MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 25

Wednesday, December 8

Crossing Number of a Graph, and Plane/Planar Graphs

From last time

Definition 1

Given a (finite) graph G, we define its *crossing number* to be the minimum number of edge crossings that a (permissible) pictorial representation of G can have.

We usually denote this number by $\mathrm{cr}(G)$.

From last time

Definition 1

Given a (finite) graph G, we define its *crossing number* to be the minimum number of edge crossings that a (permissible) pictorial representation of G can have.

We usually denote this number by $\mathrm{cr}(G)$.

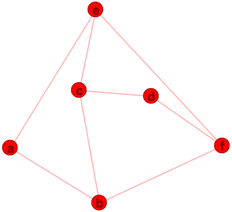
Definition 2

A graph $G = (V, E)$ is called *planar* if we can draw it in the plane without any edge crossings. In other words, if $\mathrm{cr}(G) = 0$.

Any such pictorial representation of G will be called a *planar embedding*, or equivalently a *plane representation*, of G.

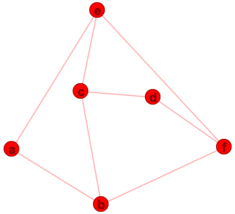
Faces of a planar graph

Consider now a plane representation G_0 of a planar finite graph (note that G_0 for now denotes the specific plane representation we are considering); for instance, G_0 could be



Faces of a planar graph

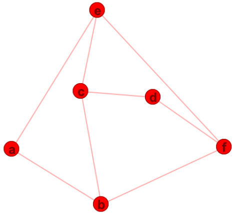
Consider now a plane representation G_0 of a planar finite graph (note that G_0 for now denotes the specific plane representation we are considering); for instance, G_0 could be



Consider also a point p which is NOT one of the vertices of G_0, NOR is it found in the relative interior of any of the edges of G_0. Then $p \in \mathbb{R}^2 \setminus G_0$, which is an open set. Thus we can also find a **maximal** **connected** **open subset** of $\mathbb{R}^2 \setminus G_0$ which contains p.

Faces of a planar graph

Consider now a plane representation G_0 of a planar finite graph (note that G_0 for now denotes the specific plane representation we are considering); for instance, G_0 could be



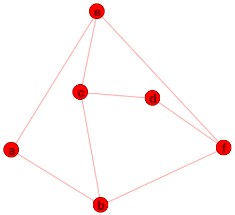
Consider also a point p which is NOT one of the vertices of G_0, NOR is it found in the relative interior of any of the edges of G_0. Then $p \in \mathbb{R}^2 \setminus G_0$, which is an open set. Thus we can also find a **maximal** **connected** **open subset** of $\mathbb{R}^2 \setminus G_0$ which contains p.

Definition 3

If G_0 is a plane representation of a planar finite graph, then the maximal connected open subsets of $\mathbb{R}^2 \setminus G_0$ are called the *faces* of G_0.

Faces of a planar graph

Consider now a plane representation G_0 of a planar finite graph (note that G_0 for now denotes the specific plane representation we are considering); for instance, G_0 could be



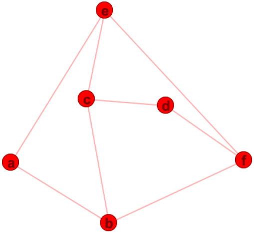
Consider also a point p which is NOT one of the vertices of G_0, NOR is it found in the relative interior of any of the edges of G_0. Then $p \in \mathbb{R}^2 \setminus G_0$, which is an open set. Thus we can also find a **maximal** **connected** **open subset** of $\mathbb{R}^2 \setminus G_0$ which contains p.

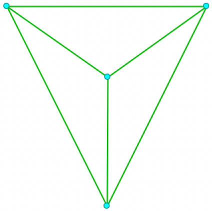
Definition 3

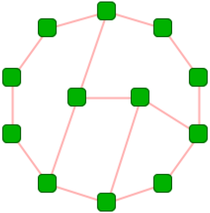
If G_0 is a plane representation of a planar finite graph, then the maximal connected open subsets of $\mathbb{R}^2 \setminus G_0$ are called the *faces* of G_0.

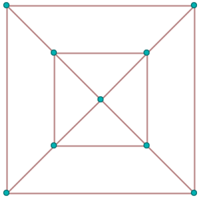
Note that since G_0 is finite, it will have exactly one unbounded face, the area 'outside' of/surrounding the graph/diagram, while all the other faces will be bounded sets.

Testing the definition on examples

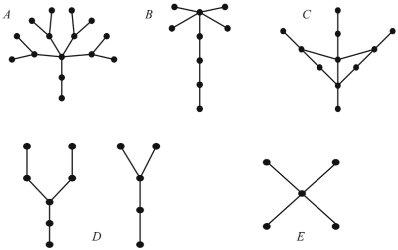




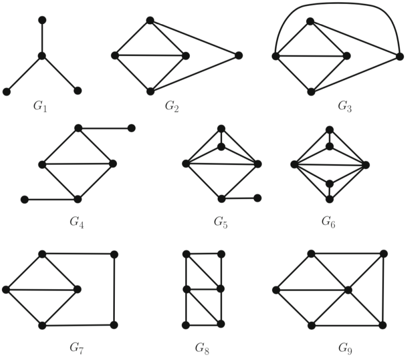




Testing the definition on examples (cont.)



Examples and non-examples
of trees and forests



The forbidden subgraphs
from Beineke's thm (see Lec 8)

Results about the faces

Definition 4

Let G_0 be a plane representation of a planar finite graph, and let f_0 be a face of G_0. We define the *length* $\ell(f_0)$ of the face f_0 to be

the length of a closed walk which bounds f_0 (in other words, which traces the boundary of f_0) and has minimum possible length.

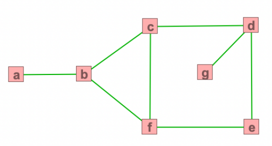
Results about the faces

Definition 4

Let G_0 be a plane representation of a planar finite graph, and let f_0 be a face of G_0. We define the *length* $\ell(f_0)$ of the face f_0 to be

the length of a closed walk which bounds f_0 (in other words, which traces the boundary of f_0) and has minimum possible length.

Example. The plane graph below has 3 faces: the unbounded face f_1, the face f_2 which is bounded by the 3-cycle $b\ c\ f\ b$, and the face f_3 which is contained within the 4-cycle $c\ d\ e\ f\ c$.

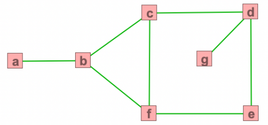


Definition 4

Let G_0 be a plane representation of a planar finite graph, and let f_0 be a face of G_0. We define the *length* $\ell(f_0)$ of the face f_0 to be

the length of a closed walk which bounds f_0 (in other words, which traces the boundary of f_0) and has minimum possible length.

Example. The plane graph below has 3 faces: the unbounded face f_1, the face f_2 which is bounded by the 3-cycle $b\ c\ f\ b$, and the face f_3 which is contained within the 4-cycle $c\ d\ e\ f\ c$.



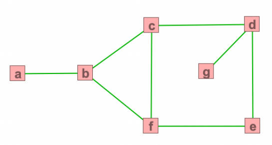
– Note that a closed walk tracing the boundary of f_2 is the 3-cycle $b\ c\ f\ b$ (and this is also best possible), thus f_2 has length 3.

Definition 4

Let G_0 be a plane representation of a planar finite graph, and let f_0 be a face of G_0. We define the *length* $\ell(f_0)$ of the face f_0 to be

the length of a closed walk which bounds f_0 (in other words, which traces the boundary of f_0) and has minimum possible length.

Example. The plane graph below has 3 faces: the unbounded face f_1, the face f_2 which is bounded by the 3-cycle $b\ c\ f\ b$, and the face f_3 which is contained within the 4-cycle $c\ d\ e\ f\ c$.



– Note that a closed walk tracing the boundary of f_2 is the 3-cycle $b\ c\ f\ b$ (and this is also best possible), thus f_2 has length 3.
– On the other hand, a closed walk tracing the entire boundary of f_3 is the walk

$$g\ d\ c\ f\ e\ d\ g.$$

Note also that we cannot find any other closed walks (tracing the boundary of f_3) which will have even shorter length, and thus $\ell(f_3) =$ length of the above walk $= 6$.

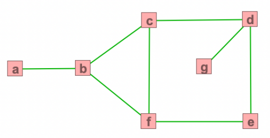
Results about the faces

Definition 4

Let G_0 be a plane representation of a planar finite graph, and let f_0 be a face of G_0. We define the *length* $\ell(f_0)$ of the face f_0 to be

the length of a closed walk which bounds f_0 (in other words, which traces the boundary of f_0) and has minimum possible length.

Example. The plane graph below has 3 faces: the unbounded face f_1, the face f_2 which is bounded by the 3-cycle $b\ c\ f\ b$, and the face f_3 which is contained within the 4-cycle $c\ d\ e\ f\ c$.



– Note that a closed walk tracing the boundary of f_2 is the 3-cycle $b\ c\ f\ b$ (and this is also best possible), thus f_2 has length 3.

– On the other hand, a closed walk tracing the entire boundary of f_3 is the walk

$$g\ d\ c\ f\ e\ d\ g.$$

Note also that we cannot find any other closed walks (tracing the boundary of f_3) which will have even shorter length, and thus $\ell(f_3) =$ length of the above walk $= 6$.

– Finally, similarly we can find that $\ell(f_1) = 7$. *(which closed walk would you use here?)*

Results about the faces

Theorem 1

Let G_0 be a plane representation of a planar finite graph, and let $f_1, f_2, \ldots, f_s$ be the faces of G_0. Then we have that

$$\sum_{i=1}^{s} \ell(f_i) = 2e(G_0).$$

Observe that this result is reminiscent of the Handshaking Lemma (and its proof is also done in the same spirit).

Results about the faces

Theorem 1

Let G_0 be a plane representation of a planar finite graph, and let $f_1, f_2, \ldots, f_s$ be the faces of G_0. Then we have that

$$\sum_{i=1}^{s} \ell(f_i) = 2e(G_0).$$

Observe that this result is reminiscent of the Handshaking Lemma (and its proof is also done in the same spirit).

Theorem 2 (Euler's formula, 1758)

Let G_0 be a plane representation of a connected planar finite graph, and write n for the number of vertices of G_0, e for the number of edges of G_0, and $\tilde{f}$ for the number of faces of G_0.

Results about the faces

Theorem 1

Let G_0 be a plane representation of a planar finite graph, and let $f_1, f_2, \ldots, f_s$ be the faces of G_0. Then we have that

$$\sum_{i=1}^{s} \ell(f_i) = 2e(G_0).$$

Observe that this result is reminiscent of the Handshaking Lemma (and its proof is also done in the same spirit).

Theorem 2 (Euler's formula, 1758)

Let G_0 be a plane representation of a connected planar finite graph, and write n for the number of vertices of G_0, e for the number of edges of G_0, and $\tilde{f}$ for the number of faces of G_0. Then (regardless of what exactly G_0 looks like), we will always have that

$$n - e + \tilde{f} = 2.$$

Proof of Euler's formula

We will be relying on a very famous theorem from set-theoretic topology:

The Jordan curve theorem

A curve $\mathcal{C}$ in $\mathbb{R}^2$ which is **simple** and **closed** divides its complement $\mathbb{R}^2 \setminus \mathcal{C}$ into two maximal connected open subsets: a bounded (or 'interior') region (which is enclosed by $\mathcal{C}$), and an unbounded (or 'exterior') region.

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n - 1$.

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n - 1$.

Base Case: $e = n - 1$.

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n - 1$.

Base Case: $e = n - 1$. Then we have that G_0 is a tree, and thus (as we also verified on examples) it will only have one face, the unbounded one. But then $\tilde{f} = 1$, and $n - e + \tilde{f} = n - (n - 1) + 1 = 2$, as we wanted.

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n - 1$.
Base Case: $e = n - 1$. Then we have that G_0 is a tree, and thus (as we also verified on examples) it will only have one face, the unbounded one. But then $\tilde{f} = 1$, and $n - e + \tilde{f} = n - (n - 1) + 1 = 2$, as we wanted.

Induction Step: Assume now that we have already confirmed Euler's formula for connected plane graphs with n vertices and $\tilde{e}$ edges, where $\tilde{e}$ is some integer $\geqslant n - 1$. Consider then a connected plane graph $\widetilde{G}_0$ with n vertices and $\tilde{e} + 1$ edges.

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n-1$.
Base Case: $e = n-1$. Then we have that G_0 is a tree, and thus (as we also verified on examples) it will only have one face, the unbounded one. But then $\tilde{f} = 1$, and $n - e + \tilde{f} = n - (n-1) + 1 = 2$, as we wanted.

Induction Step: Assume now that we have already confirmed Euler's formula for connected plane graphs with n vertices and $\tilde{e}$ edges, where $\tilde{e}$ is some integer $\geqslant n-1$. Consider then a connected plane graph $\widetilde{G}_0$ with n vertices and $\tilde{e}+1$ edges. Then we definitely have $e(\widetilde{G}_0) > n-1$, and $\widetilde{G}_0$ is not a tree. Thus we can find an edge z_0 of $\widetilde{G}_0$ which is NOT a bridge of $\widetilde{G}_0$ (in other words, z_0 is contained in some cycle of $\widetilde{G}_0$, which also implies that z_0 is part of the boundary of **at least one** <u>bounded</u> face f_0 of $\widetilde{G}_0$).

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n-1$.
Base Case: $e = n-1$. Then we have that G_0 is a tree, and thus (as we also verified on examples) it will only have one face, the unbounded one. But then $\tilde{f} = 1$, and $n - e + \tilde{f} = n - (n-1) + 1 = 2$, as we wanted.

Induction Step: Assume now that we have already confirmed Euler's formula for connected plane graphs with n vertices and $\tilde{e}$ edges, where $\tilde{e}$ is some integer $\geqslant n-1$. Consider then a connected plane graph $\widetilde{G}_0$ with n vertices and $\tilde{e}+1$ edges. Then we definitely have $e(\widetilde{G}_0) > n-1$, and $\widetilde{G}_0$ is not a tree. Thus we can find an edge z_0 of $\widetilde{G}_0$ which is NOT a bridge of $\widetilde{G}_0$ (in other words, z_0 is contained in some cycle of $\widetilde{G}_0$, which also implies that z_0 is part of the boundary of **at least one** <u>bounded</u> face f_0 of $\widetilde{G}_0$). But then $\widetilde{G}_0 - z_0$ is a connected subgraph of $\widetilde{G}_0$ (thus it is also a plane graph), and moreover we have that

– $\widetilde{G}_0 - z_0$ has the same number of vertices as $\widetilde{G}_0$ (in fact, it has the same vertices),

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n-1$.
Base Case: $e = n-1$. Then we have that G_0 is a tree, and thus (as we also verified on examples) it will only have one face, the unbounded one. But then $\tilde{f} = 1$, and $n - e + \tilde{f} = n - (n-1) + 1 = 2$, as we wanted.

Induction Step: Assume now that we have already confirmed Euler's formula for connected plane graphs with n vertices and $\tilde{e}$ edges, where $\tilde{e}$ is some integer $\geqslant n-1$. Consider then a connected plane graph $\widetilde{G}_0$ with n vertices and $\tilde{e}+1$ edges. Then we definitely have $e(\widetilde{G}_0) > n-1$, and $\widetilde{G}_0$ is not a tree. Thus we can find an edge z_0 of $\widetilde{G}_0$ which is NOT a bridge of $\widetilde{G}_0$ (in other words, z_0 is contained in some cycle of $\widetilde{G}_0$, which also implies that z_0 is part of the boundary of **at least one** <u>bounded</u> face f_0 of $\widetilde{G}_0$). But then $\widetilde{G}_0 - z_0$ is a connected subgraph of $\widetilde{G}_0$ (thus it is also a plane graph), and moreover we have that

– $\widetilde{G}_0 - z_0$ has the same number of vertices as $\widetilde{G}_0$ (in fact, it has the same vertices),

– and $e(\widetilde{G}_0 - z_0) = e(\widetilde{G}_0) - 1 = \tilde{e}$.

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n-1$.
Base Case: $e = n-1$. Then we have that G_0 is a tree, and thus (as we also verified on examples) it will only have one face, the unbounded one. But then $\tilde{f} = 1$, and $n - e + \tilde{f} = n - (n-1) + 1 = 2$, as we wanted.

Induction Step: Assume now that we have already confirmed Euler's formula for connected plane graphs with n vertices and $\tilde{e}$ edges, where $\tilde{e}$ is some integer $\geqslant n-1$. Consider then a connected plane graph $\widetilde{G}_0$ with n vertices and $\tilde{e}+1$ edges. Then we definitely have $e(\widetilde{G}_0) > n-1$, and $\widetilde{G}_0$ is not a tree. Thus we can find an edge z_0 of $\widetilde{G}_0$ which is NOT a bridge of $\widetilde{G}_0$ (in other words, z_0 is contained in some cycle of $\widetilde{G}_0$, which also implies that z_0 is part of the boundary of **at least one** bounded face f_0 of $\widetilde{G}_0$). But then $\widetilde{G}_0 - z_0$ is a connected subgraph of $\widetilde{G}_0$ (thus it is also a plane graph), and moreover we have that

– $\widetilde{G}_0 - z_0$ has the same number of vertices as $\widetilde{G}_0$ (in fact, it has the same vertices),

– and $e(\widetilde{G}_0 - z_0) = e(\widetilde{G}_0) - 1 = \tilde{e}$.

– Finally, observe that, by removing edge z_0, we definitely broke the cycle that contained it (in fact, any cycle that contained it), and we have also united two faces of $\widetilde{G}_0$ into one maximal connected open subset now. In other words, we have that

$$\tilde{f}(\widetilde{G}_0 - z_0) = \tilde{f}(\widetilde{G}_0) - 1$$

(note that this exact relation about the faces of the original graph and its connected subgraph can be rigorously justified by the Jordan curve theorem).

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n-1$.
Base Case: $e = n-1$. Then we have that G_0 is a tree, and thus (as we also verified on examples) it will only have one face, the unbounded one. But then $\tilde{f} = 1$, and $n - e + \tilde{f} = n - (n-1) + 1 = 2$, as we wanted.

Induction Step: Assume now that we have already confirmed Euler's formula for connected plane graphs with n vertices and $\tilde{e}$ edges, where $\tilde{e}$ is some integer $\geqslant n-1$. Consider then a connected plane graph $\widetilde{G}_0$ with n vertices and $\tilde{e}+1$ edges. Then we definitely have $e(\widetilde{G}_0) > n-1$, and $\widetilde{G}_0$ is not a tree. Thus we can find an edge z_0 of $\widetilde{G}_0$ which is NOT a bridge of $\widetilde{G}_0$ (in other words, z_0 is contained in some cycle of $\widetilde{G}_0$, which also implies that z_0 is part of the boundary of **at least one** <u>bounded</u> face f_0 of $\widetilde{G}_0$). But then $\widetilde{G}_0 - z_0$ is a connected subgraph of $\widetilde{G}_0$ (thus it is also a plane graph), and moreover we have that

– $\widetilde{G}_0 - z_0$ has the same number of vertices as $\widetilde{G}_0$ (in fact, it has the same vertices),

– and $e(\widetilde{G}_0 - z_0) = e(\widetilde{G}_0) - 1 = \tilde{e}$.

– Finally, observe that, by removing edge z_0, we definitely broke the cycle that contained it (in fact, any cycle that contained it), and we have also united <u>two</u> faces of $\widetilde{G}_0$ into <u>one</u> maximal connected open subset now. In other words, we have that

$$\tilde{f}(\widetilde{G}_0 - z_0) = \tilde{f}(\widetilde{G}_0) - 1$$

(note that this exact relation about the faces of the original graph and its connected subgraph can be rigorously justified by the Jordan curve theorem).

But then, by the Inductive Hypothesis, we can write

$$2 = |\widetilde{G}_0 - z_0| - e(\widetilde{G}_0 - z_0) + \tilde{f}(\widetilde{G}_0 - z_0)$$

Furthermore, we will proceed by induction in the number e of edges. Note that, since G_0 is connected, we must have $e = e(G_0) \geqslant n-1$.

Base Case: $e = n-1$. Then we have that G_0 is a tree, and thus (as we also verified on examples) it will only have one face, the unbounded one. But then $\tilde{f} = 1$, and $n - e + \tilde{f} = n - (n-1) + 1 = 2$, as we wanted.

Induction Step: Assume now that we have already confirmed Euler's formula for connected plane graphs with n vertices and $\tilde{e}$ edges, where $\tilde{e}$ is some integer $\geqslant n-1$. Consider then a connected plane graph $\widetilde{G}_0$ with n vertices and $\tilde{e}+1$ edges. Then we definitely have $e(\widetilde{G}_0) > n-1$, and $\widetilde{G}_0$ is not a tree. Thus we can find an edge z_0 of $\widetilde{G}_0$ which is NOT a bridge of $\widetilde{G}_0$ (in other words, z_0 is contained in some cycle of $\widetilde{G}_0$, which also implies that z_0 is part of the boundary of **at least one** <u>bounded</u> face f_0 of $\widetilde{G}_0$). But then $\widetilde{G}_0 - z_0$ is a connected subgraph of $\widetilde{G}_0$ (thus it is also a plane graph), and moreover we have that

– $\widetilde{G}_0 - z_0$ has the same number of vertices as $\widetilde{G}_0$ (in fact, it has the same vertices),

– and $e(\widetilde{G}_0 - z_0) = e(\widetilde{G}_0) - 1 = \tilde{e}$.

– Finally, observe that, by removing edge z_0, we definitely broke the cycle that contained it (in fact, any cycle that contained it), and we have also united <u>two</u> faces of $\widetilde{G}_0$ into <u>one</u> maximal connected open subset now. In other words, we have that

$$\tilde{f}(\widetilde{G}_0 - z_0) = \tilde{f}(\widetilde{G}_0) - 1$$

(note that this exact relation about the faces of the original graph and its connected subgraph can be rigorously justified by the Jordan curve theorem).

But then, by the Inductive Hypothesis, we can write

$$2 = |\widetilde{G}_0 - z_0| - e(\widetilde{G}_0 - z_0) + \tilde{f}(\widetilde{G}_0 - z_0) = n - \big(e(\widetilde{G}_0) - 1\big) + \big(\tilde{f}(\widetilde{G}_0) - 1\big) = n - e(\widetilde{G}_0) + \tilde{f}(\widetilde{G}_0),$$

as we wanted.

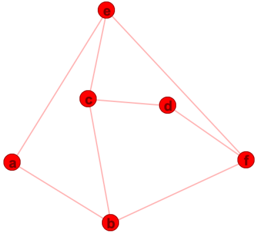
One immediate and very useful consequence of Euler's formula

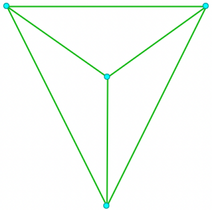
Important corollary of Euler's formula

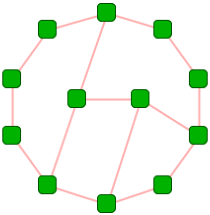
Let $G = (V, E)$ be a (connected) finite planar graph.

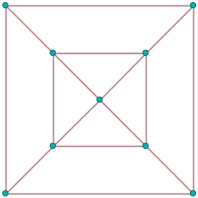
Then the number of faces of a planar embedding of G does NOT depend on the particular embedding we will consider. In other words, we can talk about the faces of G even before drawing G in the plane in a way that results in no edge crossings.

Testing Euler's formula on our examples

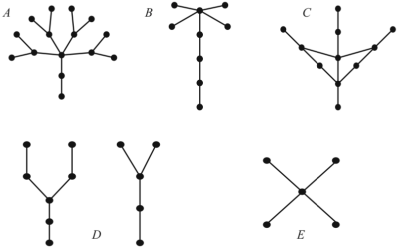


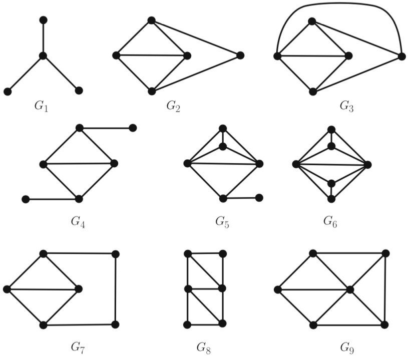






Testing Euler's formula on our examples (cont.)





Other important consequences of Euler's formula

Theorem 3

It G is a planar graph with at least 3 vertices, then

$$e(G) \leqslant 3|G| - 6.$$

Other important consequences of Euler's formula

Theorem 3

It G is a planar graph with at least 3 vertices, then

$$e(G) \leqslant 3|G| - 6.$$

Moreover, if G is also **triangle-free**, then

$$e(G) \leqslant 2|G| - 4.$$

Other important consequences of Euler's formula

Theorem 3

It G is a planar graph with at least 3 vertices, then

$$e(G) \leqslant 3|G| - 6.$$

Moreover, if G is also **triangle-free**, then

$$e(G) \leqslant 2|G| - 4.$$

Justification. It suffices to consider connected graphs (because otherwise we could just add some edges (without introducing any edge crossings, or any new cycles) in order to get a connected supergraph $\widetilde{G}$; but then we would have $|\widetilde{G}| = |G|$, while $e(G) \leqslant e(\widetilde{G})$, and thus if we could establish the inequalities for $\widetilde{G}$, then the corresponding inequalities for G would also hold).

Other important consequences of Euler's formula

Theorem 3

It G is a planar graph with at least 3 vertices, then

$$e(G) \leqslant 3|G| - 6.$$

Moreover, if G is also **triangle-free**, then

$$e(G) \leqslant 2|G| - 4.$$

Justification. It suffices to consider connected graphs (because otherwise we could just add some edges (without introducing any edge crossings, or any new cycles) in order to get a connected supergraph $\widetilde{G}$; but then we would have $|\widetilde{G}| = |G|$, while $e(G) \leqslant e(\widetilde{G})$, and thus if we could establish the inequalities for $\widetilde{G}$, then the corresponding inequalities for G would also hold).

Moreover, if we have any vertex in G which is a *leaf* (that is, it has degree 1), then by removing it we only decrease the quantities $3|G| - e(G) - 6$ or $2|G| - e(G) - 4$ (now calculated for the resulting subgraph we get).

Other important consequences of Euler's formula

Theorem 3

It G is a planar graph with at least 3 vertices, then

$$e(G) \leqslant 3|G| - 6.$$

Moreover, if G is also **triangle-free**, then

$$e(G) \leqslant 2|G| - 4.$$

Justification. It suffices to consider connected graphs (because otherwise we could just add some edges (without introducing any edge crossings, or any new cycles) in order to get a connected supergraph $\widetilde{G}$; but then we would have $|\widetilde{G}| = |G|$, while $e(G) \leqslant e(\widetilde{G})$, and thus if we could establish the inequalities for $\widetilde{G}$, then the corresponding inequalities for G would also hold).

Moreover, if we have any vertex in G which is a *leaf* (that is, it has degree 1), then by removing it we only decrease the quantities $3|G| - e(G) - 6$ or $2|G| - e(G) - 4$ (now calculated for the resulting subgraph we get). Thus, if we could still show that these quantities are $\geqslant 0$, then the original inequalities we want would also hold.

Other important consequences of Euler's formula

Theorem 3

It G is a planar graph with at least 3 vertices, then

$$e(G) \leqslant 3|G| - 6.$$

Moreover, if G is also **triangle-free**, then

$$e(G) \leqslant 2|G| - 4.$$

Justification. It suffices to consider connected graphs (because otherwise we could just add some edges (without introducing any edge crossings, or any new cycles) in order to get a connected supergraph $\widetilde{G}$; but then we would have $|\widetilde{G}| = |G|$, while $e(G) \leqslant e(\widetilde{G})$, and thus if we could establish the inequalities for $\widetilde{G}$, then the corresponding inequalities for G would also hold).

Moreover, if we have any vertex in G which is a *leaf* (that is, it has degree 1), then by removing it we only decrease the quantities $3|G| - e(G) - 6$ or $2|G| - e(G) - 4$ (now calculated for the resulting subgraph we get). Thus, if we could still show that these quantities are $\geqslant 0$, then the original inequalities we want would also hold.

Important Note. If we started with a graph G which has exactly 3 vertices, then we shouldn't be able to remove any of those vertices *(attention to our assumptions!)*. But for graphs with exactly 3 vertices, we can verify the desired inequalities simply by inspection.

Based on the above, it suffices to verify the desired inequalities

for an arbitrary connected planar graph G
which has at least 3 vertices, and which has NO leaves.

Based on the above, it suffices to verify the desired inequalities

for an arbitrary connected planar graph G
which has at least 3 vertices, and which has NO leaves.

– Since we are now assuming G is connected, it will have more than one face (thus it will have at least one bounded face).

Based on the above, it suffices to verify the desired inequalities

for an arbitrary connected planar graph G
which has at least 3 vertices, and which has NO leaves.

– Since we are now assuming G is connected, it will have more than one face (thus it will have at least one bounded face).
– Moreover, the boundary of each bounded face will contain at least 3 edges, and thus the same will be true for the unbounded face.

Based on the above, it suffices to verify the desired inequalities

for an arbitrary connected planar graph G
which has at least 3 vertices, and which has NO leaves.

– Since we are now assuming G is connected, it will have more than one face (thus it will have at least one bounded face).
– Moreover, the boundary of each bounded face will contain at least 3 edges, and thus the same will be true for the unbounded face. But then the length of each face is also at least 3.

Based on the above, it suffices to verify the desired inequalities

for an arbitrary connected planar graph G
which has at least 3 vertices, and which has NO leaves.

– Since we are now assuming G is connected, it will have more than one face (thus it will have at least one bounded face).
– Moreover, the boundary of each bounded face will contain at least 3 edges, and thus the same will be true for the unbounded face. But then the length of each face is also at least 3.

Let $f_1, f_2, \ldots, f_s$ be the faces of G. Then, by Theorem 1 we have that

$$2e(G) = \sum_{i=1}^{s} \ell(f_i) \geqslant \sum_{i=1}^{s} 3 = 3\tilde{f}(G) \tag{1}$$

$$\Rightarrow \quad \tilde{f}(G) \leqslant \frac{2}{3} e(G).$$

Based on the above, it suffices to verify the desired inequalities

for an arbitrary connected planar graph G
which has at least 3 vertices, and which has NO leaves.

– Since we are now assuming G is connected, it will have more than one face (thus it will have at least one bounded face).
– Moreover, the boundary of each bounded face will contain at least 3 edges, and thus the same will be true for the unbounded face. But then the length of each face is also at least 3.

Let $f_1, f_2, \ldots, f_s$ be the faces of G. Then, by Theorem 1 we have that

$$2e(G) = \sum_{i=1}^{s} \ell(f_i) \geqslant \sum_{i=1}^{s} 3 = 3\tilde{f}(G) \tag{1}$$

$$\Rightarrow \quad \tilde{f}(G) \leqslant \frac{2}{3}e(G).$$

But then, combining the last inequality with Euler's formula, we get

$$|G| - e(G) + \frac{2}{3}e(G) \geqslant |G| - e(G) + \tilde{f}(G) = 2,$$

which implies the 1st inequality in Theorem 3.

Based on the above, it suffices to verify the desired inequalities

for an arbitrary connected planar graph G
which has at least 3 vertices, and which has NO leaves.

– Since we are now assuming G is connected, it will have more than one face (thus it will have at least one bounded face).
– Moreover, the boundary of each bounded face will contain at least 3 edges, and thus the same will be true for the unbounded face. But then the length of each face is also at least 3.

Let $f_1, f_2, \ldots, f_s$ be the faces of G. Then, by Theorem 1 we have that

$$2e(G) = \sum_{i=1}^{s} \ell(f_i) \geqslant \sum_{i=1}^{s} 3 = 3\tilde{f}(G) \tag{1}$$

$$\Rightarrow \quad \tilde{f}(G) \leqslant \frac{2}{3}e(G).$$

But then, combining the last inequality with Euler's formula, we get

$$|G| - e(G) + \frac{2}{3}e(G) \geqslant |G| - e(G) + \tilde{f}(G) = 2,$$

which implies the 1st inequality in Theorem 3.

Finally, if G is also triangle-free, then each face will instead have length at least 4, and thus we will be able to replace (1) by the stronger

$$2e(G) = \sum_{i=1}^{s} \ell(f_i) \geqslant \sum_{i=1}^{s} 4 = 4\tilde{f}(G).$$

Based on the above, it suffices to verify the desired inequalities

for an arbitrary connected planar graph G
which has at least 3 vertices, and which has NO leaves.

– Since we are now assuming G is connected, it will have more than one face (thus it will have at least one bounded face).
– Moreover, the boundary of each bounded face will contain at least 3 edges, and thus the same will be true for the unbounded face. But then the length of each face is also at least 3.

Let $f_1, f_2, \ldots, f_s$ be the faces of G. Then, by Theorem 1 we have that

$$2e(G) = \sum_{i=1}^{s} \ell(f_i) \geqslant \sum_{i=1}^{s} 3 = 3\tilde{f}(G) \tag{1}$$

$$\Rightarrow \quad \tilde{f}(G) \leqslant \frac{2}{3} e(G).$$

But then, combining the last inequality with Euler's formula, we get

$$|G| - e(G) + \frac{2}{3} e(G) \geqslant |G| - e(G) + \tilde{f}(G) = 2,$$

which implies the 1st inequality in Theorem 3.

Finally, if G is also triangle-free, then each face will instead have length at least 4, and thus we will be able to replace (1) by the stronger

$$2e(G) = \sum_{i=1}^{s} \ell(f_i) \geqslant \sum_{i=1}^{s} 4 = 4\tilde{f}(G).$$

Then, continuing as before, we will obtain the 2nd inequality of the theorem.

Immediate Corollaries of Thm 3

Corollary 1

If G is a planar (not-necessarily complete) bipartite graph with order $|G| = n \geqslant 3$, then G has at most $2n - 4$ edges.

Immediate Corollaries of Thm 3

Corollary 1

If G is a planar (not-necessarily complete) bipartite graph with order $|G| = n \geqslant 3$, then G has at most $2n - 4$ edges.

Corollary 2

Neither of the graphs $K_{3,3}$ and K_5 is planar.

Immediate Corollaries of Thm 3

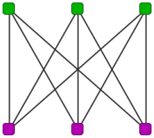
Corollary 1

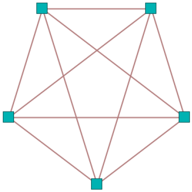
If G is a planar (not-necessarily complete) bipartite graph with order $|G| = n \geqslant 3$, then G has at most $2n - 4$ edges.

Corollary 2

Neither of the graphs $K_{3,3}$ and K_5 is planar.

In other words, $\mathrm{cr}(K_{3,3}) \geqslant 1$, and similarly $\mathrm{cr}(K_5) \geqslant 1$.





These examples of non-planar graphs are the most characteristic ones

Reminder from last time: Definition of the graph operation of *'subdivision'*

Let $G = (V, E)$ be a graph. A ***subdivision*** of G is a graph H which satisfies the following:

- H contains all the vertices of G, and perhaps a few more vertices;
- every edge $e = \{v_1, v_2\}$ in G
  - either appears as an edge of H as well,

These examples of non-planar graphs are the most characteristic ones

Reminder from last time: Definition of the graph operation of *'subdivision'*

Let $G = (V, E)$ be a graph. A ***subdivision*** of G is a graph H which satisfies the following:

- H contains all the vertices of G, and perhaps a few more vertices;
- every edge $e = \{v_1, v_2\}$ in G
  - either appears as an edge of H as well,
  - or is replaced in H by the path $v_1\, w_1\, w_2\, \ldots\, w_{s-1}\, w_s\, v_2$, where $w_1\, (= w_1(v_1, v_2)), w_2\, (= w_2(v_1, v_2)), \ldots,\ w_s\, (= w_s(v_1, v_2))$ are new vertices that we add to the vertex set of H **(and are only meant as *internal* vertices of this new path in H, that is, they will only be used in relation to vertices v_1 and v_2, to 'elongate' the previous edge $\{v_1, v_2\}$).**

These examples of non-planar graphs are the most characteristic ones

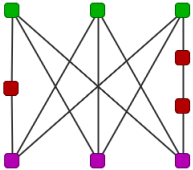
Reminder from last time: Definition of the graph operation of *'subdivision'*

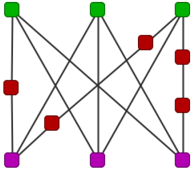
Let $G = (V, E)$ be a graph. A ***subdivision*** of G is a graph H which satisfies the following:

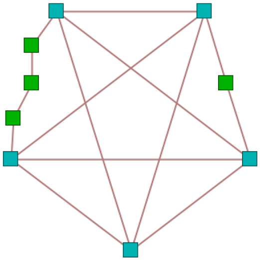
- H contains all the vertices of G, and perhaps a few more vertices;
- every edge $e = \{v_1, v_2\}$ in G
  - either appears as an edge of H as well,
  - or is replaced in H by the path $v_1\, w_1\, w_2\, \ldots\, w_{s-1}\, w_s\, v_2$, where $w_1\,(= w_1(v_1, v_2)), w_2\,(= w_2(v_1, v_2)), \ldots, w_s\,(= w_s(v_1, v_2))$ are new vertices that we add to the vertex set of H **(and are only meant as *internal* vertices of this new path in H, that is, they will only be used in relation to vertices v_1 and v_2, to 'elongate' the previous edge $\{v_1, v_2\}$).**

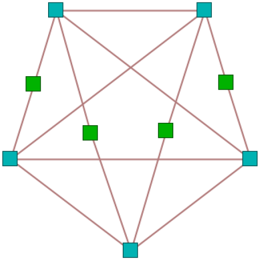
    In other words, in the second case, $e = \{v_1, v_2\}$ is replaced by the edges $\{v_1, w_1\}$, $\{w_1, w_2\}$, $\ldots$, $\{w_{s-1}, w_s\}$ and $\{w_s, v_2\}$.

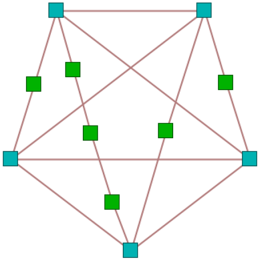
Some subdivisions of $K_{3,3}$ and of K_5











Some immediate remarks about subdivisions?

Kuratowski's theorem on planar graphs

Theorem (Kuratowski, 1930)

A finite graph G is planar **if and only if** none of its subgraphs is a subdivision of $K_{3,3}$ or of K_5.

Kuratowski's theorem on planar graphs

Theorem (Kuratowski, 1930)

A finite graph G is planar **if and only if** none of its subgraphs is a subdivision of $K_{3,3}$ or of K_5.

Terminology

If G contains a subgraph H which is a subdivision of $K_{3,3}$ or of K_5, then H is called a *Kuratowski subgraph* of G.

Kuratowski's theorem on planar graphs

Theorem (Kuratowski, 1930)

A finite graph G is planar **if and only if** none of its subgraphs is a subdivision of $K_{3,3}$ or of K_5.

Terminology

If G contains a subgraph H which is a subdivision of $K_{3,3}$ or of K_5, then H is called a *Kuratowski subgraph* of G.

With this terminology at hand, Kuratowski's theorem can be more simply stated as:

a finite graph G is planar if and only if
G does not have any Kuratowski subgraphs.