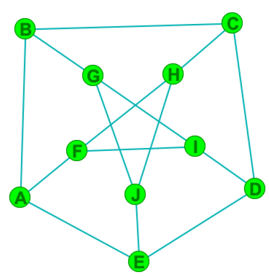
Math 322
Homework Problem Set 1

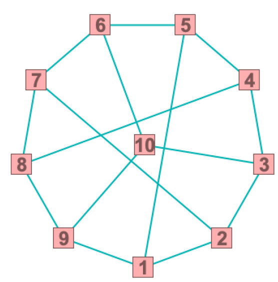
Remark. As was already mentioned in class, the adjacency matrix of a graph allows us to obtain plenty of information about the graph in efficient ways. For example, we discussed how to quickly determine the degree of each vertex of the graph by just looking at the adjacency matrix. The following question suggests one more use of the adjacency matrix (which we will expand more on later in the term).

Problem 1. Let $G = (V, E)$ be a finite graph of order n, with $V = \{v_1, v_2, \ldots, v_n\}$. Write A for the adjacency matrix of G. Show that the (j, j)-th entry of the matrix A^2 equals the degree of the vertex v_j.

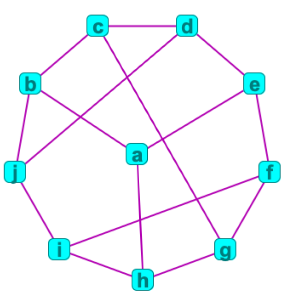
Problem 2. Two of the graphs in the image below are isomorphic.



Graph G_1



Graph G_2



Graph G_3

(a) Determine which two graphs are isomorphic, and give an appropriate graph isomorphism that confirms this.

(b) Explain why the remaining graph is not isomorphic to the other two.

Problem 3. There are 11, essentially different, unlabelled graphs on 4 vertices, or in other words, there are 11 pairwise non-isomorphic graphs of order 4. Draw all of them *(and try to find, if possible, a systematic way to list them so that you avoid drawing the same graph twice or more, and also avoid missing any of the 11 graphs; explain briefly your approach).*

Remark. We will see in class that, for every finite graph G which has at least two vertices, we can find (at least) one pair of vertices which have equal degrees.

The problem below asks you to study a bit more those graphs which have **exactly one** pair of vertices sharing the same degree.

Problem 4. (i) Let $G = (V, E)$ be a finite graph of order n, with $V = \{v_1, v_2, \ldots, v_n\}$, and let

$$\big(\deg(v_1),\, \deg(v_2),\, \ldots,\, \deg(v_n)\big) = (d_1, d_2, \ldots, d_n)$$

be the degree sequence of G (note that, here, we do not rewrite this in a decreasing way; the order in which the degrees are given corresponds to the ordering of the vertices).

What is the degree sequence of the complement $\overline{G}$ of the graph G? Determine how it depends on the degree sequence of G.

(ii) Based on part (i), show that if G has **exactly one** pair of vertices which have equal degrees, then $\overline{G}$ also has this property.

(iii) **For every n between 2 and 8** (with 2 and 8 included), find two non-isomorphic graphs each of which has exactly one pair of vertices sharing the same degree. Moreover, draw (or describe using the (V, E)-notation) these graphs (and briefly explain why they are not isomorphic).

[It might help to start by looking at the lists of pairwise non-isomorphic graphs of order 3 and of order 4 that you have been asked to find (for the graphs of order 3, see the Lecture 3 slides from Thursday September 9).]

Remark. As we have already briefly mentioned in class, for every $n \geqslant 3$, there is a 2-regular graph on n vertices, that is, a graph of order n all of whose vertices have degree 2 (the simplest such graph is the cycle C_n, but for large enough n there are other examples too). We will similarly discuss in the next lectures the families of 0-regular graphs, and of 1-regular graphs.

The last two problems of HW1 ask analogous questions about the existence of 3-regular graphs and 4-regular graphs on n vertices.

Problem 5. (i) What is the minimum possible order a 3-regular graph could have? Justify your answer fully (that is, find what the minimum order $n_{3,\min}$ should be, and draw (or describe) a 3-regular graph on $n_{3,\min}$ vertices).

(ii) Is there a maximum possible order for a finite 3-regular graph? Justify your answer.

(iii) Just as in part (i), write $n_{3,\min}$ for the minimum possible order of a 3-regular graph. Can you find a 3-regular graph of order n for every $n \geqslant n_{3,\min}$? Justify your answer (and if the answer is no, try to determine the integers n which could be orders of 3-regular graphs).

Problem 6. Analogous to Problem 5, but for 4-regular graphs.
(i) What is the minimum possible order a 4-regular graph could have? Justify your answer fully (that is, find what the minimum order $n_{4,\min}$ should be, and draw (or describe) a 4-regular graph on $n_{4,\min}$ vertices).

(ii) Is there a maximum possible order for a finite 4-regular graph? Justify your answer.

(iii) Just as in part (i), write $n_{4,\min}$ for the minimum possible order of a 4-regular graph. Can you find a 4-regular graph of order n for every $n \geqslant n_{4,\min}$? Justify your answer (and if the answer is no, try to determine the integers n which could be orders of 4-regular graphs).