MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 24

Tuesday, December 7

From Last Time

Definition: *Vertex Colourings*

Let $G = (V, E)$ be a graph. A *vertex colouring* of G is any function

$$\xi : V(G) \to \mathbb{N}_+.$$

The vertex subsets $V_i = \{v \in V(G) : \xi(v) = i\}$ will be called the *colour classes* of the vertex colouring ξ. Observe that the **non-empty** colour classes form a partition of $V(G)$.

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— A vertex colouring of G is called a *proper (vertex) colouring* if

no two adjacent vertices belong to the same colour class.

In other words, if each of the colour classes is an independent set of vertices.

Reminder: Chromatic number of a graph

For the most part, we want to deal with proper colourings of graphs.

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Let $G = (V, E)$ be a graph. A proper vertex colouring ξ of G is called an *n-colouring* if there are exactly n non-empty colour classes of ξ. In other words, if the range of the function ξ contains exactly n positive integers.

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G will be called n-colourable if we can find a (proper) n-colouring of G *(pictorially we can think of this as follows: G is n-colourable if n colours are enough for us to find a way to colour the vertices of G so that no two adjacent vertices will end up having the same colour)*.

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Definition 2 of Lecture 23

The *chromatic number* of a graph G is equal to the **smallest** integer n for which we can find a (proper) n-colouring of G.

If n_0 is this smallest integer, then we say that G is *n_0-chromatic*. We denote this smallest integer by $\chi(G)$ *(in other words, G is $\chi(G)$-chromatic)*.

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Finally, a $\chi(G)$-colouring of G, that is, a proper colouring of G in $\chi(G)$ colours, is called *minimal*.

Last time we also discussed some important results about the chromatic number

One of them was:

Theorem 1 of Lecture 23

For every graph G, we have that $\chi(G) \leqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

Remark. Note that the upper bound given by the theorem is best possible: **if G_0 is a complete graph, or G_0 is an odd cycle**, then

$$\chi(G_0) = \Delta(G_0) + 1.$$

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Theorem 2 of Lecture 23 (Brooks' theorem, 1941)

Let G be a connected graph which is neither a complete graph nor an odd cycle. Then $\chi(G) \leqslant \Delta(G)$.

Proving Theorem 1: A greedy colouring algorithm

Theorem 1

For every graph G, we have that $\chi(G) \leqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

Recall that we proved Theorem 1 by introducing/using a *greedy colouring algorithm.* This algorithm takes an ordering of the vertices, and allows us to colour the vertices one by one based on this ordering in such a way that the outcome is always a proper colouring of G requiring at most $\Delta(G) + 1$ colours. Hence, we must have

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Important Remark. The ordering of the vertices that we start with is important: different orderings of the vertices can give us proper colourings which use fewer or more colours.

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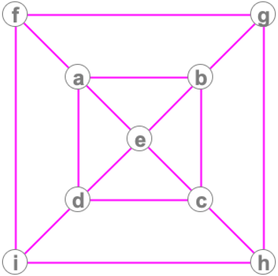
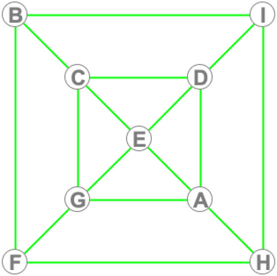
Important Remark. The ordering of the vertices that we start with is important: different orderings of the vertices can give us proper colourings which use fewer or more colours.

In particular, some of the orderings are much preferable than others because they will lead to a minimal colouring. However, figuring out which orderings are optimal is at least as hard as coming up with a minimal colouring.

Applying the algorithm to an example

Apply the greedy colouring algorithm to the following two labellings of the given graph to find proper colourings of the graph (consider an alphabetical ordering of the vertices).





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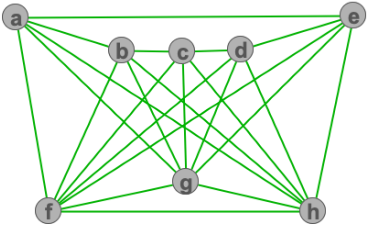
Note that the inequality here can sometimes be strict, as the following problem shows.

Past Homework Problem. Let G_0 be the following graph. Verify that (i) $\omega(G_0) = 5$, while (ii) $\chi(G_0) = 6$.



One more result about chromatic numbers

Proposition 1

Let G be a graph that does not contain any odd cycles.

Then $\chi(G) \leqslant 2$. More specifically, $\chi(G) = 1$ if G is a null graph (that is, if it contains no edges), otherwise $\chi(G) = 2$.

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The proof relies on a result we saw early in the term (in Lecture 8; see page 5 of that slide presentation).

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A graph G is a (not necessarily complete) bipartite graph if and only if G does NOT contain any odd cycles.

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Then, if we colour all vertices of G that end up in the partite set A with colour 1, and similarly we colour all vertices of G that end up in the partite set B with colour 2, we will have a proper colouring of G. This shows that $\chi(G) \leqslant 2$.

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Question. Could we now find precisely the chromatic number of any given tree?

Last Main Topic:
Plane and Planar Graphs

Crossing Number of a Graph

Recall that we can pictorially represent any graph G by some diagram on the plane drawn as follows:

- the vertices of G are drawn as distinct bullet points (or distinct very small circles),
- while two of these points are joined by a (simple) curve (usually, if possible, a straight line segment) if and only if the corresponding vertices are joined by an edge in G.

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- Finally, we also usually avoid having the relative interiors of more than two edges 'cross'/intersect at the same point.

Clearly these basic rules allow for infinitely many diagrams/pictorial representations for each graph G, and even though for much of our study of graphs all these can be thought of as equivalent (that is, equally useful), there are also instances where some 'secondary' features of these pictorial representations start having more importance to us. Then some representations of G become 'nicer'/more useful than others.

Crossing Number of a Graph

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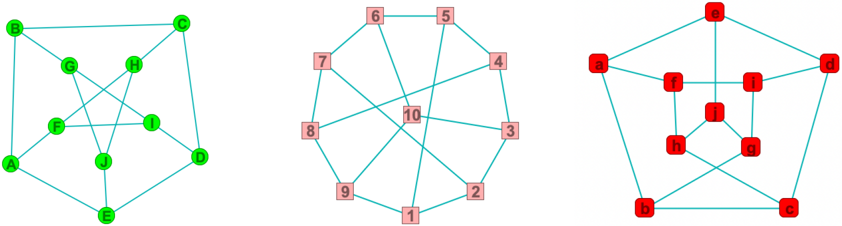
Example 1.



Note that all three diagrams here represent the same graph (or more accurately, they represent isomorphic graphs (given that the labels of vertices in each diagram are not the same)), and that they are all realisations/copies of the Petersen graph. However, the number of edge crossings in each diagram is different (what is it in each case?).

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Definition

Given a (finite) graph G, we define its *crossing number* to be the minimum number of edge crossings that a (permissible) pictorial representation of G can have.

We usually denote this number by $\mathrm{cr}(G)$.

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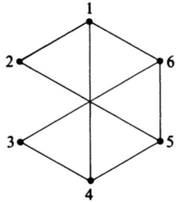
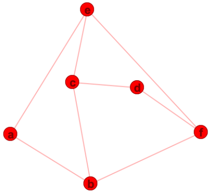
By the previous example, we obtain that $\mathrm{cr}(\text{Petersen graph}) \leqslant 2$. In fact, it can be shown that $\mathrm{cr}(\text{Petersen graph}) = 2$.

Other examples

Example 2. Early in the term we saw that the following two graphs:



from Bollobás' book



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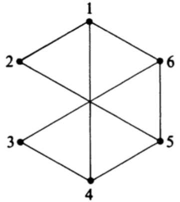

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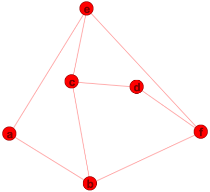

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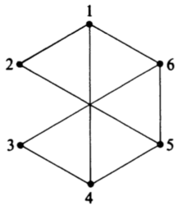

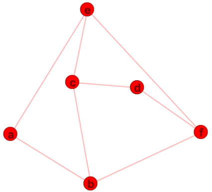

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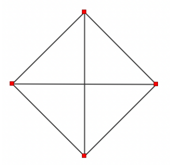
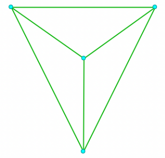
Example 3. Observe that both diagrams below are pictorial representations of K_4:





Thus $\mathrm{cr}(K_4) = 0$.

Why should we care about the crossing number?

Crossing numbers, and representations with a number of edge crossings close to the crossing number of the corresponding graph, are very useful in applications.

E.g. (from Wallis' book): "an early use was in the design of railway yards, where it is inconvenient to have the different lines crossing, and it is better to have longer track rather than extra intersections.

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More recently, small crossing numbers have proven important in the design of VLSI (Very Large Scale Integration) chips; if two parts of a circuit are not to be connected electrically, but they cross, a costly insulation process is necessary."

The crossing number of some basic graphs

Another interesting application (again from Wallis' book): "In 1944, during the 2nd World War, Pál Turán *(a mathematician whose work has been primarily in Extremal Combinatorics)* was forced to work in a brick factory, using hand-pulled carts that ran on tracks to move bricks from kilns to stores [little ovens on one side of the factory (for processing/shaping the bricks) to storage sites on the other side].

When tracks crossed, several bricks fell from the carts and had to be replaced by hand. The tracks [could be] modelled by a complete bipartite graph with one set of vertices representing kilns and the other representing stores, so to minimise the hours lost in replacing bricks, it was necessary to find $\mathrm{cr}(K_{m,n})$ (and also find a representation of $K_{m,n}$ that realised the minimum number of edge crossings)."

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This problem is now called "Turán's brick factory problem", and it is still unsolved. The best-known bound, which is also conjectured to be optimal (with the optimality still being unknown though), is given in the following

Theorem 1

We have that

$$\mathrm{cr}(K_{m,n}) \leqslant \lfloor \frac{m}{2} \rfloor \cdot \lfloor \frac{m-1}{2} \rfloor \cdot \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor.$$

A similar result

Theorem 2

For every $n \geqslant 3$, we have that

$$\mathrm{cr}(K_n) \leqslant \frac{1}{4}\lfloor\frac{n}{2}\rfloor \cdot \lfloor\frac{n-1}{2}\rfloor \cdot \lfloor\frac{n-2}{2}\rfloor \cdot \lfloor\frac{n-3}{2}\rfloor.$$

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Practice on an example: What would $\mathrm{cr}(K_5)$ be? Can you also find a representation of it which realises the minimum number of edge crossings?

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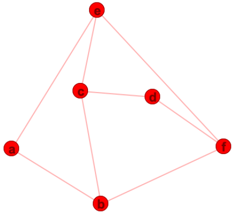
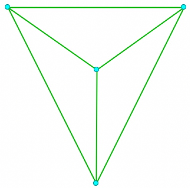
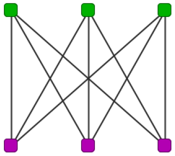
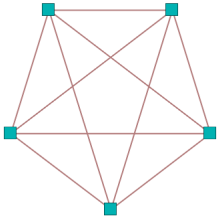




Some more examples (or non-examples)



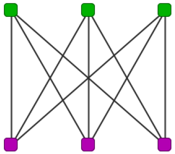
The complete bipartite graph $K_{3,3}$



The complete graph K_5

Clearly these representations of $K_{3,3}$ and of K_5 are not planar embeddings of them (in fact, even the number of edge crossings in each of them is not the best possible: indeed, (according to the theorems we stated earlier) it is certainly $>$ the crossing number of the corresponding graph).

Some more examples (or non-examples)



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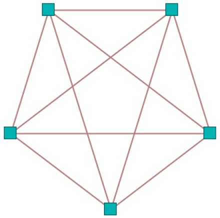

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Clearly these representations of $K_{3,3}$ and of K_5 are not planar embeddings of them (in fact, even the number of edge crossings in each of them is not the best possible: indeed, (according to the theorems we stated earlier) it is certainly $>$ the crossing number of the corresponding graph). As we will see, none of these graphs is planar, so we would NOT be able to find any plane representations of them.

These examples are the most characteristic ones

Before we properly state what this means, we need to introduce one more operation on graphs:

Definition

Let $G = (V, E)$ be a graph. A ***subdivision*** of G is a graph H which satisfies the following:

- H contains all the vertices of G, and perhaps a few more vertices;
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  - either appears as an edge of H as well,
  - or is replaced in H by the path $v_1\, w_1\, w_2\, \ldots\, w_{s-1}\, w_s\, v_2$, where $w_1\,(= w_1(v_1, v_2)), w_2\,(= w_2(v_1, v_2)), \ldots,\ w_s\,(= w_s(v_1, v_2))$ are new vertices that we add to the vertex set of H **(and are only meant as *internal* vertices of this new path in H, that is, they will only be used in relation to vertices v_1 and v_2, to 'elongate' the previous edge $\{v_1, v_2\}$).**

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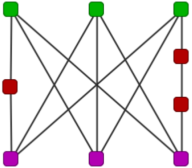
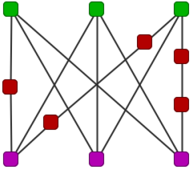
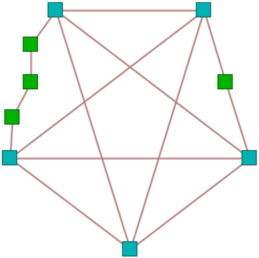
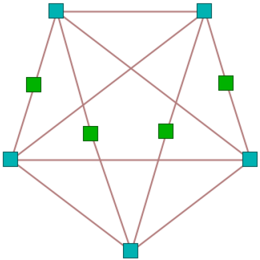
    In other words, in the second case, $e = \{v_1, v_2\}$ is replaced by the edges $\{v_1, w_1\}$, $\{w_1, w_2\}$, $\ldots$, $\{w_{s-1}, w_s\}$ and $\{w_s, v_2\}$.

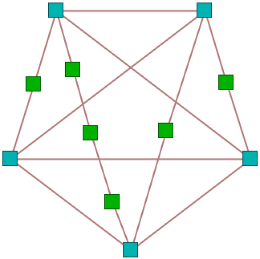
Some subdivisions of $K_{3,3}$ and of K_5











Kuratowski's theorem on planar graphs

Theorem (Kuratowski, 1930)

A finite graph G is planar **if and only if** none of its subgraphs is a subdivision of $K_{3,3}$ or of K_5.

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With this terminology at hand, Kuratowski's theorem can be more simply stated as:

a finite graph G is planar if and only if
G does not have any Kuratowski subgraphs.