MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 4

Tuesday, September 14

Reminder:
Important examples of (families of) graphs

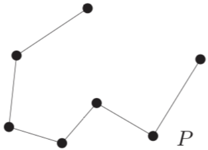
Paths

Definition. A path P is a graph of the form

$$\Big(\{x_0, x_1, x_2, \ldots, x_l\},\ \{x_0x_1, x_1x_2, \ldots, x_{l-1}x_l\}\Big)$$

where l is an integer $\geqslant 1$.

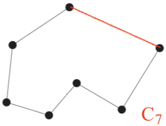
The number l is called the *length* of the path P (note that it is also the number of edges of P, that is, it is equal to the size of P).



A path P on 7 vertices, thus of length 6

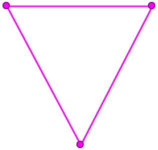
Cycles

If we 'closed' the path P by joining the initial and the terminal vertex as well, then we would get what we call a *cycle graph.*

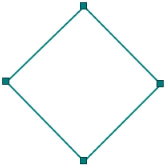


A cycle on 7 vertices, or equivalently a 7-cycle

In other words, $C_7 = \left(\{x_0, x_1, x_2, \dots, x_6\},\ \{x_0x_1, x_1x_2, \dots, x_5x_6,\ x_6x_0\}\right)$.



A 3-cycle, denoted by C_3

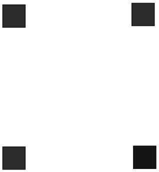


A 4-cycle, denoted by C_4

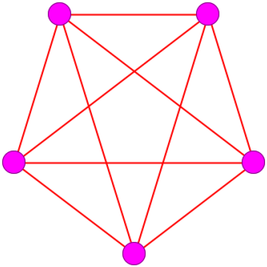
Null Graphs / Complete Graphs

Given a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$, the two 'extreme' cases of graphs with vertex set V are:

- the *null graph* on V, that is, the graph on V that has no edges at all,
- the *complete graph* on V, that is, the graph $\left(V, [V]^2\right)$, in which any two elements of V are joined.

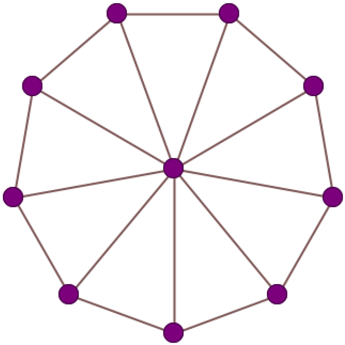


The null graph on 4 vertices, denoted by N_4



The complete graph on 5 vertices, denoted by K_5

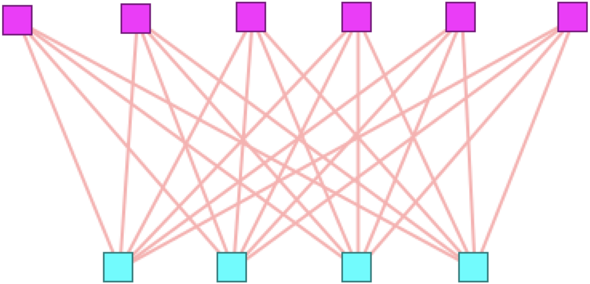
Wheels



The wheel graph on 10 vertices, denoted by W_{10}

Question. How do we form a wheel graph?
Answer. If we want to form a wheel graph on n vertices, then we start with the graph C_{n-1} (that is, an $(n-1)$-cycle), and then we add one more vertex which we join with every other vertex in our graph.
Notice that this implies that, in the wheel graph W_n, one vertex has degree $n-1$, while all the other vertices have degree $= 3$.

Bipartite Graphs



The bipartite graph $K_{6,4}$

Question. How do we construct a bipartite graph?
Answer. As the name suggests, the vertex set V of a bipartite graph is divided into **two parts**, say part V_1 and part V_2. Every vertex in V_1 is joined with every vertex in V_2; on the other hand, there is NO edge joining two vertices in V_1, and similarly there is NO edge joining two vertices in V_2.

If the cardinality $|V_1|$ of the part V_1 is m, and the cardinality $|V_2|$ of the part V_2 is n, then the bipartite graph we just described is denoted by $K_{m,n}$ (or equivalently, $K_{n,m}$).

d-Regular Graphs

Definition. A graph $G = (V, E)$ is called **regular** if all its vertices have the same degree.

If all these degrees are equal to the non-negative integer d, then we say that G is d**-regular**.

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Examples: 1. Any n-cycle is a 2-regular graph.

2. The bipartite graph $K_{5,5}$ is 5-regular.

3. Are there any paths which are regular graphs? YES, any path which has exactly two vertices and one edge joining them:

that is, any path of length 1, is a 1-regular graph.

On the other hand, note that paths of length > 1 are not regular graphs.

Unions of Graphs

With the examples of graphs that we have seen so far, we can now construct plenty more of **different** examples by using simple operations/processes that we will define.

One of the simplest such operations is taking the *disjoint union* of two graphs we have already defined.

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In particular, if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs with **disjoint vertex sets**, namely such that

$$V_1 \cap V_2 = \emptyset,$$

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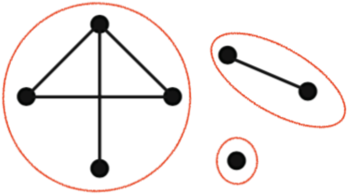
We denote this new graph by $G_1 \oplus G_2$

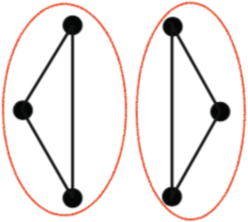
(note also that, in the graph $G_1 \oplus G_2$, none of the vertices in V_1 is joined with a vertex in V_2, and vice versa; this is because $E_1 \subseteq [V_1]^2$ and $E_2 \subseteq [V_2]^2$, so each edge of $G_1 \oplus G_2$ is either an unordered pair of elements of V_1, or an unordered pair of elements of V_2).

Unions of Graphs

Examples. The graphs below can be viewed as **disjoint unions** of their connected components.



Disjoint union of 3 graphs



Disjoint union of two 3-cycles

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Remark. It follows that a 1-regular graph can only have <u>even order</u>, that is, an even number of vertices (since each copy of the above graph contributes 2 vertices).

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– Answer. For every $n \geqslant 3$, the cycle C_n on n vertices is a 2-regular graph. Moreover, disjoint unions of cycles are also 2-regular graphs. These are all the (finite) 2-regular graphs.

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– **Question 4.** Given an arbitrary positive integer n, could you find an $(n-1)$-regular graph on n vertices?

What can we say about the degrees of different vertices of a graph?

Degree Sequences and Graphical Sequences

- Recall how we defined the *degree sequence* of a graph $G = (V(G), E(G))$ in the last lecture: if, say, we have $V = \{v_1, v_2, \ldots, v_n\}$, then the degree sequence of G is the sequence

$$\big(\deg(v_1),\, \deg(v_2),\, \ldots,\, \deg(v_n)\big).$$

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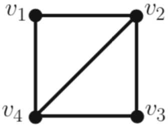
Remark. We often reorder this sequence so that it becomes decreasing.

- Suppose now that you are given a decreasing sequence of n integers,
$$(d_1, d_2, \ldots, d_n) \qquad \text{with } d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n.$$
We say that this sequence is a *graphical sequence* if there exists (at least) one graph $H = (V(H), E(H))$ of order n such that the sequence $(d_1, d_2, \ldots, d_n)$ will coincide with the degree sequence of H *(after reordering if needed).*

Degree Sequences and Graphical Sequences

E.g. the sequence $(3, 3, 2, 2)$ is graphical, since it can be viewed as the degree sequence of the graph



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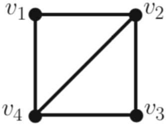

Immediate Restrictions on Graphical Sequences. 1. If a decreasing sequence $(d_1, d_2, \dots, d_n)$ is a graphical sequence, then necessarily all the integers in the sequence must be non-negative. Equivalently we must have $d_n \geqslant 0$.

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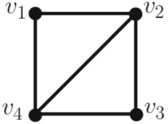

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2. If a decreasing sequence $(d_1, d_2, \ldots, d_n)$ of n integers is a graphical sequence, then necessarily all the integers are $\leqslant n-1$. Equivalently we must have $n-1 \geqslant d_1$. Indeed, recall that the sequence should coincide with the degree sequence of a graph **on n vertices**. But the largest possible degree in such a graph would be $n-1$ (and this would be attained only if there is (at least) one vertex connected to all other vertices).

Important Remark

We say that a decreasing sequence $(d_1, d_2, \ldots, d_n)$ of integers is a graphical sequence if there exists a graph $H = (V(H), E(H))$ (of order n) such that the sequence $(d_1, d_2, \ldots, d_n)$ will coincide with the degree sequence of H.

However, here H does NOT have to be unique,
or in other words, two different graphs H_1, H_2
realising $(d_1, d_2, \ldots, d_n)$ as their degree sequence
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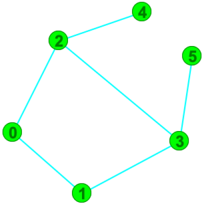

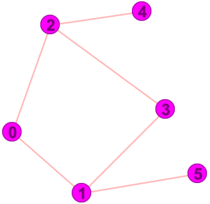

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Example from Lecture 3.

Recall that we discussed some reasons why these two graphs are **not** isomorphic.

Still, their degree sequences coincide: both have degree sequence $(3, 3, 2, 2, 1, 1)$.

The Handshaking Lemma

Let $G = (V, E)$ be a finite graph. Recall that the order $|G|$ of G is the cardinality of its vertex set V, and the size $e(G)$ of G is the cardinality of its edge set E.

For convenience here, let's say that $V = \{v_1, v_2, \ldots, v_n\}$ (and thus $|G| = n$).

Lemma

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$$\sum_{v_i \in V} \deg(v_i) = 2e(G)\,.$$

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Remark. The moniker of the lemma comes from the following real-life application: suppose that at a party each guest V_i shakes hands with some of the other guests, say with d_{V_i} guests. If we add all these non-negative integers d_{V_i}, how does this sum relate to the number of handshakes that took place?

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Won't we count each handshake **twice**?

Proof of the Handshaking Lemma

We can write

$$\begin{aligned}\sum_{v_i \in V} \deg(v_i) &= \sum_{v_i \in V} \left(\sum_{v_j \in N(v_i)} 1 \right) \\ &= \sum_{v_i \in V} \left(\sum_{\substack{e_j \in E \\ v_i \text{ is incident with } e_j}} 1 \right)\end{aligned}$$

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&= \sum_{e_j \in E} \left(\sum_{\substack{v_i \in V \\ v_i \text{ is incident with } e_j}} 1 \right) \qquad \text{(here, all inner sums are equal to 2)} \\
&= \sum_{e_j \in E} 2 = 2|E| = 2e(G)\,.
\end{aligned}
$$

An immediate consequence of the Handshaking Lemma

Corollary of the Handshaking Lemma

Let $G = (V, E)$ be a finite graph, and write

- V_{even} for the subset of all the vertices that have *even* degree,
- and V_{odd} for the subset of the remaining vertices, that is, the vertices that have *odd* degree.

The cardinality of V_{odd} will necessarily be **even**!

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By the Handshaking Lemma, we have

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\Rightarrow \quad \sum_{v_i \in V_{\text{odd}}} \deg(v_i) &= \sum_{v_i \in V} \deg(v_i) \; - \; \sum_{v_i \in V_{\text{even}}} \deg(v_i) \\
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Thus, the sum $\sum_{v_i \in V_{\mathrm{odd}}} \deg(v_i)$, which is a sum of only odd numbers (why?), must equal an even number $\Rightarrow$ there must be an even number of summands in this sum, or in other words we must have an even number of vertices in V_{odd}.

Two more interesting consequences

Problem 1

Let $G = (V, E)$ be a finite graph, and suppose that G has **exactly two** vertices with odd degree, say the vertices v_i and v_j.

Then we must have a path from v_i to v_j.

In other words, if G has only two vertices with odd degree, then these must belong to the same connected component of G.

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Observe now that both G_1 and G_2 are subgraphs of G, thus they are graphs in their own right.

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Note that the vertices of G_1 are also vertices of the original graph G, and their degrees in G_1 are the same as their degrees in G (why? this is because for each vertex v of G_1, all its neighbours are also included in G_1, given that G_1 is a maximal connected subgraph of G).

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Note that the vertices of G_1 are also vertices of the original graph G, and their degrees in G_1 are the same as their degrees in G (why? this is because for each vertex v of G_1, all its neighbours are also included in G_1, given that G_1 is a maximal connected subgraph of G).

Thus, all the vertices of G_1 will have even degree, **except for the vertex v_i which has odd degree** (note that the only other vertex of G which would have odd degree, that is, the vertex v_j, is not contained in G_1).

In other words, $V_{\mathrm{odd}}(G_1)$ will have cardinality 1, which contradicts the Corollary of the Handshaking Lemma!

Solution to Problem 1

We will do a proof by contradiction.
Assume that there is no path in G from v_i to v_j.

Then, given how we defined the connected components of G, v_i and v_j are found in different connected components of G, say components G_1 and G_2 respectively.

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Thus, our assumption that there is no path in G from v_i to v_j, or equivalently that v_i and v_j are contained in different connected components of G, was incorrect.

Two more interesting consequences (cont.)

Problem 2

In any finite graph $G = (V, E)$ **which has at least two vertices**, we can find at least one pair of (different) vertices $v_i, v_j \in V$ which have equal degrees.

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Equivalent way of thinking about this problem

In any group of two or more people *(each of whom may be friends with some of the people in the group)*, there are at least two people who have the same number of friends. *(attention: not necessarily the same friends, but the same number of friends)*

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Again we do a proof by contradiction.
Assume that our graph has n vertices, and that no two of them have equal degrees.

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For each vertex v of G, the possible values for $\deg(v)$ are between 0 and $n-1$, with 0 and $n-1$ included *(recall the most basic restrictions on degree sequences and graphical sequences that we discussed earlier).*

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These are n possible values, and we have n vertices.
Thus, given that, according to our assumption, no two vertices can have the same degree, or, in other words, no two vertices can correspond to the same value, we get that **each of these n values must appear in the degree sequence of G.**

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We have reached an absurd conclusion:

- on the one hand, we should have one vertex, say vertex v_1, which has degree $n-1$, and thus is joined with each one of the remaining $n-1$ vertices;
- on the other hand, one of the remaining $n-1$ vertices, say vertex v_2, must be an isolated vertex (this will be the vertex that has degree 0), and thus cannot be adjacent to v_1.

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We conclude that our assumption that there is no pair of vertices that have the same degree was incorrect.

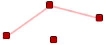
How can we use all the previous observations?

Consider the following decreasing sequences of integers. Which ones are graphical, which ones are not? *(Note that, for some of them, we might still have to guess or not give a definite answer yet, but we'll be able to come back by the end of the lecture.)*

- $(2, 1, 1, 0)$
- $(4, 3, 2, 1, 0, 0)$
- $(4, 4, 3, 3, 2, 1)$
- $(4, 3, 3, 2, 2, 1, 1)$
- $(3, 3, 2, 1, 1)$
- $(7, 6, 5, 4, 3, 3, 2)$
- $(6, 6, 5, 4, 3, 3, 1)$

(Partial) Answer

- $(2, 1, 1, 0)$ Graphical. Justification: Realised by the graph



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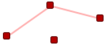

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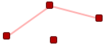

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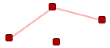

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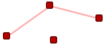

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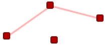

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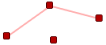

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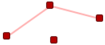

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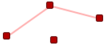

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Let's now look at an algorithmic method of determining whether a decreasing sequence of integers is graphical of not.

The Havel-Hakimi theorem and algorithm

Theorem

Consider a decreasing sequence $S_1 = (d_1, d_2, \ldots, d_n)$ of n non-negative integers.

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Suppose that S_1 is graphical, that is, suppose that S_1 is the degree sequence of a graph G with n vertices:

$$S_1 = \left(\deg_1 \geqslant \deg_2 \geqslant \cdots \geqslant \deg_{n-1} \geqslant \deg_n\right).$$

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Rewrite S_1 as follows *(the reason for this will become clear right away):*

$$S_1 = \big(\deg_1, \deg_2, \deg_3, \ldots, \deg_{\deg_1+1}, \\ \deg_{\deg_1+2}, \deg_{\deg_1+3}, \ldots, \deg_{n-1}, \deg_n\big)$$

(note that there are $\deg_1$ purple-coloured terms here).

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(note that there are $\deg_1$ purple-coloured terms here).

The assumption that S_1 is graphical is **equivalent** to the sequence

$$S_1' = \big(\deg_2 - 1, \deg_3 - 1, \dots, \deg_{\deg_1+1} - 1, \\ \deg_{\deg_1+2}, \deg_{\deg_1+3}, \dots, \deg_{n-1}, \deg_n\big)$$

being graphical.

The Havel-Hakimi theorem and algorithm

Written more succinctly...

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Consider a decreasing sequence $S_1 = (d_1, d_2, \ldots, d_n)$ of n non-negative integers.
Then S_1 is graphical **if and only if** the sequence

$$S_1' = \big(d_2 - 1,\ d_3 - 1, \ldots,\ d_{d_1+1} - 1,\ d_{d_1+2},\ d_{d_1+3}, \ldots,\ d_{n-1},\ d_n\big)$$

of $n-1$ integers is graphical *(note that the purple-coloured terms here are d_1 in total).*

(For the proof, see Wallis' book, Theorem 1.2 on page 14.)

Why is this useful? And why the word "algorithm" as well?

- Suppose that we can't guess right away whether a sequence

$$S_1 = (d_1, d_2, \dots, d_n)$$

of n integers is graphical by coming up with a graph G of order n that would realise it (and we can't find a quick justification either for why no graph G would have this sequence as its degree sequence).

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- The theorem then allows us to look at the sequence

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- Besides, if looking at S_1' instead hasn't made the problem much simpler, we can keep going and applying the theorem again and again, until we end up with a significantly simpler sequence to work with *(this suggests an algorithm for answering whether the original sequence S_1 is graphical or not).*

Examples from before

Recall that, in the case of the problem posed just before the slide of the Havel-Hakimi theorem, about whether certain sequences of integers are graphical or not, we still haven't given a definite answer for two of the sequences:

- $(4, 3, 3, 2, 2, 1, 1)$
- $(3, 3, 2, 1, 1)$

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We can now try to apply the theorem in order to conclude whether these sequences are graphical or not.

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For the 1st sequence, we have

$$(4, 3, 3, 2, 2, 1, 1) \quad \text{is graphical}$$

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$$(4, \color{red}{3, 3, 2, 2,} \color{black}{1, 1}) \quad \text{is graphical} \quad \textbf{if and only if}$$

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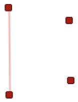

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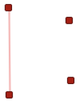

Thus, based on the theorem, all the other sequences are also graphical.

Examples from before

Similarly, for the 2nd sequence, we have

$$(3, 3, 2, 1, 1) \quad \text{is graphical}$$

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$$(3, 3, 2, 1, 1) \text{ is graphical} \quad \textbf{if and only if}$$

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Examples from before

Similarly, for the 2nd sequence, we have

$(3, 3, 2, 1, 1)$ is graphical **if and only if**

$(2, 1, 0, 1)$ is graphical or equivalently if
$(2, 1, 1, 0)$ is graphical.

Examples from before

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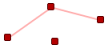
But recall now that we have already seen (in the slide giving a partial answer to the problem just before the Havel-Hakimi theorem) that the last sequence, sequence $(2,1,1,0)$, is graphical because it is realised by the graph



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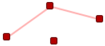

The theorem allows us to conclude that sequence $(3, 3, 2, 1, 1)$ is also graphical.

One more example

Determine whether the sequence

$$(8, 8, 7, 7, 6, 6, 4, 3, 2, 1, 1, 1)$$

is graphical or not *(left as a practice exercise).*