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Matrix and Tensor Models for Spatiotemporal Traffic Data Imputation and Forecasting

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Outline

- 1. Background**
- 2. Literature Review**
- 3. Nonstationary Temporal Matrix Factorization (NoTMF)**
- 4. Laplacian Convolutional Representation (LCR)**
- 5. Hankel Tensor Factorization (HTF)**
- 6. Conclusion**

Spatiotemporal Traffic Data

Many spatiotemporal traffic time series data are in the form of **matrix**.

- Portland highway traffic data¹



- $X \in \mathbb{R}^{N \times T}$ with N spatial locations \times T time steps
- Traffic volume/speed shows strong spatial/temporal dependencies

¹<https://portal.its.pdx.edu/home>

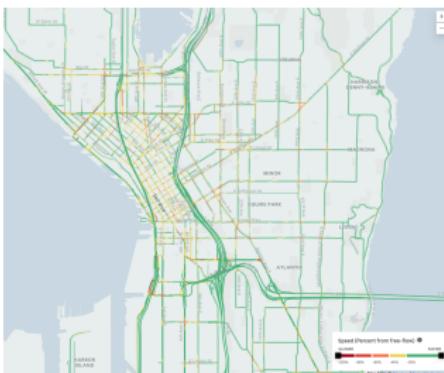
Urban Movement Data

High-dimensional & sparse

- Uber (hourly) movement speed data



NYC movement



Seattle movement

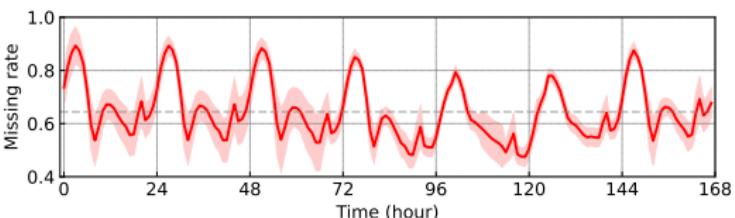
- {road segment, time slot (hour), average speed}
- Computing hourly speed: Road segments have 5+ unique trips.

Issue: Insufficient sampling of ridesharing vehicles on the road network!

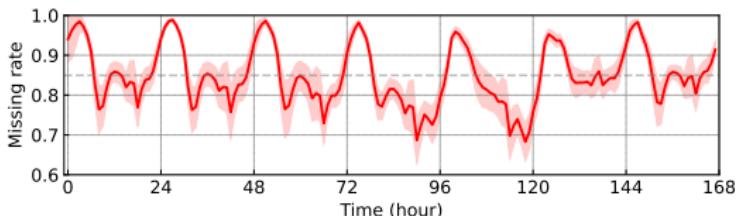
Urban Movement Data

High-dimensional & sparse

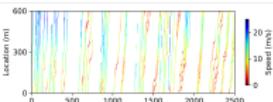
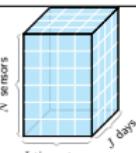
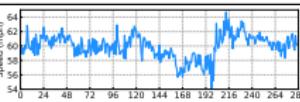
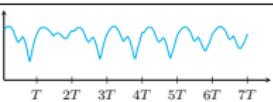
- **NYC** movement speed data (2019)
 - **98,210** road segments & 8,760 time steps (hours)
 - Overall missing rate: **64.43%**



- **Seattle** movement speed data (2019)
 - **63,490** road segments & 8,760 time steps (hours)
 - Overall missing rate: **84.95%**



Spatiotemporal Traffic Data

| | |
|-------------------|---|
| Sparse |  |
| High-dimensional |  |
| Multidimensional |  |
| Noises & outliers |  |
| Nonstationary |  |

Traffic data show complicated spatiotemporal patterns and correlations.

Problem Formulation

Objective A: Impute missing values in the data matrix $\mathbf{Y} \in \mathbb{R}^{N \times T}$ (or tensor $\mathcal{Y} \in \mathbb{R}^{M \times N \times T}$).



- Matrix completion (Observed index set Ω)

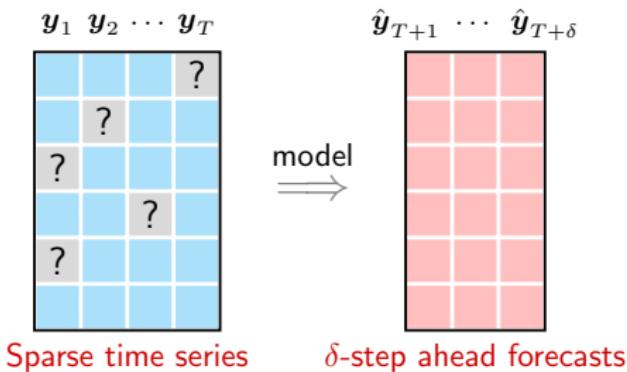
$$\underbrace{\mathcal{P}_\Omega(\mathbf{Y})}_{\text{Partially observed}} \xrightarrow{\text{Estimate}} \underbrace{\mathcal{P}_\Omega^\perp(\mathbf{Y})}_{\text{Unobserved}}$$

Modeling process:

- How to make use of spatiotemporal traffic patterns?
- How to make use of traffic time series dynamics?

Problem Formulation

Objective B: Given a partially observed data $\mathbf{Y} \in \mathbb{R}^{N \times T}$ consisting of time series $\mathbf{y}_1, \dots, \mathbf{y}_T \in \mathbb{R}^N$, forecast data points $\hat{\mathbf{y}}_{T+\delta}, \delta \in \mathbb{N}^+$.

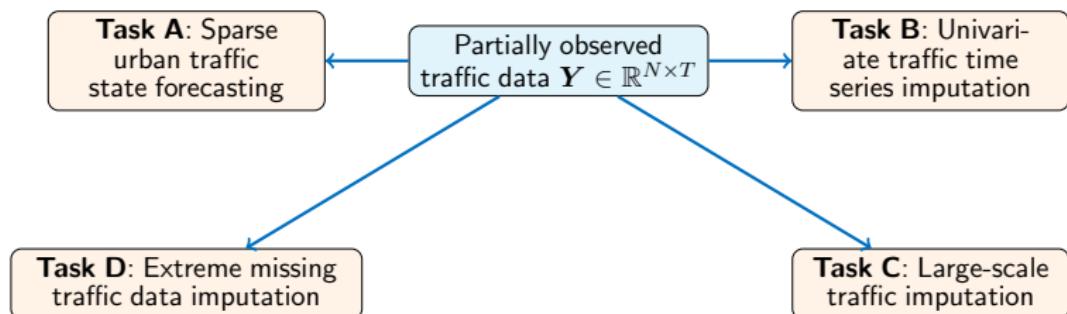


Modeling process:

- How to characterize time series dynamics in high-dimensional and sparse traffic data?

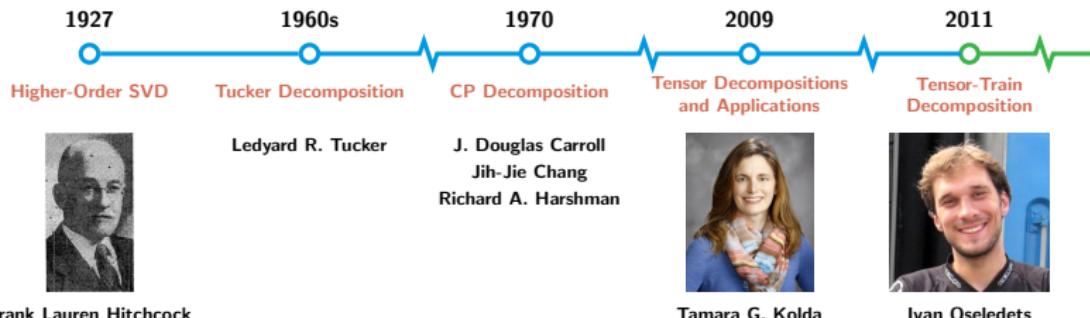
Tasks

We are working on **spatiotemporal traffic data imputation and forecasting**.

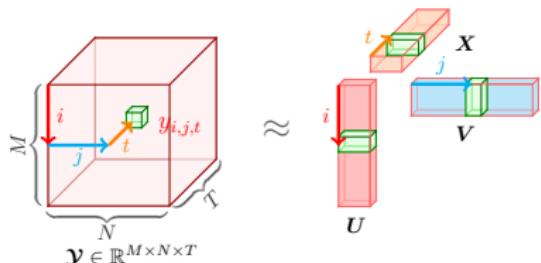


Tensor Factorization

- Revisit tensor factorization

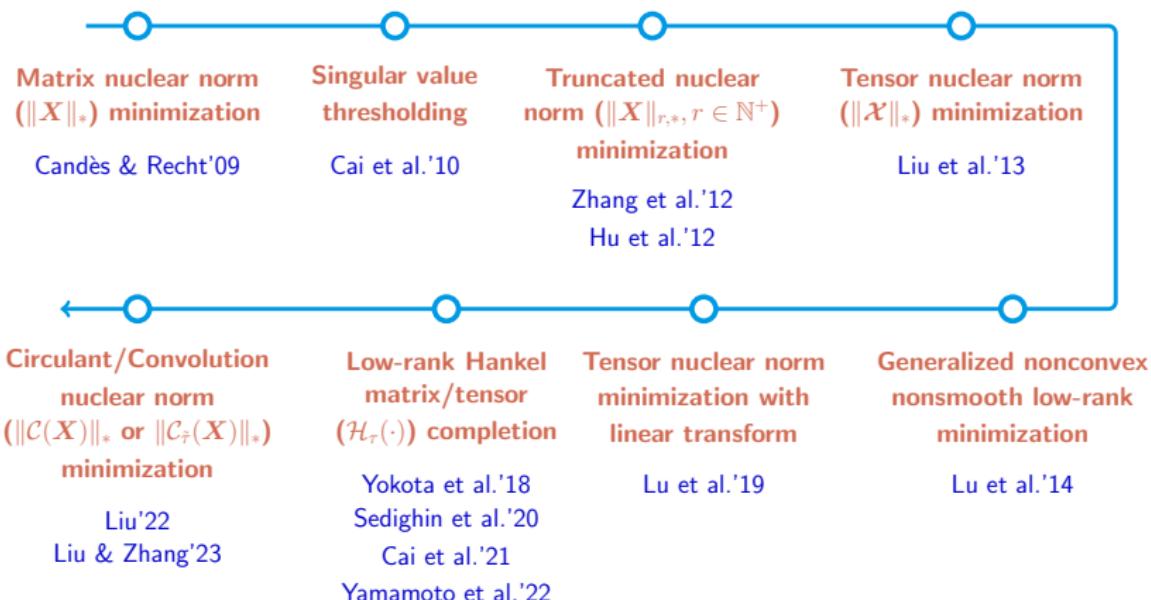


- CP tensor factorization:** Factorize \mathcal{Y} into the combination of three rank- R factor matrices (i.e., low-dimensional latent factors).



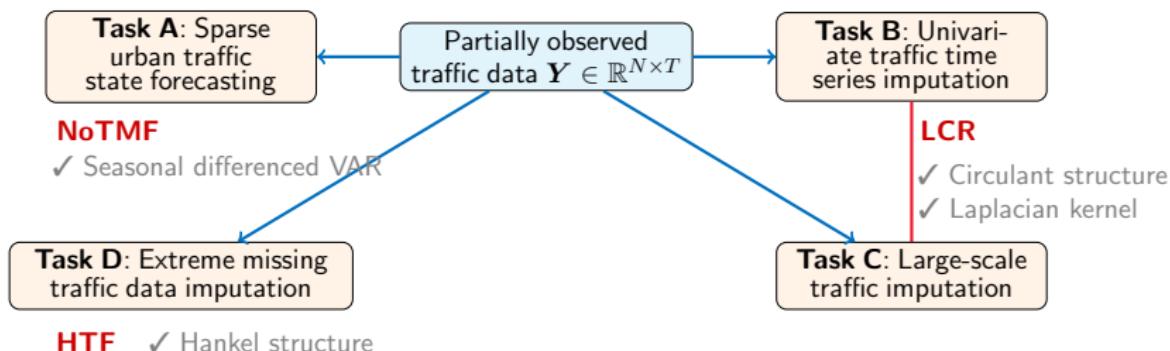
$$\left\{ \begin{array}{l} y_{i,j,t} \approx \sum_{r=1}^R u_{i,r} v_{j,r} x_{t,r} \\ \mathcal{Y} \approx \sum_{r=1}^R \mathbf{u}_r \otimes \mathbf{v}_r \otimes \mathbf{x}_r \end{array} \right.$$

Matrix/Tensor Completion



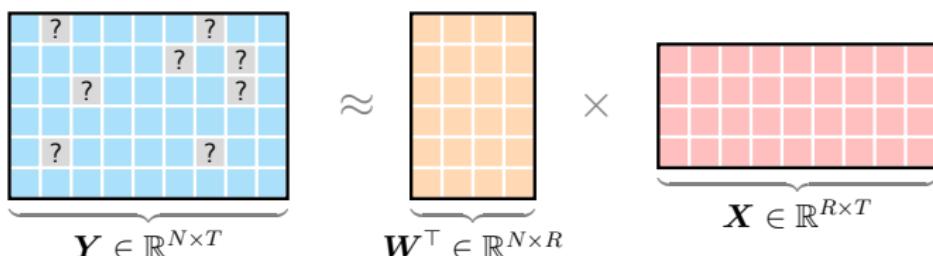
Overview

Matrix/Tensor methods + temporal modeling (e.g., smoothing & autoregression)



Matrix Factorization

A simple approach to reconstruct missing values.



MF (Koren et al.'09)

Estimating low-dimensional \mathbf{W}, \mathbf{X} :

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2$$

on data \mathbf{Y} w/ observed index set Ω .

- ✓ Learn from sparse data
- ✓ Spatial factor matrix \mathbf{W}
- ✓ Temporal factor matrix \mathbf{X}

How to build temporal correlations on MF?

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

$$\begin{matrix} \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \end{matrix} \underbrace{\quad}_{\mathbf{Y} \in \mathbb{R}^{N \times T}} \approx \begin{matrix} \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \end{matrix} \underbrace{\quad}_{\mathbf{W}^\top \in \mathbb{R}^{N \times R}} \times \begin{matrix} \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \end{matrix} \underbrace{\quad}_{\mathbf{X} \in \mathbb{R}^{R \times T}}$$

$$\begin{matrix} \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \end{matrix} \underbrace{\quad}_{\mathbf{Y} \in \mathbb{R}^{N \times T}} \approx \begin{matrix} \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \\ \text{?} & \text{?} \end{matrix} \underbrace{\quad}_{\mathbf{W}^\top \in \mathbb{R}^{N \times R}} \times \begin{matrix} \mathbf{x}_{t-2} & \mathbf{x}_{t-1} & \mathbf{x}_t & \mathbf{x}_{t+1} & \mathbf{x}_{t+2} \\ \dots & & & & \dots \end{matrix} \left. \right\} R$$

time step

\Downarrow **\mathbf{X} is time series?**

Why? Temporal factor matrix $\mathbf{X} \in \mathbb{R}^{R \times T}$ is the low-dimensional representation of time series dynamics of $\mathbf{Y} \in \mathbb{R}^{N \times T}$.

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

MF (Koren et al.'09)

Estimating low-dimensional \mathbf{W}, \mathbf{X} :

$$\min_{\mathbf{W}, \mathbf{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2$$

on data \mathbf{Y} w/ observed index set Ω .

dth-order VAR

$$\mathbf{x}_t = \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} + \underbrace{\boldsymbol{\epsilon}_t}_{\mathcal{N}(\mathbf{0}, \mathbf{I})}$$

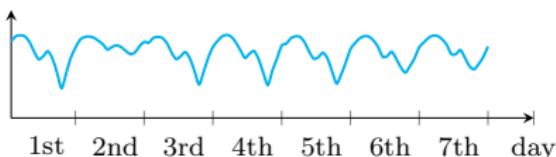
w/ coefficients $\{\mathbf{A}_k\}$.

↓ Yu et al.'16
Chen & Sun'21

$$\min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} \frac{1}{2} \underbrace{\|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2}_{\text{MF on data } \mathbf{Y}} + \frac{\gamma}{2} \underbrace{\sum_{t=d+1}^T \left\| \mathbf{x}_t - \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} \right\|_2^2}_{\text{VAR on temporal factors } \mathbf{X}}$$

Nonstationary Temporal Matrix Factorization

Traffic data are nonstationary due to daily patterns of traffic flow.



- Season- m differencing ($m \in \mathbb{N}^+$, e.g., daily/weekly):

$$\mathbf{x}_t \approx \sum_{k=1}^d \mathbf{A}_k \mathbf{x}_{t-k} \quad \Rightarrow \quad \mathbf{x}_t - \mathbf{x}_{t-m} \approx \sum_{k=1}^d \mathbf{A}_k (\mathbf{x}_{t-k} - \mathbf{x}_{t-m-k})$$

- (Ours) Optimization problem:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{X}, \{\mathbf{A}_k\}_{k=1}^d} & \frac{1}{2} \underbrace{\|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2}_{\text{MF on data } \mathbf{Y}} + \frac{\rho}{2} \underbrace{(\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)}_{\text{Regularization}} \\ & + \frac{\gamma}{2} \underbrace{\sum_{t=d+m+1}^T \left\| (\mathbf{x}_t - \mathbf{x}_{t-m}) - \sum_{k=1}^d \mathbf{A}_k (\mathbf{x}_{t-k} - \mathbf{x}_{t-m-k}) \right\|_2^2}_{\text{VAR on seasonal differenced temporal factors}} \end{aligned}$$

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF

- Optimization problem:²

$$\min_{\mathbf{W}, \mathbf{X}, \mathbf{A}} \frac{1}{2} \underbrace{\|\mathcal{P}_\Omega(\mathbf{Y} - \mathbf{W}^\top \mathbf{X})\|_F^2}_{\text{MF on data } \mathbf{Y}} + \frac{\rho}{2} \underbrace{(\|\mathbf{W}\|_F^2 + \|\mathbf{X}\|_F^2)}_{\text{Regularization}} + \frac{\gamma}{2} \underbrace{\|\mathbf{X} \Psi_0^\top - \mathbf{A} (\mathbf{I}_d \otimes \mathbf{X}) \Psi^\top\|_F^2}_{\text{VAR on } \mathbf{X}}$$

where $\Psi_0 \in \mathbb{R}^{(T-d-m) \times T}$, $\Psi \in \mathbb{R}^{(T-d-m) \times (dT)}$ are temporal operators.

- Alternating minimization (let f be the obj.):

$$\left\{ \begin{array}{ll} \text{Spatial factors} & \mathbf{W} := \{\mathbf{W} \mid \frac{\partial f}{\partial \mathbf{W}} = \mathbf{0}\} \quad (\text{least squares}) \\ \text{Temporal factors} & \mathbf{X} := \{\mathbf{X} \mid \frac{\partial f}{\partial \mathbf{X}} = \mathbf{0}\} \quad (\text{conjugate gradient}) \\ \text{VAR coefficients} & \mathbf{A} := \{\mathbf{A} \mid \frac{\partial f}{\partial \mathbf{A}} = \mathbf{0}\} \quad (\text{least squares}) \end{array} \right.$$

² $\mathbf{A} \triangleq [\mathbf{A}_1 \quad \cdots \quad \mathbf{A}_d] \in \mathbb{R}^{R \times (dR)}$ (coefficient matrix).

Nonstationary Temporal Matrix Factorization

NoTMF forecasting?

Implementation

- Estimate $\mathbf{W}, \mathbf{X}, \mathbf{A}$
- Forecast $\hat{\mathbf{x}}_{t+1}$ with VAR
- Return $\hat{\mathbf{y}}_{t+1} = \mathbf{W}^\top \hat{\mathbf{x}}_{t+1}$

- ✓ Sparse input \mathbf{Y}_t
- ✓ Forecast in latent spaces

$$\underbrace{\mathbf{Y}_t \in \mathbb{R}^{N \times t}}_{\text{Matrix } \mathbf{Y}_t \text{ with missing values marked by question marks}}$$

$$R \left\{ \begin{array}{c} \mathbf{x}_{t-3} \quad \mathbf{x}_{t-2} \quad \mathbf{x}_{t-1} \quad \mathbf{x}_t \quad \mathbf{x}_{t+1} \\ | \qquad | \qquad | \qquad | \qquad | \\ t-3 \quad t-2 \quad t-1 \quad t \quad t+1 \end{array} \right. \quad \hat{\mathbf{x}}_{t+1} = \mathbf{x}_{t+1-m} + \text{VAR}(d, m)$$

time step

Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

- Online forecasting (Gultekin & Paisley'18):
 - Fix the spatial factor matrix \mathbf{W}
 - Use input data \mathbf{Y}_{t+1} to update the temporal factor matrix \mathbf{X} and the coefficient matrix \mathbf{A}

Implementation

- Estimate \mathbf{X}, \mathbf{A}
- Forecast $\hat{\mathbf{x}}_{t+2}$ with VAR
- Return $\hat{\mathbf{y}}_{t+2} = \mathbf{W}^\top \hat{\mathbf{x}}_{t+2}$

- ✓ Sparse input \mathbf{Y}_{t+1}
- ✓ Forecast in latent spaces

$$\mathbf{Y}_{t+1} \in \mathbb{R}^{N \times (t+1)}$$

A 4x(t+1) grid of cells. The last column is red and the rest are blue. Cells in the last column are marked with question marks.

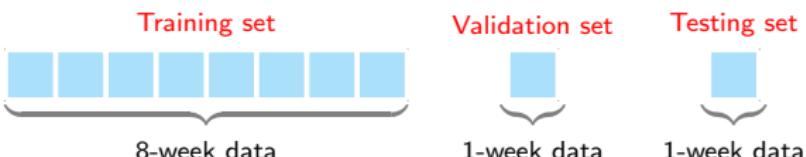
$$R \left\{ \begin{array}{c} \mathbf{x}_{t-3} \quad \mathbf{x}_{t-2} \quad \mathbf{x}_{t-1} \quad \mathbf{x}_t \quad \mathbf{x}_{t+1} \quad \mathbf{x}_{t+2} \\ | \qquad | \qquad | \qquad | \qquad | \qquad | \\ t-3 \quad t-2 \quad t-1 \quad t \quad t+1 \quad t+2 \end{array} \right. \quad \hat{\mathbf{y}}_{t+2} = \mathbf{x}_{t+2-m} + \text{VAR}(d, m)$$

A diagram showing a sequence of vectors $\mathbf{x}_{t-3}, \mathbf{x}_{t-2}, \mathbf{x}_{t-1}, \mathbf{x}_t, \mathbf{x}_{t+1}, \mathbf{x}_{t+2}$ corresponding to time steps $t-3, t-2, t-1, t, t+1, t+2$. The vector \mathbf{x}_{t+2} is highlighted in pink. Below the vectors is a horizontal arrow labeled "time step".

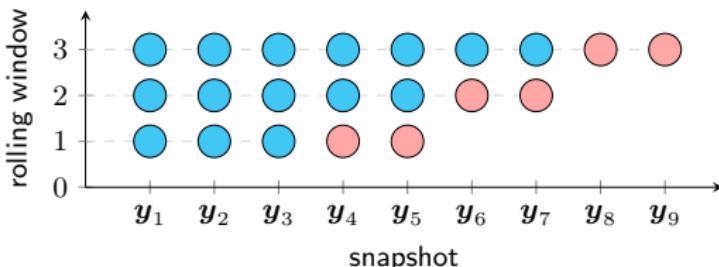
Sparse Urban Traffic State Forecasting

NoTMF forecasting

- NYC Uber movement speed dataset:
 - 10-week data of size 98210×1680 ; **66.56%** missing values
- Rolling forecasting setup (Time horizon $\delta = 1, 2, 3, 6$):



- Weight parameter $\gamma \in \{10^{-2}, 10^{-1}, 10^0, 10^1, 10^2\}$
- Weight parameter $\rho = \{10^{-1}\gamma, 5 \times 10^{-1}\gamma, \gamma, 5\gamma, 10\gamma\}$
- Rolling forecasting illustration ($\delta = 2$):



Sparse Urban Traffic State Forecasting

NoTMF vs. baseline (in MAPE/RMSE)

- On the NYC Uber movement speed dataset

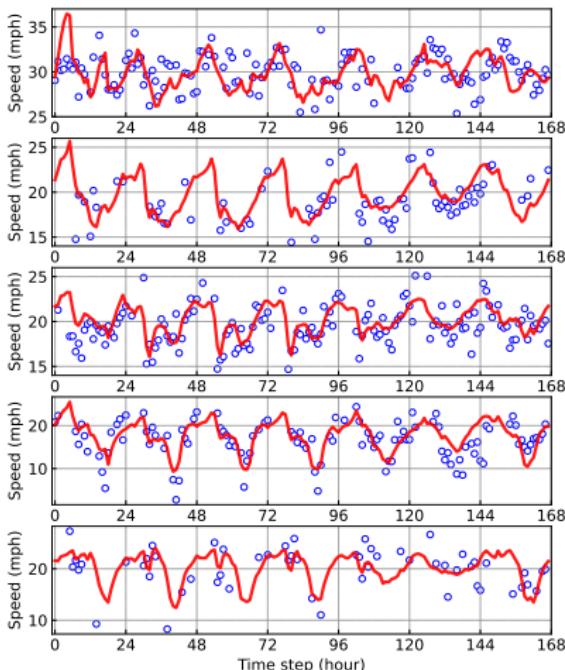
| δ | d | NoTMF ($m = 24$) | NoTMF ($m = 168$) | NoTMF-1st ($m = 168$) | TRMF | BTMF | BTRMF |
|----------|-----|-----------------------|------------------------|----------------------------|------------|------------|------------|
| 1 | 1 | 13.63/2.88 | 13.53/2.86 | 13.45/2.85 | 14.50/3.12 | 14.94/3.13 | 15.93/3.33 |
| | 2 | 13.47/2.84 | 13.41/2.84 | 13.42/2.84 | 14.14/3.05 | 15.70/3.41 | 15.90/3.35 |
| | 3 | 13.46/2.84 | 13.39/2.83 | 13.43/2.84 | 13.87/2.96 | 15.80/3.34 | 16.08/3.43 |
| | 6 | 13.41/2.83 | 13.39/2.83 | 13.41/2.83 | 14.00/2.98 | 15.45/3.27 | 16.26/3.48 |
| 2 | 1 | 13.91/2.96 | 13.76/2.94 | 13.70/2.92 | 15.85/3.43 | 15.33/3.21 | 16.85/3.56 |
| | 2 | 13.77/2.92 | 13.63/2.89 | 13.72/2.92 | 15.04/3.31 | 15.87/3.32 | 17.27/3.71 |
| | 3 | 13.72/2.91 | 13.61/2.89 | 13.73/2.92 | 15.25/3.36 | 15.69/3.33 | 17.24/3.74 |
| | 6 | 13.59/2.87 | 13.57/2.88 | 13.68/2.91 | 14.92/3.24 | 15.91/3.39 | 18.18/3.97 |
| 3 | 1 | 14.30/3.05 | 14.06/3.02 | 14.02/3.00 | 17.52/3.83 | 15.86/3.32 | 18.61/3.91 |
| | 2 | 14.01/2.98 | 13.84/2.94 | 13.96/2.98 | 17.32/4.00 | 16.30/3.40 | 18.90/4.10 |
| | 3 | 13.95/2.97 | 13.79/2.93 | 13.98/2.98 | 16.91/3.71 | 16.56/3.49 | 18.68/4.05 |
| | 6 | 13.78/2.92 | 13.73/2.92 | 13.91/2.96 | 16.72/3.65 | 15.49/3.27 | 20.45/4.66 |
| 6 | 1 | 14.61/3.11 | 14.67/3.20 | 14.98/3.32 | 21.20/4.70 | 15.99/3.32 | 22.40/4.69 |
| | 2 | 14.30/3.03 | 14.33/3.09 | 14.90/3.28 | 20.87/5.01 | 16.04/3.33 | 23.56/5.63 |
| | 3 | 14.26/3.03 | 14.28/3.09 | 14.86/3.26 | 20.08/4.65 | 15.67/3.28 | 24.27/5.72 |
| | 6 | 14.06/2.97 | 14.16/3.06 | 14.80/3.23 | 20.40/4.35 | 16.38/3.50 | 26.34/6.60 |

NoTMF performs better than other TMF models (e.g., TRMF, BTMF & BTRMF).

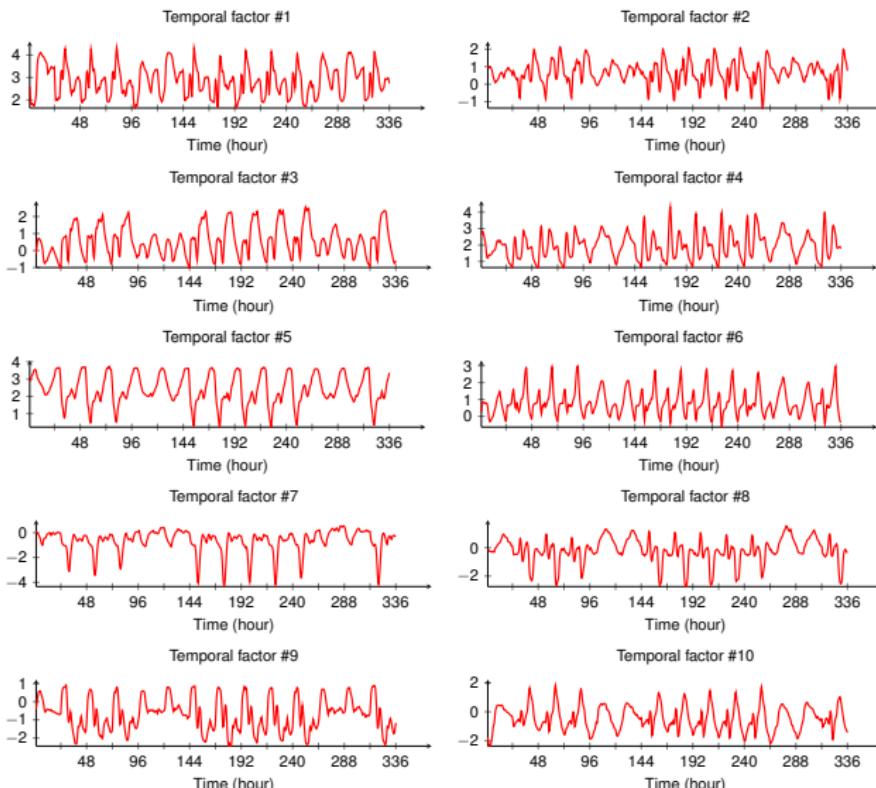
Sparse Urban Traffic State Forecasting

NoTMF forecasting ($\delta = 6$)

- On the NYC Uber movement speed dataset



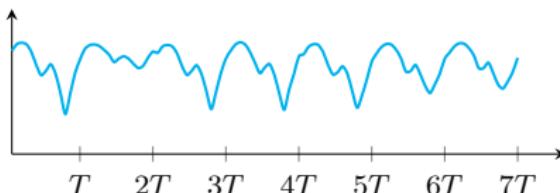
Sparse Urban Traffic State Forecasting



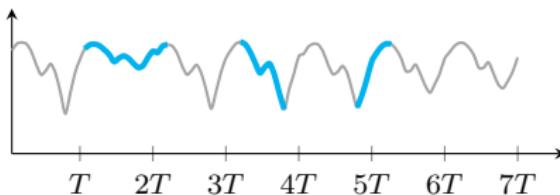
Laplacian Convolutional Representation

Motivation: Time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



- Local trends (e.g., short-term time series trends):

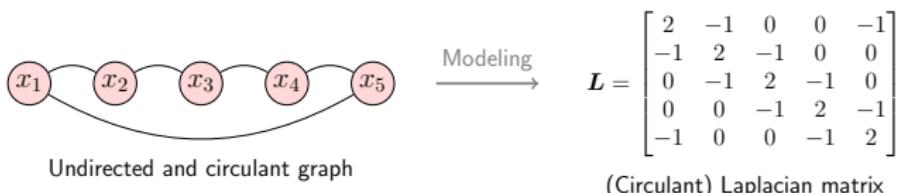


How to characterize both global and local trends in sparse time series?

Laplacian Convolutional Representation

Local trend modeling

- Intuition of (circulant) Laplacian matrix



- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (2, -1, 0, 0, -1)^\top$$

↓

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \dots, -1}_{\tau}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$.

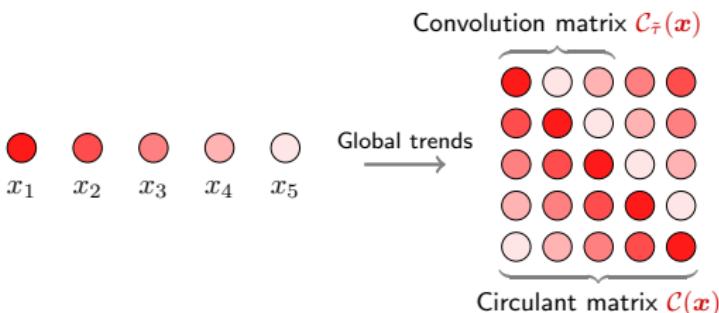
- (Laplacian) Temporal regularization:

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2$$

Reformulate temporal regularization with circular convolution.

Laplacian Convolutional Representation

Global trend modeling: Circulant matrix $\mathcal{C}(\mathbf{x})$ vs. convolution matrix $\mathcal{C}_{\tilde{\tau}}(\mathbf{x})$



- Circulant/Convolution nuclear norm minimization
 - A balance between global and local trends modeling?

CircNNM (Liu'22, Liu & Zhang'23)

Estimating \mathbf{x} :

$$\min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_*$$

$$\text{s.t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$

on data \mathbf{y} w/ observed index set Ω .

ConvNNM (Liu'22, Liu & Zhang'23)

Estimating \mathbf{x} :

$$\min_{\mathbf{x}} \|\mathcal{C}_{\tilde{\tau}}(\mathbf{x})\|_*$$

$$\text{s.t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$

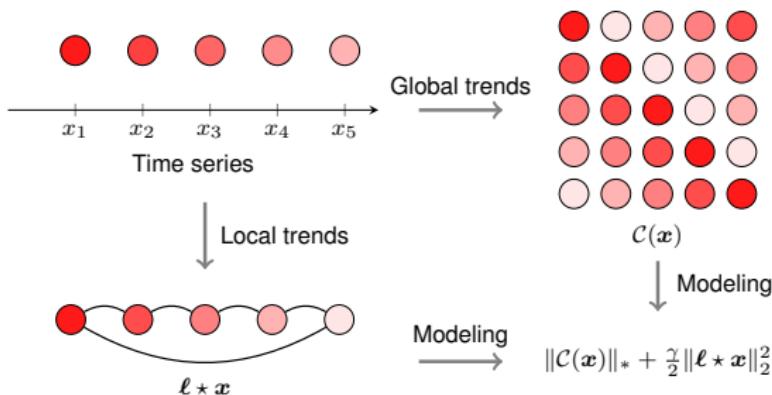
on data \mathbf{y} w/ observed index set Ω .

Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\mathbf{y} \in \mathbb{R}^T$ with observed index set Ω , LCR utilizes **circulant matrix** and **Laplacian kernel** to characterize global and local trends in time series, respectively, i.e.,

$$\begin{aligned} \min_{\mathbf{x}} \quad & \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\ell * \mathbf{x}\|_2^2}_{\text{local}} \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$



Laplacian Convolutional Representation

- Augmented Lagrangian function:³

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2$$

- The ADMM scheme:

$$\begin{cases} \mathbf{x} := \arg \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Nuclear norm minimization)} \\ \mathbf{z} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Closed-form solution)} \\ \mathbf{w} := \mathbf{w} + \lambda(\mathbf{x} - \mathbf{z}) & \text{(Standard update)} \end{cases}$$

- Optimize \mathbf{x} ?

$$\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1 \quad \& \quad \frac{1}{2} \|\boldsymbol{\ell} * \mathbf{x}\|_2^2 = \frac{1}{2T} \|\mathcal{F}(\boldsymbol{\ell}) \circ \mathcal{F}(\mathbf{x})\|_2^2$$

Nuclear norm minimization \Rightarrow **ℓ_1 -norm minimization with FFT** in $\mathcal{O}(T \log T)$ time.

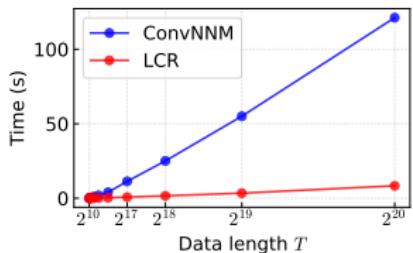
³ $\mathbf{w} \in \mathbb{R}^T$ (Lagrange multiplier); $\langle \cdot, \cdot \rangle$ (inner product).

Laplacian Convolutional Representation

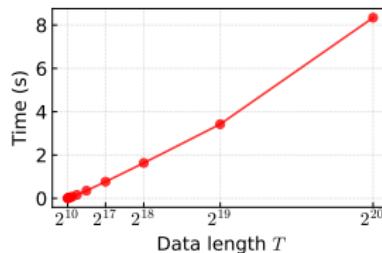
Empirical time complexity

On the synthetic data $\mathbf{y} \in \mathbb{R}^T$ with $T \in \{2^{10}, 2^{11}, \dots, 2^{20}\}$

- Ours: **LCR**
 - An FFT implementation in $\mathcal{O}(T \log T)$
 - The logarithmic factor $\log T$ makes the FFT highly efficient
- Baseline: **ConvNNM⁴** ([Liu'22](#), [Liu & Zhang'23](#))
 - Convolution matrix $C_{\tilde{\tau}}(\mathbf{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$ with kernel size $\tilde{\tau} = 2^4$
 - Singular value thresholding in $\mathcal{O}(\tilde{\tau}^2 T)$



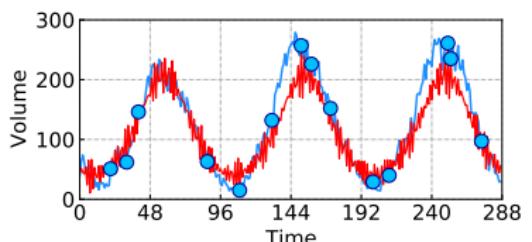
ConvNNM vs. LCR



LCR

⁴Convolution nuclear norm minimization.

Univariate Traffic Time Series Imputation



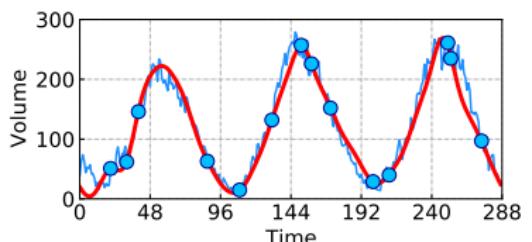
CircNNM:

$$\min_{\mathbf{x}} \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}}$$

$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$



Plus temporal regularization

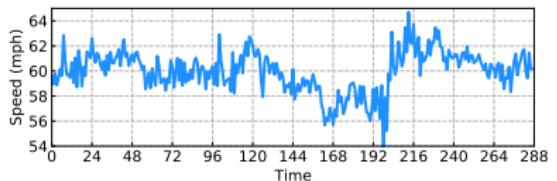


LCR:

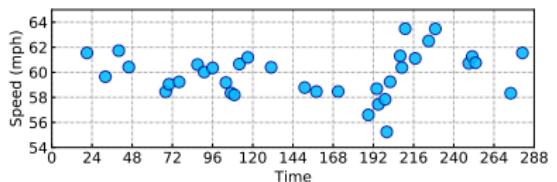
$$\min_{\mathbf{x}} \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\ell * \mathbf{x}\|_2^2}_{\text{local}}$$

$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$

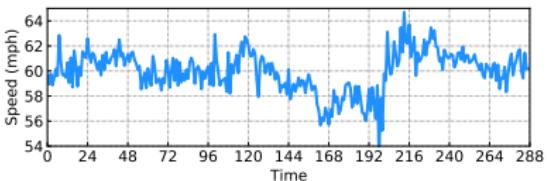
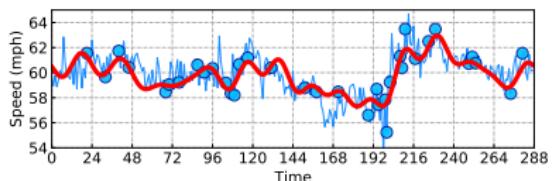
Univariate Traffic Time Series Imputation



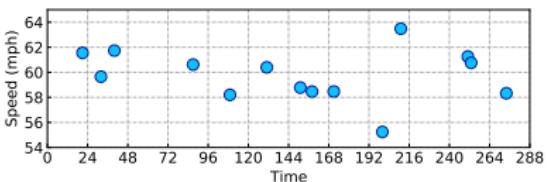
↓ Mask 90% observations



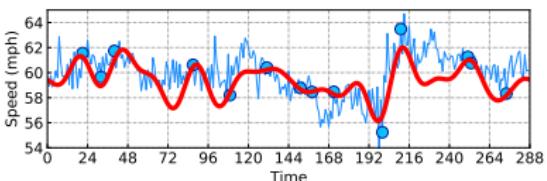
↓ Reconstruct time series



↓ Mask 95% observations



↓ Reconstruct time series



LCR can reconstruct traffic time series from very sparse data.

Large-Scale Traffic Data Imputation

LCR vs. baseline (in MAPE/RMSE)

- PeMS-4W: California freeway traffic speed dataset ($Y \in \mathbb{R}^{11160 \times 8064}$)

| Model | Missing rate | | | |
|------------------|------------------|------------------|------------------|------------------|
| | 30% | 50% | 70% | 90% |
| LCR-2D | 1.50/1.49 | 1.76/1.69 | 2.07/2.06 | 3.19/3.05 |
| LCR _N | 1.48/1.50 | 1.73/1.73 | 2.07/2.12 | 3.24/3.22 |
| LCR | 1.50/1.49 | 1.76/1.69 | 2.08/2.07 | 3.21/3.06 |
| CTNNM | 2.26/1.84 | 2.67/2.14 | 3.40/2.66 | 5.22/3.90 |
| CircNNM | 2.26/1.84 | 2.69/2.15 | 3.43/2.67 | 5.34/3.96 |
| LRMC | 2.04/1.80 | 2.43/2.12 | 3.08/2.66 | 6.05/4.43 |
| HaLRTC | 1.98/1.73 | 2.22/1.98 | 2.84/2.49 | 4.39/3.66 |
| LRTC-TNN | 1.68/1.55 | 1.93/1.77 | 2.33/2.14 | 3.40/3.10 |
| NoTMF | 2.95/2.65 | 3.05/2.73 | 3.33/2.97 | 5.22/4.71 |

Results

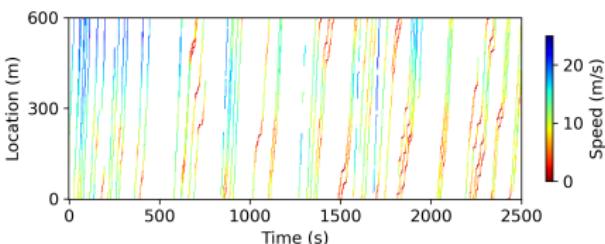
- LCR-2D > CTNNM: The importance of temporal regularization.
- CTNNM \geq CircNNM: Cyclic tensor is superior to circulant matrix.
- LCR > LRMC/LRTC: The importance of global/local modeling.

$\mathcal{O}(NT \log(NT))$ (FFT) vs. $\mathcal{O}(\min\{N^2T, NT^2\})$ (SVD)

Hankel Tensor Factorization

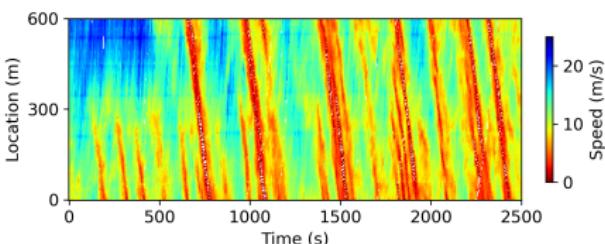
Motivation: Spatiotemporal data reconstruction

- Sparse speed field reconstruction problem in vehicular traffic flow.



200-by-500 matrix
(NGSIM)

Reconstruct speed field from
5% sparse trajectories?



How to characterize both spatial and temporal dependencies?

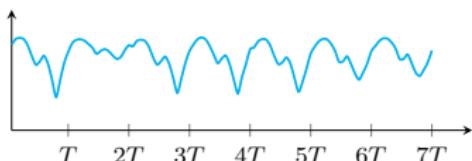
Hankel Tensor Factorization

- Hankel matrix

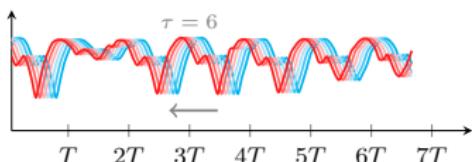
- Given $\mathbf{x} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4 \times 2}$$

- Automatic temporal modeling



Traffic time series



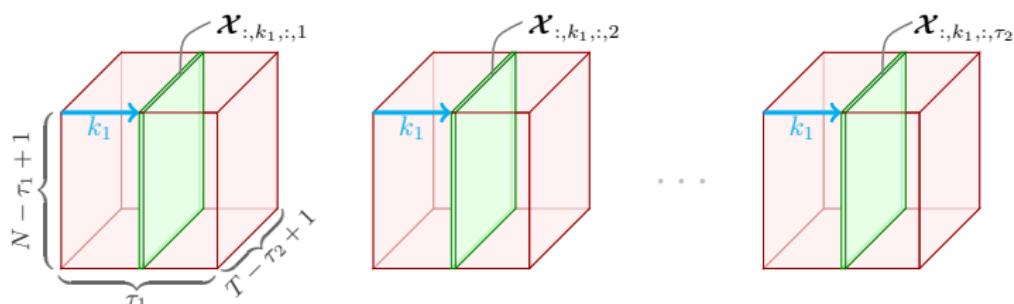
Hankel matrix

Hankel Tensor Factorization

- Hankel tensor: Given any matrix $\mathbf{X} \in \mathbb{R}^{N \times T}$, we have

$$\mathcal{X} \triangleq \mathcal{H}_{\tau_1, \tau_2}(\mathbf{X})$$

- Window lengths: $\tau_1, \tau_2 \in \mathbb{N}^+$;
- Tensor size: $(N - \tau_1 + 1) \times \tau_1 \times (T - \tau_2 + 1) \times \tau_2$;



(Figure) 4th-order Hankel tensor: A sequence of third-order tensors.

- Slice: $\mathcal{X}_{:,k_1,:,:k_2}$, $\forall k_1, k_2$;
- Slice size: $(N - \tau_1 + 1) \times (T - \tau_2 + 1)$.

Hankel Tensor Factorization

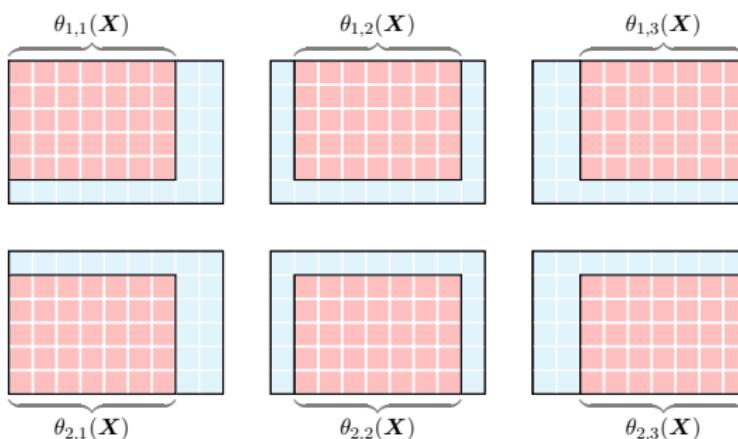
Hankel indexing

- Sampling function for the Hankel tensor:

$$\theta_{k_1, k_2}(\mathbf{X}) \triangleq [\mathcal{H}_{\tau_1, \tau_2}(\mathbf{X})]_{:, k_1, :, k_2},$$

referring to as the tensor slice with $k_1 \in \{1, \dots, \tau_1\}$, $k_2 \in \{1, \dots, \tau_2\}$.

- [Importance] Developing memory-efficient algorithms



- Tensor slices $\theta_{k_1, k_2}(\mathbf{X})$ vs. data matrix \mathbf{X}

Hankel Tensor Factorization

Ours:

- Convolutional tensor decomposition (circular convolution \star_{row}):

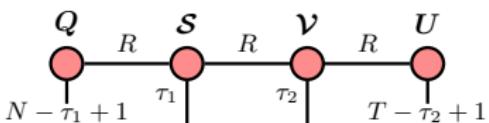
$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \star_{\text{row}} \mathbf{s}_{k_1}^{\top})(\mathbf{U} \star_{\text{row}} \mathbf{v}_{k_2}^{\top})^{\top}$$

Baselines:

- Tensor-train decomposition:

$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \mathbf{S}_{k_1})(\mathbf{U} \mathbf{V}_{k_2})^{\top}$$

- $\{\mathbf{S}_{k_1}, \mathbf{V}_{k_2}\}$ are **circulant matrices** \Rightarrow convolutional decomposition
- $\{\mathbf{S}_{k_1}, \mathbf{V}_{k_2}\}$ are **diagonal matrices** \Rightarrow CP decomposition



- CP tensor decomposition (Khatri-Rao product \odot):

$$\theta_{k_1, k_2}(\mathbf{Y}) \approx (\mathbf{Q} \odot \mathbf{s}_{k_1}^{\top})(\mathbf{U} \odot \mathbf{v}_{k_2}^{\top})^{\top}$$

Hankel Tensor Factorization

HTF (convolutional decomposition)

- Optimization problem:

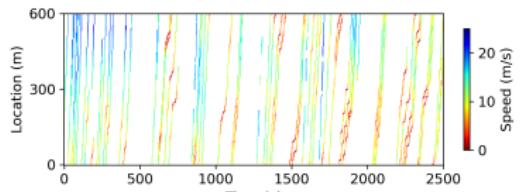
$$\min_{Q, S, U, V} \frac{1}{2} \underbrace{\sum_{k_1, k_2} \left\| \mathcal{P}_{\Omega_{k_1, k_2}} (\theta_{k_1, k_2}(\mathbf{Y}) - (Q \star_{\text{row}} s_{k_1}^{\top})(U \star_{\text{row}} v_{k_2}^{\top})^{\top}) \right\|_F^2}_{\text{Tensor decomposition on Hankel tensor slices}} + \frac{\rho}{2} (\|Q\|_F^2 + \|S\|_F^2 + \|U\|_F^2 + \|V\|_F^2)$$

- Alternating minimization (let f be the obj.):

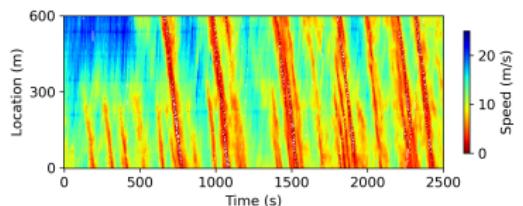
$$\begin{cases} Q := \{Q \mid \frac{\partial f}{\partial Q} = \mathbf{0}\} & \text{(conjugate gradient)} \\ s_{k_1} := \{s_{k_1} \mid \frac{\partial f}{\partial s_{k_1}} = \mathbf{0}\}, \forall k_1 & \text{(conjugate gradient)} \\ U := \{U \mid \frac{\partial f}{\partial U} = \mathbf{0}\} & \text{(conjugate gradient)} \\ v_{k_2} := \{v_{k_2} \mid \frac{\partial f}{\partial v_{k_2}} = \mathbf{0}\}, \forall k_2 & \text{(conjugate gradient)} \end{cases}$$

- Memory-efficient but still computationally costly!

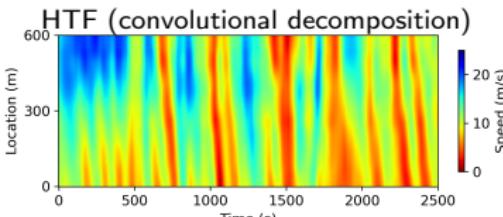
Extreme Missing Traffic Data Imputation



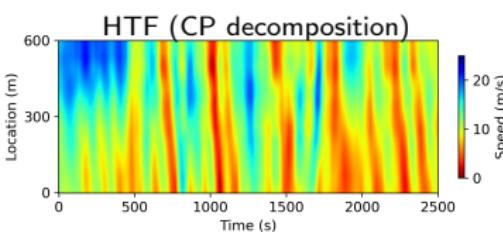
Sparse speed field



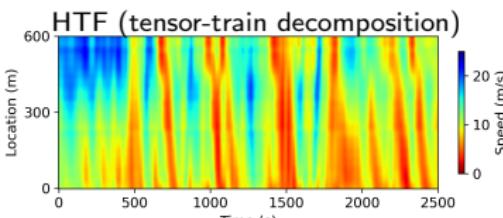
Ground truth speed field



MAPE = 51.92%



MAPE = 53.93%



MAPE = 56.48%

Extreme Missing Traffic Data Imputation

HTF vs. baseline (in MAPE/RMSE)

- On the Seattle freeway traffic speed dataset ($Y \in \mathbb{R}^{323 \times 8064}$)

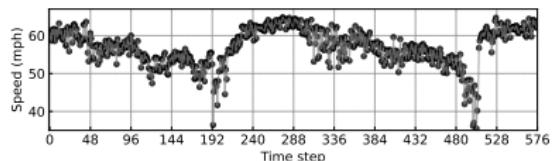
| Model | Random missing rate | | | |
|--------------------|---------------------|------------------|------------------|------------------|
| | 80% | 85% | 90% | 95% |
| HTF (convolution) | 6.21/3.88 | 6.51/4.06 | 6.98/4.30 | 8.02/4.84 |
| HTF (tensor-train) | 8.75/5.16 | 9.86/5.76 | 9.24/5.36 | 9.89/5.70 |
| HTF (CP) | 7.25/4.33 | 7.93/4.66 | 8.61/4.96 | 9.25/5.20 |
| LATC | 6.50/4.00 | 6.90/4.21 | 7.47/4.51 | 8.75/5.05 |
| LRTC-TNN | 6.97/4.24 | 7.43/4.43 | 8.19/4.81 | 9.60/5.55 |
| LCR-2D | 6.75/4.15 | 7.31/4.38 | 7.96/4.71 | 9.78/5.39 |
| BTMF | 6.85/4.17 | 7.36/4.42 | 8.13/4.79 | 9.63/5.48 |

Results

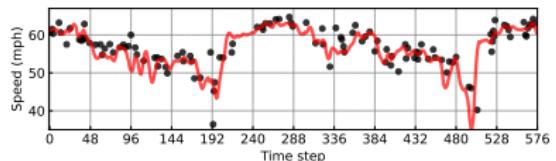
- Convolutional tensor decomposition outperforms both tensor-train and CP tensor decomposition.
- Our HTF model performs better than state-of-the-art baseline models.

Extreme Missing Traffic Data Imputation

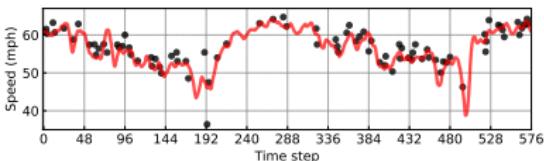
- Example: 1st time series within two days.



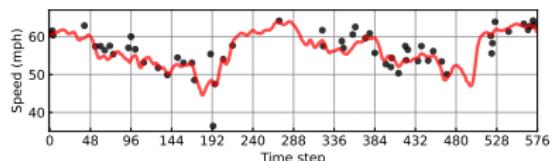
Original time series



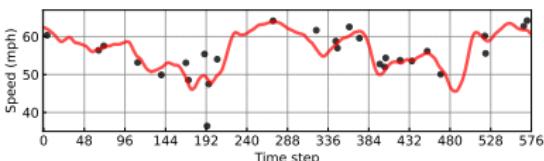
80% missing rate



85% missing rate

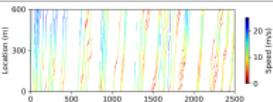
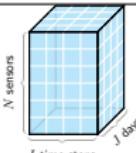
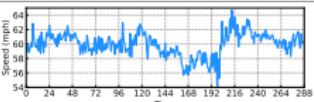
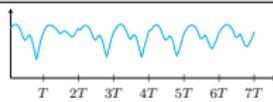


90% missing rate



95% missing rate

Conclusion

| | |
|-------------------|---|
| Sparse |  |
| High-dimensional |  |
| Multidimensional |  |
| Noises & outliers |  |
| Nonstationary |  |

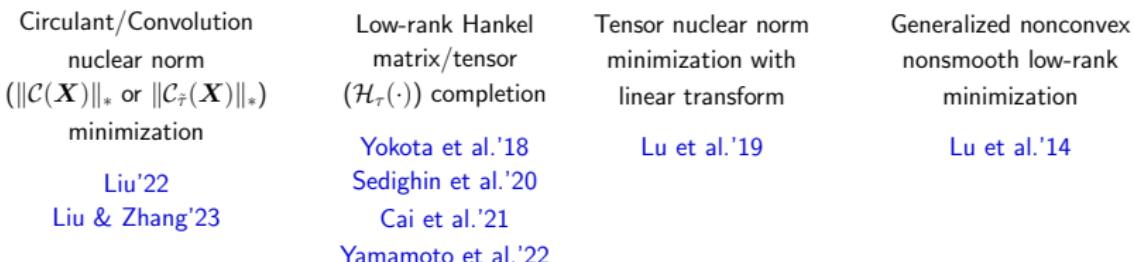
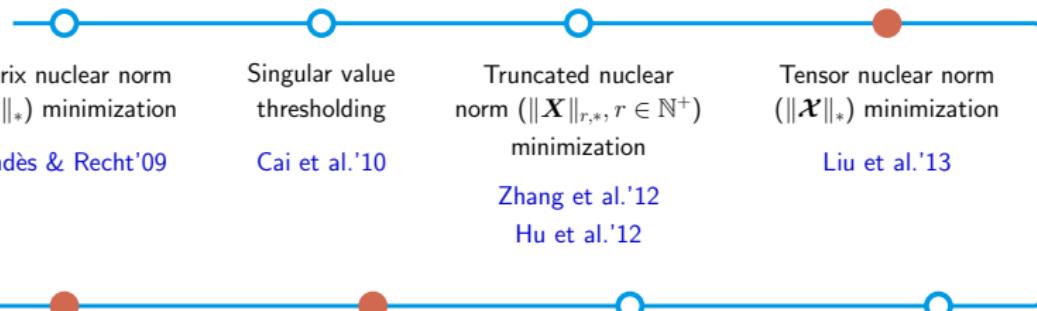
Low-rank framework:

- NoTMF: matrix factorization
- LCR: circulant matrix nuclear norm minimization
- HTF: tensor factorization

⇒ Temporal modeling:

- NoTMF: seasonal differenced vector autoregression
- LCR: temporal smoothing
- HTF: automatic temporal modeling with Hankel tensor

Highlights & Contributions



(Ours) LCR:

- ✓ Local trend modeling
- ✓ An FFT implementation

(Ours) HTF:

- ✓ Memory-efficient
- ✓ Conv. para.

References

A short list:

- ([Candès & Recht'09](#)) "Exact matrix completion via convex optimization." *Foundations of Computational Mathematics*. 2009, 9(6): 717-772.
- ([Cai et al.'10](#)) "A singular value thresholding algorithm for matrix completion." *SIAM Journal on optimization*. 2010, 20(4): 1956-1982.
- ([Zhang et al.'12](#)) "Matrix completion by truncated nuclear norm regularization." *IEEE Conference on computer vision and pattern recognition*. 2012.
- ([Hu et al.'12](#)) "Fast and accurate matrix completion via truncated nuclear norm regularization." *IEEE transactions on pattern analysis and machine intelligence*. 2012, 35(9): 2117-2130.
- ([Lu et al.'14](#)) "Generalized nonconvex nonsmooth low-rank minimization." *Proceedings of the IEEE conference on computer vision and pattern recognition*. 2014.
- ([Gultekin & Paisley'18](#)) "Online forecasting matrix factorization." *IEEE Transactions on Signal Processing*. 2018, 67(5): 1223-1236.
- ([Yokota et al.'18](#)) "Missing slice recovery for tensors using a low-rank model in embedded space." *Proceedings of the IEEE conference on computer vision and pattern recognition*. 2018.
- ([Lu et al.'19](#)) "Tensor robust principal component analysis with a new tensor nuclear norm." *IEEE transactions on pattern analysis and machine intelligence*. 2019, 42(4): 925-938.
- ([Cai et al.'21](#)) "Accelerated structured alternating projections for robust spectrally sparse signal recovery." *IEEE Transactions on Signal Processing*. 2021, 69: 809-821.
- ([Chen & Sun'22](#)) "Bayesian temporal factorization for multidimensional time series prediction." *IEEE Transactions on Pattern Analysis and Machine Intelligence*. 2022, 44(9): 4659-4673.
- ([Liu'22](#)) "Time series forecasting via learning convolutionally low-rank models." *IEEE Transactions on Information Theory*. 2022, 68(5): 3362-3380.
- ([Liu & Zhang'23](#)) "Recovery of future data via convolution nuclear norm minimization." *IEEE Transactions on Information Theory*. 2023, 69(1): 650-665.



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Thanks for your attention!

Any Questions?

Slides: <https://xinychen.github.io/slides/sustech23.pdf>

About me:

- Homepage: <https://xinychen.github.io>
- How to reach me: chenxy346@gmail.com

Research Interests

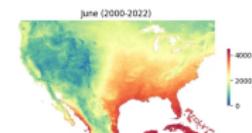
- Machine learning & spatiotemporal data modeling



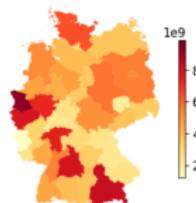
Transportation



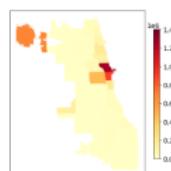
Mobile service



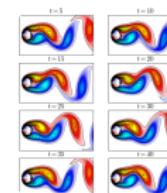
Climate



Energy



Mobility



Dynamical system

- Urban science
- Intelligent transportation systems

Past Works

Spatiotemporal traffic data imputation:

1. Xinyu Chen, Zhaocheng He, Jiawei Wang (2018). Spatial-temporal traffic speed patterns discovery and incomplete data recovery via SVD-combined tensor decomposition. *Transportation Research Part C: Emerging Technologies*. 86: 59-77. (100+ citations)
2. Xinyu Chen, Zhaocheng He, Lijun Sun (2019). A Bayesian tensor decomposition approach for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*. 98: 73-84. (200+ citations, ESI highly cited paper)
3. Xinyu Chen, Zhaocheng He, Yixian Chen, Yuhuan Lu, Jiawei Wang (2019). Missing traffic data imputation and pattern discovery with a Bayesian augmented tensor factorization model. *Transportation Research Part C: Emerging Technologies*. 104: 66-77. (100+ citations)
4. Xinyu Chen, Jinming Yang, Lijun Sun (2020). A nonconvex low-rank tensor completion model for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*. 117: 102673. (100+ citations)
5. Xinyu Chen, Yixian Chen, Nicolas Saunier, Lijun Sun (2021). Scalable low-rank tensor learning for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*. 129: 103226.

Past Works

Spatiotemporal traffic data imputation:

6. Xinyu Chen, Mengying Lei, Nicolas Saunier, Lijun Sun (2022). Low-rank autoregressive tensor completion for spatiotemporal traffic data imputation. *IEEE Transactions on Intelligent Transportation Systems*. 23 (8): 12301-12310. (50+ citations, ESI hot paper)

Spatiotemporal time series forecasting:

7. Xinyu Chen, Lijun Sun (2022). Bayesian temporal factorization for multidimensional time series prediction. *IEEE Transactions on Pattern Analysis and Machine Intelligence*. 44 (9): 4659-4673. (150+ citations, ESI hot paper & ESI highly cited paper)

Spatiotemporal pattern discovery:

8. Xinyu Chen, Chengyuan Zhang, Xiaoxu Chen, Nicolas Saunier, Lijun Sun (2023). Discovering dynamic patterns from spatiotemporal data with time-varying low-rank autoregression. *IEEE Transactions on Knowledge and Data Engineering*. Early access.

Past Works

A strong advocate of open-source and reproducible research:

Algorithms: e.g., **transdim** (1.1k stars)

- Machine learning for transportation data imputation and prediction.
- <https://github.com/xinychen/transdim>

Tools: e.g., **awesome-latex-drawing** (1.2k stars)

- Drawing Bayesian networks, graphical models, technical frameworks, and illustrations in LaTeX.
- <https://github.com/xinychen/awesome-latex-drawing>

Tutorials: e.g., **latex-cookbook** (1.1k stars)

- Academic writing with LaTeX.
- <https://github.com/xinychen/latex-cookbook>
- Published in Tsinghua University Press.

Future Plan

Research directions:

- Urban science
- Human mobility modeling
- Geospatial data analysis
- Intelligent & sustainable urban systems
- Optimization & decision making

Goals: Solving many scientific, mathematical, and engineering problems with AI algorithms.

Website: <https://spatiotemporal-data.github.io>