



**POLYTECHNIQUE  
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# Laplacian Convolutional Representation for Traffic Time Series Imputation

**Xinyu Chen**

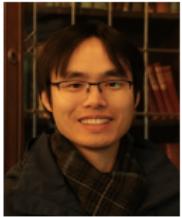
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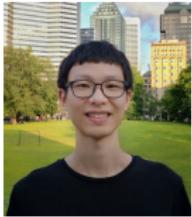
July 11, 2024



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### Publication:

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<https://doi.org/10.1109/TKDE.2024.3419698>

### Open source:

- Spatiotemporal data modeling initiative:  
<https://spatiotemporal-data.github.io> (coming soon)

# Outline

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- **Motivation**

- Data-Driven ITS
- Time Series Imputation
- Speed Field Reconstruction
- Marginal Idea Works

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- Revisit Circular Convolution
- Reformulate Laplacian regularization

- **Global Trend Modeling**

- **Laplacian Convolutional Representation**

- Model Description
- Solution Algorithm
- Empirical Time Complexity

- **Experiments**

- Traffic Volume & Speed Imputation
- Speed Field Reconstruction

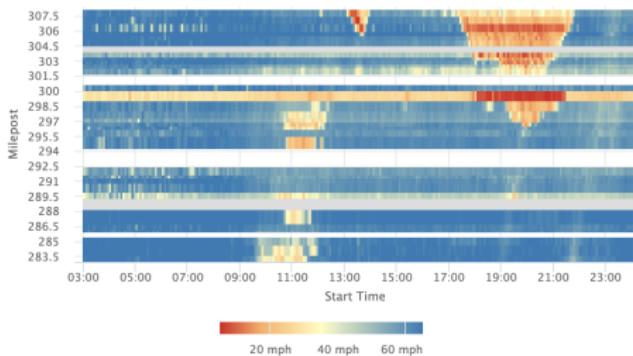
- **Conclusion**

# Motivation

- Portland highway traffic flow data<sup>1</sup>



Highway network &  $N$  sensors



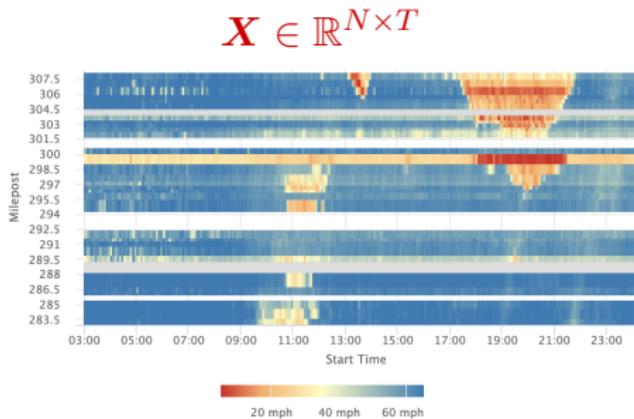
Traffic speed field

# Motivation

- Portland highway traffic flow data<sup>1</sup>



Highway network &  $N$  sensors

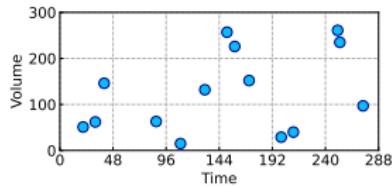


Traffic speed field

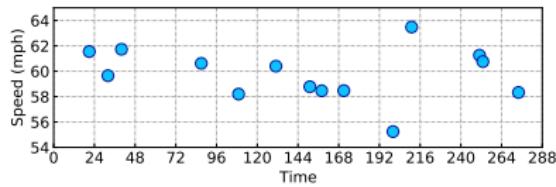
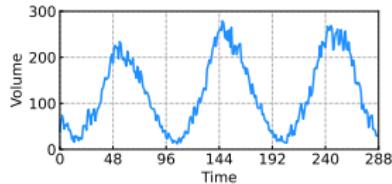
<sup>1</sup><https://portal.its.pdx.edu/home>

# Motivation

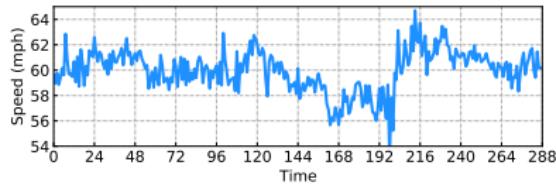
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↓ Reconstruct  
traffic volume?



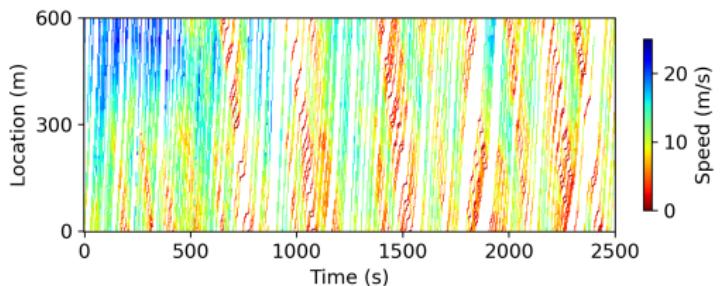
↓ Reconstruct  
traffic speed?



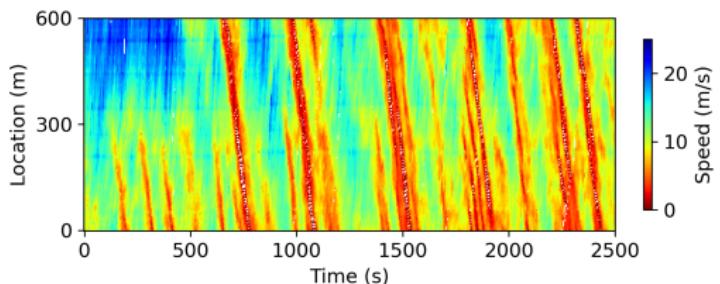
- How to utilize the global trends of traffic time series?
- How to produce local consistency of traffic data?

# Motivation

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200-by-500 matrix       $\Downarrow$       Reconstruct speed field from  
(NGSIM)                          20% sparse trajectories?



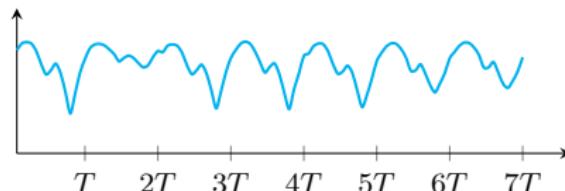
- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal local dependencies?

# Motivation

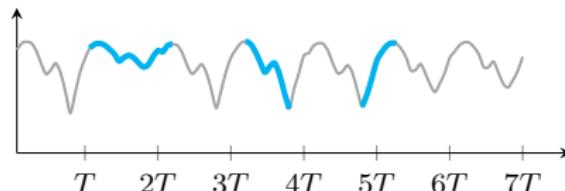
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## Sparse time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm)



- Local trends (e.g., short-term time series trends)



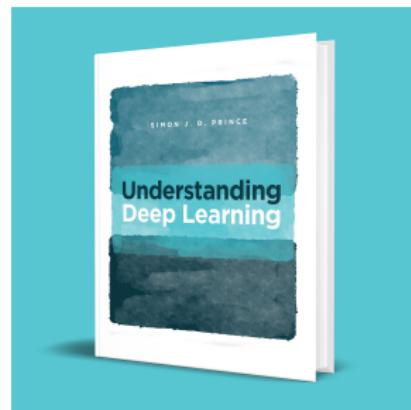
Modeling global & local trends simultaneously?

# Revisit Laplacian Matrix

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- The graph Laplacian matrix is

$$L = \underbrace{D}_{\text{degree matrix}} - \underbrace{A}_{\text{adjacency matrix}}$$

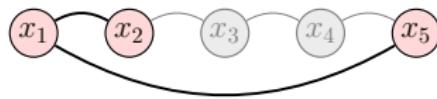


<https://udlbook.github.io/udlbook/>

# Revisit Laplacian Matrix

---

- Intuition of (circulant) Laplacian matrix.



Modeling

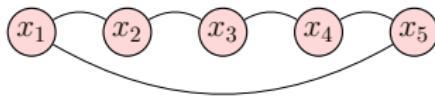
$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

## Revisit Laplacian Matrix

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- Intuition of (circulant) Laplacian matrix.



Modeling

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

# Revisit Circular Convolution

Reformulate Laplacian regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.

Undirected and circulant graph

$$\xrightarrow{\text{Modeling}} \mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Laplacian kernel:  $\ell = (2, -1, 0, 0, -1)^\top$ .

$$\mathbf{L}\mathbf{x} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \star \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \ell \star \mathbf{x}$$

where  $\star$  denotes the circular convolution.

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\ell \star \mathbf{x}\|_2^2$$

“... The circulant graph has an adjacency matrix that is a circulant matrix.”

— “*Circulant graph*” on Wikipedia

## Reformulate Laplacian Regularization

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Reformulate Laplacian regularization with circular convolution.

- Define Laplacian kernel:

$$\ell \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \dots, -1}_{\tau}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series  $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$ .

---

<sup>2</sup>It refers to the Convolution theorem.

# Reformulate Laplacian Regularization

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Reformulate Laplacian regularization with circular convolution.

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for any time series  $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$ .

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\ell * \mathbf{x}\|_2^2$$

- Property with discrete Fourier transform (denoted by  $\mathcal{F}(\cdot)$ )<sup>2</sup>:

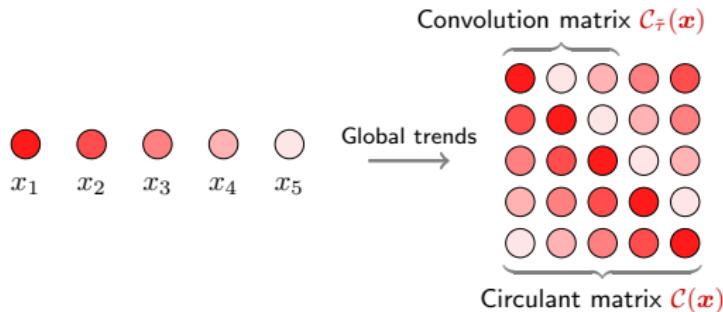
$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\ell * \mathbf{x}\|_2^2 = \underbrace{\frac{1}{2T} \|\mathcal{F}(\ell) \circ \mathcal{F}(\mathbf{x})\|_2^2}_{\text{w/ FFT in } \mathcal{O}(T \log T) \text{ time}}$$

---

<sup>2</sup>It refers to the Convolution theorem.

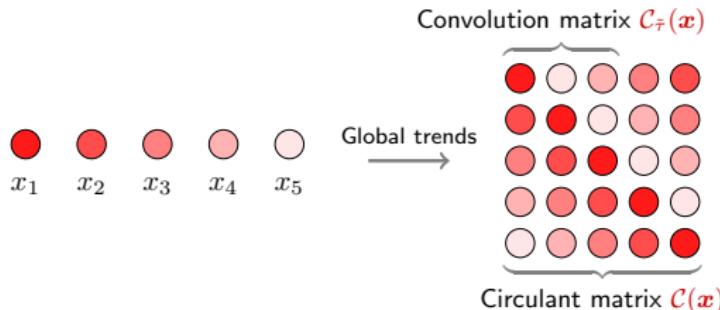
# Global Trend Modeling

Circulant matrix  $\mathcal{C}(\mathbf{x})$  vs. convolution matrix  $\mathcal{C}_{\tilde{\tau}}(\mathbf{x})$



# Global Trend Modeling

Circulant matrix  $\mathcal{C}(\mathbf{x})$  vs. convolution matrix  $\mathcal{C}_{\tilde{\tau}}(\mathbf{x})$



- Circulant/Convolution nuclear norm minimization (w/  $\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1$ )
  - A balance between global and local trends modeling?

CircNNM (Liu'22, Liu & Zhang'23)

Estimating  $\mathbf{x}$ :

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

on data  $\mathbf{y}$  w/ observed index set  $\Omega$ .

ConvNNM (Liu'22, Liu & Zhang'23)

Estimating  $\mathbf{x}$ :

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}_{\tilde{\tau}}(\mathbf{x})\|_* \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

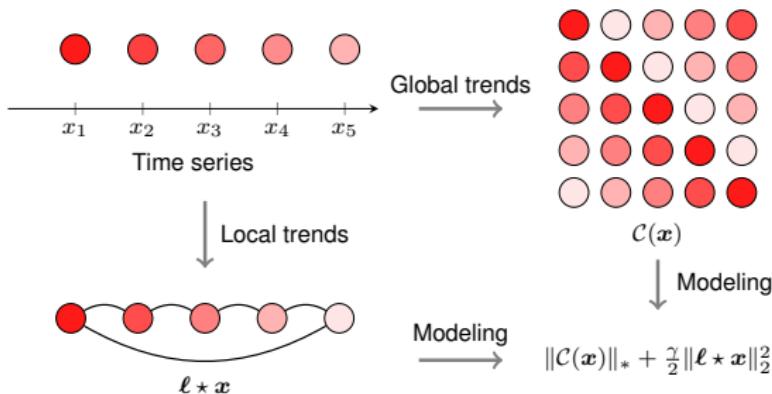
on data  $\mathbf{y}$  w/ observed index set  $\Omega$ .

# Global + Local Trends?

## Laplacian Convolutional Representation (LCR)

For any partially observed time series  $\mathbf{y} \in \mathbb{R}^T$  with observed index set  $\Omega$ , LCR utilizes **circulant matrix** and **Laplacian kernel** to characterize global and local trends in time series, respectively, i.e.,

$$\begin{aligned} \min_{\mathbf{x}} \quad & \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\ell * \mathbf{x}\|_2^2}_{\text{local}} \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$



## Laplacian Convolutional Representation

---

- LCR model:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{x}) \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

- Augmented Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2$$

where  $\mathbf{w} \in \mathbb{R}^T$  is the Lagrange multiplier, and  $\langle \cdot, \cdot \rangle$  denotes the inner product.

# Laplacian Convolutional Representation

- LCR model:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{x}) \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

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where  $\mathbf{w} \in \mathbb{R}^T$  is the Lagrange multiplier, and  $\langle \cdot, \cdot \rangle$  denotes the inner product.

- The ADMM scheme:

$$\left\{ \begin{array}{ll} \mathbf{x} := \arg \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Nuclear norm minimization)} \\ \mathbf{z} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) & \text{(Closed-form solution)} \\ = \frac{1}{\lambda + \eta} \mathcal{P}_\Omega(\lambda \mathbf{x} + \mathbf{w} + \eta \mathbf{y}) + \frac{1}{\lambda} \mathcal{P}_\Omega^\perp(\lambda \mathbf{x} + \mathbf{w}) & \\ \mathbf{w} := \mathbf{w} + \lambda(\mathbf{x} - \mathbf{z}) & \text{(Standard update)} \end{array} \right.$$

- Optimize  $\mathbf{x}$ ?

$$\underbrace{\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1}_{\text{property of circulant matrix}} \quad \& \quad \underbrace{\frac{1}{2} \|\ell * \mathbf{x}\|_2^2 = \frac{1}{2T} \|\mathcal{F}(\ell) \circ \mathcal{F}(\mathbf{x})\|_2^2}_{\text{property of circular convolution}}$$

## Laplacian Convolutional Representation

---

- Optimize  $\mathbf{x}$  via FFT (in  $\mathcal{O}(T \log T)$  time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell \star \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2\end{aligned}$$

where we introduce  $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$  (i.e., FFT).

# Laplacian Convolutional Representation

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where we introduce  $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$  (i.e., FFT).

## $\ell_1$ -norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of  $\ell_1$ -norm minimization in complex space:

$$\min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\delta}{2} \|\hat{\mathbf{x}} - \hat{\mathbf{h}}\|_2^2$$

with complex-valued  $\hat{\mathbf{x}}, \hat{\mathbf{h}} \in \mathbb{C}^T$  and weight parameter  $\delta$ , element-wise, the solution is given by

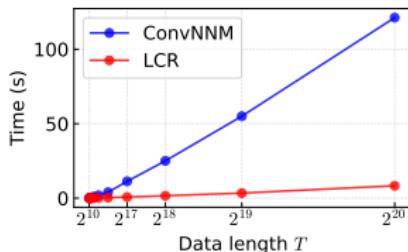
$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|} \cdot \max\{0, |\hat{h}_t| - 1/\delta\}, t = 1, \dots, T.$$

# Laplacian Convolutional Representation

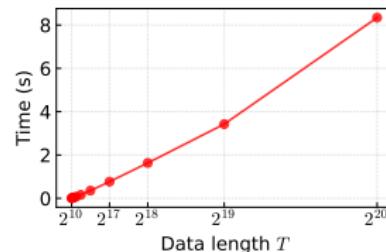
## Empirical time complexity

On the synthetic data  $\mathbf{y} \in \mathbb{R}^T$  with  $T \in \{2^{10}, 2^{11}, \dots, 2^{20}\}$

- Ours: **LCR**
  - An FFT implementation in  $\mathcal{O}(T \log T)$
  - The logarithmic factor  $\log T$  makes the FFT highly efficient
- Baseline: **ConvNNM** (Liu'22, Liu & Zhang'23)
  - Convolution matrix  $\mathcal{C}_{\tilde{\tau}}(\mathbf{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$  with kernel size  $\tilde{\tau} = 2^4$
  - Singular value thresholding in  $\mathcal{O}(\tilde{\tau}^2 T)$



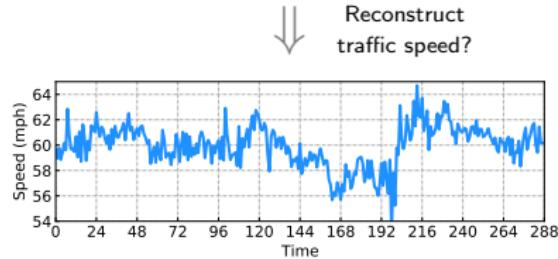
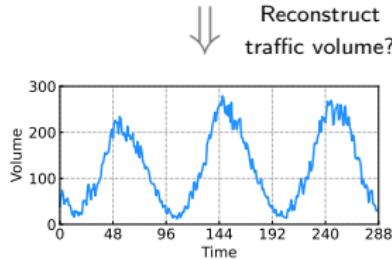
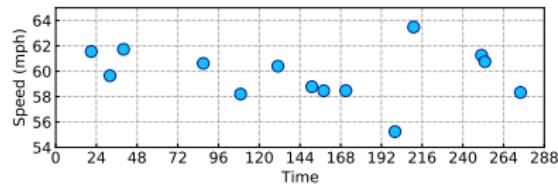
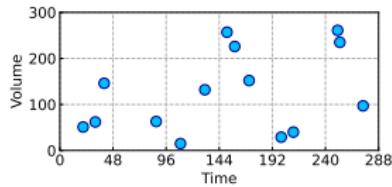
ConvNNM vs. LCR



LCR

# Experiments

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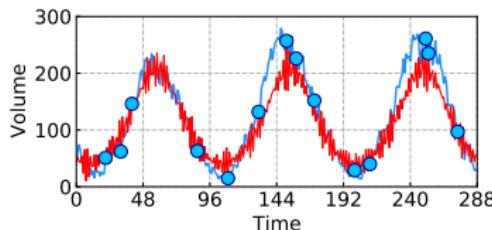


- How to utilize the global trends of traffic time series?
- How to produce local consistency of traffic data?

# Experiments

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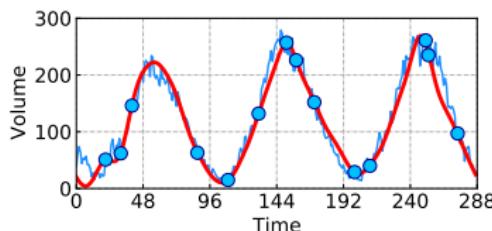
- Substantial performance gains?



**CircNNM:**

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* \\ \text{s. t. } \quad & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

↓ Plus **local** time series trends

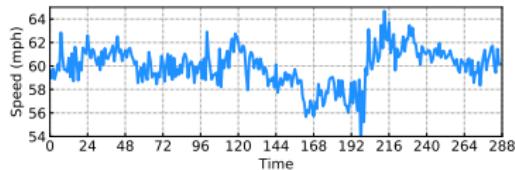


**LCR:**

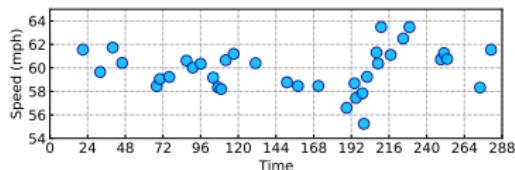
$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2} \|\boldsymbol{\ell} * \boldsymbol{x}\|_2^2 \\ \text{s. t. } \quad & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

# Experiments

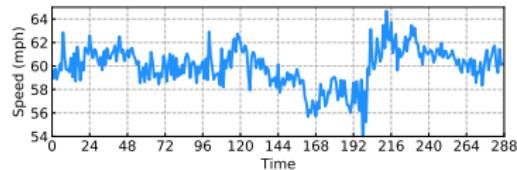
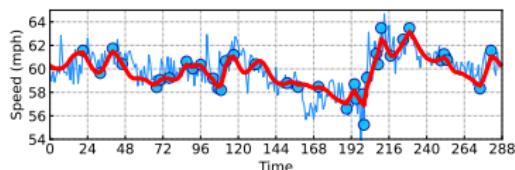
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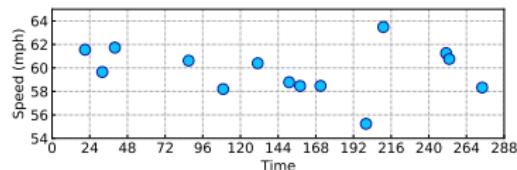
↓ Mask 90% observations



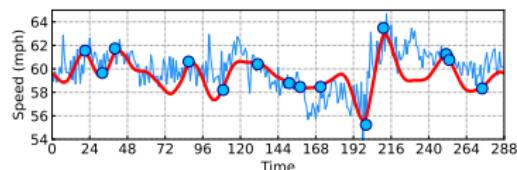
↓ Reconstruct time series



↓ Mask 95% observations



↓ Reconstruct time series



# Experiments

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- The start data points and end data points are connected?

Undirected and circulant graph

Modeling →

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Flipping operation on  $\mathbf{x} \in \mathbb{R}^5$ :

$$\mathbf{x}_{\text{new}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{J}\mathbf{x} \end{bmatrix} = (\underbrace{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5}_{\text{original time series}}, \underbrace{\mathbf{x}_5, \mathbf{x}_4, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1}_{\text{flipped time series}})^{\top} \in \mathbb{R}^{10}$$

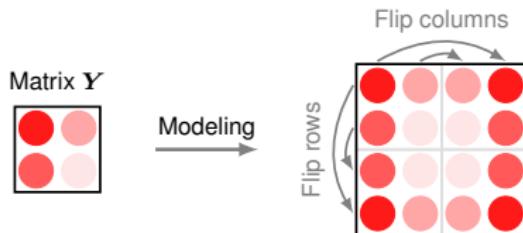
where  $\mathbf{J} \in \mathbb{R}^{5 \times 5}$  is the exchange matrix.

# Experiments

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## Speed field reconstruction<sup>3</sup>

- Flipping operation on a matrix:



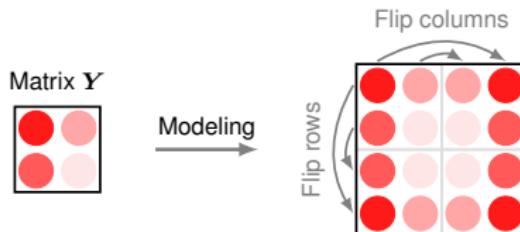
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<sup>3</sup>Highway Drone (HighD) dataset at <https://www.hightd-dataset.com/>

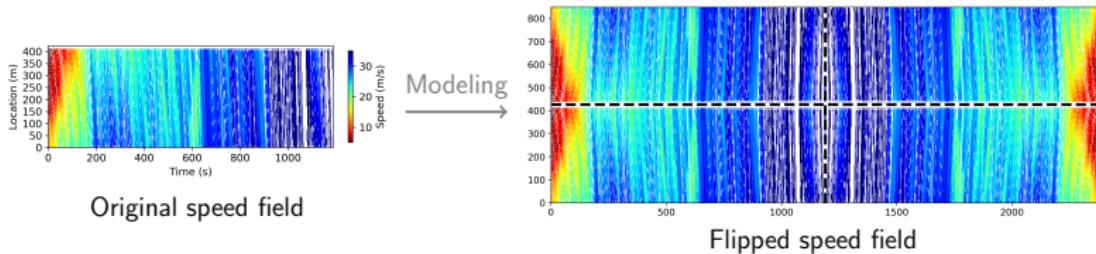
# Experiments

## Speed field reconstruction<sup>3</sup>

- Flipping operation on a matrix:



- Flipping operation on a speed field of vehicular traffic flow:



<sup>3</sup>Highway Drone (HighD) dataset at <https://www.hightd-dataset.com/>

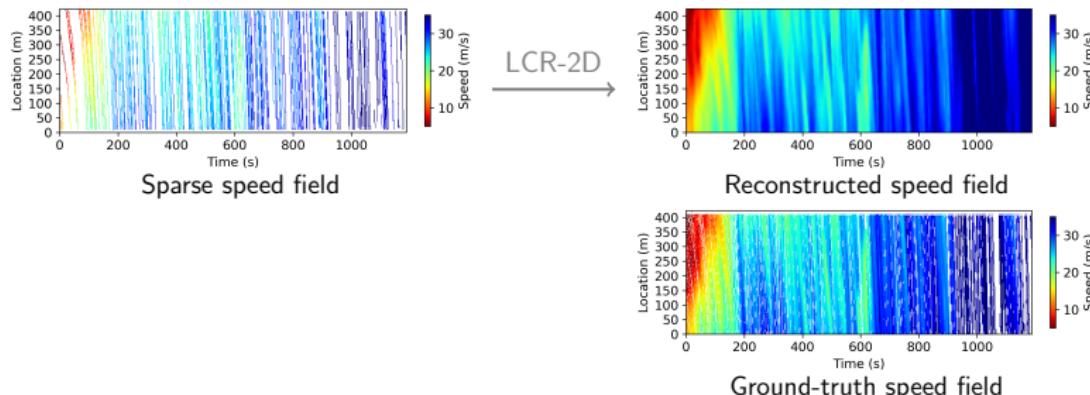
# Experiments

## Speed field reconstruction<sup>4</sup>

- Scenario: Mask trajectories of 70% vehicles
- LCR-2D on partially observed  $\mathbf{Y} \in \mathbb{R}^{N \times T}$ :

$$\min_{\mathbf{X}} \underbrace{\|\mathcal{C}(\mathbf{X})\|_*}_{\text{global trend}} + \frac{\gamma}{2} \underbrace{\|(\ell_s \ell^\top) * \mathbf{X}\|_F^2}_{\text{local trend}}$$

s.t.  $\|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \epsilon$



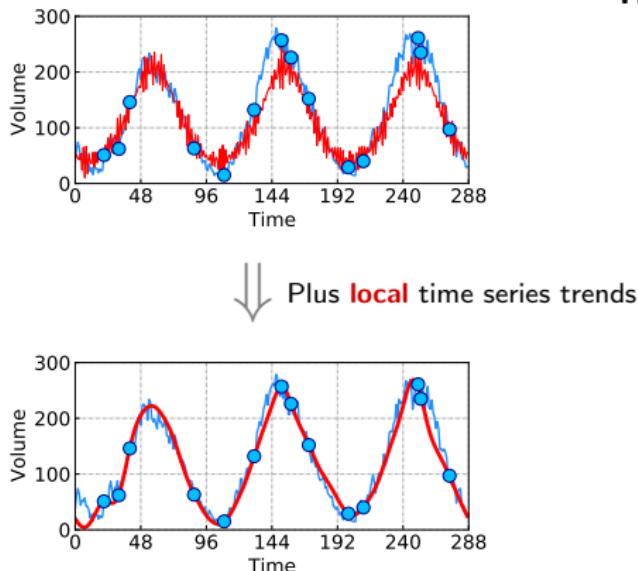
<sup>4</sup>Highway Drone (HighD) dataset at <https://www.hightd-dataset.com/>

# Contributions

Matrix nuclear norm ( $\ \mathbf{X}\ _*$ ) minimization	Singular value thresholding	Truncated nuclear norm ( $\ \mathbf{X}\ _{r,*}, r \in \mathbb{Z}^+$ ) minimization	Tensor nuclear norm ( $\ \mathcal{X}\ _*$ ) minimization
Candès & Recht'09	Cai et al.'10	Zhang et al.'12 Hu et al.'12	Liu et al.'13
←			
Circulant/Convolution nuclear norm ( $\ \mathcal{C}(\mathbf{x})\ _*$ or $\ \mathcal{C}_{\tilde{\tau}}(\mathbf{x})\ _*$ ) minimization	Low-rank Hankel matrix/tensor ( $\mathcal{H}_r(\cdot)$ ) completion	Tensor nuclear norm minimization with linear transform	Generalized nonconvex nonsmooth low-rank minimization
Liu'22 Liu & Zhang'23	Yokota et al.'18 Sedighin et al.'20 Cai et al.'21 Yamamoto et al.'22	Lu et al.'19	Lu et al.'14

(Ours) LCR:

- ✓ Local trend modeling
- ✓ An FFT implementation



## Highlights:

- Rethinking the importance of local trend modeling in traffic data imputation tasks.
- Finding a unified **global and local trend** modeling framework whose optimization can be efficiently solved by **FFT**:

$$\min_{\mathbf{x}} \underbrace{\|\mathcal{C}(\mathbf{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\ell * \mathbf{x}\|_2^2}_{\text{local}}$$

$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$

## Concluding Remarks

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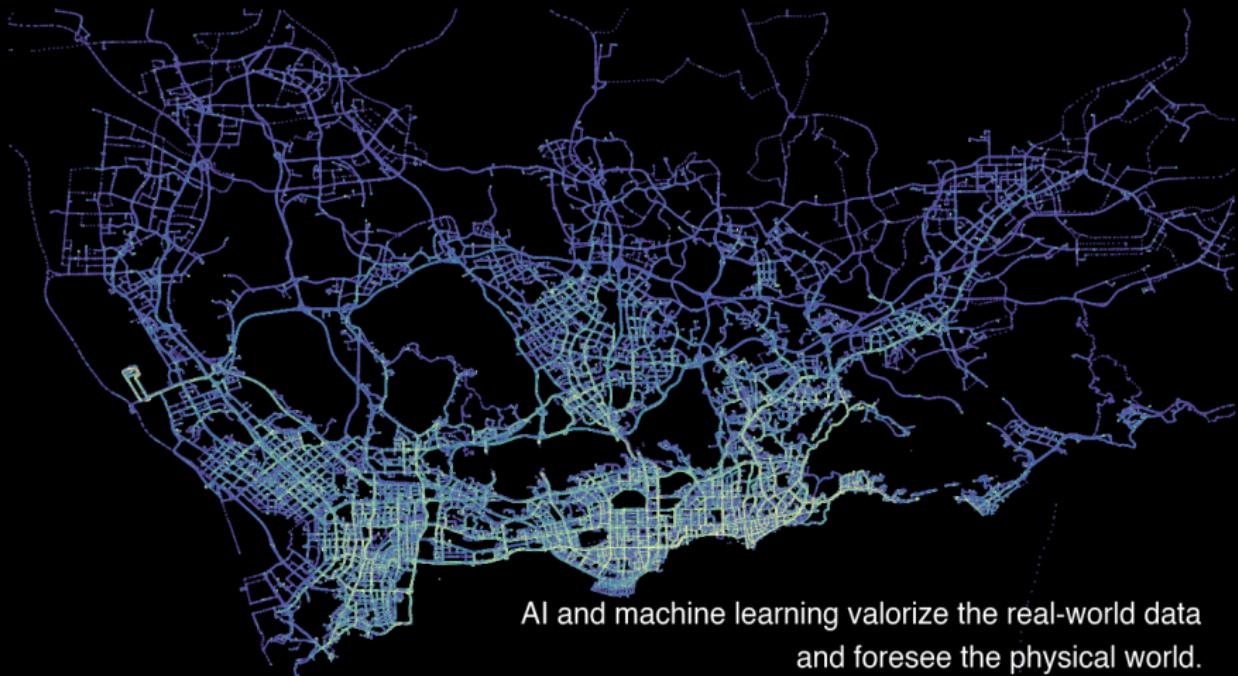
- **(Starting point)** How to impute traffic time series?
  - ✓ Low-rank models   ✓ Temporal regularization
- **(Solution)** Time series trend modeling in the low-rank framework?
  - Global time series trend modeling (low-rank model):
$$\min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_*$$
$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$
  - Local time series trend modeling (temporal regularization):
$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\boldsymbol{\ell} \star \mathbf{x}\|_2^2$$
- **(Highlight)** A unified framework with the **FFT** implementation.

## References

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### A short list:

- [Liu'22] G. Liu (2022). Time series forecasting via learning convolutionally low-rank models. *IEEE Transactions on Information Theory*, 68(5): 3362–3380.
- [Liu & Zhang'23] G. Liu and W. Zhang (2023). Recovery of future data via convolution nuclear norm minimization. *IEEE Transactions on Information Theory*, 69(1): 650–665.



AI and machine learning valorize the real-world data  
and foresee the physical world.

Source: <https://spatiotemporal-data.github.io>



# Thanks for your attention!

Any Questions?

Slides: <https://xinychen.github.io/slides/LCR24.pdf>

## About me:

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