MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 9

Tuesday, October 5

Connectivity; Reminder of Basic Definitions

Let $G = (V, E)$ be a connected graph.

1. A vertex v of G is called a *cutvertex* of G if we have that

$$G - v$$

   is no longer connected.

2. More generally, a subset V' of the vertex set V of G is called a *vertex cut* if we have that

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   is disconnected. We call it a *k-vertex cut* if the cardinality $|V'|$ of V' is equal to k (that is, if V' contains k vertices of G). V' is also called a *separating set of vertices* of G.

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1. An edge e in G is called a *bridge* (or a *cutedge*) of G if we have that
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The parameter of Edge Connectivity

The parameter $\lambda(G)$

Let G be a connected graph of order $\geqslant 2$. We define the *edge connectivity $\lambda(G)$* of G to be **the minimum cardinality of an edge cut** of G.

What about Vertex Connectivity?

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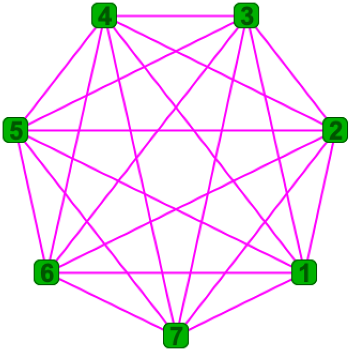
Completely analogously to edge connectivity, **we would like** *to define the parameter of 'vertex connectivity' of a connected graph G to be the minimum cardinality of a vertex cut of G.*

But...

complete graphs don't have vertex cuts



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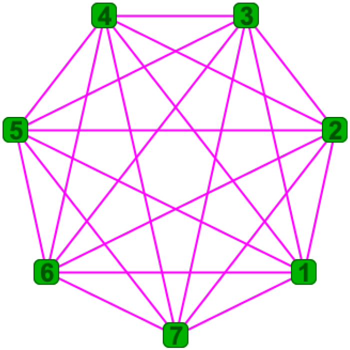
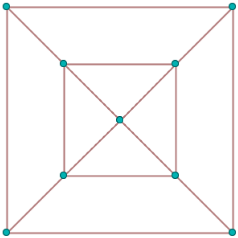


Important Remark. Given $n \geqslant 2$, we have that, for every (proper) subset V' of the vertex set V of K_n, the graph $K_n - V'$ is again a complete graph (on the vertices $V \setminus V'$ now), and hence it cannot be disconnected.

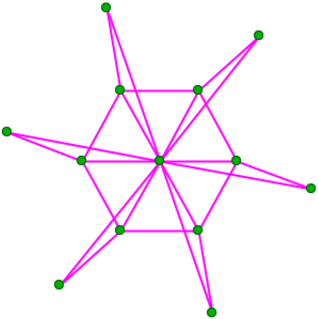
$\rightsquigarrow$ a complete graph does not have any vertex cuts.

In all other cases however...

Any connected graph of order $\geqslant 2$ <u>which is not a complete graph</u> will have vertex cuts. *(Practice Exercise from last time: For each of the examples below, find a vertex cut; if possible, try to find one with smallest possible cardinality.)*



Graph G_1



Graph G_2

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Useful Remark. Note that,

- if a connected graph H of order $n \geqslant 3$ has an s-vertex cut with $s \leqslant n-2$,
- then it also has t-vertex cuts for each cardinality t between s and $n-2$ (this is because if, by removing certain s vertices of H, we end up with a disconnected graph, then clearly by removing $t = s + (t-s)$ vertices of H in a suitable way (that is, by including the s vertices from the s-vertex cut we found before) we will end up again with a disconnected graph.

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Thus a k-vertex connected graph will have no s-vertex cuts for any $s < k-1$ either. In other words, a k-vertex connected graph G is also a t-vertex connected graph for every $0 < t \leqslant k$.

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According to how we gave Definition 1, we get that

the complete graph K_n is $(n-1)$-vertex connected.

Vertex Connectivity (cont.)

Definition 2: the parameter $\kappa(G)$

Let G be a connected graph of order $\geqslant 2$. We define the *vertex connectivity $\kappa(G)$* of G to be the maximum integer k such that G is k-vertex connected.

Note that, since we start with a connected graph G with at least two vertices, we will have that G is 1-vertex connected, and hence $\kappa(G) \geqslant 1$.

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Very Useful Remark

Let G be a connected graph on n vertices which is different from K_n. Then the vertex connectivity $\kappa(G)$ of G coincides with **the minimum cardinality of a vertex cut** of G.

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Remarks. 1. We have already proved (see Lecture 8) part of the theorem, that is, that $\lambda(G) \leqslant \delta(G)$.

2. We have seen by now that the conclusion of the theorem is true when $G = K_n$ with $n \geqslant 2$.

The conclusion of the theorem tested on 'extreme' examples

- In the case of the complete graph K_n ($n \geqslant 2$) we have seen that:
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Also, what can we say about $\kappa(T_n)$?

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As a consequence, we get that every edge of an acyclic graph (that is, a graph which is a tree or a forest) **is a bridge**.

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Indeed, for any two different vertices u, v in $V(G - e_0) = V(G)$, we can find a $u - v$ path in G. But then, if this path contains the edge $e_0 = \{w_1, w_2\}$, we can replace it by the path $w_2\, \cdots\, w_k\, w_1$ (or, if needed, by the reverse path $w_1\, w_k\, w_{k-1}\, \cdots\, w_2$); this sub-path would contain only different edges from e_0, but it would take us again from w_2 to w_1 (or from w_1 to w_2).

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Given that $G - e_0$ is connected, we can find a $u_0 - v_0$ path P in $G - e_0$. Suppose that

$$P \text{ is the path } \; u_0 \, w_1 \, w_2 \, \cdots \, w_{k-1} \, v_0$$

where $w_1, w_2, \ldots, w_{k-1}$ are $k - 1$ vertices of G different from both u_0 and v_0.

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Note that in $G - e_0$ the endvertices u_0, v_0 of the path P are not joined by an edge, but in the original graph G they are joined by the edge e_0. Thus in the original graph G we can 'close' the path P and get the cycle

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Proof of Proposition 1 (cont.)

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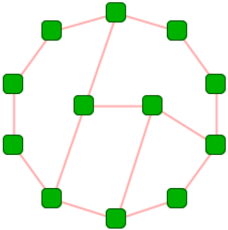
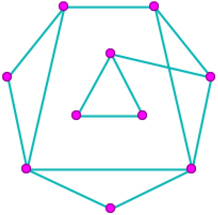
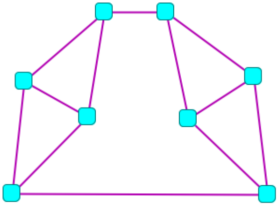
In other words, the edge $e_0 = \{u_0, v_0\}$ of G (which, as was assumed, is NOT a bridge of G) is contained in some cycle of G.

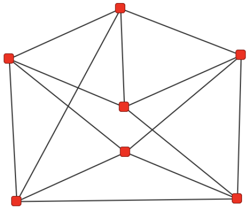
Checking this on examples

Find the edges that are bridges and the edges that are not.









Useful facts about connectivity (cont.)

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Proof. We need to show that $G - v$ is a connected graph.

Easy Case: If G is a connected graph of order 2 exactly, then $G = K_2$ and neither of its two vertices is a cutvertex (moreover, $K_2 - v$ is essentially K_1, no matter which vertex v of K_2 we consider here).

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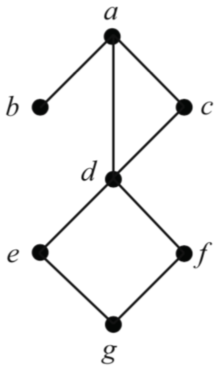
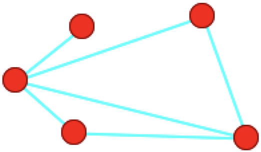
Note that v cannot appear in this path, because otherwise it would be one of the intermediate vertices, so it would have at least two neighbours. This contradicts the assumption that $\deg_G(v) = 1$.

Thus the $u_1 - w_1$ path we have considered is also a path of the subgraph $G - v$. Since u_1, w_1 were arbitrary vertices of $G - v$, the proof is complete.

Checking this on examples





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Assume now that $|T| \geqslant 3$. Let's write n for the number of vertices of T, and let's write V' for the subset of the vertices of T which have degree $\geqslant 2$.

Let us also set $m = |V'|$; then the number of leaves in T (the number that we are interested in) is $n - m$.

Proof of the corollary (cont.)

We now recall some of our key observations from the proof of Theorem 1a:

- T is connected and has 3 or more vertices, so we can't have $\delta(T) = 0$.
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Combining the two, we see that

$$2n - 2 > 2m \quad \Rightarrow \quad 2(n - m) > 2 \quad \Rightarrow \quad n - m > 1 \quad \Rightarrow \quad \boldsymbol{n - m \geqslant 2}.$$

Another useful characterisation of trees

Proposition 3

Let T be a tree on at least two vertices. Then, for every two different vertices u, v of T, there is a **unique** path in T starting at u and ending at v.

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But then the walk $u_0\, w_1\, w_2\, \cdots\, w_{k-1}\, v_0\, u_0$ contains at least 3 different vertices, and thus it is a cycle on $k+1 \geqslant 3$ vertices **contained in T.**

This contradicts the assumption that T is a tree (and thus an acyclic graph).

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Case 1 and Case 2 combined complete the proof.

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We conclude that each of the neighbours of v_0 in T will end up in a different connected component of $T - v_0$.

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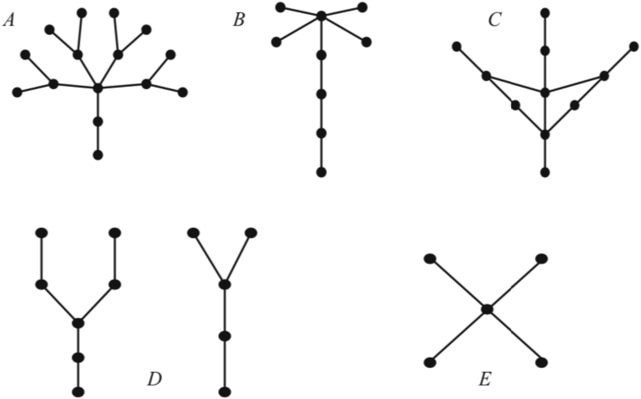

Question. In each of the graphs above, which of the vertices are cutvertices, and which are not?

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  - that T_n must have leaves, and therefore $\delta(T_n) = 1$,
  - and that every edge is a bridge, and thus $\lambda(T_n) = 1$.

  - Finally, such a tree will have vertices of degree $\geqslant 2$, and each such vertex will be a cutvertex. Thus we will have $\kappa(T_n) = 1$.