MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 16

Tuesday, November 2

The travelling salesman problem

Let G_0 be a weighted connected graph whose vertices represent different cities that a salesman wants to visit, which are connected by, say, roads and highways, or by train routes, or by airline routes, represented by the edges of the graph *(with each edge weight capturing the cost or distance of travel from one city - endvertex to the other city - endvertex joined by the corresponding edge).*

Question 1. What is the most cost-efficient (or time-efficient) way for the salesman to visit all the cities and finally return to the city which he is supposed to start from?

Question 2. Is there a way for the salesman to visit all the cities but not pass by any city more than once (except perhaps in the case that he returns to the city where he starts from at the end of his trip)?

An efficient scenic route...

The city of Königsberg, Prussia was set on the Pregel River, and included two large islands that were connected to each other and the mainland by seven bridges.

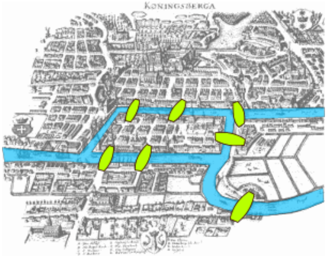


Image from Wikipedia: Map of the city in Leonhard Euler's time showing the actual layout of the seven bridges, and highlighting the river Pregel and the bridges.

People spent time trying to discover a way in which they could cross each bridge exactly once before returning to the point / place in the city that they started from.

Paths, cycles, trails and circuits in graphs

Let $G = (V, E)$ be a graph.

- walks A *walk* of length k in G is a sequence of (not necessarily distinct) vertices $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ from V, such that $v_i v_{i+1} \in E(G)$ for every $i = 0, 1, 2, \ldots, k-1$. The vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.

  Recall that, since G here is a graph (and thus, according to the definitions in this course, it does not contain multiple edges), we can completely describe the walk by simply writing the vertices it passes through, one next to the other, in the correct order: $v_{i_0}\ v_{i_1}\ v_{i_2} \cdots v_{i_k}$.

Paths, cycles, trails and circuits in graphs

Let $G = (V, E)$ be a graph.

- walks A *walk* of length k in G is a sequence of (not necessarily distinct) vertices $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ from V, such that $v_i v_{i+1} \in E(G)$ for every $i = 0, 1, 2, \ldots, k-1$. The vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.

  Recall that, since G here is a graph (and thus, according to the definitions in this course, it does not contain multiple edges), we can completely describe the walk by simply writing the vertices it passes through, one next to the other, in the correct order: $v_{i_0}\ v_{i_1}\ v_{i_2} \cdots v_{i_k}$.

- paths A path in G is simply a walk in which **all the vertices are distinct**.
- cycles A cycle is a 'closed path', that is, a walk in which all vertices are distinct except for the terminal vertex which coincides with the initial vertex.

- trails If **all the edges in a walk are distinct** (but not necessarily all the vertices), we call this walk a *trail*.

- circuits A *circuit* is a 'closed trail', that is, a walk in which **all edges are distinct**, and also the endvertices coincide.

Same objects in multigraphs?

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \ldots, v_n\}$, while $E = \{e_1, e_2, \ldots, e_m\}$ *(the latter set may include <u>different</u> edges which have <u>the same pair of endvertices</u> (and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex, as e.g. we did in an example last time, or as in a HW4 example).*

Same objects in multigraphs?

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \ldots, v_n\}$, while $E = \{e_1, e_2, \ldots, e_m\}$ *(the latter set may include* *different* *edges which have* *the same pair of endvertices* *(and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex, as e.g. we did in an example last time, or as in a HW4 example).*

- walks A *walk* of length k in H is now represented by a sequence of the form

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

  where $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are vertices from $V(H)$ (not necessarily distinct), $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ are edges from $E(H)$ (not necessarily distinct), **and for every $s = 1, 2, \ldots, k$ we have that $e_{j_s} = \{v_{i_{s-1}}, v_{i_s}\}$** (or in other words, e_{j_s} joins the vertices $v_{i_{s-1}}$ and v_{i_s};

Same objects in multigraphs?

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \dots, v_n\}$, while $E = \{e_1, e_2, \dots, e_m\}$ *(the latter set may include different edges which have the same pair of endvertices (and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex, as e.g. we did in an example last time, or as in a HW4 example).*

- walks A *walk* of length k in H is now represented by a sequence of the form

  $$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

  where $v_{i_0}, v_{i_1}, v_{i_2}, \dots, v_{i_k}$ are vertices from $V(H)$ (not necessarily distinct), $e_{j_1}, e_{j_2}, \dots, e_{j_k}$ are edges from $E(H)$ (not necessarily distinct), **and for every $s = 1, 2, \dots, k$ we have that $e_{j_s} = \{v_{i_{s-1}}, v_{i_s}\}$** (or in other words, e_{j_s} joins the vertices $v_{i_{s-1}}$ and v_{i_s}; note that now we may have cases where consecutive vertices are equal, that is, cases where $v_{i_{s-1}} = v_{i_s}$, and then e_{j_s} should be a loop at vertex v_{i_s}).

Same objects in multigraphs?

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \ldots, v_n\}$, while $E = \{e_1, e_2, \ldots, e_m\}$ *(the latter set may include* *<u>different</u>* *edges which have* *<u>the same pair of endvertices</u>* *(and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex, as e.g. we did in an example last time, or as in a HW4 example).*

- walks A *walk* of length k in H is now represented by a sequence of the form

  $$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

  where $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are vertices from $V(H)$ (not necessarily distinct), $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ are edges from $E(H)$ (not necessarily distinct), **and for every** $\boldsymbol{s = 1, 2, \ldots, k}$ **we have that** $\boldsymbol{e_{j_s} = \{v_{i_{s-1}}, v_{i_s}\}}$ (or in other words, e_{j_s} joins the vertices $v_{i_{s-1}}$ and v_{i_s}; note that now we may have cases where consecutive vertices are equal, that is, cases where $v_{i_{s-1}} = v_{i_s}$, and then e_{j_s} should be a loop at vertex v_{i_s}).

  As before, the vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.

Same objects in multigraphs?

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \ldots, v_n\}$, while $E = \{e_1, e_2, \ldots, e_m\}$ *(the latter set may include different edges which have the same pair of endvertices (and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex, as e.g. we did in an example last time, or as in a HW4 example).*

- walks A *walk* of length k in H is now represented by a sequence of the form

  $$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

  where $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are vertices from $V(H)$ (not necessarily distinct), $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ are edges from $E(H)$ (not necessarily distinct), **and for every $s = 1, 2, \ldots, k$ we have that $e_{j_s} = \{v_{i_{s-1}}, v_{i_s}\}$** (or in other words, e_{j_s} joins the vertices $v_{i_{s-1}}$ and v_{i_s}; note that now we may have cases where consecutive vertices are equal, that is, cases where $v_{i_{s-1}} = v_{i_s}$, and then e_{j_s} should be a loop at vertex v_{i_s}).

  As before, the vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.

- paths A path in H is simply a walk in which **all the vertices are distinct** *(note that, as a consequence of this definition, a path will not contain any loops)*.

Same objects in multigraphs?

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \ldots, v_n\}$, while $E = \{e_1, e_2, \ldots, e_m\}$ *(the latter set may include* *different* *edges which have* *the same pair of endvertices* *(and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex, as e.g. we did in an example last time, or as in a HW4 example).*

- walks A *walk* of length k in H is now represented by a sequence of the form

  $$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

  where $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are vertices from $V(H)$ (not necessarily distinct), $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ are edges from $E(H)$ (not necessarily distinct), **and for every $s = 1, 2, \ldots, k$ we have that $e_{j_s} = \{v_{i_{s-1}}, v_{i_s}\}$** (or in other words, e_{j_s} joins the vertices $v_{i_{s-1}}$ and v_{i_s}; note that now we may have cases where consecutive vertices are equal, that is, cases where $v_{i_{s-1}} = v_{i_s}$, and then e_{j_s} should be a loop at vertex v_{i_s}).

  As before, the vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.
- paths A path in H is simply a walk in which **all the vertices are distinct** *(note that, as a consequence of this definition, a path will not contain any loops)*.
- cycles A cycle is a closed path.

Same objects in multigraphs?

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \ldots, v_n\}$, while $E = \{e_1, e_2, \ldots, e_m\}$ *(the latter set may include* <u>*different*</u> *edges which have* <u>*the same pair of endvertices*</u> *(and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex, as e.g. we did in an example last time, or as in a HW4 example).*

- walks A *walk* of length k in H is now represented by a sequence of the form

  $$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

  where $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are vertices from $V(H)$ (not necessarily distinct), $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ are edges from $E(H)$ (not necessarily distinct), **and for every $s = 1, 2, \ldots, k$ we have that $e_{j_s} = \{v_{i_{s-1}}, v_{i_s}\}$** (or in other words, e_{j_s} joins the vertices $v_{i_{s-1}}$ and v_{i_s}; note that now we may have cases where consecutive vertices are equal, that is, cases where $v_{i_{s-1}} = v_{i_s}$, and then e_{j_s} should be a loop at vertex v_{i_s}).

  As before, the vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.
- paths A path in H is simply a walk in which **all the vertices are distinct** *(note that, as a consequence of this definition, a path will not contain any loops)*.
- cycles A cycle is a closed path.
- trails A trail in H is a walk in which **all the edges are distinct** (but not necessarily all the vertices).

Same objects in multigraphs?

Let $H = (V, E)$ be a multigraph in H, and suppose that $V = \{v_1, v_2, \ldots, v_n\}$, while $E = \{e_1, e_2, \ldots, e_m\}$ *(the latter set may include <u>different</u> edges which have <u>the same pair of endvertices</u> (and thus form groups of multiple edges), as well as loops; it's even possible that we have 'parallel' loops, that is, distinct loops which are attached to the same vertex, as e.g. we did in an example last time, or as in a HW4 example).*

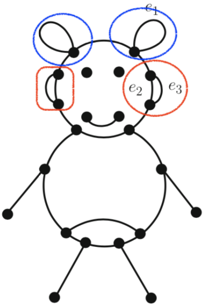
- walks A *walk* of length k in H is now represented by a sequence of the form

  $$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

  where $v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are vertices from $V(H)$ (not necessarily distinct), $e_{j_1}, e_{j_2}, \ldots, e_{j_k}$ are edges from $E(H)$ (not necessarily distinct), **and for every $s = 1, 2, \ldots, k$ we have that $e_{j_s} = \{v_{i_{s-1}}, v_{i_s}\}$** (or in other words, e_{j_s} joins the vertices $v_{i_{s-1}}$ and v_{i_s}; note that now we may have cases where consecutive vertices are equal, that is, cases where $v_{i_{s-1}} = v_{i_s}$, and then e_{j_s} should be a loop at vertex v_{i_s}).

  As before, the vertices v_{i_0} and v_{i_k} are called the *endvertices* of the walk, and we sometimes say that this is a $v_{i_0} - v_{i_k}$ walk.
- paths A path in H is simply a walk in which **all the vertices are distinct** *(note that, as a consequence of this definition, a path will not contain any loops)*.
- cycles A cycle is a closed path.
- trails A trail in H is a walk in which **all the edges are distinct** (but not necessarily all the vertices).
- circuits Finally, a circuit is a closed trail.

Reminder: Degree of a vertex in a multigraph

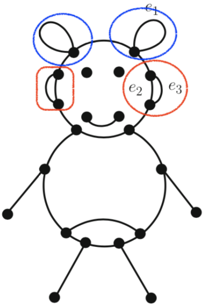


'Alternative' Definition

In a multigraph G, we define the degree of a vertex v_0 of G to be the number of edges which are incident to v_0.

By convention, if v_0 has loops attached to it, then each such loop contributes 2 to the degree of v_0.

Reminder: Degree of a vertex in a multigraph



'Alternative' Definition

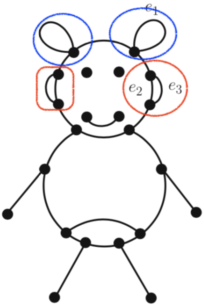
In a multigraph G, we define the degree of a vertex v_0 of G to be the number of edges which are incident to v_0.

By convention, if v_0 has loops attached to it, then each such loop contributes 2 to the degree of v_0.

One way to correctly determine the degree of a vertex v_0 in a multigraph is to examine how many different line segments / arcs we could 'walk' on moving away from v_0;

thinking in this way, it becomes clearer why each loop contributes 2 to the degree of v_0.

Reminder: Degree of a vertex in a multigraph



'Alternative' Definition

In a multigraph G, we define the degree of a vertex v_0 of G to be the number of edges which are incident to v_0.

By convention, if v_0 has loops attached to it, then each such loop contributes 2 to the degree of v_0.

One way to correctly determine the degree of a vertex v_0 in a multigraph is to examine how many different line segments / arcs we could 'walk' on moving away from v_0;

thinking in this way, it becomes clearer why each loop contributes 2 to the degree of v_0.

With this definition, the Handshaking Lemma, as well as its first Corollary, that we saw earlier in the term, continue to hold in a multigraph.

Handshaking Lemma

Let G be a finite multigraph, with vertex set V and size $e(G)$. Then $\sum_{v_i \in V} \deg(v_i) = 2e(G)$.

If V_{odd} is the subset of the vertices in G which have odd degree, then V_{odd} must have **even cardinality**.

Eulerian graphs (and multigraphs)

Definition

Let G be a connected graph (or multigraph) which is non-trivial, that is, it contains at least one edge.

Eulerian graphs (and multigraphs)

Definition

Let G be a connected graph (or multigraph) which is non-trivial, that is, it contains at least one edge.

- An *Euler trail* in G is a trail that passes by **all** edges in G (and hence, given that it is a trail, it passes by each edge exactly once).

Eulerian graphs (and multigraphs)

Definition

Let G be a connected graph (or multigraph) which is non-trivial, that is, it contains at least one edge.

- An *Euler trail* in G is a trail that passes by **all** edges in G (and hence, given that it is a trail, it passes by each edge exactly once).
- An *Euler circuit* in G is a circuit (that is, a closed trail) that passes by **all** edges in G.

Eulerian graphs (and multigraphs)

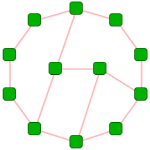
Definition

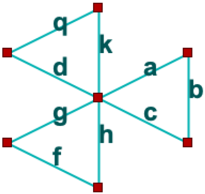
Let G be a connected graph (or multigraph) which is non-trivial, that is, it contains at least one edge.

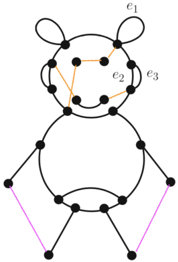
- An *Euler trail* in G is a trail that passes by **all** edges in G (and hence, given that it is a trail, it passes by each edge exactly once).
- An *Euler circuit* in G is a circuit (that is, a closed trail) that passes by **all** edges in G.

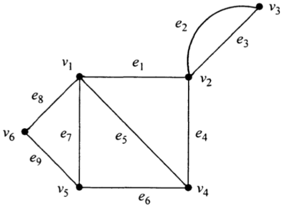
G is called *Eulerian* if we can find (at least) one Euler circuit in G.

Examples and non-examples









bottom row from the Balakrishnan-Ranganathan book (1st image modified)

Hamiltonian graphs

Definition

Let G be a connected graph.

Hamiltonian graphs

Definition

Let G be a connected graph.

- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).

Hamiltonian graphs

Definition

Let G be a connected graph.

- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).
- A *Hamilton cycle* in G is a cycle (that is, a closed path) that passes through **all** vertices in G.

Hamiltonian graphs

Definition

Let G be a connected graph.

- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).
- A *Hamilton cycle* in G is a cycle (that is, a closed path) that passes through **all** vertices in G.

G is called *Hamiltonian* if we can find (at least) one Hamilton cycle in G.

Hamiltonian graphs

Definition

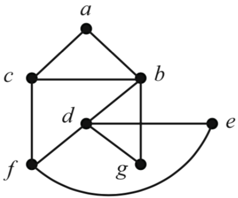
Let G be a connected graph.

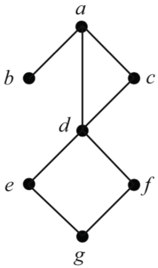
- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).
- A *Hamilton cycle* in G is a cycle (that is, a closed path) that passes through **all** vertices in G.

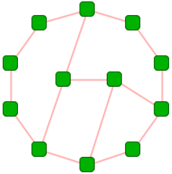
G is called *Hamiltonian* if we can find (at least) one Hamilton cycle in G.

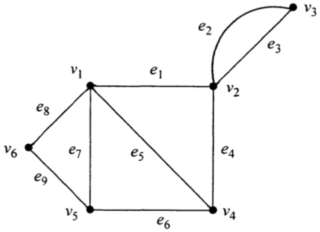
The name is in honour of the mathematician William Hamilton who introduced the idea of looking for Hamilton cycles in graphs (with the first graph he considered being (the 'frame' of) the solid dodecahedron) as a new board game!

Examples and non-examples









Eulerian graphs: necessary and sufficient conditions

Eulerian graphs: necessary and sufficient conditions

Theorem 1

Let G be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** every vertex of G has even degree.

Eulerian graphs: necessary and sufficient conditions

Theorem 1

Let G be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** every vertex of G has even degree.

Proposition 1

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Eulerian graphs: necessary and sufficient conditions

Theorem 2

Let $G = (V, E)$ be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** its edge set E can be written as the disjoint union of subsets $E_1, E_2, \ldots, E_s$ each of which forms a cycle in G.

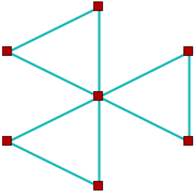
Eulerian graphs: necessary and sufficient conditions

Theorem 2

Let $G = (V, E)$ be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** its edge set E can be written as the disjoint union of subsets $E_1, E_2, \dots, E_s$ each of which forms a cycle in G.

Example. In the graph below we can write the edge set as the disjoint union of three cycles:



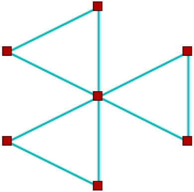
Eulerian graphs: necessary and sufficient conditions

Theorem 2

Let $G = (V, E)$ be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** its edge set E can be written as the disjoint union of subsets $E_1, E_2, \dots, E_s$ each of which forms a cycle in G.

Example. In the graph below we can write the edge set as the disjoint union of three cycles:



Immediate Corollary

Every Eulerian graph G is **bridgeless** (that is, it satisfies $\lambda(G) \geqslant 2$).

Applying Theorem 2 to multigraphs?

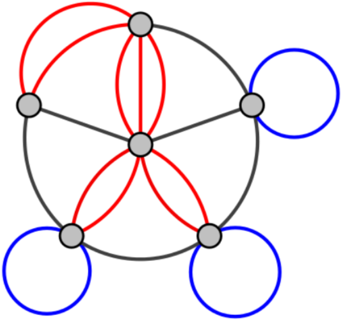
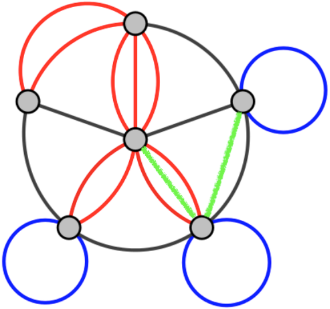


Image from Wikipedia

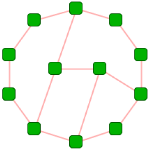


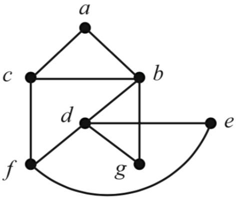
Question. Can you decompose the edge set of any of these multigraphs into disjoint cycles?

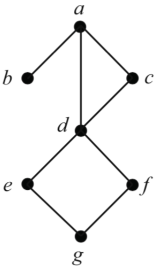
Applying Theorems 1 and 2 to examples

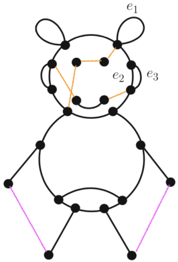
Image from Wikipedia

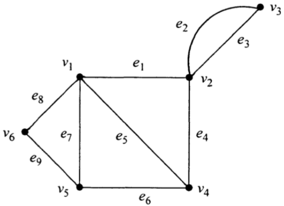
Applying Theorems 1 and 2 to examples



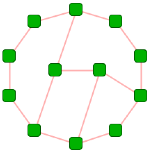


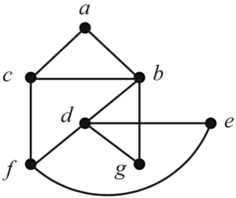


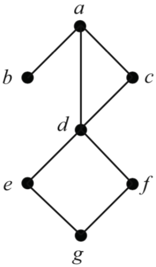




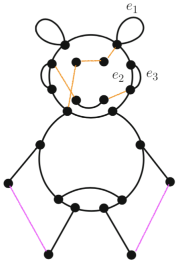
Applying Theorems 1 and 2 to examples

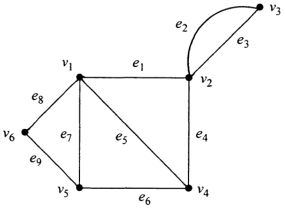






Applying Theorems 1 and 2 to examples





One more application of Thm 1: The notions of Eulerianity and Hamiltonicity are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

One more application of Thm 1: The notions of Eulerianity and Hamiltonicity are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Suppose that G is Eulerian. Suppose also that the size of G is m, and that

$$u_{i_0}\ e_{j_1}\ u_{i_1}\ e_{j_2}\ u_{i_2} \cdots\cdots u_{i_{m-2}}\ e_{j_{m-1}}\ u_{i_{m-1}}\ e_{j_m}\ u_{i_m} = u_{i_0}$$

is an Euler circuit of G (that is, u_{i_s}, $0 \leqslant s \leqslant m$, are vertices of G, not necessarily distinct, with the last vertex being equal to the first vertex as written above, while e_{j_t}, $1 \leqslant t \leqslant m$, are all the m edges of G, clearly each one appearing only once).

One more application of Thm 1: The notions of Eulerianity and Hamiltonicity are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Suppose that G is Eulerian. Suppose also that the size of G is m, and that

$$u_{i_0}\ e_{j_1}\ u_{i_1}\ e_{j_2}\ u_{i_2} \cdots\cdots u_{i_{m-2}}\ e_{j_{m-1}}\ u_{i_{m-1}}\ e_{j_m}\ u_{i_m} = u_{i_0}$$

is an Euler circuit of G (that is, u_{i_s}, $0 \leqslant s \leqslant m$, are vertices of G, not necessarily distinct, with the last vertex being equal to the first vertex as written above, while e_{j_t}, $1 \leqslant t \leqslant m$, are all the m edges of G, clearly each one appearing only once).

We note now that, for each $t < m$, the edges e_{j_t} and $e_{j_{t+1}}$ are adjacent since they have a common endvertex, the vertex u_{i_t}. Thus, e_{j_t} and $e_{j_{t+1}}$, as vertices of $L(G)$ now, will be neighbouring vertices.

One more application of Thm 1: The notions of Eulerianity and Hamiltonicity are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Suppose that G is Eulerian. Suppose also that the size of G is m, and that

$$u_{i_0}\ e_{j_1}\ u_{i_1}\ e_{j_2}\ u_{i_2}\cdots\cdots u_{i_{m-2}}\ e_{j_{m-1}}\ u_{i_{m-1}}\ e_{j_m}\ u_{i_m}=u_{i_0}$$

is an Euler circuit of G (that is, u_{i_s}, $0\leqslant s\leqslant m$, are vertices of G, not necessarily distinct, with the last vertex being equal to the first vertex as written above, while e_{j_t}, $1\leqslant t\leqslant m$, are all the m edges of G, clearly each one appearing only once).

We note now that, for each $t<m$, the edges e_{j_t} and $e_{j_{t+1}}$ are adjacent since they have a common endvertex, the vertex u_{i_t}. Thus, e_{j_t} and $e_{j_{t+1}}$, as vertices of $L(G)$ now, will be neighbouring vertices.
In other words, the walk in $L(G)$ given by

$$e_{j_1}\ e_{j_2}\cdots\cdots e_{j_{m-1}}\ e_{j_m}$$

is a Hamilton path in $L(G)$.

Moreover, we observe that e_{j_m} and e_{j_1} are neighbouring too in $L(G)$ since they have a common endvertex, the vertex $u_{i_m}=u_{i_0}$ of G.

One more application of Thm 1: The notions of Eulerianity and Hamiltonicity are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Suppose that G is Eulerian. Suppose also that the size of G is m, and that

$$u_{i_0}\ e_{j_1}\ u_{i_1}\ e_{j_2}\ u_{i_2} \cdots\cdots u_{i_{m-2}}\ e_{j_{m-1}}\ u_{i_{m-1}}\ e_{j_m}\ u_{i_m} = u_{i_0}$$

is an Euler circuit of G (that is, u_{i_s}, $0 \leqslant s \leqslant m$, are vertices of G, not necessarily distinct, with the last vertex being equal to the first vertex as written above, while e_{j_t}, $1 \leqslant t \leqslant m$, are all the m edges of G, clearly each one appearing only once).

We note now that, for each $t < m$, the edges e_{j_t} and $e_{j_{t+1}}$ are adjacent since they have a common endvertex, the vertex u_{i_t}. Thus, e_{j_t} and $e_{j_{t+1}}$, as vertices of $L(G)$ now, will be neighbouring vertices.
In other words, the walk in $L(G)$ given by

$$e_{j_1}\ e_{j_2} \cdots\cdots e_{j_{m-1}}\ e_{j_m}$$

is a Hamilton path in $L(G)$.

Moreover, we observe that e_{j_m} and e_{j_1} are neighbouring too in $L(G)$ since they have a common endvertex, the vertex $u_{i_m} = u_{i_0}$ of G. We conclude that

$$e_{j_1}\ e_{j_2} \cdots\cdots e_{j_{m-1}}\ e_{j_m}\ e_{j_1}$$

is a Hamilton cycle of $L(G)$, and thus $L(G)$ is a Hamiltonian graph.

The notions are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Next we check that $L(G)$ is Eulerian too.

The notions are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Next we check that $L(G)$ is Eulerian too.

We will use Theorem 1, which relies on what the degrees of the vertices are.

The notions are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Next we check that $L(G)$ is Eulerian too.

We will use Theorem 1, which relies on what the degrees of the vertices are.

- Since G is Eulerian, by Theorem 1 we know that all its vertices have even degree.

The notions are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Next we check that $L(G)$ is Eulerian too.

We will use Theorem 1, which relies on what the degrees of the vertices are.

- Since G is Eulerian, by Theorem 1 we know that all its vertices have even degree.

- Consider a vertex e_{j_t} of $L(G)$, or in other words an edge e_{j_t} of G. Let u_1, u_2 be its two endvertices.
  Question. What is the degree of e_{j_t} in $L(G)$?

The notions are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Next we check that $L(G)$ is Eulerian too.

We will use Theorem 1, which relies on what the degrees of the vertices are.

- Since G is Eulerian, by Theorem 1 we know that all its vertices have even degree.

- Consider a vertex e_{j_t} of $L(G)$, or in other words an edge e_{j_t} of G. Let u_1, u_2 be its two endvertices.
  Question. What is the degree of e_{j_t} in $L(G)$?
  Answer. We recall that

  $$\deg_{L(G)}(e_{j_t}) = \big(\deg_G(u_1) - 1\big) + \big(\deg_G(u_2) - 1\big) = \deg_G(u_1) + \deg_G(u_2) - 2.$$

The notions are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Next we check that $L(G)$ is Eulerian too.

We will use Theorem 1, which relies on what the degrees of the vertices are.

- Since G is Eulerian, by Theorem 1 we know that all its vertices have even degree.

- Consider a vertex e_{j_t} of $L(G)$, or in other words an edge e_{j_t} of G. Let u_1, u_2 be its two endvertices.
  Question. What is the degree of e_{j_t} in $L(G)$?
  Answer. We recall that

$$\deg_{L(G)}(e_{j_t}) = \big(\deg_G(u_1) - 1\big) + \big(\deg_G(u_2) - 1\big) = \deg_G(u_1) + \deg_G(u_2) - 2.$$

  Thus, if $\deg_G(u_1)$ and $\deg_G(u_2)$ are even, as follows from the assumptions of the theorem, then $\deg_{L(G)}(e_{j_t})$ will be even as well.

The notions are related

Theorem

Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.

Justification. Next we check that $L(G)$ is Eulerian too.

We will use Theorem 1, which relies on what the degrees of the vertices are.

- Since G is Eulerian, by Theorem 1 we know that all its vertices have even degree.
- Consider a vertex e_{j_t} of $L(G)$, or in other words an edge e_{j_t} of G. Let u_1, u_2 be its two endvertices.
  Question. What is the degree of e_{j_t} in $L(G)$?
  Answer. We recall that
  $$\deg_{L(G)}(e_{j_t}) = \big(\deg_G(u_1) - 1\big) + \big(\deg_G(u_2) - 1\big) = \deg_G(u_1) + \deg_G(u_2) - 2.$$
  Thus, if $\deg_G(u_1)$ and $\deg_G(u_2)$ are even, as follows from the assumptions of the theorem, then $\deg_{L(G)}(e_{j_t})$ will be even as well.
- Since e_{j_t} was an arbitrary vertex of $L(G)$, we conclude that every vertex of $L(G)$ has even degree. But then, by Theorem 1, we see that $L(G)$ is Eulerian too.

Proving Theorems 1 and 2?

Proving Theorems 1 and 2?

We begin with a few observations:

1. Suppose that $H = (V, E)$ is a multigraph, and consider a walk

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_9}\ v_{i_9}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

   in H. Let's say e.g. that e_{j_2} and e_{j_9} here are loops of H.

Proving Theorems 1 and 2?

We begin with a few observations:

1. Suppose that $H = (V, E)$ is a multigraph, and consider a walk

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_9}\ v_{i_9}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

in H. Let's say e.g. that e_{j_2} and e_{j_9} here are loops of H. Then we can safely 'delete' them and get a new, shorter walk in H which passes by the same vertices: note that

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

is again a walk in H (in this case, of length $k-2$), given that we must have had $v_{i_1} = v_{i_2}$ from the beginning (because e_{j_2} was a loop) and similarly we had $v_{i_8} = v_{i_9}$.

Proving Theorems 1 and 2?

We begin with a few observations:

1. Suppose that $H = (V, E)$ is a multigraph, and consider a walk

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_9}\ v_{i_9}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

in H. Let's say e.g. that e_{j_2} and e_{j_9} here are loops of H. Then we can safely 'delete' them and get a new, shorter walk in H which passes by the same vertices: note that

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

is again a walk in H (in this case, of length $k-2$), given that we must have had $v_{i_1} = v_{i_2}$ from the beginning (because e_{j_2} was a loop) and similarly we had $v_{i_8} = v_{i_9}$.

Obviously we can generalise this to any walk in H which contains some loops, and to 'deleting' any number of these loops from the given walk.

Proving Theorems 1 and 2?

We begin with a few observations:

1. Suppose that $H = (V, E)$ is a multigraph, and consider a walk

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_9}\ v_{i_9}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

   in H. Let's say e.g. that e_{j_2} and e_{j_9} here are loops of H. Then we can safely 'delete' them and get a new, shorter walk in H which passes by the same vertices: note that

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

   is again a walk in H (in this case, of length $k-2$), given that we must have had $v_{i_1} = v_{i_2}$ from the beginning (because e_{j_2} was a loop) and similarly we had $v_{i_8} = v_{i_9}$.
   Obviously we can generalise this to any walk in H which contains some loops, and to 'deleting' any number of these loops from the given walk.

2. If $\quad u_{i_0}\ \widetilde{e_{j_1}}\ u_{i_1}\ \widetilde{e_{j_2}}\ u_{i_2}\ \widetilde{e_{j_3}}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e_{j_9}}\ u_{i_9}\ \widetilde{e_{j_{10}}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e_{j_k}}\ u_{i_k} = u_{i_0}$

   is a closed walk in a (multi)graph G, then we can rewrite this walk so that it starts at any vertex we want out of the ones contained in the walk (and of course so that it ends at that same vertex).

Proving Theorems 1 and 2?

We begin with a few observations:

1. Suppose that $H = (V, E)$ is a multigraph, and consider a walk

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_2}\ v_{i_2}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_9}\ v_{i_9}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

   in H. Let's say e.g. that e_{j_2} and e_{j_9} here are loops of H. Then we can safely 'delete' them and get a new, shorter walk in H which passes by the same vertices: note that

$$v_{i_0}\ e_{j_1}\ v_{i_1}\ e_{j_3}\ v_{i_3} \cdots v_{i_8}\ e_{j_{10}}\ v_{i_{10}} \cdots v_{i_{k-1}}\ e_{j_k}\ v_{i_k}$$

   is again a walk in H (in this case, of length $k-2$), given that we must have had $v_{i_1} = v_{i_2}$ from the beginning (because e_{j_2} was a loop) and similarly we had $v_{i_8} = v_{i_9}$.
   Obviously we can generalise this to any walk in H which contains some loops, and to 'deleting' any number of these loops from the given walk.

2. If $\quad u_{i_0}\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} = u_{i_0}$

   is a closed walk in a (multi)graph G, then we can rewrite this walk so that it starts at any vertex we want out of the ones contained in the walk (and of course so that it ends at that same vertex).

   E.g. if we would prefer to start our walk at vertex u_{i_9}, then we would be able to do so and get a new closed walk which traverses the same edges as before: in this case, this could be the walk

$$u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} (= u_{i_0})\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ .$$

Proving Theorems 1 and 2?

3. Let G be a (multi)graph, and let

$$\mathrm{CI}_0 : u_{i_0}\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} = u_{i_0}$$

be a **circuit** of G (that is, now we assume that all the edges $\widetilde{e}_{j_1}, \widetilde{e}_{j_2}, \ldots, \widetilde{e}_{j_k}$ are distinct).

Proving Theorems 1 and 2?

3. Let G be a (multi)graph, and let

$$\mathrm{CI}_0 : u_{i_0}\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} = u_{i_0}$$

be a **circuit** of G (that is, now we assume that all the edges $\widetilde{e}_{j_1}, \widetilde{e}_{j_2}, \ldots, \widetilde{e}_{j_k}$ are distinct).
Then, if we view this circuit as a subgraph of G (or in other words, if we consider the subgraph of G which contains all the vertices and all the edges appearing in CI_0), then every vertex in CI_0 will have **even degree** within this subgraph.

Proving Theorems 1 and 2?

3. Let G be a (multi)graph, and let

$$\mathrm{CI}_0 : u_{i_0}\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} = u_{i_0}$$

be a **circuit** of G (that is, now we assume that all the edges $\widetilde{e}_{j_1}, \widetilde{e}_{j_2}, \ldots, \widetilde{e}_{j_k}$ are distinct).

Then, if we view this circuit as a subgraph of G (or in other words, if we consider the subgraph of G which contains all the vertices and all the edges appearing in CI_0), then every vertex in CI_0 will have **even degree** within this subgraph.

Indeed, if one of the edges appearing is a loop, say edge $\widetilde{e}_{j_2}$, then we must have that $u_{i_1} = u_{i_2}$ (and thus $\widetilde{e}_{j_2}$ can only affect the degree of vertex $u_{i_1} (= u_{i_2})$).

Proving Theorems 1 and 2?

3. Let G be a (multi)graph, and let

$$\mathrm{CI}_0 : u_{i_0}\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} = u_{i_0}$$

be a **circuit** of G (that is, now we assume that all the edges $\widetilde{e}_{j_1}, \widetilde{e}_{j_2}, \ldots, \widetilde{e}_{j_k}$ are distinct).
Then, if we view this circuit as a subgraph of G (or in other words, if we consider the subgraph of G which contains all the vertices and all the edges appearing in CI_0), then every vertex in CI_0 will have **even degree** within this subgraph.

Indeed, if one of the edges appearing is a loop, say edge $\widetilde{e}_{j_2}$, then we must have that $u_{i_1} = u_{i_2}$ (and thus $\widetilde{e}_{j_2}$ can only affect the degree of vertex $u_{i_1}(= u_{i_2})$).
We also recall that, by convention, a loop contributes 2 to the degree of the vertex it is attached to. Thus, if we first disregard all loops in CI_0, then, for every vertex appearing in CI_0, either its degree (within CI_0) is not affected at all, or it goes down by a positive multiple of 2 $\rightsquigarrow$ if we find that it is an even integer in the shortened version of CI_0, it would have been even before as well.

Proving Theorems 1 and 2?

3. Let G be a (multi)graph, and let

$$\mathrm{CI}_0 : u_{i_0}\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} = u_{i_0}$$

be a **circuit** of G (that is, now we assume that all the edges $\widetilde{e}_{j_1}, \widetilde{e}_{j_2}, \ldots, \widetilde{e}_{j_k}$ are distinct).
Then, if we view this circuit as a subgraph of G (or in other words, if we consider the subgraph of G which contains all the vertices and all the edges appearing in CI_0), then every vertex in CI_0 will have **even degree** within this subgraph.

Indeed, if one of the edges appearing is a loop, say edge $\widetilde{e}_{j_2}$, then we must have that $u_{i_1} = u_{i_2}$ (and thus $\widetilde{e}_{j_2}$ can only affect the degree of vertex $u_{i_1}(= u_{i_2})$).
We also recall that, by convention, a loop contributes 2 to the degree of the vertex it is attached to. Thus, if we first disregard all loops in CI_0, then, for every vertex appearing in CI_0, either its degree (within CI_0) is not affected at all, or it goes down by a positive multiple of 2 $\rightsquigarrow$ if we find that it is an even integer in the shortened version of CI_0, it would have been even before as well.

It remains to observe that, in the shortened version of CI_0, every time a vertex u_{i_s} appears (or reappears), it is incident with **two different edges** which are NOT loops, so each of these edges would contribute exactly 1 to the degree of u_{i_s} (within CI_0);

Proving Theorems 1 and 2?

3. Let G be a (multi)graph, and let

$$\mathrm{CI}_0 : u_{i_0}\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} = u_{i_0}$$

be a **circuit** of G (that is, now we assume that all the edges $\widetilde{e}_{j_1}, \widetilde{e}_{j_2}, \ldots, \widetilde{e}_{j_k}$ are distinct).
Then, if we view this circuit as a subgraph of G (or in other words, if we consider the subgraph of G which contains all the vertices and all the edges appearing in CI_0), then every vertex in CI_0 will have **even degree** within this subgraph.

Indeed, if one of the edges appearing is a loop, say edge $\widetilde{e}_{j_2}$, then we must have that $u_{i_1} = u_{i_2}$ (and thus $\widetilde{e}_{j_2}$ can only affect the degree of vertex $u_{i_1}(= u_{i_2})$).
We also recall that, by convention, a loop contributes 2 to the degree of the vertex it is attached to. Thus, if we first disregard all loops in CI_0, then, for every vertex appearing in CI_0, either its degree (within CI_0) is not affected at all, or it goes down by a positive multiple of 2 $\rightsquigarrow$ if we find that it is an even integer in the shortened version of CI_0, it would have been even before as well.

It remains to observe that, in the shortened version of CI_0, every time a vertex u_{i_s} appears (or reappears), it is incident with **two different edges** which are NOT loops, so each of these edges would contribute exactly 1 to the degree of u_{i_s} (within CI_0); moreover, since all edges in CI_0 are distinct, every time u_{i_s} reappears, we visit it by traversing a new edge, different from the ones we have already 'counted', and then we leave u_{i_s} by traversing yet another edge.

Proving Theorems 1 and 2?

3. Let G be a (multi)graph, and let

$$\mathrm{CI}_0 : u_{i_0}\ \widetilde{e}_{j_1}\ u_{i_1}\ \widetilde{e}_{j_2}\ u_{i_2}\ \widetilde{e}_{j_3}\ u_{i_3} \cdots u_{i_8}\ \widetilde{e}_{j_9}\ u_{i_9}\ \widetilde{e}_{j_{10}}\ u_{i_{10}} \cdots u_{i_{k-1}}\ \widetilde{e}_{j_k}\ u_{i_k} = u_{i_0}$$

be a **circuit** of G (that is, now we assume that all the edges $\widetilde{e}_{j_1}, \widetilde{e}_{j_2}, \ldots, \widetilde{e}_{j_k}$ are distinct).
Then, if we view this circuit as a subgraph of G (or in other words, if we consider the subgraph of G which contains all the vertices and all the edges appearing in CI_0), then every vertex in CI_0 will have **even degree** within this subgraph.

Indeed, if one of the edges appearing is a loop, say edge $\widetilde{e}_{j_2}$, then we must have that $u_{i_1} = u_{i_2}$ (and thus $\widetilde{e}_{j_2}$ can only affect the degree of vertex $u_{i_1}(= u_{i_2})$).
We also recall that, by convention, a loop contributes 2 to the degree of the vertex it is attached to. Thus, if we first disregard all loops in CI_0, then, for every vertex appearing in CI_0, either its degree (within CI_0) is not affected at all, or it goes down by a positive multiple of 2 $\rightsquigarrow$ if we find that it is an even integer in the shortened version of CI_0, it would have been even before as well.

It remains to observe that, in the shortened version of CI_0, every time a vertex u_{i_s} appears (or reappears), it is incident with **two different edges** which are NOT loops, so each of these edges would contribute exactly 1 to the degree of u_{i_s} (within CI_0); moreover, since all edges in CI_0 are distinct, every time u_{i_s} reappears, we visit it by traversing a new edge, different from the ones we have already 'counted', and then we leave u_{i_s} by traversing yet another edge.
$\rightsquigarrow$ Each appearance (or reappearance) of vertex u_{i_s} in the shortened version of CI_0 contributes 2 to its degree within CI_0.