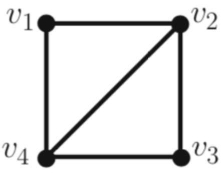
MATH 322 – Graph Theory
Fall Term 2021

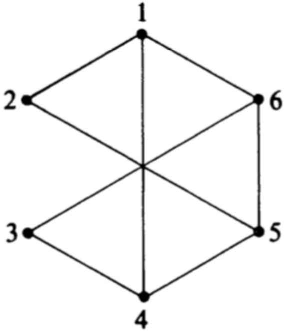
Notes for Lecture 3

Thursday, September 9

Examples/practice questions from last time



Graph G_1



Graph G_3

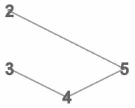
Question 1. Can we view G_1 as a subgraph of G_3? (here of course you do need to and you can relabel the vertices of G_1 using four of the numbers from $\{1, 2, 3, 4, 5, 6\}$)

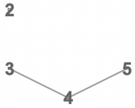
Question 2. How many subgraphs of G_3 do we have on the vertices $\{2, 3, 4, 5\}$? Can you draw them? What is the induced subgraph here?

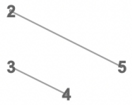
Question 3. What is the maximum length of a path in G_3? How many paths can you find with this length?

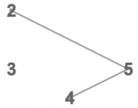
Isomorphic and Non-Isomorphic Graphs

Recall the subgraphs of G_3 that we found for Question 2:



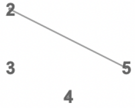






Induced Subgraph

. . . .



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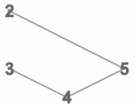

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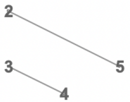

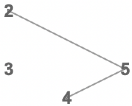

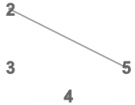

In general, if a graph $H = \big(V(H), E(H)\big)$ has m edges, and we want to find all subgraphs of H with vertex set the entire $V(H)$ (that is, if we don't want to remove any vertex), **then we will have** 2^m **different subgraphs** (because 2^m is the number of all subsets of a set with m elements, in this case the edge set E).

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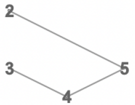

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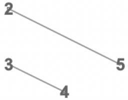

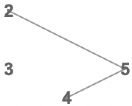

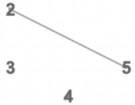

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Question. But will all these subgraphs be essentially different?

Isomorphic Graphs: Definition

Let G, H be two graphs. An *isomorphism* from the graph G to the graph H is a bijective function

$$f : V(G) \to V(H)$$

that preserves adjacencies. That is, f has to be 1-1 and onto, and we must have that

$$e \in E(G) \text{ and has endvertices } v_i, v_j \in V(G)$$
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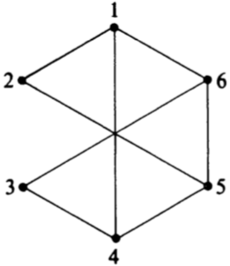
Terminology. If two graphs are not isomorphic (that is, if no such bijection from the vertex set of the first graph onto the vertex set of the second graph exists), we say that the graphs are *non-isomorphic*.

An example

Recall one of our previous examples:



Graph G_3

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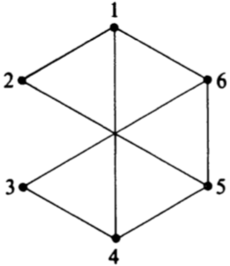

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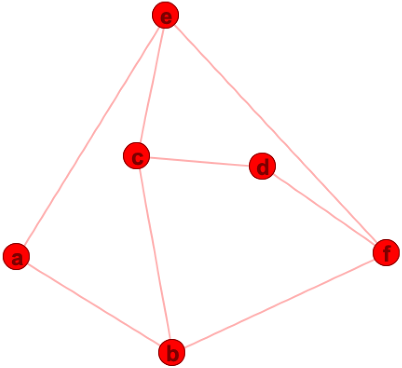

Graph G_4

We have that $G_3 \cong G_4$ (that is, the graphs G_3 and G_4 are isomorphic).

Indeed, observe that the bijective function $f : V(G_3) \to V(G_4)$ given by

$$f(1) = b, \qquad f(2) = a, \qquad f(3) = d$$
$$f(4) = f, \quad f(5) = e, \quad \text{and} \;\; f(6) = c,$$

which we can also encode more simply as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ b & a & d & f & e & c \end{pmatrix}$$

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To verify this, note that

$$E(G_3) = \{12,\ 14,\ 16,\ 25,\ 34,\ 36,\ 45,\ 56\}$$

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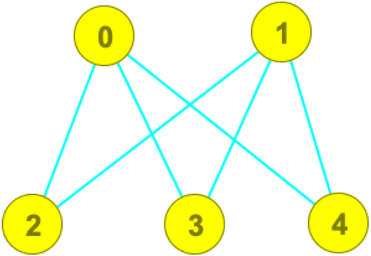
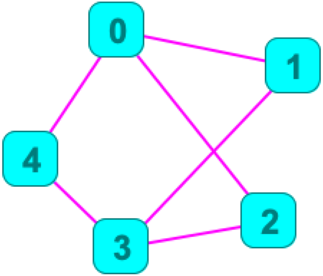
$$\begin{aligned} E(G_4) &= \{ab, bf, bc, ae, df, cd, ef, ce\} \\ &= \{f(1)f(2), f(1)f(4), f(1)f(6), f(2)f(5), \\ &\qquad f(3)f(4), f(3)f(6), f(4)f(5), f(5)f(6)\}. \end{aligned}$$

Another pair of isomorphic graphs



Graph G_5



Graph G_6

Left as an exercise: Find an isomorphism from G_5 to G_6.

Important Remarks on Isomorphic Graphs

- Isomorphic graphs must have the same order (that is, they must have the same number of vertices).

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- Isomorphic graphs must have the same size (that is, they must have the same number of edges).
- Isomorphic graphs must have the same degree sequences, up to reordering of the sequences *(much more about degree sequences very soon)*.

Just as a small clarification for now, if $G = (V, E)$ is a finite graph, with $V = \{v_1, v_2, \ldots, v_n\}$, then the degree sequence of G is the sequence $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ *(as we will see, we often reorder this sequence so that it becomes decreasing; however there are instances where it's preferable to keep it like this)*.

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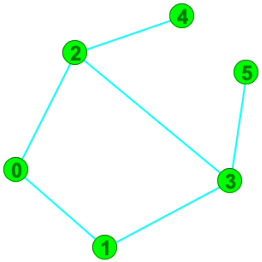

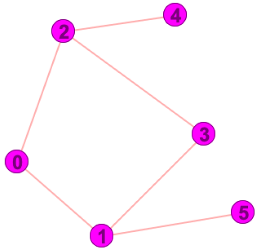

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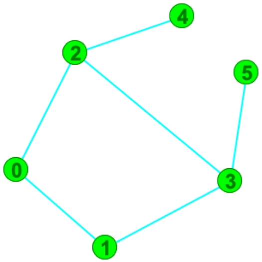
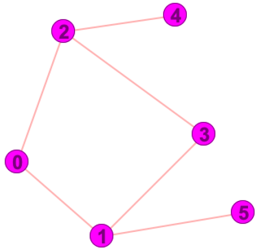
However they are not isomorphic (can you find one reason why?).





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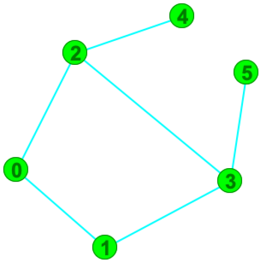
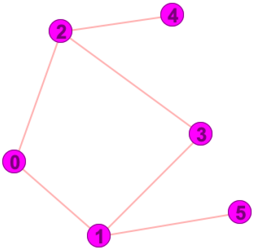
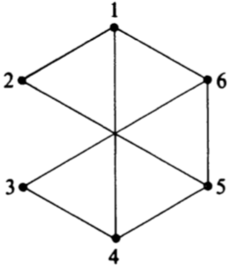




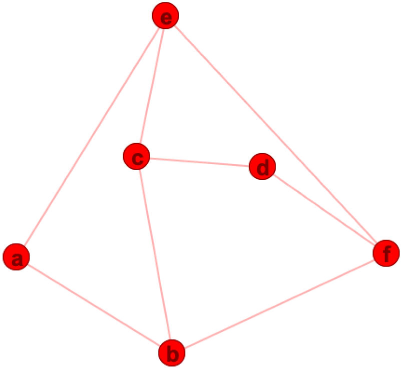
One possible approach: What is the maximum length of a path in each graph? Shouldn't it be the same if two graphs are isomorphic?

Isomorphic Graphs and Adjacency Matrix

Back to the graphs G_3 and G_4 which we saw are isomorphic:



Graph G_3



Graph G_4

Isomorphic Graphs and Adjacency Matrix

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$$A_{G_3} = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\text{and} \quad A_{G_4} = \begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

Note that the matrices are different.

Isomorphic Graphs and Adjacency Matrix

However, if two graphs G and H are isomorphic, then there exists a permutation matrix P (that is, a 0-1 matrix which has exactly one entry equal to 1 in each row, and also exactly one entry equal to 1 in each column) such that

$$A_G = P \cdot A_H \cdot P^T.$$

In other words, A_H can be turned into A_G by permuting the rows of A_H in a suitable way, and by permuting its columns too <u>in the same way</u>.

Related question: How many *essentially different* graphs on 3 vertices?

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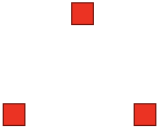
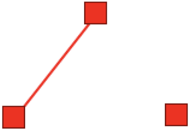
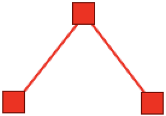


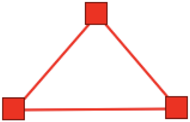
No edges



1 edge

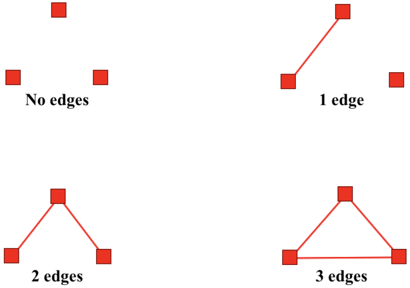


2 edges



3 edges

Related question: How many *essentially different* graphs on 3 vertices?



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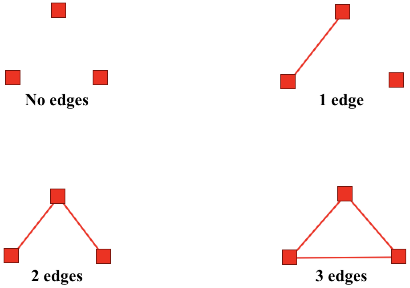

Takeaway here: In order to avoid 'repetitions' of graph structures that cannot really offer us any new information, **most of the time:**

- we work with unlabelled graphs (or start with an unlabelled graph, and consider some labelling of it mainly for clarity),
- we identify two such graphs if they are isomorphic,
- and we consider only one 'representative' from each class of isomorphic graphs.

A few more important concepts concerning graphs

Incidence Matrix of a Graph

Recall the adjacency matrix of a (finite) graph, which encodes all information about the graph.

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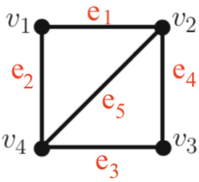
In other words, if the set of vertices $V(G)$ of G is, say, the set $\{v_1, v_2, v_3, \ldots, v_n\}$, and if the set of edges $E(G)$ of G is the set $\{e_1, e_2, e_3, \ldots, e_m\}$, then the (unoriented) incidence matrix of G is an $n \times m$ $0 - 1$ matrix, such that

- the (i, j)-th entry is equal to 1 if the vertex v_i is incident with the edge e_j,
- the (i, j)-th entry is equal to 0 otherwise.

Example: The incidence matrix of G_1

Recall G_1:



Recall its adjacency matrix:

$$
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 & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\
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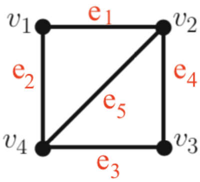

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On the other hand, its incidence matrix is

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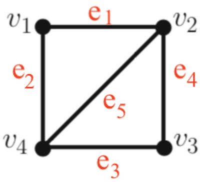

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Just as with the adjacency matrix, the incidence matrix <u>depends on the ordering</u> we choose <u>for the vertices</u>. Moreover, the incidence matrix <u>depends on the ordering</u> we choose <u>for the edges</u> too.

Complement of a Graph

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We can construct a new graph H on the same set of vertices by setting

$$H = \big(V,\ [V]^2 \setminus E\big),$$

that is, by setting the edge set of H to be the *complement* of E in $[V]^2$.

Complement of a Graph

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Definition. The new graph is called the *complement* of G, and is denoted by $\overline{G}$.

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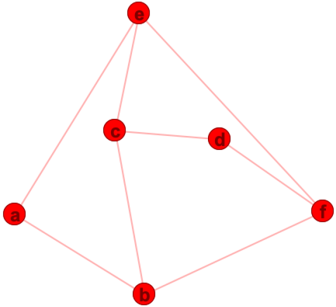

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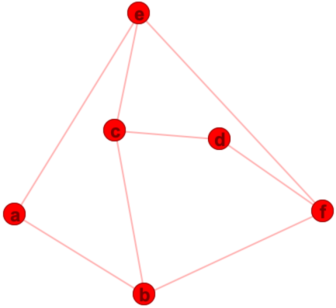

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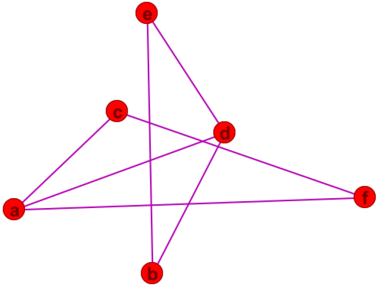

Complement and Adjacency Matrix

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Attention. In contrast, there is not such a straightforward way to get the incidence matrix of $\overline{G}$ from the incidence matrix of G (can you find one reason why?).

Connected components of a graph

Let $G = (V, E)$ be a graph. Define a relation on the vertex set V of G by setting

$$v_i \sim v_j \quad \textbf{if and only if}$$

$$v_i = v_j \quad \text{or} \quad \text{there is a path in } G \text{ from } v_i \text{ to } v_j.$$

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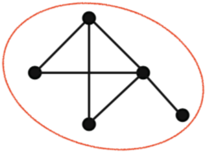
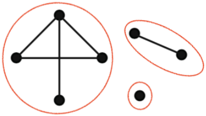
For each such equivalence class, the induced subgraph of G that we get is one of the so-called connected components of G (and as the name indicates, it is a (maximal) connected subgraph of G).

Connected components of a graph

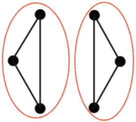
Examples



1 connected component



3 connected components



2 connected components

Important examples of (families of) graphs

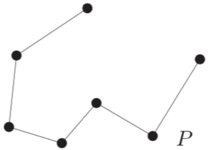
Reminder: Paths

Definition. A path P is a graph of the form

$$\Big(\{x_0, x_1, x_2, \ldots, x_l\},\ \{x_0x_1, x_1x_2, \ldots, x_{l-1}x_l\}\Big)$$

where l is an integer $\geqslant 1$.

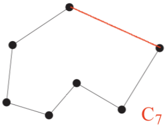
The number l is called the *length* of the path P (note that it is also the number of edges of P, that is, it is equal to the size of P).



A path P on 7 vertices, thus of length 6

Cycles

If we 'closed' the path P by joining the initial and the terminal vertex as well, then we would get what we call a *cycle graph.*



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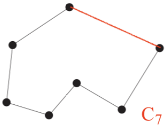

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In other words, $C_7 = \Big(\{x_0, x_1, x_2, \ldots, x_6\},\ \{x_0x_1, x_1x_2, \ldots, x_5x_6,\ x_6x_0\}\Big)$.

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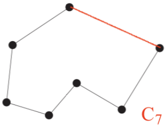

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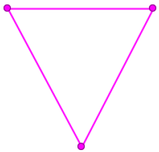
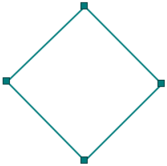


A 3-cycle, denoted by C_3



A 4-cycle, denoted by C_4

Null Graphs / Complete Graphs

Given a set of vertices $V = \{v_1, v_2, \ldots, v_n\}$, the two 'extreme' cases of graphs with vertex set V are:

- the *null graph* on V, that is, the graph on V that has no edges at all,
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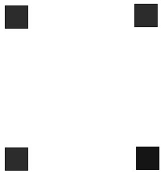
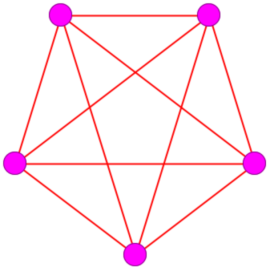
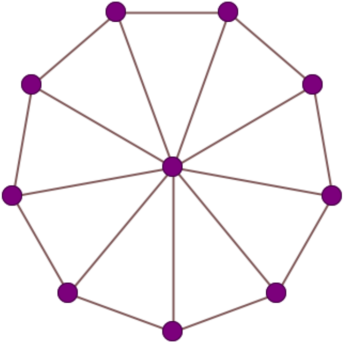


The null graph on 4 vertices, denoted by N_4



The complete graph on 5 vertices, denoted by K_5

Wheels



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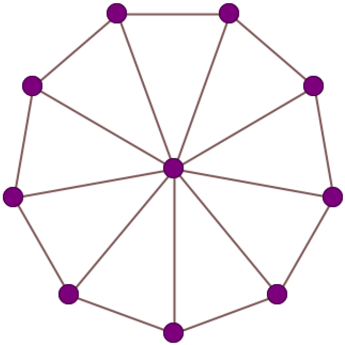

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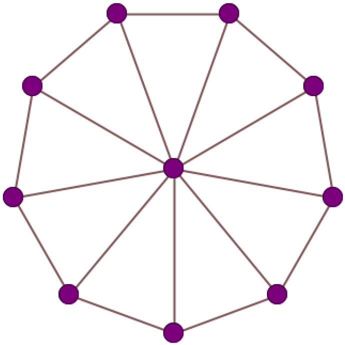

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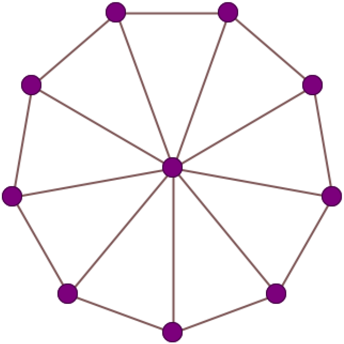

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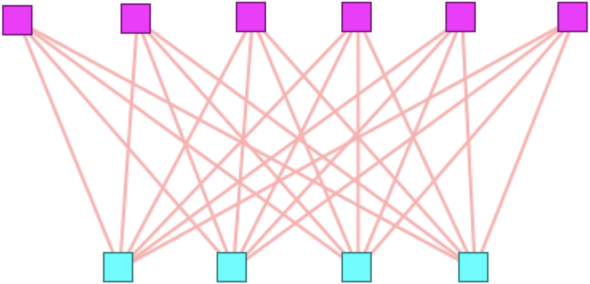
Notice that this implies that, in the wheel graph W_n, one vertex has degree $n-1$, while all the other vertices have degree $=$?

Bipartite Graphs



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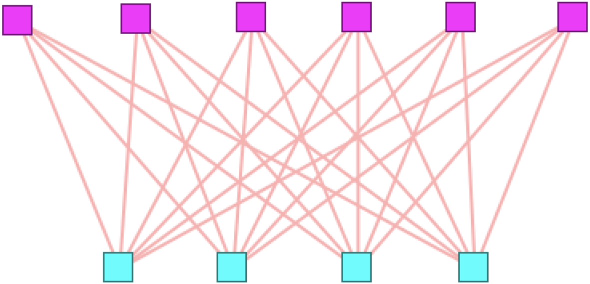

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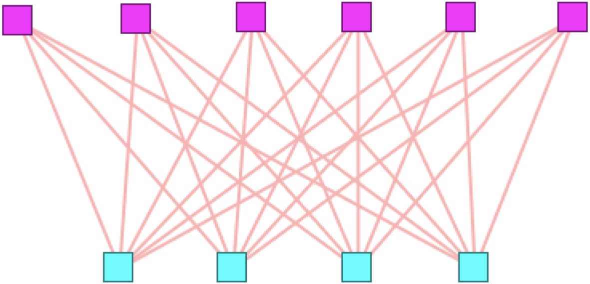

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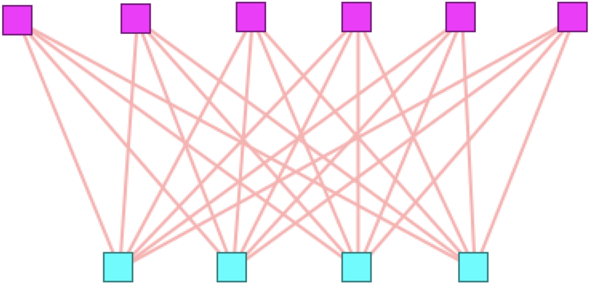

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If the cardinality $|V_1|$ of the part V_1 is m, and the cardinality $|V_2|$ of the part V_2 is n, then the bipartite graph we just described is denoted by $K_{m,n}$ (or equivalently, $K_{n,m}$).

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