MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 1

Thursday, September 2

Outline for Lecture 1

1. Introduction to the main object/concept of the course
2. Motivating examples/problems

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Most Obvious/Easy/Direct Answer: It is the area of Mathematics that studies *Graphs*!

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Question: What is Graph Theory? What is it about?

Most Obvious/Easy/Direct Answer: It is the area of Mathematics that studies *Graphs*!

- But what is a *graph*?
- From the syllabus posted on eClass: *"Graphs are constructions that encode "links" (or lack of a link) between objects of a certain set"*.

Definition of the notion of "graph"

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- the set $V(G)$ can be any (nonempty) set we want (see examples below);

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Important point to pay attention to here:

- the set $V(G)$ can be any (nonempty) set we want (see examples below);
- on the other hand, the set $E(G)$ of edges of G has to contain (unordered) pairs of elements of $V(G)$, namely it has to be a subset of **the set of 2-element subsets** of $V(G)$.

(Abstract) Example 1

If $V(G) = \{v_1, v_2, v_3, v_4\}$, then $E(G)$ has to be a subset of the set

$$\Big\{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\Big\}.$$

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$$\begin{aligned} G &= (V, E) \\ &= \Big(\{v_1, v_2, v_3, v_4\},\ \{\{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}\Big). \end{aligned}$$

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Much easier to convey all this information by <u>drawing</u> G!

How do we do this?

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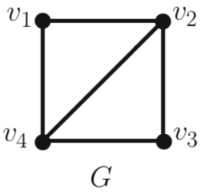

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Remark. We will soon see that the "names" of the vertices are not important pieces of information in **such a graph**, but rather what is most important to keep track of here is how the vertices are connected to each other. Thus, as we will explain, in many cases it will make sense to omit these "names" or labels.

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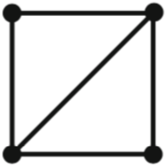
Let's draw again the graph G as an <u>unlabelled</u> graph:



(Abstract) Example 2

Suppose $V(G) = \{a, b, c, d, e, f, g, h\}$ and

$$E(G) = \Big\{\{a, d\}, \{a, e\}, \{b, c\}, \{b, e\}, \{b, g\}, \{c, f\}, \{d, f\}, \{d, g\}, \{g, h\}\Big\}.$$

Then we can draw G as follows:

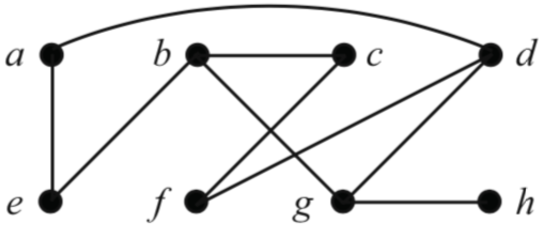


Figure: from the Harris-Hirst-Mossinghoff book

(Abstract) Example 3

Let $G = \Big(\{1,2,3,4,5,6\},\ \{12,14,16,25,34,36,45,56\}\Big)$.

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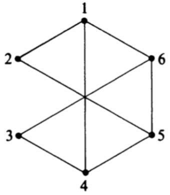

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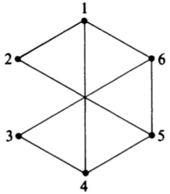

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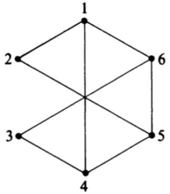

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Notes. (i) Omit the curly brackets (that is, write 12 instead of $\{1,2\}$ only when it's clear there won't be confusion.

(ii) Here 12 is the same as 21, since the edges are **unordered** pairs.

A more complicated example

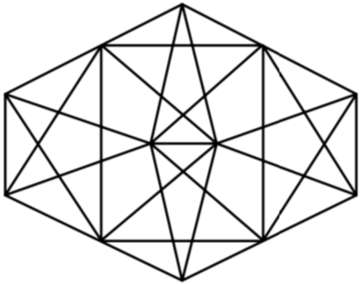

Figure: from Marcus' book

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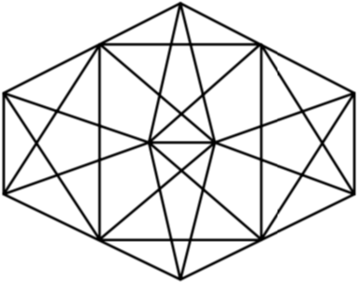

Figure: from Marcus' book

Besides the fact that it's more practical ('less of an effort') to draw an unlabelled graph instead of a labelled one, we will soon also give some mathematical intuition regarding why in certain cases it makes more sense to work with unlabelled graphs.

One more example: is this a graph???

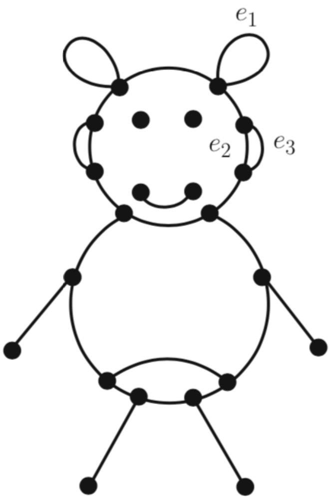

Figure: from the Balakrishnan-Ranganathan book

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Not according to the definition we have given. Here

- there are *loops* (see at the top of the stick figure),
- and also there are sets of *multiple edges* or *parallel edges* (see the edges that form the "ears" of the stick figure).

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Now let's go back to the definition of a graph that we gave: we see that the set $E(G)$ of edges of a graph G has to be

	a <u>set</u> of unordered pairs,
or in other words	**a set of 2-element subsets of $V(G)$.**

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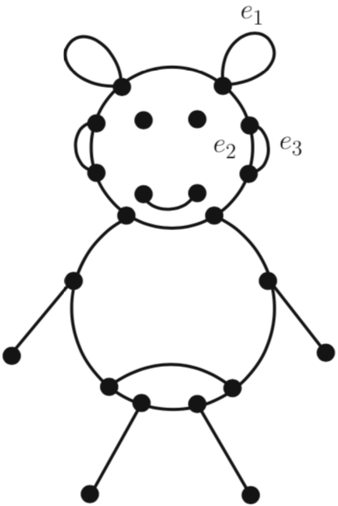
- Since $E(G)$ is a set, its elements cannot be repeated $\rightsquigarrow$ we cannot have multiple edges in a graph G (that is, two or more edges with the same endpoints);
- since $E(G)$ only contains 2-element subsets of $V(G)$, we cannot have loops!



Such a construction is still very closely related to graphs; in fact, it is one of the main "variations" of the concept of 'graph' that is also studied in Graph Theory.

In this course, we will call such a construction a *multigraph*.

Real-life examples: Transportation Networks

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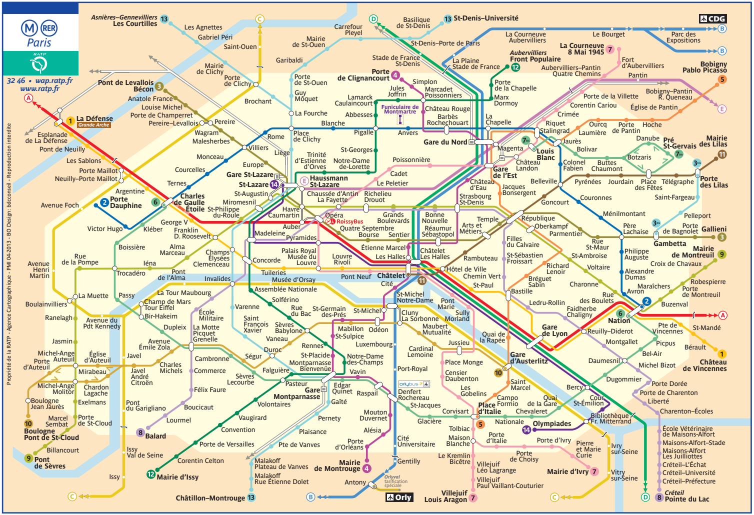

Figure: Map of the Paris Subway-Regional Train system

The Paris Subway-Regional Train system

There is a natural way to think of this as a graph:

- the vertices are the different stations;
- an edge exists between two stations if you can get from one station to the other by taking one of the trains (one of the different lines) without stopping at any other station.

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By studying the map, we can see that we can get from any given station to any other given station; we will see that a graph that has this property is called **connected**.

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Given this property, useful questions to ask here are (and these indeed come up in Graph Theory):

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- given two stations, what is the shortest route or *path* (a term which we will properly define soon) that takes us from one station to the other? How long is this path? (That is, how many intermediate stations would we have to pass through?)

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– given two stations, what is the shortest route or *path* (a term which we will properly define soon) that takes us from one station to the other? How long is this path? (That is, how many intermediate stations would we have to pass through?)

– Obviously for some pairs of stations this shortest path will be quite long (say, if you start from a station in Northwest Paris, and want to go to a station in Southeast Paris). But how long (at the most) is this shortest path for the average pair of stations/vertices (or for a large enough portion of pairs of vertices)?

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- the set of vertices here are the different accounts;
- two accounts/vertices are connected by an edge if one of them follows the other.

Facebook vs Twitter?

Question: Are the two constructions of the exact same type?

That is, do they **both** satisfy the definition that we gave above, or is it more natural to say that one of them does not fit the definition so well?