MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 5

Thursday, September 16

(More) Operations on Graphs

Operations we have seen

- disjoint union of graphs
- complement of a graph
- induced subgraph

To discuss today

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- *join* of graphs
- *line graph* of a graph
- *vertex deletion*
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and edge set

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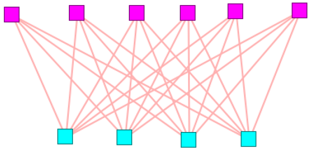
Examples



The bipartite graph $K_{6,4}$ can be viewed as the join of the null graphs N_6 and N_4:

$K_{6,4} = N_6 \vee N_4$.

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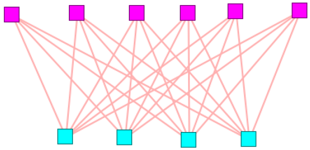

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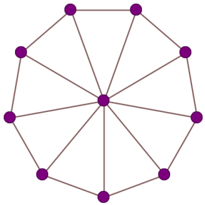

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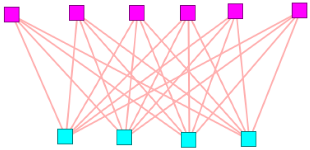

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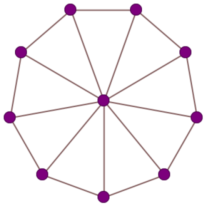

The wheel graph W_{10} can be viewed as the join of the cycle C_9 and the null graph N_1:

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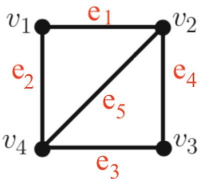
2) Given a vertex in $L(G)$, can we describe what its degree should be (again, based on parameters of the graph G)? *Exercise: Consider a vertex $e \in L(G)$; then, by definition, e is an edge of G; write v, w for its endvertices in G, and try to relate the degree of e in $L(G)$ to the degrees of v and w in G.*

An example

Recall the graph G_1:



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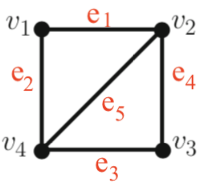
Recall the graph G_1: 

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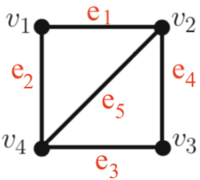

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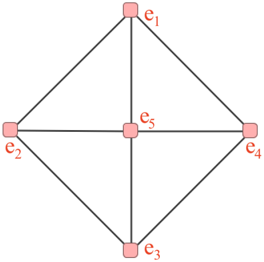

Line Graph, and Incidence and Adjacency Matrices

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Moreover, when V'' contains **only one vertex of G**, say vertex v_0, we will also sometimes write $G - v_0$ instead of $G - \{v_0\}$.

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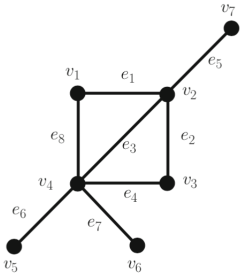
We can construct a new graph, which will also be a subgraph of G, **if we simply delete the edges of G which are contained in the edge subset E'.** In other words, we can consider the subgraph

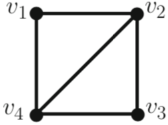
$$\big(V,\, E \setminus E'\big)$$

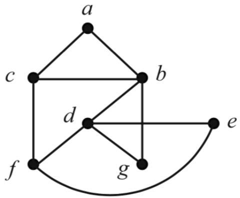
of G, which we will denote by $G - E'$.

Again, if E' contains **only one element, in this case one edge of** G, say edge e_0, we will more simply write $G - e_0$ instead of $G - \{e_0\}$.

Practise on three examples.







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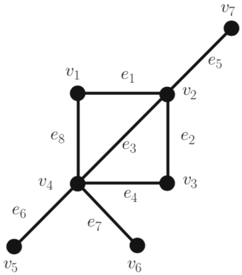

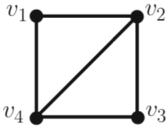

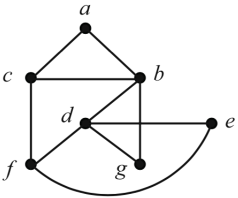

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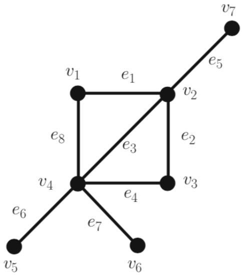

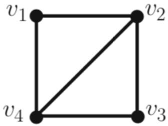

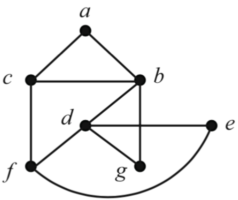

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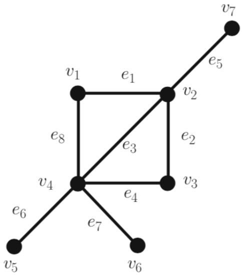

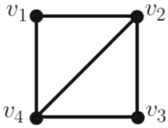

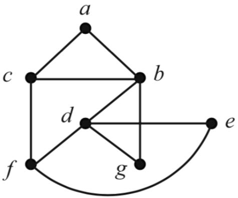

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3) Consider different subsets V' of the vertices of the third graph G_2, and similarly different subsets E' of its edges, and practise drawing $G_2 - V'$ and $G_2 - E'$ for these subsets.

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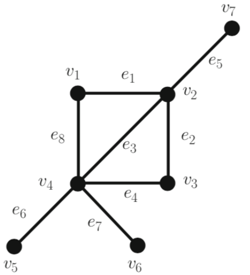

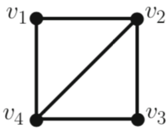

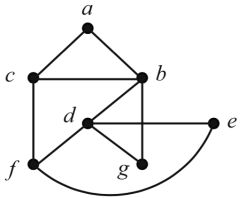

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2) Could you find a subset E' of the edges of G_0 so that $G_1 = G_0 - E'$?

3) Consider different subsets V' of the vertices of the third graph G_2, and similarly different subsets E' of its edges, and practise drawing $G_2 - V'$ and $G_2 - E'$ for these subsets.

4) What is the largest number of edges you could remove from G_0 so that you still have a connected graph? Same question for G_2.

Back to the last main result of Lecture 4: the Havel-Hakimi Theorem

The Havel-Hakimi theorem and algorithm

Theorem

Consider a decreasing sequence $S_1 = (d_1, d_2, \ldots, d_n)$ of n non-negative integers.

Suppose that S_1 is graphical, that is, suppose that S_1 is the degree sequence of a graph G with n vertices:

$$S_1 = \left(\deg_1 \geqslant \deg_2 \geqslant \cdots \geqslant \deg_{n-1} \geqslant \deg_n\right).$$

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Rewrite S_1 as follows *(the reason for this will become clear right away):*

$$S_1 = \left(\deg_1, \deg_2, \deg_3, \ldots, \deg_{\deg_1+1}, \deg_{\deg_1+2}, \deg_{\deg_1+3}, \ldots, \deg_{n-1}, \deg_n\right)$$

(note that there are $\deg_1$ purple-coloured terms here).

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The assumption that S_1 is graphical is **equivalent** to the sequence

$$S_1' = \left(\deg_2 - 1, \deg_3 - 1, \ldots, \deg_{\deg_1+1} - 1, \deg_{\deg_1+2}, \deg_{\deg_1+3}, \ldots, \deg_{n-1}, \deg_n\right)$$

being graphical.

The Havel-Hakimi theorem and algorithm

Written more succinctly...

Theorem

Consider a decreasing sequence $S_1 = (d_1, d_2, \ldots, d_n)$ of n non-negative integers.
Then S_1 is graphical **if and only if** the sequence

$$S_1' = \big(d_2 - 1,\ d_3 - 1, \ldots,\ d_{d_1+1} - 1,\ d_{d_1+2},\ d_{d_1+3}, \ldots,\ d_{n-1},\ d_n\big)$$

of $n - 1$ integers is graphical *(note that the purple-coloured terms here are d_1 in total).*

Reminder: How to apply the theorem

For the sequence $(4, 3, 3, 2, 2, 1, 1)$, we have

$$(4, 3, 3, 2, 2, 1, 1) \text{ is graphical}$$

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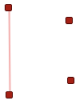
We can now verify directly that the last sequence, sequence $(1,1,0,0)$, is indeed graphical by drawing the following graph which has the sequence as its degree sequence:



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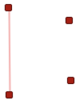

Thus, based on the theorem, all the other sequences are also graphical.

Reminder: How to apply the theorem

Let us also apply the theorem with the sequence $(6, 6, 5, 4, 3, 3, 1)$, which we have already seen is NOT graphical (see Lecture 4), and let's see what we get here.

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$(6, 6, 5, 4, 3, 3, 1)$ is graphical **if and only if**

$(5, 4, 3, 2, 2, 0)$ is graphical **if and only if**

$(3, 2, 1, 1, -1)$ is graphical.

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Clearly the last sequence is NOT graphical, and thus by the Havel-Hakimi theorem we get that the previous sequences are NOT graphical either.

Proving the theorem?

Theorem

Consider a decreasing sequence $S_1 = (d_1, d_2, \ldots, d_n)$ of n non-negative integers.

Then S_1 is graphical **if and only if** the sequence

$$S_1' = \left(d_2 - 1,\ d_3 - 1, \ldots,\ d_{d_1+1} - 1,\ d_{d_1+2},\ d_{d_1+3}, \ldots,\ d_{n-1},\ d_n\right)$$

of $n - 1$ integers is graphical *(note that the purple-coloured terms here are d_1 in total).*

Proof Strategy

We have two directions to prove:

1st Direction: we have to show that, if the shorter sequence S_1' is graphical, then the longer sequence S_1 is graphical too.

In other words, we have to show that, if there is a graph H realising the sequence S_1', then we can also construct a graph G realising the sequence S_1.

Proof of the theorem

We first assume that the sequence

$$S_1' = \left(d_2 - 1,\ d_3 - 1, \ldots,\ d_{d_1+1} - 1,\ d_{d_1+2},\ d_{d_1+3}, \ldots,\ d_{n-1},\ d_n\right)$$

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Let's assume that the vertices of H are labelled as

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in such a way that

$$\deg_H(v_i) = d_{i+1} - 1 \quad \text{for every } 1 \leqslant i \leqslant d_1,$$
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Construct a new graph G in such a way that:

- the vertex set $V(G)$ of G will consist of the vertices of H and one new vertex, vertex v_0: $V(G) = V(H) \cup \{v_0\}$;
- H will be a subgraph of G (so, in other words, any two vertices joined in H will also be joined in G), while in addition the new vertex v_0 will be joined with each of the 'purple-coloured' vertices of H: in short,

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How to use the theorem, as well as its proof

For the sequence $(4, 3, 3, 2, 2, 1, 1)$, we have

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Proof Strategy (cont.)

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2nd Direction: we have to show that, if the sequence S_1 is graphical, then the sequence S_1' is graphical too.

In other words, we have to show that, if there is a graph G realising the sequence S_1, then we can also construct a graph H realising the sequence S_1'.

Proof of the theorem (cont.)

We now deal with the reverse direction: we assume that the sequence

$$S_1 = \left(d_1,\, d_2,\, d_3, \ldots,\, d_{d_1+1},\, d_{d_1+2},\, d_{d_1+3}, \ldots,\, d_{n-1},\, d_n\right)$$

is graphical, and, based on this, we try to show that the sequence

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Since S_1 is graphical, we can find a graph G whose degree sequence coincides with S_1. Assume that the vertices of G are labelled as

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in such a way that $\deg_G(v_i) = d_{i+1}$ for every $0 \leqslant i \leqslant n-1$.

Here we have to distinguish cases. Our main criterion will be

- whether the neighbours of the vertex v_0 are among the 'purple-coloured' vertices *(call this Case 1),*
- or whether <u>some</u> of the neighbours of v_0 are also among the 'green-coloured' vertices *(call this Case 2).*

Proof of the theorem (cont.)

"Easiest" case (or Case 1). The neighbours of v_0 are among the 'purple-coloured' vertices.

Proof of the theorem (cont.)

"Easiest" case (or Case 1). The neighbours of v_0 are among the 'purple-coloured' vertices.

Then, given that v_0 must have d_1 neighbours in total, we see that **each of the 'purple-coloured' vertices of G is a neighbour of v_0** (whereas clearly **none of the 'green-coloured' vertices can be a neighbour of v_0** according to our assumption in this "easy" case).

Proof of the theorem (cont.)

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But then, check that the graph $G - v_0$ realises the sequence S_1':

Proof of the theorem (cont.)

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Then, given that v_0 must have d_1 neighbours in total, we see that **each of the 'purple-coloured' vertices of G is a neighbour of v_0** (whereas clearly **none of the 'green-coloured' vertices can be a neighbour of v_0** according to our assumption in this "easy" case).

But then, check that the graph $G - v_0$ realises the sequence S_1':

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- in $G - v_0$ each of the 'purple-coloured' vertices has one less neighbour than it did before: each such vertex v_i was adjacent to v_0 in G, but cannot be anymore since v_0 is not contained in $G - v_0$; at the same time all the other neighbours of v_i are still in $G - v_0$, and hence v_i is still connected to them *(based on how $G - v_0$ is defined, see earlier slides in this lecture)*;

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- in $G - v_0$ each of the 'green-coloured' vertices has exactly the same neighbours as it did in G: indeed, for each such vertex v_j, all its neighbours in G were different from v_0, so v_j is still connected to them in $G - v_0$ *(again, based on how $G - v_0$ is defined)*.

Proof of the theorem (cont.)

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Summarising, we have that, in $G - v_0$,

$$\deg(v_i) = d_{i+1} - 1 \quad \text{for every } 1 \leqslant i \leqslant d_1,$$

$$\text{and } \deg(v_j) = d_{j+1} \quad \text{for every } d_1 + 1 \leqslant j \leqslant n - 1\,,$$

showing that S_1' is the degree sequence of $G - v_0$, and thus that it is graphical.

Now we deal with Case 2, where some of the neighbours of v_0 are among the 'green-coloured' vertices.

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Recall that the vertices of G are labelled as

$$v_0\,,\; v_1\,,\; v_2, \ldots,\; v_{d_1}\,,\; v_{d_1+1}\,,\; v_{d_1+2}, \ldots,\; v_{n-2}, v_{n-1}\,,$$

and that according to this labelling the degree sequence of G is

$$d_1 \geqslant d_2 \geqslant d_3 \geqslant \cdots \geqslant d_{d_1+1} \geqslant d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_{n-1} \geqslant d_n\,.$$

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We consider two subcases of Case 2, based on what happens with the degrees of the neighbours of v_0:

Subcase 1: The sum of the degrees of all the neighbours of v_0 equals the sum of the degrees of the 'purple-coloured' vertices.

In other words, $\displaystyle\sum_{v_j \in N(v_0)} \deg(v_j) = \sum_{i=2}^{d_1+1} d_i\,.$

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Subcase 2: The sum of the degrees of all the neighbours of v_0 is strictly less than the sum of the degrees of the 'purple-coloured' vertices.

In other words, $\displaystyle\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i\ .$

Remark. Note that, because we have written the degree sequence of G in a decreasing manner, these are the only possible subcases.

That is, we know for sure that $\displaystyle\sum_{v_j \in N(v_0)} \deg(v_j) \leqslant \sum_{i=2}^{d_1+1} d_i\ .$

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Observe that we can write

$$\sum_{v_j \in N(v_0)} \deg(v_j) = \sum_{\substack{\text{the 'purple-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j) \quad + \sum_{\substack{\text{the 'green-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j)\,,$$

and similarly

$$\sum_{i=2}^{d_1+1} d_i = \sum_{\substack{\text{the 'purple-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j) \quad + \sum_{\substack{\text{the remaining} \\ \text{'purple-coloured' vertices } v_j}} \deg(v_j)\,.$$

Subcase 1: $\sum_{v_j \in N(v_0)} \deg(v_j) = \sum_{i=2}^{d_1+1} d_i$ (cont.)

But then we must have

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Also, observe that each summand of the first sum is <u>less than or equal to</u> each summand of the second sum, so, in order to have equality of the sums here, **these summands must all be equal.**

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But then we can relabel / reorder / 're-colour' the vertices of G, so that all the (previously) 'green-coloured' neighbours of v_0 become 'purple-coloured', and all the (previously) 'purple-coloured' vertices which are **<u>not</u>** neighbours of v_0 become 'green-coloured', **and we can do so without ruining the assumption that the (decreasing) degree sequence of G is the sequence**

$$d_1 \geqslant d_2 \geqslant d_3 \geqslant \cdots \geqslant d_{d_1+1} \geqslant d_{d_1+2} \geqslant d_{d_1+3} \geqslant \ldots \geqslant d_{n-1} \geqslant d_n\,.$$

Subcase 1: $\sum_{v_j \in N(v_0)} \deg(v_j) = \sum_{i=2}^{d_1+1} d_i$ (cont.)

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$$\sum_{\substack{\text{the 'green-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j) \quad = \sum_{\substack{\text{the remaining} \\ \text{'purple-coloured' vertices } v_j}} \deg(v_j)\,.$$

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Note that, **after having relabelled / reordered the vertices, we end up in Case 1:** the neighbours of v_0 are exactly the (new) 'purple-coloured' vertices.
Thus, as we did before, we can use the graph $G - v_0$ to conclude that the sequence

$$S_1' = (d_2 - 1,\, d_3 - 1, \ldots,\, d_{d_1+1} - 1,\, d_{d_1+2},\, d_{d_1+3}, \ldots,\, d_{n-1},\, d_n)$$

is graphical too.

Subcase 1: $\sum_{v_j \in N(v_0)} \deg(v_j) = \sum_{i=2}^{d_1+1} d_i$ (cont.)

But then we must have

$$\sum_{\substack{\text{the 'green-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j) \quad = \quad \sum_{\substack{\text{the remaining} \\ \text{'purple-coloured' vertices } v_j}} \deg(v_j)\,.$$

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But then we can relabel / reorder / 're-colour' the vertices of G, so that all the (previously) 'green-coloured' neighbours of v_0 become 'purple-coloured', and all the (previously) 'purple-coloured' vertices which are **not** neighbours of v_0 become 'green-coloured', **and we can do so without ruining the assumption that the (decreasing) degree sequence of G is the sequence**

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is graphical too. **This finishes the proof in Case 2, Subcase 1.**

Subcase 2: $\displaystyle\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$

Again we write

$$\sum_{v_j \in N(v_0)} \deg(v_j) = \sum_{\substack{\text{the 'purple-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j) \quad + \sum_{\substack{\text{the 'green-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j)\,,$$

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which shows that

$$\sum_{\substack{\text{the 'green-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j) \quad < \sum_{\substack{\text{the remaining} \\ \text{'purple-coloured' vertices } v_j}} \deg(v_j)\,.$$

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which shows that

$$\sum_{\substack{\text{the 'green-coloured'} \\ \text{neighbours } v_j \text{ of } v_0}} \deg(v_j) < \sum_{\substack{\text{the remaining} \\ \text{'purple-coloured' vertices } v_j}} \deg(v_j)\,.$$

Thus, in this subcase we must be able to find one summand of the first sum which is strictly less than at least one summand of the second sum. In other words, we can find a 'green-coloured' neighbour v_{j_1} of v_0, and a 'purple-coloured' vertex v_{j_2} which is **not** a neighbour of v_0 such that

$$\deg_G(v_{j_1}) < \deg_G(v_{j_2})\,.$$

Subcase 2: $\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$ (cont.)

- We have found a 'green-coloured' neighbour v_{j_1} of v_0, and a 'purple-coloured' vertex v_{j_2} which is **not** a neighbour of v_0 such that

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$$\deg_G(v_{j_1}) < \deg_G(v_{j_2}).$$

- Note that our assumptions give

$$v_0 v_{j_1} \in E(G) \qquad \text{while} \quad v_0 v_{j_2} \notin E(G).$$

Moreover, since $\deg_G(v_{j_1}) < \deg_G(v_{j_2})$, the vertex v_{j_2} must have at least one neighbouring vertex $\boldsymbol{w}$ which is **not** adjacent to v_{j_1}.

Subcase 2: $\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$ (cont.)

Gathering all the information again, we have that

$$v_0 v_{j_1} \text{ and } w v_{j_2} \in E(G), \qquad \text{while } w v_{j_1} \text{ and } v_0 v_{j_2} \notin E(G).$$

Subcase 2: $\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$ (cont.)

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We construct a new graph G' from G by removing the edges $v_0 v_{j_1}$ and $w v_{j_2}$, and then by adding the edges $w v_{j_1}$ and $v_0 v_{j_2}$. Note that, in the case of v_{j_1}, what we just did is remove one of its neighbours and replace it with a new neighbour, thus

$$\deg_{G'}(v_{j_1}) = \deg_G(v_{j_1}).$$

Similarly, we see that $\deg_{G'}(v_{j_2}) = \deg_G(v_{j_2})$, $\deg_{G'}(w) = \deg_G(w)$, and finally $\deg_{G'}(v_0) = \deg_G(v_0)$.

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Gathering all the information again, we have that

$$v_0 v_{j_1} \text{ and } w v_{j_2} \in E(G), \qquad \text{while } w v_{j_1} \text{ and } v_0 v_{j_2} \notin E(G).$$

We construct a new graph G' from G by removing the edges $v_0 v_{j_1}$ and $w v_{j_2}$, and then by adding the edges $w v_{j_1}$ and $v_0 v_{j_2}$. Note that, in the case of v_{j_1}, what we just did is remove one of its neighbours and replace it with a new neighbour, thus

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Important property of this construction: The degree sequence of G' is the same as the degree sequence of G, so it is the sequence S_1.

Subcase 2: $\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$ (cont.)

Gathering all the information again, we have that

$$v_0 v_{j_1} \text{ and } w v_{j_2} \in E(G), \qquad \text{while } w v_{j_1} \text{ and } v_0 v_{j_2} \notin E(G).$$

We construct a new graph G' from G by removing the edges $v_0 v_{j_1}$ and $w v_{j_2}$, and then by adding the edges $w v_{j_1}$ and $v_0 v_{j_2}$. Note that, in the case of v_{j_1}, what we just did is remove one of its neighbours and replace it with a new neighbour, thus

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Important property of this construction: The degree sequence of G' is the same as the degree sequence of G, so it is the sequence S_1.

At the same time, we have just replaced a 'green-coloured' neighbour of v_0 by a 'purple-coloured' vertex, <u>which also has bigger degree than the previous neighbour</u>. Thus we now have

$$\sum_{v_j \in N_G(v_0)} \deg_G(v_j) < \sum_{v_s \in N_{G'}(v_0)} \deg_{G'}(v_s),$$

while, given the important property above, we still have $\sum_{v_s \in N_{G'}(v_0)} \deg_{G'}(v_s) \leqslant \sum_{i=2}^{d_1+1} d_i$.

Subcase 2: $\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$ (cont.)

We can now finish the proof: we just constructed a new graph G'

- which first of all realises the sequence S_1 again (in fact, with the vertices $v_0, v_1, v_2, \ldots, v_{d_1}, v_{d_1+1}, v_{d_1+2}, \ldots, v_{n-2}, v_{n-1}$ labelled just as before),
- and secondly satisfies

$$\sum_{v_j \in N_G(v_0)} \deg_G(v_j) < \sum_{v_s \in N_{G'}(v_0)} \deg_{G'}(v_s) \leqslant \sum_{i=2}^{d_1+1} d_i \,.$$

Subcase 2: $\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$ (cont.)

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- If G' falls into Case 1, or into Case 2, Subcase 1, then we are done: the graph $G' - v_0$ will realise the sequence S_1'.

Subcase 2: $\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$ (cont.)

We can now finish the proof: we just constructed a new graph G'

- which first of all realises the sequence S_1 again (in fact, with the vertices $v_0,\ v_1,\ v_2, \ldots,\ v_{d_1},\ v_{d_1+1},\ v_{d_1+2}, \ldots,\ v_{n-2}, v_{n-1}$ labelled just as before),
- and secondly satisfies

$$\sum_{v_j \in N_G(v_0)} \deg_G(v_j) < \sum_{v_s \in N_{G'}(v_0)} \deg_{G'}(v_s) \leqslant \sum_{i=2}^{d_1+1} d_i \,.$$

• If G' falls into Case 1, or into Case 2, Subcase 1, then we are done: the graph $G' - v_0$ will realise the sequence S_1'.

• If G' falls into Case 2, Subcase 2, then we repeat the previous step and get yet another graph G'' realising the sequence S_1 and satisfying

$$\sum_{v_j \in N_G(v_0)} \deg_G(v_j) < \sum_{v_s \in N_{G'}(v_0)} \deg_{G'}(v_s) < \sum_{v_t \in N_{G''}(v_0)} \deg_{G''}(v_t) \leqslant \sum_{i=2}^{d_1+1} d_i \,.$$

Subcase 2: $\sum_{v_j \in N(v_0)} \deg(v_j) < \sum_{i=2}^{d_1+1} d_i$ (cont.)

We can now finish the proof: we just constructed a new graph G'

- which first of all realises the sequence S_1 again (in fact, with the vertices $v_0,\ v_1,\ v_2, \ldots,\ v_{d_1},\ v_{d_1+1},\ v_{d_1+2}, \ldots,\ v_{n-2}, v_{n-1}$ labelled just as before),
- and secondly satisfies

$$\sum_{v_j \in N_G(v_0)} \deg_G(v_j) < \sum_{v_s \in N_{G'}(v_0)} \deg_{G'}(v_s) \leqslant \sum_{i=2}^{d_1+1} d_i \,.$$

• If G' falls into Case 1, or into Case 2, Subcase 1, then we are done: the graph $G' - v_0$ will realise the sequence S_1'.

• If G' falls into Case 2, Subcase 2, then we repeat the previous step and get yet another graph G'' realising the sequence S_1 and satisfying

$$\sum_{v_j \in N_G(v_0)} \deg_G(v_j) < \sum_{v_s \in N_{G'}(v_0)} \deg_{G'}(v_s) < \sum_{v_t \in N_{G''}(v_0)} \deg_{G''}(v_t) \leqslant \sum_{i=2}^{d_1+1} d_i \,.$$

In other words: **whenever we end up in Case 2, Subcase 2, this method gives us a new graph in which the sum of the degrees of all the neighbours of v_0 increases, but still does not exceed the value $\sum_{i=2}^{d_1+1} d_i$.**
Thus we only need to repeat this process a finite number of times, and eventually we are guaranteed to end up in one of the 'nicer' cases, Case 1 or Case 2, Subcase 1.