Math 322
Suggested solutions to Homework Set 6

Problem 1. Observe that the event we are trying to design can be modelled by a factorization $\mathcal{F}_1$ of the complete graph K_{10}, where $\mathcal{F}_1$ will contain $\ell_1 + 1$ factors, ℓ_1 of which will be one-factors (each of the desired day sessions can be modelled by a one-factor of K_{10} given that the new hires must be divided into pairs), while the final one will be a Hamilton cycle of K_{10} (corresponding to the circular seating arrangement at the special dinner).

Moreover, we need these $\ell_1 + 1$ factors to be edge-disjoint, and also every edge of K_{10} to appear in one of them (these conditions correspond to the requirement that every pair we can form from the 10 new hires must have exactly one occasion where they are 'neighbours' (either literally, at the special dinner table, or by engaging in an activity for two during one of the day sessions)); hence we are looking for a factorization of K_{10}.

Write $F_1, F_2, \ldots, F_{\ell_1}$ for the one-factors in the factorization $\mathcal{F}_1$ of K_{10} that we are looking for, and write H_0 for the single Hamilton cycle of K_{10} that will be contained in $\mathcal{F}_1$. For every vertex u_0 of K_{10} we will have

$$9 = \deg_{K_{10}}(u_0) = \sum_{i=1}^{\ell_1} \deg_{F_i}(u_0) \ + \ \deg_{H_0}(u_0) = \sum_{i=1}^{\ell_1} 1 \ + \ 2 = \ell_1 + 2,$$

thus we need $\ell_1 = 7$. In other words, the managers of the event need to plan for 7 different day sessions.

We now give a specific factorization of K_{10} with the above properties. By Proposition 1 of Lecture 20, we know that K_{10} will have a factorization $\widetilde{F_1}$ consisting of 4 Hamilton cycles and 1 one-factor. In fact, we can choose $\widetilde{F_1}$ to consist of the following 4 Hamilton cycles:

Z_1 : AN BC AY KY IM MM HF WS JT NC AN,

Z_2 : AN KY BC MM AY WS IM NC HF JT AN,

Z_3 : AN MM KY WS BC NC AY JT IM HF AN,

Z_4 : AN WS MM NC KY JT BC HF AY IM AN,

and the following one-factor:

$$\widetilde{Z}_0 = \Big\{ V(K_{10}),\ \{\{\text{AN, AY}\}, \{\text{BC, IM}\}, \{\text{KY, HF}\}, \{\text{MM, JT}\}, \{\text{WS, NC}\}\} \Big\}$$

(note that here we use the initials of the new hires as labels for the vertices of K_{10}).

Now to get the type of factorization we want, we keep the Hamilton cycle Z_4 (and thus we will set $H_0 = Z_4$), and we also keep the one-factor $\widetilde{Z}_0$ (and we will set $F_7 = \widetilde{Z}_0$). Moreover, we will break each of the remaining Hamilton cycles above into two edge-disjoint one-factors of K_{10}. In particular,

- with our basis being the Hamilton cycle Z_1, we can get the one-factors

$$\begin{aligned} & F_1 = \Big\{V(K_{10}),\ \big\{\{\text{AN, BC}\}, \{\text{AY, KY}\}, \{\text{IM, MM}\}, \{\text{HF, WS}\}, \{\text{JT, NC}\}\big\}\Big\} \\ \text{and}\ \ & F_2 = \Big\{V(K_{10}),\ \big\{\{\text{BC, AY}\}, \{\text{KY, IM}\}, \{\text{MM, HF}\}, \{\text{WS, JT}\}, \{\text{NC, AN}\}\big\}\Big\}. \end{aligned}$$

- With our basis being the Hamilton cycle Z_2, we can get the one-factors

$$\begin{aligned} & F_3 = \Big\{V(K_{10}),\ \big\{\{\text{AN, KY}\}, \{\text{BC, MM}\}, \{\text{AY, WS}\}, \{\text{IM, NC}\}, \{\text{HF, JT}\}\big\}\Big\} \\ \text{and}\ \ & F_4 = \Big\{V(K_{10}),\ \big\{\{\text{KY, BC}\}, \{\text{MM, AY}\}, \{\text{WS, IM}\}, \{\text{NC, HF}\}, \{\text{JT, AN}\}\big\}\Big\}. \end{aligned}$$

- Finally, with our basis being the Hamilton cycle Z_3, we can get the one-factors

$$\begin{aligned} & F_5 = \Big\{V(K_{10}),\ \big\{\{\text{AN, MM}\}, \{\text{KY, WS}\}, \{\text{BC, NC}\}, \{\text{AY, JT}\}, \{\text{IM, HF}\}\big\}\Big\} \\ \text{and}\ \ & F_6 = \Big\{V(K_{10}),\ \big\{\{\text{MM, KY}\}, \{\text{WS, BC}\}, \{\text{NC, AY}\}, \{\text{JT, IM}\}, \{\text{HF, AN}\}\big\}\Big\}. \end{aligned}$$

We conclude that one plan for the desired event is modelled by the following factorization of K_{10}:

$$\mathcal{F}_1 = \big\{F_1,\ F_2,\ F_3,\ F_4,\ F_5,\ F_6,\ F_7\,(=\widetilde{Z}_0),\ H_0\,(=Z_4)\big\},$$

where each of the one-factors F_i corresponds to a day session, while the Hamilton cycle H_0 gives the seating arrangement for the special dinner.

Problem 2. Let $\mathcal{F}_0$ be a three-factorization of G, and suppose that $\mathcal{F}_0$ contains ℓ_0 different three-factors of G, say the subgraphs $H_1, H_2, \ldots, H_{\ell_0}$.

Let u be an arbitrary vertex of G. Then, in each of the factors $H_1, H_2, \ldots, H_{\ell_0}$, u is incident with exactly 3 edges. Moreover, since any two of these factors must be edge-disjoint, the 3 edges that u is incident with in H_1 are different from the 3 edges that u is incident with in H_2, and so on. Thus, this gives in total $\ell_0 \cdot 3$ different edges that u is incident with.

Finally, we recall that, since $\mathcal{F}_0$ is a factorization of G, every edge in G should be contained in one of the factors in $\mathcal{F}_0$. More specifically, every edge incident to vertex u should belong to one of these factors, and thus it will be among the $\ell_0 \cdot 3$ edges that we have already counted.

We conclude that

$$\deg_G(u) = \text{number of edges that } u \text{ is incident with} = 3\ell_0.$$

Moreover, since u was an arbitrary vertex of G, this is true for every vertex of G. In other words, all vertices in G have the same degree, and thus G is s-regular with $s = 3\ell_0$ (in other words, with s being a multiple of 3).

It remains to show that $m = e(G) =$ the number of edges in G is a multiple of 3 as well. If we write $v_1, v_2, \ldots, v_n$ for the vertices of G, by the Handshaking Lemma we have that

$$2m = \sum_{i=1}^{n} \deg_G(v_i) = \sum_{i=1}^{n} 3\ell_0 = 3\ell_0 \cdot n.$$

We conclude that $2m$ is a multiple of 3 too. Moreover, since 3 is a prime number which divides the product $2m$, it must divide one of the factors of the product too. Since 3 does not divide 2, we conclude that it must divide m (or equivalently that m is a multiple of 3).

Problem 3. Given that the students will essentially be matched with the different positions that the teachers are offering (so, from a graph-theoretical point of view, we are looking for a stable matching of positions and students, since these two sets have the same cardinality), it will be convenient to replace the first table of preferences by the following table:

T_1, Post 1	T_1, Post 2	T_1, Post 3	T_2, Post 1	T_2, Post 2	T_2, Post 3	T_2, Post 4	T_3, Post 1	T_3, Post 2
S_3	S_3	S_3	S_3	S_3	S_3	S_3	S_1	S_1
S_4	S_4	S_4	S_2	S_2	S_2	S_2	S_4	S_4
S_1	S_1	S_1	S_7	S_7	S_7	S_7	S_3	S_3
S_2	S_2	S_2	S_8	S_8	S_8	S_8	S_5	S_5
S_7	S_7	S_7	S_9	S_9	S_9	S_9	S_9	S_9
S_9	S_9	S_9	S_4	S_4	S_4	S_4	S_8	S_8
S_8	S_8	S_8	S_1	S_1	S_1	S_1	S_6	S_6
S_5	S_5	S_5	S_6	S_6	S_6	S_6	S_7	S_7
S_6	S_6	S_6	S_5	S_5	S_5	S_5	S_2	S_2

We now see that in the 1st round of the algorithm, T_1 will make an offer to S_3, T_2 will also make an offer to S_3, while T_3 will make an offer to S_1.

Next, we see that S_1 will provisionally accept the offer by T_3, while S_3 will provisionally accept the offer by T_1 (given that S_3 prefers T_1 over T_2).

Thus, by the end of the 1st round we have the provisional engagements:

$$(T_1, S_3) \quad \text{and} \quad (T_3, S_1).$$

Note that, even though both T_1 and T_3 have already been **provisionally** assigned a student, they will still make offers in the next round since they still have available positions (recall that essentially we are matching the positions with the students, so the above provisional engagements can be read as "S_3 has provisionally accepted the 1st student position offered by T_1, while S_1 has provisionally accepted the 1st student position offered by T_3").

In the 2nd round of the algorithm, T_1 will make an offer to S_4, T_2 will make an offer to S_2, while T_3 will make an offer to S_4.

Next, we see that S_2 will provisionally accept the offer by T_2, while S_4 will provisionally accept the offer by T_3 (given that S_4 prefers T_3 over T_1).

Thus, by the end of the 2nd round we have the provisional engagements:

$$(T_1, S_3), \quad (T_2, S_2), \quad (T_3, S_1) \quad \text{and} \quad (T_3, S_4).$$

We observe that, at the end of this round, all the positions that T_3 is planning to offer have been filled, so T_3 will NOT be making an offer in the next round (however, T_3 may again make an offer in a later round, if it happens that one of the students who have already chosen T_3 rejects the provisional offer for a more preferred one, and thus that position becomes available again).

In the 3rd round, T_1 makes an offer to S_1, while T_2 makes an offer to S_7.

S_7 has not received any other offers so far, so the student provisionally accepts this offer. On the other hand, S_1 has already accepted an offer made by T_3; given that S_1 prefers T_1, the student rejects the previous offer, and provisionally accepts the offer by T_1.

Thus, by the end of the 3rd round we have the provisional engagements:

$$(T_1, S_1), \quad (T_1, S_3), \quad (T_2, S_2), \quad (T_2, S_7) \quad \text{and} \quad (T_3, S_4).$$

In the 4th round, T_1 makes an offer to S_2, T_2 makes an offer to S_8, while T_3 makes an offer to S_3.

S_8 has not received any other offers so far, so the student provisionally accepts the offer made by T_2. On the other hand, S_2 has provisionally accepted an offer by T_2; since S_2 prefers T_2 over T_1, the student stays with T_2.

Similarly, S_3 has provisionally accepted an offer by T_1; since S_3 prefers T_3 over T_1, the student rejects the previous offer and accepts the new one made by T_3.

Thus, by the end of the 4th round we have the provisional engagements:

$$(T_1, S_1), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

Again, we see that T_3 has no more positions to fill, so this teacher will not be making an offer in the next round.

In the 5th round, T_1 makes an offer to S_7, while T_2 makes an offer to S_9.

S_9 has not received any other offers so far, so the student provisionally accepts this offer. On the other hand, S_7 has already accepted an offer made by T_2; given that S_7 prefers T_2 over T_1, the student rejects the new offer and stays with T_2.

Thus, by the end of the 5th round we have the provisional engagements:

$$(T_1, S_1), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

We see that both T_2 and T_3 have no more positions to fill, so in the next round only T_1 will make an offer.

In the 6th round, T_1 makes an offer to S_9. This student has already accepted an offer by T_2, and since S_9 prefers T_2 over T_1, the new offer by T_1 gets rejected.

Thus, by the end of the 6th round we have the same provisional engagements as in the previous round:

$$(T_1, S_1), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

In the 7th round, T_1 makes an offer to S_8. This student has already accepted an offer by T_2, and since S_8 prefers T_2 over T_1, this new offer by T_1 also gets rejected.

Again, we have the same provisional engagements as in the previous round:

$$(T_1, S_1), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

In the 8th round, T_1 makes an offer to S_5. This student has not received any other offers so far, so this offer gets accepted.

Thus, by the end of the 8th round we have the following provisional engagements:

$$(T_1, S_1), \quad (T_1, S_5), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

Finally, in the 9th round, T_1 makes an offer to S_6. This student has not received any other offers so far, so this offer gets accepted.

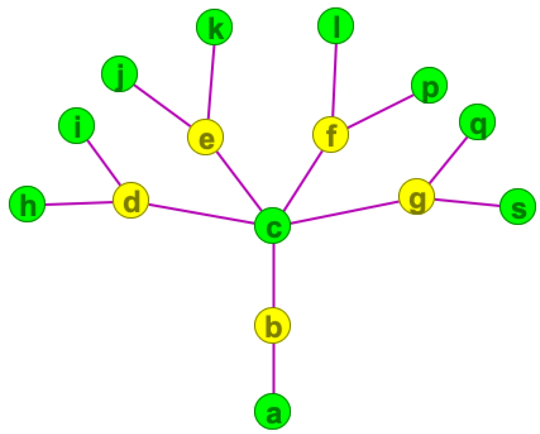
Thus, by the end of the 9th round we have the following provisional engagements:

$$(T_1, S_1), \quad (T_1, S_5), \quad (T_1, S_6), \quad (T_2, S_2), \quad (T_2, S_7), \quad (T_2, S_8), \quad (T_2, S_9), \quad (T_3, S_3) \quad \text{and} \quad (T_3, S_4).$$

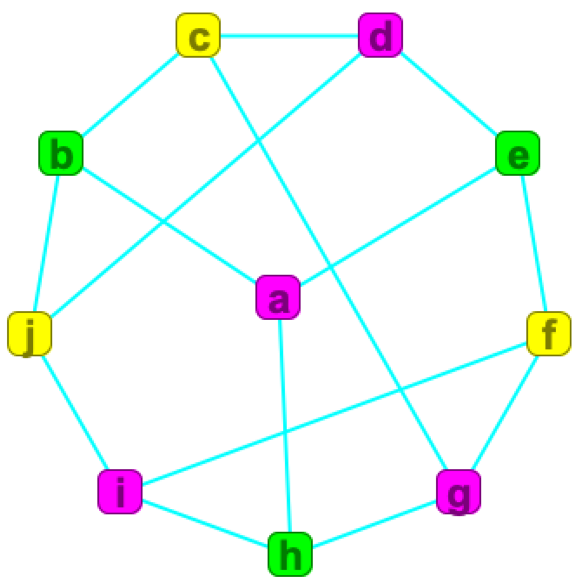
We now observe that each student has accepted a position offered by a teacher, and all positions have been filled, therefore the engagements from the last round become permanent, and the algorithm terminates.

Problem 4. We observe that G_1 is a tree (with 15 vertices and 14 edges), and hence G_1 contains no cycles (and in particular, no odd cycles). Thus, as we saw in Lecture 24, $\chi(G_1) = 2$.

We also give a minimal colouring of G_1:



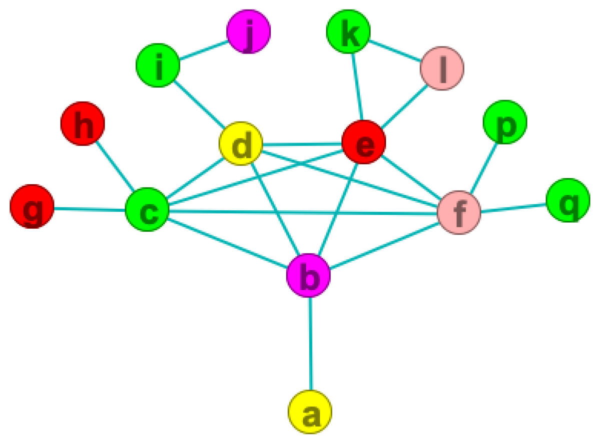
Graph G_2 contains the odd cycle $b\,a\,h\,i\,j\,b$, and hence $\chi(G_2) \geqslant 3$. We will now give a 3-colouring of G_2, which will show that $\chi(G_2) = 3$:



Graph G_3 contains the clique $(\{b, c, d, e, f\}, \{bc, bd, be, bf, cd, ce, cf, de, df, ef\})$, and hence

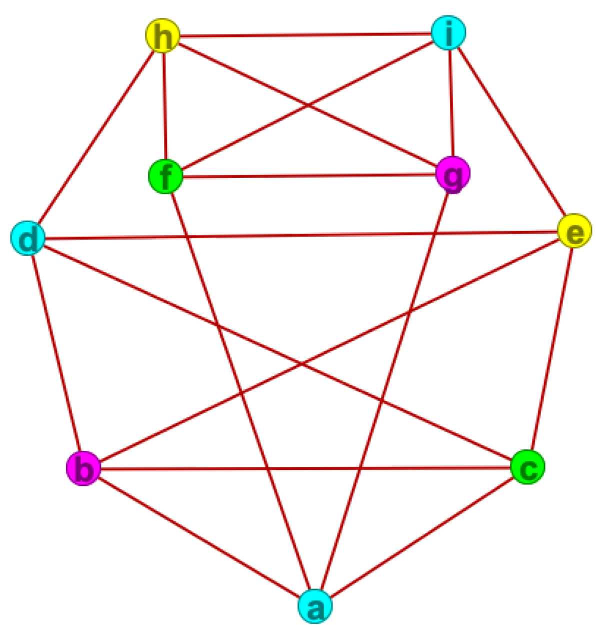
$\chi(G_3) \geqslant \omega(G_3) \geqslant 5$ (we could also check that $\omega(G_3) = 5$).

We will now give a 5-colouring of G_3, which will show that $\chi(G_3) = 5$:

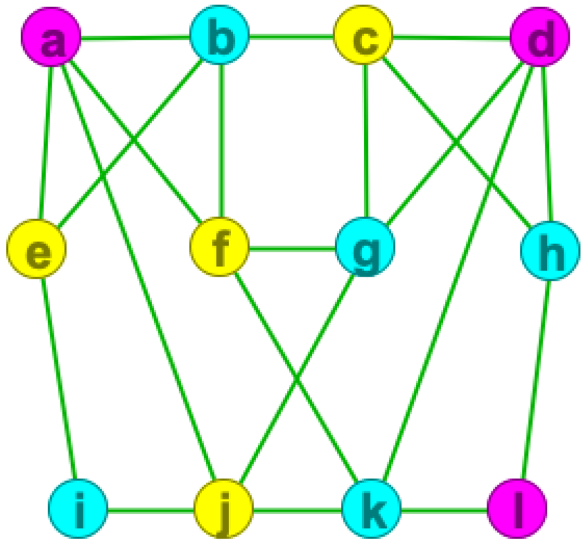


Graph G_4 contains the clique $(\{b, c, d, e\}, \{bc, bd, be, cd, ce, de\})$, and hence $\chi(G_4) \geqslant \omega(G_4) \geqslant 4$ (we could also check that $\omega(G_4) = 4$).

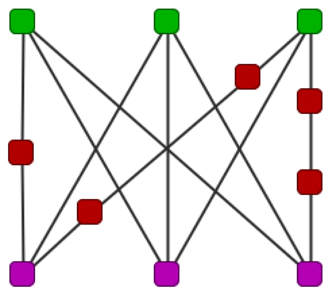
We will now give a 4-colouring of G_4, which will show that $\chi(G_4) = 4$:



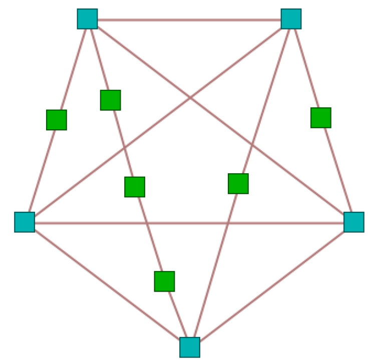
Finally, Graph G_5 contains the odd cycle $a\,b\,e\,a$, and hence $\chi(G_5) \geqslant 3$. We will now give a 3-colouring of G_5, which will show that $\chi(G_5) = 3$:



Problem 5. To make it as clear as possible which subgraphs we are considering, and also which subdivision of $K_{3,3}$ or of K_5 each such subgraph is, we will colour the vertices of a subdivision of $K_{3,3}$ in a similar way to the colouring in the next image, and analogously for the vertices of a subdivision of K_5.

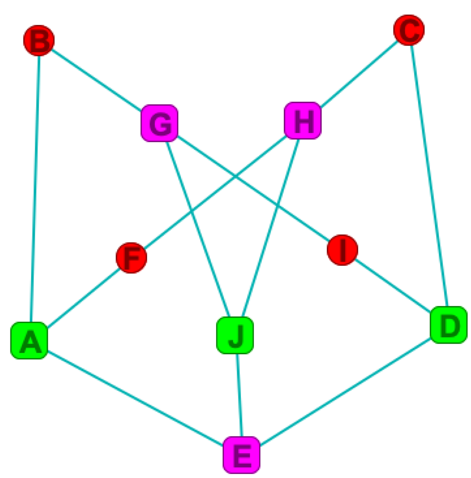


Example of a subdivision of $K_{3,3}$ with coloured vertices



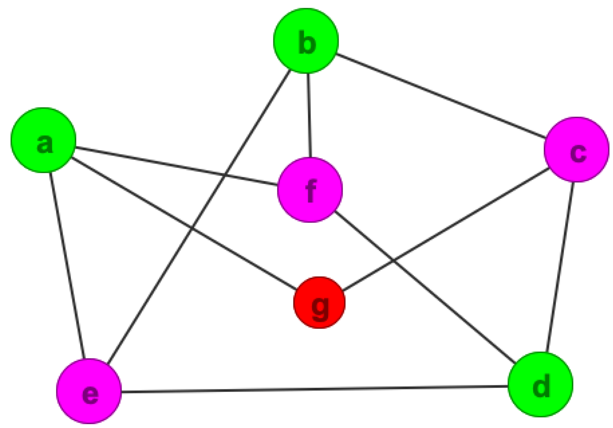
Example of a subdivision of K_5 with coloured vertices

We now have: graph H_1 contains the following subgraph



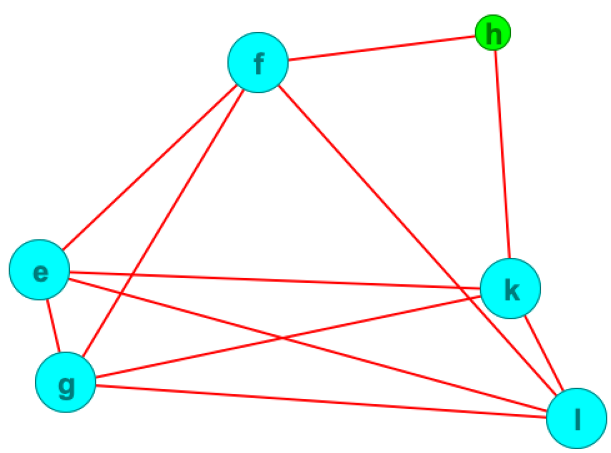
This is a subdivision of $K_{3,3}$ (which we can view as arising from subdividing preexisting edges AG, AH, DG and DH, and replacing each of them by a path of length 2).

Graph H_2 contains the following subgraph



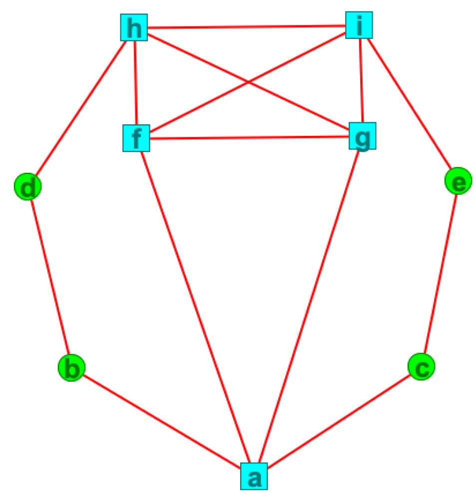
This is a subdivision of $K_{3,3}$ (which we can view as arising from subdividing a preexisting edge ac, and replacing it by a path of length 2).

Graph H_3 contains the following subgraph



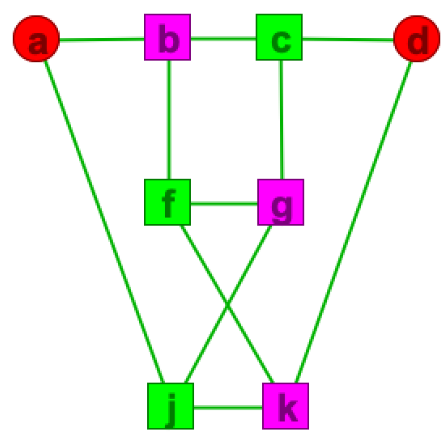
This is a subdivision of K_5 (which we can view as arising from subdividing a preexisting edge fk, and replacing it by a path of length 2).

Graph G_4 contains the following subgraph



This is a subdivision of K_5 (which we can view as arising from subdividing preexisting edges ha and ia, and replacing each of them by a path of length 3).

Finally, graph G_5 contains the following subgraph



This is a subdivision of $K_{3,3}$ (which we can view as arising from subdividing preexisting edges bj and ck, and replacing each of them by a path of length 2).