MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 8

Tuesday, September 28

A detour: 'Forbidden' subgraphs

Many important results in Graph Theory are stated as follows: a graph G has a certain property if and only if we cannot find some 'not so nice' graphs $H_1, H_2, \ldots, H_n, \ldots$ among the **subgraphs** of G, or, even more 'strictly' sometimes, if we cannot find some given graphs $H_1, H_2, \ldots, H_n, \ldots$ among the **induced subgraphs** of G.

(we'll see some initial examples in the next two slides)

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Special Terminology. We call a graph H a *forbidden subgraph for a property P* of graphs if the following holds true: given any graph G,

property P holds true for G **if and only if**

G does not contain H as an **induced subgraph**

(in other words, if H is not isomorphic to an induced subgraph of G).

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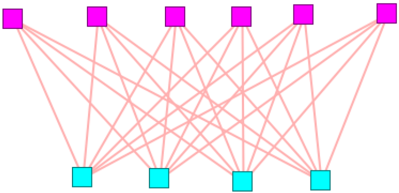

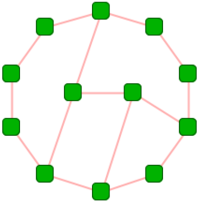

No odd cycles;
can view this as a subgraph of $K_{6,6}$ (how?)

Update on Terminology

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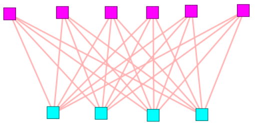
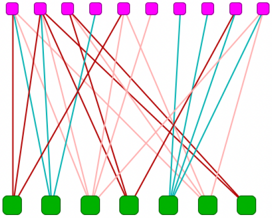


Graph $K_{6,4}$



Bipartite Graph; Subgraph of $K_{10,7}$

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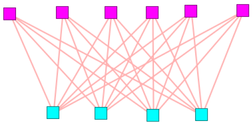

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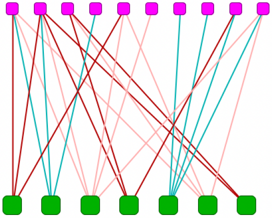

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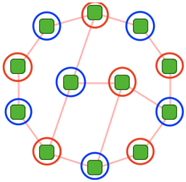

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Justification. We can directly check *(say, by induction on the number k of vertices of a cycle that we consider each time)* that a complete bipartite graph does NOT contain any odd cycles, so each of its subgraphs will also not contain any odd cycles.

(In other words, first check that any $K_{m,n}$ does not contain 3-cycles, and then that it cannot contain 5-cycles and so on; this is because the parts of $K_{m,n}$ that the vertices of a cycle belong to have to be alternating, that is, if we start from a vertex in, say, the 'upper' part of $K_{m,n}$, we then have to move to a vertex in the 'lower' part, and so on.)

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- If ℓ_w is an even integer, then 'colour' the vertex w blue.
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Claim. The blue vertices of G form one part, and the red vertices another part. There is NO edge in G joining two blue vertices, and also NO edge joining two red vertices, so G can be viewed as a bipartite graph.

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Find a $v_0 - w_1$ path P_1 in G which has length ℓ_{w_1} (that is, the smallest possible), and similarly find a $v_0 - w_2$ path P_2 in G which has length ℓ_{w_2}. **Recall that both ℓ_{w_1} and ℓ_{w_2} are even.**

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$$\ell_{w_1} - \ell_{z_0} \; + \; 1 \; + \; \ell_{w_2} - \ell_{z_0} \, .$$

This is an odd number, contradicting the assumption that G contains NO odd cycles. Thus the 'blue' vertices w_1, w_2 cannot be joined by an edge.

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Similarly, we can show that two 'red' vertices cannot be joined.

A criterion for line graphs

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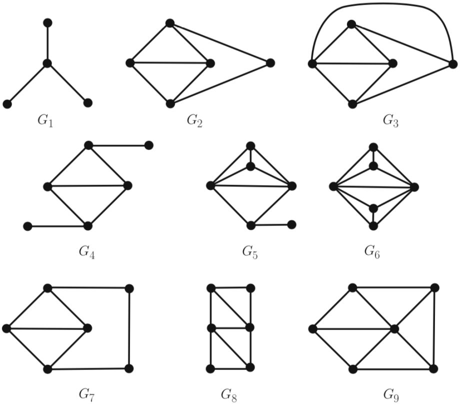
Theorem (Beineke, 1968)

A graph G is the *line graph* of some other graph H **if and only if** the following 9 graphs are forbidden subgraphs for G *(in other words, if and only if none of the following graphs is an induced subgraph of G).*



from the Balakrishnan-Ranganathan book

Back to Connectivity

Definitions

Let G be a connected graph.

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is no longer connected.
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Definition again, more generally formulated

If we start with a graph H which is not necessarily connected, then

1. a vertex $\tilde{v}$ of H is called a *cutvertex* of H if, by deleting $\tilde{v}$ (and of course all the edges $\tilde{v}$ is incident with), we increase the number of connected components of H.
2. an edge $\tilde{e}$ in H is called a *bridge* of H if, by deleting $\tilde{e}$, we increase the number of connected components of H.

Definitions

Let $G = (V, E)$ be a connected graph.

1. A subset V' of the vertex set V of G is called a *vertex cut* if we have that

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The parameters of
Vertex and Edge Connectivity

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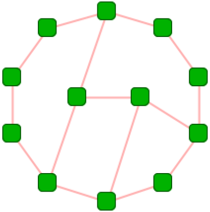
Edge Connectivity

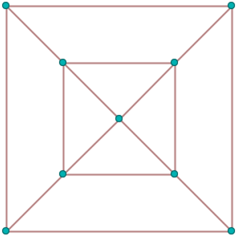
The parameter $\lambda(G)$

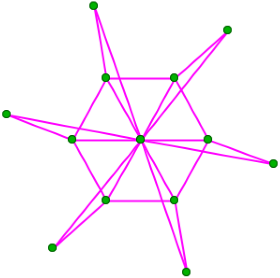
Let G be a connected graph of order $\geqslant 2$. We define the *edge connectivity $\lambda(G)$* of G to be **the minimum cardinality of an edge cut** of G.

Testing our understanding on examples

Question. What is $\lambda(G)$ for each of the following graphs G?







A “flower” graph

A very important and useful observation

Part of an important theorem we will state shortly

Let G be a connected graph of order $\geqslant 2$ *(which implies that G has at least one edge).*

We have that: $$\lambda(G) \leqslant \delta(G)\,.$$

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Then v_0 has d_0 neighbours in G, say the vertices $w_1, w_2, \ldots, w_{d_0}$, and is incident with exactly d_0 edges of G, the edges

$$e_1 = v_0 w_1,\ e_2 = v_0 w_2, \ldots\ldots,\ e_{d_0} = v_0 w_{d_0}.$$

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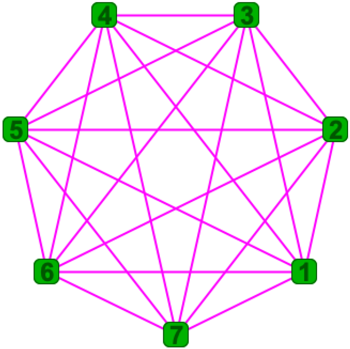
If we delete the edges $e_1, e_2, \ldots, e_{d_0}$, then v_0 is an isolated vertex in the resulting subgraph $G - \{e_1, e_2, \ldots, e_{d_0}\}$ (which still has all the other vertices too), and thus $G - \{e_1, e_2, \ldots, e_{d_0}\}$ is disconnected.

This shows that $\{e_1, e_2, \ldots, e_{d_0}\}$ is an edge cut of G, and since it has cardinality $d_0 = \delta(G)$, we must have

$$\lambda(G) := \text{minimum cardinality of an edge cut of } G \leqslant \delta(G)\,.$$

Examples (cont.)

Question. What is the edge connectivity of a complete graph? That is, what is $\lambda(K_n)$?



Complete graph on vertices $\{1, 2, 3, 4, 5, 6, 7\}$

An equivalent way of thinking about the last question

Given that every graph on n vertices is a subgraph of the complete graph K_n, and given that we are asking for the minimum number of edges that we would have to remove from K_n in order to get a disconnected graph, we could ask this question in a 'complementary' way too:

What is the maximum size (that is, largest number of edges) of a disconnected subgraph of K_n which contains all n vertices?

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It's worth comparing this question with the 'opposite' question we discussed in the last lectures, about what the minimum size of a connected graph H on n vertices is. Recall that we have found this minimum size to be $n-1$ *(because any graph with size $< n-1$ will be disconnected, whereas paths (or any other trees) on n vertices are connected (by definition) and have precisely $n-1$ edges as we showed)*.

Answer to our two 'complementary' questions

What is the maximum size (that is, largest number of edges) of a disconnected subgraph of K_n which contains all n vertices?

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Answer. We know that the size $e(K_n)$ of K_n is

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

(since the edge set of K_n contains all 2-element subsets of the vertex set of K_n); if $e_{\max}$ is the maximum size of a disconnected subgraph of K_n which contains all n vertices, then

$$\lambda(K_n) = \binom{n}{2} - e_{\max}.$$

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Based on these, we see that G is the disjoint union of two complete graphs:

$$G = K_k \oplus K_{n-k}$$

for some $1 \leqslant k \leqslant n-1$.

Maximum size of a disconnected graph on n vertices (cont.)

We have already confirmed that

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It remains to check that, when we only consider $k \in \{1, 2, \ldots, n-2, n-1\}$, this expression for $e(G)$ takes its maximum value when $k = 1$ or $k = n-1$ (in fact, the function $k \mapsto \binom{k}{2} + \binom{n-k}{2}$ is decreasing for $1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$, and then it becomes increasing).

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Plugging $k = n-1$ above, we obtain that

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is the maximum possible size of a disconnected graph on n vertices. In fact, we have also found that such a graph should be **the disjoint union of a complete graph on $n-1$ vertices and an isolated vertex.**

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We also conclude that $\lambda(K_n) = e(K_n) - \binom{n-1}{2} = \binom{n}{2} - \binom{n-1}{2} = n-1$.

Conclusions from all related discussions so far

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More generally, we can show...

Theorem

Let G be a graph of order n which has exactly k connected components (where $1 \leqslant k \leqslant n$).

Then the maximum possible size of G is $\binom{n-k+1}{2}$, and the minimum possible size is $n - k$.

See video lecture 5.2 (Edge estimates) from Dr. Seidon Alsaody's list.

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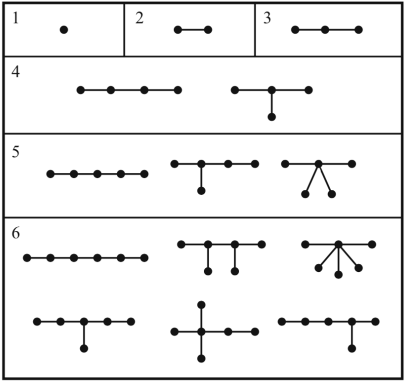
- By the observation, we get that $\lambda(G) \leqslant \delta(G) < n - 1$.

The other extreme

Question. Consider a tree T on n vertices. What is $\lambda(T)$? Does it depend on which tree on n vertices we start with?



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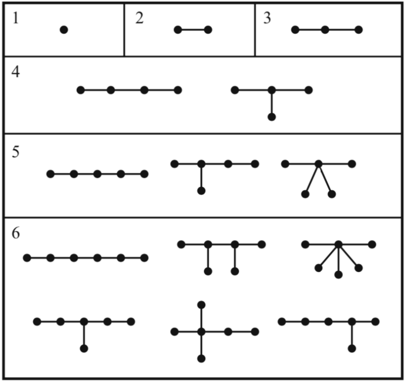

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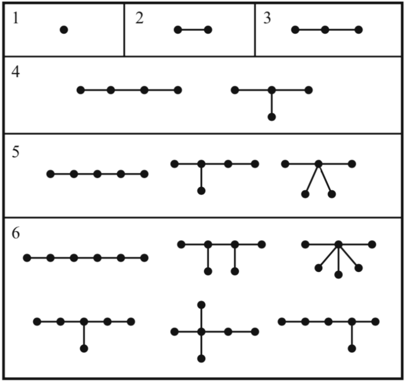

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We always have $\lambda(T) = 1$ for any tree T with at least two vertices. Moreover, we have that **every edge of T is a bridge** (or equivalently, a cutedge). *(this is because, by deleting an arbitrary edge e from T, we are left with a subgraph $T - e$ on n vertices which has $n - 2$ edges)*

What about Vertex Connectivity?

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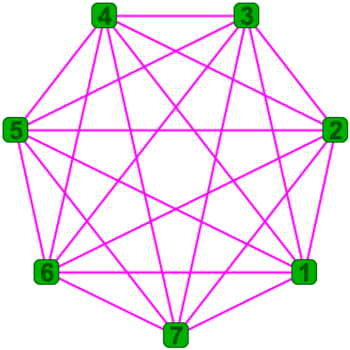
Completely analogously to edge connectivity, **we would like** *to define the parameter of 'vertex connectivity' of a connected graph G to be the minimum cardinality of a vertex cut of G.*

But...

complete graphs don't have vertex cuts



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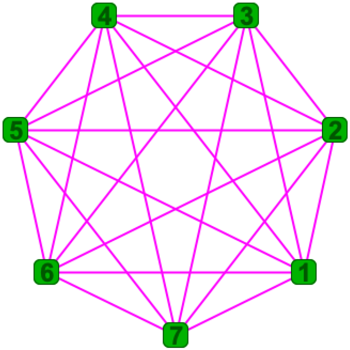
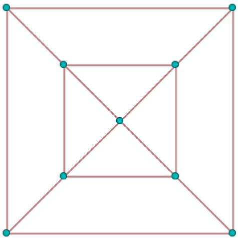


Important Remark. Given $n \geqslant 2$, we have that, for every (proper) subset V' of the vertex set V of K_n, the graph $K_n - V'$ is again a complete graph (on the vertices $V \setminus V'$ now), and hence it cannot be disconnected.

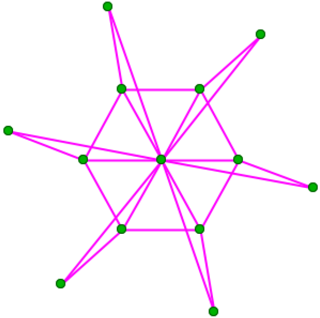
$\rightsquigarrow$ a complete graph does not have any vertex cuts.

In all other cases however...

Any connected graph of order $\geqslant 2$ which is not a complete graph will have vertex cuts. *(For each of the examples below, find a vertex cut; if possible, try to find one with smallest possible cardinality.)*



Graph G_1



Graph G_2

How do we reconcile these two facts, in order to give a useful definition?

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to be discussed next time