

Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 4: Introduction to Python Programming: Part II

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Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

"Class Participation Quiz 8"

Time slot: **2:30PM – 3:00PM**

on Canvas.

Python Functions

Why use functions?

- **Reusability:** Write once, use many times
- **Modularity:** Break code into manageable blocks
- **Abstraction:** Hide complexity behind simple interfaces
- **Testing & Debugging:** Isolate and test individual components

Basic Function Syntax

```
1 def function_name(parameters):  
2     """Optional docstring"""  
3     # Function body  
4     return value # Optional
```

Basic Function Syntax

Engineering example.

- Definition of normal stress:

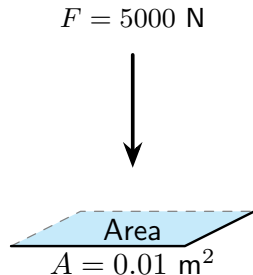
$$\sigma = \frac{F}{A}$$

```
1 def normal_stress(F, A):  
2     return F / A
```

where

- $F = 5000 \text{ N}$ (force)
- $A = 0.01 \text{ m}^2$ (area)

```
1 force = 5000    # N  
2 area = 0.01     # m^2  
3 stress = normal_stress(force, area)  
4 print('stress = {}'.format(stress))
```



Lambda Functions

Quick, one-line functions:

- Example: Quadratic function

$$y = x^2$$

```
1 # Syntax: lambda arguments: expression
2 square = lambda x: x**2
3 print(square(5))      # 25
4
5 # Equivalent def function:
6 def square_func(x):
7     return x**2
8 print(square_func(5)) # 25
```

Lambda Functions

Engineering example.

- Definition of normal stress:

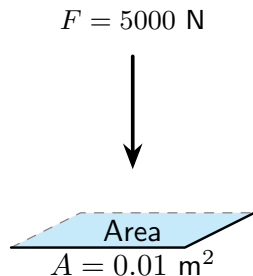
$$\sigma = \frac{F}{A}$$

```
1 stress_lam = lambda F, A: F / A
```

where

- $F = 5000$ N (force)
- $A = 0.01$ m² (area)

```
1 force = 5000 # N
2 area = 0.01 # m^2
3 stress = stress_lam(force, area)
4 print('stress = {}'.format(stress))
```



Lambda Functions

- Example:

$$g(r) = \frac{\pi r^2}{4}$$

```
1 import numpy as np
2
3 g = lambda r: np.pi * r**2 / 4
```

- Evaluate it for $r = 1.5$ and $r = 2.78$

```
1 print(g(1.5))
2 print(g(2.78))
```


Multiple Returns

- Given $ax^2 + bx + c = 0$ ($a \neq 0$), the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

```
1 import numpy as np
2
3 def quad_formula(a, b, c):
4     term = np.sqrt(b**2 - 4*a*c)
5     x1 = (-b + term) / (2*a)
6     x2 = (-b - term) / (2*a)
7     return x1, x2
```

- Case study: Solve $9x^2 + 3x - 2 = (3x - 1)(3x + 2) = 0$.

```
1 a, b, c = 9, 3, -2
2 x1, x2 = quad_formula(a, b, c)
3 print(x1)
4 print(x2)
```

Recursive Functions

Functions that call themselves

- **Factorial of a non-negative integer** n is the product of all positive integers less than or equal to n :

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1$$
$$= \begin{cases} 1 & \text{if } n = 1 \\ n \times \underbrace{(n - 1)!}_{\text{factorial}} & \text{if } n > 1 \end{cases}$$

```
1 def factorial(n):  
2     f = 1  
3     for i in range(1, n + 1):  
4         f = f * i  
5     return f
```

- Toy example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

```
1 print(factorial(5))
```

Recursive Functions

Functions that call themselves

- **Factorial of a non-negative integer** n is the product of all positive integers less than or equal to n :

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$
$$= \begin{cases} 1 & \text{if } n = 1 \\ n \times \underbrace{(n-1)!}_{\text{factorial}} & \text{if } n > 1 \end{cases}$$

```
1 def factorial_r(n):  
2     if n == 0:  
3         return 1  
4     else:  
5         return n * factorial_r(n-1)
```

- Toy example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

```
1 print(factorial_r(5))
```

Factorial with NumPy

- **Factorial of a non-negative integer** n is the product of all positive integers less than or equal to n :

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1$$

```
1 def factorial_numpy(n):  
2     if n == 0:  
3         return 1  
4     else:  
5         return np.prod(np.arange(1, n+1))
```

- Toy example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

```
1 print(factorial_numpy(5))  
2 print(np.prod(np.arange(1, 6)))
```

Factorial with NumPy

- **Factorial of a non-negative integer** n is the product of all positive integers less than or equal to n :

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 3 \times 2 \times 1$$

```
1 def factorial_numpy(n):  
2     if n == 0:  
3         return 1  
4     else:  
5         return np.prod(np.arange(1, n+1))
```

- Toy example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

```
1 print(factorial_numpy(5))  
2 print(np.prod(np.arange(1, 6)))
```

- Any other built-in function?

```
1 import math  
2  
3 print(math.factorial(5))
```

Approximation for Sine Function

Taylor series expansion for $\sin(x)$:

- Formula

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \dots$$

- Denominator is factorial of odd numbers
- More terms = better approximation

Approximation for Sine Function

Taylor series expansion for $\sin(x)$:

- Formula

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \dots$$

- Denominator is factorial of odd numbers
- More terms = better approximation
- Python programming:

$$\begin{aligned} \sin(x) &= \underbrace{\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}}_{n \text{ starts from } 1} \\ &= \underbrace{\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}_{n \text{ starts from } 0 \text{ (Python!)}} \end{aligned}$$

Approximation for Sine Function

- Python programming:

$$\sin(x) = \underbrace{\sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}_{n \text{ starts from } 0 \text{ (Python!)}$$

```
1 import numpy as np
2
3 def sin_taylor(x, num_term):
4     result = 0
5     for n in range(num_term):
6         # Term index: 0, 1, 2, ... corresponds to x^1,
6             x^3, x^5, ...
7         exp = 2*n + 1
8         factorial = np.prod(np.arange(1, exp + 1))
9         result += ((-1) ** n) * (x ** exp) / factorial
10    return result
```


Approximation for Sine Function

Test case: $\sin(0.9)$

- Ground-truth value:

```
1 print(np.sin(0.9))           # 0.7833269096274834
```

- 1 term:

```
1 print(sin_taylor(0.9, 1))    # 0.9
```

Approximation for Sine Function

Test case: $\sin(0.9)$

- Ground-truth value:

```
1 print(np.sin(0.9))           # 0.7833269096274834
```

- 1 term:

```
1 print(sin_taylor(0.9, 1))    # 0.9
```

- 2 terms:

```
1 print(sin_taylor(0.9, 2))    # 0.7785
```

- 3 terms:

```
1 print(sin_taylor(0.9, 3))    # 0.78342075
```

- 4 terms:

```
1 print(sin_taylor(0.9, 4))    # 0.7833258498214286
```

- 5 terms:

```
1 print(sin_taylor(0.9, 5))    # 0.7833269174484375
```

Quick Summary

Monday's Class:

- Basic function syntax
- Lambda function
- Multiple returns
- Recursive functions
- Two examples: Factorial and Taylor series expansion for $\sin(x)$

Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

“Class Participation Quiz 9”

Time slot: **2:30PM – 3:00PM**

on Canvas.

Norms

What are “**norms**” in mathematics?

- Mathematical rulers for measuring vector and matrix properties
- Distance measures in multi-dimensional space
- Essential tools for **error analysis**, **optimization**, and **stability**

Why civil engineers needs “**norms**”?

- Error quantification in numerical solutions
- Convergence checking in iterative methods
- Optimization criteria (least squares)
- Stability analysis of structures

Norms

Some important norms:

- ℓ_1 -norm
- ℓ_2 -norm (vector) vs. Frobenius norm (matrix)
- ℓ_∞ -norm

ℓ_1 -Norm

The ℓ_1 -norm measures the total absolute value.

- Mathematical expression:

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$$

for any vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$$

ℓ_1 -Norm

The ℓ_1 -norm measures the total absolute value.

- Mathematical expression:

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$$

for any vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$$

- Example:

$$\mathbf{a} = (1, 2, 3, 4)^\top \Rightarrow \|\mathbf{a}\|_1 = 10$$

```
1 import numpy as np
2
3 ell_1 = lambda x: np.sum(np.abs(x))
4 a = np.arange(1, 5)
5 print(a)
6 print(ell_1(a))
```

- How to use NumPy?

```
1 print(np.linalg.norm(a, 1))
```


ℓ_1 -Norm

The ℓ_1 -norm is also called Manhattan norm.

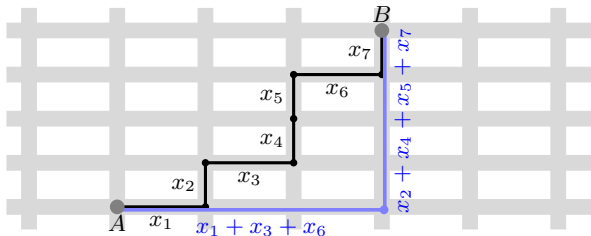
- Mathematical expression:

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$$

for any vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$$

- “Walking along city blocks” - only horizontal/vertical moves



ℓ_1 -Norm

- Physical meaning in engineering:
 - Total absolute error across all measurements
 - Resource consumption (total material used)
 - Cost summation across multiple components
- Error analysis: **Mean Absolute Error (MAE)** such that

$$\text{MAE} = \frac{1}{n} \|\boldsymbol{\varepsilon}\|_1 = \frac{1}{n} \sum_{i=1}^n |\varepsilon_i| = \frac{1}{n} \sum_{i=1}^n |\hat{x}_i - x_i|$$

with the errors:

$$\varepsilon_i = \underbrace{\hat{x}_i}_{\text{approximate}} - \underbrace{x_i}_{\text{true}} \quad i = 1, 2, \dots, n$$

- It represents the “average” absolute deviation in the same units as the data

ℓ_1 -Norm

Example: Deflection

- Step-by-step computations:

$$\text{MAE} = \frac{|0.2| + |-0.4| + |0.3| + |-0.2| + |0.3|}{5} \approx 0.28$$

```
1 import numpy as np
2
3 # True vs measured deflections (mm)
4 true = np.array([12.3, 15.7, 18.2, 14.9, 16.5])
5 measured = np.array([12.5, 15.3, 18.5, 14.7, 16.8])
6
7 # Absolute errors at each point
8 abs_errors = np.abs(measured - true)
9
10 # L1 norm of error = total absolute error
11 total_abs_error = np.sum(abs_errors)
```

- Using NumPy

```
1 np.linalg.norm(measured - true, 1)
```

ℓ_2 -Norm

- Mathematical expression:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

for any vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$$

ℓ_2 -Norm

- Mathematical expression:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

for any vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$$

- Example:

$$\mathbf{a} = (1, 2, 3, 4)^\top \Rightarrow \|\mathbf{a}\|_2 = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$

```
1 import numpy as np
2
3 ell_2 = lambda x: np.sqrt(np.sum(x ** 2))
4 a = np.arange(1, 5)
5 print(a)
6 print(ell_2(a))
```

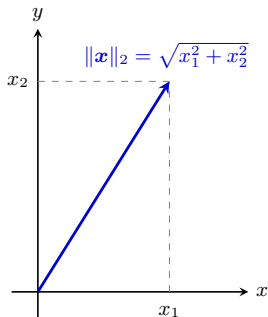
- How to use NumPy?

```
1 print(np.linalg.norm(a, 2))
```

ℓ_2 -Norm

Intuitive understanding?

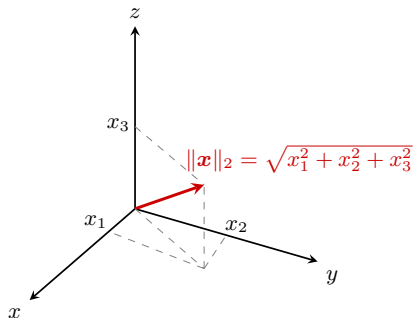
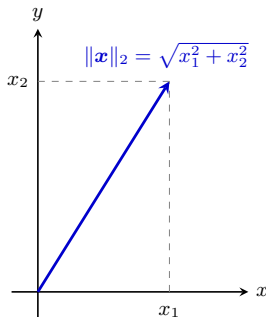
- ℓ_2 -norm is the Euclidean distance in space.
- Vectors $\mathbf{x} = (x_1, x_2)^\top$ vs. $\mathbf{x} = (x_1, x_2, x_3)^\top$



ℓ_2 -Norm

Intuitive understanding?

- ℓ_2 -norm is the Euclidean distance in space.
- Vectors $\mathbf{x} = (x_1, x_2)^\top$ vs. $\mathbf{x} = (x_1, x_2, x_3)^\top$



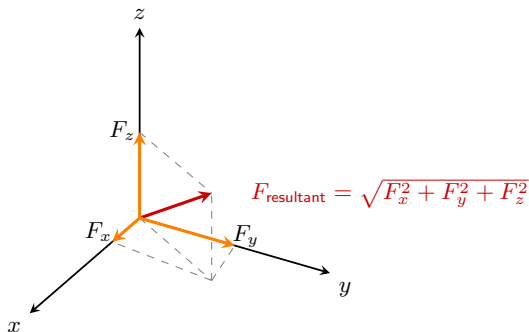
ℓ_2 -Norm

- If forces F_x, F_y, F_z act on a joint, resultant force magnitude:

$$F_{\text{resultant}} = \sqrt{F_x^2 + F_y^2 + F_z^2}$$

- Example: $F_x = 3, F_y = 4, F_z = 12$ kN, then

$$F_{\text{resultant}} = \sqrt{3^2 + 4^2 + 12^2} = 13 \text{ kN}$$

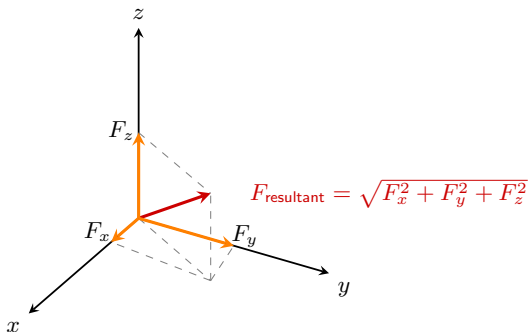


ℓ_2 -Norm

- Example: $F_x = 3.5, F_y = 2.1, F_z = 4.8$ kN, then

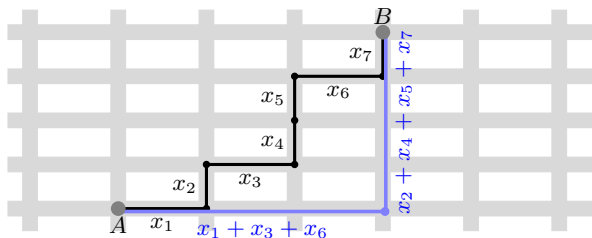
$$F_{\text{resultant}} = \sqrt{3.5^2 + 2.1^2 + 4.8^2} \approx 6.30 \text{ kN}$$

```
1 import numpy as np
2
3 F = np.array([3.5, 2.1, 4.8])
4 print(np.linalg.norm(F, 2))
```



ℓ_1 -Norm vs. ℓ_2 -Norm

In a city grid, walking from $(0, 0)$ to $(3, 4)$:



- ℓ_1 distance = $|3| + |4| = 7$ blocks
- ℓ_2 distance = $\sqrt{3^2 + 4^2} = 5$ blocks (not walkable!)

Frobenius Norm

- ℓ_2 -norm:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

for any **vector**

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$$

- **Frobenius norm:**

$$\|\mathbf{X}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n x_{ij}^2}$$

for any **matrix**

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \quad m \text{ rows \& } n \text{ columns}$$

Frobenius Norm

- Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \|\mathbf{A}\|_F = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$

```
1 import numpy as np
2
3 frob = lambda X: np.sqrt(np.sum(X ** 2))
4 A = np.array([[1, 2], [3, 4]])
5 print(frob(A))
```

- How to use NumPy?

```
1 print(np.linalg.norm(A, 'f'))
```

ℓ_∞ -Norm

- Mathematical expression of ℓ_∞ -norm (“Worst-case” or “maximum” distance):

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

- Example:

```
1 import numpy as np
2
3 a = np.array([-1, 2, -3, 4])
4 print(np.linalg.norm(a, np.inf))
```

- Write a function?

```
1 ell_inf = lambda x: np.max(np.abs(x))
2 print(ell_inf(a))
```

- Physical meaning in engineering:
 - Maximum stress in a structure
 - Peak deflection in a beam
 - Worst-case error in measurements
 - Safety factor based on extreme values

l_∞ -Norm

Example: Worst-case prediction error

- Focuses only on the worst-case element - conservative design
- Python codes

```
1 # Errors in temperature predictions at different
   locations
2 errors = np.array([-1.2, 0.8, -2.1, 1.5, -0.3, 1.9])
3
4 # L_infinity norm = maximum absolute error
5 max_abs_error = np.max(np.abs(errors))
6 worst_location = np.argmax(np.abs(errors))
```

- Maximum absolute error: -2.1
- Location: the 3rd value

ℓ_∞ -Norm

Example: Safety factor based on extreme values

- In design codes, the **maximum stress** must not exceed allowable stress:

$$\sigma_{\max} = \max\{|\sigma_1|, |\sigma_2|, \dots\} = \|\boldsymbol{\sigma}\|_\infty$$

- If measured stresses = $[120, -150, 130]$ MPa,

$$\|\boldsymbol{\sigma}\|_\infty = 150 \text{ MPa}$$

Compare to allowable stress (e.g., 200 MPa) for safety.

Quick Summary

Wednesday's Class:

- ℓ_1 -norm: Sum of absolute values \rightarrow total deviation
- ℓ_2 -norm: magnitude in space
- ℓ_∞ -norm: Maximum absolute value \rightarrow worst-case measure
- Frobenius Norm: For matrices, like ℓ_2 -norm for vectors

Assignment 2 & Exam 1 (Coding Part)

Assignment 2:

- Students:
 - Complete your coding tasks on Colab
 - Comment the question number (e.g., # Question 2.2)
 - Download “.ipynb” from Colab
- TA's task:
 - Upload your “.ipynb” to Colab
 - Run all codes
 - Grade your results

Exam 1 (coding part):

- Format: Give you Python codes, please write down the output
- Review class: Give **10-15 sample questions** (some of them will be selected for test)

Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

“Class Participation Quiz 10”

Time slot: **2:30PM – 3:00PM**

on Canvas.

- Online engagement (graded quizzes)

“Quiz 10”

Deadline: **11:59PM, February 6, 2026**

on Canvas.

Inner Product

- Mathematical expression:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^n x_i y_i$$

for any vectors

$$\begin{cases} \mathbf{x} = (x_1, x_2, \dots, x_n)^\top \\ \mathbf{y} = (y_1, y_2, \dots, y_n)^\top \end{cases}$$

- Example:

$$\begin{aligned} \mathbf{x} &= (1, 2, 3, 4)^\top & \mathbf{y} &= (6, 7, 8, 9)^\top \\ \Rightarrow \langle \mathbf{x}, \mathbf{y} \rangle &= 1 \times 6 + 2 \times 7 + 3 \times 8 + 4 \times 9 = 80 \end{aligned}$$

```
1 import numpy as np
2
3 inner = lambda x, y: np.sum(x * y)
4 x = np.arange(1, 5)
5 y = np.arange(6, 10)
6 print(inner(x, y))
```

Inner Product

- Mathematical expression:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^n x_i y_i$$

for any vectors

$$\begin{cases} \mathbf{x} = (x_1, x_2, \dots, x_n)^\top \\ \mathbf{y} = (y_1, y_2, \dots, y_n)^\top \end{cases}$$

- Example:

$$\begin{aligned} \mathbf{x} &= (1, 2, 3, 4)^\top & \mathbf{y} &= (6, 7, 8, 9)^\top \\ \Rightarrow \langle \mathbf{x}, \mathbf{y} \rangle &= 1 \times 6 + 2 \times 7 + 3 \times 8 + 4 \times 9 = 80 \end{aligned}$$

- How to use NumPy?

```
1 print(np.inner(x, y))
```

- Other options?

```
1 print(x @ y)
```

Inner Product

- For any vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$$

what is the inner product $\langle \mathbf{x}, \mathbf{x} \rangle$?

```
1 import numpy as np
2
3 x = np.arange(1, 10)
4 print(np.inner(x, x))
```

Inner Product

- For any vector

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$$

what is the inner product $\langle \mathbf{x}, \mathbf{x} \rangle$?

```
1 import numpy as np
2
3 x = np.arange(1, 10)
4 print(np.inner(x, x))
```

- Recall that

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^n x_i \cdot x_i = \|\mathbf{x}\|_2^2$$

```
1 print(np.linalg.norm(x, 2) ** 2)
```

Outer Product

- Mathematical expression:

$$\mathbf{xy}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_my_1 & x_my_2 & \cdots & x_my_n \end{bmatrix}$$

- Example (column vector \times row vector = matrix):

$$\mathbf{x} = (1, 2, 3)^\top \quad \mathbf{y} = (4, 5)^\top \quad \Rightarrow \quad \mathbf{xy}^\top = \begin{bmatrix} 1 \times 4 & 1 \times 5 \\ 2 \times 4 & 2 \times 5 \\ 3 \times 4 & 3 \times 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix}$$

```
1 import numpy as np
2
3 x = np.arange(1, 4)
4 y = np.arange(4, 6)
5 print(np.outer(x, y))
```

Kronecker Product \otimes

- Mathematical expression:

$$\mathbf{X} \otimes \mathbf{Y} = \begin{bmatrix} x_{11}\mathbf{Y} & x_{12}\mathbf{Y} & \cdots & x_{1n}\mathbf{Y} \\ x_{21}\mathbf{Y} & x_{22}\mathbf{Y} & \cdots & x_{2n}\mathbf{Y} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1}\mathbf{Y} & x_{m2}\mathbf{Y} & \cdots & x_{mn}\mathbf{Y} \end{bmatrix}$$

- Example:

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

```
1 import numpy as np
2
3 X = np.array([[1, 2], [3, 4]])
4 Y = np.array([[5, 6, 7], [8, 9, 10]])
5 print(np.kron(X, Y))
```


Kronecker Product \otimes

- Verify that the Kronecker product of

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \mathbf{Y} = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$

is

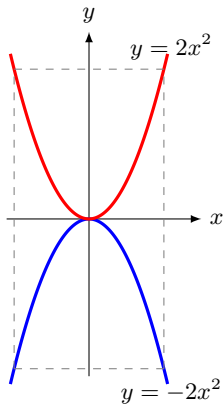
$$\begin{aligned} \mathbf{X} \otimes \mathbf{Y} &= \begin{bmatrix} 1 \times \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} & 2 \times \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \\ 3 \times \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} & 4 \times \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 5 & 6 & 7 & 10 & 12 & 14 \\ 8 & 9 & 10 & 16 & 18 & 20 \\ 15 & 18 & 21 & 20 & 24 & 28 \\ 24 & 27 & 30 & 32 & 36 & 40 \end{bmatrix} \end{aligned}$$

- Size: 4 rows & 6 columns

Positive Definite Matrix

Revisit quadratic functions $y = ax^2$:

If $a > 0$, then it always holds that $ax^2 > 0$ for any $x \neq 0$.



Positive Definite Matrix

Extension from $y = ax^2$ to $y = \mathbf{x}^\top \mathbf{A} \mathbf{x}$:

If \mathbf{A} is a **positive definite matrix**, then it always holds that $\mathbf{x}^\top \mathbf{A} \mathbf{x} > 0$ for any $\mathbf{x} \neq \mathbf{0}$.

Positive Definite Matrix

Extension from $y = ax^2$ to $y = \mathbf{x}^\top \mathbf{A} \mathbf{x}$:

If \mathbf{A} is a **positive definite matrix**, then it always holds that $\mathbf{x}^\top \mathbf{A} \mathbf{x} > 0$ for any $\mathbf{x} \neq \mathbf{0}$.

- **Example:** Is $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ a positive definite matrix?
- **Solution:** For any nonzero vector $\mathbf{x} = (x_1, x_2)^\top$, we have
 - matrix-vector multiplication:

$$\mathbf{A} \mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- inner product:

$$\mathbf{x}^\top (\mathbf{A} \mathbf{x}) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 > 0$$

So \mathbf{A} is a positive definite matrix.

Positive Definite Matrix

- **Example:** Is $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ a positive definite matrix?
- **Solution:** For any nonzero vector $\mathbf{x} = (x_1, x_2, x_3)^\top$, we have
 - matrix-vector multiplication:

$$\mathbf{Ax} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ -x_2 + 2x_3 \end{bmatrix}$$

Positive Definite Matrix

- **Example:** Is $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ a positive definite matrix?
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- inner product:

$$\begin{aligned} \mathbf{x}^\top (\mathbf{Ax}) &= x_1(2x_1 - x_2) + x_2(-x_1 + 2x_2 - x_3) + x_3(-x_2 + 2x_3) \\ &= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2 \\ &= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0 \end{aligned}$$

So A is a positive definite matrix.

Angle between Two Vectors

Building connection between inner product and vector's ℓ_2 -norm:

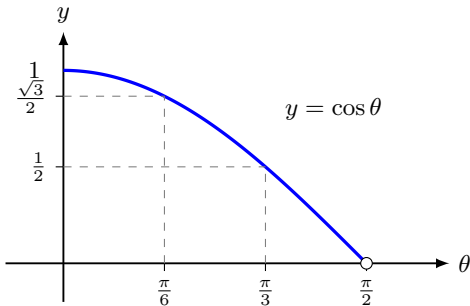
- Mathematical expression:

$$\cos(\theta) = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|_2 \cdot \|\mathbf{b}\|_2}$$

for any vectors

$$\mathbf{a} = (a_1, a_2, \dots, a_n)^\top \quad \mathbf{b} = (b_1, b_2, \dots, b_n)^\top$$

- Cosine function:



Angle between Two Vectors

- Mathematical expression:

$$\cos(\theta) = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|_2 \cdot \|\mathbf{b}\|_2}$$

- Proof** (optional)

- From geometry:

$$\|\mathbf{a} - \mathbf{b}\|_2^2 = \|\mathbf{a}\|_2^2 + \|\mathbf{b}\|_2^2 - 2\|\mathbf{a}\|_2 \cdot \|\mathbf{b}\|_2 \cdot \cos(\theta)$$

- From algebra:

$$\|\mathbf{a} - \mathbf{b}\|_2^2 = \|\mathbf{a}\|_2^2 + \|\mathbf{b}\|_2^2 - 2\mathbf{a}^\top \mathbf{b}$$

- Solve for $\cos(\theta)$

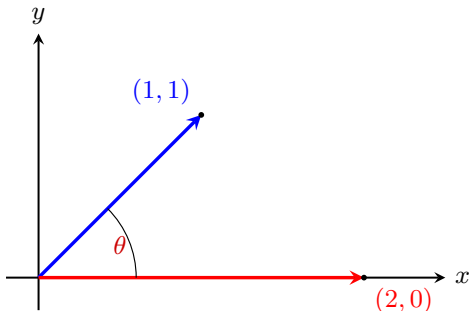
Angle between Two Vectors

- Mathematical expression:

$$\cos(\theta) = \frac{\mathbf{a}^\top \mathbf{b}}{\|\mathbf{a}\|_2 \cdot \|\mathbf{b}\|_2}$$

- Example: Given $\mathbf{a} = (1, 1)^\top$ and $\mathbf{b} = (2, 0)^\top$, we have

$$\cos(\theta) = \frac{1 \times 2 + 1 \times 0}{\sqrt{1^2 + 1^2} \times \sqrt{2^2 + 0^2}} = \frac{2}{\sqrt{2} \times 2} = \frac{1}{\sqrt{2}}$$



Angle between Two Vectors

- **Example:** For vector $x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and matrix $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, compute the angle between x and Ax .
- Inner product:

$$Ax = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

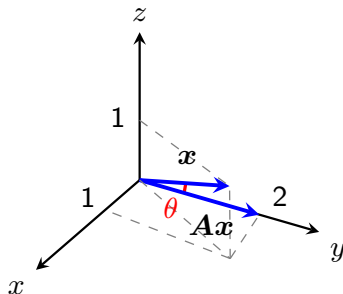
- Angle:

$$\cos(\theta) = \frac{x^\top (Ax)}{\|x\|_2 \cdot \|Ax\|_2} = \frac{1 \times 0 + 2 \times 2 + 1 \times 0}{\sqrt{1^2 + 2^2 + 1^2} \times \sqrt{0^2 + 2^2 + 0^2}} = \frac{\sqrt{6}}{3}$$

```
1 import numpy as np
2
3 x = np.array([1, 2, 1])
4 A = np.array([[2, -1, 0], [-1, 2, -1], [0, -1, 2]])
5 theta = np.arccos(x @ A @ x / (np.linalg.norm(x, 2) * np.
    linalg.norm(A @ x, 2)))
```

Angle between Two Vectors

- **Example:** For vector $x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and matrix $Ax = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, compute the angle between x and Ax .



Visualization with Python

Using Matlab in Python?

- Matplotlib is a comprehensive library for creating static, animated, and interactive visualizations in Python.
- import convention

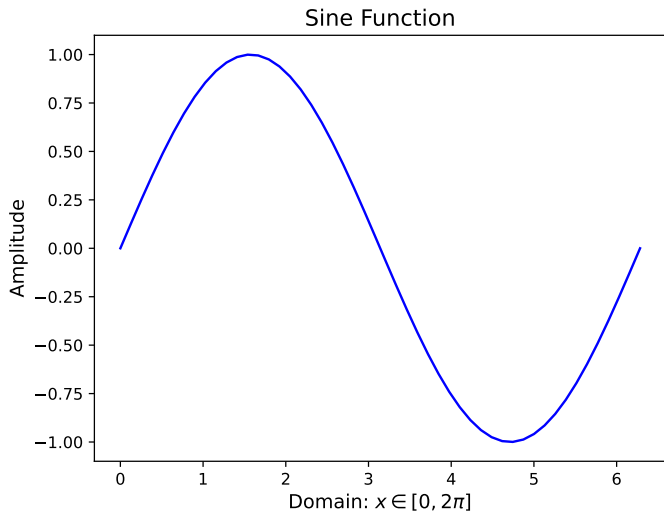
```
1 import matplotlib.pyplot as plt
```

Example: Sine Function

- Python code

```
1 import matplotlib.pyplot as plt
2 import numpy as np
3
4 # Step 1: Generate Data
5 x = np.linspace(0, 2 * np.pi, 50)
6 y = np.sin(x)
7
8 # Step 2: Plot
9 plt.plot(x, y, color = 'blue', linestyle = '-')
10
11 # Step 3: Add Labels and Title
12 plt.title('Sine Function', fontsize = 14)
13 plt.xlabel(r'Domain:  $x \in [0, 2\pi]$ ', fontsize = 12)
14 plt.ylabel('Amplitude', fontsize = 12)
15
16 # Step 4: Save & Show
17 plt.savefig('sin_func.pdf')
18 plt.show()
```

Example: Sine Function



Visualization with Python

Recommended material: <https://matplotlib.org>

Quick Summary

Friday's Class:

- Inner product e.g., $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y}$
- Outer product e.g., $\mathbf{x} \mathbf{y}^\top$
- Kronecker product \otimes
- Positive definite matrix
- Angle between two vectors
- Plot figures in Python