The Relevance of t-Statistics for Small Sample Sizes

An Introductory Class to Higher Statistics

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Outline

Answering a lot of questions, e.g.,

1. How was t-statistic developed?
2. Normal distribution vs. student t-distribution?
3. What is t-statistic?
4. How to calcuate a t-test?
5. What are the hypotheses and the assumptions?
6. How to interpret results?

Development

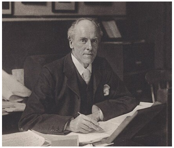
The problems of **small sample sizes**

1876 — *t*-distribution as a posterior distribution

1895 — *t*-distribution as Pearson type IV distribution

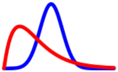
1908 — Student *t*-distribution
"The Probable Error of a Mean" (Biometrika)

F. R. Helmert **J. Lüroth**



K. Pearson

W. S. Gosset

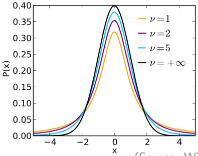


(Source: Wiki)

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

(Source: Wiki)

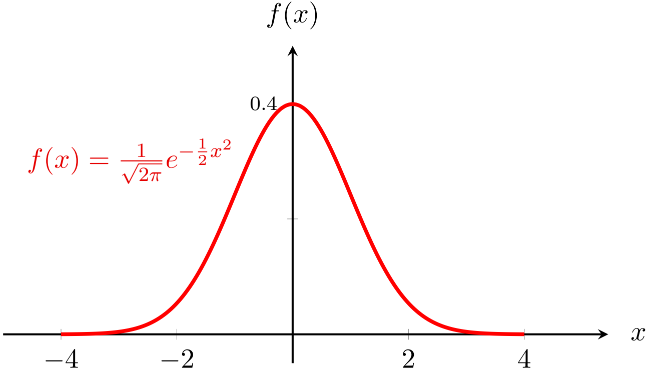
Diagram of the Pearson system



(Source: Wiki)

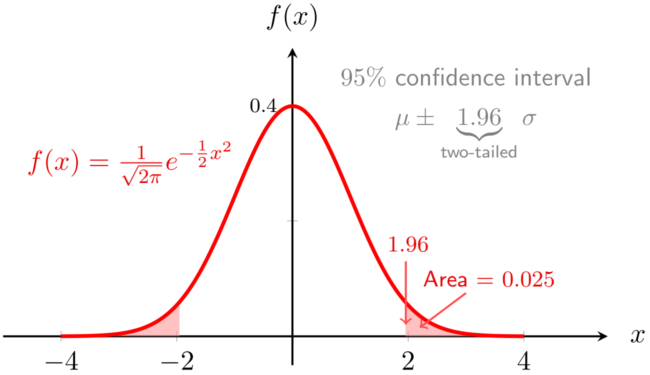
Probability density function

Revisiting Normal Distribution



Probability density function of the standard normal distribution

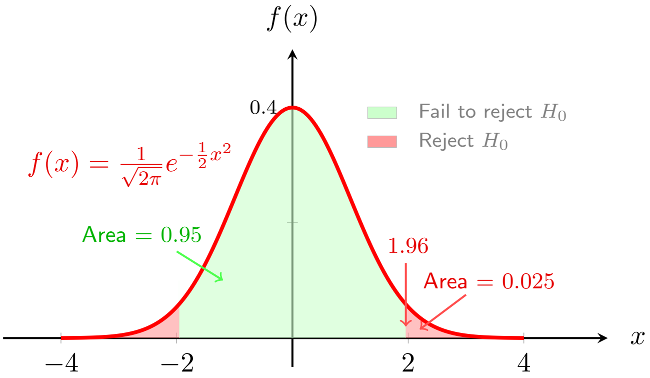
Revisiting Normal Distribution



Probability density function of the standard normal distribution

Connecting with Hypothesis Test

- Hypothesis test
  - Population: mean μ, standard deviation σ
  - Sample: mean $\bar{x}$, sample size n
  - Null hypothesis (H_0): The population mean is μ
  - z-test: $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ ($z \uparrow$ implies statistically significant difference)
- 95% confidence interval



Implementing z-Test

Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh with a population standard deviation of 5 kWh. A random sample of 40 households has an average daily energy consumption of 32 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

- $\bar{x} = 32$ (sample mean)
- $\mu = 30$ (population mean)
- $n = 40$ (sample size)
- $\sigma = 5$ (population standard deviation)

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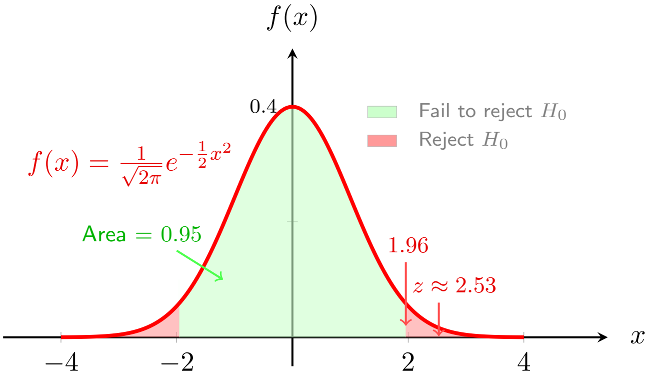
Steps:

❶ Formulate Hypotheses

- Null Hypothesis (H_0): The population mean is $\mu = 30\,\text{kWh}$.
- Alternative Hypothesis (H_a): The population mean is $\mu \neq 30\,\text{kWh}$.

❷ Use the z-test formula since the population standard deviation is known:

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{32 - 30}{5/\sqrt{40}} = \frac{2}{5/6.32} = \frac{2}{0.79} \approx 2.53$$



Test statistic $|z| > 1.96 \Rightarrow$ Reject the null hypothesis

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❸ Decision rule at a 95% confidence interval

- Reject H_0 if $|z| > 1.96$.
- Otherwise, fail to reject H_0.

❹ Interpretation

- The test statistic $|z| = 2.53 > 1.96$ (exceeding the critical value).
- Thus, we reject the null hypothesis.
- The sample provides sufficient evidence to conclude that the average daily energy consumption is not 30 kWh.

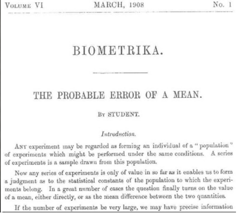
Student t-Distribution

In the case of small sample sizes?

- Switch to student t-distribution and t-test
- Probability density function:

$$f(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

  - $x \in \mathbb{R}$: random variable
  - $\nu \in \mathbb{Z}^+$: degrees of freedom
  - $\Gamma(\cdot)$: Gamma function



VOLUME VI MARCH, 1908 No. 1

BIOMETRIKA.

THE PROBABLE ERROR OF A MEAN.

BY STUDENT.

Introduction.

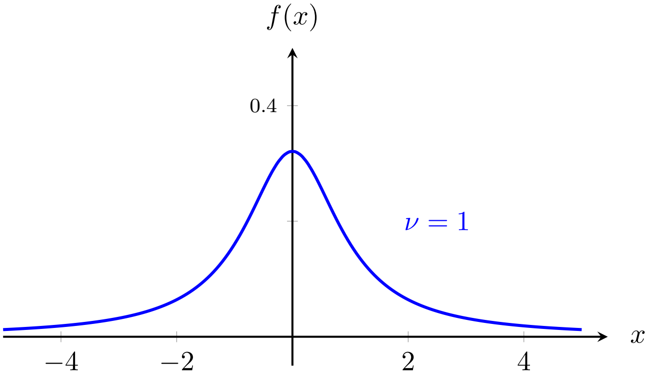
ANY experiment may be regarded as forming an individual of a "population" of experiments which might be performed under the same conditions. A series of experiments is a sample drawn from this population.

Now any series of experiments is only of value in so far as it enables us to form a judgment as to the statistical constants of the population to which the experiments belong. In a great number of cases the question finally turns on the value of a mean, either directly, or as the mean difference between the two quantities.

If the number of experiments be very large, we may have precise information

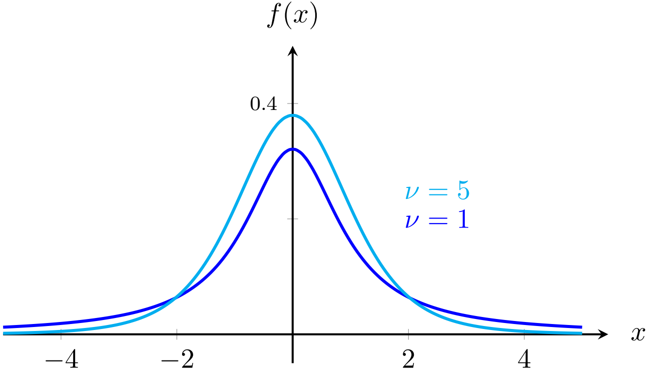
Gossset'1908 (known as "student")

Student t-Distribution



Student t-distribution of ν degrees of freedom

Student t-Distribution



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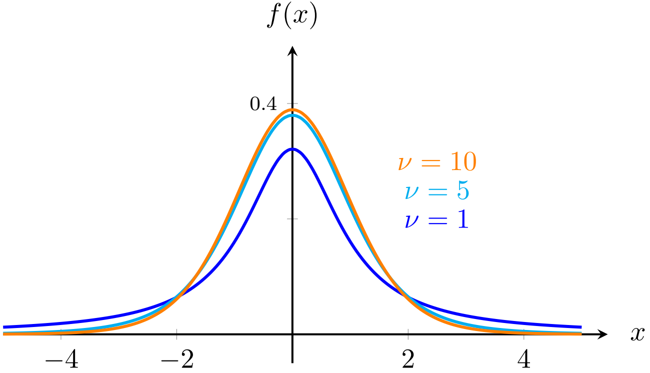

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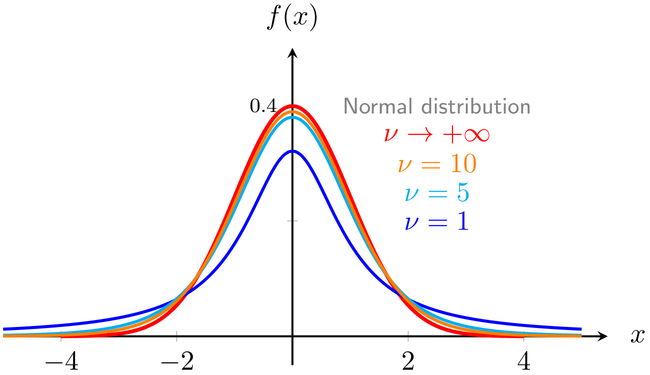

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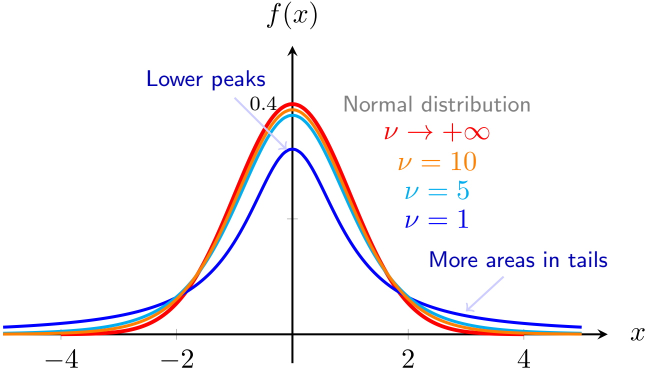


Student t-distribution of ν degrees of freedom

95% **Confidence Interval**

For the population mean μ (✓) and standard deviation σ (✓/✗)

- If population standard deviation σ is known

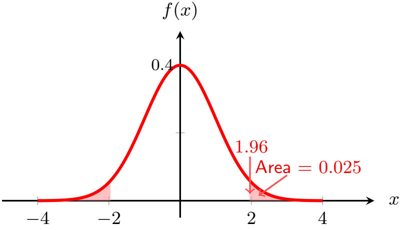
$$\bar{x} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

- Use z-test

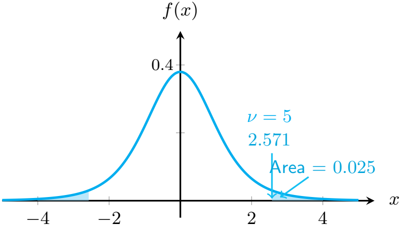
- If σ is unknown, using sample standard deviation s instead

$$\bar{x} \pm \, ? \times \frac{s}{\sqrt{n}}$$

- Use t-test



Standard normal distribution



Student t-distribution

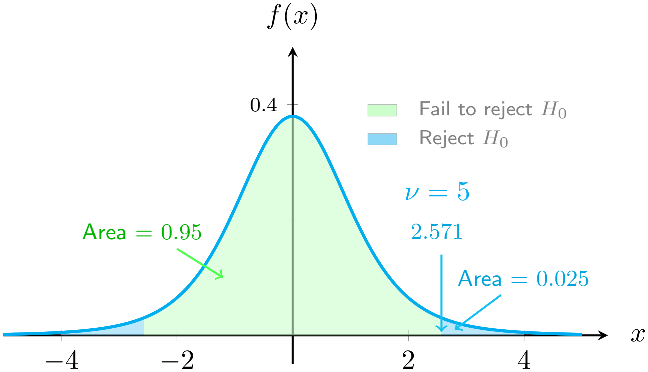
- **Heavy tail** in student t-distribution ($\nu = n - 1$ degrees of freedom) is important for small sample size n

Definition of t-Statistic

- Formula of t-statistic for small sample sizes

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

  - Population: mean μ
  - Sample: mean $\bar{x}$, standard deviation s, sample size n (small value)
- A high absolute value of t suggests a statistically significant difference.



Implementing t-Test for Small Sample Size

Problem Statement

A company claims that the average daily energy consumption of households is 30 kWh. A random sample of 6 households has an average daily energy consumption of 32 kWh, with a sample standard deviation of 6 kWh. Conduct a two-tailed hypothesis test at a 95% confidence interval to determine if the sample provides sufficient evidence to reject the company's claim.

- $\bar{x} = 32$ (sample mean)
- $s = 6$ (sample standard deviation)
- $n = 6$ (sample size)
- $\mu = 30$ (population mean)

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Steps:

1. Formulate Hypotheses
   - Null Hypothesis (H_0): The population mean is $\mu = 30\,\text{kWh}$.
   - Alternative Hypothesis (H_a): The population mean is $\mu \neq 30\,\text{kWh}$.
2. Use the t-test formula since the population standard deviation is not known:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{32 - 30}{6/\sqrt{6}} = \frac{2}{6/2.449} \approx 0.816$$

Critical Values in t-Table

Small sample sizes

- Degrees of freedom for a t-test:

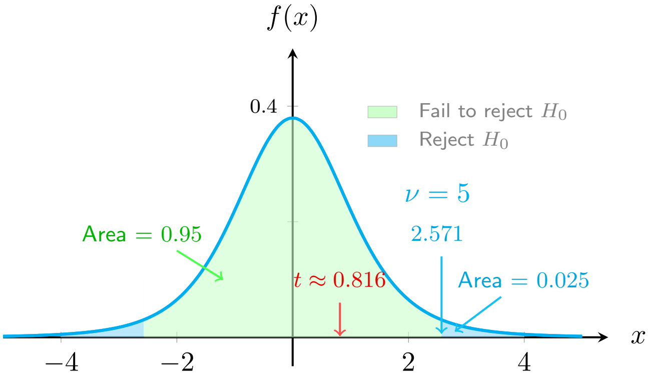
$$\nu = \underbrace{n}_{\text{sample size}} -1 = 6 - 1 = 5$$

- t-distributions with ν degrees of freedom at a 95% confidence interval (two-tailed)

$\nu = 1$	$\nu = 5$	$\nu = 10$	$\nu \to +\infty$
12.706	2.571	2.228	1.960

- The critical t-value

$$t_{\nu,(1-0.95)/2} = t_{5,0.025} = 2.571$$



Test statistic $|t| < 2.571 \Rightarrow$ Fail to reject the null hypothesis

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❸ Decision rule at a 95% confidence interval

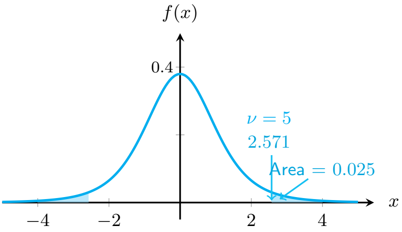
- Reject H_0 if $|t| > 2.571$.
- Otherwise, fail to reject H_0.

❹ Interpretation

- The test statistic $|t| = 0.816 < 2.571$.
- Thus, we fail to reject the null hypothesis.
- There is not enough evidence to conclude that the average daily energy consumption differs from the company's claim of 30 kWh.

Summary

- Student t-distribution of ν degrees of freedom



Student t-distribution

W. S. Gosset in Guinness

- Population: mean μ (✓), standard deviation σ (✗)
- Sample: mean $\bar{x}$, standard deviation s, and small sample size n
- t-statistic: $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}} \quad \Rightarrow \quad$ t-test
- 95% confidence interval: $\bar{x} \pm \underbrace{t_{\nu,0.025}}_{\nu=n-1} \times \dfrac{s}{\sqrt{n}}$

Thank you!

Any Questions?

Slides: https://xinychen.github.io/slides/t_stat.pdf

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