Math 322
Suggested solutions to Homework Set 1

Problem 1. Let us write $a_{i,j}$ for the (i,j)-th entry of the adjacency matrix A of G, and $b_{i,j}$ for the (i,j)-th entry of the matrix $A^2 = A \cdot A$; in other words, we write $A = \big(a_{i,j}\big)_{1\leqslant i,j\leqslant n}$ and $A^2 = \big(b_{i,j}\big)_{1\leqslant i,j\leqslant n}$.

We recall that
$$a_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E(G) \\ 0 & \text{otherwise} \end{cases} .$$

We also observe that the (i,j)-th entry of the matrix A^2 is equal to the dot product of the i-th row and the j-th column of A:
$$b_{i,j} = \langle \mathrm{Row}_i(A),\, \mathrm{Col}_j(A)\rangle.$$

In particular, for every $1 \leqslant j \leqslant n$,
$$\begin{aligned} b_{j,j} &= \langle \mathrm{Row}_j(A),\, \mathrm{Col}_j(A)\rangle \\ &= \sum_{s=1}^{n} a_{j,s} \cdot a_{s,j} \\ &= \sum_{s=1}^{n} a_{j,s}^2 \end{aligned}$$
where the last equality follows because the matrix A is symmetric.

But the last sum has summands equal only to 0 or 1 (depending on whether $a_{j,s}$ is equal to 0 or 1), so it is equal to the total number of indices $s \in \{1, 2, \ldots, n\}$ such that $a_{j,s} = 1$. But each such index corresponds to a vertex v_s of G which is adjacent to the vertex v_j, while at the same time the remaining indices s' (for which we have $a_{j,s'} = 0$) correspond to vertices of G which are **not** neighbours of v_j.

Based on all the above, we see that
$$\begin{aligned} b_{j,j} &= \sum_{s=1}^{n} a_{j,s}^2 \\ &= |\{s \in \{1, 2, \ldots, n\} : a_{j,s} = 1\}| \\ &= |\{s \in \{1, 2, \ldots, n\} : v_s \text{ is adjacent to the vertex } v_j\} \\ &= \deg(v_j)\,, \end{aligned}$$
which is what we needed to show.

Problem 2. (a) We will show that the graphs G_1 and G_2 are isomorphic.

To find an appropriate bijection from $V(G_1)$ to $V(G_2)$, we could first notice that the 'inner' vertices of the graph G_1 form a 5-cycle, the cycle $F\,H\,J\,G\,I\,F$.

Thus, we could first try to see which subgraph of G_2 could be isomorphic to the 5-cycle: we could choose the 5-cycle 3 4 8 9 10 3.

Thus we can start by defining a bijection τ from the vertex set of the first cycle to the vertex set of the second cycle:

$$\tau(F) = 3, \quad \tau(H) = 4, \quad \tau(J) = 8, \quad \tau(G) = 9, \quad \text{and } \tau(I) = 10.$$

We can then extend this to a bijection $\tau : V(G_1) \to V(G_2)$ by setting $\tau(A) = 2$ (given that A is the only neighbour of the vertex F in G_1 outside the cycle $F\,H\,J\,G\,I\,F$, and similarly 2 is the only neighbour of the vertex 3 in G_2 outside the cycle 3 4 8 9 10 3).

Analogously, we set

$$\tau(B) = 1, \quad \tau(C) = 5, \quad \tau(D) = 6, \quad \text{and } \tau(E) = 7.$$

In other words, the bijection τ we have defined is given by

$$\tau : \begin{pmatrix} A & B & C & D & E & & F & G & H & I & J \\ 2 & 1 & 5 & 6 & 7 & & 3 & 9 & 4 & 10 & 8 \end{pmatrix}.$$

Let us now check that τ is a graph isomorphism from G_1 to G_2. It is not hard to see that τ restricted to the first five vertices of G_1 is an isomorphism from the 5-cycle $A\,B\,C\,D\,E\,A$ to the 5-cycle 2 1 5 6 7 2 (indeed, the neighbours of A in the first cycle are the vertices B and E, while the vertices of $2 = \tau(A)$ in the second cycle are the vertices $1 = \tau(B)$ and $7 = \tau(E)$; we can argue analogously about the neighbours of the other vertices in the two cycles).

Similarly, we can check that τ restricted to the last five vertices of G_1 is an isomorphism from the 5-cycle $F\,H\,J\,G\,I\,F$ to the 5-cycle 3 4 8 9 10 3 (in fact, recall that this is what we based our definition of τ on, that is, this is the first thing we tried to make sure will hold): indeed, the neighbours of G in the first cycle are the vertices H and I, while the vertices of $3 = \tau(F)$ in the second cycle are the vertices $4 = \tau(H)$ and $10 = \tau(I)$; analogously we argue about the neighbours of the rest of the vertices.

Finally we can check that

A and F are neighbours in G_1, and similarly 2 and 3 are neighbours in G_2;

B and G are neighbours in G_1, and similarly 1 and 9 are neighbours in G_2;

C and H are neighbours in G_1, and similarly 5 and 4 are neighbours in G_2;

D and I are neighbours in G_1, and similarly 6 and 10 are neighbours in G_2;

E and J are neighbours in G_1, and similarly 7 and 8 are neighbours in G_2.

Based on the above, τ is a graph isomorphism from G_1 to G_2.

(b) We now justify why G_3 is NOT isomorphic to G_1 (or equivalently, why it is NOT isomorphic to G_2). One idea here is to look for a subgraph of G_3 **which is not isomorphic to any subgraph of G_1.** *(Note that, if the two graphs G_1 and G_3 were instead isomorphic, we would have that every subgraph of G_3 is isomorphic to some subgraph of G_1; why?)*

The subgraph we choose here is the 4-cycle $b\,c\,d\,j\,b$ in G_3. As already mentioned, if G_1 and G_3 were isomorphic, then G_1 should also contain a 4-cycle.

We will now show that G_1 **does NOT contain any 4-cycles.** Assume towards a contradiction that $C_0 : v_1\,v_2\,v_3\,v_4\,v_1$ is a 4-cycle in G_1. Note that we can't have all the v_i to be among the 'outer' vertices of G_1 (because there is no way to form a 4-cycle using only 'outer' vertices of G_1), and similarly we cannot have all the v_i to be among the 'inner' vertices of G_1.

Thus we must have a vertex v_i on C_0 which is among the vertices A, B, C, D and E, and such that v_{i+1} is among the vertices F, G, H, I and J. In fact, given that we can consider any vertex of the cycle to be the initial vertex, we can assume without loss of generality that this is true for $i = 1$, or in other words that v_1 is an 'outer' vertex of G_1 and v_2 is an 'inner' vertex of G_1.

Observe finally that, because of **the many symmetries** of the graph G_1, we can assume that $v_1 = A$ and thus $v_2 = F$.

We can now try to determine which are the shortest cycles in G_1 which start with the vertices A and F. We have the following;

if after the vertex F we move to the vertex H, then we can get

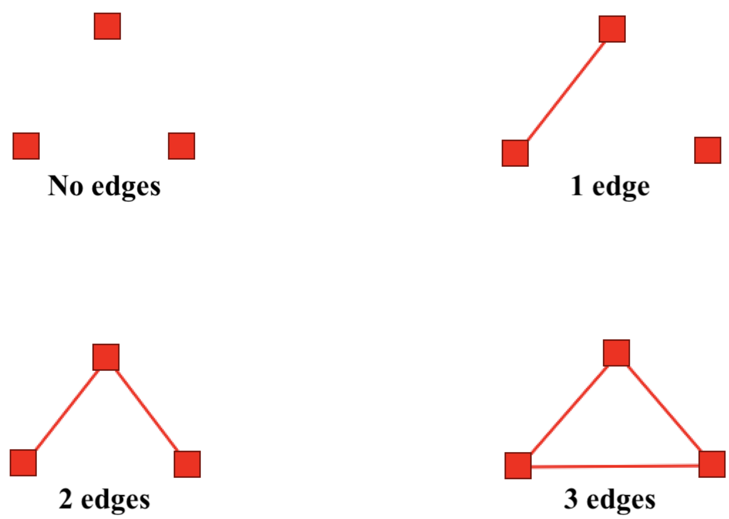
- the 5-cycle $A\,F\,H\,C\,B\,A$,
- the 5-cycle $A\,F\,H\,J\,E\,A$,
- as well as some even longer cycles;

if after the vertex F we move to the vertex I, then we can get

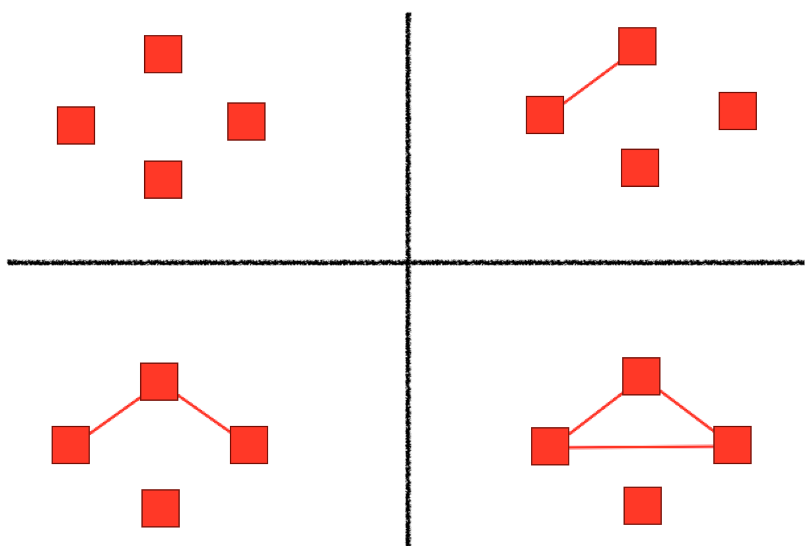
- the 5-cycle $A\,F\,I\,D\,E\,A$,
- the 5-cycle $A\,F\,I\,G\,B\,A$,
- as well as some even longer cycles.

The above show that there is no 4-cycle in G_1 starting with the vertices A and F, which contradicts our initial assumption that a 4-cycle C_0 in G_1 exists. This implies that G_1 has no 4-cycles, and therefore it cannot be isomorphic to G_3.

Problem 3. Recall that there are four unlabelled graphs on 3 vertices, which can be seen in the following picture:

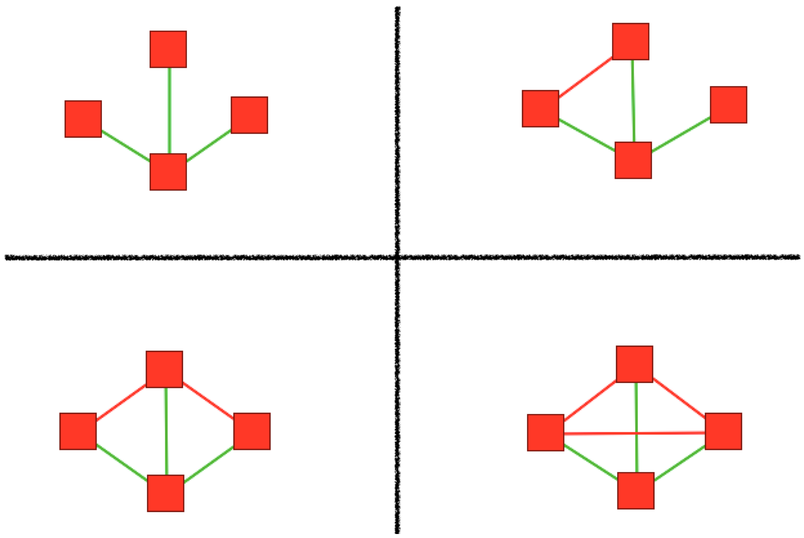


We can immediately construct some of the graphs on 4 vertices that we are asked to find by adding one more vertex to each of these graphs (without joining it to any of the vertices we have already):



Note that all four graphs here are different, given that each has a different degree sequence. In particular, the first one has degree sequence $(0, 0, 0, 0)$, the second one has degree sequence $(1, 1, 0, 0)$, the third one has degree sequence $(2, 1, 1, 0)$, and the fourth one has degree sequence $(2, 2, 2, 0)$.

Moreover, we could construct four more graphs on 4 vertices by adding a vertex to the graphs on 3 vertices that we have found, and by joining this time the new vertex with any other vertex:



Again, these are four different graphs since each of them has a different degree sequence. In particular, the first graph here has degree sequence $(3, 1, 1, 1)$, the second one has degree sequence $(3, 2, 2, 1)$, the third one has degree sequence $(3, 3, 2, 2)$, and the fourth one has degree sequence $(3, 3, 3, 3)$. In addition, these are different from the previous four graphs we found (why?).

We thus see that we have already found eight of the eleven graphs we are looking for. To find the remaining three, we examine which degree sequences we could still consider: by the Corollary of the Handshaking Lemma, which states that V_{odd} must have even cardinality, we can conclude that there are no other graphical sequences with 4 terms in which at least one term would be equal to 3. Indeed, if we have at least one term equal to 3, then an odd number of the remaining terms must be odd numbers as well, whose value must be $\leqslant 4 - 1 = 3$; thus an odd number of the remaining terms must be equal to 1 or 3, which leads to only four possibilities:

- either the sequence consists of four terms equal to 3;
- or the sequence has three terms equal to 3 and one term equal to 1 (it is not hard to check now that this would not be a graphical sequence);

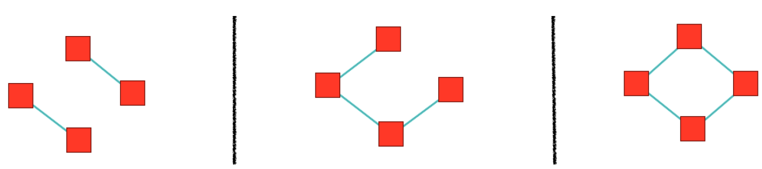
- or the sequence has two terms equal to 3 and two terms equal to 1 (again, it is not hard to check that this would not be a graphical sequence);
- or the sequence has two terms equal to 3 and no terms equal to 1 (and then, since we cannot have a term equal to 0 if we want the sequence to be graphical, both remaining terms will be equal to 2);
- or the sequence has exactly one term equal to 3 and three terms equal to 1;
- or the sequence has exactly one term equal to 3 and one term equal to 1 (and then, since we cannot have a term equal to 0 if we want the sequence to be graphical, both remaining terms will be equal to 2).

Similarly, we can see that there are no other graphical sequences with 4 terms in which at least one term is equal to 0. Indeed, if we consider such a graphical sequence, and a graph H that realises it, then by removing one isolated vertex of H (which we are guaranteed to have in the setting we are considering right now), we would end up with a graph on 3 vertices whose degree sequence has to be one of the four sequences we can find in the first picture. But the degrees of the vertices of this subgraph of H would be the same as their degrees within H (why?), and thus the degree sequence of H would be one of the sequences we can find in the second picture.

It follows that, if we are to find three additional graphical sequences with 4 terms, different from all the above, then the values we can choose for the terms are only 1 or 2 (and we also need to have an even number of 1s). This leads to three possibilities:

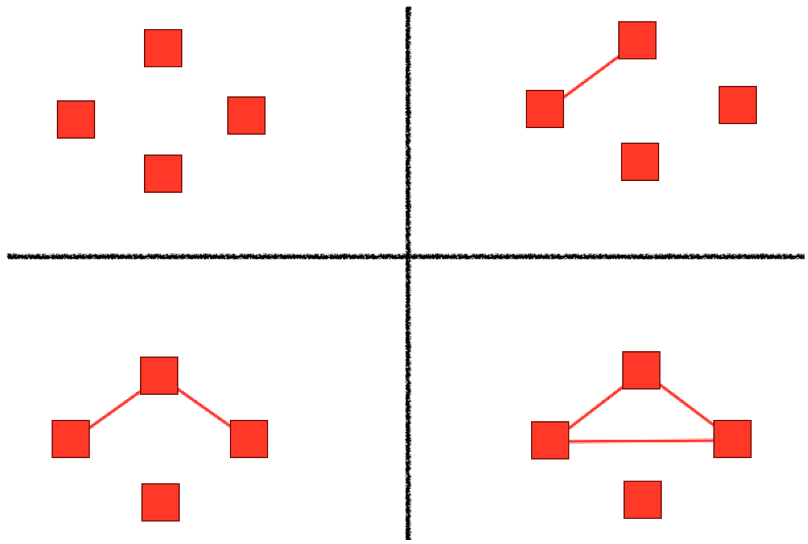
- the sequence contains four terms equal to 1, and thus it is of the form $(1, 1, 1, 1)$;
- the sequence contains two terms equal to 1, which implies that the remaining two terms equal 2, so the sequence is of the form $(2, 2, 1, 1)$;
- the sequence does not contain terms equal to 1, so it consists of four terms equal to $2:$ it is the sequence $(2, 2, 2, 2)$.

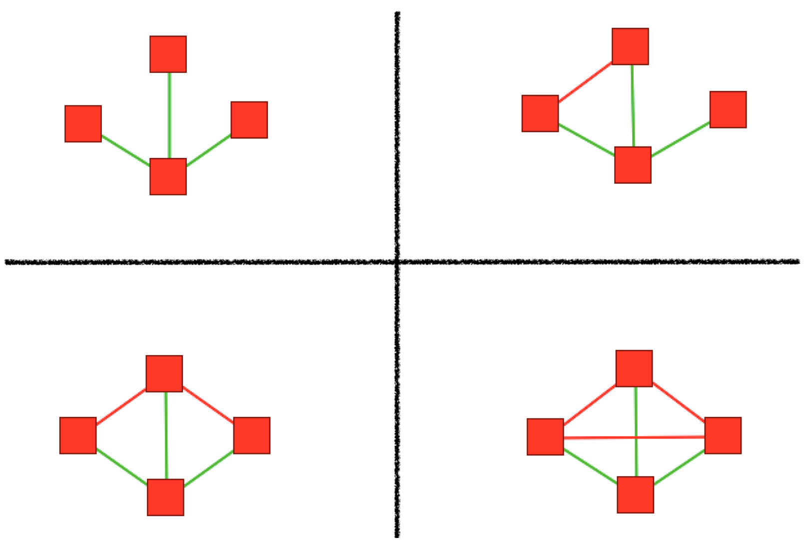
We can now check that all these three sequences are graphical and realised by the following graphs respectively:

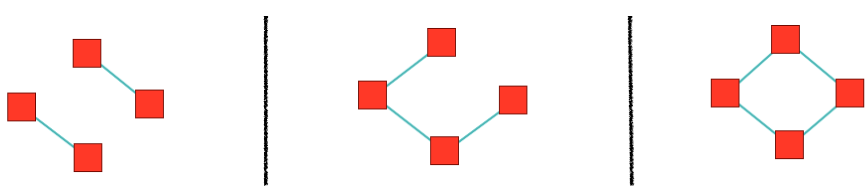


Given also that they are all different, including being different from the previous eight sequences we found, we can conclude that these last graphs are the remaining graphs on 4 vertices that we needed to find.

We list all eleven graphs again:







Problem 4. (i) Let us consider an index $j \in \{1, 2, \ldots, n\}$. Then the degree $\deg_G(v_j)$ of the vertex v_j in G is equal to the cardinality of the (open) neighbourhood $N_G(v_j)$ of v_j in G, that is, of that subset of the vertex set $\{v_1, v_2, \ldots, v_n\} \setminus \{v_j\}$ which contains only the neighbours of v_j.

Clearly the number of the remaining vertices in $\{v_1, v_2, \ldots, v_n\} \setminus \{v_j\}$, that is, the vertices which are not neighbours of v_j in G, is

$$\big|\{v_1, v_2, \ldots, v_n\} \setminus \{v_j\}\big| - \deg_G(v_j) = (n-1) - \deg_G(v_j).$$

But these vertices are exactly the vertices that will be neighbours of v_j in the complement $\overline{G}$ of G.

We can conclude that

$$\deg_{\overline{G}}(v_j) = (n-1) - \deg_G(v_j).$$

This gives us that the degree sequence of $\overline{G}$ is the sequence

$$\begin{aligned} &\big((n-1) - \deg_G(v_1),\ (n-1) - \deg_G(v_2),\ \ldots,\ (n-1) - \deg_G(v_n)\big) \\ = &\big((n-1) - d_1,\ (n-1) - d_2,\ \ldots,\ (n-1) - d_n\big). \end{aligned}$$

(ii) As we just saw in part (i), we have that, if the degree sequence of G is the sequence

$$(d_1, d_2, \ldots, d_n)$$

(where the ordering here is according to the labelling of the vertices of G, and not necessarily monotonic), then the degree sequence of $\overline{G}$ is the sequence

$$\big((n-1) - d_1,\ (n-1) - d_2,\ \ldots,\ (n-1) - d_n\big).$$

But then, for any two indices i, j in $\{1, 2, \ldots, n\}$ such that $d_i = d_j$, we will also have that $(n-1) - d_i = (n-1) - d_j$, and similarly, if we know that $(n-1) - d_i = (n-1) - d_j$, we will get that $d_i = d_j$.

In other words, the two degree sequences above have repeated terms in the same positions.

This implies that, if there is exactly one pair of indices i, j from $\{1, 2, \ldots, n\}$ such that $d_i = d_j$, then the sequence

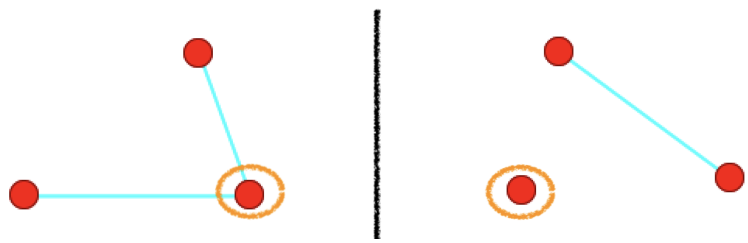
$$\big((n-1) - d_1,\ (n-1) - d_2,\ \ldots,\ (n-1) - d_n\big).$$

will also have the same property: any two of its terms will be different, except for the i-th and the j-term, which will be equal.

(iii) When the order n of the graph is 2, there are only two unlabelled graphs:

(these clearly are non-isomorphic). Both of these graphs satisfy the required property: the degree sequence of the first one is $(0, 0)$, while the degree sequence of the second one is $(1, 1)$, and thus the only two terms in either sequence have equal values.

Similarly, by looking at the list of unlabelled graphs of order 3, we see that only two of them have the required property:



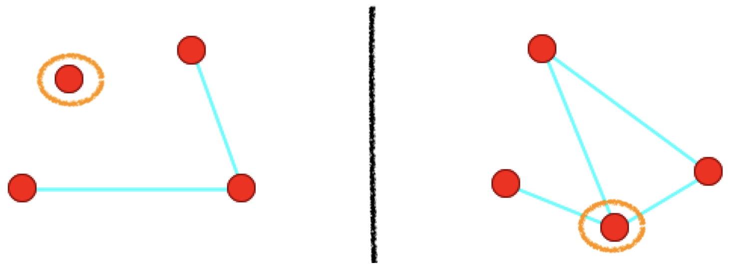
[note that here we have drawn them in such a way that each of them corresponds to one of the previous two graphs, and can be viewed as constructed based on that graph:

- the graph on the left is constructed by adding one vertex to the previous graph on the left and then joining this new vertex with every other vertex,

- while the graph on the right is constructed by adding an isolated vertex to the previous graph on the right;

the vertices that have been added are enclosed in an orange circle, while the choice of which scheme of construction to use (that is, whether to add a vertex and connect it to every other vertex, or whether to add an isolated vertex) follows from trying to preserve the required property for the degree sequences (e.g. if the previous degree sequence already had a term equal to 0, then we don't add an isolated vertex, because if we did so, our new sequence would have more than one pair of equal terms)].

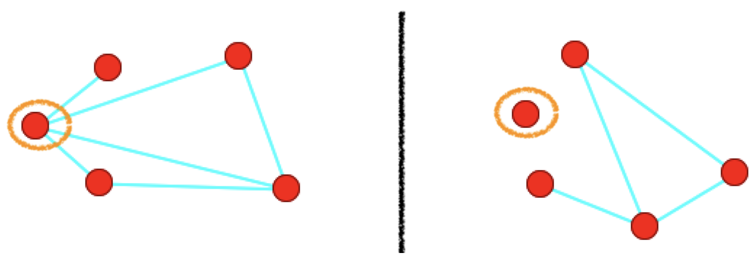
We can continue like this. Again, by looking at the list of unlabelled graphs of order 4 that we found in the previous problem (and at their degree sequences), we see that only two of them have the required property:



(and again we can view these as constructed from the previous graphs we found according to one of the schemes we described above, that is, by either adding one vertex to the graph we already have, and then joining this new vertex with every other vertex, or by adding an isolated vertex to the graph we already have).

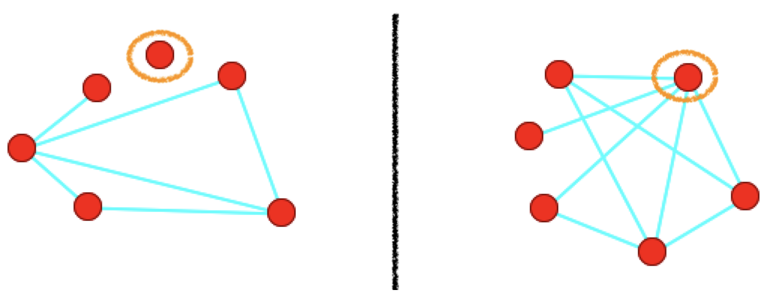
Just to double check, the degree sequences of these two graphs are $(2, 1, 1, 0)$ and $(3, 2, 2, 1)$ respectively, so they have the desired property. Moreover, one graph is the complement of the other.

We continue analogously: two graphs of order 5 that have the required property are



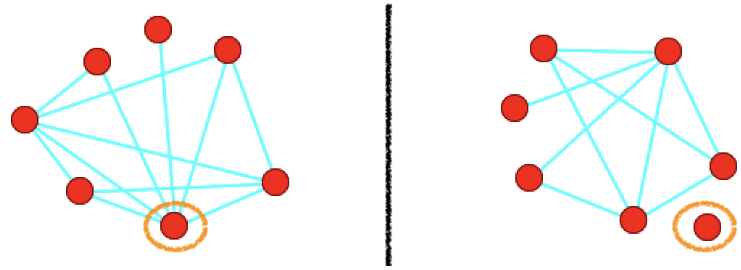
(their degree sequences are $(4, 3, 2, 2, 1)$ and $(3, 2, 2, 1, 0)$ respectively, and we can check that one is the complement of the other; also these two graphs are definitely non-isomorphic since they have different degree sequences).

Two graphs of order 6 that have the required property are



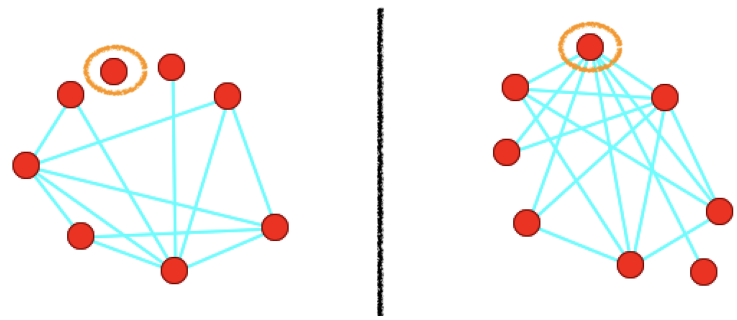
(their degree sequences are $(4, 3, 2, 2, 1, 0)$ and $(5, 4, 3, 3, 2, 1)$ respectively, and we can check that one is the complement of the other).

Two graphs of order 7 that have the required property are



(their degree sequences are $(6, 5, 4, 3, 3, 2, 1)$ and $(5, 4, 3, 3, 2, 1, 0)$ respectively, and we can check that one is the complement of the other).

Finally, two graphs of order 8 that have the required property are



(their degree sequences are $(6, 5, 4, 3, 3, 2, 1, 0)$ and $(7, 6, 5, 4, 4, 3, 2, 1)$ respectively, and we can check that one is the complement of the other).

In all the above pictures we have circled in orange the vertex that we add to the corresponding graph found in the picture right before (where this vertex then either becomes an isolated vertex, or is joined with every other vertex, depending on what the degree sequence of the previous graph was).

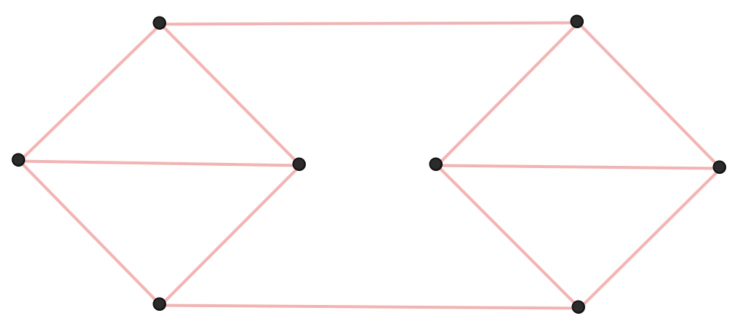
Problem 5. (i) For a graph G to have vertices with degree 3, we need G to contain at least 4 vertices; thus $n_{3,\min} \geqslant 4$.

At the same time, the complete graph K_4 on 4 vertices is a 3-regular graph, so $n_{3,\min} \leqslant |K_4| = 4$.

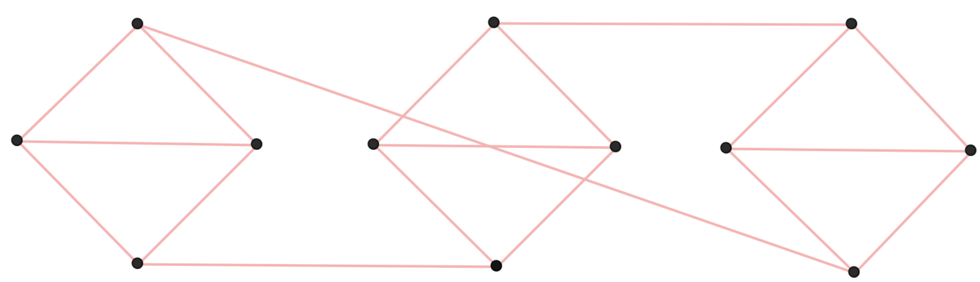
We conclude that $n_{3,\min} = 4$.

(ii) Observe that, for every $k \geqslant 2$, we can take the disjoint union of k copies of K_4, and thus get a 3-regular graph on $4k$ vertices (which has k connected components). This shows that there are 3-regular graphs with order $4k$ for every $k \geqslant 2$, and thus that there is not a maximum possible order of a finite 3-regular graph.

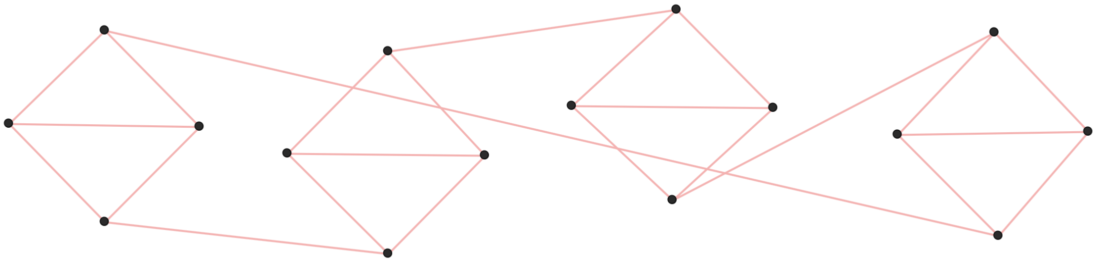
(iii) We have already seen in part (ii) that, for every $k \geqslant 1$, there exists a 3-regular graph with $4k$ vertices. In fact, the graphs we found are disconnected, except when $k = 1$, but we could have also come up with connected 3-regular graphs with $4k$ vertices: see some initial examples in the pictures below.



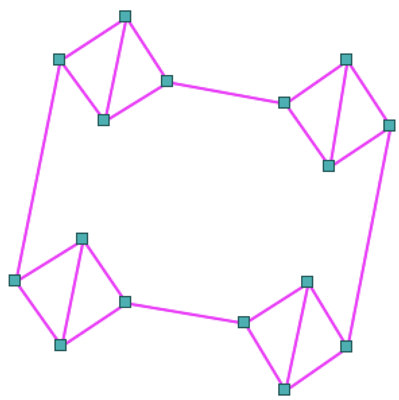
Connected 3-regular graph of order 8



Connected 3-regular graph of order 12



Connected 3-regular graph of order 16



One more representation of the above 3-regular graph of order 16

Next we note that there are no 3-regular graphs of order n when n is odd. This is because of the Corollary of the Handshaking Lemma, which tells us that V_{odd} (the subset of the vertices of the graph that have odd degree) must have even cardinality. But here V_{odd} will coincide with the entire vertex set V of the graph (since we want every vertex to have degree 3), so the cardinality of V, or equivalently the order of the graph, must be even.

We finally examine whether there are 3-regular graphs of order $4k+2$ for $k \geqslant 1$. We start with the smallest order we could have in this case, which is 6; in other words, we want to determine whether the sequence

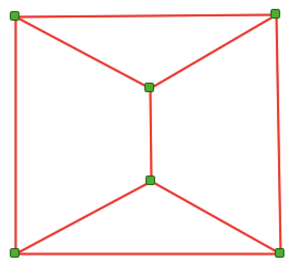
$$(3, 3, 3, 3, 3, 3)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$(3,3,3,3,3,3) \text{ is graphical}$$
$$\text{if and only if}$$
$$(2,2,2,3,3) \text{ is graphical,} \quad \text{or in other words } (3,3,2,2,2) \text{ is graphical,}$$
$$\text{if and only if}$$
$$(2,1,1,2) \text{ is graphical,} \quad \text{or in other words } (2,2,1,1) \text{ is graphical.}$$

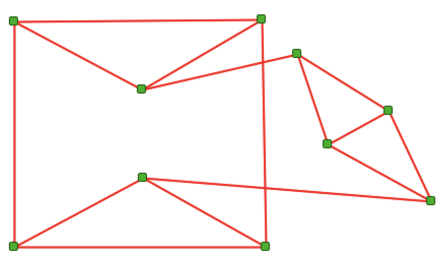
We now note that the last sequence is graphical (it coincides with the degree sequence of a path on 4 vertices), and thus all the previous sequences will be graphical too.

In fact, if we reverse the process (and recall how we were showing in the proof of the theorem that, if the shorter sequence is graphical, then the longer one is graphical too; see recording of Lecture 5, and the comments made about page 76), we can come up with a 3-regular graph of order 6:



3-regular graph of order 6

Finally, for any other $k > 1$, we can construct a 3-regular graph of order $4k + 2$ if we take the disjoint union of this graph of order 6 and of $k - 1$ copies of K_4. Or, analogously to above, we can also come up with connected constructions: e.g.



Connected 3-regular graph of order 10

We conclude that there are 3-regular graphs of order $n \geqslant 4$ for every $n = 4k$ or $= 4k+2$ (where k can be any integer $\geqslant 1$), and there are no 3-regular graphs of any other order.

Problem 6. (i) For a graph G to have vertices with degree 4, we need G to contain at least 5 vertices; thus $n_{4,\min} \geqslant 5$.

At the same time, the complete graph K_5 on 5 vertices is a 4-regular graph, so $n_{4,\min} \leqslant |K_5| = 5$.

We conclude that $n_{4,\min} = 5$.

(ii) Observe that, for every $k \geqslant 2$, we can take the disjoint union of k copies of K_5, and thus get a 4-regular graph on $5k$ vertices (which has k connected components). This shows that there are 4-regular graphs with order $5k$ for every $k \geqslant 2$, and thus that there is not a maximum possible order of a finite 4-regular graph.

(iii) We have already seen in part (ii) that, for every $k \geqslant 1$, there exists a 4-regular graph with $5k$ vertices. We now examine whether we can find 4-regular graphs of order n whenever (a) $n = 5k+1$, or (b) $n = 5k+2$, or (c) $n = 5k+3$, or finally (d) $n = 5k+4$ (where $k \geqslant 1$ is a positive integer).

Case of $5k+1$: We begin with the smallest order we could have in this case, which is 6; in other words, we want to determine whether the sequence

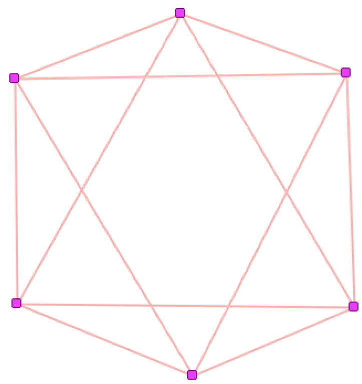
$$(4, 4, 4, 4, 4, 4)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$(4, 4, 4, 4, 4, 4) \text{ is graphical}$$
$$\text{if and only if}$$
$$(3, 3, 3, 3, 4) \text{ is graphical, or in other words } (4, 3, 3, 3, 3) \text{ is graphical,}$$
$$\text{if and only if}$$
$$(2, 2, 2, 2) \text{ is graphical.}$$

We now note that the last sequence is graphical (it coincides with the degree sequence of a cycle on 4 vertices), and thus all the previous sequences will be graphical too.

Again, if we reverse the process (and recall how we were showing in the proof of the theorem that, if the shorter sequence is graphical, then the longer one is graphical too; see recording of Lecture 5, and the comments made about page 76), we can come up with a 4-regular graph of order 6:



4-regular graph of order 6

Next we note that, for any other $k > 1$, we can construct a graph of order $5k+1$ by taking the disjoint union of this graph and of $k-1$ copies of K_5.

Case of $5k+2$: We begin with the smallest order we could have in this case, which is 7; in other words, we want to determine whether the sequence

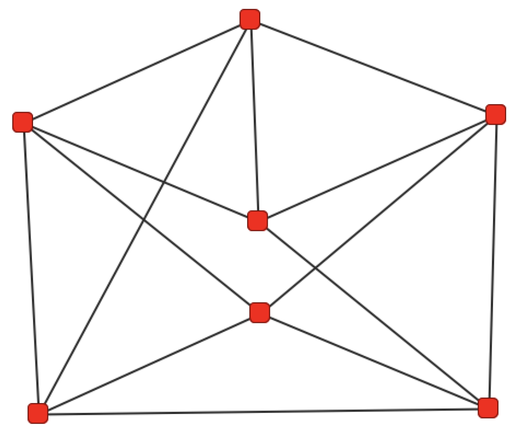
$$(4, 4, 4, 4, 4, 4, 4)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$(4, 4, 4, 4, 4, 4, 4) \text{ is graphical}$$
$$\text{if and only if}$$
$$(3, 3, 3, 3, 4, 4) \text{ is graphical, } \text{ or in other words } (4, 4, 3, 3, 3, 3) \text{ is graphical,}$$
$$\text{if and only if}$$
$$(3, 2, 2, 2, 3) \text{ is graphical, } \text{ or in other words } (3, 3, 2, 2, 2) \text{ is graphical,}$$
$$\text{if and only if}$$
$$(2, 1, 1, 2) \text{ is graphical, } \text{ or in other words } (2, 2, 1, 1) \text{ is graphical.}$$

We now note that the last sequence is graphical (it coincides with the degree sequence of a path on 4 vertices), and thus all the previous sequences will be graphical too.

We can also come up with an instance of a 4-regular graph on 7 vertices (by 'reversing' the process):



4-regular graph of order 7

Finally, observe that, for any other $k > 1$, we can construct a graph of order $5k+2$ by taking the disjoint union of this graph and of $k-1$ copies of K_5.

Case of $5k+3$: We begin with the smallest order we could have in this case, which is 8; in other words, we want to determine whether the sequence

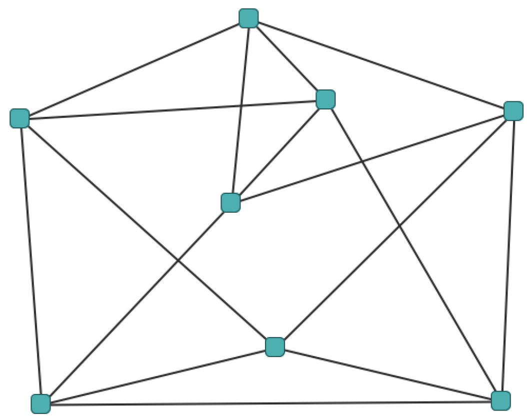
$$(4, 4, 4, 4, 4, 4, 4, 4)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$(4, 4, 4, 4, 4, 4, 4, 4) \text{ is graphical}$$
$$\text{if and only if}$$
$$(3, 3, 3, 3, 4, 4, 4) \text{ is graphical, or in other words } (4, 4, 4, 3, 3, 3, 3) \text{ is graphical,}$$
$$\text{if and only if}$$
$$(3, 3, 2, 2, 3, 3) \text{ is graphical, or in other words } (3, 3, 3, 3, 2, 2) \text{ is graphical,}$$
$$\text{if and only if}$$
$$(2, 2, 2, 2, 2) \text{ is graphical.}$$

We now note that the last sequence is graphical (it coincides with the degree sequence of a cycle on 5 vertices), and thus all the previous sequences will be graphical too.

We can also come up with an instance of a 4-regular graph on 8 vertices:



4-regular graph of order 8

Finally, observe that, for any other $k > 1$, we can construct a graph of order $5k + 3$ by taking the disjoint union of this graph and of $k - 1$ copies of K_5.

Case of $5k + 4$: We begin with the smallest order we could have in this case, which is 9; in other words, we want to determine whether the sequence

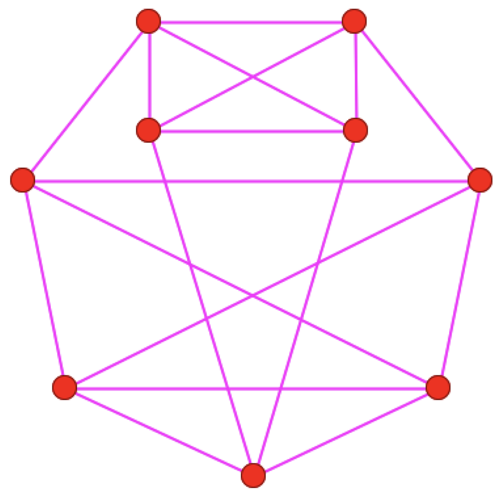
$$(4, 4, 4, 4, 4, 4, 4, 4, 4)$$

is graphical. By the Havel-Hakimi theorem, we have that

$$\begin{gathered}
(4,4,4,4,4,4,4,4,4) \text{ is graphical} \\
\text{if and only if} \\
(3,3,3,3,4,4,4,4) \text{ is graphical, } \text{ or in other words } (4,4,4,4,3,3,3,3) \text{ is graphical,} \\
\text{if and only if} \\
(3,3,3,2,3,3,3) \text{ is graphical, } \text{ or in other words } (3,3,3,3,3,3,2) \text{ is graphical,} \\
\text{if and only if} \\
(2,2,2,3,3,2) \text{ is graphical, } \text{ or in other words } (3,3,2,2,2,2) \text{ is graphical,} \\
\text{if and only if} \\
(2,1,1,2,2) \text{ is graphical, } \text{ or in other words } (2,2,2,1,1) \text{ is graphical.}
\end{gathered}$$

We now note that the last sequence is graphical (it coincides with the degree sequence of a path on 5 vertices), and thus all the previous sequences will be graphical too.

We can also come up with an instance of a 4-regular graph on 9 vertices:



4-regular graph of order 9

Finally, observe that, for any other $k > 1$, we can construct a graph of order $5k+4$ by taking the disjoint union of this graph and of $k-1$ copies of K_5.

We conclude that there are 4-regular graphs of every order $n \geqslant 5$.

Natural follow-up question; For practice/fun: Can you also find **connected** 4-regular graphs of every order $n \geqslant 5$? Could you 'manipulate' the examples mentioned above (that is, the different disjoint unions we came up with in either case) to construct connected examples instead?