(一) Lec 13 - 25 (After Midterm)

(二) Lec 1 - 12 (Summary) Midterm exam

HW1 - HW 6

Lecture notes 的 example

概念

1) 证明

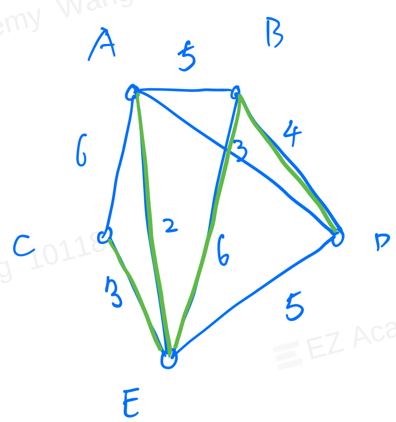
2) 算法

(一) Tree, Spanning, Minimum Connected Problem

Tree n $\begin{cases} \text{none - cycle} \\ \text{connected} \end{cases}$ $n-1 = m$

minimal spanning tree:

	A	B	C	D	E
A	0	5	6	3	2
B		0		4	6
C			0		3
D				0	5
E					0



$n=5$

$m = 5-1 = 4$

The shortest path problem

Dijkstra's algorithm

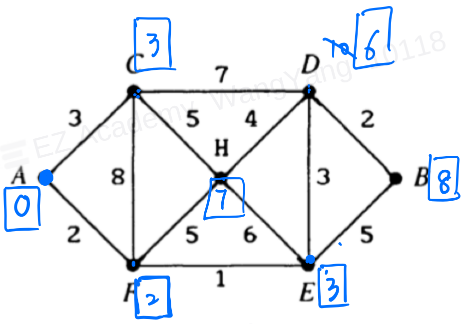
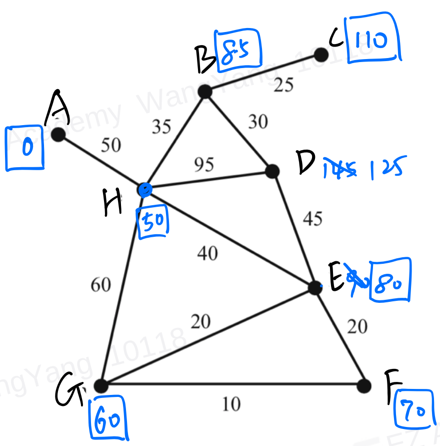


TABLE 10.5. *Steps of algorithm for shortest path from A to B*

	A	*B*	*C*	*D*	*E*	*F*	*H*
Iteration 0	$\boxed{0}$	∞	∞	∞	∞	∞	∞
Iteration 1	$\boxed{0}$	∞	$\boxed{3}$	∞	∞	$\boxed{2}$	∞
Iteration 2	$\boxed{0}$	∞	$\boxed{3}$	∞	$\boxed{3}$	$\boxed{2}$	$\boxed{7}$
Iteration 3	$\boxed{0}$	∞	$\boxed{3}$	$\boxed{10}$	$\boxed{3}$	$\boxed{2}$	$\boxed{7}$
Iteration 4	$\boxed{0}$	$\boxed{8}$	$\boxed{3}$	$\boxed{6}$	$\boxed{3}$	$\boxed{2}$	$\boxed{7}$
Iteration 5	$\boxed{0}$	$\boxed{8}$	$\boxed{3}$	$\boxed{6}$	$\boxed{3}$	$\boxed{2}$	$\boxed{7}$
Iteration 6	$\boxed{0}$	$\boxed{8}$	$\boxed{3}$	$\boxed{6}$	$\boxed{3}$	$\boxed{2}$	$\boxed{7}$



	A	*B*	*C*	*D*	*E*	*F*	G	*H*
Iteration 0	$\boxed{0}$	∞	∞	∞	∞	∞	∞	∞
Iteration 1	$\boxed{0}$	∞	∞	∞	∞	∞		$\boxed{50}$
Iteration 2	$\boxed{0}$	85	∞	145	90	∞	$\boxed{60}$	$\boxed{50}$
Iteration 3	$\boxed{0}$	85	∞	145	80	$\boxed{70}$	$\boxed{60}$	$\boxed{50}$
Iteration 4	$\boxed{0}$	85	∞	145	$\boxed{80}$	$\boxed{70}$	$\boxed{60}$	$\boxed{50}$
Iteration 5	$\boxed{0}$	$\boxed{85}$	∞	125	$\boxed{80}$	$\boxed{70}$	$\boxed{60}$	$\boxed{50}$
Iteration 6	$\boxed{0}$	$\boxed{85}$	$\boxed{110}$	125	$\boxed{80}$	$\boxed{70}$	$\boxed{60}$	$\boxed{50}$
Iteration 7	$\boxed{0}$	$\boxed{85}$	$\boxed{110}$	$\boxed{125}$	$\boxed{80}$	$\boxed{70}$	$\boxed{60}$	$\boxed{50}$

(二) Eulerian Graph & Hamiltonian Graph

Definition

Let G be a connected graph (or multigraph) which is non-trivial, that is, it contains at least one edge.

- An *Euler trail* in G is a trail that passes by **all** edges in G (and hence, given that it is a trail, it passes by each edge exactly once).
- An *Euler circuit* in G is a circuit (that is, a closed trail) that passes by **all** edges in G.

G is called *Eulerian* if we can find (at least) one Euler circuit in G.

G: ① 经过每条边
② 只经过1次
③ 回到起点
⇔ G is Eulerian graph

Definition

Let G be a connected graph.

- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).
- A *Hamilton cycle* in G is a cycle (that is, a closed path) that passes through **all** vertices in G.

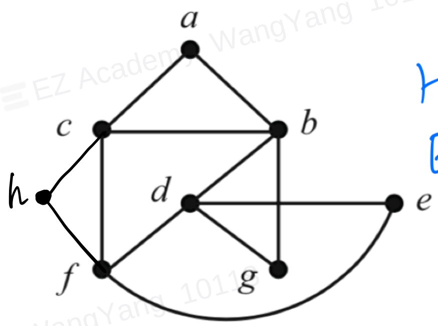
G is called *Hamiltonian* if we can find (at least) one Hamilton cycle in G.

G: ① 经过每个顶点
② 只经过1次
③ 回到起点
⇔ G is Hamiltonian graph

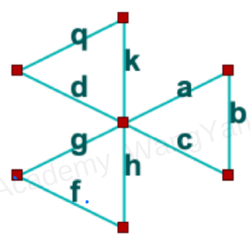
cycle 中: remove k 个顶点, 剩余的 component 个数 $\leq k$

Theorem

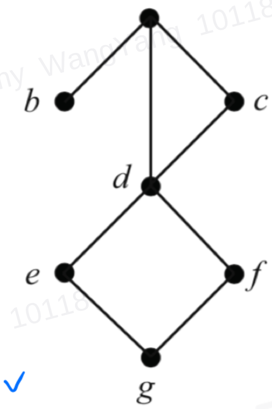
Let G be a connected graph. If G is Eulerian, then the line graph $L(G)$ of G is both Hamiltonian and Eulerian.



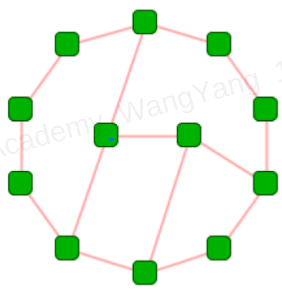
H ✓
E ✓



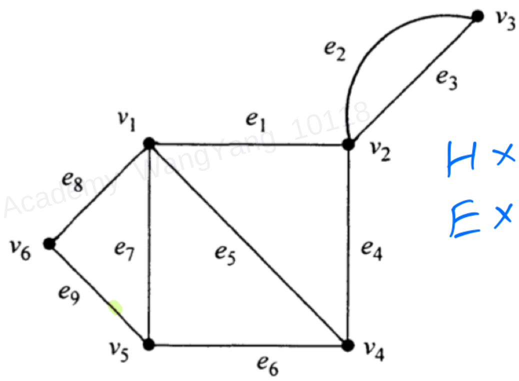
E ✓
H ✗



H ✗
E ✗



H ✓
E ✗



H ✗
E ✗

判断 Eulerian Graph 的方法：

① definition

②

Theorem 1

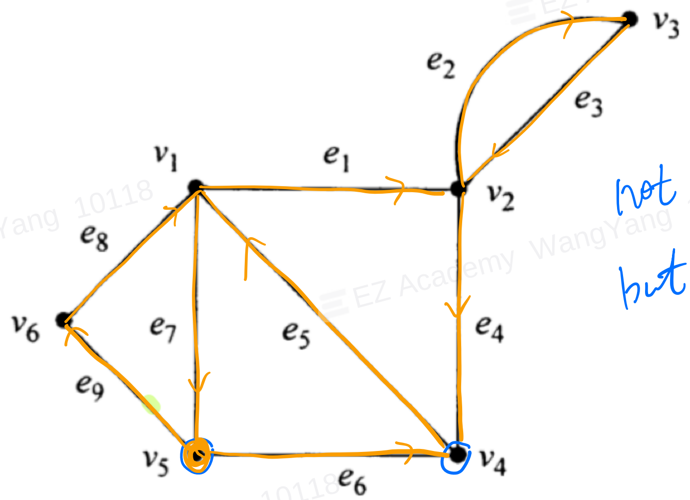
Let G be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** every vertex of G has even degree.

Proposition 1

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

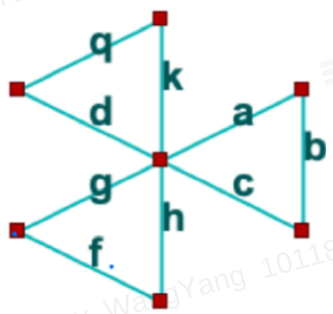


not Eulerian Graph
but has a Eulerian trail

Theorem 2

Let $G = (V, E)$ be a (non-trivial) connected graph (or multigraph).

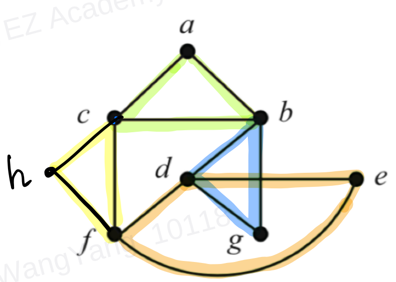
Then G is Eulerian **if and only if** its edge set E can be written as the disjoint union of subsets $E_1, E_2, \dots, E_s$ each of which forms a cycle in G.



$E_1 = \{ q, d, k \}$

$E_2 = \{ a, b, c \}$

$E_3 = \{ f, g, h \}$



cycle → 2-regular

Eulerian Graph has two factor

Eulerizations of multigraphs

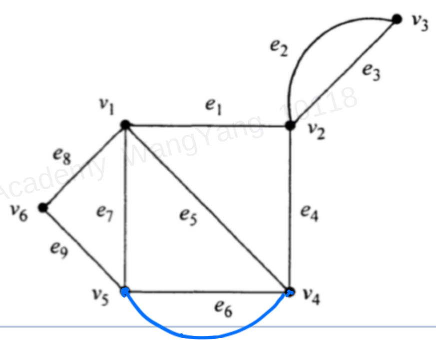
Definition

Let $G = (V(G), E(G))$ be a (non-trivial) connected multigraph, and suppose that G is NOT Eulerian.
An *Eulerization* of G is any multigraph $H = (V(H), E(H))$ which satisfies the following:

- $V(H) = V(G)$;
- $E(G) \subset E(H)$;

(note that the above already give that H is a super(multi)graph of G, and also that G is a spanning sub(multi)graph of H)

- H is Eulerian (that is, there exists an Euler circuit in H).



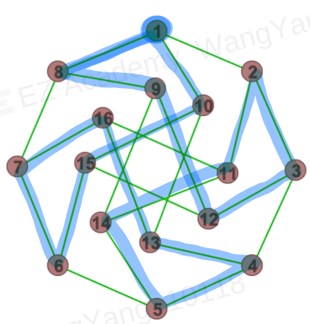
G

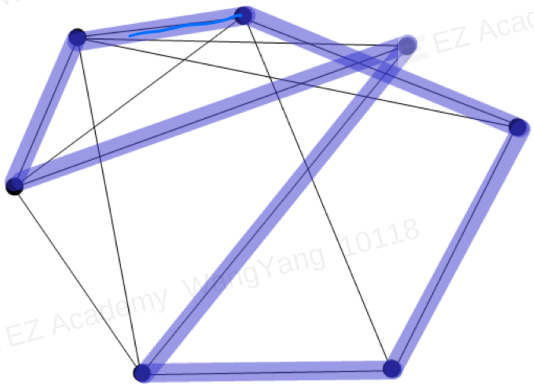
H

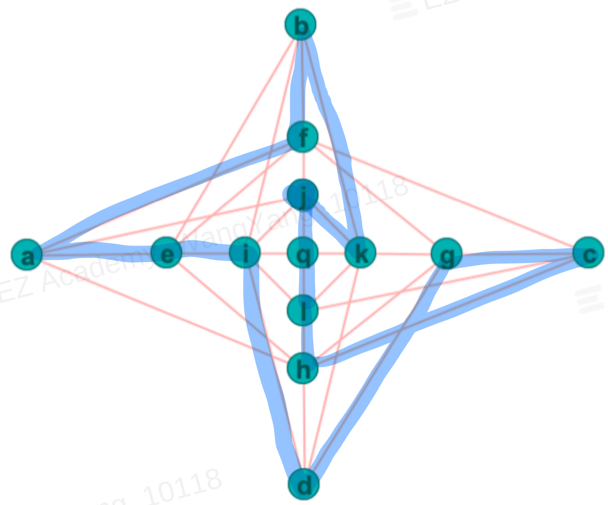
判断 Hamiltonian Graph: (HG)

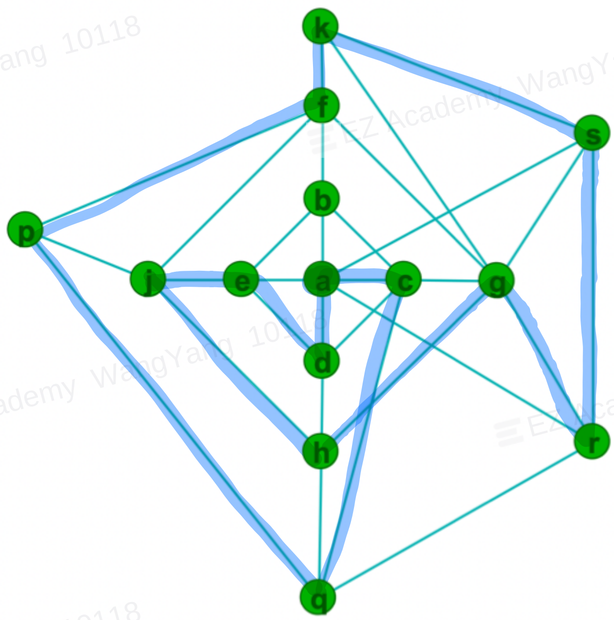
(一) 如何判断 G $\boxed{\text{is}}$ Hamiltonian Graph

① Definition G中找到 HG {① 经过每个顶点 ② 只经过1次 ③ 回到起点 ⇒ G is HG









② sufficient conditions

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$, then G is Hamiltonian

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G, we have that $\deg(u) + \deg(v) \geqslant n$.

Proposition 3

Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$m \geq \binom{n-1}{2} + 2 \qquad \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$$

edges. Then G is Hamiltonian.

Definition

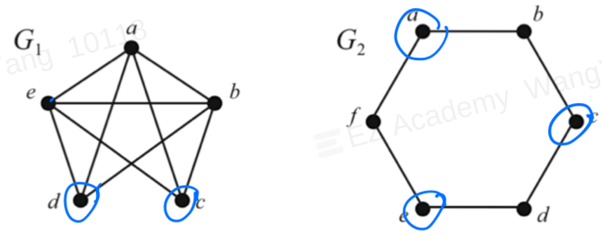
Let $G = (V, E)$ be a graph. A vertex subset $V' \subseteq V$ is called an *independent* set of vertices, if any two different vertices in V' are **non-adjacent**.

The *independence number* of G, denoted by $\alpha(G)$, is defined to be the maximum cardinality of an independent set of vertices of G.

$\delta(G)$
$\Delta(G)$
$\kappa(G)$
$\lambda(G)$
$\omega(G)$
$\alpha(G)$
$\chi(G)$

Examples. What are the independence numbers of the following two graphs?

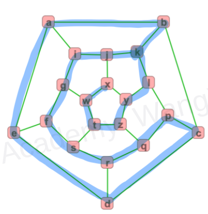
$\alpha(G_1) = 2$ $\qquad$ $\alpha(G_2) = 3$

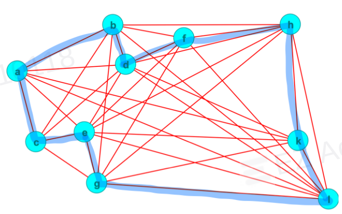


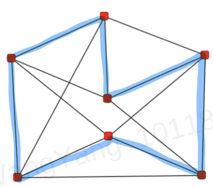
from the Harris-Hirst-Mossinghoff book

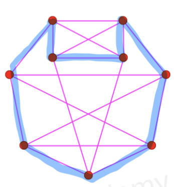
Theorem 4 (Chvátal-Erdös, 1972)

Let G be a graph of order $n \geqslant 3$ such that $\kappa(G) \geqslant \alpha(G)$. Then G is Hamiltonian.





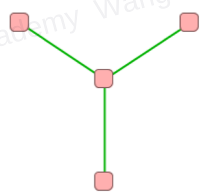




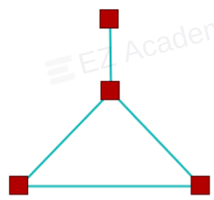
Forbidden Graph:

Family of possible forbidden subgraphs

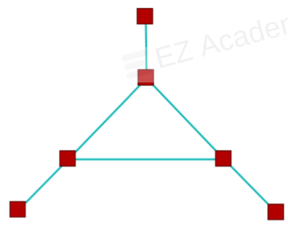
Consider the following three graphs:



Graph $K_{1,3}$



Graph Z_4



Graph Z_6

Theorem 5 (Goodman-Hedetniemi, 1974)

Let G be a graph of order $n \geqslant 3$ which is 2-vertex connected (that is, $\kappa(G) \geqslant 2$).

If G is $\{K_{1,3}, Z_4\}$-free *(that is, none of those two graphs is an induced subgraph of G)*, then G is Hamiltonian.

Theorem 6 (Duffus-Gould-Jacobson, 1980)

Let G be a $\{K_{1,3}, Z_6\}$-free graph.

- If G is connected, then G has a Hamilton path.
- If G is 2-vertex connected, then G is Hamiltonian.

(二)如何判断 G $\boxed{\text{is not}}$ Hamiltonian Graph

Necessary Condition 1

Let G be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then G has no cutvertices.
In other words, if G is Hamiltonian, then $\kappa(G) \geqslant 2$ (or equivalently G is 2-vertex connected).

$\kappa(G) \leqslant 1 \Rightarrow G$ is not HG

G has bridge

Necessary Condition 2

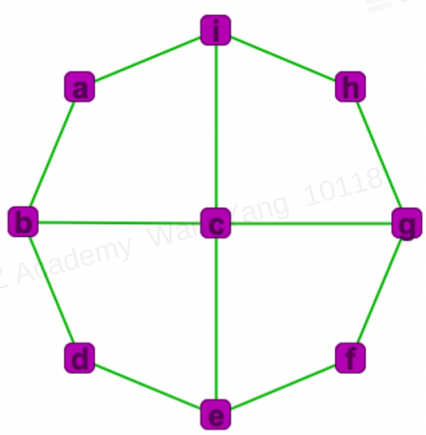
Let $G = (V, E)$ be a connected graph of order $n \geqslant 3$.

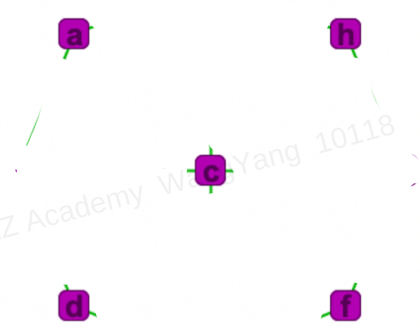
If G is Hamiltonian, then the following holds:

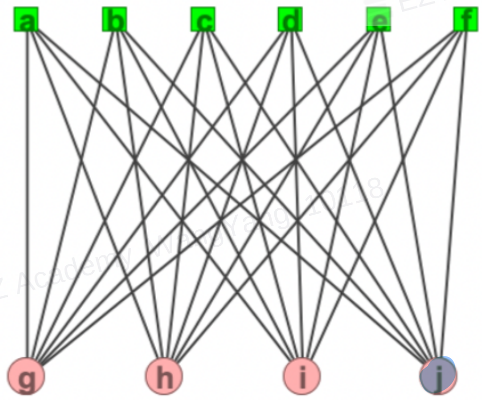
for every vertex subset $S \subsetneq V$,
the subgraph $G - S$ has at most $|S|$ connected components.

G中去掉 k个顶点之后 $\boxed{\text{剩余 components个数} > k}$

$\Rightarrow G$ is not HG







a
b
c
d
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g
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i
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a
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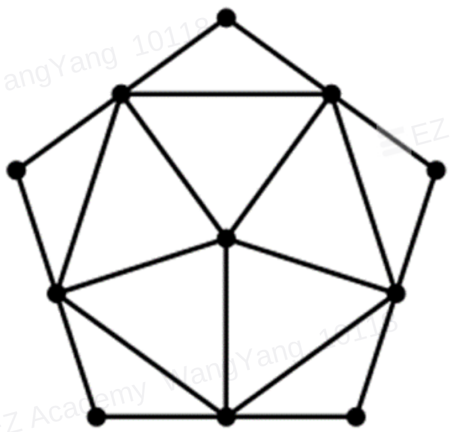

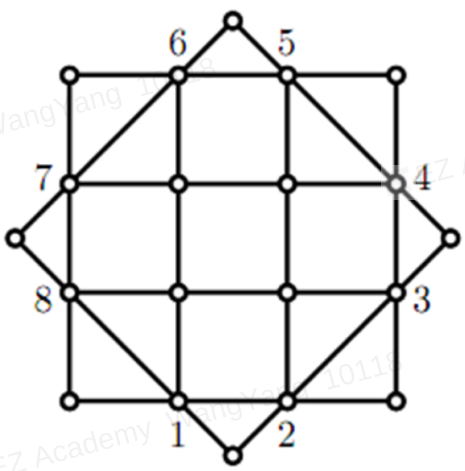

6
5
7
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8
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1
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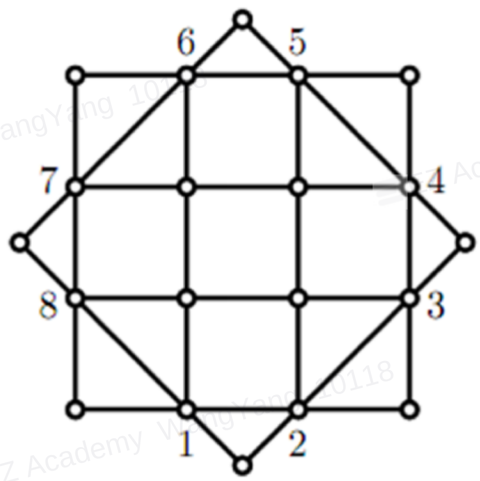


6
5
7
4
8
3
1
2

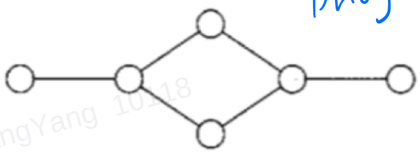
Necessary Condition 2′

Let $H = (V, E)$ be a connected graph of order $n \geqslant 2$.

If H has a Hamilton path, then the following holds:

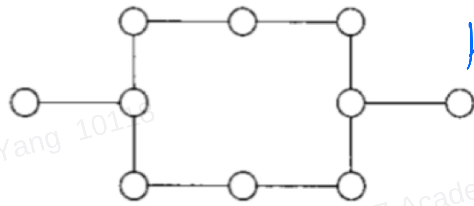
for every vertex subset $S \subsetneq V$,

the subgraph $H - S$ has at most $|S| + 1$ connected components.

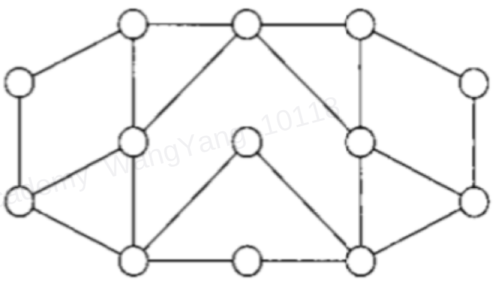


bridge

H ×



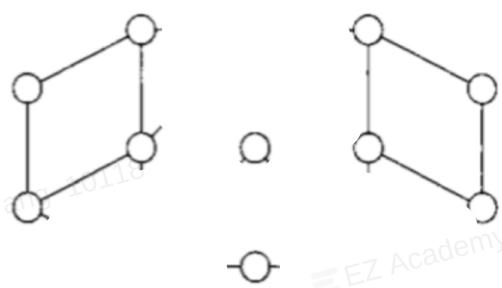
H ×



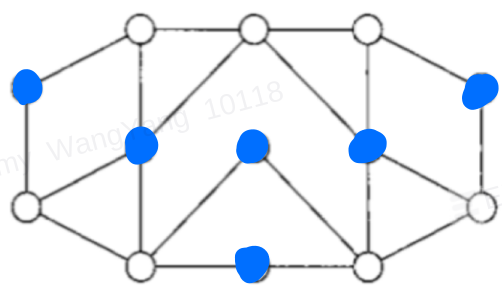
① 判断

② $\alpha(G)$

① not Hamiltonian



② $\alpha(G) = 6$



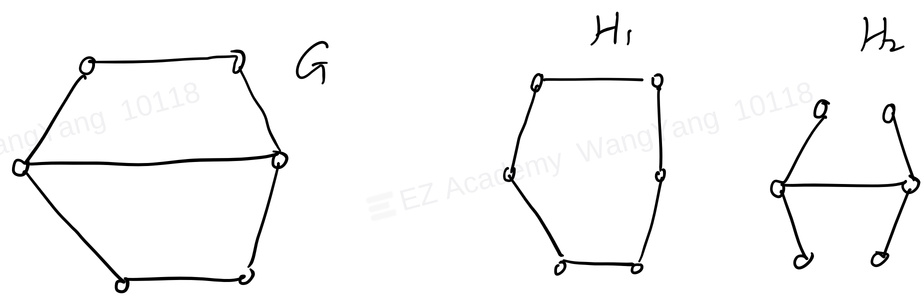
(三) factor & matching

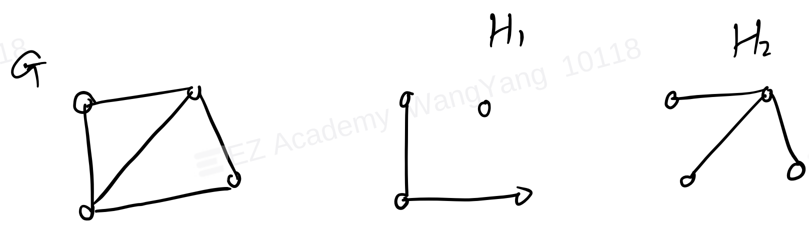
Definition

Let $G = (V, E)$ be a graph (or multigraph) of order n and size m.

1. A *factor*, or equivalently *spanning subgraph*, of G is a subgraph (or 'sub-multigraph') of G that contains all vertices of G (that is, a subgraph of order n).
2. A *factorization* of G is any collection of s factors (spanning subgraphs) $H_1, H_2, \ldots, H_s$ of G such that
   - any two different factors H_i and H_j are edge-disjoint;
   - every edge of G is contained in one of the factors $H_1, H_2, \ldots, H_s$, that is, $E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s)$.

ex:

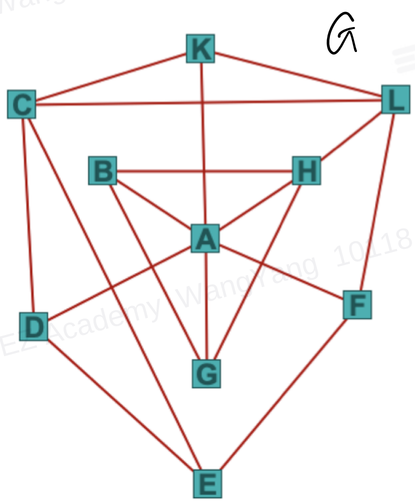


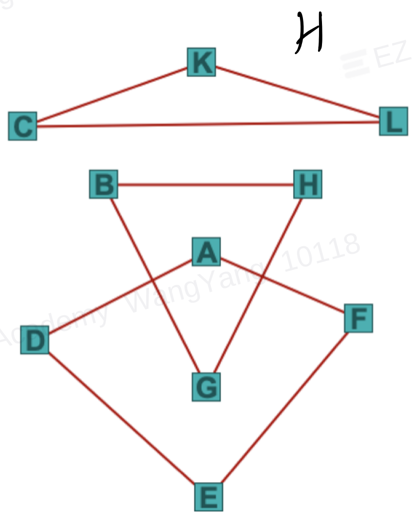


Definition

Let G be a graph (or multigraph).

- A spanning subgraph of G is called a *one-factor* of G **if it is 1-regular.**
  A *one-factorization* of G is a factorization of G consisting of one-factors of G.
- Similarly, a spanning subgraph of G is called a *two-factor* of G **if it is 2-regular**
  A *two-factorization* of G is a factorization of G consisting of two-factors of G.





Two (very simple) necessary conditions for the existence of one-factors

If a graph (or multigraph) G has a one-factor, then

(i) G has an even number of vertices.

(ii) G cannot have isolated vertices *(be careful that, if G is a multigraph, this condition is not immediately equivalent to $\delta(G) \geqslant 1$, given that some vertices of G may have no neighbours, but may have loops attached to them, so their degree will still be positive).*

Necessary conditions for the existence of a one-factorization

Let G be graph of order $n = 2k$ and size m. Suppose that G has a one-factorization. Then

(I) $k = \frac{n}{2}$ must divide m.

(II) Even more restrictively, the following property must be true for G: **the graph G must be d-regular for some d which divides m.** *(Justification? Note that, if G can be decomposed into d pairwise edge-disjoint one-factors, then each vertex v of G must be incident with precisely d edges.)*

(III) **G cannot have bridges** (except if G is a 1-regular graph itself, and hence the trivial factorization $\{G\}$ of G is a one-factorization too).

Let us now give a **necessary and sufficient** condition for a (not necessarily regular) graph to have one-factors.

Theorem (Tutte, 1947)

Let $G = (V, E)$ be a graph (or multigraph). Given a proper subset S of V, write $\mathrm{OC}(G - S)$ for the number of **odd** connected components of $G - S$ (that is, the number of those connected components of $G - S$ which have odd order).

G has a one-factor **if and only if** for every proper subset S of V, we have that $\boxed{\mathrm{OC}(G - S) \leqslant |S|}$

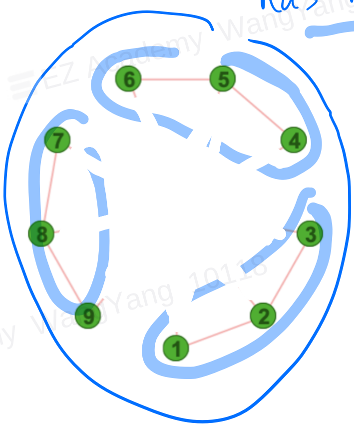
G中存在 S $\quad OC(G-S) > |S| \Rightarrow G$ has no one factors

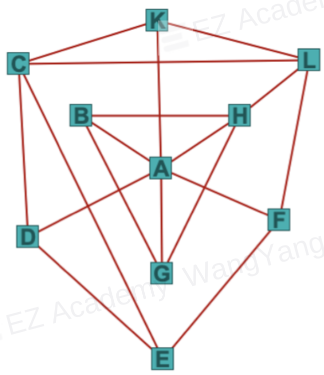
$OC(G-S) = 3 > 1$

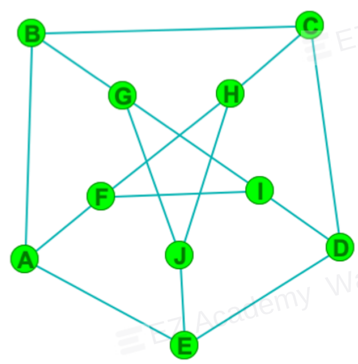
$\Downarrow$

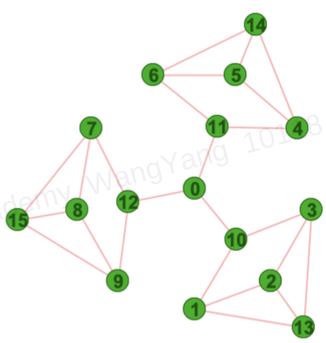
has no one factor

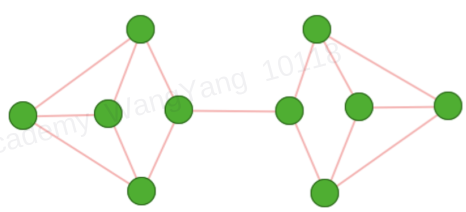
Testing Tutte's theorem on examples and non-examples











Terminology

Let G be a graph. We can obtain a directed graph (or digraph) G' from G by assigning a direction to each edge of G *[in particular, this implies that, if* $\{u, v\} \in E(G)$, *only one of the ordered pairs* (u, v) *and* (v, u) *will be contained in* $E(G')$*]*. We call the process of obtaining such a digraph G' from G, as well as G' itself, an *orientation* of G.

Moreover, an *oriented graph* H' is a digraph that can be obtained from orienting a graph H *[in other words, a digraph* H' *can be viewed as an oriented graph if and only if, for every two different vertices* x, y *of* H'*, at most one of the pairs* (x, y) *and* (y, x) *is contained in* $E(H')$*]*.

Terminology

Let $n \geqslant 2$. Any orientation of the complete graph K_n is called a *tournament*.

Theorem 0′

Every tournament on $n \geqslant 2$ vertices has a Hamilton path.

Terminology

Let H be a digraph. Two vertices u, v of H are called *strongly connected* in H if we can find

- both a (directed) path starting at u and ending at v,
- and a (directed) path starting at v and ending at u.

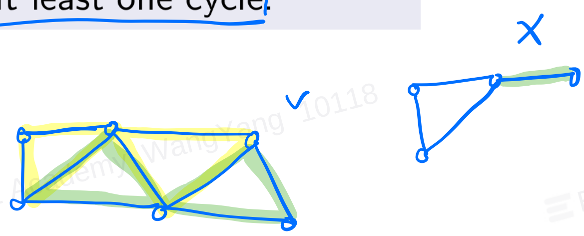
H is called *strongly connected* if any two different vertices in H are strongly connected.

Let G be a graph. An orientation of G is called a *strong orientation* if the resulting oriented graph G' is strongly connected.

G 有 strong orientation 的 充分必要条件:

Theorem 1′

Let G be a connected graph. Then G has a strong orientation **if and only if** every edge of G belongs to at least one cycle.



Theorem 1″

Let $n \geqslant 3$. A tournament H' on n vertices is strongly connected **if and only if** H' has a Hamilton cycle.

$\delta(G)$, $\Delta(G)$, $\kappa(G)$, $\lambda(G)$, $\alpha(G)$, $\nu(G)$ $\chi(G)$, $\omega(G)$

Definition

Let $G = (V, E)$ be a graph. A subset E' of E is called a *matching* in G if, for any two different edges $e_1, e_2 \in E'$, we have that e_1, e_2 are not adjacent *(that is, they don't have any common endvertex)*.

In other words, E' is a matching in G if it is the edge set of a 1-regular subgraph of G (where we consider the vertex set of the subgraph to be all the endvertices of the edges in E').

A matching E' in G is called a *perfect matching* if every vertex of G is *covered* by E', that is, if every vertex of G is the endvertex of some edge in E'. In other words, E' is a perfect matching if it is the edge set of a one-factor of G.

Definition

The maximum cardinality of a matching in G is denoted by $\nu(G)$.

Let G be a graph. Then $\nu(G)$ coincides with the independence number $\alpha(L(G))$ of the line graph $L(G)$ of G (why?).

$\nu(G) = \alpha(L(G))$

Theorem 2 (Hall, 1935)

Let $G = (V, E)$ be a (not necessarily complete) bipartite graph with partite sets A and B (that is, $V(G) = A \cup B$, and there are no edges in $E(G)$ joining two vertices in A, or two vertices in B).

Then there is a matching in G covering the set A **if and only if**

for every $S \subseteq A$, we have that $|S| \leqslant |N(S)|$,

where $N(S)$ is the union of all neighbourhoods of vertices in S.

Proposition 3 (Corollary to Hall's theorem)

Let $G = (V, E)$ be a (not necessarily complete) bipartite graph with partite sets A and B.

Assume that, for some integer $d \geqslant 1$, G satisfies the following:

for every $S \subseteq A$, we have that $|N(S)| \geqslant |S| - d$.

Then there is a matching in G of cardinality at least $|A| - d$ (in other words, a matching that covers at least $|A| - d$ of the vertices in A).

m_1	m_2	m_3
$\cancel{w_1}$	$\cancel{w_1}$	$\cancel{w_3}$
w_2	$\cancel{w_3}$	$\cancel{w_2}$
w_3	w_2	w_1

w_1	w_2	w_3
m_1	m_2	m_2
m_3	m_3	m_1
m_2	m_1	m_3

round 1. $m_1 \longrightarrow w_1$ (m_1, w_1)

$m_2 \rightarrow w_1$ (m_3, w_3)

$m_3 \longrightarrow w_3$

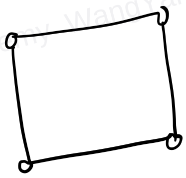
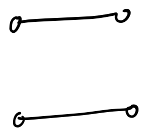
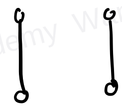
round 2 $m_2 \longrightarrow w_3$ (m_1, w_1)

(m_3, w_3)

round3 $m_3 \to w_2$

(m_1, w_1)
(m_2, w_3)
(m_3, w_2)

---

ex G   

H_1 one factor

H_2 one factor

$H_1 \cup H_2$ G one factorization

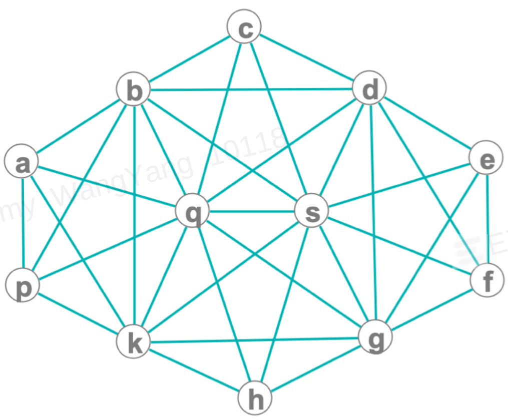


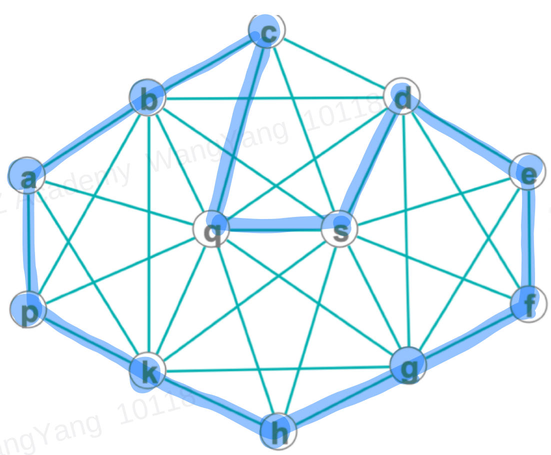
Figure 1: Graph G_0

(b) Does G_0 have a two-factor? Does it have a $\boxed{\text{two-factorization?}}$ Justify your answers fully.

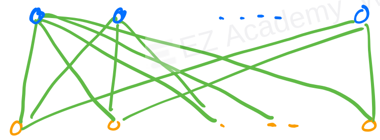
↓ yes.

↓ no.

G has two factorization $\Rightarrow$ G is d-regular



$K_{m,n}$



$K_{m,n}$ 中一定有 mn 条边

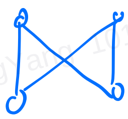
$d(V_m) = n$

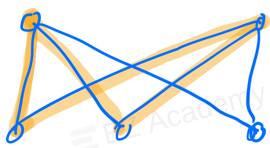
$d(V_n) = m$

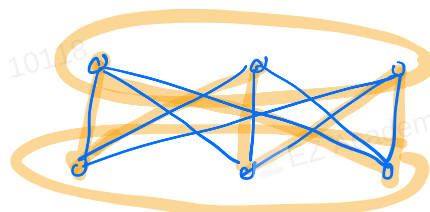
Problem 3 (*max. 20 = 10 + 10 points*) Both parts of the following problem concern complete bipartite graphs $K_{m,n}$.

(a) Find all pairs of positive integers (m, n) for which the graph $K_{m,n}$ is Eulerian. Justify your answer fully.

(b) Find all pairs of positive integers (m, n) for which the graph $K_{m,n}$ is Hamiltonian. Justify your answer fully. $m = n \geq 2$







Problem 4 (*max. 20 = 10 + 10 points*) (a) Is the following graph Hamiltonian? Justify your answer fully.

(b) Find $\kappa(G_1)$ precisely. Justify your answer fully.

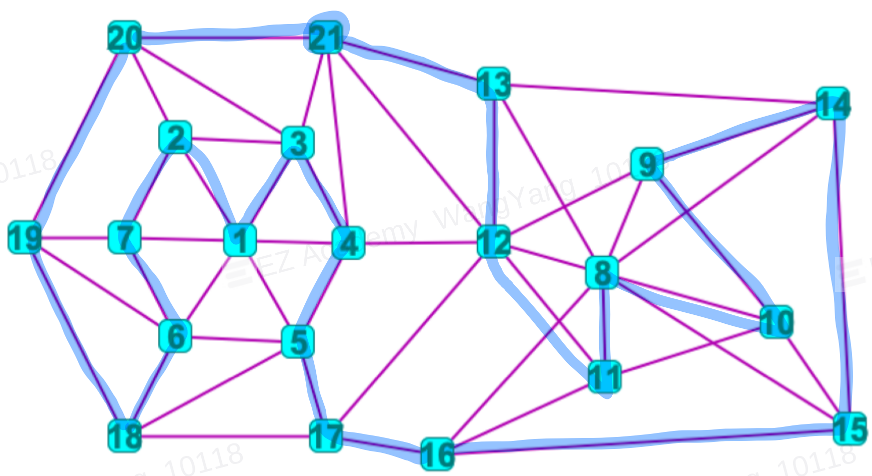


Figure 2: Graph G_1

$\kappa(G_1) \leq \delta(G_1) = 4 \quad \therefore \quad \kappa(G_1) \leq 4$

$\kappa(G_1) \leq 3$

证明 $\kappa(G_1) \geq 3$

$\kappa(G_1) \geq$

(四), Colouring Problem

colouring vertices:  adjacent 的 vertice 用不同颜色

Definition 1

Let $G = (V, E)$ be a graph. A proper vertex colouring ξ of G is called an *n-colouring* if there are exactly n non-empty colour classes of ξ. In other words, if the range of the function ξ contains exactly n positive integers.

G will be called n-colourable if we can find a (proper) n-colouring of G (*pictorially we can think of this as follows: G is n-colourable if n colours are enough for us to find a way to colour the vertices of G so that no two adjacent vertices will end up having the same colour*).

Definition 2

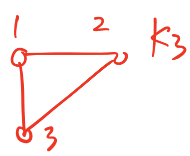
The *chromatic number* of a graph G is equal to the **smallest** integer n for which we can find a (proper) n-colouring of G.

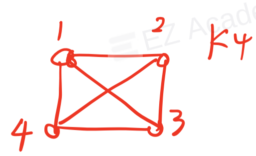
If n_0 is this smallest integer, then we say that G is *n_0-chromatic*.

$\chi(G)$

denote this smallest integer by $\chi(G)$ (*in other words, G is $\chi(G)$-chromatic*).

① K_n $\boxed{\chi(K_n)=n}$

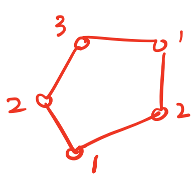




② cycle

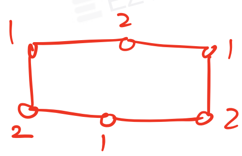
odd cycle

$\boxed{\chi(C)=3}$



even cycle

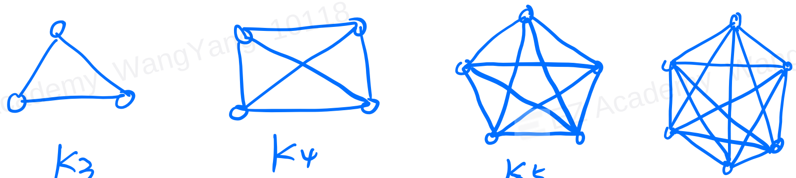
$\boxed{\chi(C)=2}$



$\boxed{\chi(K_{m,n})=2}$

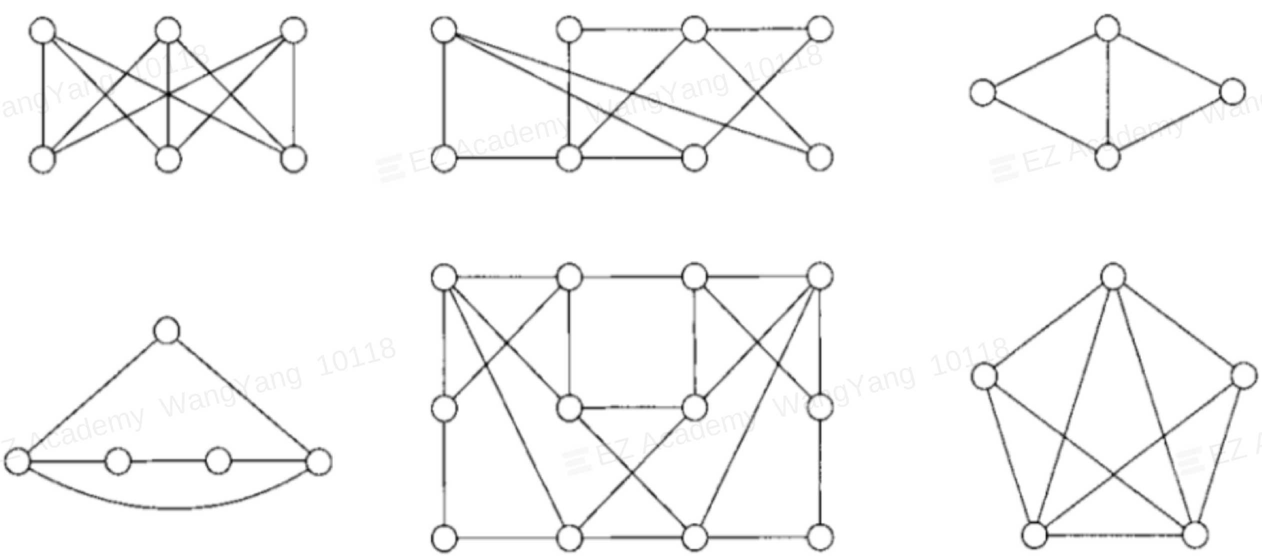
$H \subseteq G \Rightarrow \chi(G) \geq \chi(H)$

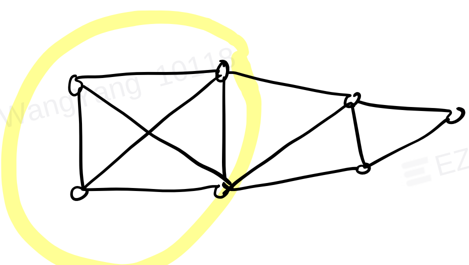
$\omega(G)$: G中所包含的最大的 Complete graph 的顶点个数



K_3 K_4 K_5 K_6

$\boxed{\chi(G) \geq \omega(G)}$





$K_4 \subseteq G$

Theorem 1

For every graph G, we have that $\chi(G) \leqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

Theorem 2 (Brooks' theorem, 1941)

Let G be a connected graph which is neither a complete graph nor an odd cycle. Then $\chi(G) \leqslant \Delta(G)$.

$\omega(G) \leq \chi(G) \leq \Delta(G)$

Theorem 3

For every graph G, we have that $\chi(G) \geqslant \omega(G)$.

Theorem 4

For every graph G of order n, we have that

$$\frac{n}{\alpha(G)} \leqslant \chi(G) \leqslant n + 1 - \alpha(G)$$

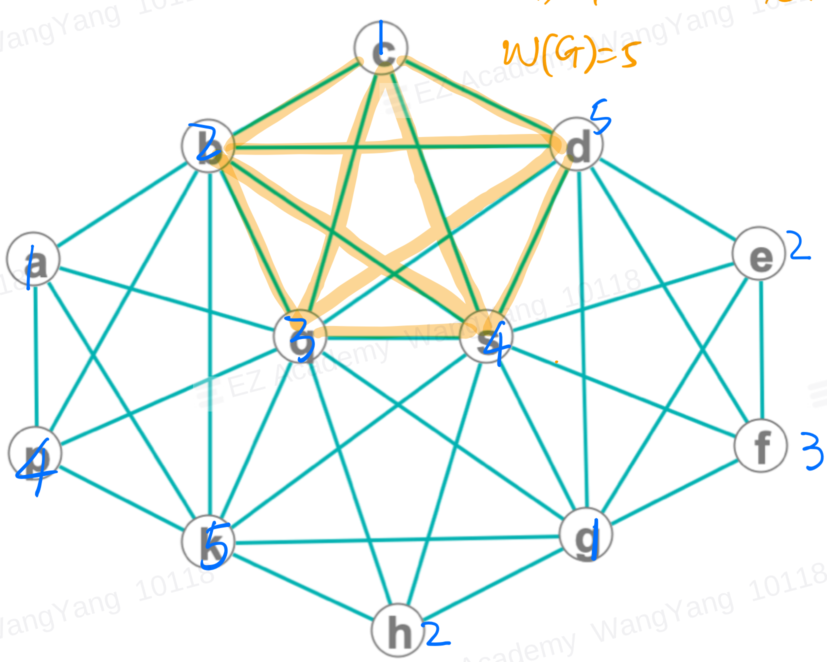
where $\alpha(G)$ is the independence number of G, that is, the maximum cardinality of an independent set of vertices in G.

$\chi(G)$

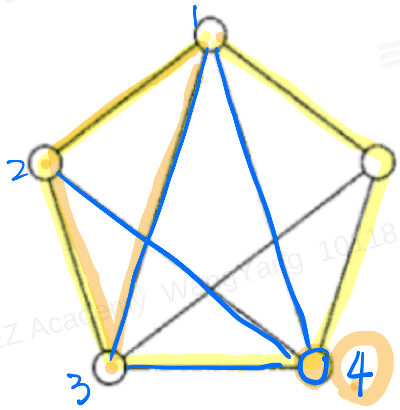
Problem 1 (*max. 20 = 10 + 10 points*) (a) Find the chromatic number of the following graph. Also, give a minimal colouring of the graph. Justify your answer fully.

$\Delta(G) = 9, \quad 5 \leq \chi(G) \leq 9$

$\omega(G) = 5$



$\therefore \chi(G) = 5$



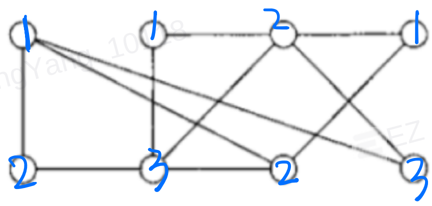
$3 \leq \chi(G)$

$\Delta(G) = 4$

$\Rightarrow 3 \leq \chi(G) \leq 5$

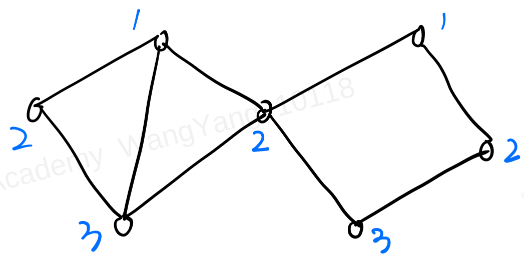
$\therefore \quad \chi(G) \leq 4$

$\chi(G) = 4$



$3 \leq \chi(G) \leq 4$

$\therefore \chi(G) = 3$



$\chi(G) = ?$

$3 \leq \chi(G) \leq 4$

$\therefore \chi(G) = 3$

Proposition 1

Let G be a graph that does not contain any odd cycles.

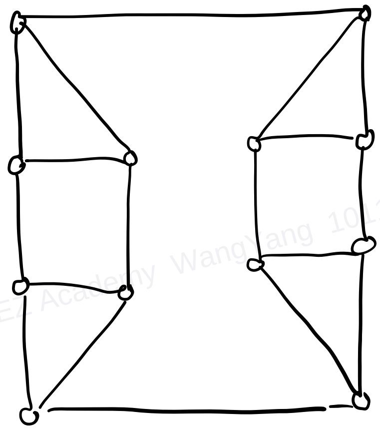
Then $\chi(G) \leqslant 2$. More specifically, $\chi(G) = 1$ if G is a null graph (that is, if it contains no edges), otherwise $\chi(G) = 2$.

The proof relies on a result we saw early in the term (in Lecture 8; see page 5 of that slide presentation).

Reminder from Lec 8: Criterion for bipartite graphs

A graph G is a (not necessarily complete) bipartite graph if and only if G does NOT contain any odd cycles.

$\chi(G) \Rightarrow$ G is bipartite

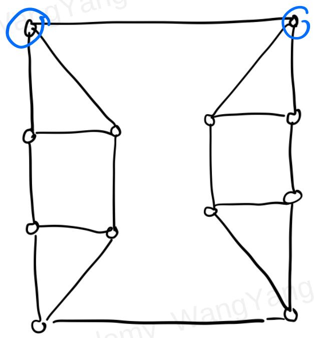


G

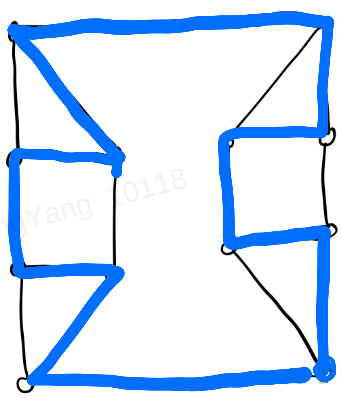
(a) Is G Eulerian, if yes,
Find an Eulerian circuit

(b) Is H Hamiltonian, if yes
Find a Hamiltonian cycle

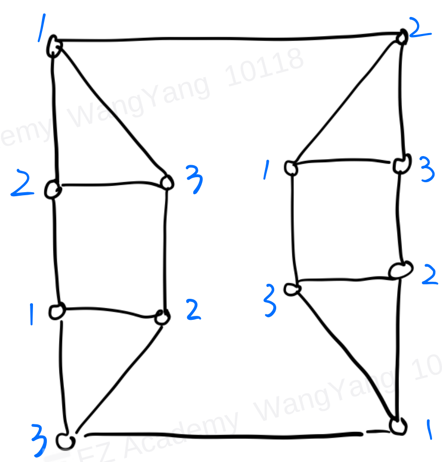
(c) $\chi(G)$, colour.



(a) not Eulerian



(b) Hamiltonian



(c) $3 \le \chi(G) \le 3$ $\qquad \chi(G) = 3$

(五) Planar Graphs

Crossing Number of a Graph

One of these 'secondary' features that it is sometimes useful to record is the **number of pairs of different edges which 'cross'**, or in other words whose relative interiors have an intersection point.

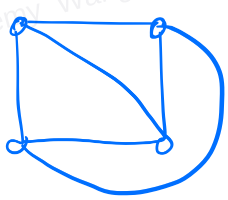
最少交叉的次数 Cr(G)

Definition

Given a (finite) graph G, we define its *crossing number* to be the minimum number of edge crossings that a (permissible) pictorial representation of G can have.

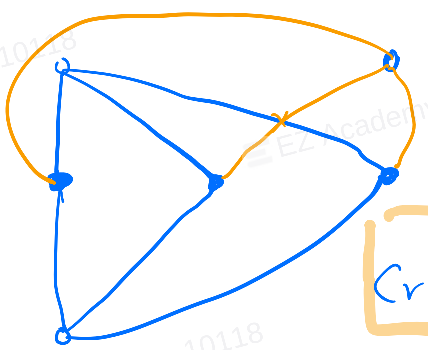
We usually denote this number by $\mathrm{cr}(G)$.

K_4



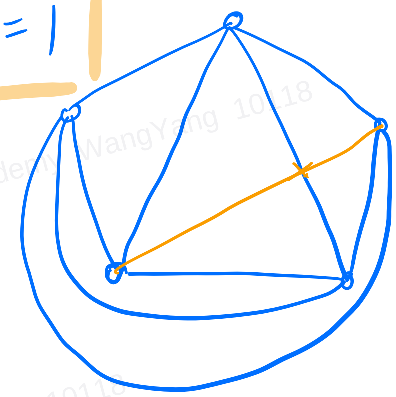
$Cr(G)=0$

$K_{3,3}$



$\boxed{Cr(K_{3,3})=1}$

$\boxed{Cr(K_5)=1}$



Theorem 1

We have that

$$\mathrm{cr}(K_{m,n}) \leqslant \lfloor \frac{m}{2} \rfloor \cdot \lfloor \frac{m-1}{2} \rfloor \cdot \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor.$$

$\lfloor n \rfloor$: 大于等于n的最小整数

ex

$m=5 \quad \lfloor \frac{5}{2} \rfloor = \lfloor 2.5 \rfloor = 3$

Theorem 2

For every $n \geqslant 3$, we have that

$$\mathrm{cr}(K_n) \leqslant \frac{1}{4} \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor \cdot \lfloor \frac{n-2}{2} \rfloor \cdot \lfloor \frac{n-3}{2} \rfloor.$$

Definition

A graph $G = (V, E)$ is called *planar* if we can draw it in the plane without any edge crossings. In other words, if $\mathrm{cr}(G) = 0$.

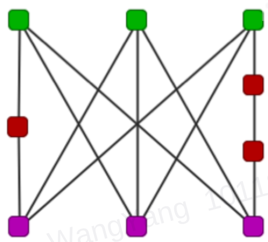
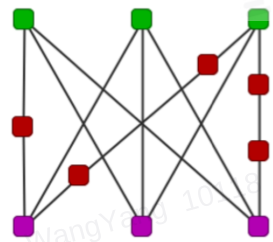
Any such pictorial representation of G will be called a *planar embedding*, or equivalently a *plane representation*, of G.

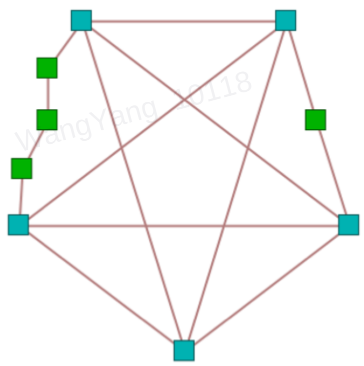
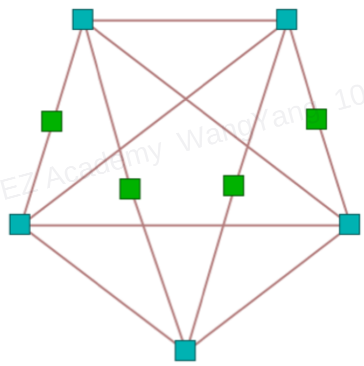
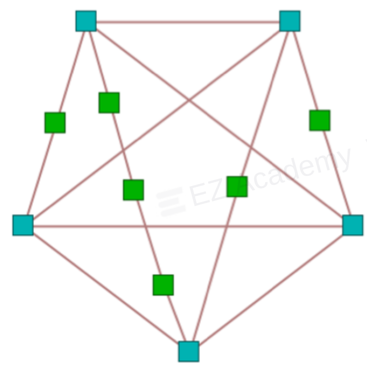
Definition

Let $G = (V, E)$ be a graph. A *subdivision* of G is a graph H which satisfies the following:

- H contains all the vertices of G, and perhaps a few more vertices;
- every edge $e = \{v_1, v_2\}$ in G
  - either appears as an edge of H as well,
  - or is replaced in H by the path $v_1\, w_1\, w_2\, \ldots\, w_{s-1}\, w_s\, v_2$, where $w_1\, (= w_1(v_1, v_2)), w_2\, (= w_2(v_1, v_2)), \ldots, w_s\, (= w_s(v_1, v_2))$ are new vertices that we add to the vertex set of H (and are only meant as *internal* vertices of this new path in H, that is, they will only be used in relation to vertices v_1 and v_2, to 'elongate' the previous edge $\{v_1, v_2\}$).

  In other words, in the second case, $e = \{v_1, v_2\}$ is replaced by the edges $\{v_1, w_1\}, \{w_1, w_2\}, \ldots, \{w_{s-1}, w_s\}$ and $\{w_s, v_2\}$.

Theorem (Kuratowski, 1930)

A finite graph G is planar **if and only if** none of its subgraphs is a subdivision of $K_{3,3}$ or of K_5.

☆ G 中存在 subgraph 是 K_5 or $K_{3,3}$ 的 subdivision

$\Leftrightarrow$ G is not planar

☆

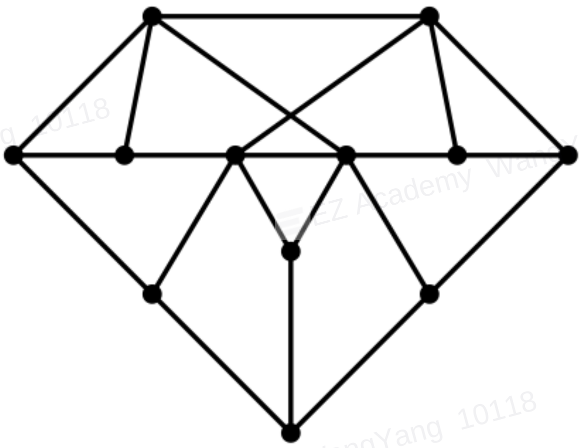
G is planar $\Leftrightarrow$ 画出 ...

Terminology

If G contains a subgraph H which is a subdivision of $K_{3,3}$ or of K_5, then H is called a *Kuratowski subgraph* of G.

(a) Planar or not? → not Planar.

(b) Cr(G) = ?

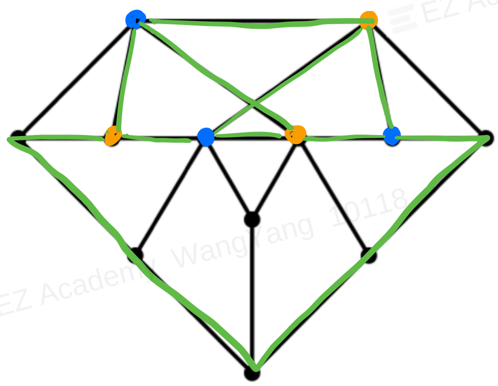


∵ not Planar

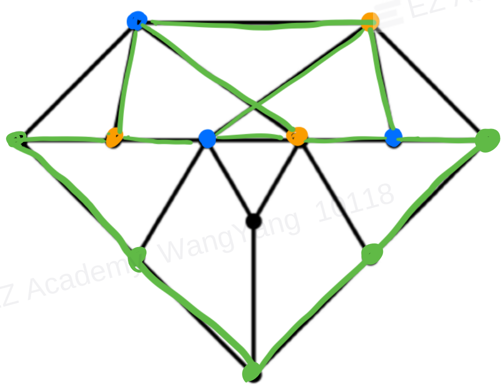
∴ Cr(G) ≥ 1

Cr(G) ≤ 1

⇒ Cr(G) = 1

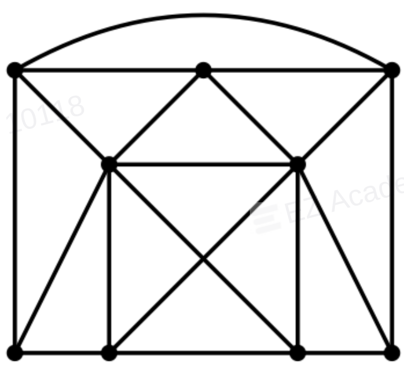


$K_{3,3}$

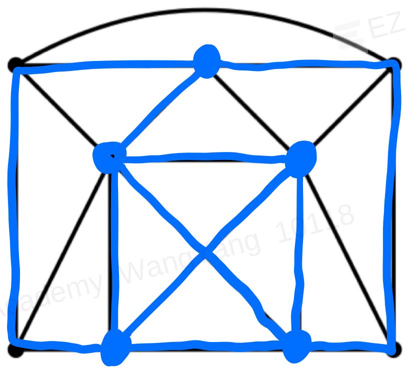


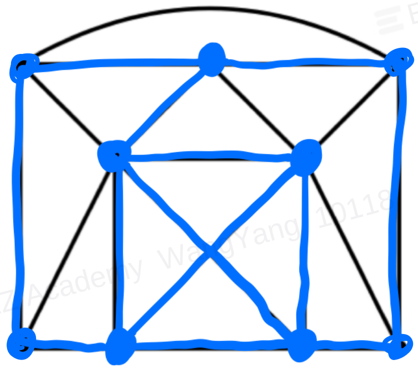
$K_{3,3}$ 的 subdivision

(a) In the graph below, find a subgraph homeomorphic to K_5.



(b) Deduce that the crossing number → not Planar ⇒ Cr(G) ≥ 1, Cr(G) ≤ 1 ⇒ Cr(G) = 1

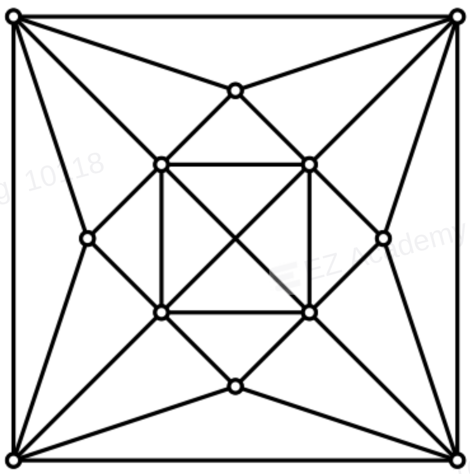




K_5 $\qquad$ K_5 的 subdivision

(a) In the graph H below, find a subgraph subdivision of K_5.

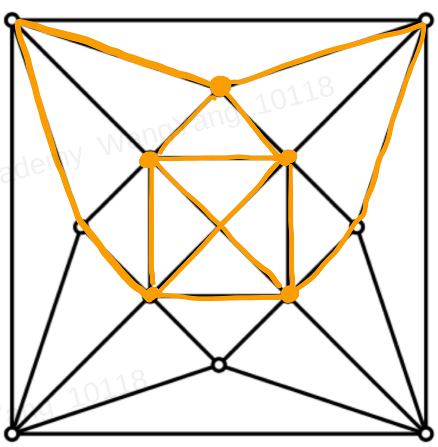
H



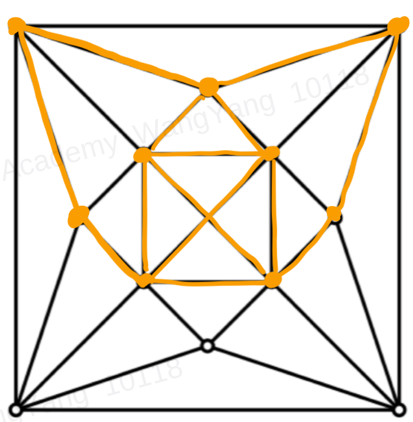
(b) Using part (a), find the crossing number of H, justifying your answer.

(c) Now find a subgraph of H subdivision to $K_{3,3}$. $Cr(H)=1$

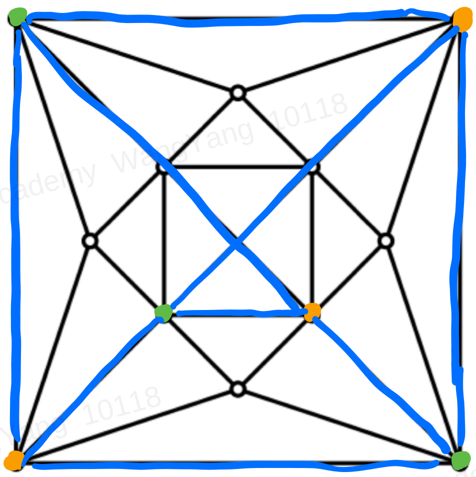
H



H

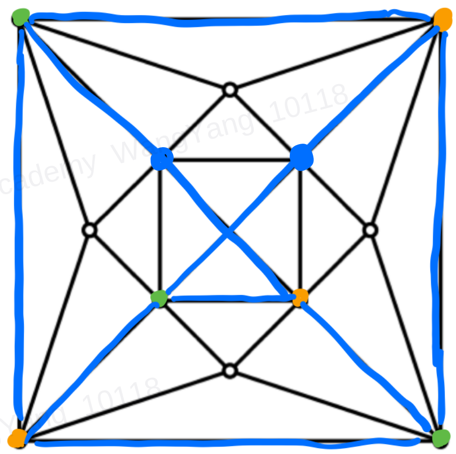


K_5



$K_{3,3}$

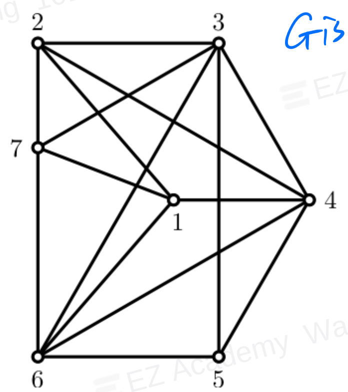
K_5 的 subdivision



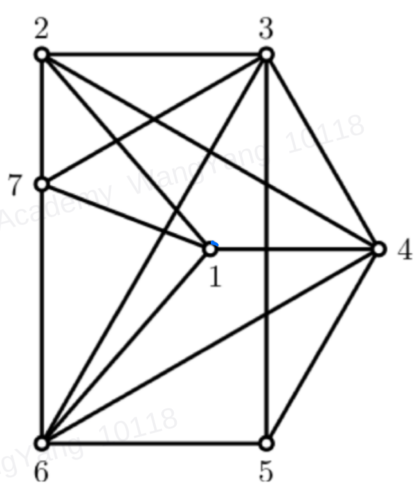
$K_{3,3}$ 的 subdivision

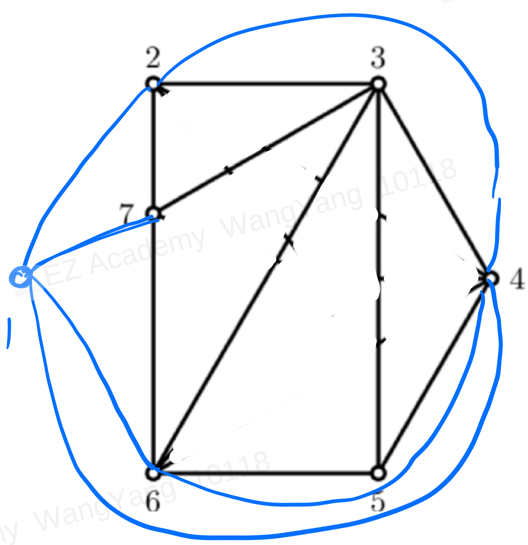
4. (a) Find the crossing number of the following graph G, justifying your answer.

G

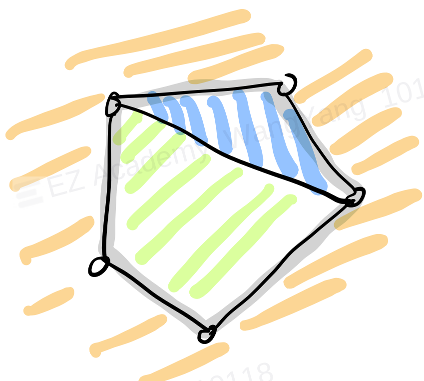


G is planar $\Rightarrow Cr(G) = 0$





G is Planar Graph $\Rightarrow$ Face



Definition 4

Let G_0 be a plane representation of a planar finite graph, and let f_0 be a face of G_0. We define the *length* $\ell(f_0)$ of the face f_0 to be

the length of a closed walk which bounds f_0 (in other words, which traces the boundary of f_0) and has minimum possible length.

Theorem 1

Let G_0 be a plane representation of a planar finite graph, and let $f_1, f_2, \ldots, f_s$ be the faces of G_0. Then we have that

$$\boxed{\sum_{i=1}^{s} \ell(f_i) = 2e(G_0).}$$

Theorem 2 (Euler's formula, 1758)

Let G_0 be a plane representation of a connected planar finite graph, and write n for the number of vertices of G_0, e for the number of edges of G_0, and $\tilde{f}$ for the number of faces of G_0. Then (regardless of what exactly G_0 looks like), we will always have that

$$n - e + \tilde{f} = 2.$$

Theorem 3

It G is a planar graph with at least 3 vertices, then

$$e(G) \leqslant 3|G| - 6.$$

$|G|$: 顶点个数
$e(G)$: 边数

Moreover, if G is also **triangle-free**, then

$$e(G) \leqslant 2|G| - 4.$$

Corollary 1

If G is a planar (not-necessarily complete) bipartite graph with order $|G| = n \geqslant 3$, then G has at most $2n - 4$ edges.

Corollary 2

Neither of the graphs $K_{3,3}$ and K_5 is planar.

$$n - e + \tilde{f} = 2.$$

G. 顶点数 - 边数 + 面的个数 = 2

Let G be a connected, 4-regular plane graph in which every face is bounded by either 3 or 4 edges.

$d(v) = 4$

(a) Use Euler's formula to show that the number of 3-sided faces in G is exactly 8.

(b) Draw such a plane graph.

(a) length = 3 for face 有 x 个

length = 4 的 face 有 y 个

$3x + 4y = 2m$ ①

Hand shaking, $\sum_{i=1}^{n} d(v_i) = 2m$ $4n = 2m$ ②

$n - m + (x+y) = 2$ ③

$n = \frac{1}{2}m = \frac{1}{4}(3x+4y)$

$$\frac{1}{4}(3x+4y)-\frac{1}{2}(3x+4y)+(x+y)=2$$

$$3x+4y-6x-8y+4x+4y=8$$

$$\underline{x=8}$$

(b) $x=8 \quad y=0 \Rightarrow m=12 \quad n=6$