = \{h_1, \dots, h_d\}$  and the coefficient matrix  $\mathbf{A} \in \mathbb{R}^{N \times d}$ .

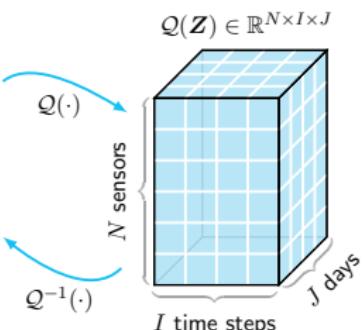


# Low-Rank Autoregressive Tensor Completion

**Local consistency** w/ autoregression



**Global consistency** w/ tensor structure



## LATC

Optimization problem:

$$\begin{aligned} \min_{\mathbf{Z}, \mathbf{A}} \quad & \|\mathcal{Q}(\mathbf{Z})\|_{r,*} + \frac{\gamma}{2} \|\mathbf{Z}\|_{\mathbf{A}, \mathcal{H}} \\ \text{s.t. } & \mathcal{P}_{\Omega}(\mathbf{Z}) = \mathcal{P}_{\Omega}(\mathbf{Y}) \end{aligned}$$

on data  $\mathbf{Y}$  w/ observed index set  $\Omega$ .

## Two subproblems

$$\Rightarrow \begin{cases} \mathbf{Z} := \underset{\mathcal{P}_{\Omega}(\mathbf{Z}) = \mathcal{P}_{\Omega}(\mathbf{Y})}{\arg \min} \|\mathcal{Q}(\mathbf{Z})\|_{r,*} + \frac{\gamma}{2} \|\mathbf{Z}\|_{\mathbf{A}, \mathcal{H}} \\ \mathbf{A} := \frac{1}{2} \|\mathbf{Z}\|_{\mathbf{A}, \mathcal{H}} \end{cases} \quad (\text{Least squares})$$

# Low-Rank Autoregressive Tensor Completion

$Z$ -subproblem:

$$\mathbf{Z} := \arg \min_{\mathcal{P}_{\Omega}(\mathbf{Z}) = \mathcal{P}_{\Omega}(\mathbf{Y})} \|\mathcal{Q}(\mathbf{Z})\|_{r,*} + \frac{\gamma}{2} \|\mathbf{Z}\|_{\mathbf{A}, \mathcal{H}}$$

- Augmented Lagrangian function:<sup>5</sup>

$$\mathcal{L}(\mathbf{X}, \mathbf{Z}, \mathbf{W}) = \|\mathbf{X}\|_{r,*} + \frac{\gamma}{2} \|\mathbf{Z}\|_{\mathbf{A}, \mathcal{H}} + \frac{\lambda}{2} \|\mathbf{X} - \mathcal{Q}(\mathbf{Z})\|_F^2 + \langle \mathbf{W}, \mathbf{X} - \mathcal{Q}(\mathbf{Z}) \rangle + \pi(\mathbf{Z})$$

## Implementation

Repeat

- Compute  $\mathbf{Z}$
- Compute  $\mathbf{A}$



## Implementation

Repeat

- Repeat
  - Compute  $\mathbf{X}$
  - Compute  $\mathbf{Z}$
  - Compute  $\mathbf{W}$
- Compute  $\mathbf{A}$

---

<sup>5</sup>  $\mathbf{W} \in \mathbb{R}^{N \times I \times J}$  (Lagrange multiplier); The indicator function:

$$\pi(\mathbf{Z}) = \begin{cases} 0, & \text{if } \mathcal{P}_{\Omega}(\mathbf{Z}) = \mathcal{P}_{\Omega}(\mathbf{Y}), \\ +\infty, & \text{otherwise.} \end{cases}$$

# Low-Rank Autoregressive Tensor Completion

- Augmented Lagrangian function:

$$\mathcal{L}(\mathbf{X}, \mathbf{Z}, \mathbf{W}) = \|\mathbf{X}\|_{r,*} + \frac{\gamma}{2} \|\mathbf{Z}\|_{A,\mathcal{H}} + \frac{\lambda}{2} \|\mathbf{X} - \mathcal{Q}(\mathbf{Z})\|_F^2 + \langle \mathbf{W}, \mathbf{X} - \mathcal{Q}(\mathbf{Z}) \rangle + \pi(\mathbf{Z})$$

- The ADMM<sup>6</sup> scheme:

$$\begin{cases} \mathbf{X} := \arg \min_{\mathbf{X}} \mathcal{L}(\mathbf{X}, \mathbf{Z}, \mathbf{W}) & \text{(Truncated nuclear norm minimization)} \\ \mathbf{Z} := \arg \min_{\mathbf{Z}} \mathcal{L}(\mathbf{X}, \mathbf{Z}, \mathbf{W}) & \text{(Generalized Sylvester equation)} \\ \mathbf{W} := \mathbf{W} + \lambda(\mathbf{X} - \mathcal{Q}(\mathbf{Z})) & \text{(Standard update)} \end{cases}$$

- ✓ Solution to  $\mathbf{X}$ : singular value thresholding
- ✓ Solution to  $\mathbf{Z}$ : conjugate gradient

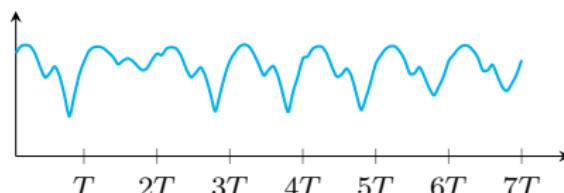
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<sup>6</sup>Alternating Direction Method of Multipliers.

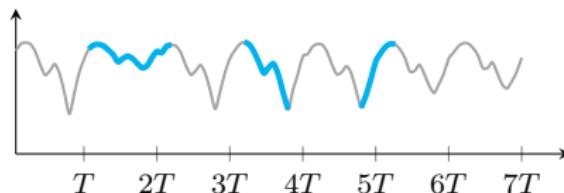
# Laplacian Convolutional Representation

## Motivation: Time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



- Local trends (e.g., short-term time series trends):



- [Question] How to characterize both global and local trends in sparse time series data?

# Laplacian Convolutional Representation

**[Local trend modeling]** Reformulate temporal regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.

Undirected and circulant graph

Modeling

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (2, -1, 0, 0, -1)^\top$$

↓

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \dots, -1}_\tau, 0, \dots, 0, \underbrace{-1, \dots, -1}_\tau)^\top \in \mathbb{R}^T$$

for any time series  $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$ .

- (Laplacian) Temporal regularization:

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2$$

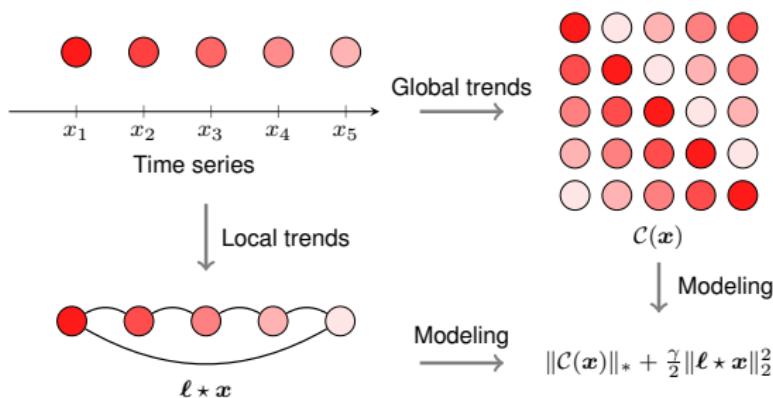
# Laplacian Convolutional Representation

## Laplacian Convolutional Representation (LCR)

For any partially observed time series  $\mathbf{y} \in \mathbb{R}^T$  with observed index set  $\Omega$ , LCR utilizes circulant matrix and Laplacian kernel to characterize **global and local trends** in time series, respectively, i.e.,

$$\begin{aligned} & \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 \\ & \text{s.t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

where  $\mathcal{C} : \mathbb{R}^T \rightarrow \mathbb{R}^{T \times T}$  denotes the circulant operator.  $\|\cdot\|_*$  denotes the nuclear norm of matrix, namely, the sum of singular values.



## Laplacian Convolutional Representation

- Augmented Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2$$

where  $\mathbf{w} \in \mathbb{R}^T$  is the Lagrange multiplier, and  $\langle \cdot, \cdot \rangle$  denotes the inner product.

- The ADMM scheme:

$$\begin{cases} \mathbf{x} := \arg \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Nuclear norm minimization)} \\ \mathbf{z} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Closed-form solution)} \\ = \frac{1}{\lambda + \eta} \mathcal{P}_\Omega(\lambda \mathbf{x} + \mathbf{w} + \eta \mathbf{y}) + \frac{1}{\lambda} \mathcal{P}_\Omega^\perp(\lambda \mathbf{x} + \mathbf{w}) \\ \mathbf{w} := \mathbf{w} + \lambda(\mathbf{x} - \mathbf{z}) & \text{(Standard update)} \end{cases}$$

- Optimize  $\mathbf{x}$ ?

$$\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1 \quad \& \quad \frac{1}{2} \|\ell * \mathbf{x}\|_2^2 = \frac{1}{2T} \|\mathcal{F}(\ell) \circ \mathcal{F}(\mathbf{x})\|_2^2$$

Nuclear norm minimization  $\Rightarrow \ell_1$ -norm minimization with FFT (in  $\mathcal{O}(T \log T)$  time).

## Laplacian Convolutional Representation

- Optimize  $\mathbf{x}$  via FFT (in  $\mathcal{O}(T \log T)$  time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2\end{aligned}$$

where we introduce  $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$  (i.e., FFT).

### $\ell_1$ -norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of  $\ell_1$ -norm minimization in complex space:

$$\min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\omega}{2} \|\hat{\mathbf{x}} - \hat{\mathbf{h}}\|_2^2$$

with complex-valued vectors  $\hat{\mathbf{x}}, \hat{\mathbf{h}} \in \mathbb{C}^T$  and weight parameter  $\omega$ , element-wise, the solution is given by

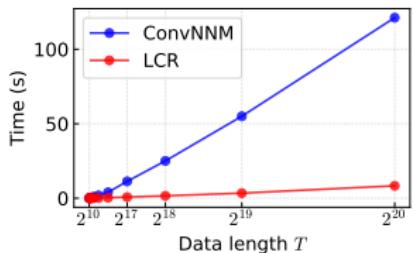
$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|} \cdot \max\{0, |\hat{h}_t| - 1/\omega\}, t = 1, \dots, T.$$

# Laplacian Convolutional Representation

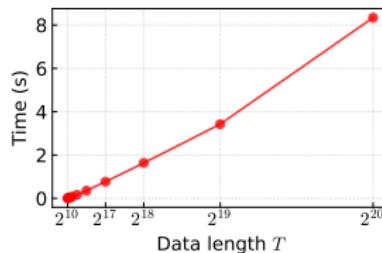
## Empirical time complexity

On the synthetic data  $\mathbf{y} \in \mathbb{R}^T$  with  $T \in \{2^{10}, 2^{11}, \dots, 2^{20}\}$

- Ours: **LCR**
  - An FFT implementation in  $\mathcal{O}(T \log T)$
  - The logarithmic factor  $\log T$  makes the FFT highly efficient
- Baseline: **ConvNNM**<sup>7</sup> ([Liu'22](#), [Liu & Zhang'23](#))
  - Convolution matrix  $C_{\tilde{\tau}}(\mathbf{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$  with kernel size  $\tilde{\tau} \in \mathbb{N}^+$
  - Singular value thresholding in  $\mathcal{O}(\tilde{\tau}^2 T)$



ConvNNM vs. LCR



LCR

<sup>7</sup>Convolution nuclear norm minimization.

## Two-Dimensional LCR (LCR-2D)

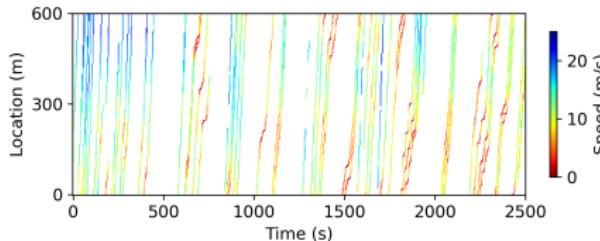
For any partially observed time series  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  with observed index set  $\Omega$ , LCR can be formulated as follows,

$$\begin{aligned} & \min_{\mathbf{X}} \|\mathcal{C}(\mathbf{X})\|_* + \frac{\gamma}{2} \|(\boldsymbol{\ell}_s \boldsymbol{\ell}^\top) \star \mathbf{X}\|_F^2 \\ & \text{s.t. } \|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \epsilon \end{aligned}$$

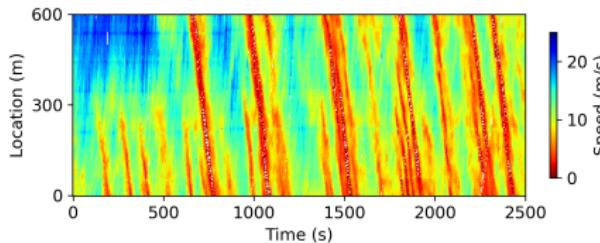
where  $\mathcal{C} : \mathbb{R}^{N \times T} \rightarrow \mathbb{R}^{N \times N \times T \times T}$  denotes the circulant operator.

## Motivation: Spatiotemporal data reconstruction

- Speed field reconstruction problem in vehicular traffic flow.



200-by-500 matrix  
(NGSIM)  $\Downarrow$  Reconstruct speed field from  
5% sparse trajectories?



- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal dependencies?

# Hankel Tensor Factorization

- Hankel matrix
  - Given  $\mathbf{x} = (1, 2, 3, 4, 5)^\top$  and window length  $\tau = 2$ , we have

$$\mathcal{H}_\tau(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4 \times 2}$$



Hankel matrix (Source: Twitter)

# Hankel Tensor Factorization

- Hankel matrix

- On time series  $\mathbf{y} = (y_1, y_2, \dots, y_5)^\top$  with  $\tau = 2$ :

$$\mathcal{H}_\tau(\mathbf{y}) = \begin{bmatrix} y_1 & y_2 \\ y_2 & y_3 \\ y_3 & y_4 \\ y_4 & y_5 \end{bmatrix} \approx \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\implies \hat{\mathbf{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \\ \hat{y}_5 \end{bmatrix} = \mathcal{H}_\tau^{-1} \left( \begin{bmatrix} v_1 x_1 & v_1 x_2 \\ v_2 x_1 & v_2 x_2 \\ v_3 x_1 & v_3 x_2 \\ v_4 x_1 & v_4 x_2 \end{bmatrix} \right) = \begin{bmatrix} v_1 x_1 \\ (\mathbf{v}_1 x_2 + \mathbf{v}_2 x_1)/2 \\ (\mathbf{v}_2 x_2 + \mathbf{v}_3 x_1)/2 \\ (\mathbf{v}_3 x_2 + \mathbf{v}_4 x_1)/2 \\ \mathbf{v}_4 x_2 \end{bmatrix}$$

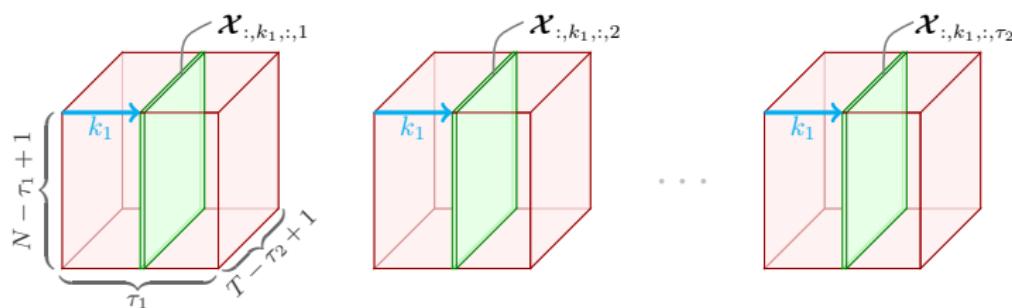
- Automatic temporal modeling.

# Hankel Tensor Factorization

- Hankel tensor: Given any matrix  $\mathbf{X} \in \mathbb{R}^{N \times T}$ , we have

$$\mathcal{X} \triangleq \mathcal{H}_{\tau_1, \tau_2}(\mathbf{X})$$

- Window lengths:  $\tau_1, \tau_2 \in \mathbb{N}^+$ ;
- Tensor size:  $(N - \tau_1 + 1) \times \tau_1 \times (T - \tau_2 + 1) \times \tau_2$ ;



(Figure) 4th order Hankel tensor: A sequence of third-order tensors.

- Slice:  $\mathcal{X}_{:,k_1,:,:,\tau_2}, \forall k_1, k_2$ ;
- Slice size:  $(N - \tau_1 + 1) \times (T - \tau_2 + 1)$ .

# Hankel Tensor Factorization

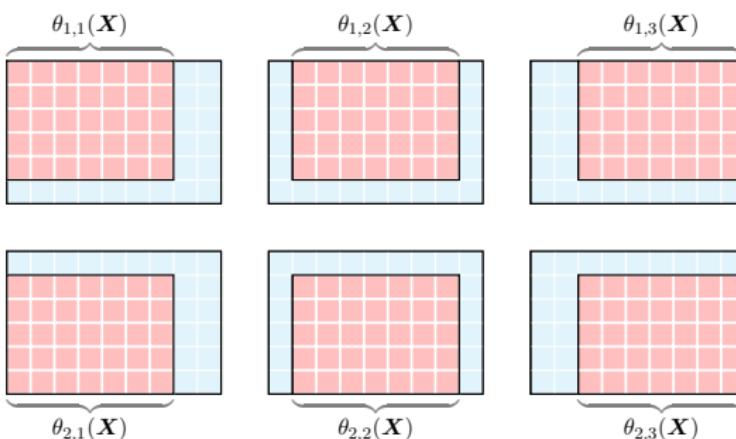
## Hankel indexing:

- Sampling function for the Hankelization:

$$\theta_{k_1, k_2}(\mathbf{X}) \triangleq [\mathcal{H}_{\tau_1, \tau_2}(\mathbf{X})]_{:, k_1, :, k_2},$$

referring to the tensor slice with  $k_1 \in \{1, \dots, \tau_1\}$ ,  $k_2 \in \{1, \dots, \tau_2\}$ .

- [Importance] Developing memory-efficient algorithms.



- Tensor slices  $\theta_{k_1, k_2}(\mathbf{X})$  vs. data matrix  $\mathbf{X}$

# Hankel Tensor Factorization

## Ours:

- Convolutional tensor decomposition (circular convolution  $\star_{\text{row}}$ ):

$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \star_{\text{row}} \mathbf{s}_{k_1}^{\top})(\mathbf{U} \star_{\text{row}} \mathbf{v}_{k_2}^{\top})^{\top}$$

## Baselines:

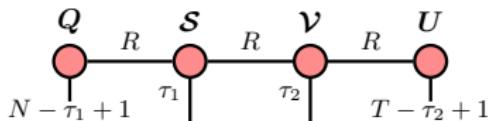
- CP tensor decomposition (Khatri-Rao product  $\odot$ ):

$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \odot \mathbf{s}_{k_1}^{\top})(\mathbf{U} \odot \mathbf{v}_{k_2}^{\top})^{\top}$$

- Tensor-train decomposition:

$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \mathbf{S}_{k_1})(\mathbf{U} \mathbf{V}_{k_2})^{\top}$$

- $\{\mathbf{S}_{k_1}, \mathbf{V}_{k_2}\}$  are **circulant matrices**  $\Rightarrow$  convolutional decomposition
- $\{\mathbf{S}_{k_1}, \mathbf{V}_{k_2}\}$  are **diagonal matrices**  $\Rightarrow$  CP decomposition



# Hankel Tensor Factorization

## HTF (convolutional decomposition)

- Optimization problem:

$$\begin{aligned} \min_{\mathbf{Q}, \mathbf{S}, \mathbf{U}, \mathbf{V}} \quad & \frac{1}{2} \sum_{k_1, k_2} \left\| \mathcal{P}_{\Omega_{k_1, k_2}} (\theta_{k_1, k_2}(\mathbf{Y}) - (\mathbf{Q} \star_{\text{row}} \mathbf{s}_{k_1})(\mathbf{U} \star_{\text{row}} \mathbf{v}_{k_2})^\top) \right\|_F^2 \\ & + \frac{\rho}{2} (\|\mathbf{Q}\|_F^2 + \|\mathbf{S}\|_F^2 + \|\mathbf{U}\|_F^2 + \|\mathbf{V}\|_F^2) \end{aligned}$$

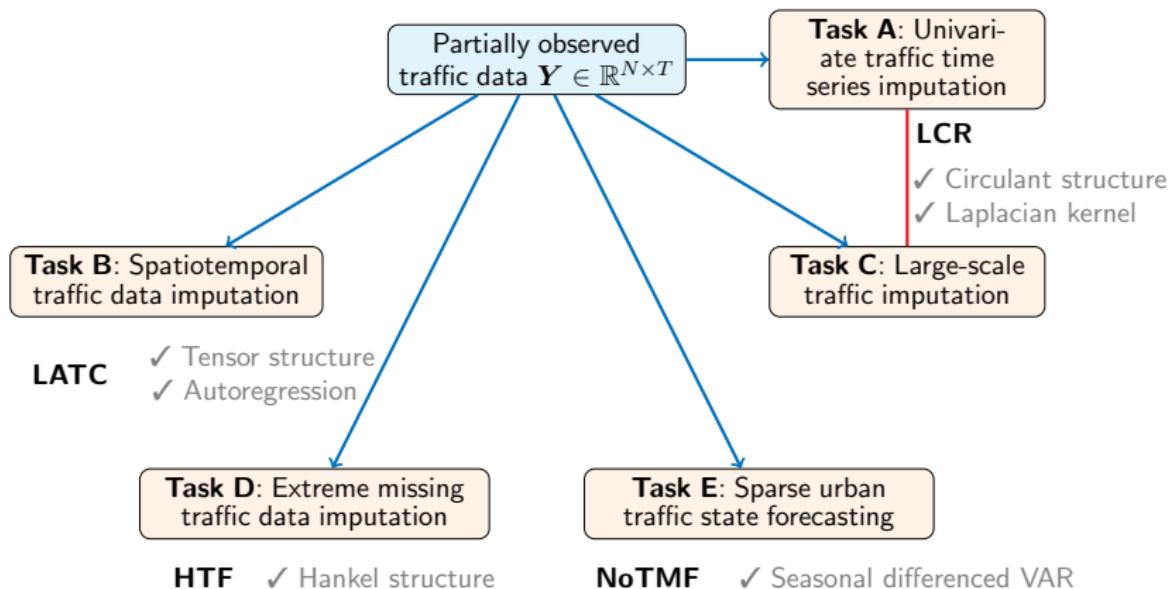
- Alternating minimization:

$$\left\{ \begin{array}{l} \mathbf{Q} := \{\mathbf{Q} \mid \frac{\partial f}{\partial \mathbf{Q}} = \mathbf{0}\} \\ \mathbf{s}_{k_1} := \{\mathbf{s}_{k_1} \mid \frac{\partial f}{\partial \mathbf{s}_{k_1}} = \mathbf{0}\}, k_1 \in \{1, 2, \dots, \tau_1\} \\ \mathbf{U} := \{\mathbf{U} \mid \frac{\partial f}{\partial \mathbf{U}} = \mathbf{0}\} \\ \mathbf{v}_{k_2} := \{\mathbf{v}_{k_2} \mid \frac{\partial f}{\partial \mathbf{v}_{k_2}} = \mathbf{0}\}, k_2 \in \{1, 2, \dots, \tau_2\} \end{array} \right.$$

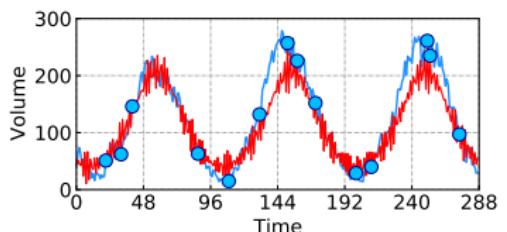


# Whole Picture

We are working on **spatiotemporal traffic data modeling**.



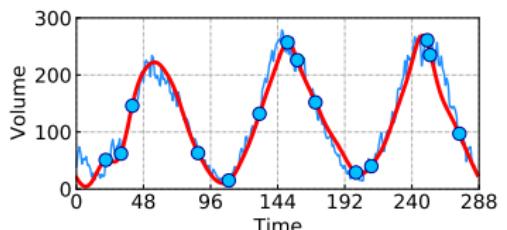
# Univariate Traffic Time Series Imputation



**CircNNM:**

$$\begin{aligned} & \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* \\ \text{s. t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

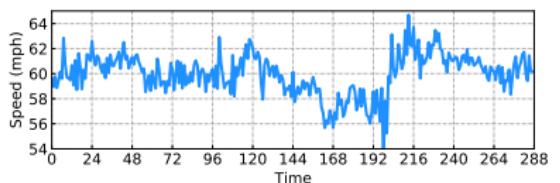
↓ Plus temporal regularization (TR)



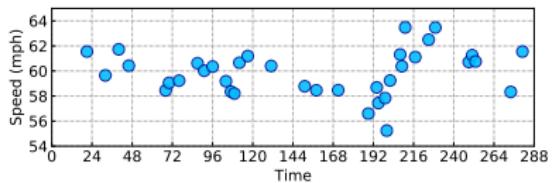
**LCR:**

$$\begin{aligned} & \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell \star \mathbf{x}\|_2^2 \\ \text{s. t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

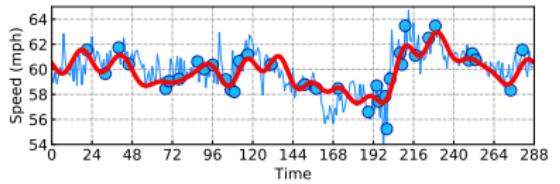
# Univariate Traffic Time Series Imputation



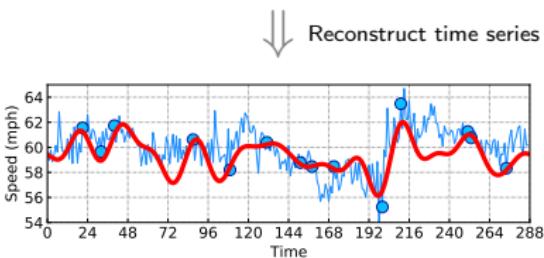
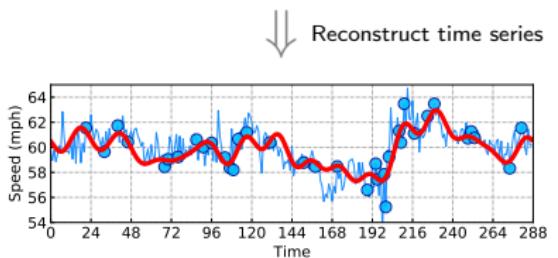
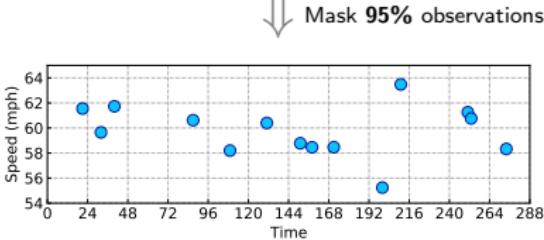
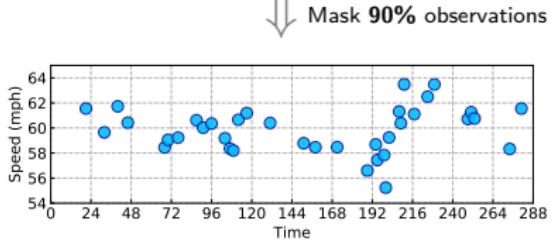
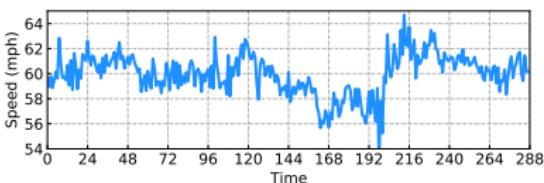
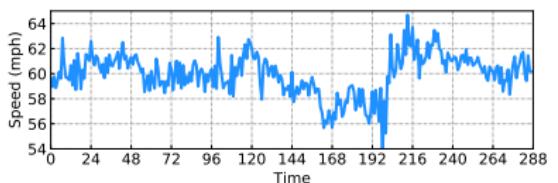
↓ Mask 90% observations



↓ Reconstruct time series



# Univariate Traffic Time Series Imputation



# Spatiotemporal Traffic Data Imputation

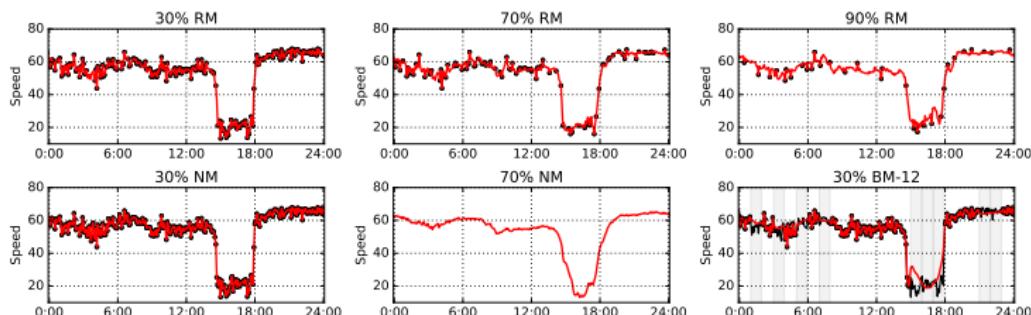
Table: Imputation performance comparison (in MAPE/RMSE) of LATC and baseline models on the Seattle freeway traffic speed dataset.

Missing rate	LATC	LAMC	LRTC-TNN	BTMF	SPC
30%, RM	<b>4.90/3.16</b>	5.98/3.73	4.99/3.20	5.91/3.72	5.92/3.62
70%, RM	<b>5.96/3.71</b>	8.02/4.70	6.10/3.77	6.47/3.98	7.38/4.30
90%, RM	<b>7.46/4.50</b>	10.56/5.91	8.08/4.80	8.17/4.81	9.75/5.31
30%, NM	7.10/4.33	6.99/4.25	<b>6.85/4.21</b>	9.26/5.36	8.87/4.99
70%, NM	9.40/5.40	9.75/5.60	<b>9.23/5.35</b>	10.47/6.15	11.32/5.92
30%, BM-12	<b>9.43/5.36</b>	27.05/13.66	9.52/5.41	14.33/13.60	11.30/5.84

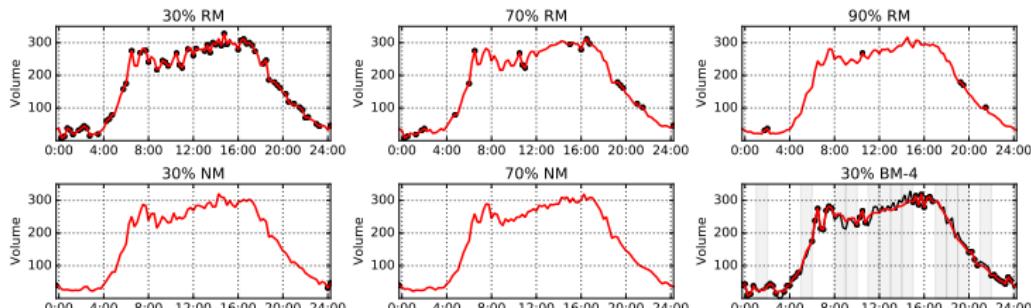
# Spatiotemporal Traffic Data Imputation

## LATC imputation

- Seattle freeway traffic speed data



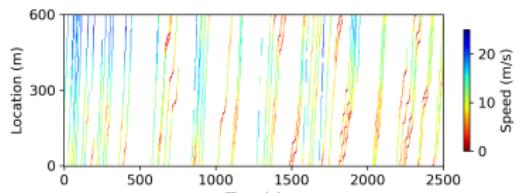
- Portland highway traffic volume data



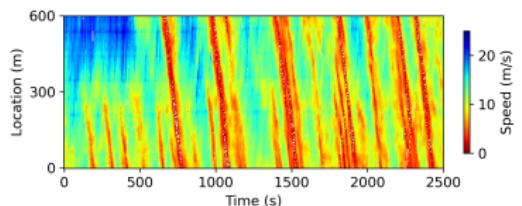
# Large-Scale Traffic Data Imputation

LCR

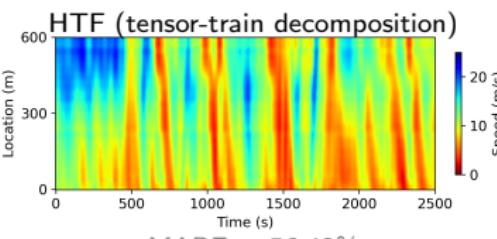
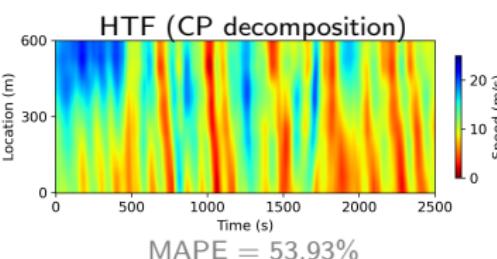
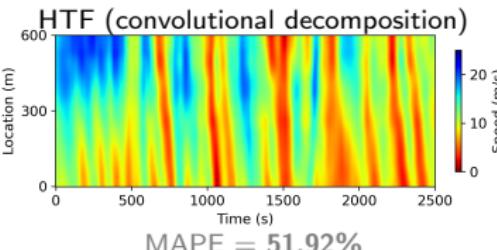
# Extreme Missing Traffic Data Imputation



Sparse speed field



Ground truth speed field



# Sparse Urban Traffic State Forecasting

NoTMF

# Conclusion

- Data (large-scale, high-dimensional, city-wide, sparse)
- Modeling (meaningfulness and importance of temporal correlations)

# Conclusion

## Time-Varying Autoregression:

- (Highlight) Interpretable model with tensor factorization.
  - ✓ Parameter compression ✓ Pattern discovery

## Laplacian Convolutional Representation:

- (Solution) Time series trend modeling in the low-rank framework?
  - Global time series trend modeling (low-rank model):

$$\begin{aligned} & \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* \\ & \text{ s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

- Local time series trend modeling (temporal regularization):

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2$$

- (Highlight) A unified framework with the **FFT** implementation.

## Hankel Tensor Factorization:

- (Highlight) Memory-efficient **Hankel indexing & convolutional parameterization**.

## **Our studies:**

- ② X. Chen, Z. Cheng, N. Saunier, L. Sun (2022). Laplacian convolutional representation for traffic time series imputation. arXiv preprint arXiv:2212.01529.
- ③ X. Chen, L. Sun (2022). Bayesian temporal factorization for multidimensional time series prediction. IEEE Transactions on Pattern Analysis and Machine Intelligence. 44 (9): 4659-4673.

## **GitHub repository:**

- **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (1,000 stars & 270 forks on GitHub)  
<https://github.com/xinychen/transdim>

## Collaborators



Dr. HanQin Cai



Xiaoxu Chen



Dr. Zhanhong Cheng



Chengyuan Zhang



Dr. Xi-Le Zhao

## References

A short list:

- [Candès & Recht'09] Exact matrix completion via convex optimization.  
Foundations of Computational Mathematics, 9 (6), 717-772.
- [Cai et al.'10] A singular value thresholding algorithm for matrix completion
- [Zhang et al.'12] Matrix completion by truncated nuclear norm regularization
- [Hu et al.'12] Fast and accurate matrix completion via truncated nuclear norm regularization
- [Lu et al.'14] Generalized Nonconvex Nonsmooth Low-Rank Minimization
- [Lu et al.'19] Tensor Robust Principal Component Analysis with A New Tensor Nuclear Norm
- [Yokota et al.'18] Missing Slice Recovery for Tensors Using a Low-rank Model in Embedded Space
- [Cai et al.'21] Accelerated Structured Alternating Projections for Robust Spectrally Sparse Signal Recovery
- [Liu'22]
- [Liu & Zhang'23]



# Thanks for your attention!

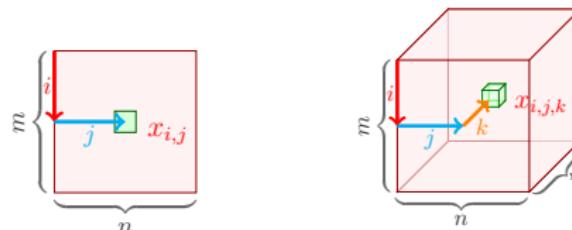
Any Questions?

## About me:

- Homepage: <https://xinychen.github.io>
- GitHub: <https://github.com/xinychen>
- How to reach me: [chenxy346@gmail.com](mailto:chenxy346@gmail.com)

# What Is Tensors?

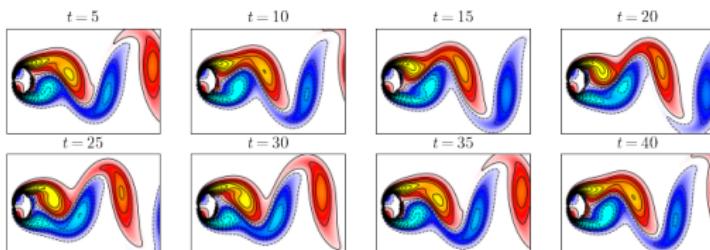
- What is tensor?  $\mathbf{X} \in \mathbb{R}^{m \times n}$  vs.  $\mathcal{X} \in \mathbb{R}^{m \times n \times t}$



- Tensors are everywhere!



Color image with  
RGB channels



Dynamical system (fluid flow)

## Convolution & FFT

## Flipping Operation in LCR

## Training, Validation & Testing

## Tuning Hyperparameters