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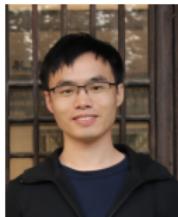
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# Laplacian Convolutional Representation for Traffic Time Series Imputation

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## Preprint:

- X. Chen, Z. Cheng, N. Saunier, L. Sun (2022). Laplacian convolutional representation for traffic time series imputation. arXiv preprint arXiv:2212.01529.  
[https://xinychen.github.io/papers/Laplacian\\_convolution.pdf](https://xinychen.github.io/papers/Laplacian_convolution.pdf)

## GitHub repository:

- **transdim**: Machine learning for spatiotemporal traffic data imputation and forecasting. (1,000+ stars & 270+ forks on GitHub)  
<https://github.com/xinychen/transdim>

## Slides:

- <https://xinychen.github.io/slides/LCR.pdf>

# Outline

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- **Motivation**

- Data-Driven ITS

- Time Series Imputation

- Speed Field Reconstruction

- **Revisit Laplacian Matrix & Circular Convolution**

- Laplacian Matrix

- Laplacian Regularization

- **Circulant Matrix Nuclear Norm Minimization**

- **Laplacian Convolutional Representation**

- Model Description

- Solution Algorithm

- **Experiments**

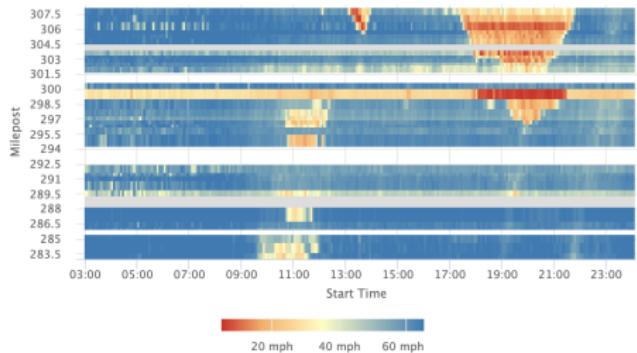
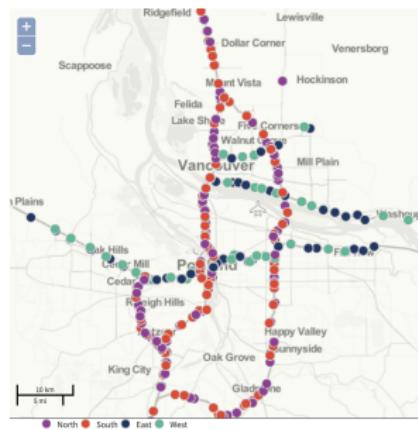
- Univariate Traffic Time Series Imputation

- Speed Field Reconstruction

- **Conclusion**

# Motivation

- Portland highway traffic flow data<sup>1</sup>



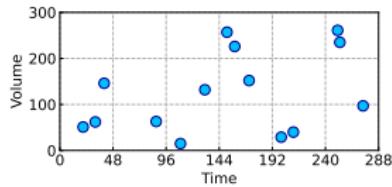
Traffic speed field

Highway network & sensor locations

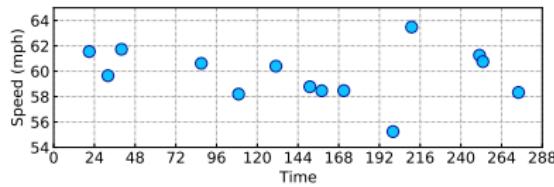
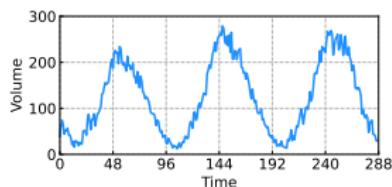
- Speed field  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  ( $N$  locations &  $T$  time steps)
- Speed field shows strong spatial/temporal dependencies

<sup>1</sup><https://portal.its.pdx.edu/home>

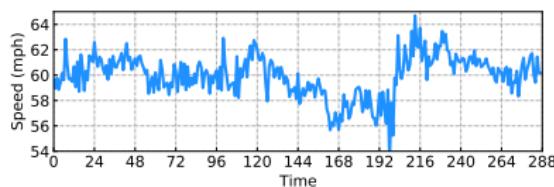
# Motivation



↓ Reconstruct  
traffic volume?



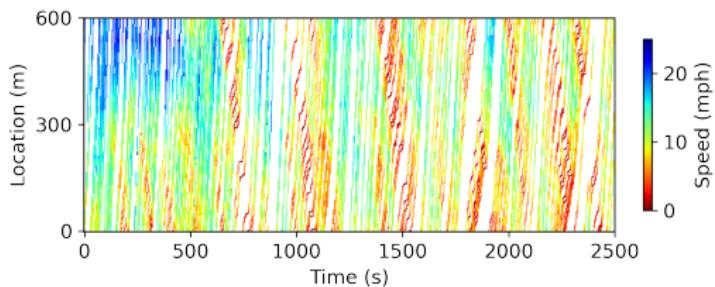
↓ Reconstruct  
traffic speed?



- How to utilize the global trends of traffic time series?
- How to produce local consistency of traffic data?

# Motivation

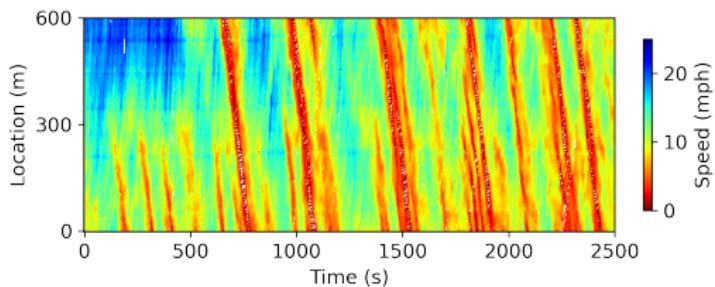
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200-by-500 matrix  
(NGSIM)



Reconstruct speed field from  
20% sparse trajectories?

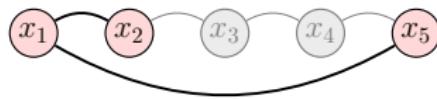


- How to learn from sparse spatiotemporal data?
- How to characterize spatial/temporal local dependencies?

## Revisit Laplacian Matrix & Circular Convolution

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- Intuition of (circulant) Laplacian matrix.



Modeling

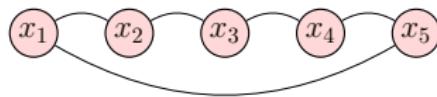
$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

## Revisit Laplacian Matrix & Circular Convolution

---

- Intuition of (circulant) Laplacian matrix.



Modeling

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

## Revisit Laplacian Matrix & Circular Convolution

Reformulate Laplacian regularization with circular convolution.

- Intuition of (circulant) Laplacian matrix.

Undirected and circulant graph

Modeling

$$\mathbf{L} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

(Circulant) Laplacian matrix

- Laplacian kernel:  $\ell = (2, -1, 0, 0, -1)^\top$ .

$$\mathbf{L}\mathbf{x} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \star \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \ell \star \mathbf{x}$$

where  $\star$  denotes the circular convolution.

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\ell \star \mathbf{x}\|_2^2$$

## Revisit Laplacian Matrix & Circular Convolution

---

Reformulate Laplacian regularization with circular convolution.

- Define Laplacian kernel:

$$\boldsymbol{\ell} \triangleq (\underbrace{2\tau}_{\text{degree}}, \underbrace{-1, \dots, -1}_{\tau}, 0, \dots, 0, \underbrace{-1, \dots, -1}_{\tau})^\top \in \mathbb{R}^T$$

for any time series  $\mathbf{x} = (x_1, \dots, x_T)^\top \in \mathbb{R}^T$ .

- Local trend modeling via (Laplacian) temporal regularization:

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\mathbf{L}\mathbf{x}\|_2^2 = \frac{1}{2} \|\boldsymbol{\ell} \star \mathbf{x}\|_2^2$$

- Property with discrete Fourier transform (denoted by  $\mathcal{F}(\cdot)$ ):

$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\boldsymbol{\ell} \star \mathbf{x}\|_2^2 = \frac{1}{2} \|\mathcal{F}(\boldsymbol{\ell}) \circ \mathcal{F}(\mathbf{x})\|_2^2$$

# Circulant Matrix Nuclear Norm Minimization

## Circulant Matrix Nuclear Norm Minimization (CircNNM)

For any partially observed time series  $\mathbf{y} \in \mathbb{R}^T$  with observed index set  $\Omega$ , the optimization problem of CircNNM for reconstructing time series is given by

$$\begin{aligned}\min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon\end{aligned}$$

where  $\mathcal{C} : \mathbb{R}^T \rightarrow \mathbb{R}^{T \times T}$  denotes the circulant operator.  $\|\cdot\|_*$  denotes the nuclear norm of matrix, namely, the sum of singular values.

- An important property:

$$\|\mathcal{C}(\mathbf{x})\|_* = \|\mathcal{F}(\mathbf{x})\|_1$$

- CircNNM shows an efficient  $\text{FFT}^2$  implementation in  $\mathcal{O}(T \log T)$  time (Liu'22, Liu & Zhang'23).

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<sup>2</sup>Fast Fourier Transform (FFT).

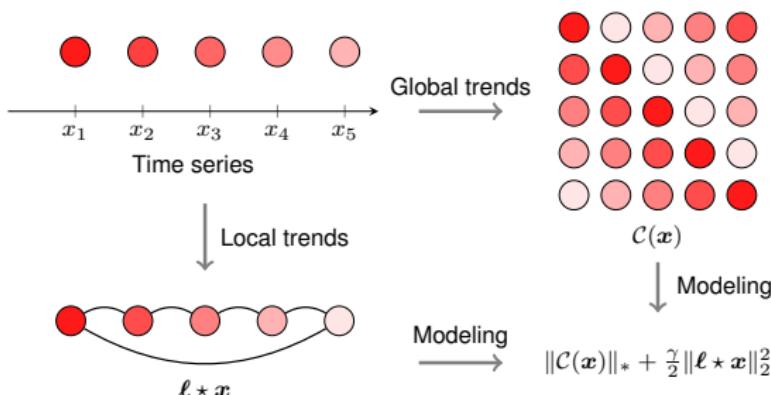
# Laplacian Convolutional Representation

## Laplacian Convolutional Representation (LCR)

For any partially observed time series  $\mathbf{y} \in \mathbb{R}^T$  with observed index set  $\Omega$ , LCR utilizes circulant matrix and Laplacian kernel to characterize **global and local trends** in time series, respectively, i.e.,

$$\min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{x})$$

$$\text{s.t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$



## Laplacian Convolutional Representation

---

- LCR model:

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\boldsymbol{x}) \\ \text{s.t. } & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

- Augmented Lagrangian function:

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{w}) = \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2} \|\boldsymbol{\ell} \star \boldsymbol{x}\|_2^2 + \frac{\lambda}{2} \|\boldsymbol{x} - \boldsymbol{z}\|_2^2 + \langle \boldsymbol{w}, \boldsymbol{x} - \boldsymbol{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\boldsymbol{z} - \boldsymbol{y})\|_2^2$$

where  $\boldsymbol{w} \in \mathbb{R}^T$  is the Lagrange multiplier, and  $\langle \cdot, \cdot \rangle$  denotes the inner product.

# Laplacian Convolutional Representation

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- LCR model:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathcal{C}(\mathbf{x})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{x}) \\ \text{s.t.} \quad & \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon \end{aligned}$$

- Augmented Lagrangian function:

$$\mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) = \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z}\|_2^2 + \langle \mathbf{w}, \mathbf{x} - \mathbf{z} \rangle + \frac{\eta}{2} \|\mathcal{P}_\Omega(\mathbf{z} - \mathbf{y})\|_2^2$$

where  $\mathbf{w} \in \mathbb{R}^T$  is the Lagrange multiplier, and  $\langle \cdot, \cdot \rangle$  denotes the inner product.

- The ADMM scheme:

$$\left\{ \begin{array}{l} \mathbf{x} := \arg \min_{\mathbf{x}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) \\ \mathbf{z} := \arg \min_{\mathbf{z}} \mathcal{L}(\mathbf{x}, \mathbf{z}, \mathbf{w}) \\ = \frac{1}{\lambda + \eta} \mathcal{P}_\Omega(\lambda \mathbf{x} + \mathbf{w} + \eta \mathbf{y}) + \frac{1}{\lambda} \mathcal{P}_\Omega^\perp(\lambda \mathbf{x} + \mathbf{w}) \\ \mathbf{w} := \mathbf{w} + \lambda(\mathbf{x} - \mathbf{z}) \end{array} \right.$$

## Laplacian Convolutional Representation

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- Optimize  $\mathbf{x}$  via FFT (in  $\mathcal{O}(T \log T)$  time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2 \\ &= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - (\lambda \hat{\mathbf{z}} - \hat{\mathbf{w}}) \oslash (\lambda \hat{\ell}^* \circ \hat{\ell} + \lambda \mathbb{1}_T)\|_2^2\end{aligned}$$

where we introduce  $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$  (i.e., FFT).

# Laplacian Convolutional Representation

- Optimize  $\mathbf{x}$  via FFT (in  $\mathcal{O}(T \log T)$  time):

$$\begin{aligned}\mathbf{x} &:= \arg \min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_* + \frac{\gamma}{2} \|\ell * \mathbf{x}\|_2^2 + \frac{\lambda}{2} \|\mathbf{x} - \mathbf{z} + \mathbf{w}/\lambda\|_2^2 \\ \implies \hat{\mathbf{x}} &:= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\gamma}{2T} \|\hat{\ell} \circ \hat{\mathbf{x}}\|_2^2 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{z}} + \hat{\mathbf{w}}/\lambda\|_2^2 \\ &= \arg \min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - (\lambda \hat{\mathbf{z}} - \hat{\mathbf{w}}) \oslash (\lambda \hat{\ell}^* \circ \hat{\ell} + \lambda \mathbb{1}_T)\|_2^2\end{aligned}$$

where we introduce  $\{\hat{\ell}, \hat{\mathbf{x}}, \hat{\mathbf{z}}, \hat{\mathbf{w}}\} \triangleq \mathcal{F}\{\ell, \mathbf{x}, \mathbf{z}, \mathbf{w}\}$  (i.e., FFT).

## $\ell_1$ -norm Minimization in Complex Space (Liu & Zhang'23)

For any optimization problem in the form of  $\ell_1$ -norm minimization in complex space:

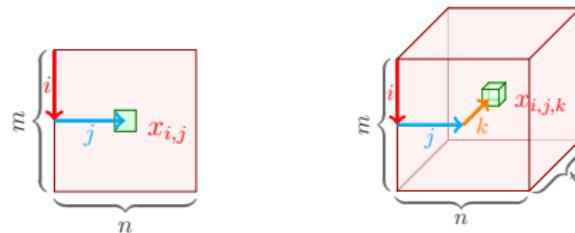
$$\min_{\hat{\mathbf{x}}} \|\hat{\mathbf{x}}\|_1 + \frac{\lambda}{2T} \|\hat{\mathbf{x}} - \hat{\mathbf{h}}\|_2^2$$

with complex-valued  $\hat{\mathbf{x}}, \hat{\mathbf{h}} \in \mathbb{C}^T$ , element-wise, the solution is given by

$$\hat{x}_t := \frac{\hat{h}_t}{|\hat{h}_t|} \cdot \max\{0, |\hat{h}_t| - T/\lambda\}, t = 1, \dots, T.$$

# Laplacian Convolutional Representation

- What is tensor?  $\mathbf{X} \in \mathbb{R}^{m \times n}$  vs.  $\mathbf{X} \in \mathbb{R}^{m \times n \times t}$



## Two-Dimensional LCR (LCR-2D)

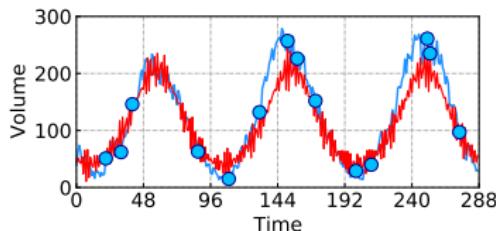
For any partially observed time series  $\mathbf{Y} \in \mathbb{R}^{N \times T}$  with observed index set  $\Omega$ , LCR can be formulated as follows,

$$\begin{aligned} \min_{\mathbf{X}} \quad & \|\mathcal{C}(\mathbf{X})\|_* + \frac{\gamma}{2} \|(\ell_s \ell^\top) \star \mathbf{X}\|_F^2 \\ \text{s.t. } & \|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \epsilon \end{aligned}$$

where  $\mathcal{C} : \mathbb{R}^{N \times T} \rightarrow \mathbb{R}^{N \times N \times T \times T}$  denotes the circulant operator.

# Experiments

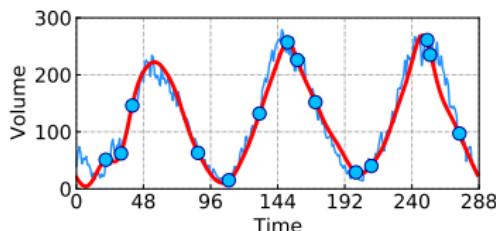
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**CircNNM:**

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* \\ \text{s. t. } \quad & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

↓ Plus temporal regularization (TR)

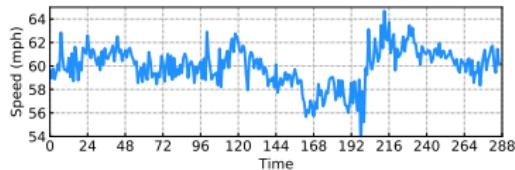


**LCR:**

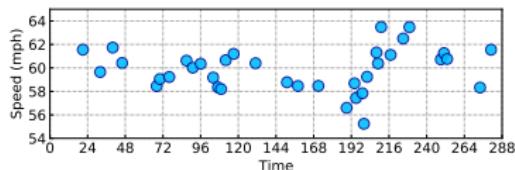
$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \|\mathcal{C}(\boldsymbol{x})\|_* + \frac{\gamma}{2} \|\boldsymbol{\ell} * \boldsymbol{x}\|_2^2 \\ \text{s. t. } \quad & \|\mathcal{P}_\Omega(\boldsymbol{x} - \boldsymbol{y})\|_2 \leq \epsilon \end{aligned}$$

# Experiments

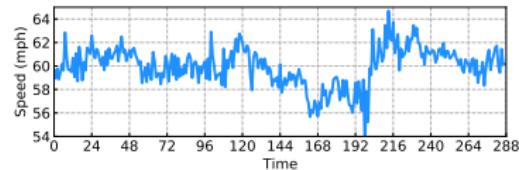
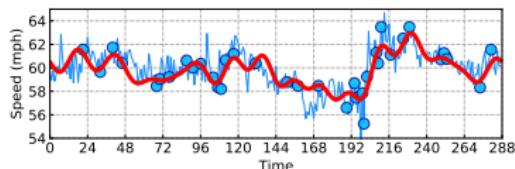
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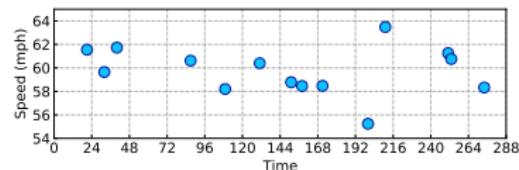
↓ Mask 90% observations



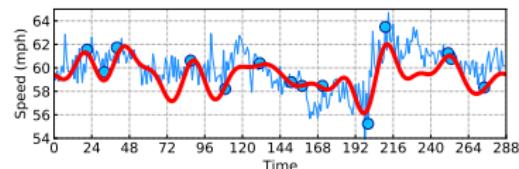
↓ Reconstruct time series



↓ Mask 95% observations

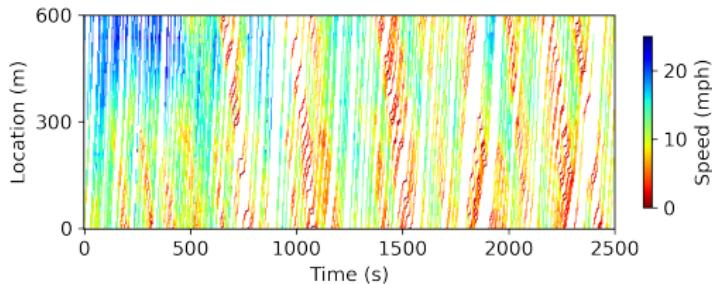


↓ Reconstruct time series

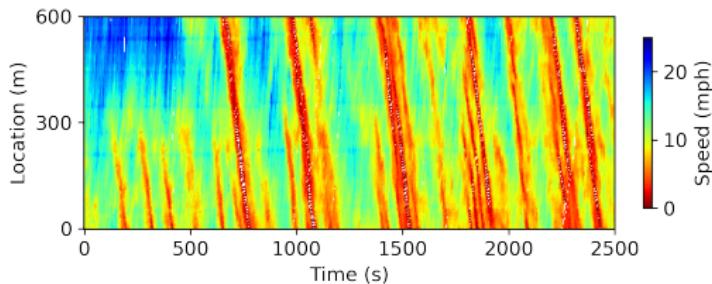


# Experiments

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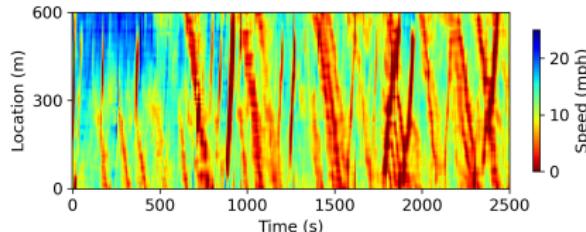


200-by-500 matrix  
(NGSIM)      ↓ Reconstruct speed field from  
                                20% sparse trajectories?

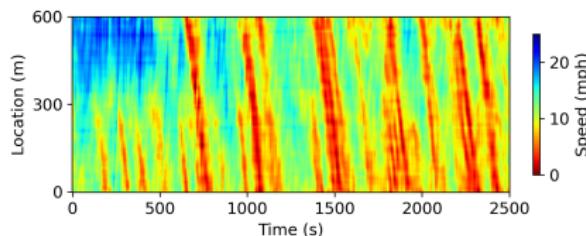


# Experiments

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↓ Plus CircNNM



**TR:**

$$\min_{\mathbf{X}} \mathcal{R}_\tau(\mathbf{X})$$

$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \epsilon$$

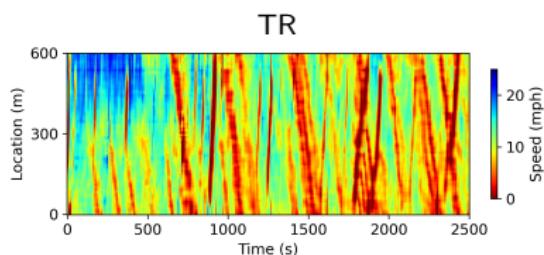
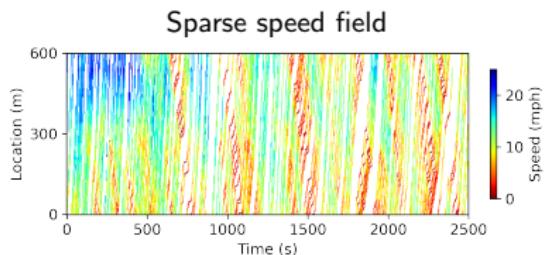
**LCR:**

$$\min_{\mathbf{X}} \|\mathcal{C}(\mathbf{X})\|_* + \gamma \cdot \mathcal{R}_\tau(\mathbf{X})$$

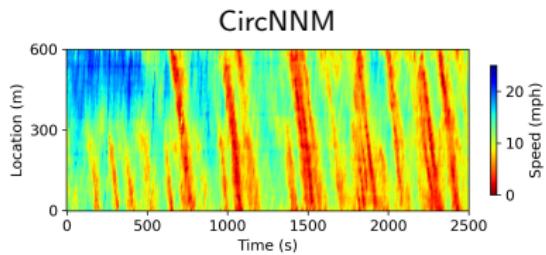
$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{X} - \mathbf{Y})\|_F \leq \epsilon$$

# Experiments

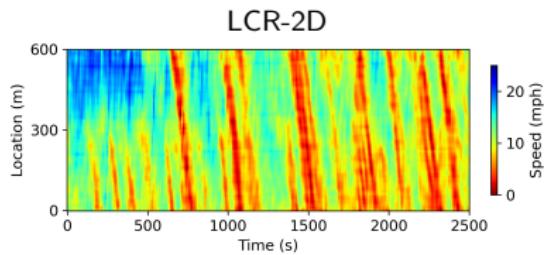
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MAPE = 46.94% & RMSE = 4.34mph



MAPE = 43.51% & RMSE = 1.65mph



MAPE = 41.29% & RMSE = 1.55mph

# Conclusion

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- **(Starting point)** How to impute traffic time series?
  - ✓ Low-rank models   ✓ Temporal regularization
- **(Solution)** Time series trend modeling in the low-rank framework?
  - Global time series trend modeling (low-rank model):
$$\min_{\mathbf{x}} \|\mathcal{C}(\mathbf{x})\|_*$$
$$\text{s. t. } \|\mathcal{P}_\Omega(\mathbf{x} - \mathbf{y})\|_2 \leq \epsilon$$
  - Local time series trend modeling (temporal regularization):
$$\mathcal{R}_\tau(\mathbf{x}) = \frac{1}{2} \|\boldsymbol{\ell} \star \mathbf{x}\|_2^2$$
- **(Highlight)** A unified framework with the **FFT** implementation.

## References

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### A short list:

- [Liu'22] G. Liu (2022). Time series forecasting via learning convolutionally low-rank models. *IEEE Transactions on Information Theory*, 68(5): 3362–3380.
- [Liu & Zhang'23] G. Liu and W. Zhang (2023). Recovery of future data via convolution nuclear norm minimization. *IEEE Transactions on Information Theory*, 69(1): 650–665.



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# Thanks for your attention!

## Any Questions?

### About me:

- 🏠 Homepage: <https://xinychen.github.io>
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- ✉️ How to reach me: [chenxy346@gmail.com](mailto:chenxy346@gmail.com)

