Matrix and Tensor Models for Spatiotemporal Traffic Data Imputation and Forecasting

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Outline

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   - Urban movement data
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   - Matrix/Tensor completion
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   - Sparse urban traffic state forecasting
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   - Univariate traffic time series imputation
   - Large-scale traffic data imputation
5. **Hankel Tensor Factorization (HTF)**
   - Model description
   - Extreme missing traffic data imputation
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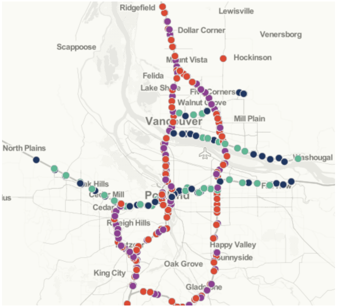
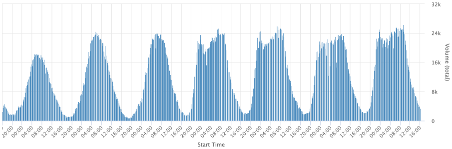
Spatiotemporal Traffic Data

Many spatiotemporal traffic time series data are in the form of **matrix**.

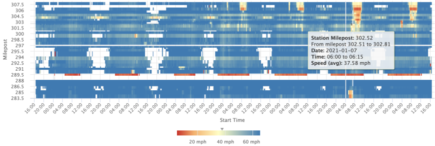
- Portland highway traffic data[1]



Highway network & sensor locations



Traffic volume



Traffic speed

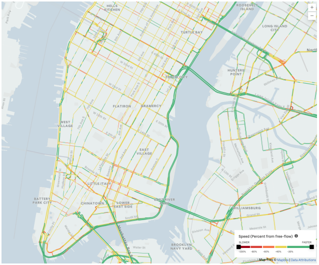
- $\boldsymbol{X} \in \mathbb{R}^{N \times T}$ with N spatial locations $\times$ T time steps
- Traffic volume/speed shows strong spatial/temporal dependencies

[1] https://portal.its.pdx.edu/home

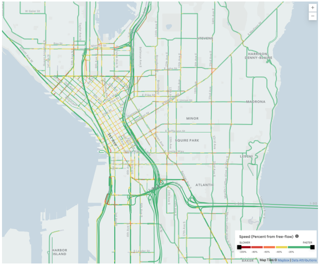
Urban Movement Data

High-dimensional & sparse

- Uber (hourly) movement speed data



NYC movement



Seattle movement

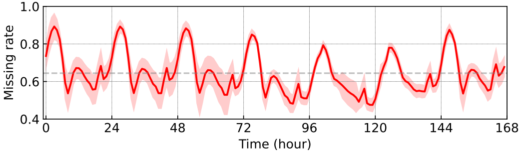
- {road segment, time slot (hour), average speed}
- Computing hourly speed: Road segments have 5+ unique trips.

Issue: Insufficient sampling of ridesharing vehicles on the road network!

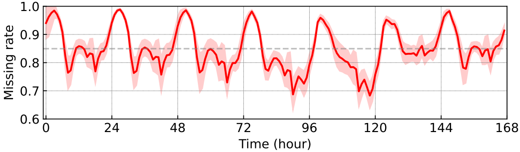
Urban Movement Data

High-dimensional & sparse

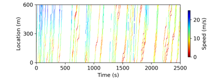
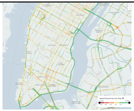
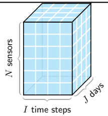
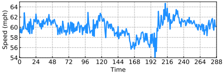
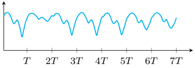
- **NYC** movement speed data (2019)
  - **98,210** road segments & 8,760 time steps (hours)
  - Overall missing rate: **64.43%**



- **Seattle** movement speed data (2019)
  - **63,490** road segments & 8,760 time steps (hours)
  - Overall missing rate: **84.95%**



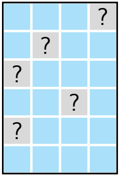
Spatiotemporal Traffic Data

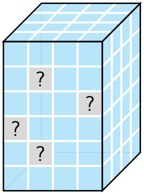
Sparse	
High-dimensional	
Multidimensional	
Noises & outliers	
Nonstationary	

Traffic data show complicated spatiotemporal patterns and correlations.

Problem Formulation

Objective A: Impute missing values in the data matrix $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ (or tensor $\boldsymbol{\mathcal{Y}} \in \mathbb{R}^{M \times N \times T}$).





- Matrix completion (Observed index set Ω)

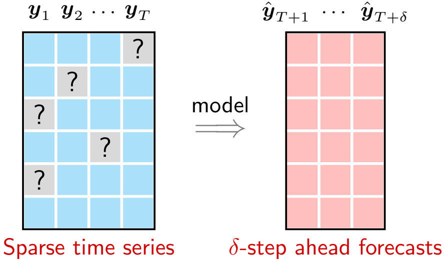
$$\underbrace{\mathcal{P}_{\Omega}(\boldsymbol{Y})}_{\text{Partially observed}} \xrightarrow{\text{Estimate}} \underbrace{\mathcal{P}_{\Omega}^{\perp}(\boldsymbol{Y})}_{\text{Unobserved}}$$

Modeling process:

- How to make use of spatiotemporal traffic patterns?
- How to make use of traffic time series dynamics?

Problem Formulation

Objective B: Given a partially observed data $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$ consisting of time series $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_T \in \mathbb{R}^N$, forecast data points $\boldsymbol{y}_{T+\delta}, \delta \in \mathbb{N}^+$.



Modeling process:

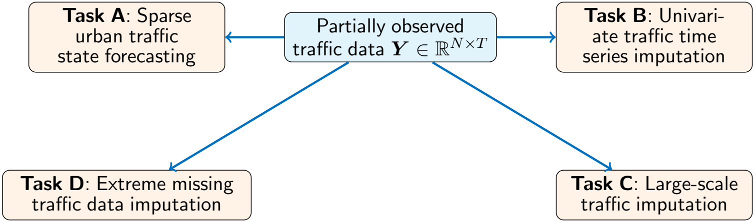
- How to characterize time series dynamics in high-dimensional and sparse traffic data?

Real-world applications:

- Forecasting urban traffic states with sparse data.

Tasks

Focus: **spatiotemporal traffic data imputation and forecasting**.



Tensor Factorization

- Revisit tensor factorization

1927 — Higher-Order SVD

Frank Lauren Hitchcock

1960s — Tucker Decomposition

Ledyard R. Tucker

1970 — CP Decomposition

J. Douglas Carroll
Jih-Jie Chang
Richard A. Harshman

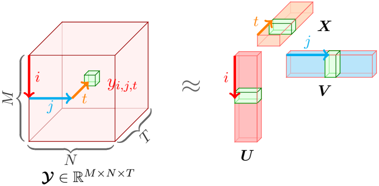
2009 — Tensor Decompositions and Applications

Tamara G. Kolda

2011 — Tensor-Train Decomposition

Ivan Oseledets

- **CP tensor factorization**: Factorize $\boldsymbol{\mathcal{Y}}$ into the combination of three rank-R factor matrices (i.e., low-dimensional latent factors).



$$\begin{cases} y_{i,j,t}\approx\sum_{r=1}^{R}u_{i,r}v_{j,r}x_{t,r} \\ \boldsymbol{\mathcal{Y}}\approx\sum_{r=1}^{R}\boldsymbol{u}_r\otimes\boldsymbol{v}_r\otimes\boldsymbol{x}_r \end{cases}$$

Matrix/Tensor Completion

Matrix nuclear norm ($\|X\|_*$) minimization

Candès & Recht'09

Singular value thresholding

Cai et al.'10

Truncated nuclear norm ($\|X\|_{r,*}, r \in \mathbb{N}^+$) minimization

Zhang et al.'12

Hu et al.'12

Tensor nuclear norm ($\|\mathcal{X}\|_*$) minimization

Liu et al.'13

Generalized nonconvex nonsmooth low-rank minimization

Lu et al.'14

Tensor nuclear norm minimization with linear transform

Lu et al.'19

Low-rank Hankel matrix/tensor ($\mathcal{H}_\tau(\cdot)$) completion

Yokota et al.'18

Sedighin et al.'20

Cai et al.'21

Yamamoto et al.'22

Circulant/Convolution nuclear norm ($\|\mathcal{C}(X)\|_*$ or $\|\mathcal{C}_{\tilde{\tau}}(X)\|_*$) minimization

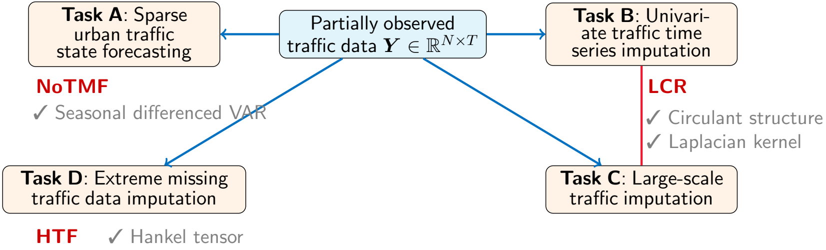
Liu'22

Liu & Zhang'23

Overview

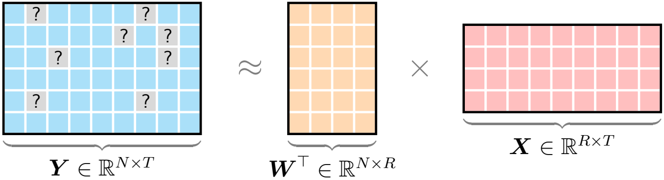
Machine learning framework

- Matrix and tensor methods: Learning from sparse data.
- Temporal modeling: Building and reinforcing temporal dependencies for time series.



Matrix Factorization

A simple approach to reconstruct missing values.



MF (Koren et al.'09)

Estimating low-dimensional $\boldsymbol{W}, \boldsymbol{X}$:

$$\min_{\boldsymbol{W}, \boldsymbol{X}} \frac{1}{2} \|\mathcal{P}_\Omega(\boldsymbol{Y} - \boldsymbol{W}^\top \boldsymbol{X})\|_F^2$$

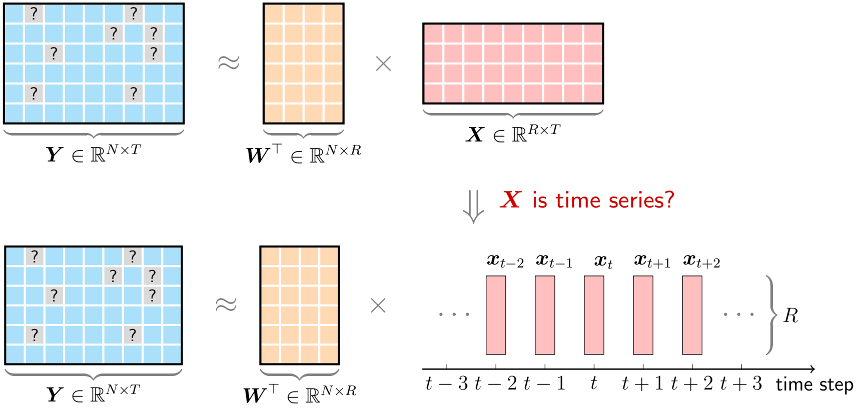
on data $\boldsymbol{Y}$ w/ observed index set Ω.

- ✓ Learn from sparse data
- ✓ Spatial factor matrix $\boldsymbol{W}$
- ✓ Temporal factor matrix $\boldsymbol{X}$

How to build temporal correlations on MF?

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.



Why? Temporal factor matrix $\boldsymbol{X} \in \mathbb{R}^{R \times T}$ is the low-dimensional representation of time series dynamics of $\boldsymbol{Y} \in \mathbb{R}^{N \times T}$.

Temporal Matrix Factorization

Vector autoregression (VAR) on the temporal factor matrix.

MF (Koren et al.'09)

Estimating low-dimensional $\boldsymbol{W}, \boldsymbol{X}$:

$$\min_{\boldsymbol{W}, \boldsymbol{X}} \frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2$$

on data $\boldsymbol{Y}$ w/ observed index set Ω.

$+$

dth-order VAR

$$\boldsymbol{x}_t = \sum_{k=1}^{d}\boldsymbol{A}_k\boldsymbol{x}_{t-k} + \underbrace{\boldsymbol{\epsilon}_t}_{\mathcal{N}(\boldsymbol{0},\boldsymbol{I})}$$

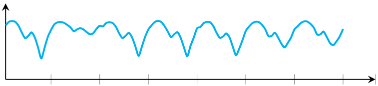
w/ coefficients $\{\boldsymbol{A}_k\}$.

$\Downarrow$ Yu et al.'16
Chen & Sun'21

$$\boxed{\min_{\boldsymbol{W}, \boldsymbol{X}, \{\boldsymbol{A}_k\}_{k=1}^{d}} \underbrace{\frac{1}{2}\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2}_{\text{MF on data } \boldsymbol{Y}} + \frac{\gamma}{2}\underbrace{\sum_{t=d+1}^{T}\left\|\boldsymbol{x}_t - \sum_{k=1}^{d}\boldsymbol{A}_k\boldsymbol{x}_{t-k}\right\|_2^2}_{\text{VAR on temporal factors } \boldsymbol{X}}}$$

Nonstationary Temporal Matrix Factorization

Traffic data are nonstationary due to daily patterns of traffic flow.



- Season-m differencing ($m \in \mathbb{N}^{+}$, e.g., daily/weekly):

$$\boldsymbol{x}_t \approx \sum_{k=1}^{d} \boldsymbol{A}_k \boldsymbol{x}_{t-k} \quad \Rightarrow \quad \boldsymbol{x}_t - \boldsymbol{x}_{t-m} \approx \sum_{k=1}^{d} \boldsymbol{A}_k (\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k})$$

- **(Ours)** Optimization problem:

$$\begin{aligned} \min_{\boldsymbol{W}, \boldsymbol{X}, \{\boldsymbol{A}_k\}_{k=1}^{d}} \quad & \frac{1}{2} \underbrace{\|\mathcal{P}_{\Omega}(\boldsymbol{Y} - \boldsymbol{W}^{\top}\boldsymbol{X})\|_F^2}_{\text{MF on data } \boldsymbol{Y}} + \frac{\rho}{2} \underbrace{(\|\boldsymbol{W}\|_F^2 + \|\boldsymbol{X}\|_F^2)}_{\text{Regularization}} \\ & + \frac{\gamma}{2} \underbrace{\sum_{t=d+m+1}^{T} \left\| (\boldsymbol{x}_t - \boldsymbol{x}_{t-m}) - \sum_{k=1}^{d} \boldsymbol{A}_k (\boldsymbol{x}_{t-k} - \boldsymbol{x}_{t-m-k}) \right\|_2^2}_{\text{VAR on seasonal differenced temporal factors}} \end{aligned}$$

Nonstationary Temporal Matrix Factorization

Rewrite NoTMF

- Optimization problem:[2]

$$\min_{\boldsymbol{W},\boldsymbol{X},\boldsymbol{A}} \frac{1}{2}\underbrace{\|\mathcal{P}_{\Omega}(\boldsymbol{Y}-\boldsymbol{W}^\top\boldsymbol{X})\|_F^2}_{\text{MF on data } \boldsymbol{Y}}+\frac{\rho}{2}\underbrace{(\|\boldsymbol{W}\|_F^2+\|\boldsymbol{X}\|_F^2)}_{\text{Regularization}}$$
$$+\frac{\gamma}{2}\underbrace{\|\boldsymbol{X}\boldsymbol{\Psi}_0^\top-\boldsymbol{A}(\boldsymbol{I}_d\otimes\boldsymbol{X})\boldsymbol{\Psi}^\top\|_F^2}_{\text{VAR on } \boldsymbol{X}}$$

where $\boldsymbol{\Psi}_0\in\mathbb{R}^{(T-d-m)\times T},\boldsymbol{\Psi}\in\mathbb{R}^{(T-d-m)\times(dT)}$ are temporal operators.

- Alternating minimization (let f be the obj.):

$$\begin{cases} \text{Spatial factors} \quad \boldsymbol{W}:=\{\boldsymbol{W}\mid\frac{\partial f}{\partial\boldsymbol{W}}=\boldsymbol{0}\} \quad \text{(least squares)} \\ \text{Temporal factors} \quad \boldsymbol{X}:=\{\boldsymbol{X}\mid\frac{\partial f}{\partial\boldsymbol{X}}=\boldsymbol{0}\} \quad \text{(conjugate gradient)} \\ \text{VAR coefficients} \quad \boldsymbol{A}:=\{\boldsymbol{A}\mid\frac{\partial f}{\partial\boldsymbol{A}}=\boldsymbol{0}\} \quad \text{(least squares)} \end{cases}$$

[2] $\boldsymbol{A}\triangleq\begin{bmatrix}\boldsymbol{A}_1 & \cdots & \boldsymbol{A}_d\end{bmatrix}\in\mathbb{R}^{R\times(dR)}$ (coefficient matrix).

Nonstationary Temporal Matrix Factorization

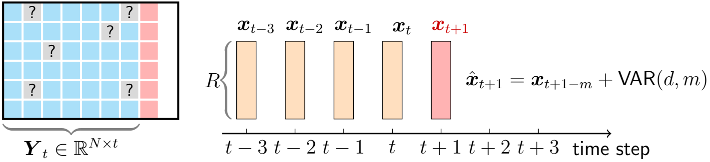
NoTMF forecasting?

Implementation

- Estimate $\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{A}$
- Forecast $\hat{\boldsymbol{x}}_{t+1}$ with VAR
- Return $\hat{\boldsymbol{y}}_{t+1} = \boldsymbol{W}^\top \hat{\boldsymbol{x}}_{t+1}$

✓ Sparse input $\boldsymbol{Y}_t$

✓ Forecast in latent spaces



Nonstationary Temporal Matrix Factorization

NoTMF forecasting on streaming data?

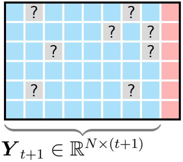
- Online forecasting (Gultekin & Paisley'18):
  - Fix the spatial factor matrix $\boldsymbol{W}$
  - Use input data $\boldsymbol{Y}_{t+1}$ to update the temporal factor matrix $\boldsymbol{X}$ and the coefficient matrix $\boldsymbol{A}$

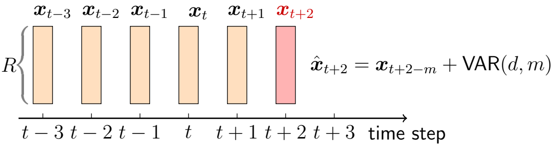
Implementation

- Estimate $\boldsymbol{X}, \boldsymbol{A}$
- Forecast $\hat{\boldsymbol{x}}_{t+2}$ with VAR
- Return $\hat{\boldsymbol{y}}_{t+2} = \boldsymbol{W}^\top \hat{\boldsymbol{x}}_{t+2}$

✓ Sparse input $\boldsymbol{Y}_{t+1}$

✓ Forecast in latent spaces

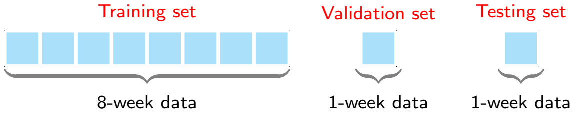




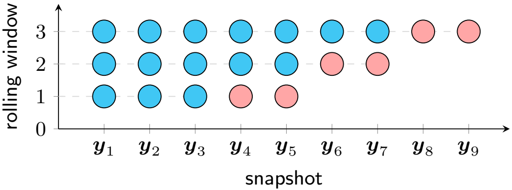
Sparse Urban Traffic State Forecasting

NoTMF forecasting

- NYC Uber movement speed dataset:
  - **10-week data** of size 98210×1680; **66.56%** missing values
- Rolling forecasting setup (Time horizon $\delta = 1, 2, 3, 6$):



  - Weight parameter $\gamma \in \{10^{-2}, 10^{-1}, 10^{0}, 10^{1}, 10^{2}\}$
  - Weight parameter $\rho = \{10^{-1}\gamma, 5 \times 10^{-1}\gamma, \gamma, 5\gamma, 10\gamma\}$
- Rolling forecasting illustration ($\delta = 2$):



Sparse Urban Traffic State Forecasting

NoTMF vs. baseline (in MAPE/RMSE)

- On the NYC Uber movement speed dataset

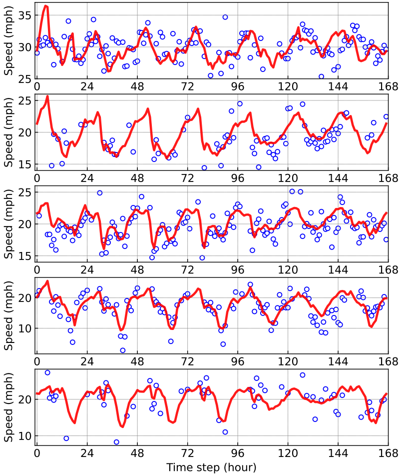
δ	d	NoTMF ($m = 24$)	NoTMF ($m = 168$)	NoTMF-1st ($m = 168$)	TRMF	BTMF	BTRMF
1	1	13.63/2.88	13.53/2.86	**13.45**/**2.85**	14.50/3.12	14.94/3.13	15.93/3.33
	2	13.47/**2.84**	**13.41**/**2.84**	13.42/**2.84**	14.14/3.05	15.70/3.41	15.90/3.35
	3	13.46/2.84	**13.39**/**2.83**	13.43/2.84	13.87/2.96	15.80/3.34	16.08/3.43
	6	13.41/**2.83**	**13.39**/**2.83**	13.41/**2.83**	14.00/2.98	15.45/3.27	16.26/3.48
2	1	13.91/2.96	13.76/2.94	**13.70**/**2.92**	15.85/3.43	15.33/3.21	16.85/3.56
	2	13.77/2.92	**13.63**/**2.89**	13.72/2.92	15.04/3.31	15.87/3.32	17.27/3.71
	3	13.72/2.91	**13.61**/**2.89**	13.73/2.92	15.25/3.36	15.69/3.33	17.24/3.74
	6	13.59/**2.87**	**13.57**/2.88	13.68/2.91	14.92/3.24	15.91/3.39	18.18/3.97
3	1	14.30/3.05	14.06/3.02	**14.02**/**3.00**	17.52/3.83	15.86/3.32	18.61/3.91
	2	14.01/2.98	**13.84**/**2.94**	13.96/2.98	17.32/4.00	16.30/3.40	18.90/4.10
	3	13.95/2.97	**13.79**/**2.93**	13.98/2.98	16.91/3.71	16.56/3.49	18.68/4.05
	6	13.78/**2.92**	**13.73**/**2.92**	13.91/2.96	16.72/3.65	15.49/3.27	20.45/4.66
6	1	**14.61**/**3.11**	14.67/3.20	14.98/3.32	21.20/4.70	15.99/3.32	22.40/4.69
	2	**14.30**/**3.03**	14.33/3.09	14.90/3.28	20.87/5.01	16.04/3.33	23.56/5.63
	3	**14.26**/**3.03**	14.28/3.09	14.86/3.26	20.08/4.65	15.67/3.28	24.27/5.72
	6	**14.06**/**2.97**	14.16/3.06	14.80/3.23	20.40/4.35	16.38/3.50	26.34/6.60

NoTMF performs better than other TMF models (e.g., TRMF, BTMF & BTRMF).

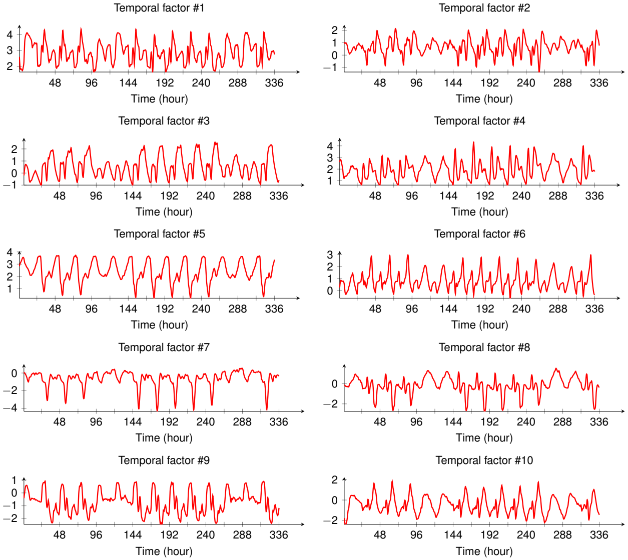
Sparse Urban Traffic State Forecasting

NoTMF forecasting ($\delta = 6$)

- On the NYC Uber movement speed dataset



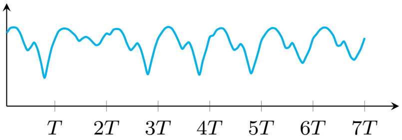
Sparse Urban Traffic State Forecasting



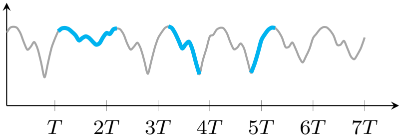
Laplacian Convolutional Representation

Motivation: Time series imputation

- Global trends (e.g., long-term quasi-seasonality & daily/weekly rhythm):



- Local trends (e.g., short-term time series trends):

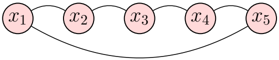


How to characterize both global and local trends in sparse time series?

Laplacian Convolutional Representation

Local trend modeling

- Intuition of (circulant) Laplacian matrix



Undirected and circulant graph

$$\xrightarrow{\text{Modeling}} \quad \boldsymbol{L}=\begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (2,-1,0,0,-1)^\top$$

$$\Downarrow$$

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1,\cdots,-1}_{\tau}, 0,\cdots,0, \underbrace{-1,\cdots,-1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series $\boldsymbol{x}=(x_1,\ldots,x_T)^\top \in \mathbb{R}^T$.

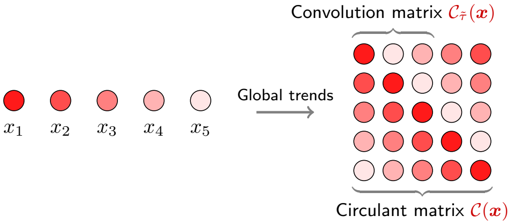
- (Laplacian) Temporal regularization:

$$\mathcal{R}_\tau(\boldsymbol{x})=\frac{1}{2}\|\boldsymbol{L}\boldsymbol{x}\|_2^2=\frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2$$

Reformulate temporal regularization with circular convolution.

Laplacian Convolutional Representation

Global trend modeling: Circulant matrix $\mathcal{C}(\boldsymbol{x})$ vs. convolution matrix $\mathcal{C}_{\tilde{\tau}}(\boldsymbol{x})$



- Circulant/Convolution nuclear norm minimization
  - A balance between global and local trends modeling?

CircNNM (Liu'22, Liu & Zhang'23)

Estimating $\boldsymbol{x}$:

$$\min_{\boldsymbol{x}} \ \|\mathcal{C}(\boldsymbol{x})\|_*$$
$$\text{s.t. } \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

on data $\boldsymbol{y}$ w/ observed index set Ω.

ConvNNM (Liu'22, Liu & Zhang'23)

Estimating $\boldsymbol{x}$:

$$\min_{\boldsymbol{x}} \ \|\mathcal{C}_{\tilde{\tau}}(\boldsymbol{x})\|_*$$
$$\text{s.t. } \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

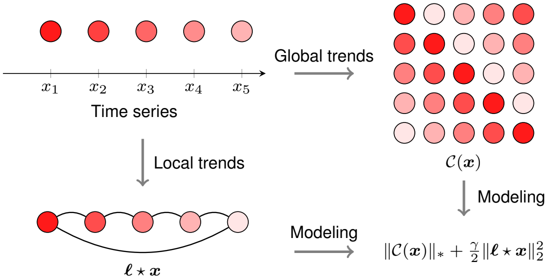
on data $\boldsymbol{y}$ w/ observed index set Ω.

Laplacian Convolutional Representation

Laplacian Convolutional Representation (LCR)

For any partially observed time series $\boldsymbol{y} \in \mathbb{R}^T$ with observed index set Ω, LCR utilizes circulant matrix and Laplacian kernel to characterize global and local trends in time series, respectively, i.e.,

$$\min_{\boldsymbol{x}} \underbrace{\|\mathcal{C}(\boldsymbol{x})\|_*}_{\text{global}} + \frac{\gamma}{2} \underbrace{\|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2}_{\text{local}}$$
$$\text{s.t. } \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon$$



Laplacian Convolutional Representation

- Augmented Lagrangian function:[3]

$$\mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) = \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 + \frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_2^2 + \langle\boldsymbol{w},\boldsymbol{x}-\boldsymbol{z}\rangle + \frac{\eta}{2}\|\mathcal{P}_\Omega(\boldsymbol{z}-\boldsymbol{y})\|_2^2$$

- The ADMM scheme:

$$\begin{cases} \boldsymbol{x} := \arg\min_{\boldsymbol{x}} \ \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) & \text{(Nuclear norm minimization)} \\ \boldsymbol{z} := \arg\min_{\boldsymbol{z}} \ \mathcal{L}(\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}) & \text{(Closed-form solution)} \\ \boldsymbol{w} := \boldsymbol{w} + \lambda(\boldsymbol{x}-\boldsymbol{z}) & \text{(Standard update)} \end{cases}$$

- Optimize $\boldsymbol{x}$?

$$\|\mathcal{C}(\boldsymbol{x})\|_* = \|\mathcal{F}(\boldsymbol{x})\|_1 \qquad \& \qquad \frac{1}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2 = \frac{1}{2T}\|\mathcal{F}(\boldsymbol{\ell})\circ\mathcal{F}(\boldsymbol{x})\|_2^2$$

Nuclear norm minimization $\Rightarrow$ ℓ_1**-norm minimization with FFT**

[3] $\boldsymbol{w}\in\mathbb{R}^T$ (Lagrange multiplier); $\langle\cdot,\cdot\rangle$ (inner product).

Laplacian Convolutional Representation

- Optimize $\boldsymbol{x}$ via FFT (in $\mathcal{O}(T\log T)$ time):

$$\boldsymbol{x}:=\arg\min_{\boldsymbol{x}}\ \|\mathcal{C}(\boldsymbol{x})\|_*+\frac{\gamma}{2}\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2+\frac{\lambda}{2}\|\boldsymbol{x}-\boldsymbol{z}+\boldsymbol{w}/\lambda\|_2^2$$

$$\Longrightarrow\hat{\boldsymbol{x}}:=\arg\min_{\hat{\boldsymbol{x}}}\ \|\hat{\boldsymbol{x}}\|_1+\frac{\gamma}{2T}\|\hat{\boldsymbol{\ell}}\circ\hat{\boldsymbol{x}}\|_2^2+\frac{\lambda}{2T}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{z}}+\hat{\boldsymbol{w}}/\lambda\|_2^2$$

where we introduce $\{\hat{\boldsymbol{\ell}},\hat{\boldsymbol{x}},\hat{\boldsymbol{z}},\hat{\boldsymbol{w}}\}\triangleq\mathcal{F}\{\boldsymbol{\ell},\boldsymbol{x},\boldsymbol{z},\boldsymbol{w}\}$ (i.e., FFT).

ℓ_1-norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of ℓ_1-norm minimization in complex space:

$$\min_{\hat{\boldsymbol{x}}}\ \|\hat{\boldsymbol{x}}\|_1+\frac{\delta}{2}\|\hat{\boldsymbol{x}}-\hat{\boldsymbol{h}}\|_2^2$$

with complex-valued $\hat{\boldsymbol{x}},\hat{\boldsymbol{h}}\in\mathbb{C}^T$ and weight parameter δ, element-wise, the solution is given by

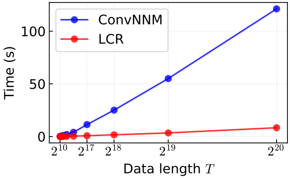
$$\hat{x}_t:=\frac{\hat{h}_t}{|\hat{h}_t|}\cdot\max\{0,|\hat{h}_t|-1/\delta\},t=1,\ldots,T.$$

Laplacian Convolutional Representation

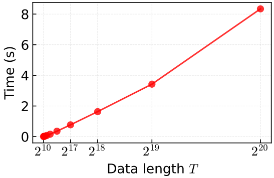
Empirical time complexity

On the synthetic data $\boldsymbol{y} \in \mathbb{R}^T$ with $T \in \{2^{10}, 2^{11}, \dots, 2^{20}\}$

- Ours: **LCR**
  - An FFT implementation in $\mathcal{O}(T \log T)$
  - The logarithmic factor $\log T$ makes the FFT highly efficient
- Baseline: **ConvNNM**[4] (Liu'22, Liu & Zhang'23)
  - Convolution matrix $\mathcal{C}_{\tilde{\tau}}(\boldsymbol{y}) \in \mathbb{R}^{T \times \tilde{\tau}}$ with kernel size $\tilde{\tau} = 2^4$
  - Singular value thresholding in $\mathcal{O}(\tilde{\tau}^2 T)$



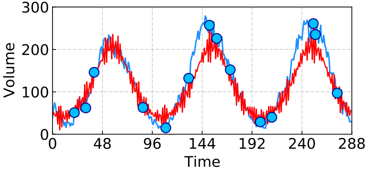
ConvNNM vs. LCR



LCR

[4] Convolution nuclear norm minimization.

Univariate Traffic Time Series Imputation

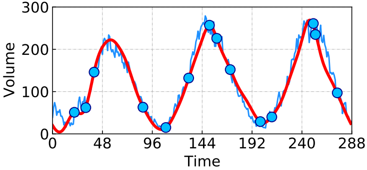


CircNNM:

$$\min_{\boldsymbol{x}} \quad \underbrace{\|\mathcal{C}(\boldsymbol{x})\|_*}_{\text{global}}$$

$$\text{s.t.} \quad \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

$\Downarrow$ Plus temporal regularization

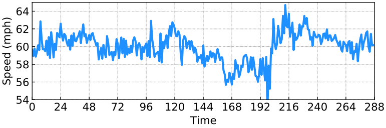


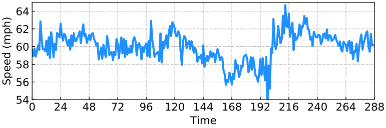
LCR:

$$\min_{\boldsymbol{x}} \quad \underbrace{\|\mathcal{C}(\boldsymbol{x})\|_*}_{\text{global}} + \frac{\gamma}{2}\underbrace{\|\boldsymbol{\ell}\star\boldsymbol{x}\|_2^2}_{\text{local}}$$

$$\text{s.t.} \quad \|\mathcal{P}_\Omega(\boldsymbol{x}-\boldsymbol{y})\|_2 \leq \epsilon$$

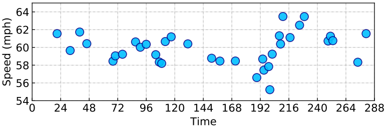
Univariate Traffic Time Series Imputation

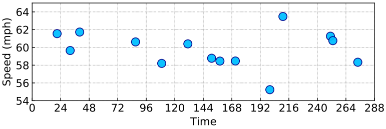




⇓ Mask **90%** observations

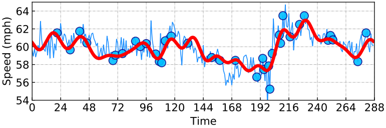
⇓ Mask **95%** observations

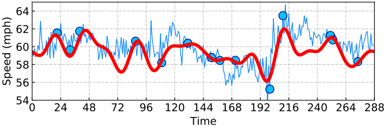




⇓ Reconstruct time series

⇓ Reconstruct time series





LCR can reconstruct traffic time series from very sparse data.

Large-Scale Traffic Data Imputation

LCR vs. baseline (in MAPE/RMSE)

- PeMS-4W: California freeway traffic speed dataset ($\boldsymbol{Y} \in \mathbb{R}^{11160 \times 8064}$)

Model	Missing rate			
	30%	50%	70%	90%
LCR-2D	1.50/**1.49**	1.76/**1.69**	**2.07**/**2.06**	**3.19**/**3.05**
LCR$_N$	**1.48**/1.50	**1.73**/1.73	**2.07**/2.12	3.24/3.22
LCR	1.50/**1.49**	1.76/**1.69**	2.08/2.07	3.21/3.06
CTNNM	2.26/1.84	2.67/2.14	3.40/2.66	5.22/3.90
CircNNM	2.26/1.84	2.69/2.15	3.43/2.67	5.34/3.96
LRMC	2.04/1.80	2.43/2.12	3.08/2.66	6.05/4.43
HaLRTC	1.98/1.73	2.22/1.98	2.84/2.49	4.39/3.66
LRTC-TNN	1.68/1.55	1.93/1.77	2.33/2.14	3.40/3.10
NoTMF	2.95/2.65	3.05/2.73	3.33/2.97	5.22/4.71

Results

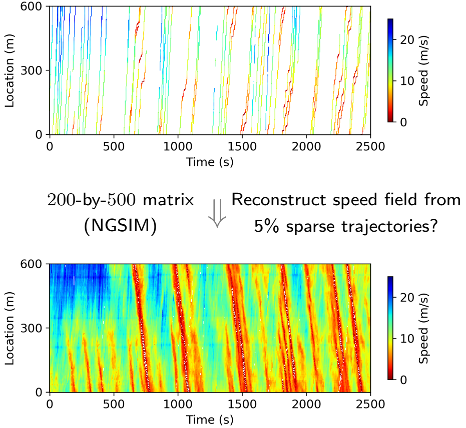
- LCR-2D $>$ CTNNM: The importance of temporal regularization.
- CTNNM $\geq$ CircNNM: Ciculant tensor is superior to circulant matrix.
- LCR $>$ LRMC/LRTC: The importance of global/local modeling.

$$\mathcal{O}(NT\log(NT)) \quad \text{(FFT)} \quad \text{vs.} \quad \mathcal{O}(\min\{N^2T, NT^2\}) \quad \text{(SVD)}$$

Hankel Tensor Factorization

Motivation: Spatiotemporal data reconstruction

- Sparse speed field reconstruction problem in vehicular traffic flow.



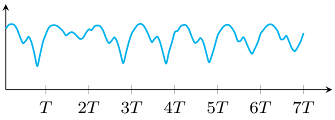
How to characterize both spatial and temporal dependencies?

Hankel Tensor Factorization

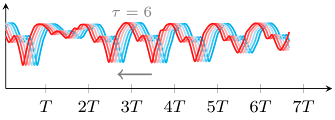
- Hankel matrix
  - Given $\boldsymbol{x} = (1, 2, 3, 4, 5)^\top$ and window length $\tau = 2$, we have

$$\mathcal{H}_\tau(\boldsymbol{x}) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \in \mathbb{R}^{4\times 2}$$

  - Automatic temporal modeling



Traffic time series



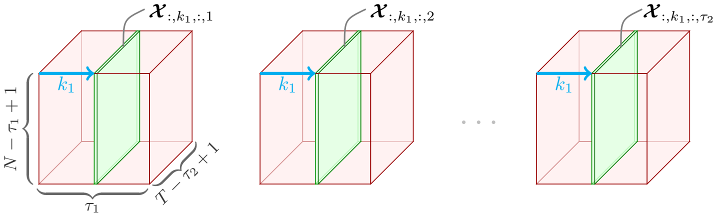
Hankel matrix

Hankel Tensor Factorization

- Hankel tensor: Given any matrix $\boldsymbol{X} \in \mathbb{R}^{N \times T}$, we have

$$\boldsymbol{\mathcal{X}} \triangleq \mathcal{H}_{\tau_1, \tau_2}(\boldsymbol{X})$$

  - Window lengths: $\tau_1, \tau_2 \in \mathbb{N}^{+}$;
  - Tensor size: $(N - \tau_1 + 1) \times \tau_1 \times (T - \tau_2 + 1) \times \tau_2$;



(Figure) 4th-order Hankel tensor: A sequence of third-order tensors.

  - Slice: $\boldsymbol{\mathcal{X}}_{:,k_1,:,k_2}, \forall k_1, k_2$;
  - Slice size: $(N - \tau_1 + 1) \times (T - \tau_2 + 1)$.

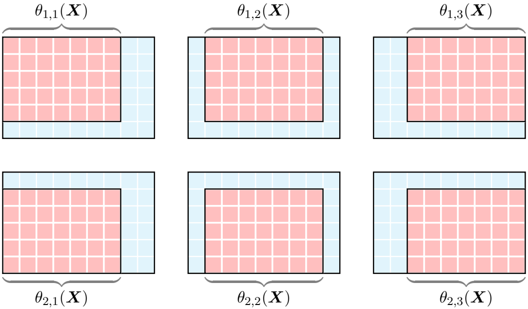
Hankel Tensor Factorization

Hankel indexing

- Sampling function for the Hankel tensor:

$$\theta_{k_1,k_2}(\boldsymbol{X}) \triangleq [\mathcal{H}_{\tau_1,\tau_2}(\boldsymbol{X})]_{:,k_1,:,k_2},$$

  referring to as the tensor slice with $k_1 \in \{1,\ldots,\tau_1\}$, $k_2 \in \{1,\ldots,\tau_2\}$.
- **[Importance]** Developing memory-efficient algorithms



- Tensor slices $\theta_{k_1,k_2}(\boldsymbol{X})$ vs. data matrix $\boldsymbol{X}$

Hankel Tensor Factorization

Ours:

- Convolutional tensor decomposition (circular convolution $\star_{\text{row}}$):

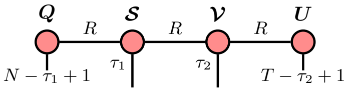
$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q} \star_{\text{row}} \boldsymbol{s}_{k_1}^\top)(\boldsymbol{U} \star_{\text{row}} \boldsymbol{v}_{k_2}^\top)^\top$$

Baselines:

- Tensor-train decomposition:

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q}\boldsymbol{S}_{k_1})(\boldsymbol{U}\boldsymbol{V}_{k_2})^\top$$

  - $\{\boldsymbol{S}_{k_1}, \boldsymbol{V}_{k_2}\}$ are **circulant matrices** $\Rightarrow$ convolutional decomposition
  - $\{\boldsymbol{S}_{k_1}, \boldsymbol{V}_{k_2}\}$ are **diagonal matrices** $\Rightarrow$ CP decomposition



- CP tensor decomposition (Khatri-Rao product $\odot$):

$$\theta_{k_1,k_2}(\boldsymbol{Y}) \approx (\boldsymbol{Q} \odot \boldsymbol{s}_{k_1}^\top)(\boldsymbol{U} \odot \boldsymbol{v}_{k_2}^\top)^\top$$

Hankel Tensor Factorization

HTF (convolutional decomposition)

- Optimization problem:

$$\min_{\boldsymbol{Q},\boldsymbol{S},\boldsymbol{U},\boldsymbol{V}} \ \underbrace{\frac{1}{2}\sum_{k_1,k_2}\left\|\mathcal{P}_{\Omega_{k_1,k_2}}(\theta_{k_1,k_2}(\boldsymbol{Y})-(\boldsymbol{Q}\star_{\text{row}}\boldsymbol{s}_{k_1}^\top)(\boldsymbol{U}\star_{\text{row}}\boldsymbol{v}_{k_2}^\top)^\top)\right\|_F^2}_{\text{Tensor decomposition on Hankel tensor slices}}$$

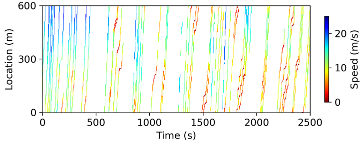
$$+\frac{\rho}{2}(\|\boldsymbol{Q}\|_F^2+\|\boldsymbol{S}\|_F^2+\|\boldsymbol{U}\|_F^2+\|\boldsymbol{V}\|_F^2)$$

- Alternating minimization (let f be the obj.):

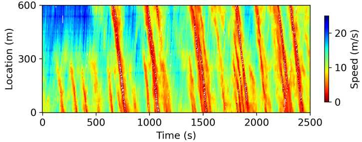
$$\begin{cases}\boldsymbol{Q}:=\{\boldsymbol{Q}\mid\frac{\partial f}{\partial\boldsymbol{Q}}=\boldsymbol{0}\} & \text{(conjugate gradient)}\\ \boldsymbol{s}_{k_1}:=\{\boldsymbol{s}_{k_1}\mid\frac{\partial f}{\partial\boldsymbol{s}_{k_1}}=\boldsymbol{0}\},\ \forall k_1 & \text{(conjugate gradient)}\\ \boldsymbol{U}:=\{\boldsymbol{U}\mid\frac{\partial f}{\partial\boldsymbol{U}}=\boldsymbol{0}\} & \text{(conjugate gradient)}\\ \boldsymbol{v}_{k_2}:=\{\boldsymbol{v}_{k_2}\mid\frac{\partial f}{\partial\boldsymbol{v}_{k_2}}=\boldsymbol{0}\},\ \forall k_2 & \text{(conjugate gradient)}\end{cases}$$

- Memory-efficient but still computationally costly!

Extreme Missing Traffic Data Imputation

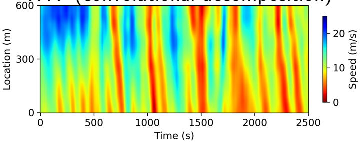


Sparse speed field



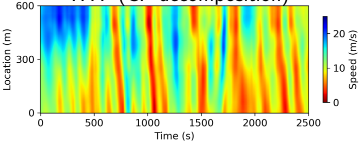
Ground truth speed field

HTF (convolutional decomposition)



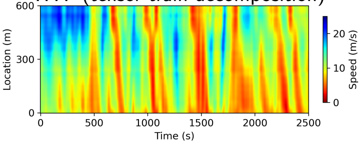
MAPE = **51.92%**

HTF (CP decomposition)



MAPE = 53.93%

HTF (tensor-train decomposition)



MAPE = 56.48%

Extreme Missing Traffic Data Imputation

HTF vs. baseline (in MAPE/RMSE)

- On the Seattle freeway traffic speed dataset ($\boldsymbol{Y} \in \mathbb{R}^{323 \times 8064}$)

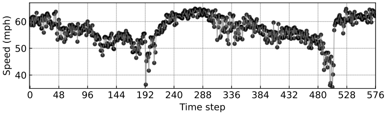
Model	Random missing rate			
	80%	85%	90%	95%
HTF (`convolution`)	**6.21/3.88**	**6.51/4.06**	**6.98/4.30**	**8.02/4.84**
HTF (`tensor-train`)	8.75/5.16	9.86/5.76	9.24/5.36	9.89/5.70
HTF (`CP`)	7.25/4.33	7.93/4.66	8.61/4.96	9.25/5.20
LATC	6.50/4.00	6.90/4.21	7.47/4.51	8.75/5.05
LRTC-TNN	6.97/4.24	7.43/4.43	8.19/4.81	9.60/5.55
LCR-2D	6.75/4.15	7.31/4.38	7.96/4.71	9.78/5.39
BTMF	6.85/4.17	7.36/4.42	8.13/4.79	9.63/5.48

Results

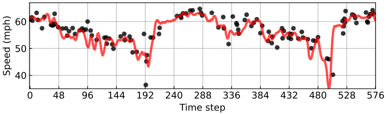
- Convolutional tensor decomposition outperforms both tensor-train and CP tensor decomposition.
- Our HTF model performs better than state-of-the-art baseline models.

Extreme Missing Traffic Data Imputation

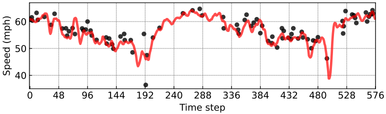
- Example: 1st time series within two days.



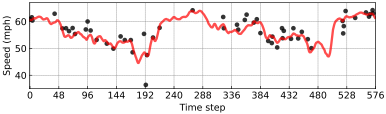
Original time series



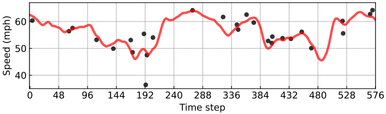
80% missing rate



85% missing rate

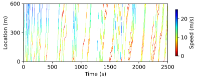
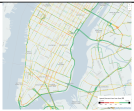
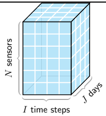
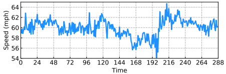
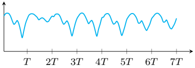


90% missing rate



95% missing rate

Conclusion

Sparse	
High-dimensional	
Multidimensional	
Noises & outliers	
Nonstationary	

$\Rightarrow$

Low-rank framework:

- NoTMF: matrix factorization
- LCR: circulant matrix nuclear norm minimization
- HTF: tensor factorization

Temporal modeling:

- NoTMF: seasonal differenced vector autoregression
- LCR: temporal smoothing
- HTF: automatic temporal modeling with Hankel tensor

Highlights & Contributions

Matrix nuclear norm ($\|\boldsymbol{X}\|_*$) minimization

Candès & Recht'09

Singular value thresholding

Cai et al.'10

Truncated nuclear norm ($\|\boldsymbol{X}\|_{r,*}, r \in \mathbb{N}^+$) minimization

Zhang et al.'12
Hu et al.'12

Tensor nuclear norm ($\|\boldsymbol{\mathcal{X}}\|_*$) minimization

Liu et al.'13

Generalized nonconvex nonsmooth low-rank minimization

Lu et al.'14

Tensor nuclear norm minimization with linear transform

Lu et al.'19

Low-rank Hankel matrix/tensor ($\mathcal{H}_\tau(\cdot)$) completion

Yokota et al.'18
Sedighin et al.'20
Cai et al.'21
Yamamoto et al.'22

(Ours) **HTF**:
✓ Memory-efficient
✓ Conv. para.

Circulant/Convolution nuclear norm ($\|\mathcal{C}(\boldsymbol{X})\|_*$ or $\|\mathcal{C}_{\tilde{\tau}}(\boldsymbol{X})\|_*$) minimization

Liu'22
Liu & Zhang'23

(Ours) **LCR**:
✓ Local trend modeling
✓ An FFT implementation

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- (Cai et al.'21) "Accelerated structured alternating projections for robust spectrally sparse signal recovery." IEEE Transactions on Signal Processing. 2021, 69: 809-821.
- (Chen & Sun'22) "Bayesian temporal factorization for multidimensional time series prediction." IEEE Transactions on Pattern Analysis and Machine Intelligence. 2022, 44(9): 4659-4673.
- (Liu'22) "Time series forecasting via learning convolutionally low-rank models." IEEE Transactions on Information Theory. 2022, 68(5): 3362-3380.
- (Liu & Zhang'23) "Recovery of future data via convolution nuclear norm minimization." IEEE Transactions on Information Theory. 2023, 69(1): 650-665.

Thanks for your attention!

Any Questions?

Slides: https://xinychen.github.io/slides/sustech23.pdf

About me:

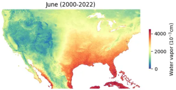
- Homepage: https://xinychen.github.io
- How to reach me: chenxy346@gmail.com

Research Interests

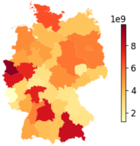
- Machine learning & spatiotemporal data modeling

Transportation

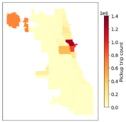
Mobile service



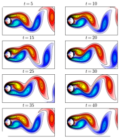
Climate



Energy



Mobility



Dynamical system

- Urban science
- Intelligent transportation systems

Past Works

Spatiotemporal traffic data imputation:

1. Xinyu Chen, Zhaocheng He, Jiawei Wang (2018). Spatial-temporal traffic speed patterns discovery and incomplete data recovery via SVD-combined tensor decomposition. *Transportation Research Part C: Emerging Technologies*. 86: 59-77. (100+ citations)
2. Xinyu Chen, Zhaocheng He, Lijun Sun (2019). A Bayesian tensor decomposition approach for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*. 98: 73-84. (200+ citations, **ESI highly cited paper**)
3. Xinyu Chen, Zhaocheng He, Yixian Chen, Yuhuan Lu, Jiawei Wang (2019). Missing traffic data imputation and pattern discovery with a Bayesian augmented tensor factorization model. *Transportation Research Part C: Emerging Technologies*. 104: 66-77. (100+ citations)
4. Xinyu Chen, Jinming Yang, Lijun Sun (2020). A nonconvex low-rank tensor completion model for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*. 117: 102673. (100+ citations)
5. Xinyu Chen, Yixian Chen, Nicolas Saunier, Lijun Sun (2021). Scalable low-rank tensor learning for spatiotemporal traffic data imputation. *Transportation Research Part C: Emerging Technologies*. 129: 103226.

Past Works

Spatiotemporal traffic data imputation:

6. Xinyu Chen, Mengying Lei, Nicolas Saunier, Lijun Sun (2022). Low-rank autoregressive tensor completion for spatiotemporal traffic data imputation. *IEEE Transactions on Intelligent Transportation Systems*. 23 (8): 12301-12310. (50+ citations, **ESI hot paper**)

Spatiotemporal time series forecasting:

7. Xinyu Chen, Lijun Sun (2022). Bayesian temporal factorization for multidimensional time series prediction. *IEEE Transactions on Pattern Analysis and Machine Intelligence*. 44 (9): 4659-4673. (150+ citations, **ESI hot paper** & **ESI highly cited paper**)

Spatiotemporal pattern discovery:

8. Xinyu Chen, Chengyuan Zhang, Xiaoxu Chen, Nicolas Saunier, Lijun Sun (2023). Discovering dynamic patterns from spatiotemporal data with time-varying low-rank autoregression. *IEEE Transactions on Knowledge and Data Engineering*. Early access.

Past Works

A strong advocate of open-source and reproducible research:

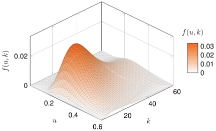
- https://github.com/xinychen

Algorithms

transdim

(1.1k stars)

Tools



awesome-latex-drawing

(1.2k stars)

Tutorials

latex-cookbook

(1.1k stars)

(THU Press)

Future Plan

Broad interests:

- Data Science
- Machine Learning
- AI for Science

Research directions:

- Urban science (e.g., connection among infrastructure, mobile/mobility activities, urban development, and economy)
- Human mobility modeling (e.g., long-range sequence prediction)
- Geospatial data analysis (e.g., orthogonal mode decomposition)
- Intelligent & sustainable urban systems
- Optimization & decision making
- ......

Goals: Solving many scientific, mathematical, and engineering problems with AI algorithms.

Website: https://spatiotemporal-data.github.io

Thanks for your attention!

Any Questions?

Slides: https://xinychen.github.io/slides/sustech23.pdf

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