MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 20

Tuesday, November 23

Proving some of the sufficient conditions for Hamiltonicity

Recall:

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$. Then G is Hamiltonian.

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of <u>distinct</u> and <u>non-adjacent</u> vertices u and v of G,
we have that $\deg(u) + \deg(v) \geqslant n$.

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Proposition 3

Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

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edges. Then G is Hamiltonian.

We have said that Dirac's theorem follows from Ore's theorem (since the condition in Dirac's theorem is stronger than the condition in Ore's theorem).

We also showed last time that Proposition 3 follows from Ore's theorem too (because the number of edges required in the statement of Prop 3 implies that the condition in Ore's theorem holds).

Proving Ore's Theorem

Consider a graph G on $n \geqslant 3$ vertices which has the property:

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1st Side Note: The given condition implies quickly that G is connected. Indeed, assume towards a contradiction that G had at least two connected components, say components G_1 and G_2. Consider a vertex u_1 in G_1 and a vertex v_1 in G_2. Assume also that G_1 contains exactly r of the n vertices of G (where $r < n$ since v_1 is not in G_1).

Then we must have $\deg_G(u_1) = \deg_{G_1}(u_1) \leqslant r - 1$, and $\deg_G(v_1) \leqslant n - r - 1$.

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Then we must have $\deg_G(u_1) = \deg_{G_1}(u_1) \leqslant r - 1$, and $\deg_G(v_1) \leqslant n - r - 1$. At the same time, u_1 and v_1 are different vertices which are NOT adjacent, so they should satisfy the degree-condition assumed above for such pairs of vertices: we should have

$$(r-1) + (n-r-1) \geqslant \deg_G(u_1) + \deg_G(v_1) \geqslant n,$$

which is absurd.

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Back to the proof of Ore's theorem: we first show that G has a Hamilton path. Let $\mathcal{P}$ be a path in G with maximum possible length, and assume towards a contradiction that $\mathcal{P}$ is NOT a Hamilton path.

Then, if we write

$$\mathcal{P} : w_1\, w_2\, \ldots\, w_{k-1}\, w_k,$$

we must have $k \leqslant n - 1$, and thus there exists at least one vertex z_0 different from the vertices appearing in $\mathcal{P}$ *(note that k here is the order of $\mathcal{P}$, while its length is $k - 1$).*

Proving Ore's theorem (cont.)

Crucial Observation: All the neighbours of the initial vertex w_1 of $\mathcal{P}$ are among the vertices $w_2, w_3, \ldots, w_k$. *(This also implies that z_0 and w_1 are non-adjacent vertices. Moreover, it implies that $\deg(w_1) \leqslant k-1$.)*

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Indeed, if we could find a neighbour y_0 of w_1 which does not appear in $\mathcal{P}$, then we could also consider the path

$$y_0 \; w_1 \, w_2 \, \ldots \, w_{k-1} \, w_k,$$

which extends $\mathcal{P}$. This would contradict the assumption that $\mathcal{P}$ has maximum possible length.

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Similarly, we observe that all the neighbours of the terminal vertex w_k are among the other vertices of $\mathcal{P}$ (and thus, again in particular, z_0 and w_k are non-adjacent).

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Similarly, we observe that all the neighbours of the terminal vertex w_k are among the other vertices of $\mathcal{P}$ (and thus, again in particular, z_0 and w_k are non-adjacent).

Key Claim: Let us write $d_1 = \deg(w_1)$, and suppose that $w_{j_1}, w_{j_2}, \ldots, w_{j_{d_1}}$ are all the neighbours of w_1, where $j_1 = 2 < j_2 < \cdots < j_{d_1} \leqslant k$.

Let us focus on the d_1 vertices in $\mathcal{P}$, each of which precedes (in $\mathcal{P}$) one of the neighbours of w_1: that is, the vertices

$$w_{j_1-1}, \;\; w_{j_2-1}, \;\; \ldots, \;\; w_{j_{d_1}-1}.$$

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It can be shown that **none of these vertices is a neighbour of z_0.**

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Proof of the Key Claim: Assume towards a contradiction that, for some $s \in \{1, 2, \ldots, d_1\}$, we have that $w_{j_s - 1}$ is a neighbour of z_0 (recall also that w_{j_s} is a neighbour of w_1).

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We conclude that the neighbours of z_0 are among the remaining $n - d_1 - 1$ vertices of G; or even better, if we exclude w_k as well, among the remaining $n - d_1 - 2$ vertices of G. In other words, $\deg(z_0) \leqslant n - d_1 - 2$.

Finishing the proof

Recall that: we started with a graph G on $n \geqslant 3$ vertices which has the property:

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– At the same time, z_0 and w_1 are distinct, non-adjacent vertices, thus, by the degree-condition assumed above, we must have

$$n \leqslant \deg(z_0) + \deg(w_1) \leqslant n - 2$$

which is absurd.

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$$\mathcal{P} : w_1\, w_2\, \ldots\, w_{k-1}\, w_k.$$

– We assumed towards a contradiction that $\mathcal{P}$ is NOT a Hamilton path, or in other words that $k \leqslant n-1$. Thus, we could find a vertex z_0 in G which does NOT appear in $\mathcal{P}$.

– We showed that z_0 cannot be adjacent to w_1 or to w_k (since we shouldn't be able to extend $\mathcal{P}$ to a longer path). We also showed that we must have
$\deg(z_0) \leqslant n - \deg(w_1) - 2$.

– At the same time, z_0 and w_1 are distinct, non-adjacent vertices, thus, by the degree-condition assumed above, we must have

$$n \leqslant \deg(z_0) + \deg(w_1) \leqslant n - 2$$

which is absurd.

– We conclude that the assumption that $\mathcal{P}$ is NOT a Hamilton path, even though it has maximum possible length, was incorrect.

Finishing the proof

Recall that: we started with a graph G on $n \geqslant 3$ vertices which has the property:

for every pair of distinct and non-adjacent vertices u and v of G,
we have that $\deg(u) + \deg(v) \geqslant n$

and we have found a Hamilton path

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– if w_1 and w_n are adjacent, then we are done (because $\mathcal{P}$ can be extended to a Hamilton cycle by adding the edge $w_n\, w_1$ at the end).

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HW5, Problem 1

Consider a graph G_0 on n vertices (where $n \geqslant 3$) which has the following property: u, z is a pair of **non-adjacent** vertices of G_0 such that $\deg_{G_0}(u) + \deg_{G_0}(z) \geqslant n$.

Prove that the graph $G_0 + uz$ (that is, the graph we get by adding the edge $\{u, z\}$ to G_0) is Hamiltonian **if and only if** G_0 is Hamiltonian.

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Prove that the graph $G_0 + uz$ (that is, the graph we get by adding the edge $\{u, z\}$ to G_0) is Hamiltonian **if and only if** G_0 is Hamiltonian.

We conclude that, since $G + \{w_1, w_n\}$ is Hamiltonian (and since $\deg_G(w_1) + \deg_G(w_n) \geqslant n$), then G must be Hamiltonian too (it's just that it contains a different Hamilton cycle).

Next Main Topic:

Factors, Matchings and (Stable) Marriages

A fun problem

Suppose that 11 new hires at a company want to get to know each other, so they plan to have a series of dinners at different houses. Their dinner plans are as follows.

(i) Each evening they will be sitting at a round table.

(ii) The seating arrangements should be such that no person has the same neighbour at any two different dinners.

Show that this can go on for 5 evenings (and hence each person will eventually sit next to any other person).

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Analysing what the problem asks for:

- Each seating arrangement can be viewed as a Hamilton cycle of K_{11}.
- Since no person can have the same neighbour twice, any two such Hamilton cycles must be **edge-disjoint**.
- **Recall** why there can be **at most** 5 such dinners:
  - Consider one of the hires, say person A. At the first dinner, person A has a pair of neighbours, who can come from the remaining 10 new hires.
  - At the second dinner, person A cannot sit next to any of the two colleagues he or she sat next to during the first dinner, so the new neighbours of person A are among the remaining $10 - 2 = 8$ new hires.
  - Continuing like this, we see that the pairs of neighbours of person A form a collection of pairwise disjoint 2-subsets of the set of 10 collleagues of person A, so this collection can have at most 5 such subsets.

Theory behind this problem

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

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Some immediate observations:

- A spanning tree (or spanning forest, in cases where G is not connected) is a factor of G.
- Every graph (or multigraph) G has a trivial factorization: since G is a factor of itself, the collection $\{G\}$ is a factorization of G.
- The problem in the previous slide asks us to find **a factorization of K_{11} consisting of Hamilton cycles of K_{11}.**

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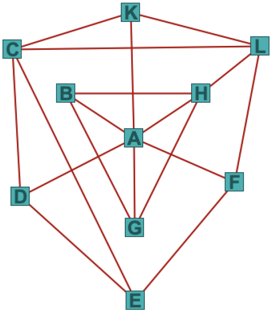
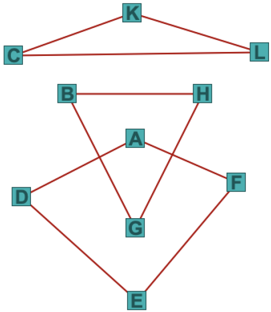
- Similarly, a spanning subgraph of G is called a *two-factor* of G **if it is 2-regular.** A *two-factorization* of G is a factorization of G consisting of two-factors of G.

A subtle point. Note that a Hamilton cycle of G is a two-factor of G, but not every two-factor needs to be a Hamilton cycle. E.g. the graph on the right below is a two-factor of the graph on the left (but not a Hamilton cycle):





One solution to the 'fun' problem

The solution will be a special case of the following proposition.

Proposition 1

(i) If $s \geqslant 3$ is an odd positive integer, then K_s can be decomposed into $\frac{s-1}{2}$ pairwise edge-disjoint Hamilton cycles.

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Proof of (i). Let $s = 2k + 1$ with $k \geqslant 1$, and suppose that the vertices of K_s are indexed by the integers $0, 1, 2, \ldots, 2k$. For each $i = 1, 2, \ldots, k$, define the walks Z_i by

$$Z_i : \; v_0 \, v_i \, v_{i+1} \, v_{i-1} \, v_{i+2} \, v_{i-2} \, \ldots \, v_{i+j} \, v_{i-j} \, \ldots \, v_{i+k} \, v_0$$

and according to the convention that, whenever $i - j \leqslant 0$ (for any of the internal vertices of the walk), then the index $i - j$ is replaced by the unique integer in $\{1, 2, \ldots, 2k-1, 2k\}$ which is congruent to $i - j$ modulo $2k$.

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We can check that these walks are Hamilton cycles of K_s, and that the collection $\{Z_1, Z_2, \ldots, Z_k\}$ is a two-factorization of K_s.

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Note that all these are Hamilton cycles of K_{11}, and any two of them are edge-disjoint.

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$$H_0 = \big(V(K_t),\, \big\{\{v_\infty, v_0\},\, \{v_1, v_{2r}\},\, \{v_2, v_{2r-1}\},\, \ldots,\, \{v_r, v_{r+1}\}\big\}\big).$$

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Then the collection $\{Z_1, Z_2, \ldots, Z_r, H_0\}$ is a factorization of K_t.

Say, in the specific case of K_8, we have...

$8 = 2 \cdot 3 + 2$, and hence $r = 3$. Moreover,

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Note that all these are factors of K_8, any two of them are edge-disjoint, and combined together they give us a factorization of K_8.

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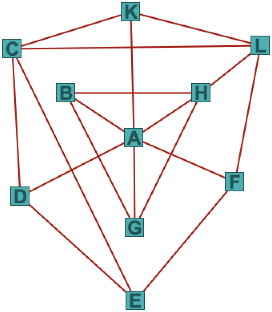
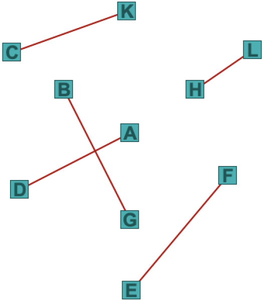
Note that all these are factors of K_8, any two of them are edge-disjoint, and combined together they give us a factorization of K_8.

Question. Can we also find a one-factorization of K_8 based on this factorization? How many one-factors will we get?

Remarks about one-factors

Question 1. What about a one-factor of the last (non-complete) graph we saw?





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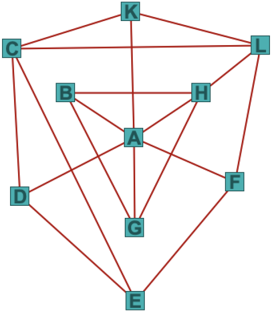

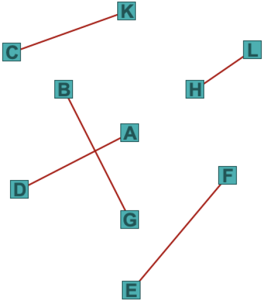

Question 2. Does the given graph have a one-factorization?

Remarks about one-factors (cont.)

Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

Remarks about one-factors (cont.)

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If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

(ii) G cannot have isolated vertices *(be careful that, if G is a multigraph, this condition is not immediately equivalent to $\delta(G) \geqslant 1$, given that some vertices of G may have no neighbours, but may have loops attached to them, so their degree will still be positive)*.

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Necessary conditions for the existence of a one-factorization

Let G be graph of order $n = 2k$ and size m. Suppose that G has a one-factorization. Then

(I) $k = \frac{n}{2}$ must divide m.

Remarks about one-factors (cont.)

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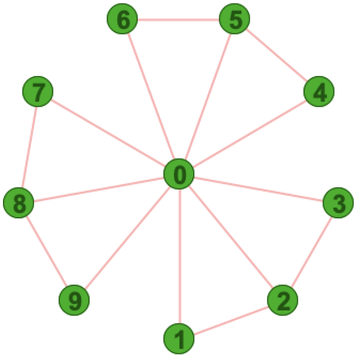
(III) ***G* cannot have bridges** (except if G is a 1-regular graph itself, and hence the trivial factorization $\{G\}$ of G is a one-factorization too).

None of these conditions are sufficient too

Example 1. The following graph has even order, and no isolated vertices, but it does not have a one-factor (why?).



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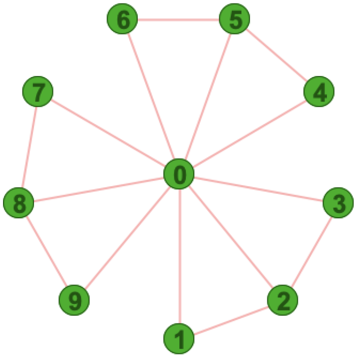

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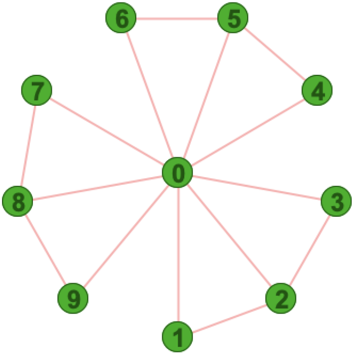
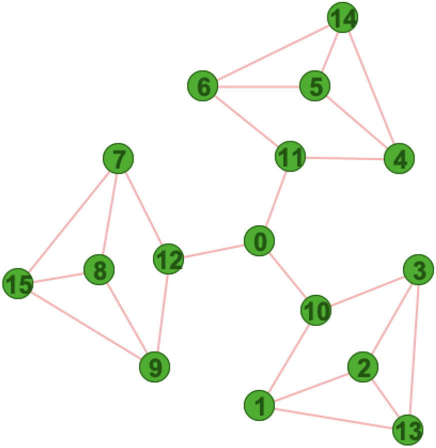


As a consequence of this, we obtain that the graph is not Hamiltonian either [why? note again that a Hamilton cycle with **an even number** of vertices has both a one-factor, and a one-factorization (in fact, it can be decomposed into two edge-disjoint one-factors)].

None of these conditions are sufficient too (cont.)

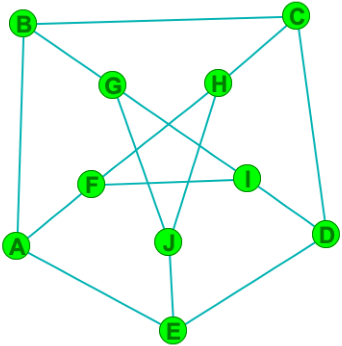
Example 2. The following graph is 3-regular (or equivalently, a *cubic graph*), but it does not have any one-factors (and of course it does not have a one-factorization).



Note that this is the smallest cubic graph without one-factors.

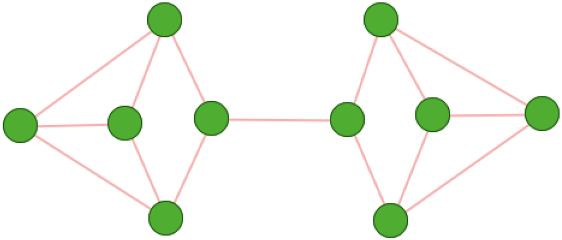
None of these conditions are sufficient too (cont.)

Example 3: The Petersen graph. As we have said, this is a cubic graph which satisfies $\kappa(G_0) = \lambda(G_0) = 3$, so it has no bridges. However, it does not have a one-factorization (although it has one-factors).



One more (non-)example

The following graph is the smallest cubic graph with no one-factorization *(can you see why it does not have a one factorization? also, can you find one factors of this graph?)*.



Let us now give a **necessary and sufficient** condition for a (not necessarily regular) graph to have one-factors.

Tutte's theorem

Theorem (Tutte, 1947)

Let $G = (V, E)$ be a graph (or multigraph). Given a proper subset S of V, write $\mathrm{OC}(G - S)$ for the number of **odd** connected components of $G - S$ (that is, the number of those connected components of $G - S$ which have odd order).

G has a one-factor **if and only if**

for every proper subset S of V, we have that $\mathrm{OC}(G - S) \leqslant |S|$.

Testing Tutte's theorem on examples and non-examples