POLYTECHNIQUE MONTRÉAL
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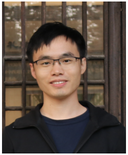
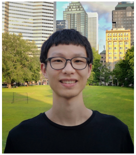
IVADO

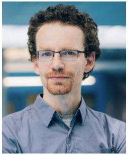
Laplacian Convolutional Representation for Traffic Time Series Imputation

Xinyu Chen

July 19, 2023







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Current work:

- X. Chen, Z. Cheng, N. Saunier, L. Sun (2022). Laplacian convolutional representation for traffic time series imputation. arXiv preprint arXiv:2212.01529.

GitHub repository:

- **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (1,000+ stars & 270+ forks on GitHub)
  https://github.com/xinychen/transdim

Slides:

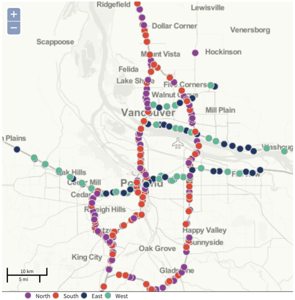
- https://xinychen.github.io/slides/LCR.pdf

Outline

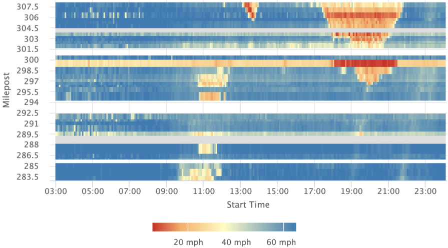
- **Motivation**
  Data-Driven ITS
  Time Series Imputation
  Speed Field Reconstruction
- **Low-Rank Laplacian Convolutional Model**
  Reformulate Laplacian Regularization
- **Experiments**
  Univariate Traffic Time Series Imputation
  Multivariate Model for Speed Field Reconstruction
- **Conclusion**

Motivation

- Portland highway traffic flow data[1]



Highway network & sensor locations



Traffic speed field

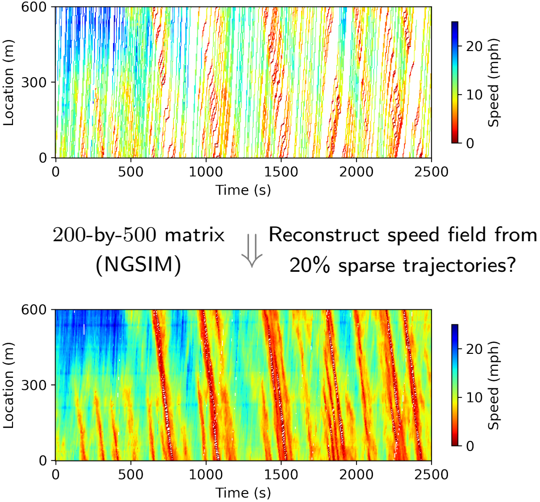
- Speed field $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ (N locations & T time steps)
- Speed field shows strong spatial/temporal dependencies

[1] https://portal.its.pdx.edu/home

Motivation

- How to reconstruct missing values from partial observations?

Motivation

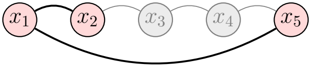


- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal local dependencies?

Low-Rank Laplacian Convolutional Model

Reformulate Laplacian regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.



Undirected and circulant graph

$$\xrightarrow{\text{Modeling}}$$

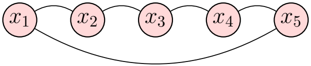
$$\boldsymbol{L}=\begin{bmatrix} \mathbf{2} & -1 & 0 & 0 & -1 \\ \mathbf{-1} & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ \mathbf{-1} & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

Low-Rank Laplacian Convolutional Model

Reformulate Laplacian regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.



Undirected and circulant graph

$\xrightarrow{\text{Modeling}}$

$$\boldsymbol{L}=\begin{bmatrix} \mathbf{2} & -1 & 0 & 0 & -1 \\ \mathbf{-1} & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ \mathbf{-1} & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

Low-Rank Laplacian Convolutional Model

Reformulate Laplacian regularization with circular convolution.

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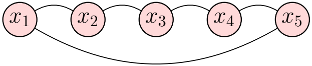


Undirected and circulant graph

$$\xrightarrow{\text{Modeling}} \quad \boldsymbol{L}=\begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Laplacian kernel: $\boldsymbol{\ell}=(2,-1,0,0,-1)^\top$.

$$\boldsymbol{L}\boldsymbol{x}=\begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\star\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}=\boldsymbol{\ell}\star\boldsymbol{x}$$

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}(\boldsymbol{x})=\frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2=\frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2$$

Low-Rank Laplacian Convolutional Model

Reformulate Laplacian regularization with circular convolution.

- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \cdots, -1}_{\tau}, 0, \cdots, 0, \underbrace{-1, \cdots, -1}_{\tau})^\top \in \mathbb{R}^T$$

  for any time series $\boldsymbol{x} = (x_1, \ldots, x_T)^\top \in \mathbb{R}^T$.

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}_\tau(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2 = \frac{1}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 = \frac{1}{2T}\|\mathcal{F}(\boldsymbol{\ell}) \circ \mathcal{F}(\boldsymbol{x})\|_2^2$$

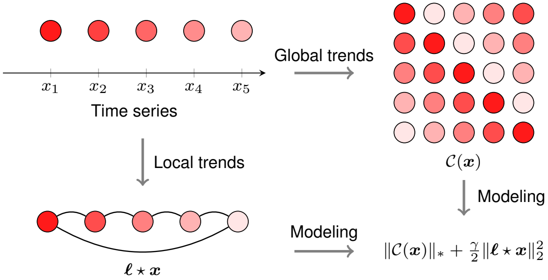
Low-Rank Laplacian Convolutional Model

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\boldsymbol{y} \in \mathbb{R}^T$ with observed index set Ω, LCR utilizes circulant matrix and Laplacian kernel to characterize **global and local trends** in time series, respectively, i.e.,

$$\begin{aligned} \min_{\boldsymbol{x}} \; & \|\mathcal{C}(\boldsymbol{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\boldsymbol{x}) \\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

where $\mathcal{C} : \mathbb{R}^T \to \mathbb{R}^{T \times T}$ denotes the circulant operator. $\|\cdot\|_*$ denotes the nuclear norm of matrix, namely, the sum of singular values.



Low-Rank Laplacian Convolutional Model

- LCR model:

$$\begin{aligned} \min_{\boldsymbol{x}} \; & \|\mathcal{C}(\boldsymbol{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\boldsymbol{x}) \\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

- Augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) = \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x} - \boldsymbol{z}\|_2^2 + \langle \boldsymbol{w}, \boldsymbol{x} - \boldsymbol{z} \rangle + \frac{\eta}{2}\|\mathcal{P}_\Omega(\boldsymbol{z} - \boldsymbol{y})\|_2^2$$

  where $\boldsymbol{w} \in \mathbb{R}^T$ is the Lagrange multiplier, and $\langle \cdot, \cdot \rangle$ denotes the inner product.

- The ADMM scheme:

$$\begin{cases} \boldsymbol{x} := \arg\min_{\boldsymbol{x}} \; \mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) \\ \boldsymbol{z} := \arg\min_{\boldsymbol{z}} \; \mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) \\ \quad = \frac{1}{\lambda + \eta}\mathcal{P}_\Omega(\lambda\boldsymbol{x} + \boldsymbol{w} + \eta\boldsymbol{y}) + \mathcal{P}_\Omega^\perp(\boldsymbol{x} + \boldsymbol{w}/\lambda) \\ \boldsymbol{w} := \boldsymbol{w} + \lambda(\boldsymbol{x} - \boldsymbol{z}) \end{cases}$$

Low-Rank Laplacian Convolutional Model

- Optimize $\boldsymbol{x}$ via fast Fourier transform (in $\mathcal{O}(T\log T)$ time):

$$\begin{aligned} \boldsymbol{x} &:= \arg\min_{\boldsymbol{x}}\ \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{w}/\lambda\|_2^2 \\ \Longrightarrow \hat{\boldsymbol{x}} &:= \arg\min_{\hat{\boldsymbol{x}}}\ \|\hat{\boldsymbol{x}}\|_1 + \frac{\gamma}{2T}\|\hat{\boldsymbol{\ell}}\circ\hat{\boldsymbol{x}}\|_2^2 + \frac{\lambda}{2T}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{z}}+\hat{\boldsymbol{w}}/\lambda\|_2^2 \\ &= \arg\min_{\hat{\boldsymbol{x}}}\ \|\hat{\boldsymbol{x}}\|_1 + \frac{\lambda}{2T}\|\hat{\boldsymbol{x}}-(\lambda\hat{\boldsymbol{z}}-\hat{\boldsymbol{w}})\oslash(\lambda\hat{\boldsymbol{\ell}}^*\circ\hat{\boldsymbol{\ell}}+\lambda\mathbb{1}_T)\|_2^2 \end{aligned}$$

  where we introduce $\{\hat{\boldsymbol{\ell}},\hat{\boldsymbol{x}},\hat{\boldsymbol{z}},\hat{\boldsymbol{w}}\}$ referring to $\{\boldsymbol{\ell},\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}\}$ in the frequency domain.

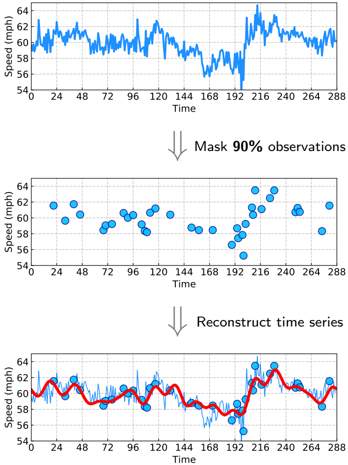
ℓ_1-norm Minimization in Complex Space (Liu & Zhang'22)

For any optimization problem in the form of ℓ_1-norm minimization in complex space:

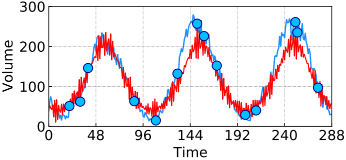
$$\min_{\hat{\boldsymbol{x}}}\ \|\hat{\boldsymbol{x}}\|_1 + \frac{\lambda}{2T}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{h}}\|_2^2$$

with complex-valued $\hat{\boldsymbol{x}},\hat{\boldsymbol{h}}\in\mathbb{C}^T$, element-wise, the solution is given by

$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|}\cdot\max\{0,|\hat{h}_t|-T/\lambda\}, t=1,\ldots,T.$$

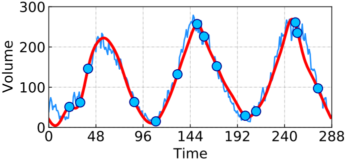

Mask 90% observations
Mask 95% observations
Reconstruct time series
Reconstruct time series
Speed (mph)
Time

Circulant matrix nuclear norm minimization



$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_* \quad \text{s. t.} \ \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

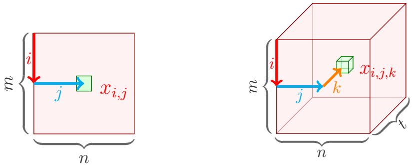
Plus temporal regularization



$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\boldsymbol{x}) \quad \text{s. t.} \ \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

Low-Rank Laplacian Convolutional Model

- What is tensor? $\boldsymbol{X} \in \mathbb{R}^{m \times n}$ vs. $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{m \times n \times t}$



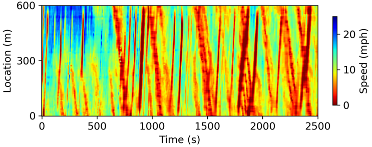
Multivariate LCR (LCR-2D)

For any partially observed time series $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ with observed index set Ω, LCR can be formulated as follows,

$$\begin{aligned} \min_{\boldsymbol{X}} \; & \|\mathcal{C}(\boldsymbol{X})\|_* + \frac{\gamma}{2}\|(\boldsymbol{\ell}_s\boldsymbol{\ell}^\top) \star \boldsymbol{X}\|_F^2 \\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{X} - \boldsymbol{Y})\|_F \leq \epsilon \end{aligned}$$

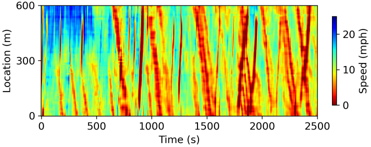
where $\mathcal{C} : \mathbb{R}^{N \times T} \to \mathbb{R}^{N \times N \times T \times T}$ denotes the circulant operator.

Quadratic variation completion (QVC)



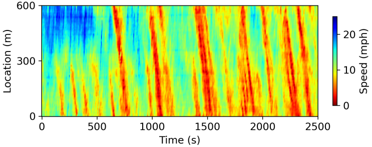
MAPE = 51.50% & RMSE = 4.86mph

Laplacian kernelized completion (LKC)



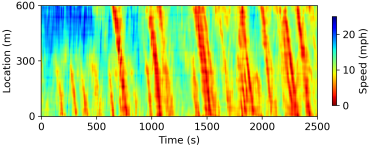
MAPE = 46.94% & RMSE = 4.34mph

Circulant tensor nuclear norm minimization (CTNNM)



MAPE = 43.51% & RMSE = 1.65mph

Laplacian convolutional representation (LCR)



MAPE = **41.29%** & RMSE = **1.55mph**

- QVC & LKC:

$$\begin{aligned} \min_{\boldsymbol{X}} \quad & \frac{\gamma}{2}\|(\boldsymbol{\ell}_s\boldsymbol{\ell}^\top)\star\boldsymbol{X}\|_F \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\boldsymbol{X}-\boldsymbol{Y})\|_F\leq\epsilon \end{aligned}$$

- CTNNM:

$$\begin{aligned} \min_{\boldsymbol{X}} \quad & \|\mathcal{C}(\boldsymbol{X})\|_* \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\boldsymbol{X}-\boldsymbol{Y})\|_F\leq\epsilon \end{aligned}$$

Conclusion

- **(Starting point)** How to reconstruct sparse speed field?
  ✓ Matrix factorization (**MF**) ✓ Tensor factorization (**TF**)
- **(Highlight)** The importance of spatiotemporal modeling in low-rank methods?
  - Spatial/temporal **smoothing** regularization:
  $$\begin{aligned}\min_{\boldsymbol{W},\boldsymbol{X}}\ &\frac{1}{2}\left\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\right\|_F^2+\frac{\rho}{2}(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)\\ &+\frac{\lambda}{2}(\|\boldsymbol{W}\boldsymbol{\Psi}_1^\top\|_F^2+\|\boldsymbol{X}\boldsymbol{\Psi}_2^\top\|_F^2)\end{aligned}$$
  - Automatic temporal modeling via **Hankelization**:
  $$\min_{\boldsymbol{U},\boldsymbol{V},\boldsymbol{X}}\ \frac{1}{2}\left\|\mathcal{P}_{\tilde{\Omega}}\left(\mathcal{H}_{\tau}(\boldsymbol{Y})-\sum_{r=1}^{R}\boldsymbol{u}_r\otimes\boldsymbol{v}_r\otimes\boldsymbol{x}_r\right)\right\|_F^2$$
  vs.
  $$\min_{\boldsymbol{Q},\boldsymbol{S},\boldsymbol{U},\boldsymbol{V}}\ \frac{1}{2}\left\|\mathcal{P}_{\dot{\Omega}}\left(\mathcal{H}_{\tau_1,\tau_2}(\boldsymbol{Y})-\sum_{r=1}^{R}\boldsymbol{q}_r\otimes\boldsymbol{s}_r\otimes\boldsymbol{u}_r\otimes\boldsymbol{v}_r\right)\right\|_F^2$$
- **(Challenge)** How to solve the high memory consumption issue in the Hankelization process?

Thanks for your attention!

Any Questions?

About me:

- Homepage: https://xinychen.github.io
- GitHub: https://github.com/xinychen (**3.4k+ stars**)
- Blog: https://medium.com/@xinyu.chen (**70k+ views**)
- How to reach me: chenxy346@gmail.com