Gu, Xinyuan ME614 HW#1

Problem 1

a) Plots of Lagrangian polynomials for $N=3$ & $N=7$ were attached at the end of problem 1.

Yes. When N moves to a large number, the fitting could be very tough (as $N=7$), which looks potentially troublesome.

b) Plots of all 12 Scenarios were attached at the end of problem 1.

After comparison, ① When they are in the same type <(C) centered, (S) biased etc.>, more points in l & r will generate more accurate results.

② Centered (S) will have the most accurate simulation results when they have the same number of points.

c) The order of accuracy doesn't always correspond to the order of the polynomial interpolant. For a given order P, the maximum order of accuracy is P-1 and the minimum order is undetermined.

please check the plots, when the discretization has n points, the trend of truncation error will be parallel to N-1 order of magnitude.

d) Pick $N=5$ to calculate Vandermode matrix.

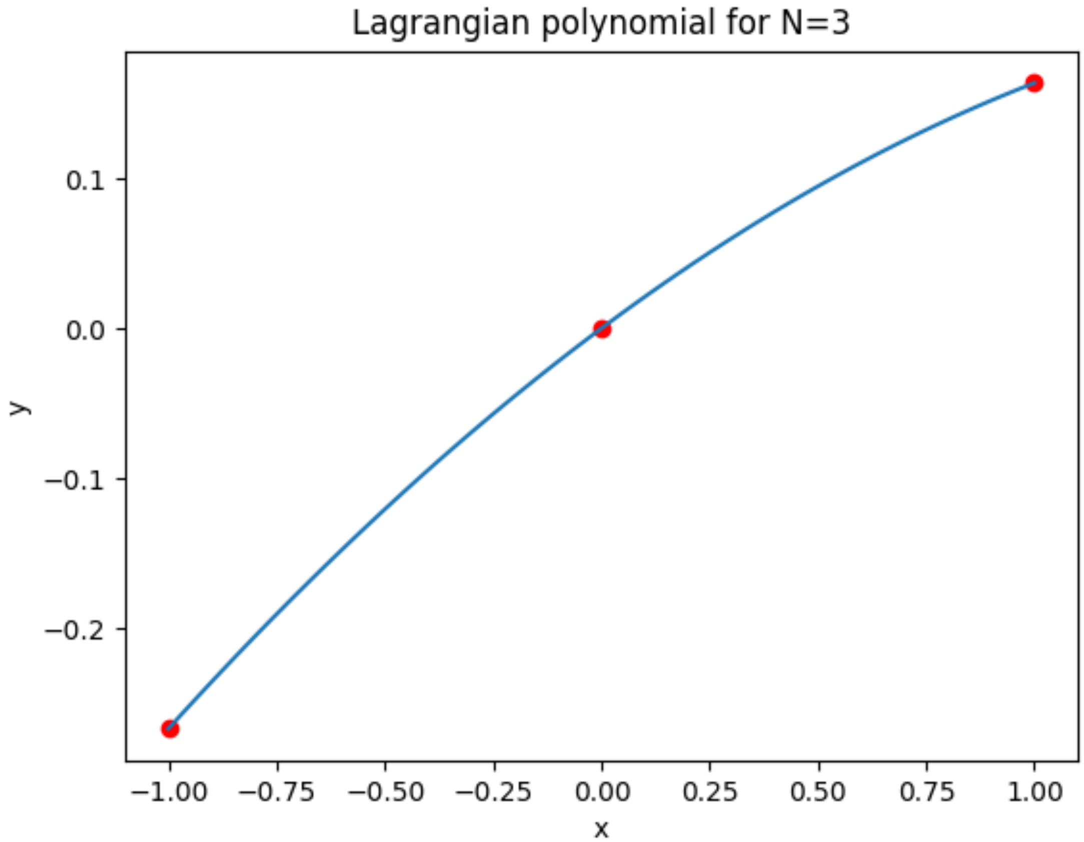
choose point for $X = -1, -0.5, 0, 0.5, 1$

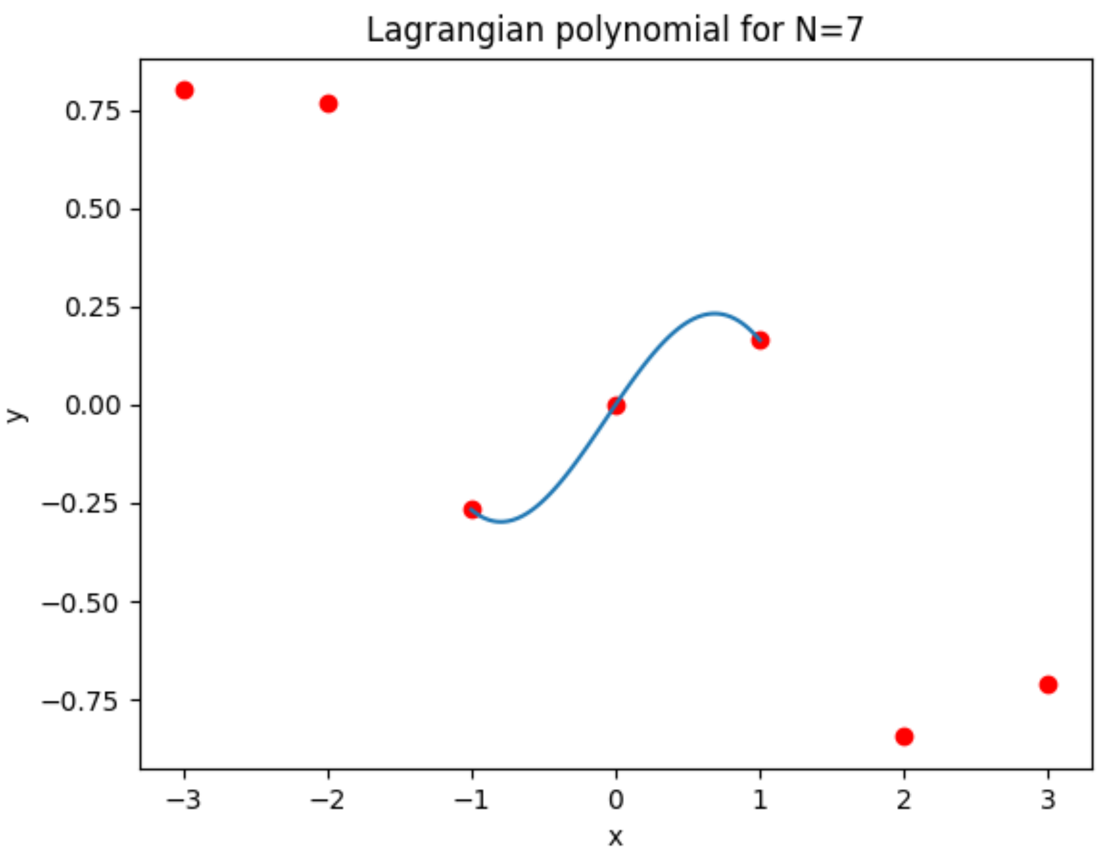
$f'(x) = -3.83, -1.91, 0.998, -2.11, 3.83$.

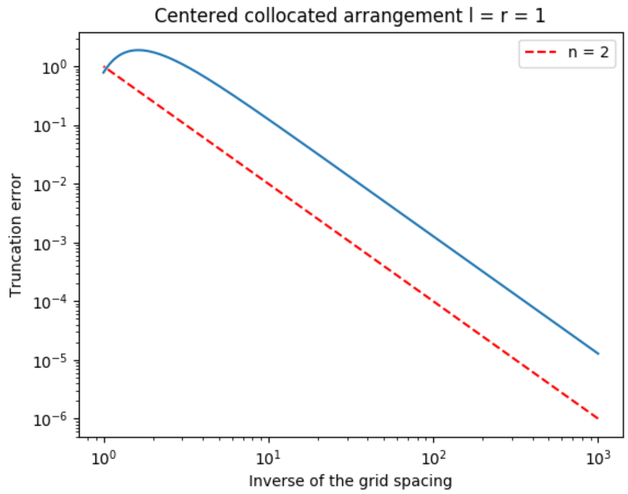
where $f'(x) = \sec^2(x)\sin(1.5+5x) + 5\cos(1.5+5x)\tanh(x)$.

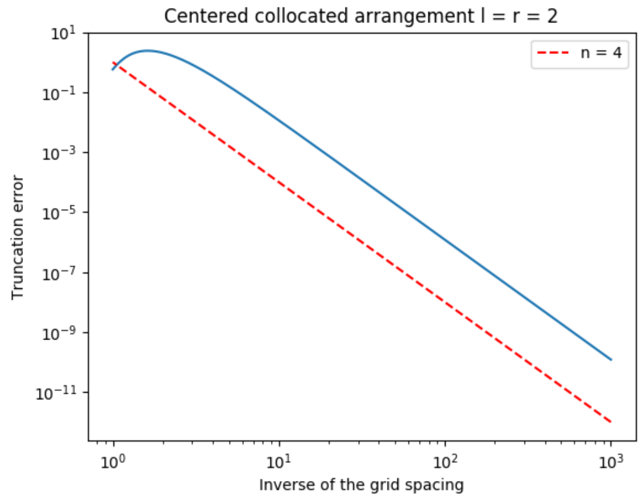
It is not allowed to directly use np.~~vande~~ Vander since that would generate the same polynomial as Lagrangian.

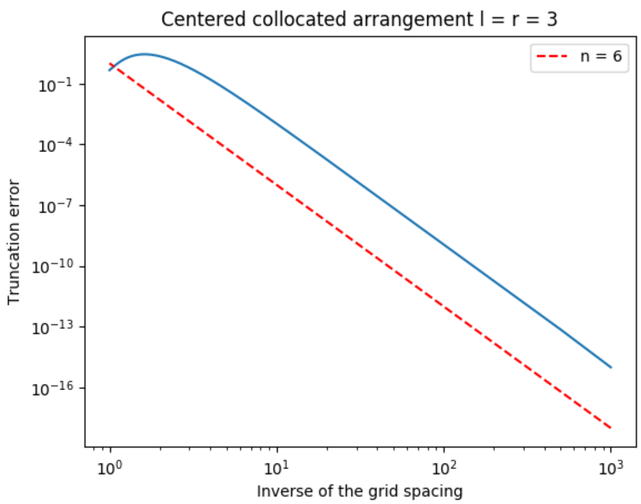
So hand calculated is required.

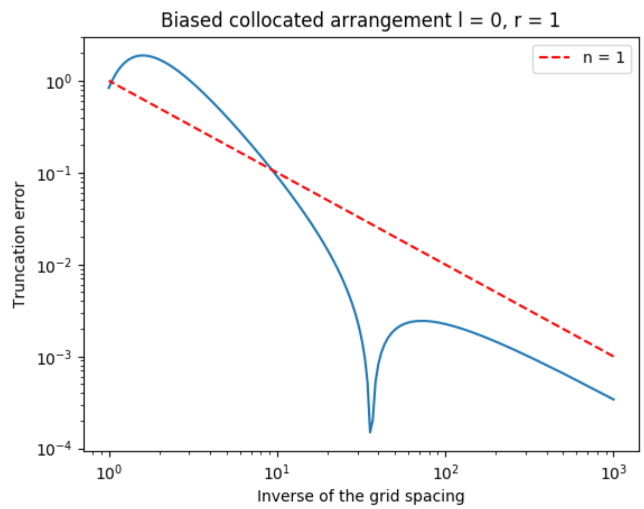

Lagrangian polynomial for N=3
y
x
0.1
0.0
−0.1
−0.2
−1.00
−0.75
−0.50
−0.25
0.00
0.25
0.50
0.75
1.00

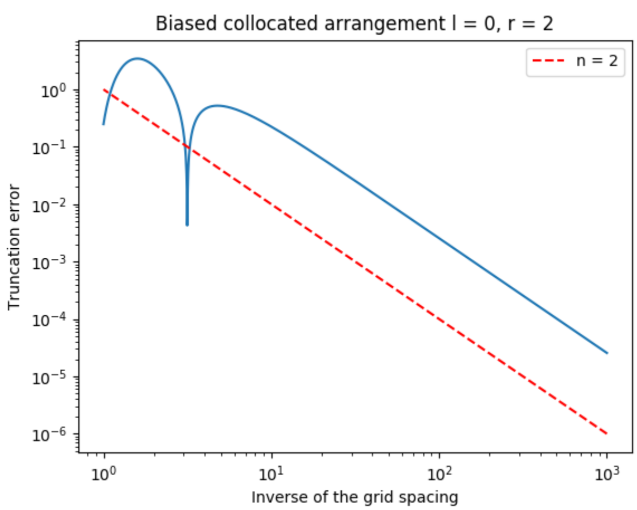

Lagrangian polynomial for N=7
y
x
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−3
−2
−1
0
1
2
3

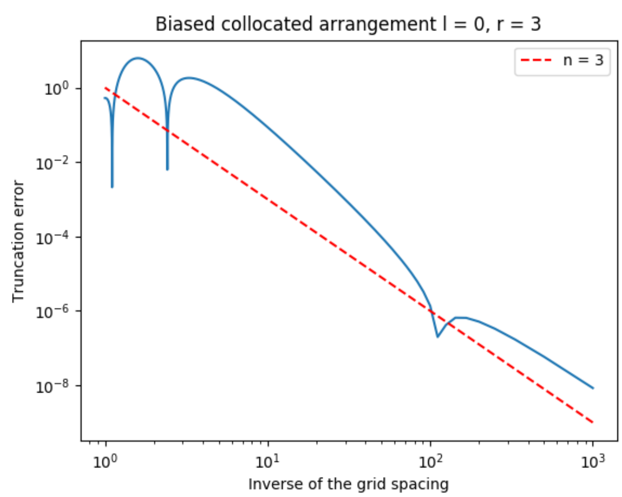

Centered collocated arrangement l = r = 1
n = 2
Truncation error
Inverse of the grid spacing

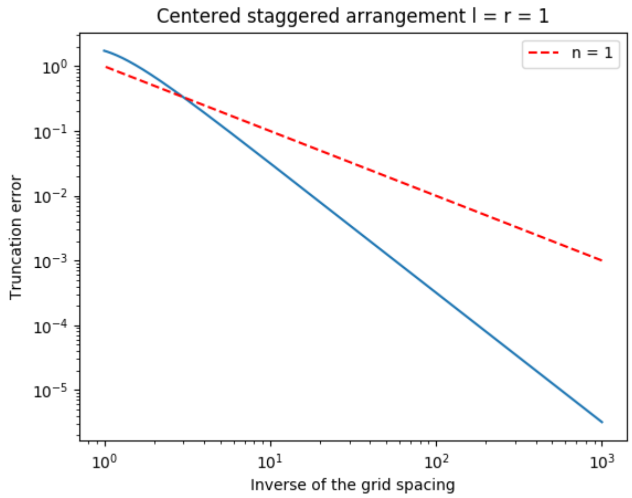

Centered collocated arrangement l = r = 2
n = 4
Truncation error
Inverse of the grid spacing

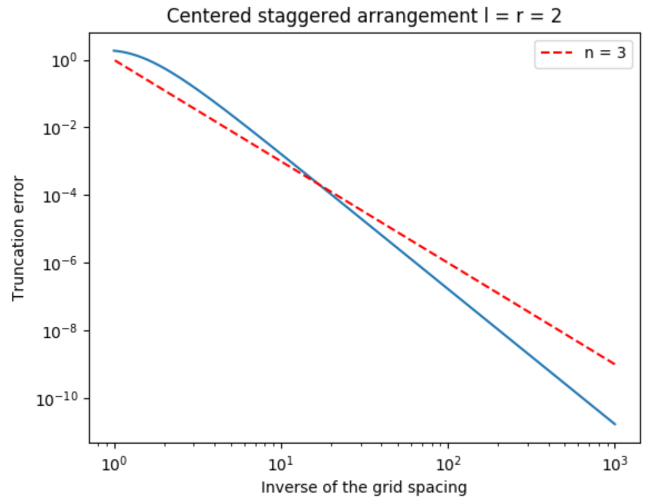

Centered collocated arrangement l = r = 3
n = 6
Truncation error
Inverse of the grid spacing

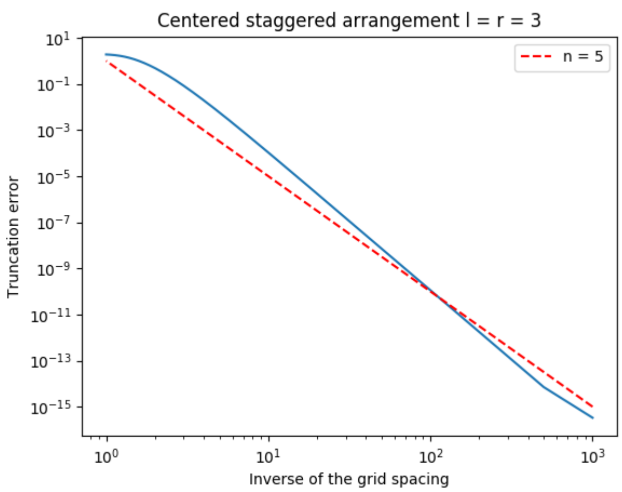

Biased collocated arrangement l = 0, r = 1
n = 1
Truncation error
Inverse of the grid spacing

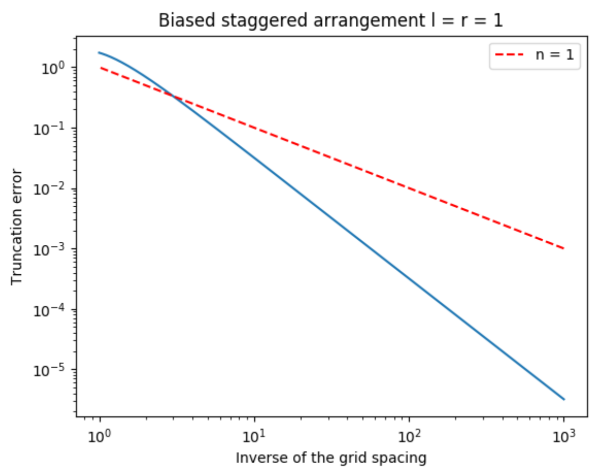

Biased collocated arrangement l = 0, r = 2
n = 2
Truncation error
Inverse of the grid spacing

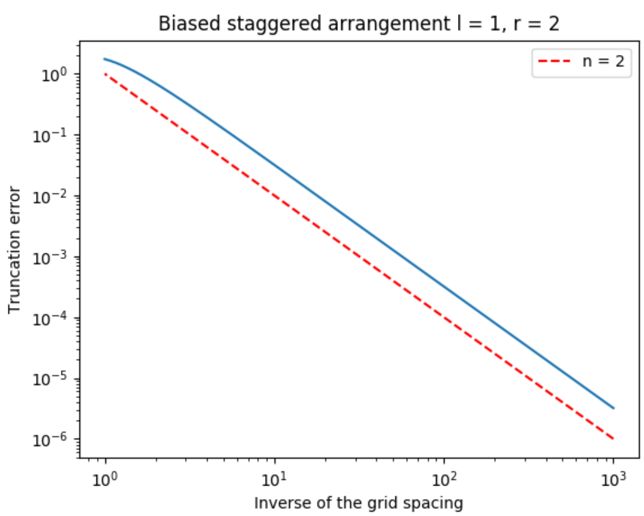

Biased collocated arrangement l = 0, r = 3
n = 3
Truncation error
Inverse of the grid spacing

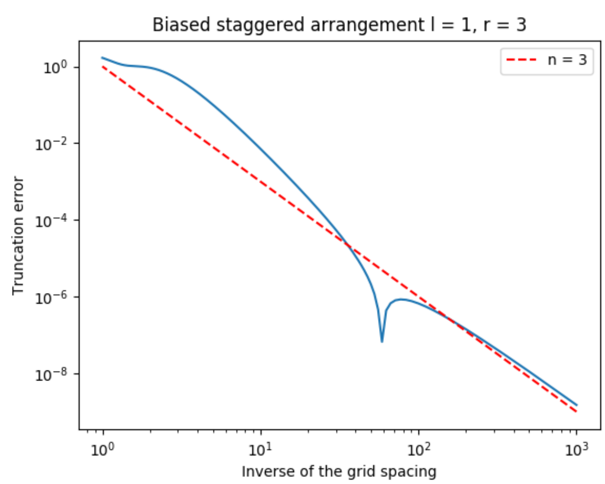

Centered staggered arrangement l = r = 1
n = 1
Truncation error
Inverse of the grid spacing


Centered staggered arrangement l = r = 2
n = 3
Truncation error
Inverse of the grid spacing


Centered staggered arrangement l = r = 3
n = 5
Truncation error
Inverse of the grid spacing


Biased staggered arrangement l = r = 1
n = 1
Truncation error
Inverse of the grid spacing


Biased staggered arrangement l = 1, r = 2
n = 2
Truncation error
Inverse of the grid spacing


Biased staggered arrangement l = 1, r = 3
n = 3
Truncation error
Inverse of the grid spacing

Problem 2.

a) For first derivative, second order,

$$\alpha \; < u_i = u_i >$$
$$\beta \; < u_{i+1} = u_i + \frac{du}{dx}\Big|_i \Delta x + \frac{d^2u}{dx^2}\Big|_i \frac{\Delta x^2}{2} + \mathcal{O}(\Delta x^3) >$$
$$\gamma \; < u_{i+2} = u_i + \frac{du}{dx}\Big|_i (2\Delta x) + \frac{d^2u}{dx^2}\Big|_i \frac{(2\Delta x)^2}{2} + \mathcal{O}(\Delta x^3). >$$

$$\hat{\frac{du}{dx}}\Big|_i = \frac{\alpha u_i + \beta u_{i+1} + \gamma u_{i+2}}{\Delta x} - \underbrace{(\alpha+\beta+\gamma)}_{0}\frac{u_i}{\Delta x} + \underbrace{(\beta+2\gamma)}_{1}\frac{du}{dx}\Big|_i + \underbrace{(\beta+4\gamma)}_{0}\frac{d^2u}{dx^2}\frac{\Delta x}{2} + \mathcal{O}(\Delta x^2).$$

$$\Longrightarrow \hat{\frac{du}{dx}}\Big|_i = \frac{-\frac{3}{2}u_i + 2u_{i+1} - \frac{1}{2}u_{i+2}}{\Delta x}$$

Apply the same rule for central difference,

$$\Longrightarrow \hat{\frac{du}{dx}}\Big|_i = \frac{-\frac{1}{2}u_{i-1} + \frac{1}{2}u_{i+1}}{\Delta x}$$

Apply the same rule for backward difference.

$$\Longrightarrow \hat{\frac{du}{dx}}\Big|_i = \frac{\frac{1}{2}u_{i-2} - 2u_{i-1} + \frac{3}{2}u_i}{\Delta x}$$

Numerical operator for 1st deriv. 2nd order,

$$\frac{1}{\Delta x}\begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} & & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & & \ddots & 0 & 0 \\ & & \ddots & & & \ddots & \\ 0 & 0 & & \ddots & & & 0 \\ & & \ddots & & \ddots & & \\ 0 & 0 & 0 & & -\frac{1}{2} & 0 & \frac{1}{2} \\ & & \ddots & & & & \\ 0 & 0 & 0 & & \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}$$

For first derivative, third order.

$$\alpha < u_i = u_i >$$
$$\beta < u_{i+1} = u_i + \frac{du}{dx}\Big|_i \Delta x + \frac{d^2u}{dx^2}\Big|_i \frac{\Delta x^2}{2} + \frac{d^3u}{dx^3}\Big|_i \frac{\Delta x^3}{6} + \mathcal{O}(\Delta x^4) >$$
$$\gamma < u_{i+2} = u_i + \frac{du}{dx}\Big|_i (2\Delta x) + \frac{d^2u}{dx^2}\Big|_i \frac{(2\Delta x)^2}{2} + \frac{d^3u}{dx^3}\Big|_i \frac{(2\Delta x)^3}{6} + \mathcal{O}(\Delta x^4) >.$$
$$\varepsilon < u_{i+3} = u_i + \frac{du}{dx}\Big|_i (3\Delta x) + \frac{d^2u}{dx^2}\Big|_i \frac{(3\Delta x)^2}{2} + \frac{d^3u}{dx^3}\Big|_i \frac{(3\Delta x)^3}{6} + \mathcal{O}(\Delta x^4) >.$$

$$\left.\frac{\widehat{d u}}{dx}\right|_i = \frac{\alpha u_i + \beta u_{i+1} + \gamma u_{i+2} + \varepsilon u_{i+3}}{\Delta x}$$

$$= \underbrace{(\alpha+\beta+\gamma+\varepsilon)}_{0}\frac{u_i}{\Delta x} + \underbrace{(\beta+2\gamma+3\varepsilon)}_{1}\left.\frac{du}{dx}\right|_i + \underbrace{(\beta+4\gamma+9\varepsilon)}_{0}\frac{d^2u}{dx^2}\frac{\Delta x}{2}$$

$$+\underbrace{(8\gamma+27\varepsilon)}_{0}\frac{d^3u}{dx^3}\frac{(\Delta x)^3}{6} + O(\Delta x^3).$$

$$\begin{cases} \alpha+\beta+\gamma+\varepsilon=0 \\ \beta+2\gamma+3\varepsilon=1 \\ \beta+4\gamma+9\varepsilon=0 \\ 8\gamma+27\varepsilon=0. \end{cases} \Rightarrow \left| \begin{array}{l} \alpha = -11/6 \\ \beta = 3 \\ \gamma = -3/2 \\ \varepsilon = 1/3 \end{array} \right. \Rightarrow \left.\frac{\widehat{du}}{dx}\right|_i = \frac{-\frac{11}{6}u_i + 3u_{i+1} - \frac{3}{2}u_{i+2} + \frac{1}{3}u_{i+3}}{\Delta x}.$$

follow the same rule for backward scheme.

$$\left.\frac{\widehat{du}}{dx}\right|_i = \frac{-\frac{1}{3}u_{i-3} + \frac{3}{2}u_{i-2} + (-3)u_{i-1} + \frac{11}{6}u_i}{\Delta x}$$

for 1 point left & 2 point right,

$$\left.\frac{\widehat{du}}{dx}\right|_i = \frac{-\frac{1}{3}u_{i-1} - \frac{1}{2}u_i + u_{i+1} - \frac{1}{6}u_{i+2}}{\Delta x}$$

for 2 point left & 1 point right,

$$\left.\frac{\widehat{du}}{dx}\right|_i = \frac{\frac{1}{6}u_{i-2} - u_{i-1} + \frac{1}{2}u_i + \frac{1}{3}u_{i+1}}{\Delta x}$$

Numerical operator

$$\frac{1}{\Delta x}\begin{bmatrix} -11/6 & 3 & -3/2 & 1/3 & & & & \\ & & & & \ddots & & & \\ & & & & -11/6 & 3 & -3/2 & 1/3 \\ & & & & -1/3 & -1/2 & 1 & -1/6 \\ & & & & -1/3 & 3/2 & -3 & 11/6 \end{bmatrix}$$

2.2

For second derivative, second order

$\alpha < u_i = u_i >$

$\beta < u_{i+1} = u_i + \frac{du}{dx}\Big|_i \Delta x + \frac{d^2u}{dx^2}\Big|_i \frac{\Delta x^2}{2} + O(\Delta x)^3 >$

$\gamma < u_{i+2} = u_i + \frac{du}{dx}\Big|_i (\Delta x \cdot 2) + \frac{d^2u}{dx^2}\Big|_i \frac{(2\Delta x)^2}{2} + O(\Delta x^3)>.$

$\varepsilon < u_{i+3} = u_i + \frac{du}{dx}\Big|_i (3\Delta x) + \frac{d^2u}{dx^2}\Big|_i \frac{(3\Delta x)^2}{2} + O(\Delta x^3)>.$

$$\frac{d^2\hat{u}}{dx^2}\Big|_i = \frac{\alpha u_i + \beta u_{i+1} + \gamma u_{i+2}}{\Delta x^2} = (\alpha+\beta+\gamma \underset{+\varepsilon}{)} \frac{u_i}{\Delta x^2} + (\beta + 2\gamma \underset{+3\varepsilon}{)} \frac{du}{dx}\Big|_i \frac{1}{\Delta x} + (\beta + 4\gamma \underset{+9\varepsilon}{)} \frac{d^2u}{dx^2} \frac{1}{2} + O(\Delta x^2).$$

$$\Rightarrow \quad \frac{d^2\hat{u}}{dx^2}\Big|_i = \frac{2u_i - 5u_{i+1} + 4u_{i+2} - u_{i+3}}{\Delta x^2}$$

Apply the same rule for central difference.

$$\frac{d^2\hat{u}}{dx^2}\Big|_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2}$$

Apply the same rule for backward difference.

$$\frac{d^2\hat{u}}{dx^2}\Big|_i = \frac{-u_{i-3} + 4u_{i-2} - 5u_{i-1} + 2u_i}{\Delta x^2}$$

Numerical operator:

$$\frac{1}{(\Delta x)^2}\begin{bmatrix} 2 & -5 & 4 & -1 & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & & 1 & -2 & 1 \\ & & & -1 & 4 & -5 & 2 \end{bmatrix}$$

For second derivative, third order,

$\alpha < u_i = u_i >$

$\beta < u_{i+1} = u_i + \frac{du}{dx}\Big|_i \Delta x + \frac{d^2u}{dx^2}\Big|_i \frac{\Delta x^2}{2} + \frac{d^3u}{dx^3}\Big|_i \frac{\Delta x^3}{6} + \frac{d^4u}{dx^4}\Big|_i \frac{\Delta x^4}{24} + O(\Delta x^5)) >$

$\gamma < (u_{i+2} = u_i + \frac{du}{dx}\Big|_i (2\Delta x) + \frac{d^2u}{dx^2}\Big|_i \frac{(2\Delta x)^2}{2} + \frac{d^3u}{dx^3}\Big|_i \frac{(2\Delta x)^3}{6} + \frac{d^4u}{dx^4}\Big|_i \frac{(2\Delta x)^4}{24} + O(\Delta x^5)) >.$

$\varepsilon < (u_{i+3} = u_i + \frac{du}{dx}\Big|_i (3\Delta x) + \frac{d^2u}{dx^2}\Big|_i \frac{(3\Delta x)^2}{2} + \frac{d^3u}{dx^3}\Big|_i \frac{(3\Delta x)^3}{6} + \frac{d^4u}{dx^4}\Big|_i \frac{(3\Delta x)^4}{24} + O(\Delta x^5) >.$

$K < (u_{i+4}) = u_i + \frac{du}{dx}\Big|_i (4\Delta x) + \frac{d^2u}{dx^2}\Big|_i \frac{(4\Delta x)^2}{2} + \frac{d^3u}{dx^3}\Big|_i \frac{(4\Delta x)^3}{6} + \frac{d^4u}{dx^4} \frac{(4\Delta x)^4}{24} + O(\Delta x^5)) >.$

$$\Rightarrow \left.\frac{d^2u}{dx^2}\right|_i = \frac{1}{(\Delta x)^2}\left[\frac{35}{12}u_i - \frac{26}{3}u_{i+1} + \frac{19}{2}u_{i+2} - \frac{14}{3}u_{i+3} + \frac{11}{12}u_{i+4}\right].$$

Apply the same rule for 1 point left 3 point right.

$$\left.\frac{\widehat{d^2u}}{dx^2}\right|_i = \frac{1}{(\Delta x)^2}\left[\frac{11}{12}u_{i-1} - \frac{20}{12}u_i + \frac{6}{12}u_{i+1} + \frac{4}{12}u_{i+2} - \frac{1}{12}u_{i+3}\right].$$

Apply the same rule for 2 point left 2 point right.

$$\left.\frac{\widehat{d^2u}}{dx^2}\right|_i = \frac{1}{(\Delta x)^2}\left[-\frac{1}{12}u_{i-2} + \frac{16}{12}u_{i-1} - \frac{30}{12}u_i + \frac{16}{12}u_{i+1} - \frac{1}{12}u_{i+2}\right].$$

Apply the same rule for 3 point left 1 point right.

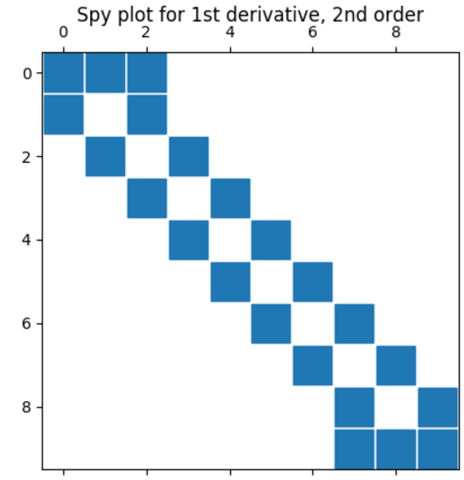
$$\left.\frac{\widehat{d^2u}}{dx^2}\right|_i = \frac{1}{(\Delta x)^2}\left[-\frac{1}{12}u_{i-3} + \frac{4}{12}u_{i-2} + \frac{6}{12}u_{i-1} - \frac{20}{12}u_i + \frac{11}{12}u_{i+1}\right].$$

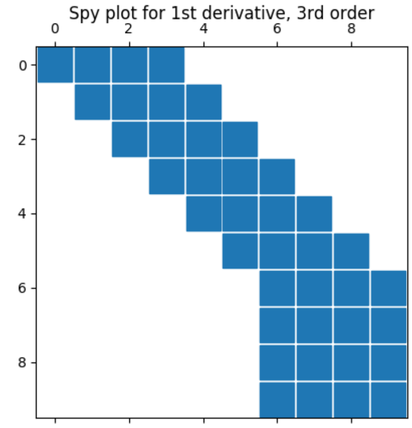
Apply the same rule for backward scheme.

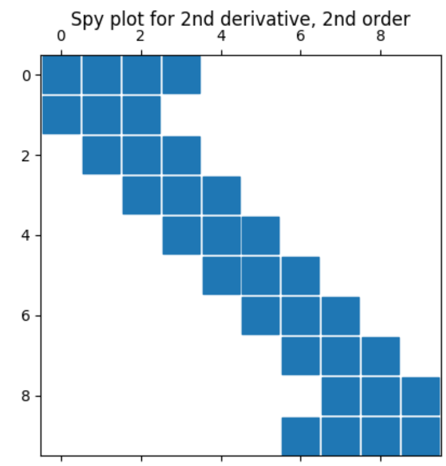
$$\left.\frac{\widehat{d^2u}}{dx^2}\right|_i = \frac{1}{(\Delta x)^2}\left[\frac{11}{12}u_{i-4} - \frac{56}{12}u_{i-3} + \frac{114}{12}u_{i-2} - \frac{104}{12}u_{i-1} + \frac{35}{12}u_i\right].$$

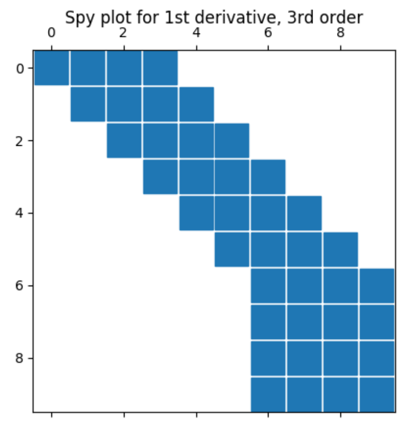
Numerical operator:

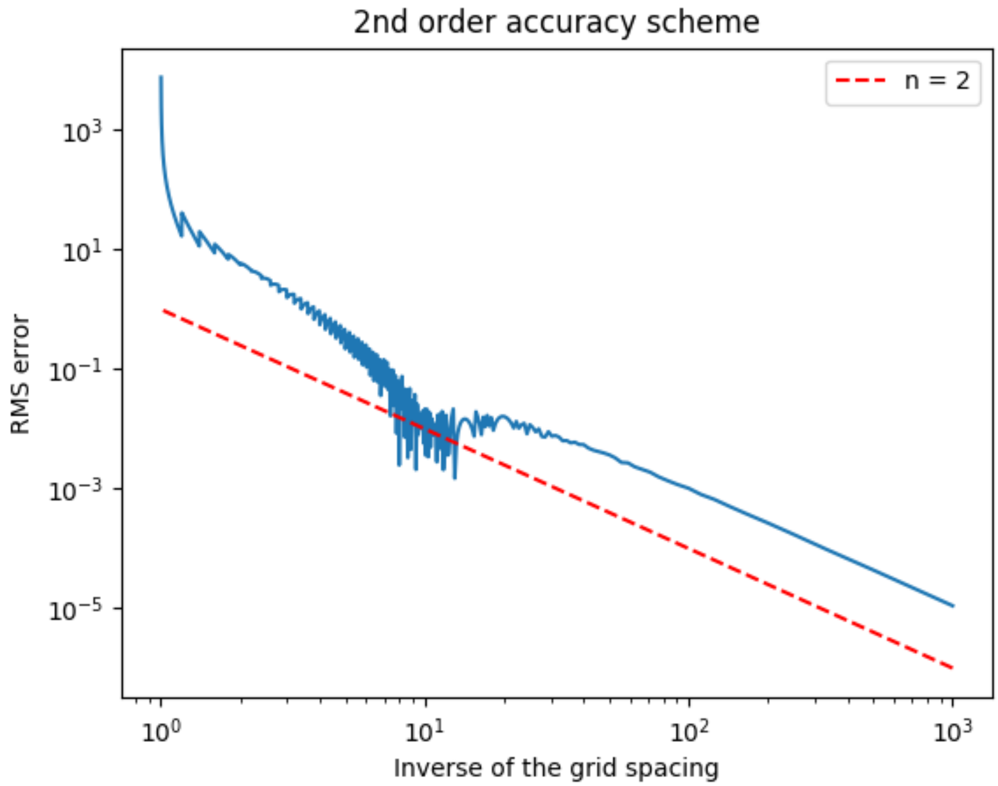
$$\frac{1}{(\Delta x)^2}\begin{bmatrix} \frac{35}{12} & -\frac{26}{3} & \frac{19}{2} & \frac{-14}{3} & \frac{11}{12} & & & & \\ & & & & & \ddots & & & \\ & & & & \frac{11}{12} & -\frac{20}{12} & \frac{6}{12} & \frac{4}{12} & \frac{-1}{12} \\ & & & & \frac{-1}{12} & \frac{16}{12} & \frac{-30}{12} & \frac{16}{12} & \frac{-1}{12} \\ & & & & \frac{-1}{12} & \frac{4}{12} & \frac{6}{12} & \frac{-20}{12} & \frac{11}{12} \\ & & & & \frac{11}{12} & \frac{-56}{12} & \frac{114}{12} & \frac{-104}{12} & \frac{35}{12} \end{bmatrix}$$

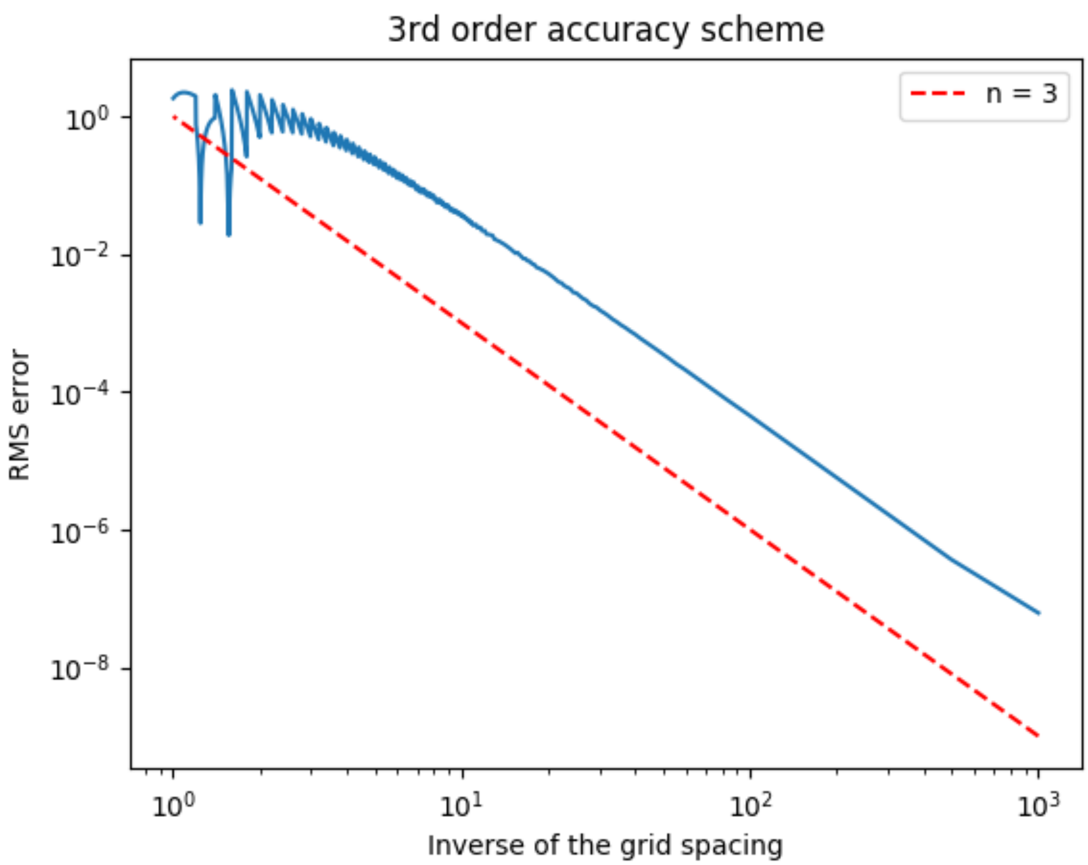

Spy plot for 1st derivative, 2nd order
0
2
4
6
8


Spy plot for 1st derivative, 3rd order
0
2
4
6
8


Spy plot for 2nd derivative, 2nd order
0
2
4
6
8


Spy plot for 1st derivative, 3rd order
0
2
4
6
8


2nd order accuracy scheme
n = 2
RMS error
Inverse of the grid spacing
10^3
10^1
10^{-1}
10^{-3}
10^{-5}
10^0
10^1
10^2
10^3


3rd order accuracy scheme
n = 3
RMS error
Inverse of the grid spacing
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^0
10^1
10^2
10^3

Problem 3.

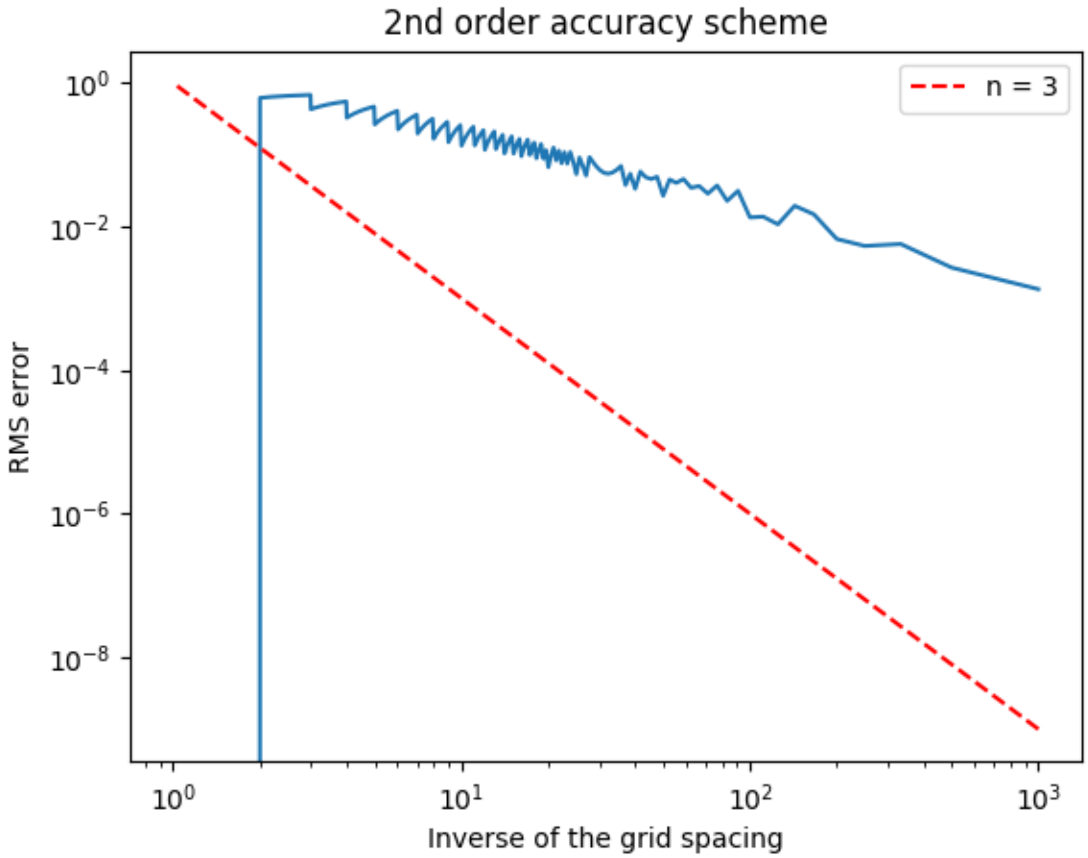
$\hat{x}_i = x_i + \bar{\Delta x}\,\mathcal{N}(0,\sigma)\ \forall i \in \{0,\cdots N-1\}$. poly. = x^3.

• Normal distribution with different standard deviation might be able to influence the RMS value.

With a higher std. RMS will increase.

plots of 2nd order accuracy shown in the next page.

Since the boundary nodes are not under consideration, so the trend won't perfectly fit the line of -2 order of magnitude.


2nd order accuracy scheme
n = 3
RMS error
Inverse of the grid spacing
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^1
10^2
10^3