Recitation 5
Neural Networks

10-601: Introduction to Machine Learning
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1 Neural Network Example

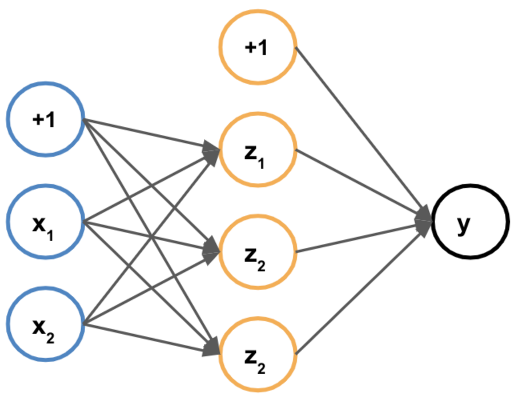

Figure 1: A One Hidden Layer Neural Network

Network Overview Consider the neural network with one hidden layer shown in Figure 1. The input layer consists of 2 features $\mathbf{x} = [x_1, x_2]^T$, the hidden layer has 3 nodes $\mathbf{z} = [z_1, z_2, z_3]^T$, and the output layer is a scalar distribution y. We also add a bias to the input, $x_0 = 1$ and the hidden layer $z_0 = 1$, both of which are fixed to 1.

$\boldsymbol{\alpha}$ is the matrix of weights from the inputs to the hidden layer and $\boldsymbol{\beta}$ is the matrix of weights from the hidden layer to the output layer. $\alpha_{j,i}$ represents the weight going *to* the node z_j in the hidden layer *from* the node x_i in the input layer (e.g. $\alpha_{1,2}$ is the weight from x_2 to z_1), and $\boldsymbol{\beta}$ is defined similarly. We will use a **tanh** activation function for the hidden layer and no activation for the output layer.

Network Details Equivalently, we define each of the following.

The input:

$$\mathbf{x} = [x_0, x_1, x_2]^T \tag{1}$$

Linear combination at the first (hidden) layer:

$$a_j = \sum_{i=0}^{2} \alpha_{j,i} * x_i, \ \forall j \in \{1, \ldots, 3\} \tag{2}$$

Activation at the first (hidden) layer:

$$z_j = tanh(a_j) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \ \forall j \in \{1, \ldots, 3\} \tag{3}$$

Linear combination at the second (output) layer:

$$\hat{y} = \sum_{j=0}^{3} \beta_j \times z_j, \tag{4}$$

Here we fold in the bias term $\alpha_{j,0}$ by thinking of $x_0 = 1$, and fold in β_0 by thinking of $z_0 = 1$.

Loss We will use quadratic loss, $\ell(\hat{y}, y)$:

$$\ell(\hat{y}, y) = \frac{1}{2}(\hat{y} - y)^2 \tag{5}$$

We initialize the network weights as:

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\boldsymbol{\beta} = \begin{bmatrix} 0 & 1 & 2 & 2 \end{bmatrix}$$

For the following questions, we use y = 3.

1. **Scalar Form:** Given $x_1 = 1$, $x_2 = 2$,
   - Forward: What are the values of a_1, ℓ?

- Backward: What are the values of $\frac{d\ell}{d\alpha_{1,1}}$, $\frac{d\ell}{d\beta_1}$ **Hint:** $\frac{d\,\text{tanh}(x)}{d\mathbf{x}} = 1 - tanh(x)^2$

Table 1: $tanh$ values

x	1	2	3	4	5	6	7	8	9
$tanh(x)$	0.76159	0.96403	0.99505	0.99933	0.99991	0.99999	0.99999	0.99999	0.99999

2. **Vector Form:** The vector form of forward computation is:

$$
\begin{aligned}
\mathbf{a} &= \boldsymbol{\alpha}\mathbf{x} \\
\mathbf{z} &= tanh(\mathbf{a}) \\
y &= \boldsymbol{\beta}\mathbf{z}
\end{aligned} \tag{6}
$$

Given $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

- Forward: Find ℓ?

- Backward: What are the values of $\nabla\boldsymbol{\alpha}$, $\nabla\boldsymbol{\beta}$?

Given $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$,

- Forward: Find ℓ?

- Backward: What are the values of $\nabla \boldsymbol{\alpha}$, $\nabla \boldsymbol{\beta}$?