Math 322
Suggested solutions to Homework Set 2

Problem 1. (i) Let us write $a_{s,t}$ for the (s,t)-th entry of the adjacency matrix A of G, and $b_{s,t}$ for the (s,t)-th entry of the matrix $A^2 = A\cdot A$; in other words, we write $A = \big(a_{s,t}\big)_{1\leqslant s,t\leqslant n}$ and $A^2 = \big(b_{s,t}\big)_{1\leqslant s,t\leqslant n}$.

We recall that

$$a_{s,t} = \begin{cases} 1 & \text{if } \{v_s, v_t\} \in E(G) \\ 0 & \text{otherwise} \end{cases}.$$

We also have that

$$\begin{aligned} b_{i,j} &= \langle \mathrm{Row}_i(A),\, \mathrm{Col}_j(A)\rangle \qquad \text{(dot product of } \mathrm{Row}_i(A) \text{ and } \mathrm{Col}_j(A)) \\ &= \sum_{s=1}^{n} a_{i,s}\cdot a_{s,j} \\ &= \big|\big\{s \in \{1,2,\ldots,n\} : a_{i,s} = a_{s,j} = 1\big\}\big| \\ &= \big|\big\{s \in \{1,2,\ldots,n\} : \{v_i, v_s\} \in E(G) \text{ and } \{v_s, v_j\} \in E(G)\big\}\big|. \end{aligned}$$

In other words, $b_{i,j}$ equals the number of vertices v_s which are adjacent both to v_i and to v_j, which is what we wanted to show.

As the subsequent remark also states, this is the same as the total number of $v_i - v_j$ walks of length 2 in the graph G.

(ii) We will use induction in k to verify the following

Claim 1. For every $s, t \in \{1,2,\ldots,n\}$ (with s not necessarily different from t), the (s,t)-th entry of A^k is equal to the total number of $v_s - v_t$ walks of length k in G (that is, walks in G which have length k and endvertices the vertex v_s and the vertex v_t).

Note that this includes the desired conclusion of part (ii), as well as the corresponding conclusion for the cases where $s = t$ (which part (ii) did not require us to consider); however it is easier to prove the entire claim using induction. Observe also that, when $s = t$ (or in other words, v_s and v_t stand for the same vertex of G), $b_{s,s}$ counts closed walks of length k.

Base Case: $k = 2$. We observe that we have verified the claim in this case in part (i) as well as in HW1, Problem 1 (where we showed that the (i,i)-th entry of A^2 equals the number of neighbours of the vertex v_i; note that, as we argue in the Remark after part (i), we can check that the number of neighbours of v_i coincides with the number of closed walks of length 2 which start (and end) at v_i).

Induction Step: Assume that the claim has been verified for some $k \geqslant 2$, and we now want to prove it for $k+1$. For convenience, let us write $w_{s,t}$ for the (s,t)-th entry of the matrix A^k (that is, $A^k = \big(w_{s,t}\big)_{1\leqslant s,t\leqslant n}$). We also keep writing $A = \big(a_{s,t}\big)_{1\leqslant s,t\leqslant n}$.

Consider $i, j \in \{1, 2, \ldots, n\}$ (not necessarily different). Then

$$\begin{aligned}(i,j)\text{-th entry of } A^{k+1} &= (i,j)\text{-th entry of } A^k \cdot A \\ &= \langle \mathrm{Row}_i(A^k),\, \mathrm{Col}_j(A)\rangle \\ &= \sum_{s=1}^{n} w_{i,s}\cdot a_{s,j} \\ &= \sum_{s: a_{s,j}=1} w_{i,s}.\end{aligned}$$

In other words, if $v_{l_1}, v_{l_2}, \ldots, v_{l_{d_j}}$ are the neighbours of v_j in G where $d_j = \deg(v_j)$ (that is, $l_1, l_2, \ldots, l_{d_j}$ are those indices s for which we have $a_{s,j} = 1$), then

the (i,j)-th entry of A^{k+1} equals

the number of walks in G of length k which start at v_i and end at v_{l_1}

$+$ the number of walks in G of length k which start at v_i and end at v_{l_2}

$+$ the number of walks in G of length k which start at v_i and end at v_{l_3}

....

$+$ the number of walks in G of length k which start at v_i and end at $v_{l_{d_j}}$.

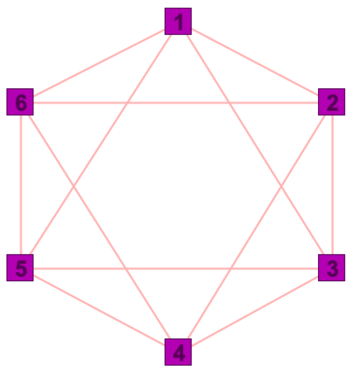
This final sum equals the total number of walks in G of length $k+1$ which start at v_i and end at v_j: indeed,

- note that, if $v_i\, u_1\, u_2 \cdots u_{k-1}\, u_k\, v_j$ is such a walk, then the vertex u_k must be a neighbour of v_j. Say it is the vertex v_{l_r} for some $1 \leqslant r \leqslant d_j$, in which case the initial part $v_i\, u_1\, u_2 \cdots u_{k-1}\, u_k$ of this walk, which is itself a walk of length k from v_i to $u_k = v_{l_r}$, is counted above by the term w_{i,l_r}.
- Conversely, for any walk $v_i\, z_1\, z_2 \cdots z_{k-1}\, v_{l_r}$ of length k which starts at v_i and ends at a neighbour v_{l_r} of v_j, we can get a walk of length $k+1$ which ends at v_j by just adding the vertex v_j at the end of the sequence $v_i\, z_1\, z_2 \cdots z_{k-1}\, v_{l_r}$; thus each such walk of length k corresponds uniquely to a $v_i - v_j$ walk of length $k+1$.

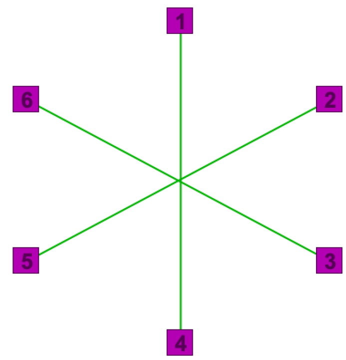
We conclude that the (i,j)-th entry of A^{k+1} equals the total number of $v_i - v_j$ walks of length $k+1$ in G. This completes the proof of the Induction Step too.

(iii) *(Practice Question)* Note that, by proving the entire Claim 1 in part (ii), we have also verified that each diagonal entry $w_{j,j}$ of A^k counts the total number of closed walks of length k which start (and end) at the vertex v_j.

Problem 2. (i) The graph G_1 below has order 6 and size 12. Thus, its complement has size 3, and is clearly disconnected based on what we have established regarding the minimum size of a connected graph (we can also verify this from the pictures; the second graph is the disjoint union of three paths of length 1).

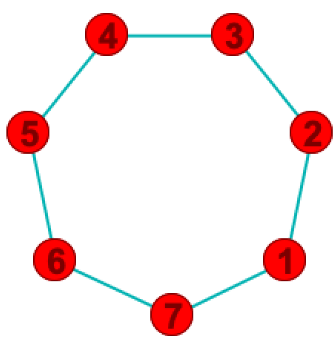


Graph G_1

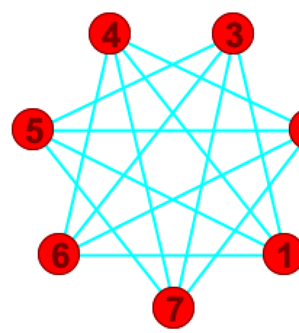


Complement of G_1

(ii) The 7-cycle below has order 7 and is connected, with a connected complement too (note that e.g. the complement contains the cycle 1 3 5 7 2 4 6 1 which passes by all the vertices).



Graph C_7



Complement of C_7

Problem 3. Let us write $v_1, v_2, \ldots, v_n$ for the vertices of G (recall that $n \geqslant 2$) and $e_1, e_2, \ldots, e_m$ for the edges of G, which are also the vertices of $L(G)$.

Consider two different edges e_i, e_j of G. We need to show that there is a $e_i\!-\!e_j$ path in $L(G)$. Let v_{i_1}, v_{i_2} be the two endvertices of e_i, and v_{j_1}, v_{j_2} be the two endvertices of e_j (note that these aren't necessarily four distinct vertices, because the edges e_i and e_j that we have considered might have a common endvertex).

We can consider two cases:

Case 1: e_i and e_j have a common endvertex; say $v_{i_1} = v_{j_1}$. Then e_i and e_j are neighbours in $L(G)$, and therefore the sequence $e_i\, e_j$ is a path from e_i to e_j.

Case 2: e_i and e_j don't have a common endvertex. Then $v_{i_1}, v_{i_2}, v_{j_1}$ and v_{j_2} are four different vertices of G. Given that G is connected, we can find a $v_{i_1}\!-\!v_{j_1}$ path in G. We write this path here, including the edges of G that it traverses:

$$v_{i_1} - e_{s_1} - w_1 - e_{s_2} - w_2 - e_{s_3} - w_3 - \cdots\cdots - w_{l-2} - e_{s_{l-1}} - w_{l-1} - e_{s_l} - v_{j_1}\,.$$

Observe that $w_1, w_2, \ldots, w_{l-1}$ are $l-1$ distinct vertices of G which are also all different from v_{i_1} and v_{j_1}. It also follows that $e_{s_1}, e_{s_2}, \ldots, e_{s_l}$ are l distinct edges of G. Moreover, consecutive edges here are adjacent (since they share one endvertex), and hence the sequence

$$e_{s_1}\, e_{s_2}\, e_{s_3}\, \cdots\, e_{s_{l-1}}\, e_{s_l}$$

is a path in $L(G)$.

We now check that this path can give us the $e_i\!-\!e_j$ path in $L(G)$ that we want. We consider a few cases:

- $e_{s_1} = e_i$ and $e_{s_l} = e_j$. This happens if we already have that $w_1 = v_{i_2}$ and $w_{l-1} = v_{j_2}$. Then the path we already found is the path we wanted.
- $e_{s_1} = e_i$ but $e_{s_l} \neq e_j$. In this case, e_{s_l} is a neighbour of e_j in $L(G)$ (given that these two edges of G both have v_{j_1} as an endvertex). Therefore, the sequence

  $$e_i = e_{s_1}\, e_{s_2}\, e_{s_3}\, \cdots\, e_{s_{l-1}}\, e_{s_l}\, e_j$$

  is also a path in $L(G)$.
- $e_{s_l} = e_j$ but $e_{s_1} \neq e_i$. We can deal with this case completely analogously to the previous one: the sequence

  $$e_i\, e_{s_1}\, e_{s_2}\, e_{s_3}\, \cdots\, e_{s_{l-1}}\, e_{s_l} = e_j$$

  is a path in $L(G)$.

– Finally, in the case that both $e_{s_1} \neq e_i$ and $e_{s_l} \neq e_j$, the $e_i - e_j$ path we want is the path

$$e_i \, e_{s_1} \, e_{s_2} \, e_{s_3} \cdots e_{s_{l-1}} \, e_{s_l} \, e_j$$

(note that, in this case, e_{s_1} and e_i are neighbours in $L(G)$ since they share the vertex v_{i_1} when viewed as edges of G, and similarly e_{s_l} and e_j are neighbours in $L(G)$ since they share the vertex v_{j_1} when viewed as edges of G).

Problem 4. (i) We have the assumption that $G_1 \cong G_2$. We recall that this means that, if $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_n\}$, then we can find a bijection

$$f : V(G_1) \to V(G_2)$$

which preserves adjacencies, in other words, such that, for every $i, j \in \{1, 2, \ldots, n\}$, it will hold that

$$\{v_i, v_j\} \in E(G_1) \quad \textbf{if and only if} \quad \{f(v_i), f(v_j)\} \in E(G_2).$$

This also implies that, if $E(G_1) = \{e_1, e_2, \ldots, e_{m-1}, e_m\}$ is the edge set of G_1 and $E(G_2) = \{d_1, d_2, \ldots, d_{m-1}, d_m\}$ is the edge set of G_2, then we also get a bijection

$$g : E(G_1) \to E(G_2).$$

Indeed, we can set $g(e_s) = d_t$ if the following holds true: the endvertices of e_s are the vertices v_i, v_j of G_1 while the endvertices of d_t are the vertices $u_k = f(v_i)$ and $u_l = f(v_j)$ of G_2 (note that this rule gives us a function from $E(G_1)$ to $E(G_2)$ which is injective, given that f is injective, and also surjective, since $\{f(v_i), f(v_j)\} \in E(G_2)$ only if $\{v_i, v_j\} \in E(G_1)$).

We now recall that $V(L(G_1)) = E(G_1) = \{e_1, e_2, \ldots, e_{m-1}, e_m\}$, while $V(L(G_2)) = E(G_2) = \{d_1, d_2, \ldots, d_{m-1}, d_m\}$. We already have the bijection

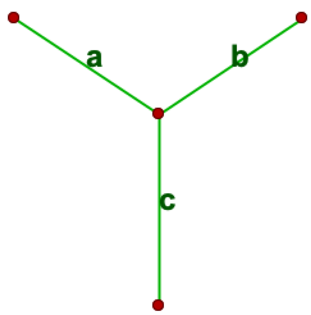
$$g : V(L(G_1)) \to V(L(G_2)).$$

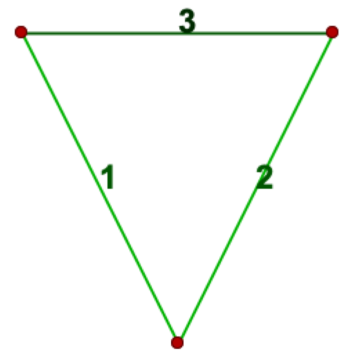
We check that this is a graph isomorphism. We have that

$\{e_i, e_j\} \in E(L(G_1))$
if and only if
the edges e_i and e_j of G_1 have a common endvertex, say vertex $v_s \in V(G_1)$
if and only if
the edges $g(e_i)$ and $g(e_j)$ of G_2 have a common endvertex, vertex $f(v_s) \in V(G_2)$
if and only if
$\{g(e_i), g(e_j)\} \in E(L(G_2))$.

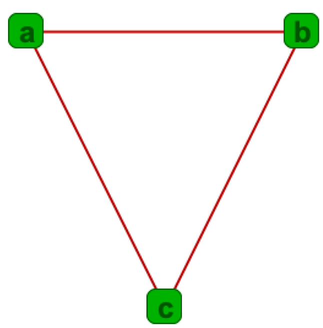
Thus g preserves adjacencies. Given that we have found a graph isomorphism from $L(G_1)$ to $L(G_2)$, we conclude that $L(G_1) \cong L(G_2)$.

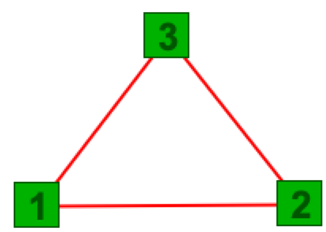
(ii) In the picture below we have labelled representations of the graphs $K_{1,3}$ and K_3 (note that we have only labelled the edges of the graphs):





We observe that each of these graphs has 3 edges, any two of which are adjacent. Therefore, the line graph of either of these graphs is isomorphic to K_3, the complete graph on 3 vertices:





Problem 5. (i) Let s_0 be the largest possible length of a path in G (clearly we can find a maximum value here, since a path in G can contain at most all the vertices of G, so it can have length at most $|G| - 1$). We need to show that $s_0 \geqslant k$.

Assume towards a contradiction that $s_0 < k$, and consider a path P_0 in G which has length s_0. Let us write $x_0, x_1, x_2, \ldots, x_{s_0-1}, x_{s_0}$ for the vertices which P_0 passes through (in the order that these vertices appear in the path, with x_0 being one of the endvertices of P_0, viewed here as its initial vertex).

Since $\delta(G) \geqslant k$, we have that $\deg(x_{s_0}) \geqslant k > s_0$, and thus we can find at least one neighbour of x_{s_0} which is not among the vertices $\{x_0, x_1, \ldots, x_{s_0-1}\}$. Say this is the vertex y_0 of G.

But then the walk

$$x_0 \, x_1 \, x_2 \, \cdots \, x_{s_0-1} \, x_{s_0} \, y_0$$

is also a path of G (given that all the vertices are distinct), and it has length $s_0 + 1 > s_0$, which contradicts the assumption that P_0 had the largest possible length among paths in G.

We conclude that the assumption that s_0, the largest possible length of a path in G, is $< k$ was incorrect.

(ii) Again, let us consider a path P_1 of G of largest possible length. Then P_1 has the form

$$z_0 \, z_1 \, z_2 \, \ldots \, z_{l-1} \, z_l$$

for some distinct vertices $z_0, z_1, z_2, \ldots, z_{l-1}$ and z_l of G, with $l \geqslant k \geqslant 2$.

Claim. All the neighbours of z_0 are included among the vertices $z_1, \ldots, z_{l-1}, z_l$. This is because if this were not the case, then we could find a neighbour y_1 of z_0 which would not be among the vertices already in the path P_1, and then the walk

$$y_1 \, z_0 \, z_1 \, z_2 \, \ldots \, z_{l-1} \, z_l$$

would be a longer path in G, contradicting the assumption that P_1 has longest possible length.

We look for the last vertex in P_1 which is a neighbour of z_0; say this is vertex z_{t_1}. Based on the claim, we know that $t_1 \geqslant \deg(z_0) \geqslant \delta(G) \geqslant k \geqslant 2$. Then the walk

$$z_0 \, z_1 \, z_2 \, \ldots \, z_{t_1-1} \, z_{t_1} \, z_0$$

is a cycle of G which passes through $t_1 + 1 \geqslant k + 1$ vertices.

Problem 6. *Note/Initial Thinking.* Suppose that we first try to find a graph with the required properties that has 11 vertices.

Then $n = 11$, and thus $\frac{n-2}{2} = \frac{9}{2} = 4.5$. Thus, if we require that $\delta(G) \geqslant 4.5$, then necessarily, given that $\delta(G)$ has to be an integer, we also get that

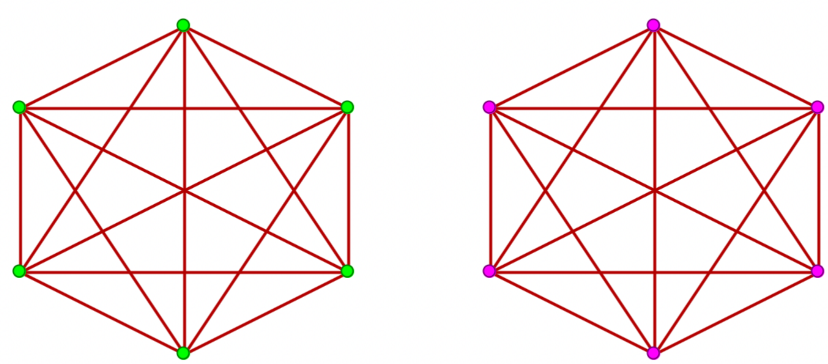
$$\delta(G) \geqslant 5 = \frac{n-1}{2}.$$

But then the assumptions of Proposition 2 from Lecture 6 would be satisfied, and thus we would get that G is connected.

In other words, it's impossible to find a graph G with the required properties that has exactly 11 vertices (analogously we can verify that it's impossible to find a graph with the required properties that has an odd number of vertices).

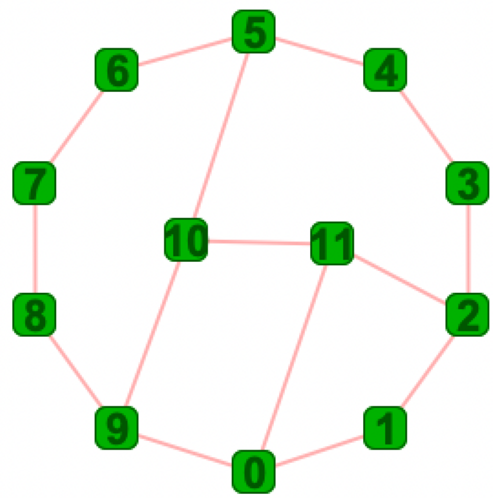
Actual Solution. However, we can find a graph G with the required properties that has 12 vertices: the graph below is the disjoint union of two copies of K_6, so

- it is disconnected,
- it contains exactly $6 + 6 = 12$ vertices,
- and the minimum degree (in fact, the common degree of all vertices, since this turns out to be a regular graph) is equal to $5 = \frac{12-2}{2}$.



Graph G: 2 connected components, each of them isomorphic to K_6

Problem 7. (i) This is false. One counterexample is the following graph that we have already discussed a few times in class:



We have seen that this graph does not have any bridges (this is because every edge is part of at least one cycle contained in the graph).

However, the graph clearly contains more than one cycle *(in fact, in this example we can even find cycles which do not have any common vertices: e.g. the cycles* 5 6 7 8 9 10 5 *and* 0 1 2 11 0*; of course this is stronger than what was asked for here).*

(ii) This is false in a very special case: consider the graph K_2. Recall that complete graphs do not have vertex cuts, and thus they definitely don't have cutvertices. In particular, we can verify this directly for K_2.

On the other hand, the only edge contained in K_2 is a bridge.

(iii) This is correct. If we take a finite, connected graph G which has at least 3 vertices, then there are two possibilities we could consider.

Case 1: G is a complete graph on at least 3 vertices, so it definitely doesn't have bridges (given that all its edges will be found on cycles). It also satisfies all the assumptions we want, given that, as remarked before too, complete graphs don't have cutvertices.

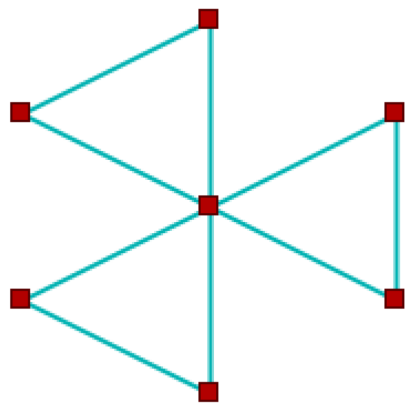
Case 2: G is not a complete graph, in which case it does have vertex cuts, but by our assumptions <u>none of those vertex cuts consists of a single vertex</u>.

In this case then,

$$\kappa(G) = \text{minimum cardinality of a vertex cut of } G \geqslant 2,$$

and thus by Whitney's theorem $\lambda(G) \geqslant 2$ too. This shows that G does not have bridges.

(iv) This is false. Consider the following graph:



Note that this graph can be seen as the 'union' (note: not disjoint union of course) of three 'triangles' (or 3-cycles) <u>with a single common vertex</u>; this common vertex is a cutvertex of the graph.

On the other hand, this graph has no bridges since every edge of it is found on a cycle.