MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 6

Tuesday, September 21

Reminder: Operations on Graphs

- disjoint union of graphs
- complement of a graph
- induced subgraph

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From last time

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Reminder: Disjoint Union of Graphs

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs with **disjoint vertex sets**, namely such that

$$V_1 \cap V_2 = \emptyset,$$

then the ordered pair

$$\big(V_1 \cup V_2,\ E_1 \cup E_2\big)$$

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is a new graph whose vertices consist of all the vertices of G_1 and all the vertices of G_2.

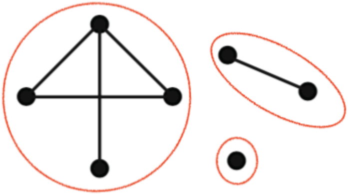
We denote this new graph by $G_1 \oplus G_2$,

and we call it the disjoint union of G_1 and G_2.

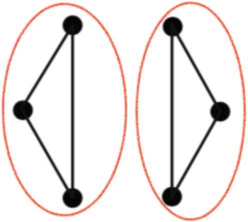
(Note also that, in the graph $G_1 \oplus G_2$, none of the vertices in V_1 is joined with a vertex in V_2, and vice versa; this is because $E_1 \subseteq [V_1]^2$ and $E_2 \subseteq [V_2]^2$, so each edge of $G_1 \oplus G_2$ is either an unordered pair of elements of V_1, or an unordered pair of elements of V_2.)

Examples

The graphs below can be viewed as **disjoint unions** of their connected components.



Disjoint union of 3 graphs



Disjoint union of two 3-cycles

Reminder: Complement of a Graph

Let $G = (V, E)$ be a graph. Recall that E is a subset of the set of 2-element subsets of V (sometimes we denote this set by $[V]^2$).

We can construct a new graph H on the same set of vertices by setting

$$H = \big(V,\ [V]^2 \setminus E\big),$$

that is, by setting the edge set of H to be the *complement* of E in $[V]^2$.

In essence, what we are doing is removing any edges/'connections' we have in G, and then we are joining any two vertices that were not joined in G.

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In essence, what we are doing is removing any edges/'connections' we have in G, and then we are joining any two vertices that were not joined in G.

Definition. The new graph is called the *complement* of G, and is denoted by $\overline{G}$.

Reminder: Subgraphs and Induced Subgraphs

Let $G = (V, E)$ be a graph.

Definition. A subgraph H of G is an ordered pair $\left(V', E'\right)$

- where $\emptyset \neq V' \subseteq V$ (that is, V' is a non-empty subset of V),
- and where $E' \subseteq E$ with every edge $e \in E'$ having both endvertices in V'.

In this case, we write $H \subseteq G$.

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Definition. If $H = \left(V', E'\right)$ is a subgraph of G, and E' contains **all the edges of G which have both endvertices in V',** then we say that H is the subgraph of G that is *induced* or *spanned* by V'.

We denote this induced subgraph by $G[V']$.

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The *join* of G_1 and G_2 is a new graph with vertex set

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and edge set

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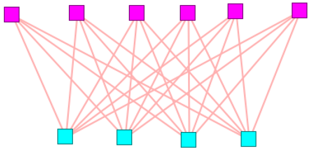
- or both v, w are vertices of G_2, and there is an edge in G_2 joining v and w,
- or finally v is a vertex of G_1 and w is a vertex of G_2 *(or conversely, v is a vertex of G_2 and w a vertex of G_1).*

Examples



The bipartite graph $K_{6,4}$ can be viewed as the join of the null graphs N_6 and N_4:

$K_{6,4} = N_6 \vee N_4$.

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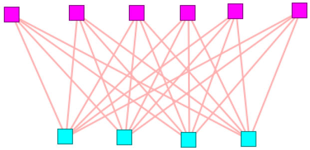

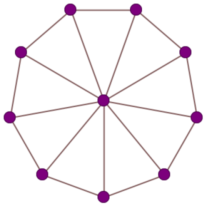

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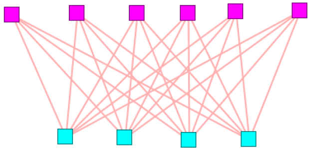

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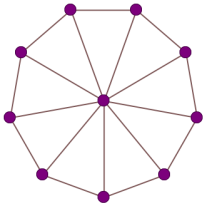

The wheel graph W_{10} can be viewed as the join of the cycle C_9 and the null graph N_1:

$W_{10} = C_9 \vee N_1.$

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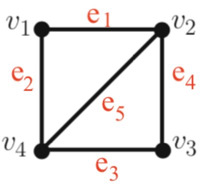
- the vertex set of $L(G)$ is the edge set of G; in other words, $V(L(G)) = E(G)$;
- two 'vertices' in $L(G)$ are joined if, when we view them as edges of G, they are adjacent, or in other words, they have a common endvertex.

An example

Recall the graph G_1: 

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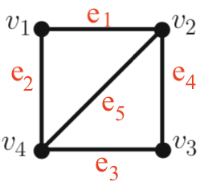
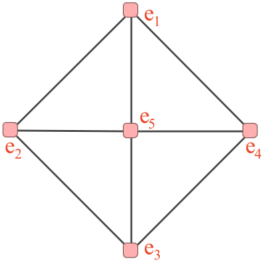


Then $L(G_1)$ is the following graph:



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We denote the *induced subgraph* of G on the vertices in V' by $G[V']$.

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Consider now a **proper** subset V'' of the vertex set V of G.

We can construct a new graph, which will also be a subgraph of G, **if we simply delete the vertices of G which are contained in V'', and of course also remove all the edges of G which have at least one endvertex in V''.**

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We will also use the notation $G - V''$ for this graph.

Moreover, when V'' contains **only one vertex of G**, say vertex v_0, we will also sometimes write $G - v_0$ instead of $G - \{v_0\}$.

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$$\big(V,\, E \setminus E'\big)$$

of G, which we will denote by $G - E'$.

Again, if E' contains **only one element, in this case one edge of** G, say edge e_0, we will more simply write $G - e_0$ instead of $G - \{e_0\}$.

Next Main Topic: Results about Connected Graphs and about Connectivity

One such result: Complement of a disconnected graph

Proposition 1

Let G be a disconnected graph, that is, a graph that has at least two connected components.

Then the complement $\overline{G}$ of G must be connected.

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Case 2: u, v belong to the same connected component of G. Then at least one of the components G_1, G_2 that we considered above must be different from the connected component of G which contains both u and v; say, G_1 is different from the component of u and v.

Consider a vertex w in G_1. Since u and v are not contained in G_1, while G_1 contains all the neighbours of w (given the way we define the connected components of G), we see that $uw \notin E(G)$ and similarly $vw \notin E(G)$.

But then $uw \in E(\overline{G})$ and similarly $vw \in E(\overline{G})$. It follows that the path

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We conclude that $\overline{G}$ is connected.

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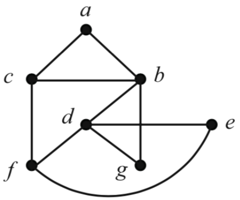
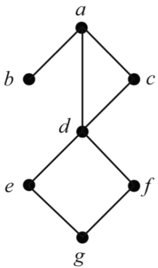
Two examples.



Here $\delta(G) = 2$ and $\Delta(G) = 4$



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One example of the significance of $\delta(G)$

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Practice Exercise for you! Justify with an example that the lower bound $\frac{n-1}{2}$ for $\delta(G)$ is **best possible** and we cannot make it smaller; that is, come up with a graph G' which

- satisfies $\delta(G') \geqslant \frac{n-2}{2}$,
- but is still disconnected.

Proof of Proposition 2

Assume towards a contradiction that G is disconnected. Then G has at least two connected components, and if we consider one of those connected components of G which has smallest possible order, say, component G_1, then we can be certain that G_1 contains $\leqslant \frac{n}{2}$ vertices of G *(why?)*.

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The above show that the assumption that G has at least two connected components was incorrect.

Another such result

One of the problems to be given in HW2

Let k be a positive integer, and let G be a graph satisfying $\delta(G) \geqslant k$.

- Show that G contains a path of length at least k.
- If $k \geqslant 2$, show that G contains a cycle of order at least $k + 1$.

One more result about connected graphs

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Then G **must** be disconnected.

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We'll justify this very soon.

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Recall the families of paths and of cycle graphs, and recall that we have seen examples where we look for graphs from these families among the subgraphs of a given graph G (with, say, vertex set $V = \{v_1, v_2, \ldots, v_n\}$).

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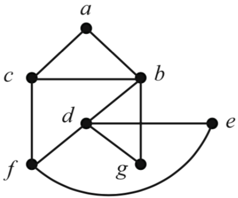
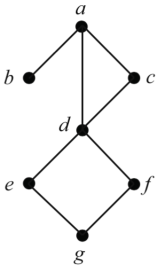
- circuits A *circuit* is a 'closed trail', that is, a walk in which **all edges are distinct**, and also <u>the endvertices coincide</u>.

Practice on examples

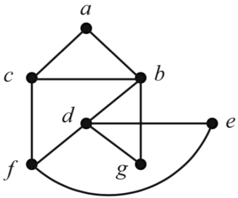


Graph G_1



Graph G_2

Practice on examples



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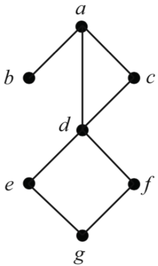

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The $a - e$ walk in graph G_1 given by $a\,b\,g\,d\,f\,c\,b\,d\,e$

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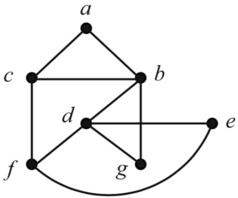

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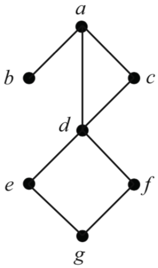

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The $a - e$ walk in graph G_1 given by $a\,b\,g\,d\,f\,c\,b\,d\,e$ is a trail of length 8, but it is not a path (because the edges are all distinct, but the vertices b and d appear in the sequence more than once).

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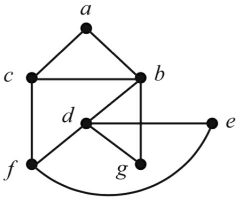

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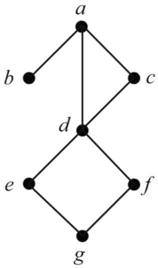

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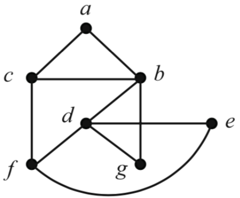

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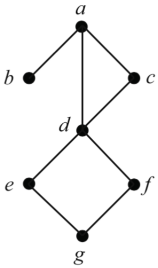

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The walk in graph G_2 given by $a\,c\,d\,e\,g\,f\,d\,a$ is a circuit of length 7, but it is not a cycle (because the edges are all distinct, and the initial and terminal vertex coincide (so it is a 'closed trail'), but at the same time the vertex d, which is not the initial or terminal vertex, also appears twice in the sequence).

Relating walks and paths

Theorem

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- If there are vertices in the sequence which are equal, then we can find $j, r \in \{0, 1, 2, \ldots, k, k+1\}$ with $j < r$ such that $w_j = w_r$. But then the sequence

$$u = w_0 \; w_1 \; w_2 \cdots w_{j-1} \; w_j \; \rightsquigarrow \; w_{r+1} \; w_{r+2} \; \cdots \; w_k \; w_{k+1} = v\,,$$

  in which we go directly from the vertex w_j to the vertex w_{r+1} (which we can do because w_{r+1} is adjacent to $w_r = w_j$), is also a $u - v$ walk which has length $< k + 1$, and thus by the Inductive Hypothesis it will contain a $u - v$ path.

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Since G has been assumed connected,

- we can find a path starting at vertex v_1 and ending at vertex v_2,
- and similarly we can find a path starting at vertex v_2 and ending at vertex v_3,
- and so on, until finally we find a path which starts at vertex v_{n-1} and ends at vertex v_n.

By simply traversing all these paths one after the other, we can construct a walk in G which starts at vertex v_1, ends at vertex v_n, and passes by every vertex of G.

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By the claim, we get that the graph G itself must contain at least $n-1$ edges, which contradicts the assumption that the size of G is $< n-1$. Thus our assumption that G is connected was incorrect.

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   It is called *acyclic* otherwise.
2. A connected acyclic graph is called a *tree*.
3. On the other hand, if an acyclic graph is disconnected, then it is called a *forest* (in other words, *forests* are disjoint unions of two or more *trees*).

Cyclic/Acyclic Graphs – Trees and Forests

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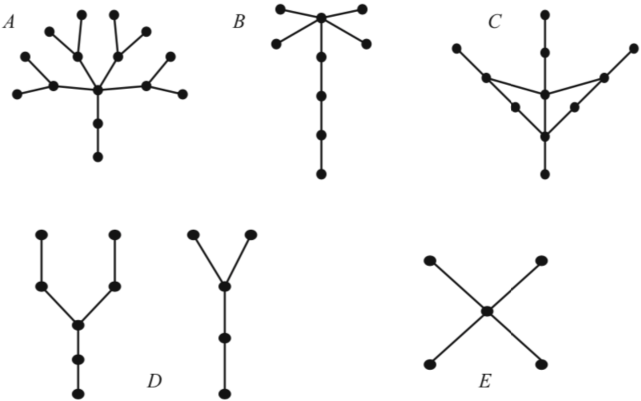
4. A vertex of degree 1 in a tree or a forest is called a *leaf*.

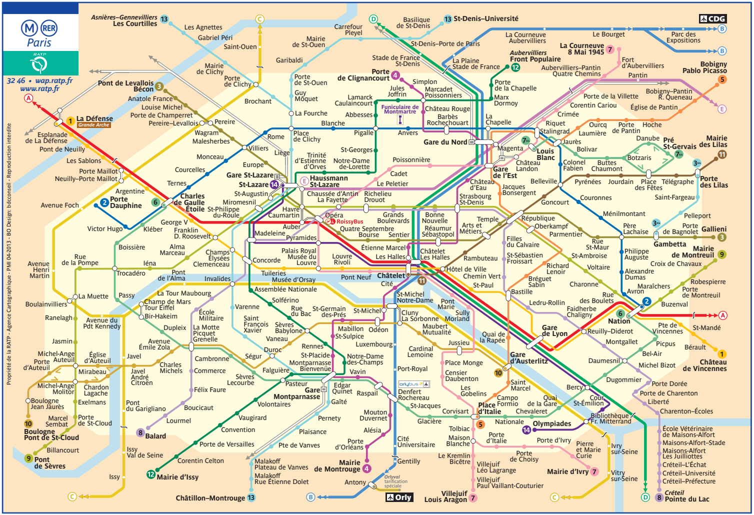
Examples



from the Harris-Hirst-Mossinghoff book

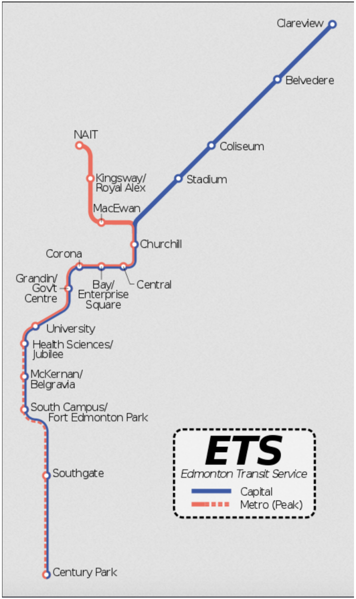
Question. Which of the above graphs are trees? Which are not, and why? Which vertices are leaves in the graphs that are trees?

Example of a cyclic graph



Map of the Paris Subway-Regional Train system

Example of an acyclic graph



One more example of a tree capturing a real-life event/situation

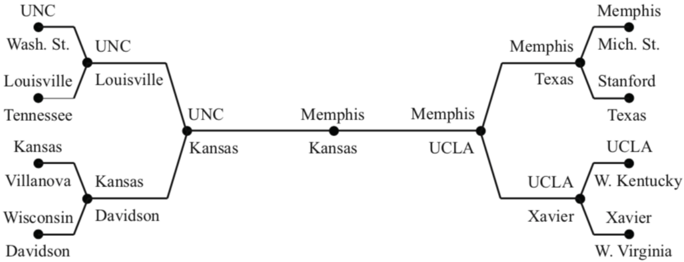


FIGURE 1.35. The 2008 Men's Sweet 16.

from the Harris-Hirst-Mossinghoff book