# MATH 322 – Graph Theory
# Fall Term 2021

**Notes for Lecture 10**

Thursday, October 7

# Important results from last time

## Proposition 1

Let $G$ be a connected graph of order $\geqslant 3$.

An edge $e$ in $G$ is NOT a bridge of $G$ **if and only if** $e$ belongs to some cycle contained in $G$.

**Remark.** Equivalently we could state the proposition as follows: "Let $H$ be a connected graph of order $\geqslant 2$. Then $e$ is a bridge of $H$ **if and only if** $e$ does NOT belong to <u>any</u> cycle contained in $H$."

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**Proposition 2**

Let $G$ be a connected graph of order $\geqslant 2$, and let $v$ be a vertex of $G$ such that $\deg(v) = 1$.

Then $v$ cannot be a cutvertex.

# Important results from last time (cont.)

## Proposition 3

Let $T$ be a tree on at least two vertices. Then, for every two different vertices $u, v$ of $T$, there is a **unique** path in $T$ starting at $u$ and ending at $v$.

# Important results from last time (cont.)

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## Proposition 4

Let $T$ be a tree **of order $\geqslant 3$**. Then $T$ has vertices of degree $\geqslant 2$, and every such vertex is a cutvertex of $T$.

# Proof of Proposition 4 (1st half from last time)

Consider a tree $T$ of order $\geqslant 3$. First we need to check that $T$ has at least one vertex of degree $\geqslant 2$.

Write $n$ for the order of $T$, that is, for the number of vertices in $T$. By Theorem 1a from Lecture 7, we know that $e(T) = n - 1$.

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Assume towards a contradiction that $T$ contained only leaves, that is, vertices of degree 1.

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Then we would have

$$\sum_{v \in V(T)} \deg(v) = \sum_{v \in V(T)} 1 = |T| = n < 2(n-1) = 2e(T),$$

where the (strict) inequality holds because $n \geqslant 3$. This contradicts the Handshaking Lemma.

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Thus $T$ must contain at least one vertex of degree $\geqslant 2$.

# Proof of Proposition 4 (cont.)

Consider now a vertex $v_0$ of $T$ which has degree $\geqslant 2$. We will show that $T - v_0$ is a disconnected subgraph.

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Indeed, let $u_1, w_1$ be two different neighbours of $v_0$ in $T$. Then the path $P_0 :\ u_1\, v_0\, w_1$ is a path of length 2 contained in $T$ which starts at $u_1$ and ends at $w_1$.

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*Recall now Proposition 3 that we proved last time (and that we just restated):* it implies that there is NO OTHER path in $T$ which starts at $u_1$ and ends at $w_1$.

In other words, we cannot find a path $P_1$ in $T - v_0$ which starts at $u_1$ and ends at $w_1$ *(because such a path would also be a path in $T$, and it would be different from the path $P_0$).*

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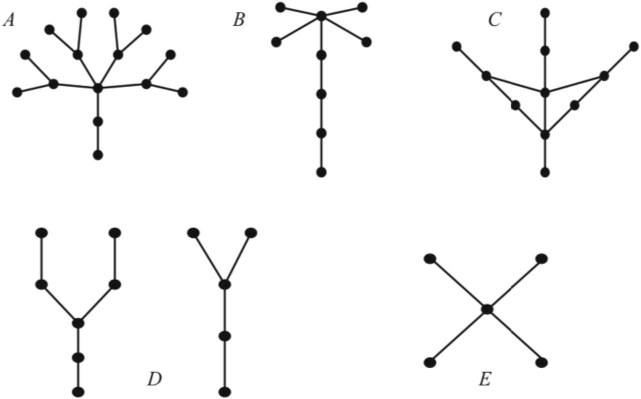
We conclude that each of the neighbours of $v_0$ in $T$ will end up in a different connected component of $T - v_0$.

# Checking these on examples




**Question.** In each of the graphs above, which of the vertices are cutvertices, and which are not?

## Reminder: Whitney's Theorem tested on 'extreme' examples

- In the case of the complete graph $K_n$ ($n \geqslant 2$) we have seen that:
  - $\kappa(K_n) = n - 1$ by definition,
  - $\lambda(K_n) = n - 1$ by the analysis we did last time,
  - and clearly $\delta(K_n) = n - 1$.

- In the case of a tree $T_n$ on $n$ vertices (where $n \geqslant 3$, because a tree on 2 vertices would just be $K_2$ and we have already handled this example), we have so far seen:

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  - that $T_n$ must have leaves, and therefore $\delta(T_n) = 1$,
  - and that every edge is a bridge, and thus $\lambda(T_n) = 1$.

  - Finally, such a tree will have vertices of degree $\geqslant 2$, and each such vertex will be a cutvertex. Thus we will have $\kappa(T_n) = 1$.

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We have already proved (see Lecture 8) part of the theorem, that is, that $\lambda(G) \leqslant \delta(G)$.

Before we justify the rest of the theorem's conclusion, we are going to take a detour, and gather a few more characterisations for trees which can be justified using the most recent results.

# Characterisations of trees

**Reminder: Theorem 1a from Lecture 7**

Let $T$ be a graph of order $n$. The following two statements are equivalent.

(i) $T$ is a tree (that is, $T$ is acyclic and connected).

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(v') For every two different vertices $u, v$ of $T$, there is a **unique** path in $T$ starting at $u$ and ending at $v$.

(vi') $T$ is acyclic, and adding to it a single new edge creates precisely one cycle (regardless of where this edge is added).

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$(i) \Rightarrow (iv')$ Since $T$ is a tree on $n$ vertices, we know that it has $n-1$ edges. Moreover, $T$ contains no cycles, and thus none of its edges belongs to a cycle. By Prop 1 from last time, this implies that each of the edges of $T$ is a bridge.

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$(iv') \Rightarrow (iii')$ Already by our assumptions we know that $T$ has $n-1$ edges. Moreover, we know that every edge of $T$ is a bridge. Note now that, if $T$ contained cycles, then all the edges belonging to any of these cycles would NOT be bridges of $T$. Thus the latter assumption about $T$ implies that $T$ is acyclic.

# Proof of Theorem 2a (cont.)

$(iii') \Rightarrow (ii')$ Again, because $T$ is assumed acyclic, we get that every edge of $T$ is a bridge. Moreover, $T$ is either a tree or a forest. In the former case, we immediately get that $T$ is connected.

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Assume that we could have a forest $T$ satisfying the given assumptions. In other words, assume that $T$ is an acyclic graph with at least two connected components, total number of vertices $n$ and total number of edges $n-1$. Write $T_1, T_2, \ldots, T_k$ for the connected components of $T$ (where $k \geqslant 2$).

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Since $T$ is a forest, each of its connected components $T_i$ is a tree, and thus we know that $e(T_i) = |T_i| - 1$ (that is, the size of $T_i$ is equal to the order of $T_i$ minus 1).

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which contradicts the assumption that $k > 1$.

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$(ii') \Rightarrow (i)$ Since we assumed that every edge of $T$ is a bridge, $T$ must be acyclic. Moreover, $T$ is connected by assumption. Thus $T$ is a tree.

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First of all, this implies that $T$ has no cycles (because, if we had a cycle, then for every pair of different vertices $u_1, v_1$ on the cycle, there would be two $u_1 - v_1$ paths contained in the cycle itself).

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By our assumption, there was already a $u_2 - v_2$ path $P_0$ in $T$. Note that this path has length $\geqslant 2$ (since $u_2$ and $v_2$ are not joined in $T$), and thus, with the addition of the new edge, we now have a cycle $C_0$ in $\widetilde{T}$ which contains the vertices $u_2$, $v_2$.

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It remains to show that $C_0$ is the only cycle of $\widetilde{T}$.

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Consider now two different vertices $u_2, v_2$ in $T$ which are not already joined by an edge in $T$ *(clearly this excludes the cases where $T = K_1$ or $T = K_2$, for which statement $(vi')$ is true in a vacuous manner, since we simply cannot add any (more) edges),* and add the edge $e_2 = \{u_2, v_2\}$ to get a supergraph $\widetilde{T}$ of $T$.

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It remains to show that $C_0$ is the only cycle of $\widetilde{T}$.

- Clearly we can't have a cycle in $\widetilde{T}$ which does not contain the edge $e_2$, because then this would also be a cycle in the original graph $T$.
- If $C_1$ were another cycle in $\widetilde{T}$ which contained the edge $e_2$, and, say, we could write $C_1 : u_2\, z_1\, z_2 \cdots z_{k-1}\, v_2\, \boldsymbol{u_2}$, then the path $P_1 : u_2\, z_1\, z_2 \cdots z_{k-1}\, v_2$ should be a $u_2 - v_2$ path in $T$ different from $P_0$. This contradicts our assumption about the uniqueness of $P_0$.

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We already have that $T$ is acyclic. Let's assume that $T$ is NOT connected. Then $T$ has at least two connected components, say components $T_1$ and $T_2$.

Consider a vertex $u$ in $T_1$ and a vertex $v$ in $T_2$, and add the edge $e_0 = \{u, v\}$.

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But then this shows that there is a $u - v$ path in the original graph $T$ (since we can 'travel' from $u$ to $v$ on the cycle without traversing the edge $e_0$), which contradicts the assumption that $u$ and $v$ belong to different connected components of the original graph $T$.

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We conclude that $T$ must be connected, and thus that $T$ is a tree.

# Recap of the characterisations for trees

**Theorem 1a from Lecture 7**

Let $T$ be a graph of order $n$. The following two statements are equivalent.

(i) $T$ is a tree (that is, $T$ is acyclic and connected).

(ii) $T$ is connected and has precisely $n-1$ edges.

**Theorem 2a**

Let $T$ be a graph of order $n$. The following statements are equivalent.

(i) $T$ is a tree.

(ii') $T$ is connected and every edge is a bridge.

(iii') $T$ is acyclic and has precisely $n-1$ edges.

(iv') $T$ has $n-1$ edges and every edge is a bridge.

(v') For every two different vertices $u, v$ of $T$, there is a **unique** path in $T$ starting at $u$ and ending at $v$.

(vi') $T$ is acyclic, and adding to it a single new edge creates precisely one cycle (regardless of where this edge is added).

# Back to Whitney's theorem

**Theorem 1 (Whitney, 1932)**

For every connected graph $G$ (of order $\geqslant 2$), we have

$$\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)\,.$$

We have proved so far (see Lecture 8) that $\lambda(G) \leqslant \delta(G)$.

## Proving that $\kappa(G) \leqslant \lambda(G)$

Consider a connected graph $G$ on at least 2 vertices, and let $n$ be the order of $G$ (as we just said, $n \geqslant 2$).

Consider also an edge cut $\{e_1, e_2, \ldots, e_{t_0}\}$ of $G$ such that $t_0 = \lambda(G)$. In other words, $G - \{e_1, e_2, \ldots, e_{t_0}\}$ is a disconnected subgraph, while $t_0$, the cardinality of this edge cut, is as small as possible. By what we have already showed, $t_0 \leqslant \delta(G) \leqslant n - 1$.

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<u>'Easier' case</u>: $t_0 = \lambda(G) = n - 1$. Then, we must have $\delta(G) = n - 1$ too, and thus $G = K_n$. But then, according to our definitions, $\kappa(G) = n - 1$ too, and the inequality $\kappa(G) \leqslant \lambda(G)$ holds (in fact, we have equality).

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Consider also an edge cut $\{e_1, e_2, \ldots, e_{t_0}\}$ of $G$ such that $t_0 = \lambda(G)$. In other words, $G - \{e_1, e_2, \ldots, e_{t_0}\}$ is a disconnected subgraph, while $t_0$, the cardinality of this edge cut, is as small as possible. By what we have already showed, $t_0 \leqslant \delta(G) \leqslant n - 1$.

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Remaining cases: Assume now that $t_0 \leqslant n - 2$. Since $G - \{e_1, e_2, \ldots, e_{t_0}\}$ is a disconnected subgraph, we can find vertices $u_1, v_1$ of $G$ which fall into different connected components of $G - \{e_1, e_2, \ldots, e_{t_0}\}$.

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Remaining cases: Assume now that $t_0 \leqslant n-2$. Since $G - \{e_1, e_2, \ldots, e_{t_0}\}$ is a disconnected subgraph, we can find vertices $u_1, v_1$ of $G$ which fall into different connected components of $G - \{e_1, e_2, \ldots, e_{t_0}\}$.

We consider two subcases:
Subcase 1: $u_1, v_1$ are NOT joined by an edge in $G$. Then any path in $G$ connecting $u_1$ and $v_1$ must have length $\geqslant 2$ (or in other words, it must contain intermediate vertices too).

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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Subcase 1: $u_1, v_1$ are NOT joined by an edge in $G$. Then any path in $G$ connecting $u_1$ and $v_1$ must have length $\geqslant 2$ (or in other words, it must contain intermediate vertices too).
**Moreover, every path in $G$ connecting $u_1$ and $v_1$ must traverse one or more of the edges $e_1, e_2, \ldots, e_{t_0}$.**

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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**Moreover, every path in $G$ connecting $u_1$ and $v_1$ must traverse one or more of the edges $e_1, e_2, \ldots, e_{t_0}$.** *(Indeed, if we could also find a $u_1 - v_1$ path in $G$ which did not contain any of the edges $e_1, e_2, \ldots, e_{t_0}$, then this would be a path in the subgraph $G - \{e_1, e_2, \ldots, e_{t_0}\}$ too, and thus $u_1$, $v_1$ would not belong to different connected components of the subgraph.)*

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Remaining cases: Assume now that $t_0 \leqslant n-2$. Since $G - \{e_1, e_2, \ldots, e_{t_0}\}$ is a disconnected subgraph, we can find vertices $u_1, v_1$ of $G$ which fall into different connected components of $G - \{e_1, e_2, \ldots, e_{t_0}\}$.

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**Moreover, every path in $G$ connecting $u_1$ and $v_1$ must traverse one or more of the edges $e_1, e_2, \ldots, e_{t_0}$.** *(Indeed, if we could also find a $u_1 - v_1$ path in $G$ which did not contain any of the edges $e_1, e_2, \ldots, e_{t_0}$, then this would be a path in the subgraph $G - \{e_1, e_2, \ldots, e_{t_0}\}$ too, and thus $u_1$, $v_1$ would not belong to different connected components of the subgraph.)*

Now, for each of the edges $e_i$, pick one of its endvertices **which is different from both $u_1$ and $v_1$** (this is always possible here because the edge $\{u_1, v_1\}$ does not exist in $G$), and denote it by $w_i$.

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Remaining cases: Assume now that $t_0 \leqslant n-2$. Since $G - \{e_1, e_2, \ldots, e_{t_0}\}$ is a disconnected subgraph, we can find vertices $u_1, v_1$ of $G$ which fall into different connected components of $G - \{e_1, e_2, \ldots, e_{t_0}\}$.

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**Moreover, every path in $G$ connecting $u_1$ and $v_1$ must traverse one or more of the edges $e_1, e_2, \ldots, e_{t_0}$.** *(Indeed, if we could also find a $u_1 - v_1$ path in $G$ which did not contain any of the edges $e_1, e_2, \ldots, e_{t_0}$, then this would be a path in the subgraph $G - \{e_1, e_2, \ldots, e_{t_0}\}$ too, and thus $u_1$, $v_1$ would not belong to different connected components of the subgraph.)*

Now, for each of the edges $e_i$, pick one of its endvertices **which is different from both $u_1$ and $v_1$** (this is always possible here because the edge $\{u_1, v_1\}$ does not exist in $G$), and denote it by $w_i$. Then the vertex set $\{w_1, w_2, \ldots, w_{t_0}\}$ is a vertex cut of $G$, since

$$G - \{w_1, w_2, \ldots, w_{t_0}\} \subseteq G - \{e_1, e_2, \ldots, e_{t_0}\},$$

and thus the vertices $u_1, v_1$, which are still left in $G - \{w_1, w_2, \ldots, w_{t_0}\}$, will again be separated.

*One more side note here: $\{w_1, w_2, \ldots, w_{t_0}\}$ contains **at most** $t_0$ vertices (because some vertex here might be a common endvertex of two or more of the edges $e_1, e_2, \ldots, e_{t_0}$, and might have been picked more than once).*

This shows that $\kappa(G) \leqslant t_0 = \lambda(G)$ in this subcase.

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Subcase 2: $u_1, v_1$ are joined by an edge in $G$. Then, because $u_1, v_1$ are separated in $G - \{e_1, e_2, \ldots, e_{t_0}\}$, one of the edges that we remove must be the edge $\{u_1, v_1\}$.

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Subcase 2: $u_1, v_1$ are joined by an edge in $G$. Then, because $u_1, v_1$ are separated in $G - \{e_1, e_2, \ldots, e_{t_0}\}$, one of the edges that we remove must be the edge $\{u_1, v_1\}$. Say without loss of generality that $\{u_1, v_1\} = e_{t_0}$.

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Then, as before, for each of the edges $e_i$ with $i \neq t_0$, pick one of its endvertices **which is different from both $u_1$ and $v_1$**, and denote it by $w_i$.

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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Then, as before, for each of the edges $e_i$ with $i \neq t_0$, pick one of its endvertices **which is different from both $\boldsymbol{u_1}$ and $\boldsymbol{v_1}$**, and denote it by $w_i$.

Note now that

$$\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0} \quad \subseteq \quad G - \{e_1, e_2, \ldots, e_{t_0}\},$$

and thus $u_1, v_1$ are separated in the smaller subgraph too.

Note also that the subgraph $\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0}$ contains **at least** $n - (t_0 - 1) = n - t_0 + 1 \geqslant n + 1 - (n - 2) = 3$ vertices (where we're using the initial assumption that $t_0 \leqslant n - 2$). Thus it contains at least one more vertex $z_0$ different from $u_1$ and $v_1$.

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Subcase 2: $u_1, v_1$ are joined by an edge in $G$. Then, because $u_1, v_1$ are separated in $G - \{e_1, e_2, \ldots, e_{t_0}\}$, one of the edges that we remove must be the edge $\{u_1, v_1\}$. Say without loss of generality that $\{u_1, v_1\} = e_{t_0}$.

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- If $z_0$ and $u_1$ are in the same connected component of $\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0}$, **which implies that $z_0$ and $v_1$ are in different connected components,** then, instead of removing the edge $e_{t_0}$, remove the vertex $u_1$. In the graph $G - \{w_1, w_2, \ldots, w_{t_0-1}, u_1\}$, the vertices $z_0$ and $v_1$ are in different connected components, and thus the subset $\{w_1, w_2, \ldots, w_{t_0-1}, u_1\}$ is a vertex cut of $G$ with cardinality $\leqslant t_0 = \lambda(G)$.

# Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Subcase 2: $u_1, v_1$ are joined by an edge in $G$. Then, because $u_1, v_1$ are separated in $G - \{e_1, e_2, \ldots, e_{t_0}\}$, one of the edges that we remove must be the edge $\{u_1, v_1\}$. Say without loss of generality that $\{u_1, v_1\} = e_{t_0}$.

Then, as before, for each of the edges $e_i$ with $i \neq t_0$, pick one of its endvertices **which is different from both $u_1$ and $v_1$**, and denote it by $w_i$.

Note now that

$$\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0} \quad \subseteq \quad G - \{e_1, e_2, \ldots, e_{t_0}\},$$

and thus $u_1, v_1$ are separated in the smaller subgraph too.

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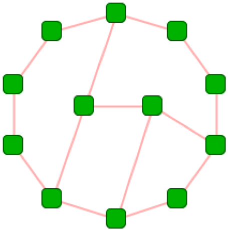
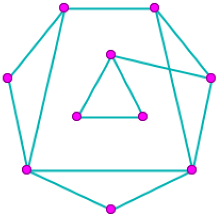
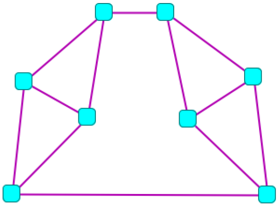
- If $z_0$ and $u_1$ are in the same connected component of $\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0}$, **which implies that $z_0$ and $v_1$ are in different connected components,** then, instead of removing the edge $e_{t_0}$, remove the vertex $u_1$. In the graph $G - \{w_1, w_2, \ldots, w_{t_0-1}, u_1\}$, the vertices $z_0$ and $v_1$ are in different connected components, and thus the subset $\{w_1, w_2, \ldots, w_{t_0-1}, u_1\}$ is a vertex cut of $G$ with cardinality $\leqslant t_0 = \lambda(G)$.
- If instead $z_0$ and $u_1$ are in **different connected components** of $\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0}$, then analogously, instead of the edge $e_{t_0}$, remove the vertex $v_1$. Now, in the graph $G - \{w_1, w_2, \ldots, w_{t_0-1}, \boldsymbol{v_1}\}$, it's the vertices $z_0$ and $u_1$ which are in different connected components.

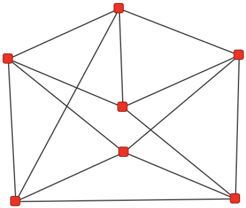
## Checking Whitney's theorem (and the proof constructions) on examples

*For each of the graphs $G$ below, find $\delta(G), \lambda(G)$ and $\kappa(G)$, as well as edge cuts and vertex cuts which 'capture' the parameters.*

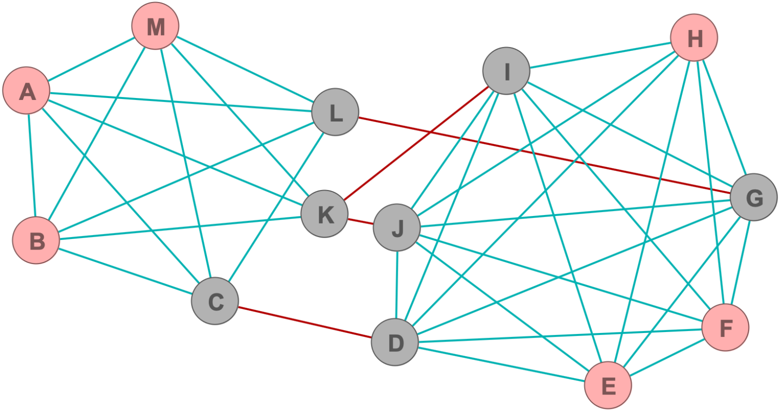








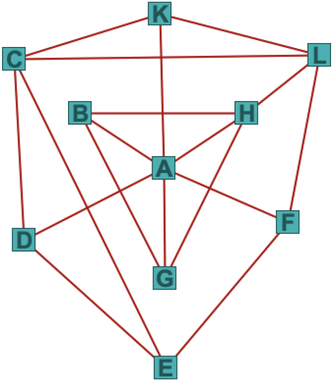
# One more practice example




Same question as before: What is $\delta(G), \lambda(G)$ and $\kappa(G)$ here?

# Past exam problem

Consider the following connected graph $G_0$.




(a) Show that $\kappa(G_0) = 2$. Give a full justification.

(b) What is $\lambda(G_0)$? Determine it precisely, and justify your answer fully.