MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 19

Thursday, November 18

Reminder

Definition

Let G be a connected graph.

- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).
- A *Hamilton cycle* in G is a cycle (that is, a closed path) that passes through **all** vertices in G.

G is called *Hamiltonian* if we can find (at least) one Hamilton cycle in G.

As we have said:

Unlike what we saw for Eulerian and non-Eulerian graphs,

there are <u>no</u> simple characterisations (that is, conditions that are <u>both</u> necessary and sufficient) for Hamiltonicity.

Thus, we will state:

some conditions that are necessary for a graph to be Hamiltonian
(that is, if any of these conditions <u>doesn't</u> hold,
then the graph <u>cannot</u> be Hamiltonian),

and some conditions that are sufficient
(that is, it suffices to check for any one of these conditions,
and if it <u>does</u> hold true, then the graph <u>will be</u> Hamiltonian).

Necessary conditions that we saw

Necessary Condition 1

Let G be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then G has no cutvertices.
In other words, if G is Hamiltonian, then $\kappa(G) \geqslant 2$ (or equivalently G is 2-vertex connected).

Necessary conditions that we saw

Necessary Condition 1

Let G be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then G has no cutvertices.
In other words, if G is Hamiltonian, then $\kappa(G) \geqslant 2$ (or equivalently G is 2-vertex connected).

Necessary Condition 2

Let $G = (V, E)$ be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $G - S$ has at most $|S|$ connected components.

Sufficient conditions that we saw

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$.
Then G is Hamiltonian.

Sufficient conditions that we saw

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$. Then G is Hamiltonian.

Dirac's theorem follows easily from another sufficient condition for Hamiltonicity which came out a little later:

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G, we have that $\deg(u) + \deg(v) \geqslant n$.

Then G is Hamiltonian.

Sufficient conditions that we saw

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$.
Then G is Hamiltonian.

Dirac's theorem follows easily from another sufficient condition for Hamiltonicity which came out a little later:

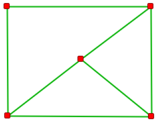
Theorem 2 (Ore, 1960)

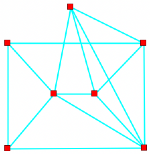
Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G,
we have that $\deg(u) + \deg(v) \geqslant n$.

Then G is Hamiltonian.

Testing Our Understanding. For each of the following graphs, determine whether it has the property stated in Ore's theorem. If yes, can you find a Hamilton cycle in it?





Sufficient conditions that we saw (cont.)

Proposition 3

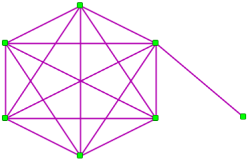
Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$\binom{n-1}{2} + 2$$

edges. Then G is Hamiltonian.

Remark 1. Note that the maximum number of edges that G could have is $\binom{n}{2}$ (in which case G would be the complete graph on n vertices). Thus, this proposition allows us to deal with graphs with size between $\binom{n-1}{2} + 2$ and $\binom{n}{2}$ *(with the endpoints included)*, and we can find quite a few examples here.

Remark 2. Again, the lower bound on the size of G is best possible: below is an example of a graph H with $\binom{n-1}{2} + 1$ edges which is not Hamiltonian (note that $n = 7$ here, but this type of example can work for other n as well).



Sufficient conditions that we saw (cont.)

Definition

Let $G = (V, E)$ be a graph. A vertex subset $V' \subseteq V$ is called an *independent* set of vertices, if any two different vertices in V' are **non-adjacent**.

Sufficient conditions that we saw (cont.)

Definition

Let $G = (V, E)$ be a graph. A vertex subset $V' \subseteq V$ is called an *independent* set of vertices, if any two different vertices in V' are **non-adjacent**.

The *independence number* of G, denoted by $\alpha(G)$, is defined to be the maximum cardinality of an independent set of vertices of G.

Sufficient conditions that we saw (cont.)

Definition

Let $G = (V, E)$ be a graph. A vertex subset $V' \subseteq V$ is called an *independent* set of vertices, if any two different vertices in V' are **non-adjacent**.

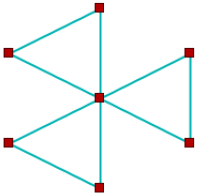
The *independence number* of G, denoted by $\alpha(G)$, is defined to be the maximum cardinality of an independent set of vertices of G.

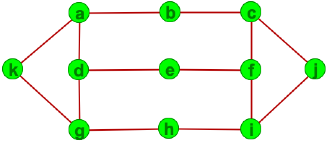
Theorem 4 (Chvátal-Erdös, 1972)

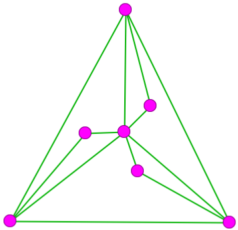
Let G be a graph of order $n \geqslant 3$ such that $\kappa(G) \geqslant \alpha(G)$. Then G is Hamiltonian.

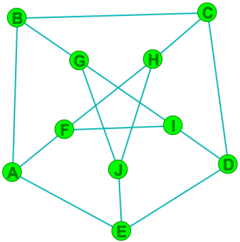
Testing these sufficient conditions on non-examples

Note that the conditions should fail here (why?).

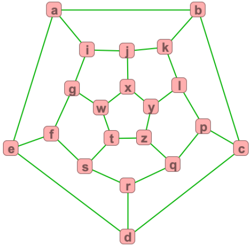


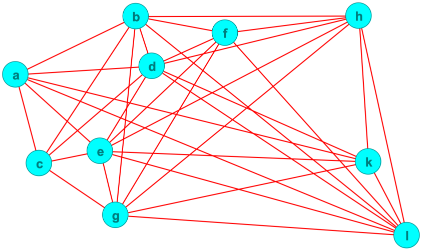


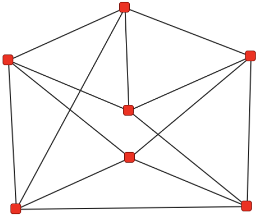


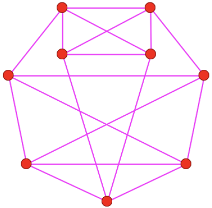


Testing these sufficient conditions on (possible) examples *(practice)*

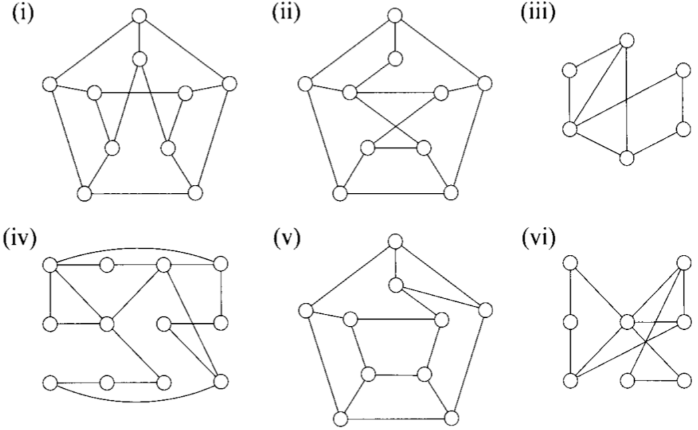








Testing these sufficient conditions on possible examples



from Wallis' book

Left as a practice exercise.

A couple more sufficient conditions for Hamiltonicity:

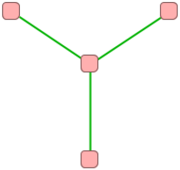
A couple more sufficient conditions for Hamiltonicity:

Both these conditions are stated in terms of **forbidden subgraphs**

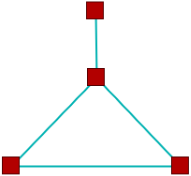
that is, a graph G will be Hamiltonian if certain, already given graphs **cannot** be viewed as induced subgraphs of G.

Family of possible forbidden subgraphs

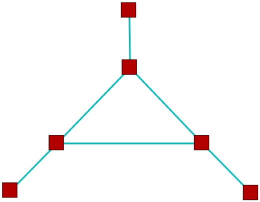
Consider the following three graphs:



Graph $K_{1,3}$



Graph Z_4



Graph Z_6

1st sufficient condition in terms of forbidden subgraphs

Theorem 5 (Goodman-Hedetniemi, 1974)

Let G be a graph of order $n \geqslant 3$ which is 2-vertex connected (that is, $\kappa(G) \geqslant 2$).

1st sufficient condition in terms of forbidden subgraphs

Theorem 5 (Goodman-Hedetniemi, 1974)

Let G be a graph (of order $n \geqslant 3$) which is 2-vertex connected (that is, $\kappa(G) \geqslant 2$).

1st sufficient condition in terms of forbidden subgraphs

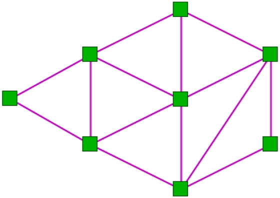
Theorem 5 (Goodman-Hedetniemi, 1974)

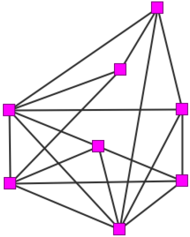
Let G be a graph (of order $n \geqslant 3$) which is 2-vertex connected (that is, $\kappa(G) \geqslant 2$).

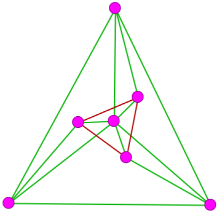
If G is $\{K_{1,3}, Z_4\}$-free *(that is, none of those two graphs is an induced subgraph of G)*, then G is Hamiltonian.

Possible examples and non-examples

Question. Are any of the following graphs $\{K_{1,3},\ Z_4\}$-free?







2nd sufficient condition in terms of forbidden subgraphs

Theorem 6 (Duffus-Gould-Jacobson, 1980)

Let G be a $\{K_{1,3},\, Z_6\}$-free graph.

2nd sufficient condition in terms of forbidden subgraphs

Theorem 6 (Duffus-Gould-Jacobson, 1980)

Let G be a $\{K_{1,3}, Z_6\}$-free graph.

- If G is connected, then G has a Hamilton path.

2nd sufficient condition in terms of forbidden subgraphs

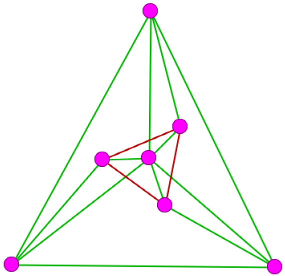
Theorem 6 (Duffus-Gould-Jacobson, 1980)

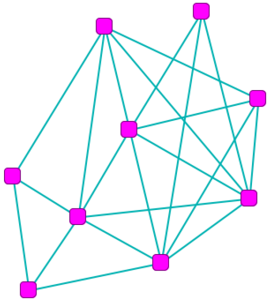
Let G be a $\{K_{1,3},\, Z_6\}$-free graph.

- If G is connected, then G has a Hamilton path.
- If G is 2-vertex connected, then G is Hamiltonian.

Possible examples and non-examples

Question. Are any of the following graphs $\{K_{1,3},\ Z_6\}$-free?





Also, a necessary condition for containing a Hamilton path

Recall: Necessary Condition 2

Let $G = (V, E)$ be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $G - S$ has at most $|S|$ connected components.

Also, a necessary condition for containing a Hamilton path

Recall: Necessary Condition 2

Let $G = (V, E)$ be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $G - S$ has at most $|S|$ connected components.

Necessary Condition 2′

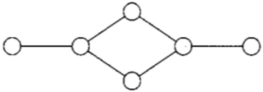
Let $H = (V, E)$ be a connected graph of order $n \geqslant 2$.

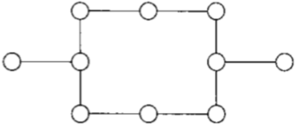
If H has a Hamilton path, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $H - S$ has at most $|S| + 1$ connected components.

Testing this necessary condition

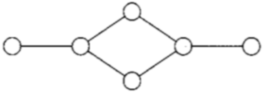
Question 1. Does any of the following graphs from Wallis' book have a Hamilton path? If it does, find one such path. If it doesn't, can you justify why not?

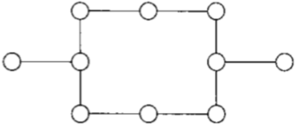




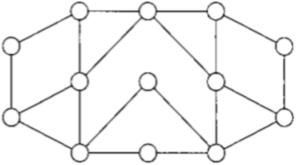
Testing this necessary condition

Question 1. Does any of the following graphs from Wallis' book have a Hamilton path? If it does, find one such path. If it doesn't, can you justify why not?





Question 2. What about this graph?



Proving some of these sufficient conditions

Recall:

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$. Then G is Hamiltonian.

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G,
we have that $\deg(u) + \deg(v) \geqslant n$.

Then G is Hamiltonian.

Proposition 3

Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$\binom{n-1}{2} + 2$$

edges. Then G is Hamiltonian.

Proving some of these sufficient conditions

Recall:

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$. Then G is Hamiltonian.

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G,
we have that $\deg(u) + \deg(v) \geqslant n$.

Then G is Hamiltonian.

Proposition 3

Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$\binom{n-1}{2} + 2$$

edges. Then G is Hamiltonian.

We have said that Dirac's theorem follows from Ore's theorem (since the condition in Dirac's theorem is stronger than the condition in Ore's theorem).

Proving some of these sufficient conditions

Recall:

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$. Then G is Hamiltonian.

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G, we have that $\overline{\deg(u) + \deg(v)} \geqslant n$.

Then G is Hamiltonian.

Proposition 3

Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$\binom{n-1}{2} + 2$$

edges. Then G is Hamiltonian.

We have said that Dirac's theorem follows from Ore's theorem (since the condition in Dirac's theorem is stronger than the condition in Ore's theorem).

We will now see that Proposition 3 also follows from Ore's theorem.

Proof of Proposition 3

We will show that our assumption about the size of G implies that G satisfies the condition in Ore's theorem.

Proof of Proposition 3

We will show that our assumption about the size of G implies that G satisfies the condition in Ore's theorem.

Consider two different vertices u and v of G (here we don't even have to make sure that u and v are not adjacent). Set

$$E' = \{e \in E(G) : e \text{ is incident to } u \text{ or to } v\}.$$

Then

$$\deg(u) + \deg(v) = \begin{cases} |E'| + 1 & \text{if } u \text{ and } v \text{ are adjacent} \\ |E'| & \text{if } u \text{ and } v \text{ are not adjacent} \end{cases}.$$

Proof of Proposition 3

We will show that our assumption about the size of G implies that G satisfies the condition in Ore's theorem.

Consider two different vertices u and v of G (here we don't even have to make sure that u and v are not adjacent). Set

$$E' = \{e \in E(G) : e \text{ is incident to } u \text{ or to } v\}.$$

Then

$$\deg(u) + \deg(v) = \begin{cases} |E'| + 1 & \text{if } u \text{ and } v \text{ are adjacent} \\ |E'| & \text{if } u \text{ and } v \text{ are not adjacent} \end{cases}.$$

Thus, no matter which case we're in, it suffices to show that $|E'| \geqslant n$.

We will show that our assumption about the size of G implies that G satisfies the condition in Ore's theorem.

Consider two different vertices u and v of G (here we don't even have to make sure that u and v are not adjacent). Set

$$E' = \{e \in E(G) : e \text{ is incident to } u \text{ or to } v\}.$$

Then

$$\deg(u) + \deg(v) = \begin{cases} |E'| + 1 & \text{if } u \text{ and } v \text{ are adjacent} \\ |E'| & \text{if } u \text{ and } v \text{ are not adjacent} \end{cases}.$$

Thus, no matter which case we're in, it suffices to show that $|E'| \geqslant n$.
Note that the remaining edges of G (which are not incident either to u or to v) can be at most $\binom{n-2}{2}$ (since each such edge will join two of the remaining $n-2$ vertices of G).

Proof of Proposition 3

We will show that our assumption about the size of G implies that G satisfies the condition in Ore's theorem.

Consider two different vertices u and v of G (here we don't even have to make sure that u and v are not adjacent). Set

$$E' = \{e \in E(G) : e \text{ is incident to } u \text{ or to } v\}.$$

Then

$$\deg(u) + \deg(v) = \begin{cases} |E'| + 1 & \text{if } u \text{ and } v \text{ are adjacent} \\ |E'| & \text{if } u \text{ and } v \text{ are not adjacent} \end{cases}.$$

Thus, no matter which case we're in, it suffices to show that $|E'| \geqslant n$.
Note that the remaining edges of G (which are not incident either to u or to v) can be at most $\binom{n-2}{2}$ (since each such edge will join two of the remaining $n-2$ vertices of G).
But then

$$\begin{aligned} |E'| &\geqslant e(G) - \binom{n-2}{2} \\ &\geqslant \binom{n-1}{2} + 2 - \binom{n-2}{2} \\ &= \frac{(n-1)(n-2)}{2} + 2 - \frac{(n-2)(n-3)}{2} \\ &= \frac{n^2 - 3n + 2 + 4 - n^2 + 5n - 6}{2} = n, \end{aligned}$$

as we wanted.

Proof of Proposition 3

We will show that our assumption about the size of G implies that G satisfies the condition in Ore's theorem.

Consider two different vertices u and v of G (here we don't even have to make sure that u and v are not adjacent). Set

$$E' = \{e \in E(G) : e \text{ is incident to } u \text{ or to } v\}.$$

Then

$$\deg(u) + \deg(v) = \begin{cases} |E'| + 1 & \text{if } u \text{ and } v \text{ are adjacent} \\ |E'| & \text{if } u \text{ and } v \text{ are not adjacent} \end{cases}.$$

Thus, no matter which case we're in, it suffices to show that $|E'| \geqslant n$.
Note that the remaining edges of G (which are not incident either to u or to v) can be at most $\binom{n-2}{2}$ (since each such edge will join two of the remaining $n-2$ vertices of G).
But then

$$\begin{aligned} |E'| &\geqslant e(G) \; - \; \binom{n-2}{2} \\ &\geqslant \binom{n-1}{2} + 2 \; - \; \binom{n-2}{2} \\ &= \frac{(n-1)(n-2)}{2} + 2 \; - \; \frac{(n-2)(n-3)}{2} \\ &= \frac{n^2 - 3n + 2 + 4 - n^2 + 5n - 6}{2} \; = \; n, \end{aligned}$$

as we wanted. Proof of Ore's theorem? To be discussed next time.

Next Main Topic:

Factors, Matchings and (Stable) Marriages

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

Analysing what the problem asks for:

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

Analysing what the problem asks for:

- Each seating arrangement can be viewed as a Hamilton cycle of K_{11}.

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

Analysing what the problem asks for:

- Each seating arrangement can be viewed as a Hamilton cycle of K_{11}.
- Since no person can have the same neighbour twice, any two such Hamilton cycles must be **edge-disjoint**.

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

Analysing what the problem asks for:

- Each seating arrangement can be viewed as a Hamilton cycle of K_{11}.
- Since no person can have the same neighbour twice, any two such Hamilton cycles must be **edge-disjoint**.
- Let us first explain why there can be **at most** 5 such dinners:

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

Analysing what the problem asks for:

- Each seating arrangement can be viewed as a Hamilton cycle of K_{11}.
- Since no person can have the same neighbour twice, any two such Hamilton cycles must be **edge-disjoint**.
- Let us first explain why there can be **at most** 5 such dinners:
  - Consider one of the hires, say person A. At the first dinner, person A has a pair of neighbours, who can come from the remaining 10 new hires.

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

Analysing what the problem asks for:

- Each seating arrangement can be viewed as a Hamilton cycle of K_{11}.
- Since no person can have the same neighbour twice, any two such Hamilton cycles must be **edge-disjoint**.
- Let us first explain why there can be **at most** 5 such dinners:
  - Consider one of the hires, say person A. At the first dinner, person A has a pair of neighbours, who can come from the remaining 10 new hires.
  - At the second dinner, person A cannot sit next to any of the two colleagues he or she sat next to during the first dinner, so the new neighbours of person A are among the remaining $10 - 2 = 8$ new hires.

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

Analysing what the problem asks for:

- Each seating arrangement can be viewed as a Hamilton cycle of K_{11}.
- Since no person can have the same neighbour twice, any two such Hamilton cycles must be **edge-disjoint**.
- Let us first explain why there can be **at most** 5 such dinners:
  - Consider one of the hires, say person A. At the first dinner, person A has a pair of neighbours, who can come from the remaining 10 new hires.
  - At the second dinner, person A cannot sit next to any of the two colleagues he or she sat next to during the first dinner, so the new neighbours of person A are among the remaining $10 - 2 = 8$ new hires.
  - Continuing like this, we see that the pairs of neighbours of person A form a collection of pairwise disjoint 2-subsets of the set of 10 collleagues of person A, so this collection can have at most 5 such subsets.

Theory behind this problem

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).

Theory behind this problem

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).

2. A *factorization* of G is any collection of s factors (spanning subgraphs) $H_1, H_2, \ldots, H_s$ of G such that
   - any two different factors H_i and H_j are edge-disjoint;

Theory behind this problem

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).

2. A *factorization* of G is any collection of s factors (spanning subgraphs) $H_1, H_2, \ldots, H_s$ of G such that
   - any two different factors H_i and H_j are edge-disjoint;
   - every edge of G is contained in one of the factors $H_1, H_2, \ldots, H_s$, that is, $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s)$.

Theory behind this problem

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).
2. A *factorization* of G is any collection of s factors (spanning subgraphs) $H_1, H_2, \ldots, H_s$ of G such that
   - any two different factors H_i and H_j are edge-disjoint;
   - every edge of G is contained in one of the factors $H_1, H_2, \ldots, H_s$, that is, $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s)$.

Some immediate observations:

Theory behind this problem

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).

2. A *factorization* of G is any collection of s factors (spanning subgraphs) $H_1, H_2, \ldots, H_s$ of G such that
   - any two different factors H_i and H_j are edge-disjoint;
   - every edge of G is contained in one of the factors $H_1, H_2, \ldots, H_s$, that is, $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s)$.

Some immediate observations:

- A spanning tree (or spanning forest, in cases where G is not connected) is a factor of G.

Theory behind this problem

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).

2. A *factorization* of G is any collection of s factors (spanning subgraphs) $H_1, H_2, \ldots, H_s$ of G such that
   - any two different factors H_i and H_j are edge-disjoint;
   - every edge of G is contained in one of the factors $H_1, H_2, \ldots, H_s$, that is, $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s)$.

Some immediate observations:

- A spanning tree (or spanning forest, in cases where G is not connected) is a factor of G.
- Every graph (or multigraph) G has a trivial factorization: since G is a factor of itself, the collection $\{G\}$ is a factorization of G.

Theory behind this problem

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).
2. A *factorization* of G is any collection of s factors (spanning subgraphs) $H_1, H_2, \ldots, H_s$ of G such that
   - any two different factors H_i and H_j are edge-disjoint;
   - every edge of G is contained in one of the factors $H_1, H_2, \ldots, H_s$, that is, $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s)$.

Some immediate observations:

- A spanning tree (or spanning forest, in cases where G is not connected) is a factor of G.
- Every graph (or multigraph) G has a trivial factorization: since G is a factor of itself, the collection $\{G\}$ is a factorization of G.
- The problem in the previous slide asks us to find **a factorization of K_{11} consisting of Hamilton cycles of K_{11}.**

Other interesting types of factors

Definition

Let G be a graph (or multigraph).

Other interesting types of factors

Definition

Let G be a graph (or multigraph).

- A spanning subgraph of G is called a *one-factor* of G **if it is 1-regular.**
  A *one-factorization* of G is a factorization of G consisting of one-factors of G.

Other interesting types of factors

Definition

Let G be a graph (or multigraph).

- A spanning subgraph of G is called a *one-factor* of G **if it is 1-regular.**
  A *one-factorization* of G is a factorization of G consisting of one-factors of G.
- Similarly, a spanning subgraph of G is called a *two-factor* of G **if it is 2-regular.**
  A *two-factorization* of G is a factorization of G consisting of two-factors of G.

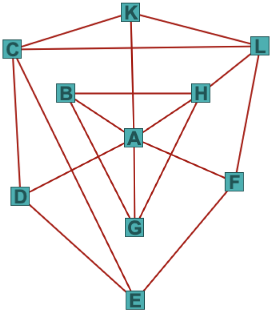
Other interesting types of factors

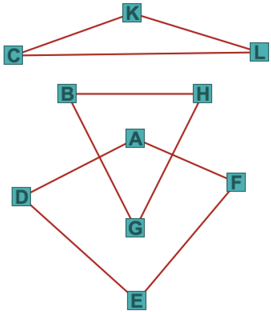
Definition

Let G be a graph (or multigraph).

- A spanning subgraph of G is called a *one-factor* of G **if it is 1-regular.** A *one-factorization* of G is a factorization of G consisting of one-factors of G.
- Similarly, a spanning subgraph of G is called a *two-factor* of G **if it is 2-regular.** A *two-factorization* of G is a factorization of G consisting of two-factors of G.

A subtle point. Note that a Hamilton cycle of G is a two-factor of G, but not every two-factor needs to be a Hamilton cycle. E.g. the graph on the right below is a two-factor of the graph on the left (but not a Hamilton cycle):



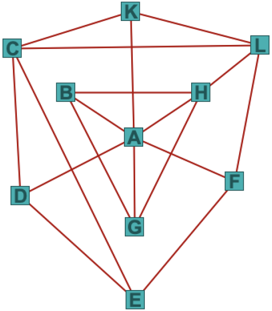


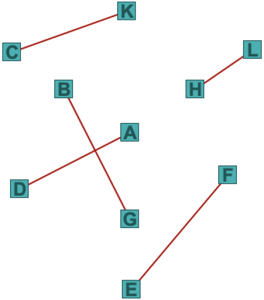
Remarks about one-factors

Question 1. Find a one-factor of the previous graph?

Remarks about one-factors

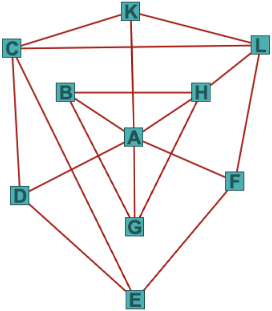
Question 1. Find a one-factor of the previous graph?

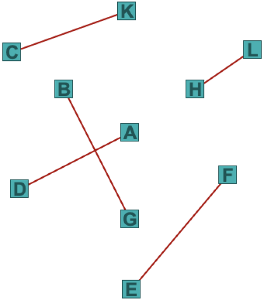




Remarks about one-factors

Question 1. Find a one-factor of the previous graph?





Question 2. Does the given graph have a one-factorization?

Remarks about one-factors (cont.)

Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

Remarks about one-factors (cont.)

Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

(ii) G cannot have isolated vertices *(be careful that, if G is a multigraph, this condition is not immediately equivalent to $\delta(G) \geqslant 1$, given that some vertices of G may have no neighbours, but may have loops attached to them, so their degree will still be positive)*.

Remarks about one-factors (cont.)

Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

(ii) G cannot have isolated vertices *(be careful that, if G is a multigraph, this condition is not immediately equivalent to $\delta(G) \geqslant 1$, given that some vertices of G may have no neighbours, but may have loops attached to them, so their degree will still be positive)*.

Necessary conditions for the existence of a one-factorization

Let G be graph of order $n = 2k$ and size m. Suppose that G has a one-factorization. Then

(I) $k = \frac{n}{2}$ must divide m.

Remarks about one-factors (cont.)

Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

(ii) G cannot have isolated vertices *(be careful that, if G is a multigraph, this condition is not immediately equivalent to $\delta(G) \geqslant 1$, given that some vertices of G may have no neighbours, but may have loops attached to them, so their degree will still be positive)*.

Necessary conditions for the existence of a one-factorization

Let G be graph of order $n = 2k$ and size m. Suppose that G has a one-factorization. Then

(I) $k = \frac{n}{2}$ must divide m.

(II) Even more restrictively, the following property must be true for G: **the graph G must be d-regular for some d which divides m.** *(Justification?*

Remarks about one-factors (cont.)

Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

- (i) G has an even number of vertices.
- (ii) G cannot have isolated vertices *(be careful that, if G is a multigraph, this condition is not immediately equivalent to $\delta(G) \geqslant 1$, given that some vertices of G may have no neighbours, but may have loops attached to them, so their degree will still be positive)*.

Necessary conditions for the existence of a one-factorization

Let G be graph of order $n = 2k$ and size m. Suppose that G has a one-factorization. Then

- (I) $k = \frac{n}{2}$ must divide m.
- (II) Even more restrictively, the following property must be true for G: **the graph G must be d-regular for some d which divides m.** *(Justification? Note that, if G can be decomposed into d pairwise edge-disjoint one-factors, then each vertex v of G must be incident with precisely d edges.)*

Remarks about one-factors (cont.)

Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

(ii) G cannot have isolated vertices *(be careful that, if G is a multigraph, this condition is not immediately equivalent to $\delta(G) \geqslant 1$, given that some vertices of G may have no neighbours, but may have loops attached to them, so their degree will still be positive)*.

Necessary conditions for the existence of a one-factorization

Let G be graph of order $n = 2k$ and size m. Suppose that G has a one-factorization. Then

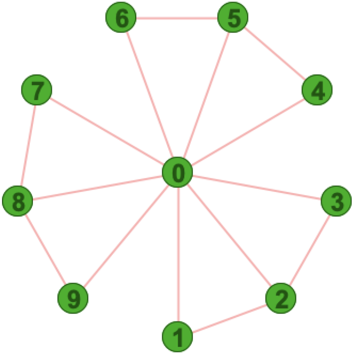
(I) $k = \frac{n}{2}$ must divide m.

(II) Even more restrictively, the following property must be true for G: **the graph G must be d-regular for some d which divides m.** *(Justification? Note that, if G can be decomposed into d pairwise edge-disjoint one-factors, then each vertex v of G must be incident with precisely d edges.)*

(III) ***G* cannot have bridges** (except if G is a 1-regular graph itself, and hence the trivial factorization $\{G\}$ of G is a one-factorization too).

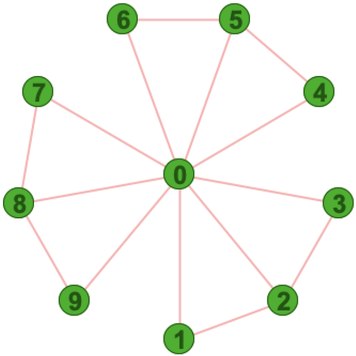
None of these conditions are sufficient too

Example 1. The following graph has even order, and no isolated vertices, but it does not have a one-factor (why?).



None of these conditions are sufficient too

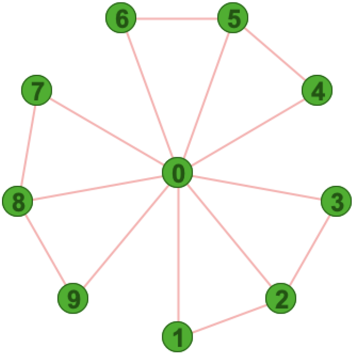
Example 1. The following graph has even order, and no isolated vertices, but it does not have a one-factor (why?).



As a consequence of this, we obtain that the graph is not Hamiltonian either [why?

None of these conditions are sufficient too

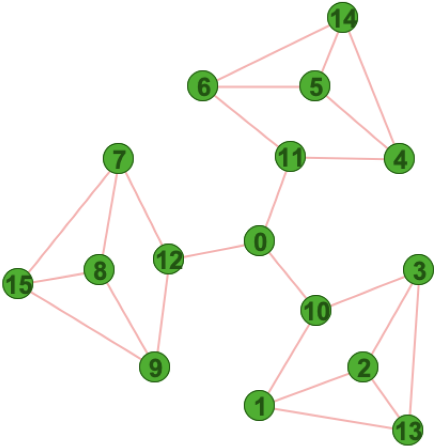
Example 1. The following graph has even order, and no isolated vertices, but it does not have a one-factor (why?).



As a consequence of this, we obtain that the graph is not Hamiltonian either [why? note that a Hamilton cycle with **an even number** of vertices has both a one-factor, and a one-factorization (in fact, it can be decomposed into two edge-disjoint one-factors)].

None of these conditions are sufficient too (cont.)

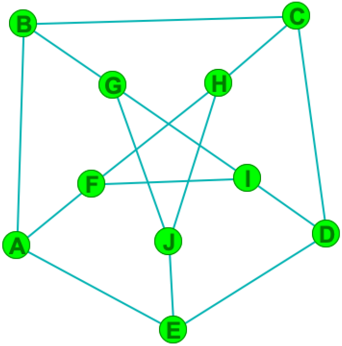
Example 2. The following graph is 3-regular (or equivalently, a *cubic graph*), but it does not have any one-factors (and of course it does not have a one-factorization).



Note that this is the smallest cubic graph without one-factors.

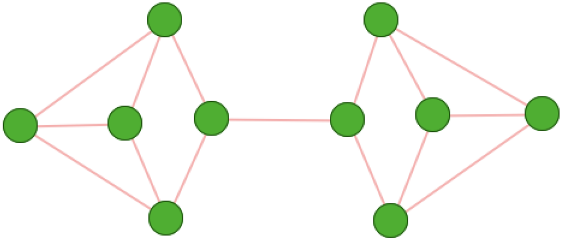
None of these conditions are sufficient too (cont.)

Example 3: The Petersen graph. As we have said, this is a cubic graph which satisfies $\kappa(G_0) = \lambda(G_0) = 3$, so it has no bridges. However, it does not have a one-factorization (although it has one-factors).



One more (non-)example

The following graph is the smallest cubic graph with no one-factorization *(can you see why it does not have a one factorization? also, can you find one factors of this graph?)*.



Let us now give a **necessary and sufficient** condition for a (not necessarily regular) graph to have one-factors.

Tutte's theorem

Theorem (Tutte, 1947)

Let $G = (V, E)$ be a graph (or multigraph). Given a proper subset S of V, write $\mathrm{OC}(G - S)$ for the number of **odd** connected components of $G - S$ (that is, the number of those connected components of $G - S$ which have odd order).

G has a one-factor **if and only if**

for every proper subset S of V, we have that $\mathrm{OC}(G - S) \leqslant |S|$.