MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 11

Tuesday, October 12

Reminder from last time: Whitney's theorem

Theorem 1 (Whitney, 1932)

For every connected graph G (of order $\geqslant 2$), we have

$$\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)\,.$$

Recall that, in Lecture 8, we had already proved that $\lambda(G) \leqslant \delta(G)$.

Proving that $\kappa(G) \leqslant \lambda(G)$

Consider a connected graph G on at least 2 vertices, and let n be the order of G (as we just said, $n \geqslant 2$).

Consider also an edge cut $\{e_1, e_2, \ldots, e_{t_0}\}$ of G such that $t_0 = \lambda(G)$. In other words, $G - \{e_1, e_2, \ldots, e_{t_0}\}$ is a disconnected subgraph, while t_0, the cardinality of this edge cut, is as small as possible. By what we have already showed, $t_0 \leqslant \delta(G) \leqslant n - 1$.

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'Easier' case: $t_0 = \lambda(G) = n - 1$. Then, we must have $\delta(G) = n - 1$ too, and thus $G = K_n$. But then, according to our definitions, $\kappa(G) = n - 1$ too, and the inequality $\kappa(G) \leqslant \lambda(G)$ holds (in fact, we have equality).

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Remaining cases: Assume now that $t_0 \leqslant n - 2$. Since $G - \{e_1, e_2, \ldots, e_{t_0}\}$ is a disconnected subgraph, we can find vertices u_1, v_1 of G which fall into different connected components of $G - \{e_1, e_2, \ldots, e_{t_0}\}$.

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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We consider two subcases:
Subcase 1: u_1, v_1 are NOT joined by an edge in G. Then any path in G connecting u_1 and v_1 must have length $\geqslant 2$ (or in other words, it must contain intermediate vertices too).

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Subcase 1: u_1, v_1 are NOT joined by an edge in G. Then any path in G connecting u_1 and v_1 must have length $\geqslant 2$ (or in other words, it must contain intermediate vertices too).
Moreover, every path in G connecting u_1 and v_1 must traverse one or more of the edges $e_1, e_2, \ldots, e_{t_0}$.

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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Moreover, every path in G connecting u_1 and v_1 must traverse one or more of the edges $e_1, e_2, \ldots, e_{t_0}$. *(Indeed, if we could also find a $u_1 - v_1$ path in G which did not contain any of the edges $e_1, e_2, \ldots, e_{t_0}$, then this would be a path in the subgraph $G - \{e_1, e_2, \ldots, e_{t_0}\}$ too, and thus u_1, v_1 would not belong to different connected components of the subgraph.)*

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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Now, for each of the edges e_i, pick one of its endvertices **which is different from both u_1 and v_1** (this is always possible here because the edge $\{u_1, v_1\}$ does not exist in G), and denote it by w_i.

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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Now, for each of the edges e_i, pick one of its endvertices **which is different from both u_1 and v_1** (this is always possible here because the edge $\{u_1, v_1\}$ does not exist in G), and denote it by w_i. Then the vertex set $\{w_1, w_2, \ldots, w_{t_0}\}$ is a vertex cut of G, since

$$G - \{w_1, w_2, \ldots, w_{t_0}\} \subseteq G - \{e_1, e_2, \ldots, e_{t_0}\},$$

and thus the vertices u_1, v_1, which are still left in $G - \{w_1, w_2, \ldots, w_{t_0}\}$, will again be separated.

*One more side note here: $\{w_1, w_2, \ldots, w_{t_0}\}$ contains **at most** t_0 vertices (because some vertex here might be a common endvertex of two or more of the edges $e_1, e_2, \ldots, e_{t_0}$, and might have been picked more than once).*

This shows that $\kappa(G) \leqslant t_0 = \lambda(G)$ in this subcase.

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Subcase 2: u_1, v_1 are joined by an edge in G. Then, because u_1, v_1 are separated in $G - \{e_1, e_2, \ldots, e_{t_0}\}$, one of the edges that we remove must be the edge $\{u_1, v_1\}$.

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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Then, as before, for each of the edges e_i with $i \neq t_0$, pick one of its endvertices **which is different from both $\boldsymbol{u_1}$ and $\boldsymbol{v_1}$**, and denote it by w_i.

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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Note now that

$$\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0} \quad \subseteq \quad G - \{e_1, e_2, \ldots, e_{t_0}\},$$

and thus u_1, v_1 are separated in the smaller subgraph too.

Note also that the subgraph $\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0}$ contains **at least** $n - (t_0 - 1) = n - t_0 + 1 \geqslant n + 1 - (n-2) = 3$ vertices (where we're using the initial assumption that $t_0 \leqslant n - 2$). Thus it contains at least one more vertex z_0 different from u_1 and v_1.

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

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- If z_0 and u_1 are in the same connected component of $\big(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\big) - e_{t_0}$, **which implies that z_0 and v_1 are in different connected components,** then, instead of removing the edge e_{t_0}, remove the vertex u_1. In the graph $G - \{w_1, w_2, \ldots, w_{t_0-1}, u_1\}$, the vertices z_0 and v_1 are in different connected components, and thus the subset $\{w_1, w_2, \ldots, w_{t_0-1}, u_1\}$ is a vertex cut of G with cardinality $\leqslant t_0 = \lambda(G)$.

Proving that $\kappa(G) \leqslant \lambda(G)$ (cont.)

Subcase 2: u_1, v_1 are joined by an edge in G. Then, because u_1, v_1 are separated in $G - \{e_1, e_2, \ldots, e_{t_0}\}$, one of the edges that we remove must be the edge $\{u_1, v_1\}$. Say without loss of generality that $\{u_1, v_1\} = e_{t_0}$.

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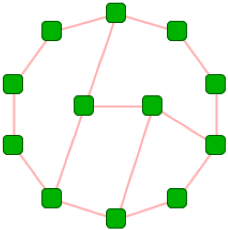
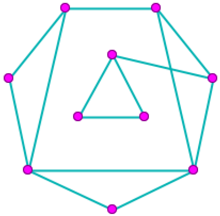
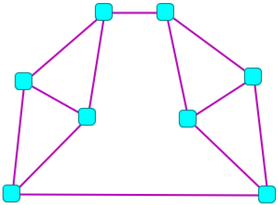
- If z_0 and u_1 are in the same connected component of $\left(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\right) - e_{t_0}$, **which implies that $\boldsymbol{z_0}$ and $\boldsymbol{v_1}$ are in different connected components,** then, instead of removing the edge e_{t_0}, remove the vertex u_1. In the graph $G - \{w_1, w_2, \ldots, w_{t_0-1}, u_1\}$, the vertices z_0 and v_1 are in different connected components, and thus the subset $\{w_1, w_2, \ldots, w_{t_0-1}, u_1\}$ is a vertex cut of G with cardinality $\leqslant t_0 = \lambda(G)$.
- If instead z_0 and u_1 are in **different connected components** of $\left(G - \{w_1, w_2, \ldots, w_{t_0-1}\}\right) - e_{t_0}$, then analogously, instead of the edge e_{t_0}, remove the vertex v_1. Now, in the graph $G - \{w_1, w_2, \ldots, w_{t_0-1}, \boldsymbol{v_1}\}$, it's the vertices z_0 and u_1 which are in different connected components.

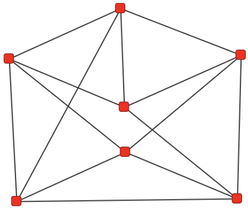
Checking Whitney's theorem (and the proof constructions) on examples

For each of the graphs G below, find $\delta(G), \lambda(G)$ and $\kappa(G)$, as well as edge cuts and vertex cuts which 'capture' the parameters.



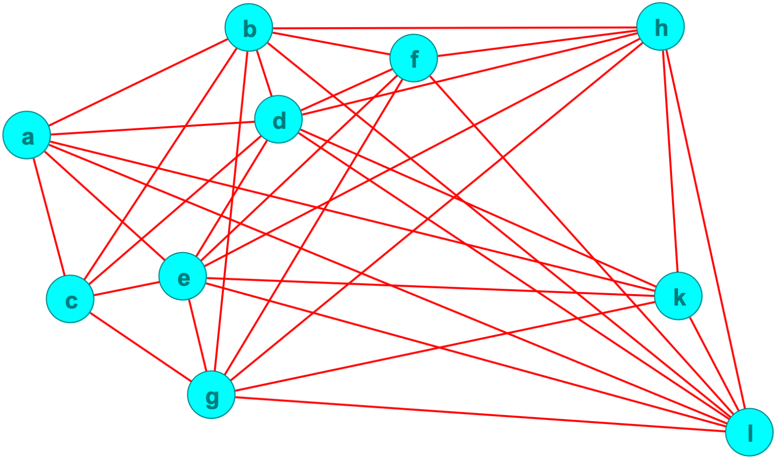






More practice examples

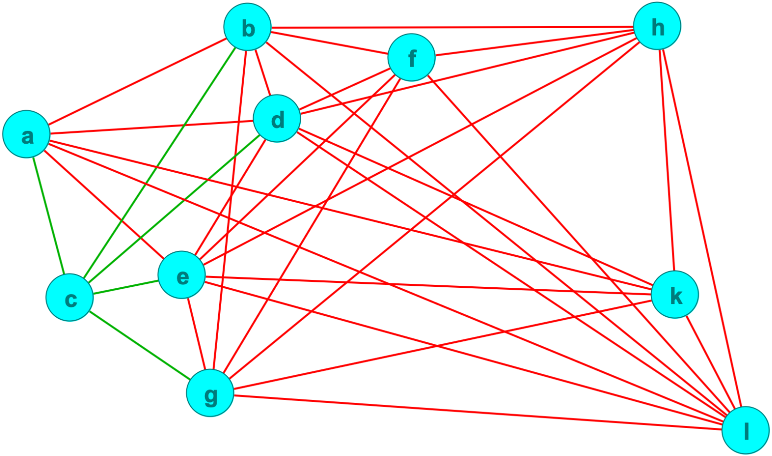
Example 6.



Same question as before: What is $\delta(G), \lambda(G)$ and $\kappa(G)$ here?

More practice examples

Example of previous slide.

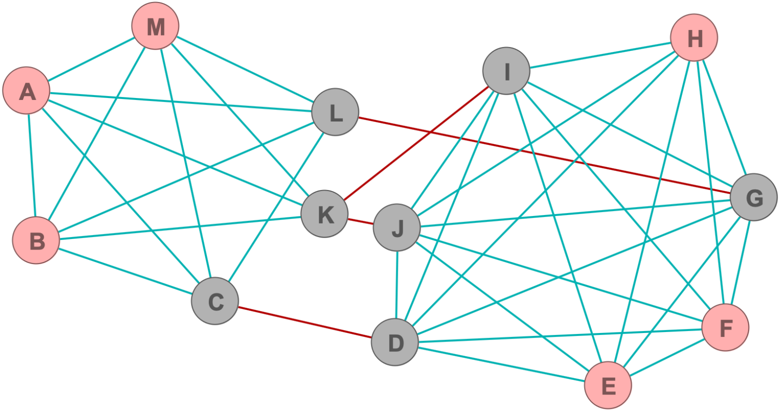


The **green-coloured** edges form an edge cut of G (by the way, does this edge cut have smallest cardinality possible? or could you find an even smaller edge cut?).

What about a vertex cut? Can you find one based on the above edge cut?

More practice examples

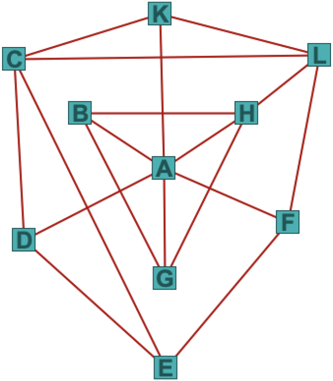
Example 7.



Same main question as before: What is $\delta(G), \lambda(G)$ and $\kappa(G)$ here?

Past exam problem

Consider the following connected graph G_0.



(a) Show that $\kappa(G_0) = 2$. Give a full justification.

(b) What is $\lambda(G_0)$? Determine it precisely, and justify your answer fully.

Other(?) criteria/methods for determining definitively the parameters $\lambda(G)$ and $\kappa(G)$

(and whether (some of) the inequalities in $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$ are strict)

Local Vertex Connectivity

Definition

Let G be a connected graph of order n (other than K_n), and let u, v be two **non-adjacent** vertices of G.

A *vertex cut for u and v* is a subset V' of $V(G) \setminus \{u, v\}$ with the property that

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Note. It's not hard to convince ourselves that $\kappa(u, v) = \kappa(v, u)$.

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Important Observation

We have that $\kappa(G)$ equals the minimum of the quantities $\kappa(u, v)$ that we obtain if we consider all pairs (u, v) of **non-adjacent** vertices of G:

$$\kappa(G) = \min\big\{\kappa(u, v) : u, v \in V(G),\ u \neq v,\ uv \notin E(G)\big\}.$$

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Note. It's not hard to convince ourselves that $\lambda(w, z) = \lambda(z, w)$.

Important Observation

We have that $\lambda(G)$ equals the minimum of the quantities $\lambda(w, z)$ that we obtain if we consider all pairs (w, z) of different vertices of G:

$$\lambda(G) = \min\{\lambda(w, z) : w, z \in V(G),\ w \neq z\}.$$

But is there an <u>efficient</u> way to compute the local vertex connectivities $\kappa(u, v)$ and the local edge connectivities $\lambda(w, z)$ for a given graph G?

Reminder from Lecture 9

Proposition 1 of Lecture 9

Let G be a connected graph of order $\geqslant 3$.

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or equivalently **if and only if** there is **no** $u{-}v$ path in H of length $\geqslant 2$ (that is, **no** $u{-}v$ path which does not contain the edge e).

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Moreover, in such a case, $\{e\}$ is also an edge cut for the vertices u and v (that is, there is no longer a $u-v$ path in the subgraph $H-e$, and thus, by removing the edge e from the original graph H, we will separate the vertices u and v;

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Moreover, in such a case, $\{e\}$ is also <u>an edge cut for the vertices u and v</u> (that is, there is no longer a $u - v$ path in the subgraph $H - e$, and thus, by removing the edge e from the original graph H, we will separate the vertices u and v; of course in such a case it follows that $\lambda(u, v) = 1$).

Another motivating example

Proposition 3 of Lecture 9

Let T be a tree on at least two vertices. Then, for every two different vertices u, v of T, there is a **unique** path in T starting at u and ending at v.

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Consider now a tree T_0 which has at least 3 vertices, and pick vertices u_0, v_0 in T_0 which are NOT adjacent.

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Consider now a tree T_0 which has at least 3 vertices, and pick vertices u_0, v_0 in T_0 which are NOT adjacent.

Then we know that T_0 contains a $u_0 - v_0$ path, and we also know that, since u_0, v_0 have been assumed non-adjacent, this path must have length $\geqslant 2$.

In other words, this path contains at least one intermediate vertex, say, vertex z_1 of T_0 (where $u_0 \neq z_1 \neq v_0$).

Another motivating example

Proposition 3 of Lecture 9

Let T be a tree on at least two vertices. Then, for every two different vertices u, v of T, there is a **unique** path in T starting at u and ending at v.

Consider now a tree T_0 which has at least 3 vertices, and pick vertices u_0, v_0 in T_0 which are NOT adjacent.

Then we know that T_0 contains a $u_0 - v_0$ path, and we also know that, since u_0, v_0 have been assumed non-adjacent, this path must have length $\geqslant 2$.

In other words, this path contains at least one intermediate vertex, say, vertex z_1 of T_0 (where $u_0 \neq z_1 \neq v_0$).

But then, if we delete the vertex z_1, the vertices u_0 and v_0 become separated in the resulting subgraph, since the $u_0 - v_0$ path that we already considered above is the only $u_0 - v_0$ path in T_0 (by Prop 3).

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The latter also implies that $\{z_1\}$ is a vertex cut for u_0 and v_0 in T_0, and thus that $\kappa(u_0, v_0) = 1$.

Takeaway from the last two examples:

Looking at paths connecting different vertices in a given connected graph G, and 'counting' how many such paths we can find for the various pairs of vertices we can consider,

and also 'measuring' 'how different' these paths are,

can give us a good idea about the parameters $\kappa(G)$ and $\lambda(G)$ (and also about the local vertex and edge connectivities of G).

Disjoint Paths

Definition 1

Let G be a graph, and let u, v be two vertices of G. Suppose that $P_1, P_2, \ldots, P_l$ are l different $u - v$ paths in G.

The collection $\{P_1, P_2, \ldots, P_l\}$ is called *internally disjoint* (or alternatively *vertex-disjoint*) if, for any two different paths in this collection, **their only common vertices** are the vertices u and v (in other words, if none of the internal vertices in any one of these paths appears in another path too).

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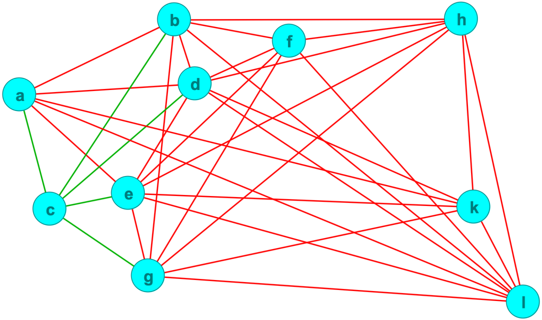
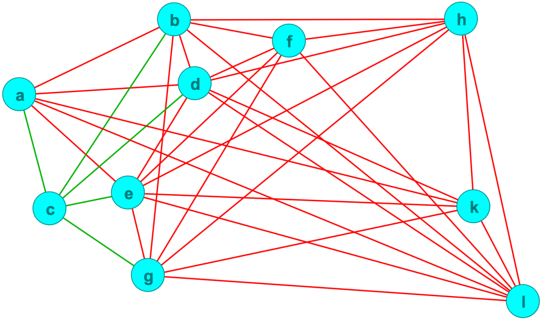
We write $\kappa'(u, v)$ for the maximum possible cardinality that an internally disjoint collection of $u - v$ paths in G can have.

Back to examples

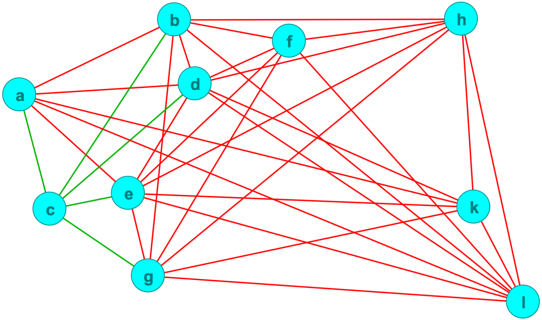


Back to examples



Question 1. Do we have $\kappa'(a, c) \geqslant 5$?

Back to examples



Question 1. Do we have $\kappa'(a, c) \geqslant 5$?

Question 2. Can you find 5 pairwise internally disjoint $c-h$ paths?

Disjoint Paths (cont.)

Definition 2

Let G be a graph, and let w, z be two vertices of G. Suppose that $Q_1, Q_2, \ldots, Q_s$ are s different $w-z$ paths in G.

The collection $\{Q_1, Q_2, \ldots, Q_s\}$ is called *edge-disjoint* if, for any two different paths Q_i, Q_j in this collection, Q_i and Q_j contain **no** common edges.

Disjoint Paths (cont.)

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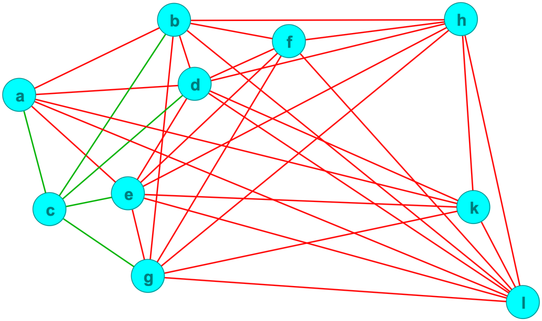
We write $\lambda'(w, z)$ for the maximum possible cardinality that an edge-disjoint collection of $w-z$ paths in G can have.

Back to examples



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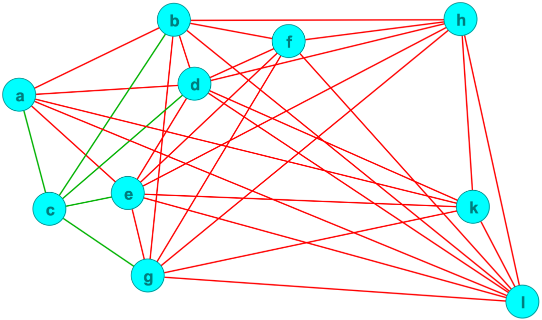

Question 3. Focusing e.g. on the vertices g and h, can you find edge-disjoint paths that connect them which are not vertex-disjoint (equivalently, internally disjoint)?

A very useful theorem

Menger's theorem *(vertex form)*

Let G be a connected graph of order n (other than K_n), and let u, v be two **non-adjacent** vertices of G.

Then the **minimum** cardinality of a vertex cut for u and v equals the **maximum** cardinality of an internally disjoint collection of $u-v$ paths in G. In other words,

$$\kappa(u, v) = \kappa'(u, v).$$

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Menger's theorem *(edge form)*

Let G be a connected graph of order n, and let w, z be two vertices of G. Then the **minimum** cardinality of an edge cut for w and z equals the **maximum** cardinality of an edge-disjoint collection of $w-z$ paths in G.

In other words,

$$\lambda(w, z) = \lambda'(w, z).$$

Why the theorem is so useful to us

Important Corollary of Menger's Theorem

- Let G be a connected graph of order n (other than K_n). Then $\kappa(G) \geqslant t$ if and only if, for any two **non-adjacent** vertices u, v of G, we can find **at least** t pairwise internally disjoint paths in G that connect u and v.

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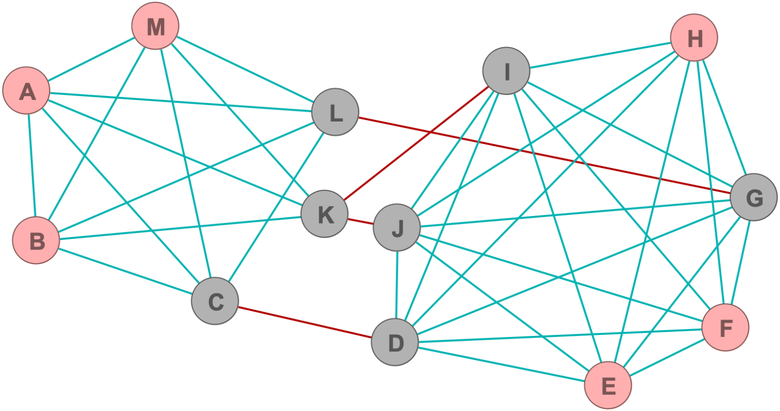
- Let H be a connected graph of order n. Then $\lambda(H) \geqslant s$ if and only if, for any two different vertices w, z of H, we can find **at least** s pairwise edge-disjoint paths in H that connect w and z.

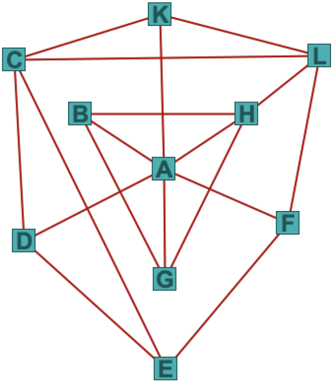
Relying on Menger's theorem for some of the earlier examples



Determine precisely $\lambda(G)$ and $\kappa(G)$.

Relying on Menger's theorem for some of the earlier examples (cont.)

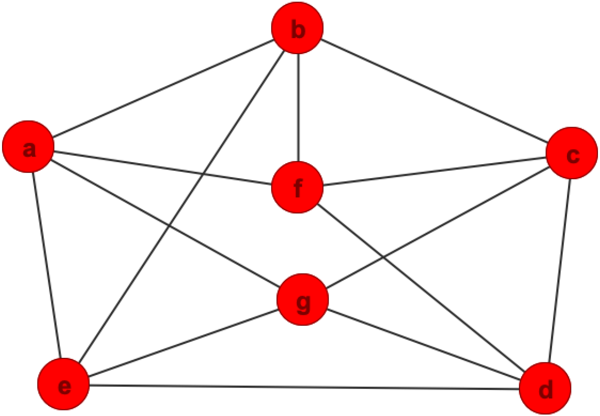
Past Exam Problem.



(a) Show that $\kappa(G_0) = 2$. Give a full justification.

(b) What is $\lambda(G_0)$? Determine it precisely, and justify your answer fully.

Relying on Menger's theorem for some of the earlier examples (cont.)



Question. What is $\kappa(G)$ and $\lambda(G)$ here?