Math 322
Homework Problem Set 2

Remark 1. In HW1, Problem 1, you confirmed that the square A_G^2 of the adjacency matrix A_G of a finite graph G can also give us information about the graph: its diagonal coincides with the degree sequence of the graph G.

The purpose of the following problem is to highlight other pieces of information that powers of the adjacency matrix can give us about the graph G, and in particular with regard to how many different walks in G we can have.

Problem 1. Let $G = (V, E)$ be a finite graph of order n, with $V = \{v_1, v_2, \ldots, v_n\}$. Write A for the adjacency matrix of G.

(i) Show that, if $1 \leqslant i, j \leqslant n$, $i \neq j$, the (i, j)-th entry of the matrix $A^2 = A \cdot A$ equals the number of common neighbours of the vertices v_i and v_j.

Observe also that this is equivalent to saying that the (i, j)-th entry of the matrix A^2 equals the number of $v_i - v_j$ walks in G that have length 2: indeed,

- for every common neighbour v_s of v_i and v_j (that is, every vertex v_s of G such that v_s is different from v_i and v_j, and such that both $v_i\, v_s$ and $v_j\, v_s$ are edges of G), we get that the sequence $v_i\, v_s\, v_j$ is a $v_i - v_j$ walk in G of length 2,
- while conversely every $v_i - v_j$ walk of length 2 must pass by one more vertex of G which has to be a neighbour of both v_i and v_j.

(ii) Show that, for every $k \geqslant 2$, the (i, j)-th entry of the matrix $A^k = A \cdot A \cdots A \cdot A$ (where the latter product has k factors) equals the number of $v_i - v_j$ walks in G that have length k.

[*Hint.* You may wish to use mathematical induction in k. In such a case, what would be your base case? Would you have already obtained it?]

(iii) *(Practice Question, not to be submitted)* What would the diagonal entries of the powers A^k of A be counting?

Remark 2. Recall that we proved in class that, if a graph G is <u>disconnected</u>, then the complement graph $\overline{G}$ of G is instead connected (see Lecture 6, Proposition 1).

The primary purpose of the following problem is to show that, in the remaining cases, that is, when G is connected, the complement $\overline{G}$ could be either connected OR disconnected.

Problem 2. (i) Give an example of a connected graph G on 6 vertices whose complement $\overline{G}$ is **disconnected** (but is **not** the null graph on 6 vertices). Verify that your example works.

(ii) Give an example of a connected graph G on 7 vertices whose complement $\overline{G}$ is **connected**. Verify that your example works.

Problem 3. Let G be a connected graph containing at least 2 vertices (by which we also obtain that G contains at least one edge; why?).

Show that $L(G)$ *(which we can consider in this case)* is a connected graph too.

Problem 4. (i) Let G_1 and G_2 be two graphs of order $n \geqslant 2$ and size $m \geqslant 1$. Show that, if G_1 and G_2 are isomorphic, then their line graphs are isomorphic as well, that is,

$$L(G_1) \cong L(G_2)$$

(it may help to start with labelled representations of G_1 and G_2).

(ii) By considering labelled representations of $K_{1,3}$ and K_3, find their respective line graphs and confirm that they are isomorphic (even though $K_{1,3} \ncong K_3$).

Remark 3. The above problem shows that, even if we restrict our attention only to connected graphs, we cannot conclude that two such graphs are isomorphic if we know that their line graphs are isomorphic.

However, a very nice theorem by a mathematician called Hassler Whitney showed that the example in part (ii) is the only example that prevents us from formulating the converse to part (i) in the case of connected graphs; in other words, we can state:

Let G_1 and G_2 be connected graphs of order $n \geqslant 2$ and size $m \geqslant 1$.
Then $G_1 \cong G_2$ if and only if $L(G_1) \cong L(G_2)$,
unless we have that one of G_1, G_2 is isomorphic to $K_{1,3}$
while the other one is isomorphic to K_3.

Problem 5. Let k be a positive integer, and let G be a finite graph satisfying $\delta(G) \geqslant k$.

(i) Show that G contains a path of length at least k.

(ii) If $k \geqslant 2$, show that G contains a cycle of order at least $k+1$ (that is, a cycle on at least $k+1$ vertices).

Remark 4. Recall that we proved in class (see Lecture 6, Proposition 2) that, if a finite graph G has order n and $\delta(G) \geqslant \frac{n-1}{2}$, then G must be connected.

Problem 6. Show with an example that the lower bound for the minimum degree $\delta(G)$ in this result is best possible. In particular, find a finite graph G on **at least 11 vertices** which satisfies the following:

- G is **disconnected**;
- $\delta(G) \geqslant \frac{n-2}{2}$.

Briefly explain why your example has all the required properties.

Problem 7. Let G be a finite, connected graph. Prove or disprove each of the following statements.

(i) If G has no bridges, then G contains **exactly one** cycle.

(ii) If G has no cutvertices, then G has no bridges.

(iii) If $|G| \geqslant 3$ and G has no cutvertices, then G has no bridges.

(iv) If G has no bridges, then G has no cutvertices.

[*Hint/Clarification.* Given any one of the above statements, **if (you believe) it is correct**, you will have to verify it for an arbitrary finite connected graph G (and not only for a specific graph of your choice).

On the other hand, **if (you believe) it is false**, then you only need to give one example of a finite connected graph G that fails to satisfy the statement; it may help in such a case to look at examples of graphs we have seen in class (or of subgraphs of such graphs).]