MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 21

Thursday, November 25

Reminders

Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).

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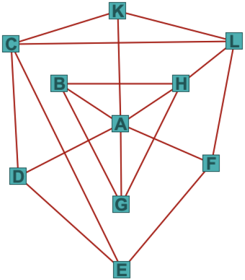
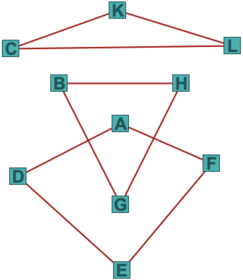
A subtle point. Note that a Hamilton cycle of G is a two-factor of G, but not every two-factor needs to be a Hamilton cycle. E.g. the graph on the right below is a two-factor of the graph on the left (but not a Hamilton cycle):





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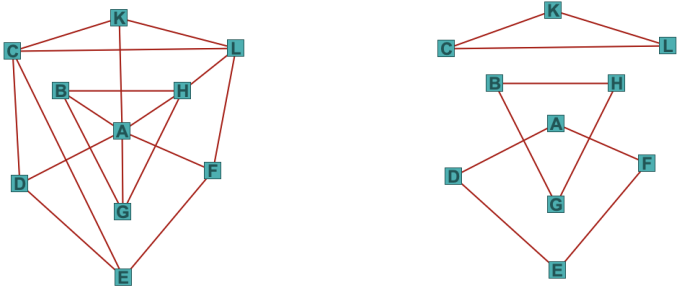
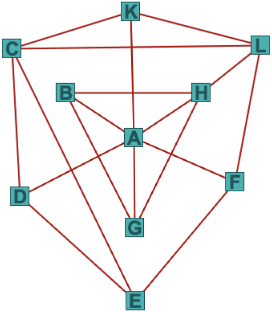
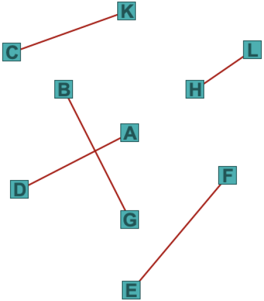


One more interesting point: Even though as we will see, the graph on the left DOES have one-factors too, we cannot come up with a one-factor of it simply by removing edges from the two-factor on the right. In other words, there is NO one-factor of the graph on the left which is also a subgraph of the two-factor on the right. *(why?)*

Remarks about one-factors

The graph on the RHS of this slide is a one-factor of the LHS graph of the previous slide (which also appears here on the left).





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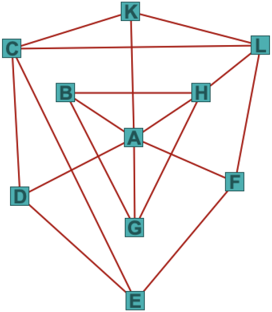

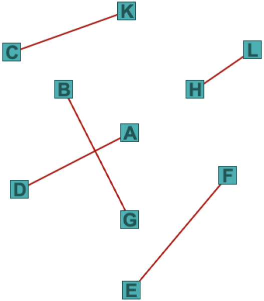

Question. Does the LHS graph have a one-factorization as well?

Remarks about one-factors (cont.)

Two (very simple) necessary conditions for the existence of one-factors

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Let G be graph of order $n = 2k$ and size m. Suppose that G has a one-factorization. Then

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(III) **G cannot have bridges** (except if G is a 1-regular graph itself, and hence the trivial factorization $\{G\}$ of G is a one-factorization too).

None of these conditions are sufficient too

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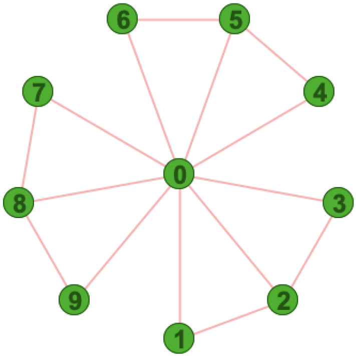

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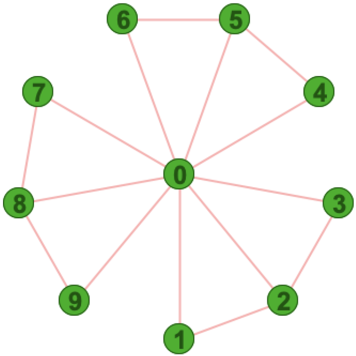

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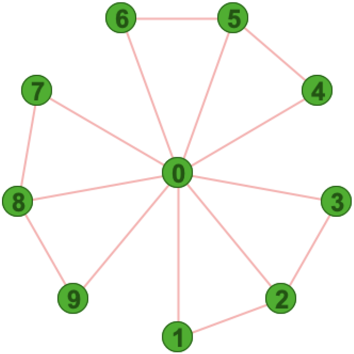
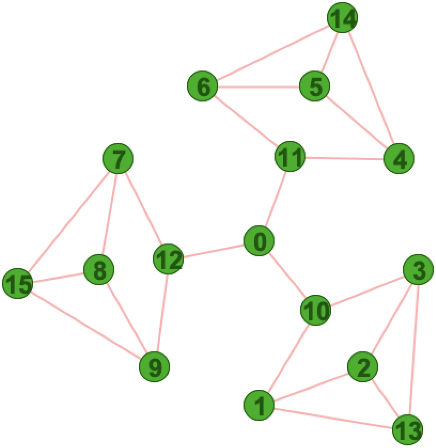


As a consequence of this, we obtain that the graph is not Hamiltonian either [why? note again that a Hamilton cycle with **an even number** of vertices has both a one-factor, and a one-factorization (in fact, it can be decomposed into two edge-disjoint one-factors)].

None of these conditions are sufficient too (cont.)

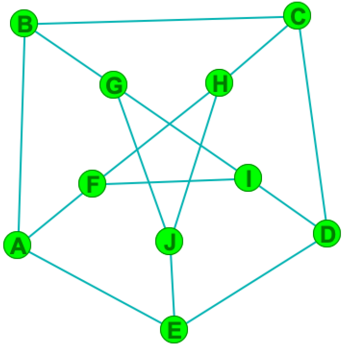
Example 2. The following graph is 3-regular (or equivalently, a *cubic graph*), but it does not have any one-factors (and of course it does not have a one-factorization).



Note that this is the smallest cubic graph without one-factors.

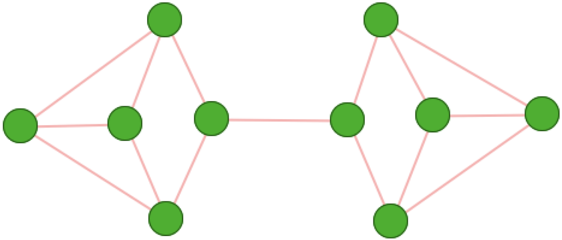
None of these conditions are sufficient too (cont.)

Example 3: The Petersen graph. As we have said, this is a cubic graph which satisfies $\kappa(G_0) = \lambda(G_0) = 3$, so it has no bridges. However, it does not have a one-factorization (although it has one-factors).



One more (non-)example

The following graph is the smallest cubic graph with no one-factorization *(can you see why it does not have a one factorization? also, can you find one factors of this graph?)*.



Let us now give a **necessary and sufficient** condition for a (not necessarily regular) graph to have one-factors.

Tutte's theorem

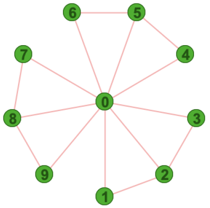
Theorem 1 (Tutte, 1947)

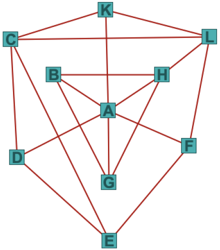
Let $G = (V, E)$ be a graph (or multigraph). Given a proper subset S of V, write $\mathrm{OC}(G - S)$ for the number of **odd** connected components of $G - S$ (that is, the number of those connected components of $G - S$ which have odd order).

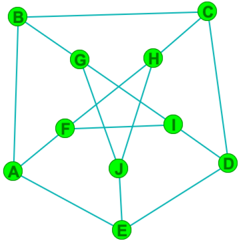
G has a one-factor **if and only if**

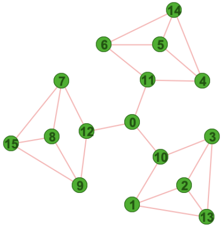
for every proper subset S of V, we have that $\mathrm{OC}(G - S) \leqslant |S|$.

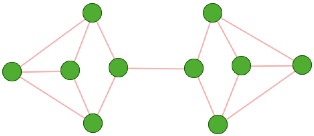
Testing Tutte's theorem on examples and non-examples











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Imagine a sports competition in which every player/team participating must play against any other player/team exactly once. This is usually called *round robin* of the competition.

Each time the players/teams are divided into different pairs of opponents, let us call this a session/stage of round robin.

— These different sessions/stages are essentially one-factors of the complete graph K_{2n} (where $2n$ is the total number of players/teams),

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In fact, the word 'tournament' has a graph-theoretical meaning too. To introduce the term, we first need to properly define the notion of 'orientation'.

Oriented graphs and tournaments

Terminology

Let G be a graph. We can obtain a directed graph (or digraph) G' from G by assigning a direction to each edge of G *[in particular, this implies that, if* $\{u, v\} \in E(G)$*, only one of the ordered pairs* (u, v) *and* (v, u) *will be contained in* $E(G')$*].* We call the process of obtaining such a digraph G' from G, as well as G' itself, an *orientation* of G.

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Moreover, an *oriented graph* H' is a digraph that can be obtained from orienting a graph H *[in other words, a digraph H' can be viewed as an oriented graph if and only if, for every two different vertices x, y of H', at most one of the pairs (x, y) and (y, x) is contained in $E(H')$].*

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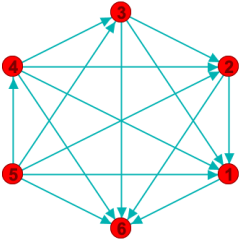
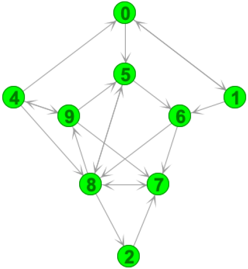
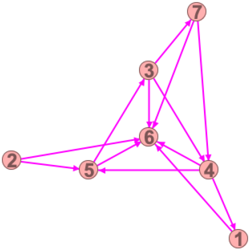
Remark. If we interpret the directed edge $x \to y$ (which formally would be written as the ordered pair (x, y)) as capturing that player/team x beat player/team y, then a one-factor of a tournament is not only a stage of a round robin, but it also encodes the results of that stage (that is, we can tell which player/team was the winner of any game that was played by looking at the direction of the corresponding edge in the one-factor).

Examples and non-examples of oriented graphs







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When working with digraphs (and especially digraphs whose 'underlying graph' is a complete graph), it makes sense to look for walks in the graph that also respect the direction of each edge which the walk traverses. That is, if H is digraph, and, say, $(x, y) \in E(H)$ but $(y, x) \notin E(H)$, then the convention is that it shouldn't be possible to find a walk in H of the form

$$v_0 \to v_1 \to v_2 \to \cdots \to y \to x \to \cdots \to v_{k-1} \to v_k,$$

because H does <u>not</u> contain the edge $y \to x$ (attention here: we don't consider the above to be a walk in H, even if $v_0\, v_1\, v_2 \cdots y\, x \cdots v_{k-1}\, v_k$ would be a walk of the underlying graph of H).

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With this in mind, it's no longer obvious that a tournament would necessarily have Hamilton paths and/or Hamilton cycles. But...

Tournaments have Hamilton paths

Theorem $0'$

Every tournament on $n \geqslant 2$ vertices has a Hamilton path.

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Proof. We will use induction in n.
Base case: $n = 2$. Then any tournament we can consider in this case will be of the form $\big(\{x, y\}, \{(x, y)\}\big)$ (after we appropriately 'rename' the vertices as x and y). Clearly, the path $x \to y$ is the Hamilton path we are looking for.

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Induction step: Assume now that, for some $n \geqslant 2$, we already know that the theorem is true. Consider a tournament K on $n + 1$ vertices, and denote its vertices by $v_1, v_2, \dots, v_n, u$ (note that, to keep the proof as clear as possible, we will use different notation for the last vertex of K).

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We conclude that K has a Hamilton path in all cases.

What about Hamilton cycles in tournaments?

Terminology

Let H be a digraph. Two vertices u, v of H are called *strongly connected* in H if we can find

- both a (directed) path starting at u and ending at v,
- and a (directed) path starting at v and ending at u.

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Let G be a graph. An orientation of G is called a *strong orientation* if the resulting oriented graph G' is strongly connected.

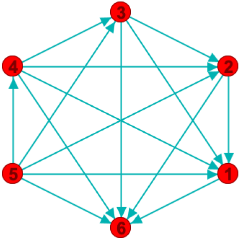
What about Hamilton cycles in tournaments?

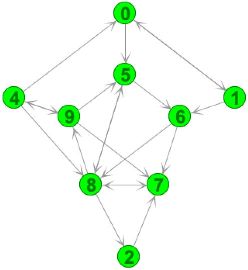
Theorem 1′

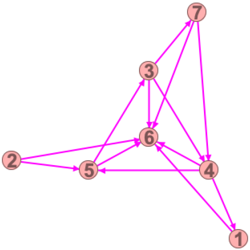
Let G be a connected graph. Then G has a strong orientation **if and only if** every edge of G belongs to at least one cycle.

Examples and non-examples

Question. Are any of the following digraphs strongly connected? (Equivalently in the case of any oriented graphs in the image below, are any of them strong orientations of the underlying graphs? And if not, could we find other orientations that are strong?)







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Note that an immediate corollary of Theorem 1′ would be the fact that a complete graph K_n with at least 3 vertices DOES have strong orientations (that is, there exist strongly connected tournaments on n vertices for all $n \geqslant 3$).

In fact, we can even state a (more or less efficient) criterion for when a tournament will be strongly connected:

Theorem 1″

Let $n \geqslant 3$. A tournament H' on n vertices is strongly connected **if and only if** H' has a Hamilton cycle.

Back to one-factors and one-factorizations of undirected graphs

A related notion

Definition

Let $G = (V, E)$ be a graph. A subset E' of E is called a *matching* in G if, for any two different edges $e_1, e_2 \in E'$, we have that e_1, e_2 are not adjacent *(that is, they don't have any common endvertex).*

In other words, E' is a matching in G if it is the edge set of a 1-regular subgraph of G (where we consider the vertex set of the subgraph to be all the endvertices of the edges in E').

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Definition

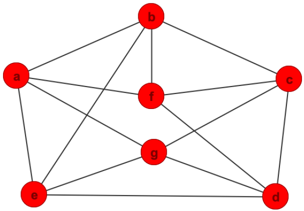
The maximum cardinality of a matching in G is denoted by $\nu(G)$.

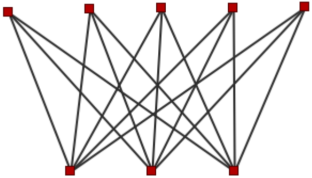
Useful observation

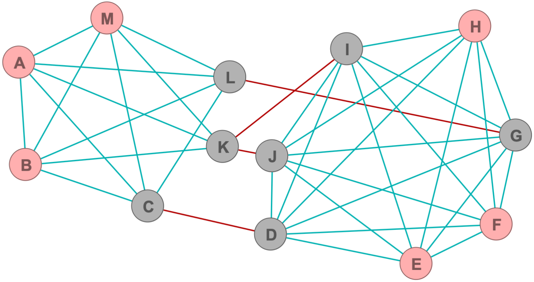
Let G be a graph. Then $\nu(G)$ coincides with the independence number $\alpha(L(G))$ of the line graph $L(G)$ of G (why?).

Examples

Practice Exercise. Find $\nu(G)$ for each graph G below (and also find a matching of the graph with cardinality $\nu(G)$).

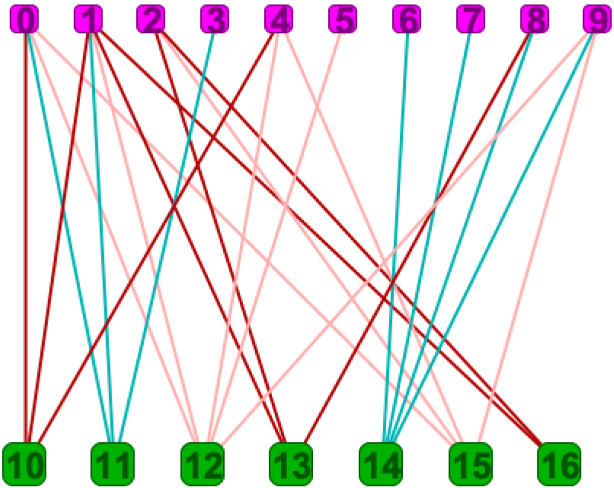






Examples (cont.)

Looking for matchings makes even more sense in bipartite graphs (here, as we will see, it's useful to discuss both complete and non-complete bipartite graphs).



Some real-life interpretations of matchings in (complete or non-complete) bipartite graphs

- Suppose one partite set of the graph represents workers in a factory, while the other partite set represents the machines they can operate (and there's an edge connecting a worker with a machine **if and only if** that worker knows how to operate that machine). Suppose also that we can't have two or more workers operating the same machine at the same time.

  We would of course want to have as many workers as possible operating (some of) the machines at the same time (ideally we would like to have all workers being able to work at the same time),

  or, **in the case that we have more workers than machines,** we would like to have as many machines as possible being in operation at the same time (ideally all the machines would be in operation).

  This corresponds to finding a matching in the graph which has maximum possible cardinality.

Some real-life interpretations of matchings in (complete or non-complete) bipartite graphs

- Suppose one partite set of the graph represents (undergraduate or graduate) positions at Canadian universities, while the other partite set represents prospective students (and there's an edge connecting a student with a university (thus in our model with all the available positions at this university) **if and only if** this student has applied to that university).

  We would like to:

  - either fill all available positions,
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  This corresponds to finding a matching in the graph which has maximum cardinality.

We will revisit a version of this problem which takes into account even more criteria a bit later.

Matching in bipartite graphs

Theorem 2 (Hall, 1935)

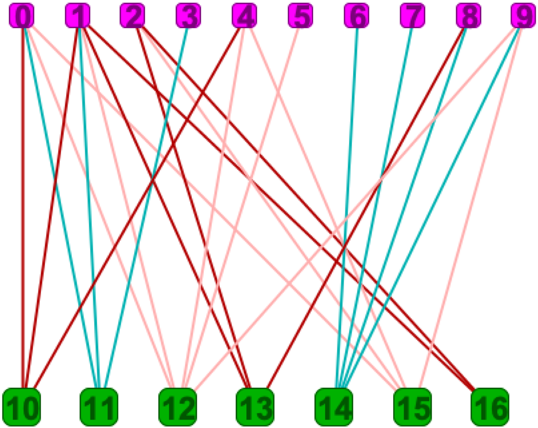
Let $G = (V, E)$ be a (not necessarily complete) bipartite graph with partite sets A and B (that is, $V(G) = A \cup B$, and there are no edges in $E(G)$ joining two vertices in A, or two vertices in B).

Then there is a matching in G covering the set A **if and only if**

for every $S \subseteq A$, we have that $|S| \leqslant |N(S)|$,

where $N(S)$ is the union of all neighbourhoods of vertices in S.

Examples (cont.)



Matching in bipartite graphs

In real-life applications, it's also very useful to have a *defect version of Hall's theorem.*

Proposition 3 (Corollary to Hall's theorem)

Let $G = (V, E)$ be a (not necessarily complete) bipartite graph with partite sets A and B.

Assume that, for some integer $d \geqslant 1$, G satisfies the following:

for every $S \subseteq A$, we have that $|N(S)| \geqslant |S| - d$.

Then there is a matching in G of cardinality at least $|A| - d$ (in other words, a matching that covers at least $|A| - d$ of the vertices in A).

Matching in bipartite graphs

Proof of Proposition 3 (defect version of Hall's theorem). We add d new vertices to the set B, say vertices $w_1, w_2, \ldots, w_d$, and we join each of these vertices **with each vertex from A.** The new graph G' that we have constructed is again a bipartite graph with partite sets A and $B \cup \{w_1, w_2, \ldots, w_d\}$.

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We now check that G' satisfies the condition of Hall's theorem: for every non-empty $S \subseteq A$, we have that $N_{G'}(S)$, that is, the neighbourhood of the set S in the graph G', contains all the vertices from G that are neighbours with vertices from S in the original graph G, as well as all the new vertices (since each of these new vertices is adjacent to every vertex in A).

In other words,

$$N_{G'}(S) = N_G(S) \cup \{w_1, w_2, \ldots, w_d\}$$

$$\Rightarrow \quad |N_{G'}(S)| = |N_G(S)| + |\{w_1, w_2, \ldots, w_d\}| = |N_G(S)| + d \geqslant \big(|S| - d\big) + d = |S|.$$

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Thus, by Hall's theorem the new graph G' has a matching M'_0 which covers A.

We now observe that M'_0 can contain **at most** d edges with one endvertex from the new vertex subset $\{w_1, w_2, \ldots, w_d\}$ that we introduced. Thus, if we remove these edges from M'_0, we get a matching M_0 of the original graph G which will have cardinality $\geqslant |M'_0| - d = |A| - d$.

Stable matchings?

Important Remark. The only (complete) bipartite graphs for which there exists a perfect matching are the graphs $K_{n,n}$ (where $n \geqslant 1$ is some integer).

In fact, in the graph $K_{n,n}$ we will have $n!$ different perfect matchings (or one-factors of $K_{n,n}$) (can you explain why this is so?).

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Thus, in such graphs it seems more useful to not just list the different matchings (of which there are plenty), but to start introducing even more criteria regarding whether a certain matching is preferable over another matching.