Applied Numerical Methods for Civil Engineering

CGN 3405 - 0002

Week 5: Modeling and Errors

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Quizzes Now!

- **Today's participation** (ungraded survey): Please check out

  "`Class Participation Quiz 11`"

  Time slot: **2:30PM – 3:00PM**

  on Canvas.

- Online engagement (graded quizzes)

  "`Quiz 11`"

  Deadline: **11:59PM, February 9, 2026**

  on Canvas.

Truncation Errors & Taylor Series

Learning objectives:

- Define truncation errors in numerical methods
- Understand the role of Taylor series in function approximation
- Derive finite difference approximations using Taylor expansions
- Apply Taylor series to estimate derivatives numerically

Truncation Errors

What is a truncation error?

- Error from using an approximation instead of an exact mathematical procedure
- Example: Approximating a derivative

$$\frac{\mathrm{d}\, v}{\mathrm{d}\, t} \approx \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

$$\text{Intuition:} \qquad \frac{\text{numerator} \rightarrow \text{difference in velocity } \Delta v}{\text{denominator} \rightarrow \text{difference in time } \Delta t}$$

The difference equation is an approximation $\rightarrow$ introduces truncation error

Truncation Errors

Why study truncation errors?

- Understand accuracy of numerical methods
- Choose appropriate approximations for given problems
- Estimate error bounds for computations
- Improve algorithms by reducing error terms

Taylor Series

Taylor theorem:

- Any smooth function can be approximated by a polynomial
- If f and its first $n+1$ derivatives are continuous on $[a, x]$, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

Recall that

- $f'(a)$: first-order derivative
- $f''(a)$: second-order derivative
- $\cdots$
- $f^{(n)}(a)$: nth-order derivative

with

- $n!$: factorial of integer n [Python: `np.prod(np.arange(1, n+1))`]

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$$

Taylor Series

Intuition: We predict $f(x_{i+1})$ using information at x_i:

- **Zeroth-order** (constant) approximation:

$$f(x_{i+1}) \approx f(x_i)$$

  Only works if f is **constant** between x_i and x_{i+1}

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- **First-order** Taylor approximation:
  Add slope information:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)\underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

  - Represents a **straight line** (linear approximation)
  - Exact if f is linear

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  - Exact if f is linear

- **Euler's formula**:

$$\underbrace{y_{i+1}}_{\text{next value}} = \underbrace{y_i}_{\text{current value}} + \underbrace{\Delta x}_{\text{step size}} \cdot \underbrace{f(x_i, y_i)}_{\text{slope}} \qquad x_{i+1} = x_i + \underbrace{\Delta x}_{\text{step size}}$$

Taylor Series

Intuition: We predict $f(x_{i+1})$ using information at x_i:

- **Second-order** Taylor approximation:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

  - Captures quadratic behavior
  - Better accuracy for smooth functions

Taylor Series

- **General Taylor polynomial**

$$
\begin{aligned}
f(x_{i+1}) \approx f(x_i) & \\
& + f'(x_i)(x_{i+1} - x_i) \\
& + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 \\
& + \cdots \\
& + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n \\
= & \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!}(x_{i+1} - x_i)^k
\end{aligned}
$$

Higher $n \to$ better approximation (if function is smooth)

Taylor Series Approximation of a Polynomial

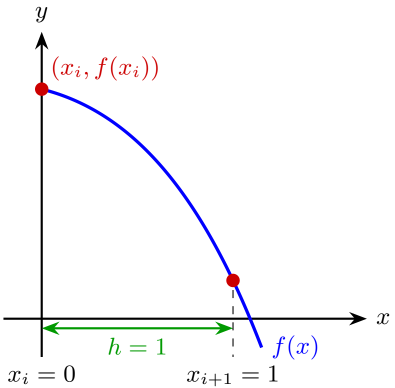
Problem statement:

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

  at $x_{i+1} = 1$ starting from $x_i = 0$ with step size $h = x_{i+1} - x_i = 1$.

- **Goal**: Predict $f(1)$ using Taylor approximations of increasing order.



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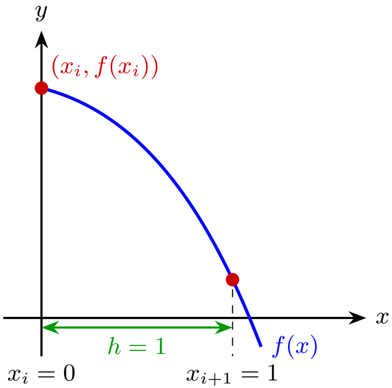
- **Goal**: Predict $f(1)$ using Taylor approximations of increasing order.

  - Function $f(x)$:

$$f(0) = 1.2$$

$$\begin{aligned} f(1) = &-0.1 - 0.15 - 0.5 \\ &-0.25 + 1.2 = 0.2 \end{aligned}$$

  - True value to predict: $f(1) = 0.2$



Taylor Series Approximation of a Polynomial

First-order approximation for $f(1)$:

- Need first-order derivative at $x_i = 0$:

$$\begin{aligned} & f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow \quad & f'(x) = -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow \quad & f'(0) = -0.25 \end{aligned}$$

- First-order Taylor series:

$$\begin{aligned} & f(x_{i+1}) \approx f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i) \\ \Rightarrow \quad & f(1) \approx 1.2 + (-0.25) \times 1 = 0.95 \end{aligned}$$

Taylor Series Approximation of a Polynomial

Second-order approximation for $f(1)$:

- Need second-order derivative at $x_i = 0$:

$$\begin{aligned} f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow \quad f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow \quad f''(x) &= -1.2x^2 - 0.9x - 1 \\ \Rightarrow \quad f''(0) &= -1 \end{aligned}$$

- Second-order Taylor series:

$$f(x_{i+1}) \approx f(x_i) + f'(x_i) \cdot (x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

$$\Rightarrow \qquad f(1) \approx 1.2 - 0.25 \times 1 + \left(\frac{-1}{2}\right) \times 1^2 = 0.45$$

Taylor Series Approximation of a Polynomial

Third-order approximation for $f(1)$:

- Need third-order derivative at $x_i = 0$:

$$\begin{aligned} f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow \quad f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow \quad f''(x) &= -1.2x^2 - 0.9x - 1 \\ \Rightarrow \quad f'''(x) &= -2.4x - 0.9 \\ \Rightarrow \quad f'''(0) &= -0.9 \end{aligned}$$

- Third-order Taylor series:

$$f(1) \approx 1.2 - 0.25 \times 1 - 0.5 \times 1^2 + \left(\frac{-0.9}{3!}\right) \times 1^3 = 0.3$$

Taylor Series Approximation of a Polynomial

Fourth-order approximation for $f(1)$:

- Need fourth-order derivative at $x_i = 0$:

$$\begin{aligned} f(x) &= -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2 \\ \Rightarrow \quad f'(x) &= -0.4x^3 - 0.45x^2 - x - 0.25 \\ \Rightarrow \quad f''(x) &= -1.2x^2 - 0.9x - 1 \\ \Rightarrow \quad f'''(x) &= -2.4x - 0.9 \\ \Rightarrow \quad f^{(4)}(x) &= -2.4 \\ \Rightarrow \quad f^{(4)}(0) &= -2.4 \end{aligned}$$

- Fourth-order Taylor series:

$$f(1) \approx 1.2 - 0.25 \times 1 - 0.5 \times 1^2 - 0.15 \times 1^3 - \left(\frac{-2.4}{4!}\right) \times 1^4 = 0.2$$

Taylor Series Approximation of a Polynomial

- Absolute error:

$$\varepsilon = |f(x_{i+1}) - \hat{f}(x_{i+1})|$$

where the approximation is

$$\hat{f}(x_{i+1}) = \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!}(x_{i+1} - x_i)^k$$

- Summary of results:

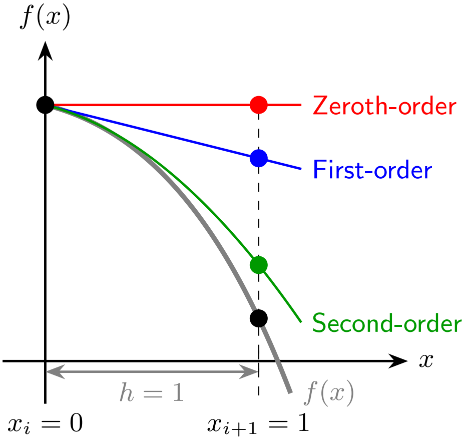
Order (n)	Approximation $\hat{f}(1)$	Absolute error ε
1	0.95	$\|0.2 - 0.95\| = 0.75$
2	0.45	$\|0.2 - 0.45\| = 0.25$
3	0.3	$\|0.2 - 0.3\| = 0.1$
4	0.2	$\|0.2 - 0.2\| = 0$

- Error decreases as order increases
- With $n = 4$, approximation is exact because $f(x)$ is a 4th-degree polynomial

Taylor Series Approximation of a Polynomial

- Use zeroth- through fourth-order Taylor series expansions to approximate:

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$



Taylor Series Approximation of a Polynomial

Concluding remarks:

- Taylor series approximates functions using derivatives at a point
- Higher-order terms improve accuracy by capturing curvature
- For an m-degree polynomial, a Taylor series of order m gives the exact function
- Truncation error quantifies the approximation error
- This example illustrates the power of Taylor series for function approximation in numerical methods

Taylor Series Approximation of $\sin(x)$

Maclaurin series:

- A Taylor series expansion of a function about 0
- Taylor series approximation:

$$f(x_{i+1}) \approx \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!}(x_{i+1} - x_i)^k$$

- Set $x_i = 0$ and $x = x_{i+1}$, then

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Taylor Series Approximation of $\sin(x)$

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- Set $x_i = 0$ and $x = x_{i+1}$, then

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

- Derivatives of $f(x) = \sin(x)$:

$$f'(x) = \underbrace{\cos(x)}_{\cos(0)=1}, \quad f''(x) = -\underbrace{\sin(x)}_{\sin(0)=0}, \quad f'''(x) = -\underbrace{\cos(x)}_{\cos(0)=1}, \quad f^{(4)}(x) = \underbrace{\sin(x)}_{\sin(0)=0}$$

- Formula:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \cdots$$

Quick Summary

Monday's Class:

- Definition of truncation error
- Taylor series of any smooth function (approximated by polynomial)
- Approximation with zeroth-, first-, second-, and higher-order information
- Example: Taylor series approximation of a polynomial function
- Revisit Taylor series approximation of $\sin(x)$

Function $f(x) = x^m$

The effect of **nonlinearity** and **step size** on the Taylor series approximation

- Function

$$f(x) = x^m$$

for $m = 1, 2, 3, 4$ over the range from $x = 1$ to 2.

Function $f(x) = x^m$

The effect of **nonlinearity** and **step size** on the Taylor series approximation

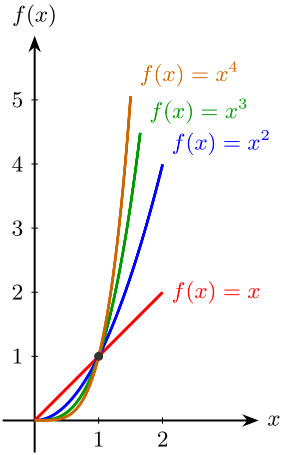
- Function

  $$f(x) = x^m$$

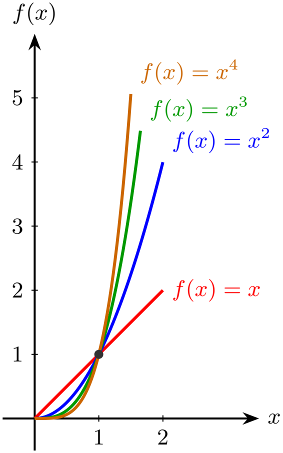
  for $m = 1, 2, 3, 4$ over the range from $x = 1$ to 2.

- **First-order** Taylor series expansion:

  $$\begin{aligned}\hat{f}(x_{i+1}) &= f(x_i) + f'(x_i)(x_{i+1} - x_i) \\ &= f(x_i) + m x_i^{m-1}(x_{i+1} - x_i)\end{aligned}$$



Function $f(x) = x^m$: Nonlinearity



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + m x_i^{m-1}(x_{i+1} - x_i)$$

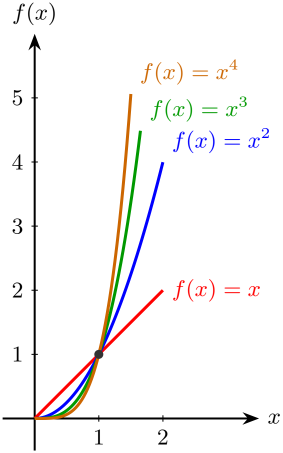
- For $m = 1$, the function is $f(x) = x$ and derivative is $f'(x) = 1$:
  - True value $f(2) = 2$
  - Approximation:

    $$\hat{f}(2) = f(1) + 1 \times (2 - 1) = 2$$

  - Absolute error:

    $$|\hat{f}(2) - f(2)| = 0$$

Function $f(x) = x^m$: Nonlinearity



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + m x_i^{m-1}(x_{i+1} - x_i)$$

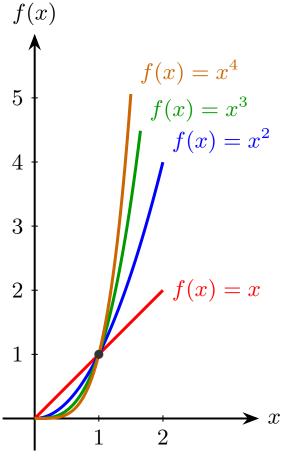
- For $m = 2$, the function is $f(x) = x^2$ and derivative is $f'(x) = 2x$:
  - True value $f(2) = 4$
  - Approximation:

    $$\hat{f}(2) = f(1) + 2 \times (2 - 1) = 3$$

  - Absolute error:

    $$|\hat{f}(2) - f(2)| = 1$$

Function $f(x) = x^m$: Nonlinearity



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + m x_i^{m-1}(x_{i+1} - x_i)$$

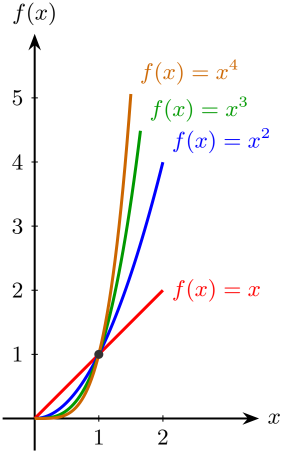
- For $m = 3$, the function is $f(x) = x^3$ and derivative is $f'(x) = 3x^2$:
  - True value $f(2) = 8$
  - Approximation:

    $$\hat{f}(2) = f(1) + 3 \times (2 - 1) = 4$$

  - Absolute error:

    $$|\hat{f}(2) - f(2)| = 4$$

Function $f(x) = x^m$: Nonlinearity



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + mx_i^{m-1}(x_{i+1} - x_i)$$

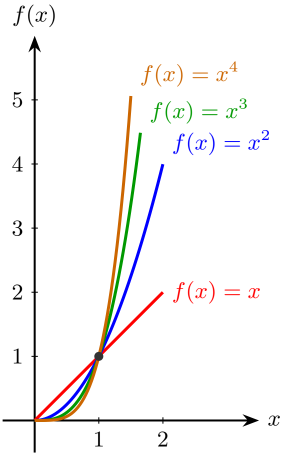
- For $m = 4$, the function is $f(x) = x^4$ and derivative is $f'(x) = 4x^3$:
  - True value $f(2) = 16$
  - Approximation:

    $$\hat{f}(2) = f(1) + 4 \times (2 - 1) = 5$$

  - Absolute error:

    $$|\hat{f}(2) - f(2)| = 11$$

Function $f(x) = x^m$: Nonlinearity



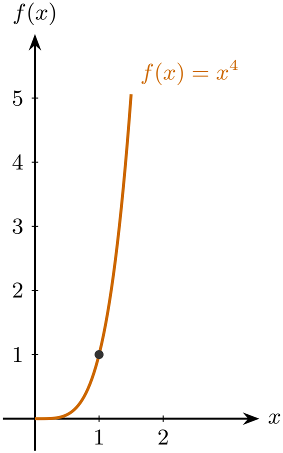
First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + m x_i^{m-1}(x_{i+1} - x_i)$$

[Effect of nonlinearity] Error increases as the function becomes more nonlinear.

Exp. m	$f(2)$	$\hat{f}(2)$	Absolute error
1	2	2	0
2	4	3	1
3	8	4	4
4	16	5	11

Function $f(x) = x^4$: Step Size h



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + 4x_i^3 \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

Evaluate $x + h$ where $x = 1$:

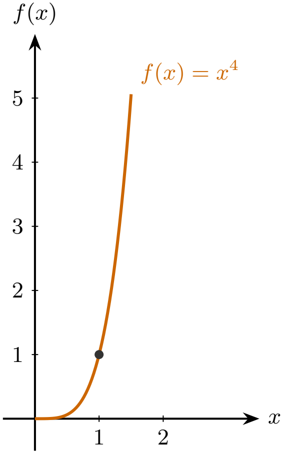
- For $h = 1$:
  - True value $f(2) = 16$
  - Approximation:

$$\hat{f}(2) = f(1) + 4 \times (2 - 1) = 5$$

- For $h = 0.5$:
  - True value $f(1.5) = 5.0625$
  - Approximation:

$$\hat{f}(1.5) = f(1) + 4 \times (1.5 - 1) = 3$$

Function $f(x) = x^4$: Step Size h



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + 4x_i^3 \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

Evaluate $x + h$ where $x = 1$:

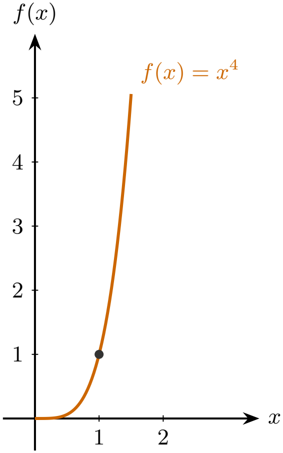
- For $h = 0.25$:
  - True value $f(1.25) = 2.441406$
  - Approximation:

    $$\hat{f}(1.25) = f(1) + 4 \times (1.25 - 1) = 2$$

- For $h = 0.125$:
  - True value $f(1.125) = 1.601807$
  - Approximation:

    $$\hat{f}(1.125) = f(1) + 4 \times (1.125 - 1) = 1.5$$

Function $f(x) = x^4$: Step Size h



First-order Taylor series expansion:

$$\hat{f}(x_{i+1}) = f(x_i) + 4x_i^3 \underbrace{(x_{i+1} - x_i)}_{\text{step size } h}$$

Evaluate $x + h$ where $x = 1$:

h	$f(x+h)$	$\hat{f}(x+h)$	Absolute error
1	16	5	11
0.5	5.0625	3	2.0625
0.25	2.441406	2	0.441406
0.125	1.601807	1.5	0.101807

[**Effect of step size**] Discrepancy will decrease as h is reduced.

Taylor Series Approximation

Maclaurin series:

- A Taylor series expansion of a function about 0
- Taylor series approximation:

$$f(x_{i+1}) \approx \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!}(x_{i+1} - x_i)^k$$

- Set $x_i = 0$ and $x = x_{i+1}$, then

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

Taylor Series Approximation

Maclaurin series:

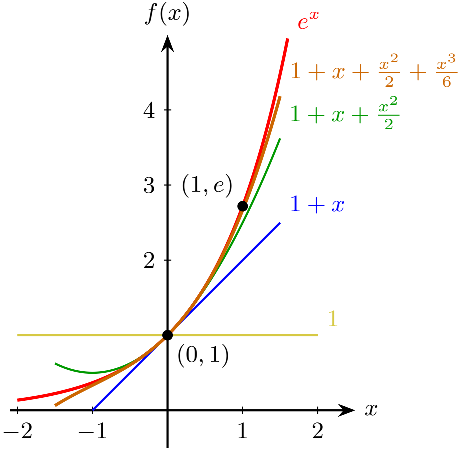
- A Taylor series expansion of a function about 0

$$f(x) \approx \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

- Derivatives of $f(x) = e^x$:

$$f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x \quad \Rightarrow \quad f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

Taylor Series Approximation of $\exp(x)$



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