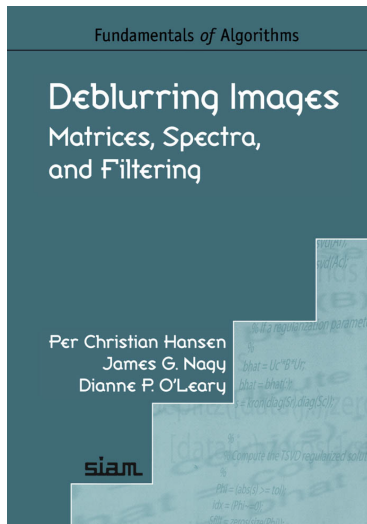


Deblurring Images Matrices, Spectra, and Filtering

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- **Chapter 1: The Image Deblurring Problem**
- **Chapter 4: Structured Matrix Computations**

The Image Deblurring Problem

About the image deblurring¹:

- **[Significance]** Image deblurring is fundamental in making pictures sharp and useful.
- **[General idea]** Recovering the original and sharp image by using a mathematical model of blurring process.
- **[Fact]** No hope to recover the original image exactly!
- **[Technical goal]** Develop efficient and reliable algorithms for recovering as much information as possible from the given data.
- **[Representation]** A digital image is a two- or three-dimensional array of numbers representing intensities on a grayscale or color scale.

¹The images and Matlab functions discussed in the book are available at <https://archive.siam.org/books/fa03/>.

The Image Deblurring Problem

A blurred picture and simple linear model.

- **Sharp image** vs. **blurred image**



- Notation: $\mathbf{X} \in \mathbb{R}^{m \times n}$ (desired **sharp** image) vs. $\mathbf{B} \in \mathbb{R}^{m \times n}$ (recorded **blurred** image)
- A simple linear model:
 - Suppose the blurring of the columns in the image is independent of the blurring of the rows.
 - **Bilinear relationship:** $\mathbf{A}_c \mathbf{X} \mathbf{A}_r^\top = \mathbf{B}$

The Image Deblurring Problem

A first attempt at deblurring.

- Recall that the simple linear model:

$$\mathbf{A}_c \mathbf{X} \mathbf{A}_r^\top = \mathbf{B} \implies \mathbf{X}_{\text{naive}} = \mathbf{A}_c^{-1} \mathbf{B} (\mathbf{A}_r^\top)^{-1} \quad (1)$$

ignores several types of errors.

- Let

$$\mathbf{B}_{\text{exact}} = \mathbf{A}_c \mathbf{X} \mathbf{A}_r^\top \quad (2)$$

be the ideal (noise-free) blurred image, ignoring all kinds of errors.

- Consider small random errors (noise) in the recorded blurred image:

$$\mathbf{B} = \mathbf{B}_{\text{exact}} + \mathbf{E} = \mathbf{A}_c \mathbf{X} \mathbf{A}_r^\top + \mathbf{E} \quad (3)$$

where $\mathbf{E} \in \mathbb{R}^{m \times n}$ is the **noise image**.

The Image Deblurring Problem

A first attempt at deblurring.

The naive reconstruction

Recall that

$$\begin{cases} X_{\text{naive}} = A_c^{-1} B (A_r^\top)^{-1} \\ B = B_{\text{exact}} + E = A_c X A_r^\top + E \end{cases} \quad (4)$$

we therefore have the naive reconstruction:

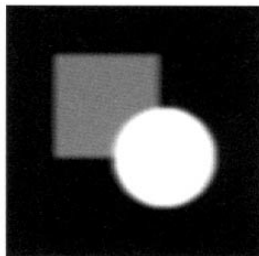
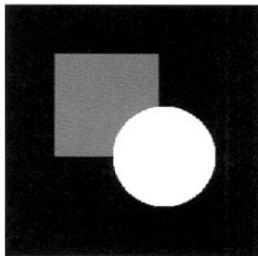
$$\begin{aligned} X_{\text{naive}} &= A_c^{-1} B (A_r^\top)^{-1} \\ &= A_c^{-1} B_{\text{exact}} (A_r^\top)^{-1} + A_c^{-1} E (A_r^\top)^{-1} \\ &= X + A_c^{-1} E (A_r^\top)^{-1} \end{aligned} \quad (5)$$

- The blurred image consists of two components: the first component is the **exact image**, and the second component is the **inverted noise**.

The Image Deblurring Problem

A first attempt at deblurring.

- A simple test: **Exact image** $X \in \mathbb{R}^{m \times n}$ vs. **blurred image** $B \in \mathbb{R}^{m \times n}$



The Image Deblurring Problem

Lemma

For the simple model $\mathbf{B} = \mathbf{A}_c \mathbf{X} \mathbf{A}_r^\top + \mathbf{E}$, the relative error in the naive reconstruction $\mathbf{X}_{\text{naive}} = \mathbf{A}_c^{-1} \mathbf{B} (\mathbf{A}_r^\top)^{-1}$ satisfies

$$\frac{\|\mathbf{X}_{\text{naive}} - \mathbf{X}\|_F}{\|\mathbf{X}\|_F} \leq \text{cond}(\mathbf{A}_c) \cdot \text{cond}(\mathbf{A}_r) \cdot \frac{\|\mathbf{E}\|_F}{\|\mathbf{B}\|_F} \quad (6)$$

where $\|\cdot\|_F$ denotes the Frobenius norm^a, and $\text{cond}(\cdot)$ denotes the conditional number^b.

^aFor any $\mathbf{X} \in \mathbb{R}^{m \times n}$, we have $\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n x_{ij}^2}$.

^bFor any $\mathbf{A} \in \mathbb{R}^{N \times N}$ whose singular values are strictly positive, namely, $\sigma_1 \geq \dots \geq \sigma_N > 0$, we have $\text{cond}(\mathbf{A}) = \sigma_1 / \sigma_N$.

The Image Deblurring Problem

Deblurring using a general linear model.

- In most situations, the blur is indeed **linear**, or at least well approximated by a linear model.
- A general linear model via **vectorization**.
 - Given sharp image $\mathbf{X} \in \mathbb{R}^{m \times n}$ and blurred image $\mathbf{B} \in \mathbb{R}^{m \times n}$, since the blurring is assumed to be a linear operation, there must exist a large **blurring matrix** $\mathbf{A} \in \mathbb{R}^{N \times N}$ ($N = mn$) such that

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{7}$$

with

$$\mathbf{x} = \text{vec}(\mathbf{X}) = \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} \in \mathbb{R}^N, \quad \mathbf{b} = \text{vec}(\mathbf{B}) = \begin{bmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} \in \mathbb{R}^N \tag{8}$$

- The naive approach to image deblurring is simply to solve this linear algebraic system.

The Image Deblurring Problem

Deblurring using a general linear model.

The naive reconstruction (matrix-form)

Recall that

$$\begin{cases} X_{\text{naive}} = A_c^{-1} B (A_r^\top)^{-1} \\ B = B_{\text{exact}} + E = A_c X A_r^\top + E \end{cases} \quad (9)$$

we therefore have the naive reconstruction:

$$\begin{aligned} X_{\text{naive}} &= A_c^{-1} B (A_r^\top)^{-1} \\ &= A_c^{-1} B_{\text{exact}} (A_r^\top)^{-1} + A_c^{-1} E (A_r^\top)^{-1} \\ &= X + A_c^{-1} E (A_r^\top)^{-1} \end{aligned} \quad (10)$$

The naive reconstruction (vector-form)

Vectorize blurred image B and noise image E as

$b_{\text{exact}} = \text{vec}(B_{\text{exact}}) = Ax$ and $e = \text{vec}(E)$, respectively, then we have

$$x_{\text{naive}} = A^{-1}b = A^{-1}b_{\text{exact}} + A^{-1}e = x + A^{-1}e \quad (11)$$

The Image Deblurring Problem

Deblurring using a general linear model.

- Relationship between matrix- and vector-form reconstruction:

$$\begin{aligned} \mathbf{X}_{\text{naive}} &= \mathbf{A}_c^{-1} \mathbf{B} (\mathbf{A}_r^\top)^{-1} \\ \implies \mathbf{x}_{\text{naive}} &= (\mathbf{A}_r^{-1} \otimes \mathbf{A}_c^{-1}) \mathbf{b} \\ &= (\mathbf{A}_r \otimes \mathbf{A}_c)^{-1} \mathbf{b} \end{aligned} \tag{12}$$

it therefore demonstrates that $\mathbf{A} \triangleq \mathbf{A}_r \otimes \mathbf{A}_c$.

- Property of Kronecker product \otimes :

Proposition

Let $\mathbf{A} \in \mathbb{R}^{m \times m}$, $\mathbf{X} \in \mathbb{R}^{m \times n}$, and $\mathbf{B} \in \mathbb{R}^{n \times n}$ be three matrices commensurate from multiplication in that order, then it holds that

$$\text{vec}(\mathbf{A} \mathbf{X} \mathbf{B}) = (\mathbf{B}^\top \otimes \mathbf{A}) \text{vec}(\mathbf{X}) \tag{13}$$

The Image Deblurring Problem

Deblurring using a general linear model.

Singular value decomposition (SVD)

For any $\mathbf{A} \in \mathbb{R}^{N \times N}$ whose singular values are strictly positive, we have

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top = \sum_{i=1}^N \sigma_i \mathbf{u}_i \mathbf{v}_i^\top \implies \mathbf{A}^{-1} = \sum_{i=1}^N \frac{1}{\sigma_i} \mathbf{u}_i \mathbf{v}_i^\top \quad (14)$$

The naive reconstruction with SVD

The naive reconstruction can be written as follows,

$$\mathbf{x}_{\text{naive}} = \mathbf{A}^{-1} \mathbf{b} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^\top \mathbf{b} = \sum_{i=1}^N \frac{\mathbf{u}_i^\top \mathbf{b}}{\sigma_i} \mathbf{v}_i \quad (15)$$

in which the inverted noise is

$$\mathbf{A}^{-1} \mathbf{e} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^\top \mathbf{e} = \sum_{i=1}^N \frac{\mathbf{u}_i^\top \mathbf{e}}{\sigma_i} \mathbf{v}_i \quad (16)$$

The Image Deblurring Problem

Deblurring using a general linear model.

- Recall that the inverted noise is

$$\mathbf{A}^{-1}\mathbf{e} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^{\top}\mathbf{e} = \sum_{i=1}^N \frac{\mathbf{u}_i^{\top}\mathbf{e}}{\sigma_i} \mathbf{v}_i$$

- Properties for image deblurring problems:
 - The error components $|\mathbf{u}_i^{\top}\mathbf{e}|$ are small and typically of roughly the same order of magnitude for all i .
 - The singular values decay to a value very close to zero. As a consequence, the condition number $\text{cond}(\mathbf{A}) = \sigma_1/\sigma_N$ is very large, indicating that **the solution is very sensitive to perturbation and rounding errors.**
 - The singular vectors corresponding to the smaller singular values typically represent high-frequency information.** That is, as i increases, the vectors \mathbf{u}_i and \mathbf{v}_i tend to have more sign changes.

The Image Deblurring Problem

Deblurring using a general linear model.

- Recall that the inverted noise is

$$\mathbf{A}^{-1}\mathbf{e} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^{\top}\mathbf{e} = \sum_{i=1}^N \frac{\mathbf{u}_i^{\top}\mathbf{e}}{\sigma_i} \mathbf{v}_i$$

Remark

For $\mathbf{A}^{-1}\mathbf{e}$, the quantities $\mathbf{u}_i^{\top}\mathbf{e}/\sigma_i$ are the expansion coefficients for the basis vectors \mathbf{v}_i . When these quantities are small in magnitude, the solution has very little contribution from \mathbf{v}_i , but when we divide by a small singular values such as σ_N , we greatly magnify the corresponding error component $\mathbf{u}_N^{\top}\mathbf{e}$ which in turn contributes a large multiple of the high-frequency information contained in \mathbf{v}_N to the reconstruction solution.

- Thus, we can remove the high-frequency components that are dominated by error.

The Image Deblurring Problem

Deblurring using a general linear model.

- The naive reconstruction with SVD:

$$\mathbf{x}_{\text{naive}} = \sum_{i=1}^N \frac{\mathbf{u}_i^\top \mathbf{b}}{\sigma_i} \mathbf{v}_i \quad (17)$$

- The truncated expansion with $k < N, k \in \mathbb{N}^+$:

$$\mathbf{x}_k = \sum_{i=1}^k \frac{\mathbf{u}_i^\top \mathbf{b}}{\sigma_i} \mathbf{v}_i \quad (18)$$

which is indeed a reduced-rank linear model.

- We may wonder if a different value for k will produce a better reconstruction!

Structured Matrix Computations

- A general linear model:

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{e} \quad (19)$$

with

$$\begin{cases} \mathbf{b} = \text{vec}(\mathbf{B}) \in \mathbb{R}^N & \text{(blurred image)} \\ \mathbf{x} = \text{vec}(\mathbf{X}) \in \mathbb{R}^N & \text{(sharp image)} \\ \mathbf{e} = \text{vec}(\mathbf{E}) \in \mathbb{R}^N & \text{(noise image)} \\ \mathbf{A} \in \mathbb{R}^{N \times N} & \text{(blurring matrix)} \end{cases}$$

- The deblurring algorithms use certain orthogonal or unitary decompositions of \mathbf{A} .
 - SVD: $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$ vs. spectral decomposition²: $\mathbf{A} = \tilde{\mathbf{U}}\mathbf{\Lambda}\tilde{\mathbf{U}}^H$
 - If \mathbf{A} has real entries, then the elements in the matrices of the SVD will be real, but the entries in the spectral decomposition may be complex.

²A matrix is unitary if $\tilde{\mathbf{U}}^H \tilde{\mathbf{U}} = \tilde{\mathbf{U}} \tilde{\mathbf{U}}^H = \mathbf{I}$ where $\tilde{\mathbf{U}}^H = \text{conj}(\tilde{\mathbf{U}})^\top$ is the complex conjugate transpose of $\tilde{\mathbf{U}}$. $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of \mathbf{A} .

Structured Matrix Computations

Basic structures.

- Convolution is a mathematical operation.
- If $p(s)$ and $x(s)$ are **continuous** functions, then the convolution of $p(s)$ and $x(s)$ is a function $b(s)$ having the form

$$b(s) = \int_{-\infty}^{\infty} p(s-t)x(t)dt \quad (20)$$

each values of $b(s)$ is essentially a weighted average of the values of $x(s)$, where the weights are given by $p(s)$.

- The **discrete** version of convolution is a summation over a finite number of terms.

Structured Matrix Computations

Basic structures.

- x