Math 322
Homework Problem Set 3

Remark 1. By Whitney's theorem, we know that, for every connected graph G on n vertices where $n \geqslant 2$,

$$\kappa(G) \leqslant \delta(G).$$

Moreover, while proving Whitney's theorem in class, we remarked that, if $\delta(G) = n - 1$, then necessarily $G = K_n$, and thus $\kappa(G) = \delta(G)$.

The first part of the following problem is about another similar case in which we would also certainly have $\kappa(G) = \delta(G)$ (and thus, as a consequence, we'd also have $\kappa(G) = \lambda(G) = \delta(G)$).

Problem 1. (i) Let G be an arbitrary connected graph on n vertices where $n \geqslant 3$. Prove that, if $\delta(G) = n - 2$, then necessarily $\kappa(G) = \delta(G)$.

(ii) Find an example of a connected graph H on **at least** 5 vertices (that is, such that $|H| \geqslant 5$), which has the following properties:

- $\delta(H) = |H| - 3$;
- $\kappa(H) < \delta(H)$ (that is, the vertex connectivity of H is **strictly** smaller than its minimum degree).

Briefly explain why your example has the required properties.

(iii) Can you find an example of a 4-regular graph M which satisfies $\kappa(M) = 2$? Justify your answer.

Problem 2. (i) Let $n \geqslant 2$, and let G be a graph of order n. Consider a vertex v of G. Show that

$$\overline{G - v} = \overline{G} - v$$

(by confirming that the two constructions give the same vertex set and the same edge set; observe that here we should understand the first construction as being the result of first removing the vertex v and then taking the complement, while for the second one we apply these operations in the reverse order).

(ii) By relying on part (i) if you want to, prove the following statement.

Let H be a graph of order $n \geqslant 3$, and let v_0 be a vertex of H. Suppose that v_0 is a cutvertex of H (that is, suppose that $H - v_0$ has more connected components than H does).

Then v_0 is NOT a cutvertex of $\overline{H}$ (in other words, by deleting the vertex v_0 from the graph $\overline{H}$, we would NOT increase the number of connected components).

(iii) Are there examples of graphs K on at least 3 vertices such that

both K and $\overline{K}$ have NO cutvertices?

Justify your answer.

(iv) Are there examples of graphs M on at least 3 vertices such that

both M and $\overline{M}$ HAVE cutvertices?

Of course note that, because of what you are asked to show in part (ii), the cutvertices of M will be different from the cutvertices of $\overline{M}$.

Justify your answer.

Remark 2. In Lectures 9 and 10, we saw that, in any tree T with at least 2 vertices, we have that

- every vertex of degree 1 (in other words, every leaf) is NOT a cutvertex *(in fact, this holds more generally for any graph G with at least 2 vertices, see Lecture 9, Proposition 2)*,
- while every vertex of T of degree > 1 is a cutvertex.

In particular, this implies that: if a connected graph G on at least 2 vertices is a tree, then every vertex of G of degree > 1 is a cutvertex.

The following problem allows us to pin down all other connected graphs which have the same property, that is, for which every vertex of degree > 1 is a cutvertex.

In fact, Problem 3 provides a criterion for when a vertex v_0 of a connected graph G will be a cutvertex of G *(in a sense, analogous to the criterion (Proposition 1 of Lecture 9) that we have for bridges).* (Of course, once again recall that, to even consider the question of whether a vertex is a cutvertex, we need the vertex to have degree > 1; this is because of Proposition 2 of Lecture 9.)

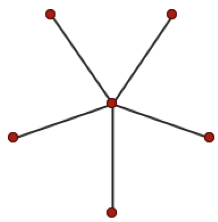
Problem 3. Let G be a connected graph on at least 3 vertices, and consider a vertex v_0 of G which has degree > 1. Show that:

v_0 is a cutvertex of G

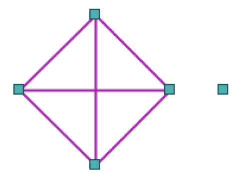
if and only if

we can find a pair of neighbours u_1, u_2 of v_0
for which there is no cycle in G containing both u_1 and u_2
(in other words, we cannot include both u_1 and u_2 in a single cycle subgraph of G).

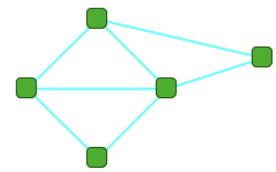
Problem 4. *(Past Exam Problem)* Consider the following 3 graphs:



Graph G_1



Graph G_2

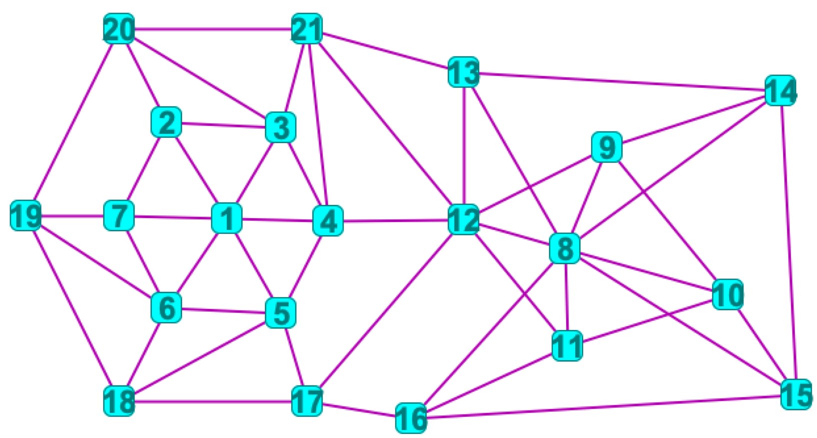


Graph G_3

Exactly two of them are line graphs of some other graphs, while the remaining one is not the line graph of any graph.

Determine which two are line graphs *(you can either pick the two correct ones and explain why there are graphs H_1 and H_2 such that the two graphs you picked are $L(H_1)$ and $L(H_2)$ respectively, or alternatively you can try to find which one graph cannot be written as a line graph and justify this).*

Problem 5. For the graph G_4 below, determine $\kappa(G_4)$ and $\lambda(G_4)$ precisely. Justify your answer fully.



Graph G_4