MATH 322 – Graph Theory
Fall Term 2021

Notes for Lecture 18

Tuesday, November 16

Reminders

Definition

Let G be a connected graph (or multigraph) which is non-trivial, that is, it contains at least one edge.

- An *Euler trail* in G is a trail that passes by **all** edges in G (and hence, given that it is a trail, it passes by each edge exactly once).
- An *Euler circuit* in G is a circuit (that is, a closed trail) that passes by **all** edges in G.

G is called *Eulerian* if we can find (at least) one Euler circuit in G.

Reminders: necessary and sufficient conditions for Eulerian graphs and multigraphs

Theorem 1 of Lecture 16

Let G be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** every vertex of G has even degree.

Theorem 2 of Lecture 16

Let $G = (V, E)$ be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** its edge set E can be written as the disjoint union of subsets $E_1, E_2, \ldots, E_s$ each of which forms a cycle in G.

Reminders: necessary and sufficient conditions for Eulerian graphs and multigraphs

Theorem 1 of Lecture 16

Let G be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** every vertex of G has even degree.

Proposition 1 of Lecture 16

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Theorem 2 of Lecture 16

Let $G = (V, E)$ be a (non-trivial) connected graph (or multigraph).

Then G is Eulerian **if and only if** its edge set E can be written as the disjoint union of subsets $E_1, E_2, \ldots, E_s$ each of which forms a cycle in G.

Proof of Proposition 1

Proposition 1 of Lecture 16

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Proof of Proposition 1

Proposition 1 of Lecture 16

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Justification (of the "if" direction). Write u_1 and u_2 for the two vertices of G which have odd degree.

Proof of Proposition 1

Proposition 1 of Lecture 16

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Justification (of the "if" direction). Write u_1 and u_2 for the two vertices of G which have odd degree. If we introduce a new edge e_0' which joins vertices u_1 and u_2, then the multigraph G_0' we get contains G as a sub(multi)graph, and moreover satisfies:

every vertex of G_0' has even degree.

Proof of Proposition 1

Proposition 1 of Lecture 16

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Justification (of the "if" direction). Write u_1 and u_2 for the two vertices of G which have odd degree. If we introduce a new edge e_0' which joins vertices u_1 and u_2, then the multigraph G_0' we get contains G as a sub(multi)graph, and moreover satisfies:

every vertex of G_0' has even degree.

Indeed, $E(G_0') = E(G) \cup \{e_0'\}$, and thus

$$\deg_{G_0'}(u_1) = \deg_G(u_1) + 1 \text{ and } \deg_{G_0'}(u_2) = \deg_G(u_2) + 1,$$

$$\text{while } \deg_{G_0'}(w) = \deg_G(w) \text{ for any other vertex } w \in V(G_0') = V(G).$$

Proof of Proposition 1

Proposition 1 of Lecture 16

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Justification *(of the "if" direction).* Write u_1 and u_2 for the two vertices of G which have odd degree. If we introduce a new edge e_0' which joins vertices u_1 and u_2, then the multigraph G_0' we get contains G as a sub(multi)graph, and moreover satisfies:

every vertex of G_0' has even degree.

Indeed, $E(G_0') = E(G) \cup \{e_0'\}$, and thus

$$\deg_{G_0'}(u_1) = \deg_G(u_1) + 1 \quad \text{and} \quad \deg_{G_0'}(u_2) = \deg_G(u_2) + 1,$$

$$\text{while } \deg_{G_0'}(w) = \deg_G(w) \text{ for any other vertex } w \in V(G_0') = V(G).$$

In other words, G_0' is (what we call) an Eulerization of G.

Proof of Proposition 1

Proposition 1 of Lecture 16

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Justification *(of the "if" direction).* Write u_1 and u_2 for the two vertices of G which have odd degree. If we introduce a new edge e_0' which joins vertices u_1 and u_2, then the multigraph G_0' we get contains G as a sub(multi)graph, and moreover satisfies:

every vertex of G_0' has even degree.

Indeed, $E(G_0') = E(G) \cup \{e_0'\}$, and thus

$$\deg_{G_0'}(u_1) = \deg_G(u_1) + 1 \text{ and } \deg_{G_0'}(u_2) = \deg_G(u_2) + 1,$$

$$\text{while } \deg_{G_0'}(w) = \deg_G(w) \text{ for any other vertex } w \in V(G_0') = V(G).$$

In other words, G_0' is (what we call) an Eulerization of G.

By Theorem 1, we know that we can find an Euler circuit CI_0 in G_0'. Recall also that we can rewrite a circuit so that it starts at any vertex we want, and has initial edge any of the edges appearing in it; thus CI_0 can be rewritten as e.g.

$$\mathrm{CI}_0 : w_{i_0} = u_1 \; e_0' \; w_{i_1} = u_2 \; e_{j_1} \; w_{i_2} \; e_{j_2} \; w_{i_3} \cdots w_{i_8} \; e_{j_8} \; w_{i_9} \; e_{j_9} \; w_{i_{10}} \cdots w_{i_m} \; e_{j_m} \; w_{i_{m+1}} = u_1 \, .$$

Observe that $e_{j_1}, e_{j_2}, \ldots, e_{j_m}$ are the m different edges of G.

Proof of Proposition 1

Proposition 1 of Lecture 16

Let G be a connected graph (or multigraph).

Then G has an Euler trail, but not an Euler circuit **if and only if** exactly two vertices of G have odd degree (and all other vertices have even degree).

Justification *(of the "if" direction).* Write u_1 and u_2 for the two vertices of G which have odd degree. If we introduce a new edge e_0' which joins vertices u_1 and u_2, then the multigraph G_0' we get contains G as a sub(multi)graph, and moreover satisfies:

every vertex of G_0' has even degree.

Indeed, $E(G_0') = E(G) \cup \{e_0'\}$, and thus

$$\deg_{G_0'}(u_1) = \deg_G(u_1) + 1 \text{ and } \deg_{G_0'}(u_2) = \deg_G(u_2) + 1,$$

$$\text{while } \deg_{G_0'}(w) = \deg_G(w) \text{ for any other vertex } w \in V(G_0') = V(G).$$

In other words, G_0' is (what we call) an Eulerization of G.

By Theorem 1, we know that we can find an Euler circuit CI_0 in G_0'. Recall also that we can rewrite a circuit so that it starts at any vertex we want, and has initial edge any of the edges appearing in it; thus CI_0 can be rewritten as e.g.

$$\mathrm{CI}_0 : w_{i_0} = u_1 \ e_0' \ w_{i_1} = u_2 \ e_{j_1} \ w_{i_2} \ e_{j_2} \ w_{i_3} \cdots w_{i_8} \ e_{j_8} \ w_{i_9} \ e_{j_9} \ w_{i_{10}} \cdots w_{i_m} \ e_{j_m} \ w_{i_{m+1}} = u_1 \,.$$

Observe that $e_{j_1}, e_{j_2}, \ldots, e_{j_m}$ are the m different edges of G. Thus,

$$w_{i_1} = u_2 \ e_{j_1} \ w_{i_2} \ e_{j_2} \ w_{i_3} \cdots w_{i_8} \ e_{j_8} \ w_{i_9} \ e_{j_9} \ w_{i_{10}} \cdots w_{i_m} \ e_{j_m} \ w_{i_{m+1}} = u_1$$

is an Euler trail of G.

Reminders: Hamiltonian graphs

Definition

Let G be a connected graph.

Reminders: Hamiltonian graphs

Definition

Let G be a connected graph.

- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).
- A *Hamilton cycle* in G is a cycle (that is, a closed path) that passes through **all** vertices in G.

G is called *Hamiltonian* if we can find (at least) one Hamilton cycle in G.

Reminders: Hamiltonian graphs

Definition

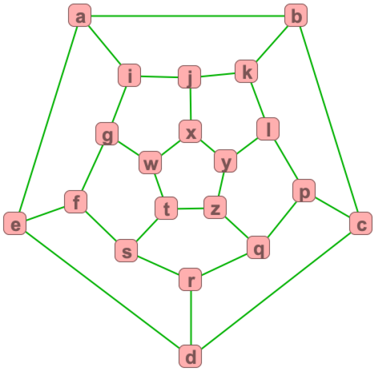
Let G be a connected graph.

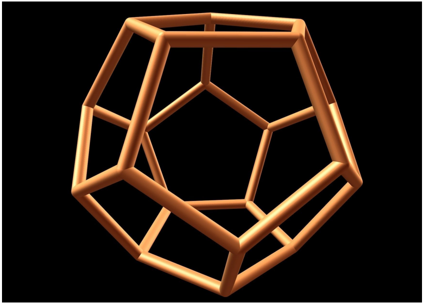
- A *Hamilton path* in G is a path that passes through **all** vertices in G (and hence, given that it is a path, it passes through each vertex exactly once).
- A *Hamilton cycle* in G is a cycle (that is, a closed path) that passes through **all** vertices in G.

G is called *Hamiltonian* if we can find (at least) one Hamilton cycle in G.

The name is in honour of the mathematician William Hamilton who introduced the idea of looking for Hamilton cycles in graphs (with the first graph he considered being (the 'frame' of) the solid dodecahedron) as a new board game!

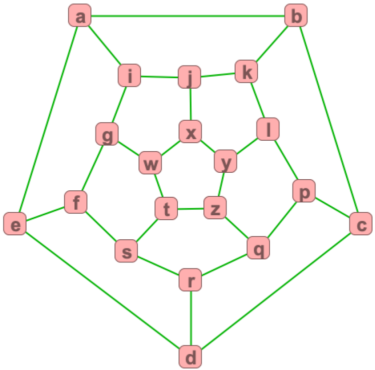
Hamilton's board game

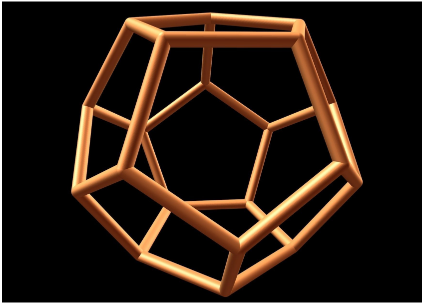




The graph G_0 on the left can be identified with the frame of the solid dodecahedron, as can be seen from the image on the right.

Hamilton's board game





The graph G_0 on the left can be identified with the frame of the solid dodecahedron, as can be seen from the image on the right.

Practice Exercise. Find a Hamilton cycle in G_0.

Unlike what we saw for Eulerian and non-Eulerian graphs,

there are <u>no</u> simple characterisations (that is, conditions that are <u>both</u> necessary and sufficient) for Hamiltonicity.

Unlike what we saw for Eulerian and non-Eulerian graphs,

there are no simple characterisations (that is, conditions that are both necessary and sufficient) for Hamiltonicity.

Thus, we will state:

some conditions that are necessary for a graph to be Hamiltonian

(that is, if any of these conditions doesn't hold,
then the graph cannot be Hamiltonian),

Unlike what we saw for Eulerian and non-Eulerian graphs,

there are no simple characterisations (that is, conditions that are both necessary and sufficient) for Hamiltonicity.

Thus, we will state:

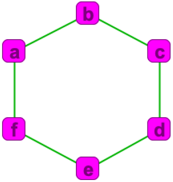
some conditions that are necessary for a graph to be Hamiltonian

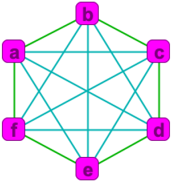
(that is, if any of these conditions doesn't hold, then the graph cannot be Hamiltonian),

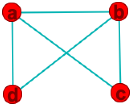
and some conditions that are sufficient

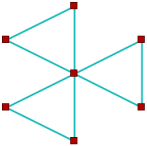
(that is, it suffices to check for any one of these conditions, and if it does hold true, then the graph will be Hamiltonian).

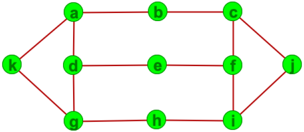
Now, degrees can be all even, all odd, or a mix

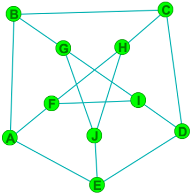








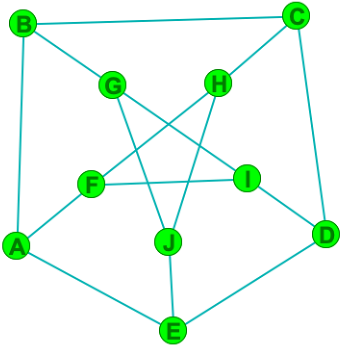




The Petersen Graph

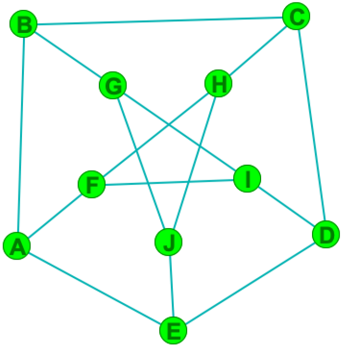
The graphs in the top row are all Hamiltonian (why?).
None of the graphs in the bottom row is Hamiltonian.

Side Note: The Petersen graph



For future reference too: this graph is **a 3-regular graph** (equivalently called **a cubic graph**) which is usually referred to as the Petersen graph, and it comes up as an interesting (and fairly easy to draw) example or counterexample in many problems in Graph Theory *(recall that you have already worked with it in HW1, Pb2).*

Side Note: The Petersen graph



For future reference too: this graph is **a 3-regular graph** (equivalently called **a cubic graph**) which is usually referred to as the Petersen graph, and it comes up as an interesting (and fairly easy to draw) example or counterexample in many problems in Graph Theory *(recall that you have already worked with it in HW1, Pb2).*

Question. Why is the Petersen graph not Hamiltonian? Can we give a justification?

Hamiltonian graphs: necessary conditions

Necessary Condition 1

Let G be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then G has no cutvertices.
In other words, if G is Hamiltonian, then $\kappa(G) \geqslant 2$ (or equivalently G is 2-vertex connected).

Hamiltonian graphs: necessary conditions

Necessary Condition 1

Let G be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then G has no cutvertices.
In other words, if G is Hamiltonian, then $\kappa(G) \geqslant 2$ (or equivalently G is 2-vertex connected).

Justification. Suppose that G is Hamiltonian. Then G contains a cycle H_0 passing through all the vertices of G *(here we can see why it's necessary to assume that $|G| \geqslant 3$)*.

Clearly H_0 is a subgraph of G: $H_0 \subseteq G$.

But then, for any two distinct vertices v_1, v_2 of G, the subgraph H_0 contains two internally disjoint $v_1 - v_2$ paths, and hence G also contains these paths.

This shows that $\kappa'_G(v_1, v_2) \geqslant 2$ for the arbitrary pair of distinct vertices v_1, v_2 of G, and hence, as we have seen in previous lectures, it follows that $\kappa(G) \geqslant 2$.

Hamiltonian graphs: necessary conditions

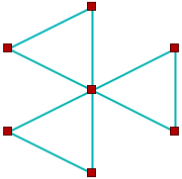
Necessary Condition 2

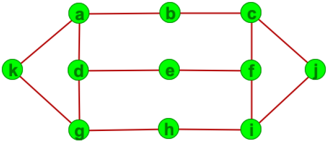
Let $G = (V, E)$ be a connected graph of order $n \geqslant 3$.

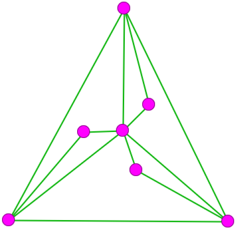
If G is Hamiltonian, then the following holds:

$$\text{for every vertex subset } S \subsetneq V,$$
$$\text{the subgraph } G - S \text{ has at most } |S| \text{ connected components.}$$

Testing the above necessary conditions







Not Hamiltonian,
but satisfies the necessary conditions;
hence these conditions are not sufficient too.

Hamiltonian graphs: some sufficient conditions

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$, then G is Hamiltonian.

Hamiltonian graphs: some sufficient conditions

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$, then G is Hamiltonian.

Remark. The lower bound $\frac{n}{2}$ for the minimum degree of G is optimal, and, perhaps surprisingly, **cannot** even be replaced by $\delta(G) \geqslant \lfloor \frac{n}{2} \rfloor$.

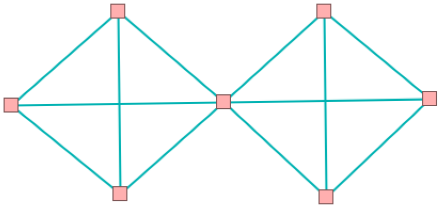
Hamiltonian graphs: some sufficient conditions

Theorem 1 (Dirac, 1952)

Let G be a graph of order $n \geqslant 3$ such that the minimum degree $\delta(G) \geqslant \frac{n}{2}$, then G is Hamiltonian.

Remark. The lower bound $\frac{n}{2}$ for the minimum degree of G is optimal, and, perhaps surprisingly, **cannot** even be replaced by $\delta(G) \geqslant \lfloor \frac{n}{2} \rfloor$.

Example. The following graph satisfies $\delta(G) \geqslant \lfloor \frac{n}{2} \rfloor$, but it is not Hamiltonian (why?).



Hamiltonian graphs: some sufficient conditions

Dirac's theorem follows easily from another sufficient condition for Hamiltonicity which came out a little later:

Hamiltonian graphs: some sufficient conditions

Dirac's theorem follows easily from another sufficient condition for Hamiltonicity which came out a little later:

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G, we have that $\deg(u) + \deg(v) \geqslant n$.

Hamiltonian graphs: some sufficient conditions

Dirac's theorem follows easily from another sufficient condition for Hamiltonicity which came out a little later:

Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of distinct and non-adjacent vertices u and v of G, we have that $\deg(u) + \deg(v) \geqslant n$.

Then G is Hamiltonian.

Hamiltonian graphs: some sufficient conditions

Dirac's theorem follows easily from another sufficient condition for Hamiltonicity which came out a little later:

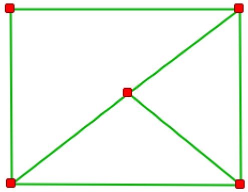
Theorem 2 (Ore, 1960)

Let G be a graph of order $n \geqslant 3$ which satisfies the following property:

for every pair of <u>distinct</u> and <u>non-adjacent</u> vertices u and v of G, we have that $\deg(u) + \deg(v) \geqslant n$.

Then G is Hamiltonian.

Testing Our Understanding. Does the following graph have the property stated in Ore's theorem? If yes, can you find a Hamilton cycle in it?



Hamiltonian graphs: some sufficient conditions

Proposition 3

Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$\binom{n-1}{2} + 2$$

edges. Then G is Hamiltonian.

Proposition 3

Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$\binom{n-1}{2} + 2$$

edges. Then G is Hamiltonian.

Remark 1. Note that the maximum number of edges that G could have is $\binom{n}{2}$ (in which case G would be the complete graph on n vertices). Thus, this proposition allows us to deal with graphs with size between $\binom{n-1}{2} + 2$ and $\binom{n}{2}$ *(with the endpoints included)*, and we can find quite a few examples here.

Proposition 3

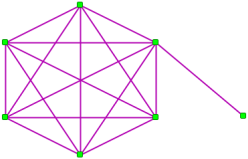
Let G be a graph of order $n \geqslant 3$, and suppose that G has at least

$$\binom{n-1}{2} + 2$$

edges. Then G is Hamiltonian.

Remark 1. Note that the maximum number of edges that G could have is $\binom{n}{2}$ (in which case G would be the complete graph on n vertices). Thus, this proposition allows us to deal with graphs with size between $\binom{n-1}{2} + 2$ and $\binom{n}{2}$ *(with the endpoints included)*, and we can find quite a few examples here.

Remark 2. Again, the lower bound on the size of G is best possible: below is an example of a graph H with $\binom{n-1}{2} + 1$ edges which is not Hamiltonian (note that $n = 7$ here, but this type of example can work for other n as well).



One more parameter of graphs

Definition

Let $G = (V, E)$ be a graph. A vertex subset $V' \subseteq V$ is called an *independent* set of vertices, if any two different vertices in V' are **non-adjacent**.

One more parameter of graphs

Definition

Let $G = (V, E)$ be a graph. A vertex subset $V' \subseteq V$ is called an *independent* set of vertices, if any two different vertices in V' are **non-adjacent**.

The *independence number* of G, denoted by $\alpha(G)$, is defined to be the maximum cardinality of an independent set of vertices of G.

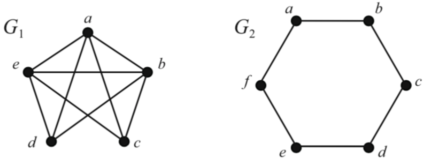
One more parameter of graphs

Definition

Let $G = (V, E)$ be a graph. A vertex subset $V' \subseteq V$ is called an *independent* set of vertices, if any two different vertices in V' are **non-adjacent**.

The *independence number* of G, denoted by $\alpha(G)$, is defined to be the maximum cardinality of an independent set of vertices of G.

Examples. What are the independence numbers of the following two graphs?



from the Harris-Hirst-Mossinghoff book

Independence number and Hamiltonicity?

Theorem 4 (Chvátal-Erdös, 1972)

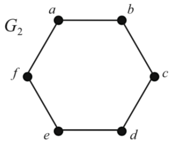
Let G be a graph of order $n \geqslant 3$ such that $\kappa(G) \geqslant \alpha(G)$. Then G is Hamiltonian.

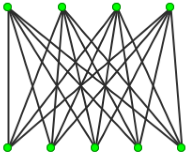
Independence number and Hamiltonicity?

Theorem 4 (Chvátal-Erdös, 1972)

Let G be a graph of order $n \geqslant 3$ such that $\kappa(G) \geqslant \alpha(G)$. Then G is Hamiltonian.

Remark. This condition again is only a sufficient condition. E.g. the graph on the left in the image below (which we saw on the previous slide too) satisfies $\kappa(G_2) = 2 < 3 = \alpha(G_2)$, but of course it's easy to see that it's Hamiltonian since it is a cycle graph:

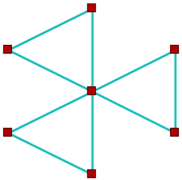


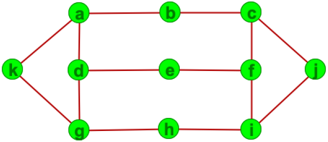


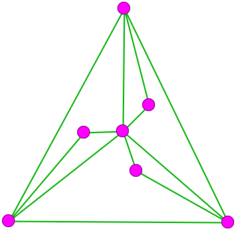
On the other hand, the graph on the right, which is the bipartite graph $K_{4,5}$, is not Hamiltonian (can you justify this?), while it satisfies $\kappa(G) = 4 = \alpha(G) - 1$. This shows that, if we even slightly relax the above condition, the theorem no longer holds.

Testing these sufficient conditions on non-examples

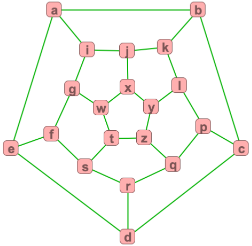
Note that the conditions should fail here (why?).

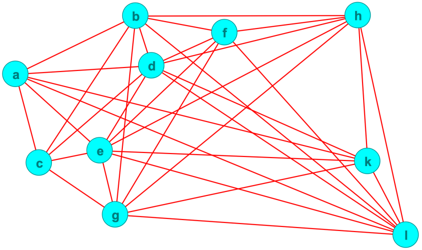


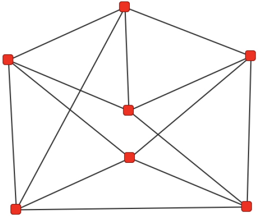


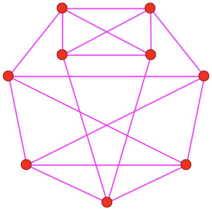


Testing these sufficient conditions on (possible) examples

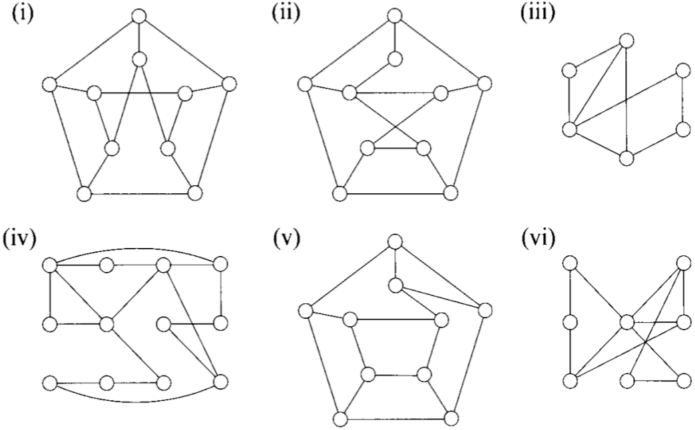








Testing these sufficient conditions on possible examples



from Wallis' book

Left as a practice exercise.

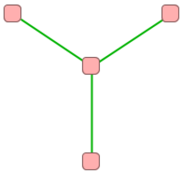
A couple more sufficient conditions for Hamiltonicity:

A couple more sufficient conditions for Hamiltonicity:

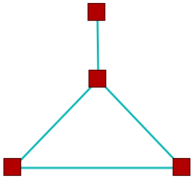
Both these conditions are stated in terms of **forbidden subgraphs**
that is, a graph G will be Hamiltonian if certain, already given graphs **cannot** be viewed as induced subgraphs of G.

Family of possible forbidden subgraphs

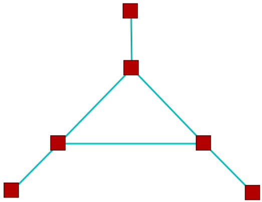
Consider the following three graphs:



Graph $K_{1,3}$



Graph Z_4



Graph Z_6

1st sufficient condition in terms of forbidden subgraphs

Theorem 5 (Goodman-Hedetniemi, 1974)

Let G be a graph of order $n \geqslant 3$ which is 2-vertex connected (that is, $\kappa(G) \geqslant 2$).

If G is $\{K_{1,3}, Z_4\}$-free *(that is, none of those two graphs is an induced subgraph of G)*, then G is Hamiltonian.

1st sufficient condition in terms of forbidden subgraphs

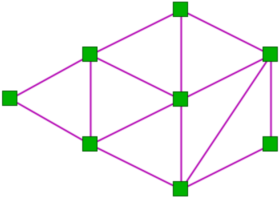
Theorem 5 (Goodman-Hedetniemi, 1974)

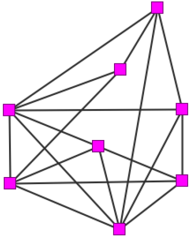
Let G be a graph (of order $n \geqslant 3$) which is 2-vertex connected (that is, $\kappa(G) \geqslant 2$).

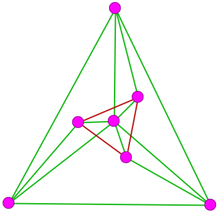
If G is $\{K_{1,3}, Z_4\}$-free *(that is, none of those two graphs is an induced subgraph of G)*, then G is Hamiltonian.

Possible examples and non-examples

Question. Are any of the following graphs $\{K_{1,3},\ Z_4\}$-free?







2nd sufficient condition in terms of forbidden subgraphs

Theorem 6 (Duffus-Gould-Jacobson, 1980)

Let G be a $\{K_{1,3}, Z_6\}$-free graph.

2nd sufficient condition in terms of forbidden subgraphs

Theorem 6 (Duffus-Gould-Jacobson, 1980)

Let G be a $\{K_{1,3}, Z_6\}$-free graph.

- If G is connected, then G has a Hamilton path.

2nd sufficient condition in terms of forbidden subgraphs

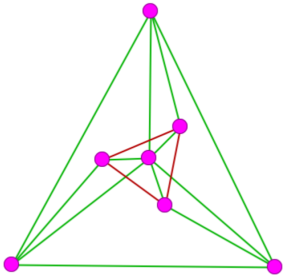
Theorem 6 (Duffus-Gould-Jacobson, 1980)

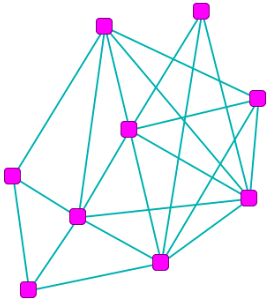
Let G be a $\{K_{1,3}, Z_6\}$-free graph.

- If G is connected, then G has a Hamilton path.
- If G is 2-vertex connected, then G is Hamiltonian.

Possible examples and non-examples

Question. Are any of the following graphs $\{K_{1,3},\ Z_6\}$-free?





Also, a necessary condition for containing a Hamilton path

Recall: Necessary Condition 2

Let $G = (V, E)$ be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $G - S$ has at most $|S|$ connected components.

Also, a necessary condition for containing a Hamilton path

Recall: Necessary Condition 2

Let $G = (V, E)$ be a connected graph of order $n \geqslant 3$.

If G is Hamiltonian, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $G - S$ has at most $|S|$ connected components.

Necessary Condition 2′

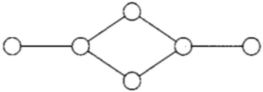
Let $H = (V, E)$ be a connected graph of order $n \geqslant 2$.

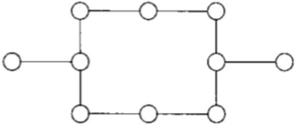
If H has a Hamilton path, then the following holds:

for every vertex subset $S \subsetneq V$,
the subgraph $H - S$ has at most $|S| + 1$ connected components.

Testing this necessary condition

Question 1. Does any of the following graphs from Wallis' book have a Hamilton path? If it does, find one such path. If it doesn't, can you justify why not?





Testing this necessary condition

Question 1. Does any of the following graphs from Wallis' book have a Hamilton path? If it does, find one such path. If it doesn't, can you justify why not?

Question 2. What about this graph?