# ME614 Spring 2018 - Homework 3

## Poisson Solver & Taylor-Green Vortex

(Due November 28, 2018)

Please submit your homework on Blackboard in the form of a (1) report in PDF (keep file size smaller than 5MB) and (2) a working code, zipped in one file. The submitted code needs to run and create all (and possibly only) the plots you are including in your report. The use of LaTeX for your report is strongly recommended (but not required) and you can start with a template from http://www.latextemplates.com/. Discussions and sharing of ideas are encouraged but individually prepared submissions (codes, figures, written reports, etc.) are required. Due to the sensitivity of some numerical results to the specific coding choices, and sometimes even hardware, it is easy for the instructor to flag homeworks as suspicious. A plagiarism detection algorithm will be run against all codes submitted. Do NOT include in your submisison any files that are not required (e.g. the syllabus, zip files with python libraries, other PDFs, sample python sessions etc).
Points will be deducted from late submissions at a rate of 20% of the overall homework value per day late. Homeworks are due at 11:59 PM of the due date.

### Problem 1

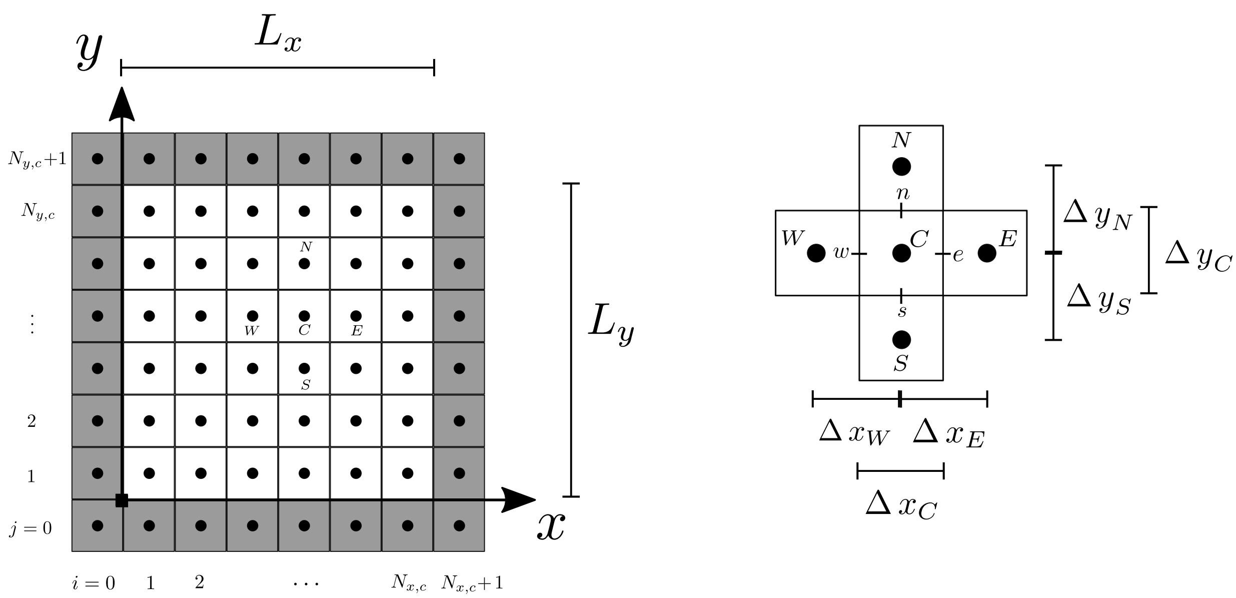


Figure 1: Computational stencil for Poisson solver illustrating second-order central schemes for divergence and gradient operators. Note that for a non-uniform grid $\Delta x_C \neq \Delta x_E \neq \Delta x_W \neq \Delta y_C \neq \Delta y_N \neq \Delta y_S$. Ghost cells (shaded) inherit their size from neighboring internal cells.

*Operator Definition*

Using the grid arrangement in Figure 1, solve the following equation

$$\frac{\partial \phi}{\partial x} - \frac{1}{Re}\nabla^2 \phi = f(x, y) \tag{1}$$

where $\phi$ will play the role of pressure in your soon-to-come Navier-Stokes solver, $f(x, y)$ is a given scalar function of the coordinates $x$ and $y$ and $Re$ is a Reynolds number. Use the following discretization for the Laplacian operator:

$$\widehat{\nabla}^2 \phi = [\mathbf{D}_x \cdot \mathbf{G}_x + \mathbf{D}_y \cdot \mathbf{G}_y] \cdot \boldsymbol{\phi} = \frac{1}{\Delta x_C}\left[\left.\frac{\partial \phi}{\partial x}\right|_e - \left.\frac{\partial \phi}{\partial x}\right|_w\right] + \frac{1}{\Delta y_C}\left[\left.\frac{\partial \phi}{\partial y}\right|_n - \left.\frac{\partial \phi}{\partial y}\right|_s\right] =$$
$$\frac{1}{\Delta x_C}\left[\frac{\phi_E - \phi_C}{\Delta x_E} - \frac{\phi_C - \phi_W}{\Delta x_W}\right] + \frac{1}{\Delta y_C}\left[\frac{\phi_N - \phi_C}{\Delta y_N} - \frac{\phi_C - \phi_S}{\Delta y_S}\right]$$

where $\mathbf{D}_{x,y}$ and $\mathbf{G}_{x,y}$ are, respectively, second-order divergence and gradient operators. Prepare the discrete Laplacian operator above for both homogeneous Neumann and homogeneous Dirichlet boundary conditions for a generic **non-uniform grid**. Show the resulting elliptic operator with the `spy` command for a small grid size.

[5%]

*Iterative Solver*

For the following class of solutions

$$\phi(x,y) = \sin(2\,\pi\,n\,x)\,\sin(2\,\pi\,n\,y) \tag{2}$$

find the expression of the corresponding $f(x,y)$ in (1).

Consider three values of $n$ in the set $\{1,\ldots,\min(N_{x,c},N_{y,c})\}$ (pick a low, medium, and high wavenumber) and solve (1) for $Re = 1$ with the Successive Over Relaxation (SOR) method applied to the Gauss-Seidel iterative scheme on a uniform grid on a square computational domain, $L_x = L_y = 1$. Find the optimum relaxation parameter $\omega$ by plotting the number of iterations required to reach the condition on the residual:

$$\frac{||\mathbf{r}^k||}{||\mathbf{r}^0||} < 10^{-5} \tag{3}$$

for different grid sizes, homogeneous Dirichlet boundary conditions, values of $n$, starting with the initial guess, $\boldsymbol{\phi} = \mathbf{0}$. Plot the history of the residual $||\mathbf{r}^k||$, versus the iteration number $k$.

[10%]

For a high wavenumber case and $N_{x,c} = N_{y,c} = 256$, still using an over-relaxed Gauss-Seidel, plot the history of the residual (i.e. $||\mathbf{r}^k||$ versus $k$) for six values of the Reynolds number in the logarithmically equispaced range $\log_{10} Re = -2, 0, 2, 4, 6, 8$. Use the optimal value of the relaxation parameter for each Reynolds number at the given grid resolution and wavenumber.

[15%]

For a given grid size $N_{x,c} = N_{y,c} > 128$ and wavenumber ($n > 1$) of your choice, for the range of Reynolds numbers $\log_{10} Re = \{-2, 0, 2, 4, 6, 8\}$, solve iteratively the nonlinear equation

$$\phi\frac{\partial\phi}{\partial x} - \frac{1}{Re}\nabla^2\phi = f(x,y) \tag{4}$$

(a) leaving only the Laplace operator on the left-hand-side:

$$-\frac{1}{Re}\nabla^2\phi^k = -\phi^{k-1}\frac{\partial}{\partial x}\phi^{k-1} + f(x,y) \tag{5}$$

(b) leaving only the nonlinear operator the left-hand-size. For the iterative method (b) use sub-iterations of the type:

$$\phi^{\ell-1,k}\frac{\partial\phi^{\ell,k}}{\partial x} = \frac{1}{Re}\nabla^2\phi^{k-1} + f(x,y) \tag{6}$$

where the subiteration $\ell$ should be carried out to convergence, $\lim_{\ell\to\infty}\phi^{\ell,k} = \phi^k$, with an arbitrary but strict criteria, and adopting the following regularization (in red):

$$\epsilon\nabla^2\phi^{\ell,k} + \phi^{\ell-1,k}\frac{\partial\phi^{\ell,k}}{\partial x} = \frac{1}{Re}\nabla^2\phi^{k-1} + f(x,y) + \epsilon\nabla^2\phi^{k-1} \tag{7}$$

via an arbitrary regularization parameter, $\epsilon$, to be picked wisely[1]. Compare the number of $k$-iterations and the total time required to reach the same condition on the residual (3).

[20%]

What are the gains in performance for method (b) if a LU pre-factorization of the (near-singular) operator on the left-hand-side is stored in preprocessing (instead of calling `spsolve` on the full, non-factorized operator at each iteration)? Does performing an LU factorization for case (b) allow you to pick $\epsilon = 0$?

[5%]

[1]Use negative values of $\epsilon$, perhaps of the order of 1/Re

## Problem 2

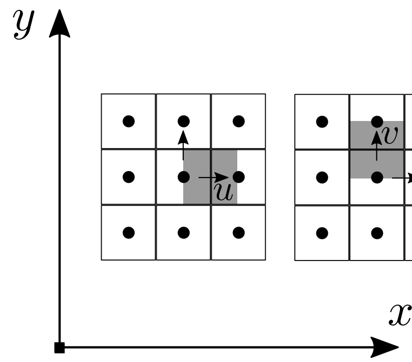


Figure 2: Staggered grid arrangement. Three different grids need to be defined, corresponding to three different nodal locations for $u$ (horizontal arrows), $v$ (vertical arrows) and $p, \rho$ (black circles) with corresponding control volumes shown from left to right respectively.

The Taylor-Green vortex is an exact solution of the fully nonlinear incompressible Navier-Stokes equations that, for unitary kinematic viscosity $\nu = 1$, is given by

$$\begin{aligned} u(x, y, t) &= -e^{-2\,t} \cos(x)\, \sin(y) && (8) \\ v(x, y, t) &= e^{-2\,t} \sin(x)\, \cos(y) && (9) \\ p(x, y, t) &= -\frac{e^{-4\,t}}{4} \left(\cos(2\,x) + \cos(2\,y)\right). && (10) \end{aligned}$$

Integrate the Navier-Stokes equations numerically, with the fractional step method explained in class, between $t = 0$ and $t = 1$ using (8), (9), (10) as initial conditions, and periodic boundary conditions over the domain $0 \le x \le 2\pi$, $0 \le y \le 2\pi$ and a uniform grid with $N_{x,c} = N_{y,c} = N$ pressure cells in each direction.

Adopt a **fully-explicit** time-advancement strategy, that is, use RK1, RK2, RK3 and RK4 for both advection and diffusive terms for the *prediction step*. Apply the same pseudo-pressure correction in all cases only after all of the Runge-Kutta sub-steps are complete. Use a staggered grid arrangement as shown in figure 2 and central second-order spatial operators as discussed in class. Briefly explain (with equations) the suggested time-advancement strategies.

Compare your results to the analytical solution for the aforementioned four time advancement strategies by plotting the RMS of the error at time $t = 1$ versus the average grid size, $h = \sqrt{\Delta x \Delta y}$, for a very small time step ($\Delta t << 1$).

*Hint*: Use the provided analytical solution to verify that **each term** in the momentum equation of the Navier-Stokes equations is individually approximated with second-order accuracy on the exact grid locations where they belong. Do not proceed implementing the discrete time advancement if this has not been verified first.

[45%]