Math 322
Homework Problem Set 4

Problem 1. Let $n \geqslant 3$, and let G be a labelled connected graph on n vertices which is NOT a tree. Prove that G has **at least 3** different spanning trees *(of course, not necessarily pairwise non-isomorphic)*.

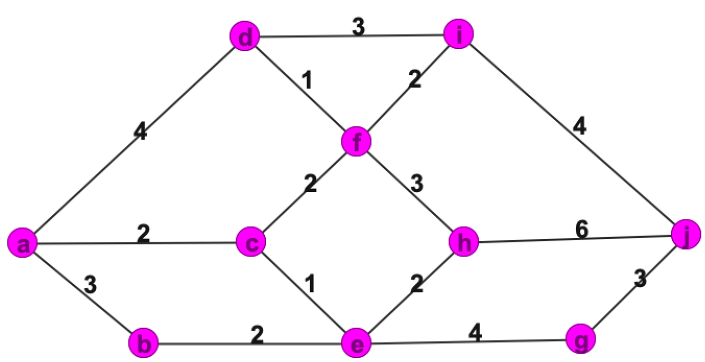
Problem 2. (a) Give an example of a (labelled) connected weighted graph G_1 which has the following properties:

- there exists a pair of edges e_1 and e_2 of G_1 which have the same weight, and some cycle C_0 of G_1 which contains both edges;
- G_1 has a **unique** minimum weight spanning tree T_0, and T_0 contains edge e_1, but not edge e_2.

Verify that your example works.

(b) Can you find a (labelled) connected weighted graph G_2 on $n \geqslant 9$ vertices and with at least $2n$ edges, which has **exactly 2** minimum weight spanning trees, and such that one of these trees is a path, while the other one is not? Justify your answer.

Problem 3. Consider the following weighted graph G_0:



(a) Using Dijkstra's algorithm, find the shortest distance from vertex a to every other vertex of G_0. Show all your work (that is, how you proceed at each stage of the algorithm).

You **don't need** to also find ~~paths of shortest length~~ minimum weight paths in this part of the problem.

(b) By relying, if you want to, on your work in part (a), find all ~~paths of shortest length~~ minimum weight paths from a to j.

Problem 4. In Lecture 16 we will see that, if a connected graph G of size $\geqslant 3$ is Eulerian, then its line graph $L(G)$ is Hamiltonian.

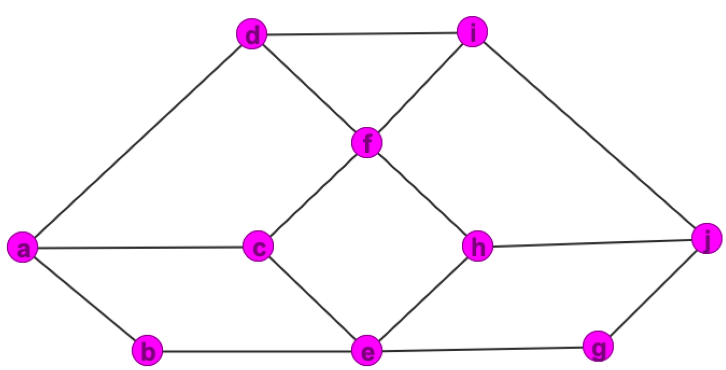
Show that the converse is not always true. That is, find a connected graph H of size $\geqslant 3$ such that

- its line graph $L(H)$ is Hamiltonian,
- but H is **not** Eulerian.

Confirm that your example has the above properties.

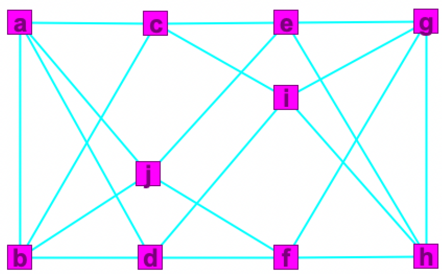
Problem 5. Let d be a positive integer $\geqslant 2$. Prove the following statement: for every connected d-regular graph G, its line graph $L(G)$ is Eulerian.

Problem 6. (a) Consider the graph G_0 from Problem 3:

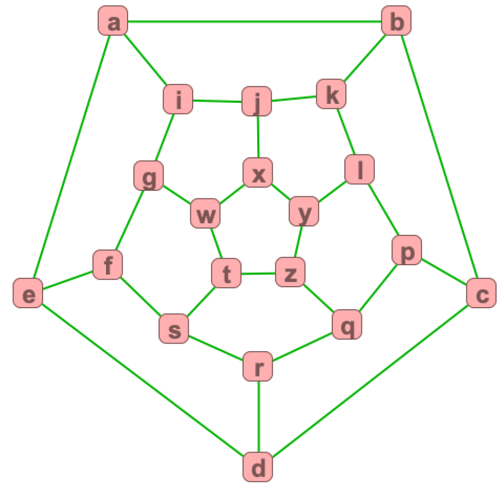


Show that G_0 has a Hamilton path. Moreover, show that G_0 is **not** Hamiltonian.

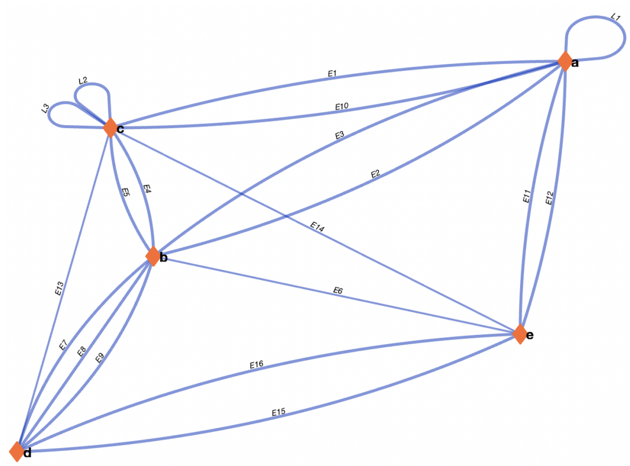
(b) For each of the graphs (or multigraphs) below, determine whether it is Eulerian or not. If the (multi)graph is Eulerian, then find an Euler circuit. If it is not, explain why it is not, and then describe some Eulerization of the (multi)graph which uses the minimum number of edges.



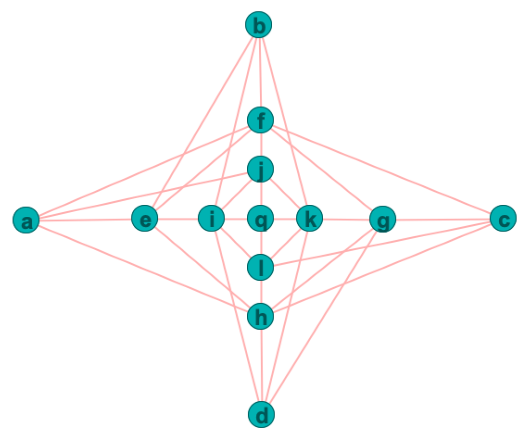
Graph H_1



Graph H_2



Multigraph H_3



Graph H_4