HW2 ME 614 Gu, Xinyuan

Problem 1. See attached figures in Appendix

Problem 2. $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} = 0$ periodic.

a) Prove that $u(x,t) = C_1 e^{-\omega_1^2 \alpha t} \sin\left[\omega_1(x-ct)-\gamma_1\right] - C_2 e^{-\omega_2^2 \alpha t}$.

$\cos\left[\omega_2(x-ct)-\gamma_2\right]$

$$\frac{\partial u}{\partial t} = C_1 e^{-\omega_1^2 \alpha t}\left\{\sin\left[\omega_1(ct-x)+\gamma_1\right] - c\omega_1\cos\left[c(t-x)+\gamma_1\right]\right\} - C_2 e^{-\omega_2^2 \alpha t}$$

$$\left\{\omega_2 c \sin\left[(ct-x)+\gamma_2\right] + \cos\left[\omega_2(ct-x)+\gamma_2\right]\right\}$$

$$\frac{\partial u}{\partial x} = C_1 e^{-\omega_1^2 \alpha t}\cos\left[\omega_1(ct-x)+\gamma_1\right] + C_2 e^{-\omega_2^2 \alpha t}\sin\left[\omega_2(ct-x)+\gamma_2\right]$$

$$\frac{\partial^2 u}{\partial x^2} = -C_1 e^{-\omega_1^2 \alpha t}\sin\left[\omega_1(x-ct)-\gamma_1\right] + C_2 e^{-\omega_2^2 \alpha t}\cos\left[\omega_2(x-ct)-\gamma\right.$$

$$\Rightarrow \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2}$$

$$= C_1 e^{-\omega_1^2 \alpha t}\left[0\right] + C_2 e^{-\omega_2^2 \alpha t}\left[0\right]$$

$$= 0$$

b) initial condition:

$$u(x,0) = c_1 \sin(\omega_1 x - \gamma_1) - c_2 \cos(\omega_2 x - \gamma_2)$$

$\omega_1 = \frac{2\pi}{L}$, $\omega_2 = \frac{2\pi m}{L}$, $T_f = (\omega_2^2 \alpha)^{-1}$

$\gamma_1, \gamma_2, c_1, c_2, m, L$ as non-zero values of your choice

For advection-diffusion eq. $c = \alpha = 1$

$L = 10$ $m = 1$, $c_1 = c_2 = 1$, $\gamma_1 = \gamma_2 = 1$ $\Rightarrow$ $\omega_1 = \omega_2 = 0.2\pi$

$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0$ $\qquad CFL = c\frac{\Delta t}{\Delta x} = \frac{\Delta t}{\Delta x} < 1$

So the analytical solution:

$$u(x,t) = \exp\left(\frac{-4\pi^2 t}{100}\right) \sin\left[0.2\pi(x-t)-1\right] - \exp\left(\frac{-4\pi^2 t}{100}\right) \cos\left[0.2\pi(x-t)-1\right]$$

$$= \exp\left(\frac{-4\pi^2 t}{100}\right)\left[\sin\left[0.2\pi(x-t)-1\right] - \cos\left[0.2\pi(x-t)-1\right]\right]$$

Initial condition: $u(x,0) = \sin(0.2\pi x - 1) - \cos(0.2\pi x - 1)$

Plot at time $T_f = \left(\frac{4\pi^2}{100}\right)^{-1} = 2.533s$ with $N = 10, 25, 50, 100$. with the reference from the analytical solution.

Plots in Appendix

c) For Crank-nicolson, rms error at $T_f$ vs. $N$ (small $\Delta t$)
$\Delta t$ (small $\Delta x$).

For $\Delta x = 0.05$, $\Delta t$ in range of $0.05 \times (0.001 \sim 0.3)$

For $\Delta t = 0.01$ $\quad \Delta x = (0.0[illegible]01 \sim 0.1)$

$N_x = 1000 \sim 1e4$

For rms_error vs. $\Delta t$.

1st-order upwind: almost linear increase with a lots of oscillations $\Rightarrow$ 1st order increase.

2nd-order central:

error is decreasing linearly with lots of oscillations

2nd-order upwind: linearly increased error.

Conclusion: Upwind Scheme provides more accurate results considering temporal terms.

For rms_error vs. $\Delta x$ (grid points).

1st-order upwind: fast decrease & stay horizontal

2nd-order central: fast increase & stay horizontal

2nd-order upwind: fast decrease & stay horizontal

Conclusion: Upwind scheme provides more accurate results considering spatial terms.

d) check spy plots in Appendix

e) $C_c = \frac{c \cdot \Delta t}{\Delta x} = \frac{\Delta t}{\Delta x}$, $C_\alpha = \frac{\alpha \Delta t}{\Delta x^2} = \frac{\Delta t}{\Delta x^2}$

$$\Rightarrow \Delta x = \frac{C_c}{C_\alpha} \qquad \Delta t = \frac{C_c^2}{C_\alpha}$$

then sweep $C_c$ & $C_\alpha$ to get $dt$, $dx$

$\Rightarrow$ find matrix $T$ $\Rightarrow$ Max.( eigenvalue) $\Rightarrow$ Contour

Bug: Since $dt$ & $Dx$ are correlated & the xx linspace required integers, I haven't find a proper way to sweep $C_c$ & $C_\alpha$ and get integer for $1/dx$ at the same time.

Problem 3.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} = \beta\frac{\partial^3 u}{\partial x^3}$$

$u(0,t) = a$ & $\left.\frac{du}{dx}\right|_{x=L} = 0$

a) $c$ is convection speed; with higher $c$, the flow moves faster in one direction.

$\alpha$ is diffusion term. with higher $\alpha$, the magnitude of the flow decays faster.

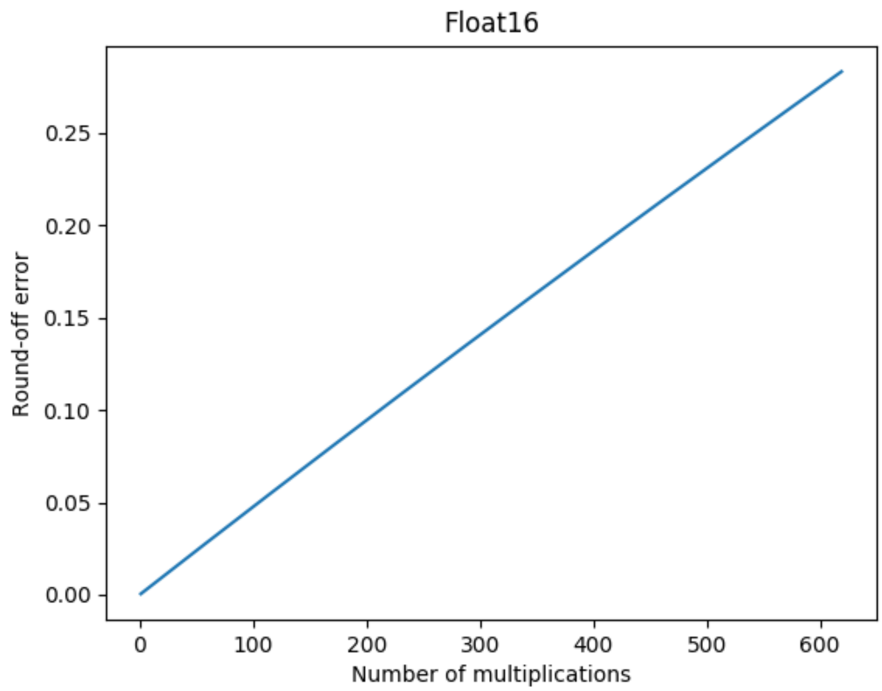
$\beta$ is the dispersion coeff, with higher $\beta$, more flow is dissipated into solid surface.

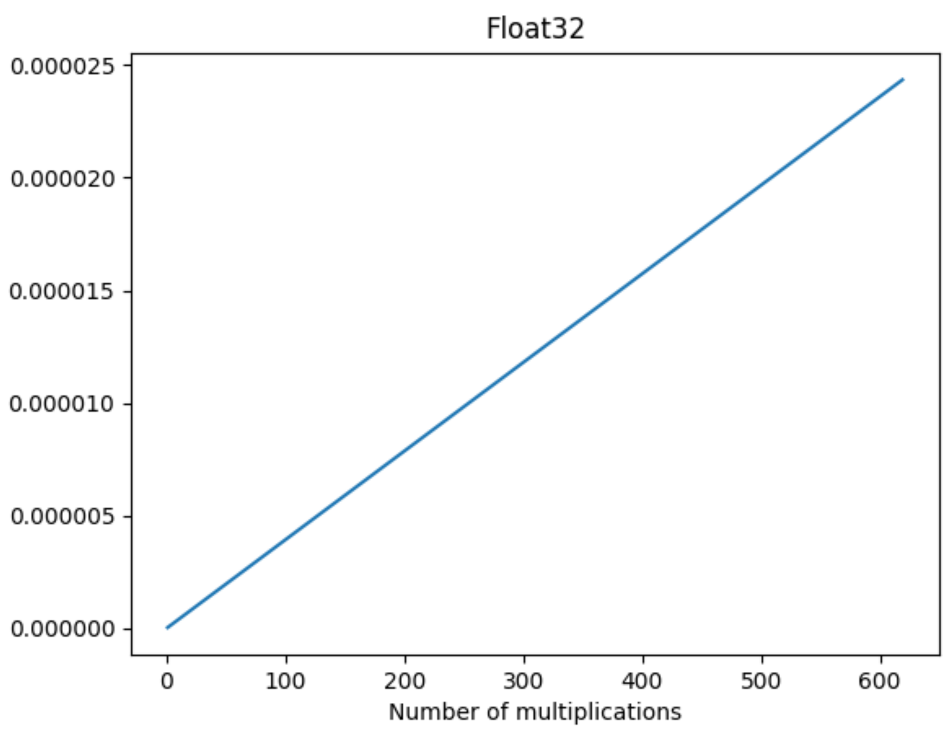
$a$ is the boundary condition value. with higher $a$, more flow coming from the boundary.

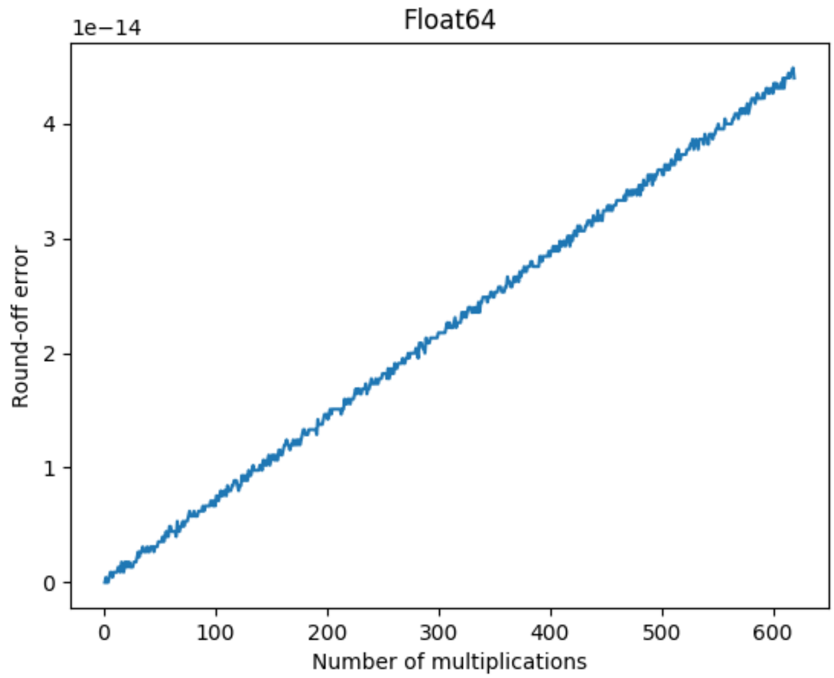
b) $CFL_\beta = \frac{\Delta t \cdot \beta}{\Delta x^3} \leq 1$

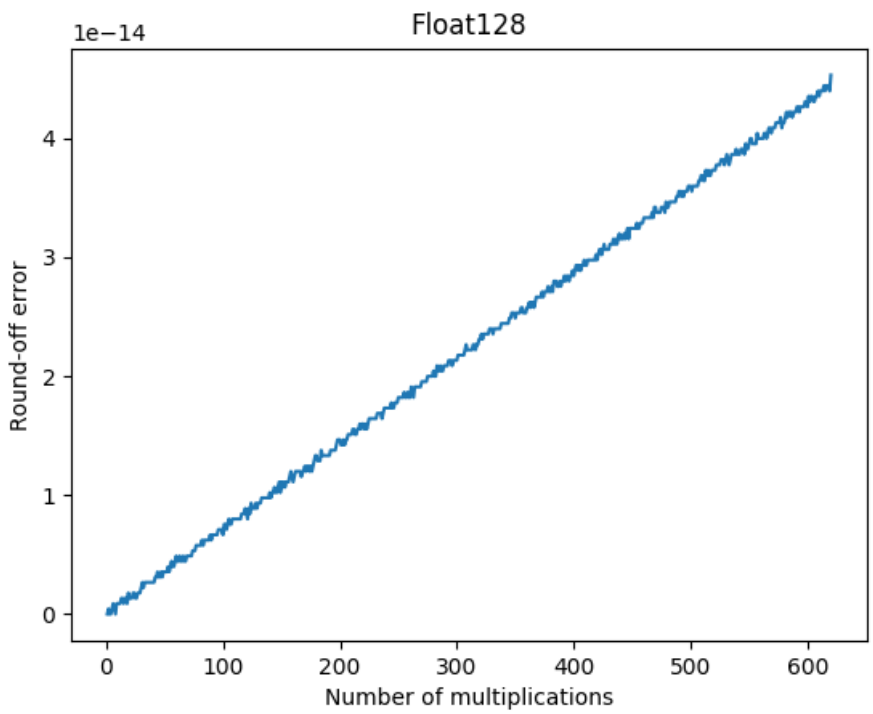
# Appendix

Problem 1




Float64
1e−14
Round-off error
Number of multiplications
0
1
2
3
4
0
100
200
300
400
500
600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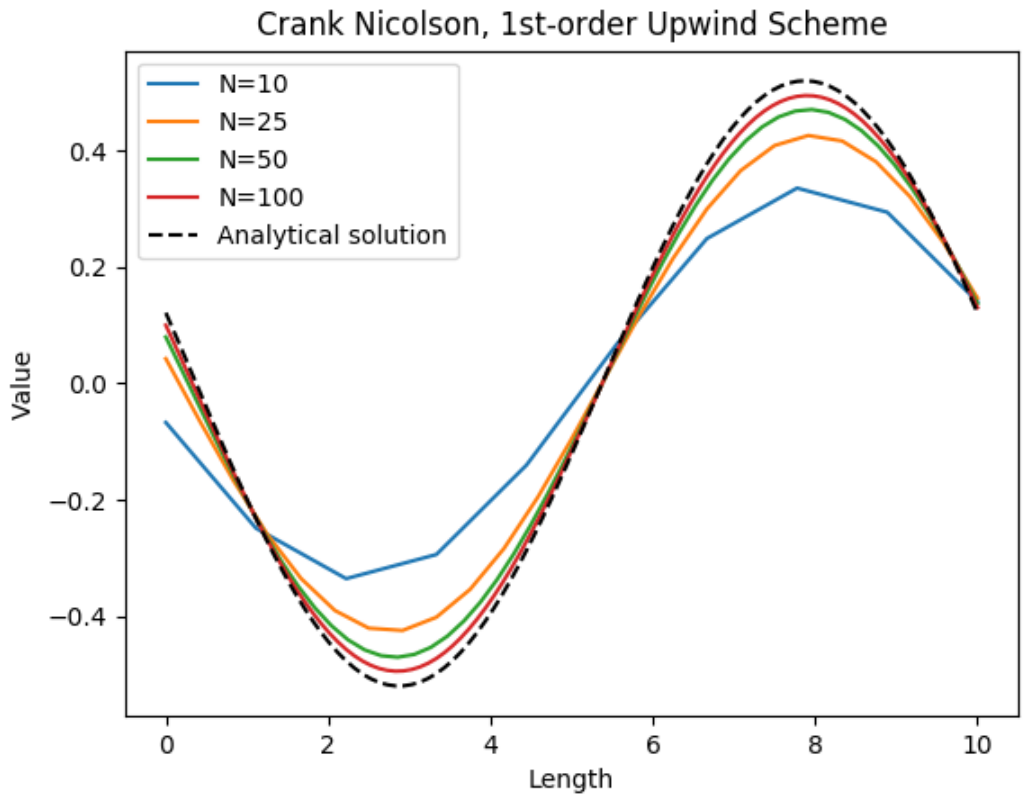
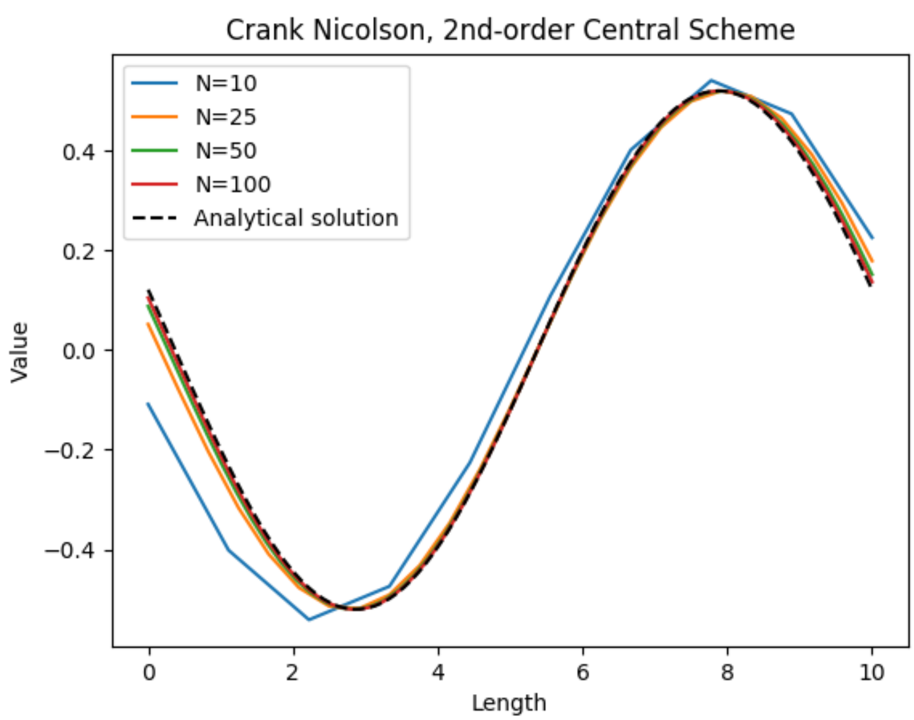
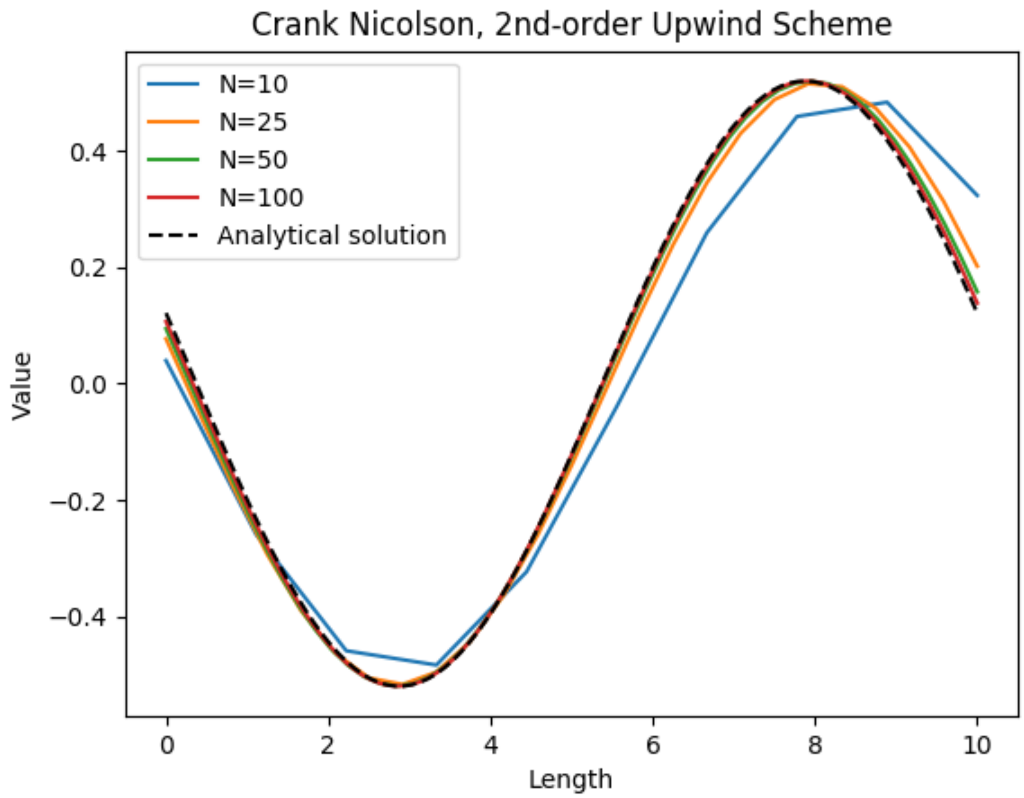
Round-off error
Number of multiplications
0
1
2
3
4
0
100
200
300
400
500
600

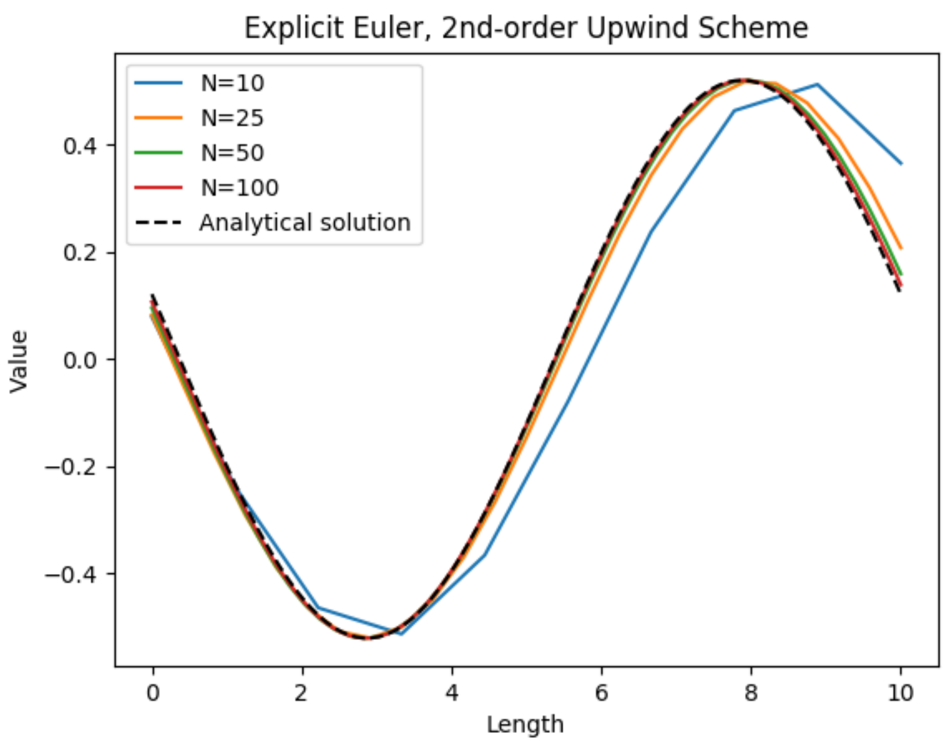
Problem 2 b


Crank Nicolson, 1st-order Upwind Scheme
N=10
N=25
N=50
N=100
Analytical solution
Value
Length
0.4
0.2
0.0
−0.2
−0.4
0
2
4
6
8
10

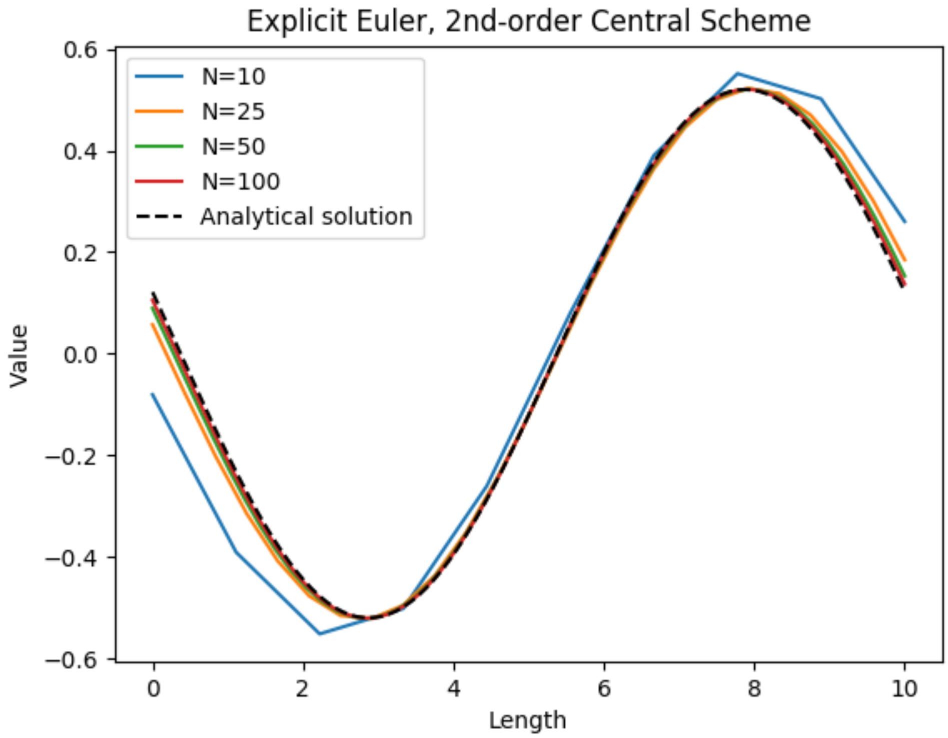

Crank Nicolson, 2nd-order Central Scheme
N=10
N=25
N=50
N=100
Analytical solution
Value
Length
0.4
0.2
0.0
−0.2
−0.4
0
2
4
6
8
10

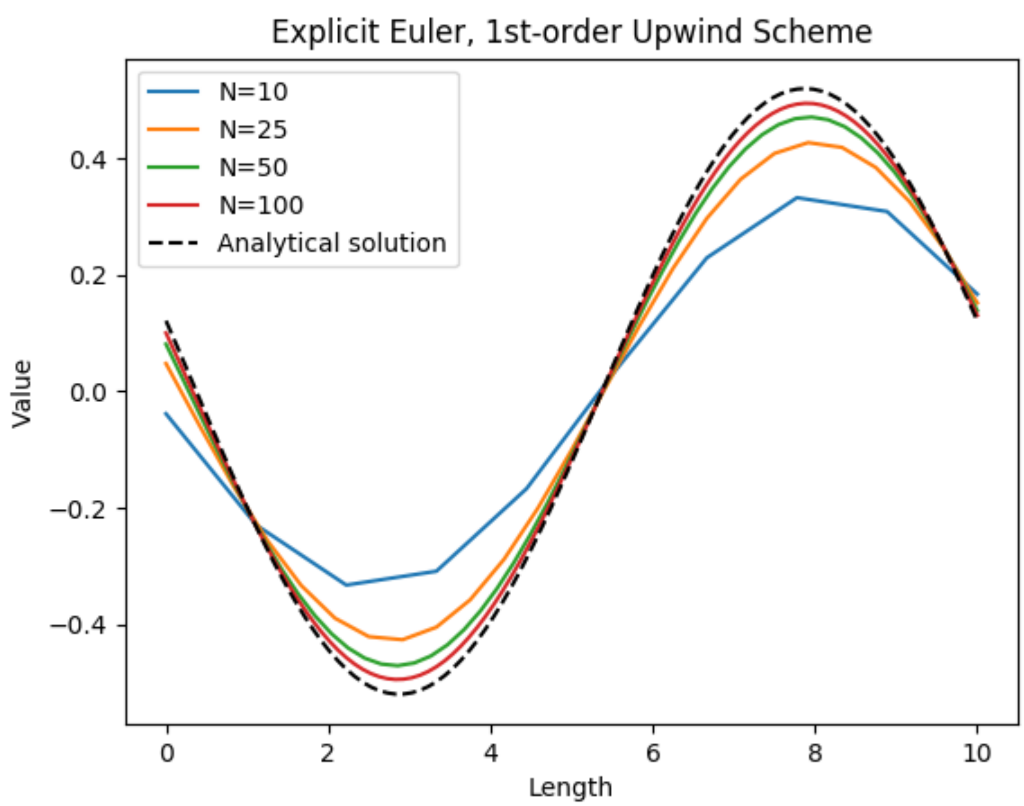


Crank Nicolson, 2nd-order Upwind Scheme
N=10
N=25
N=50
N=100
Analytical solution
Value
Length

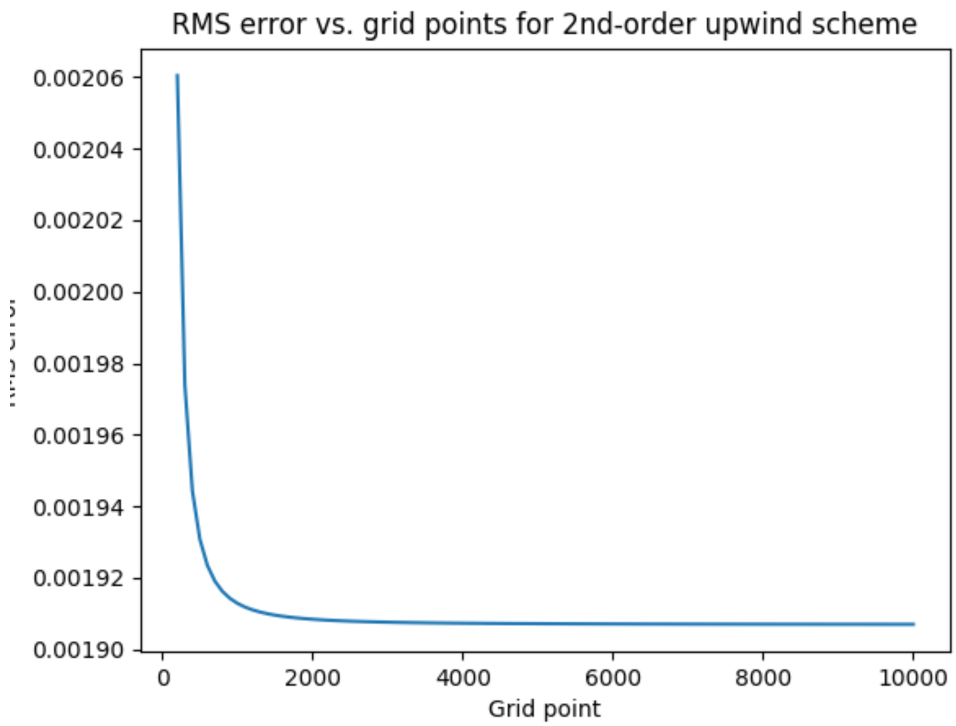



Explicit Euler, 2nd-order Upwind Scheme
N=10
N=25
N=50
N=100
Analytical solution
Value
Length

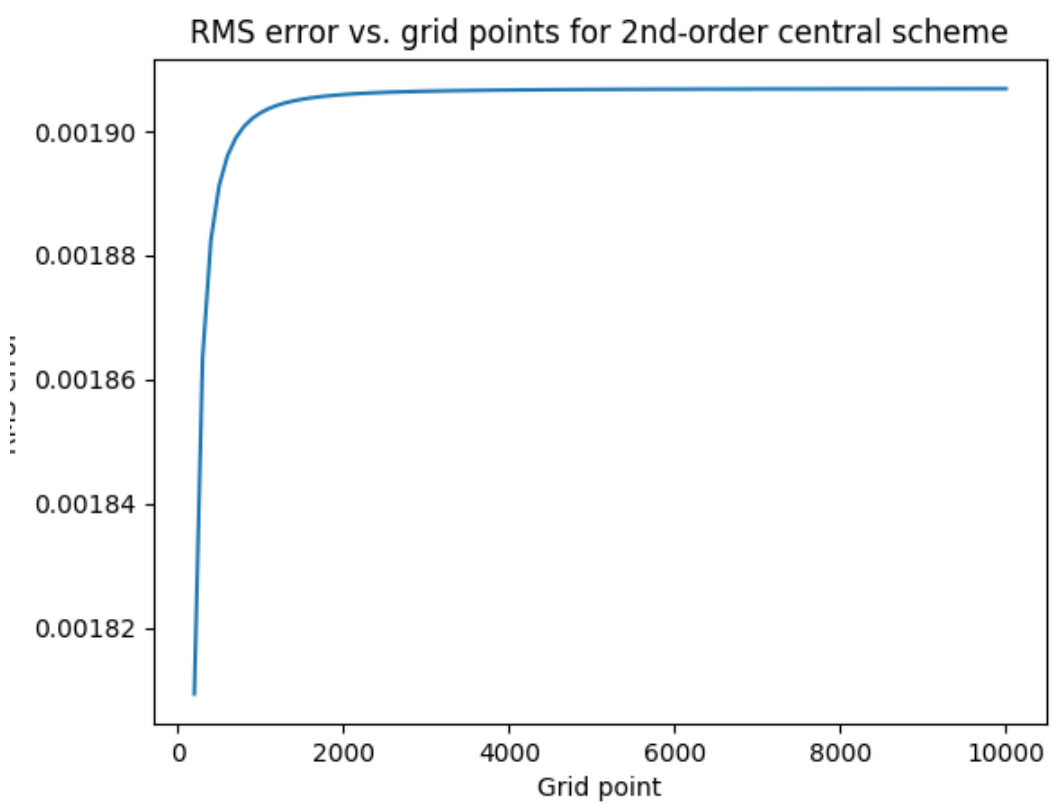

Explicit Euler, 2nd-order Central Scheme
N=10
N=25
N=50
N=100
Analytical solution
Value
Length
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0
2
4
6
8
10


Explicit Euler, 1st-order Upwind Scheme
N=10
N=25
N=50
N=100
Analytical solution
Value
Length
0.4
0.2
0.0
−0.2
−0.4
0
2
4
6
8
10

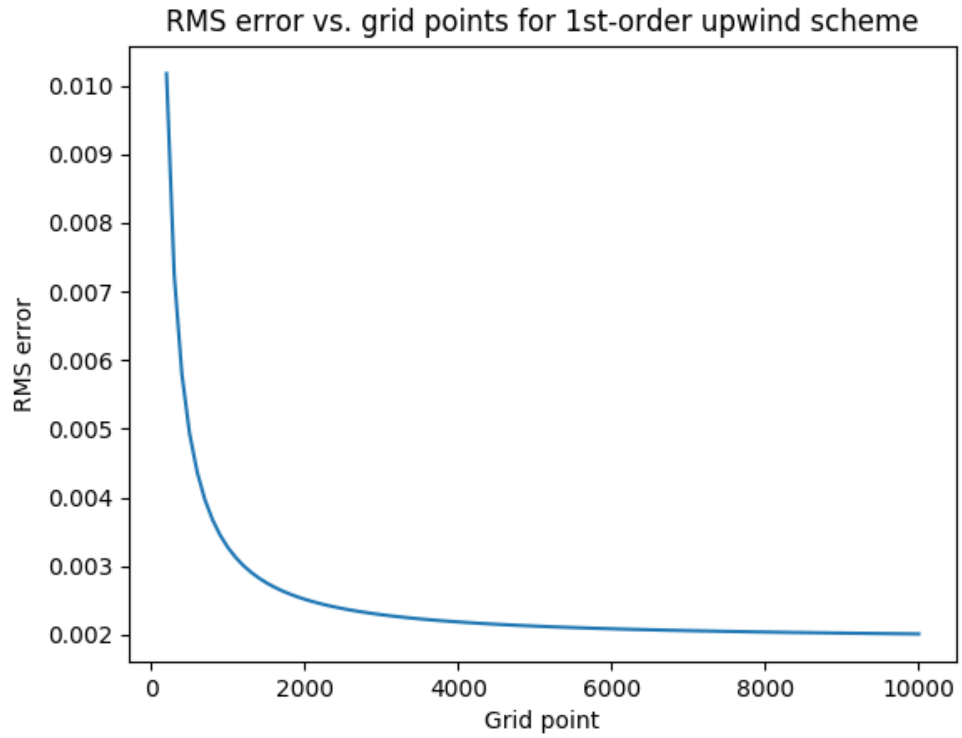
Problem 2 c

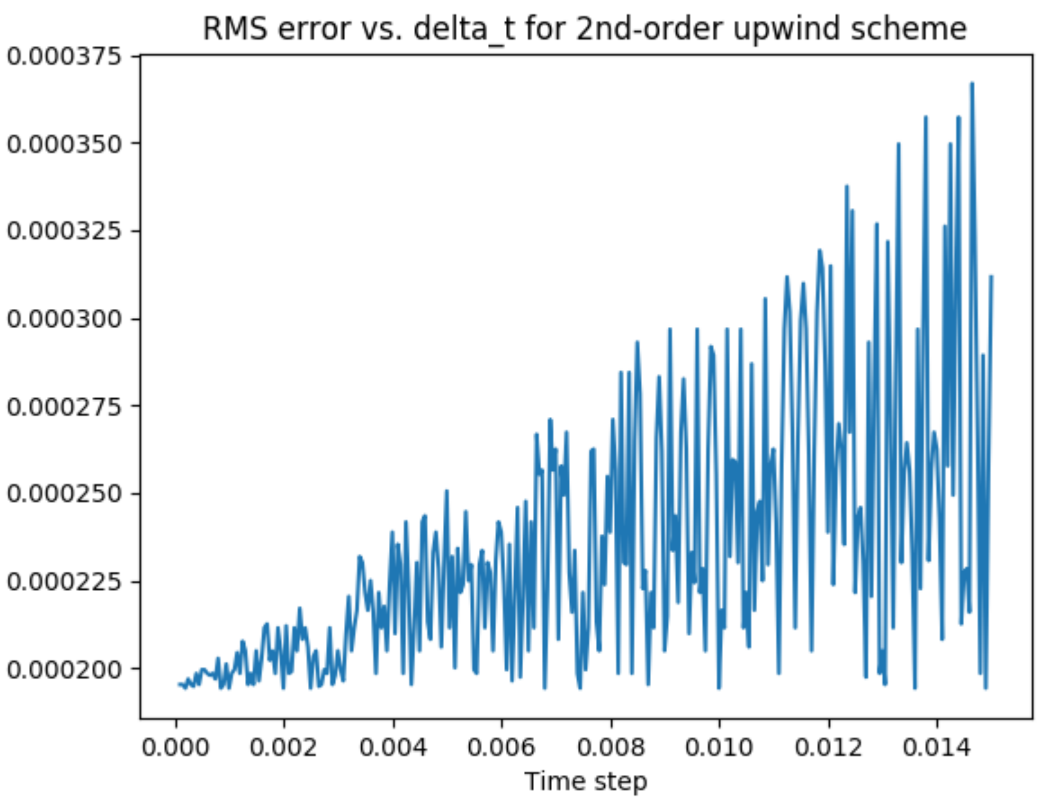


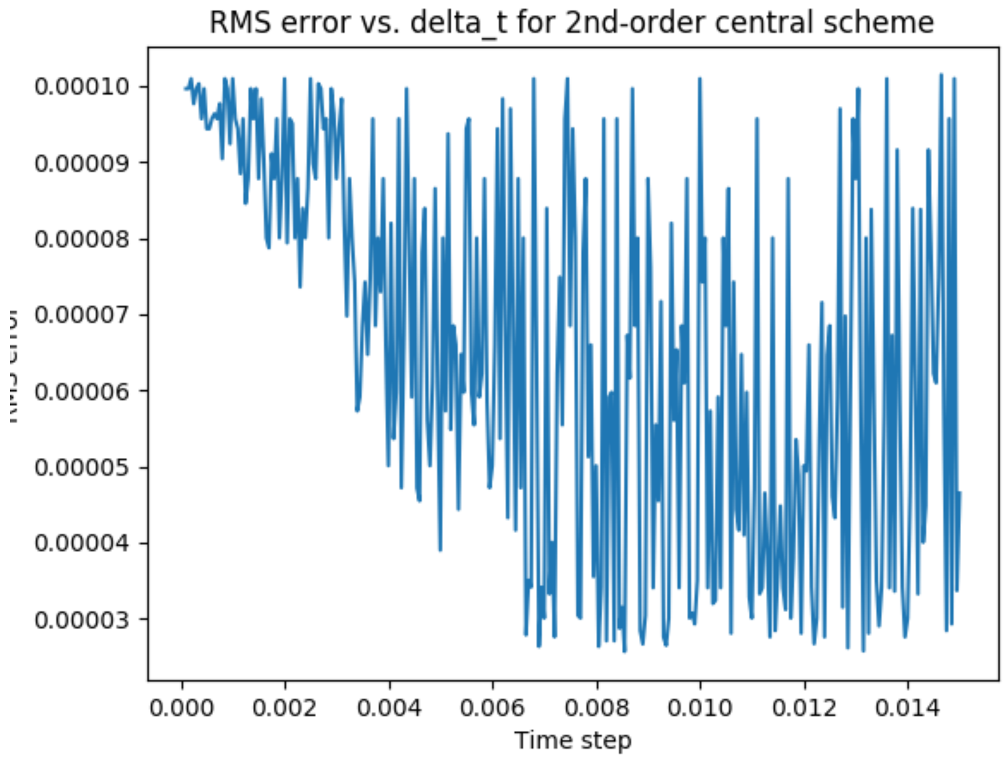
RMS error vs. grid points for 2nd-order upwind scheme
0.00206
0.00204
0.00202
0.00200
0.00198
0.00196
0.00194
0.00192
0.00190
0
2000
4000
6000
8000
10000
Grid point

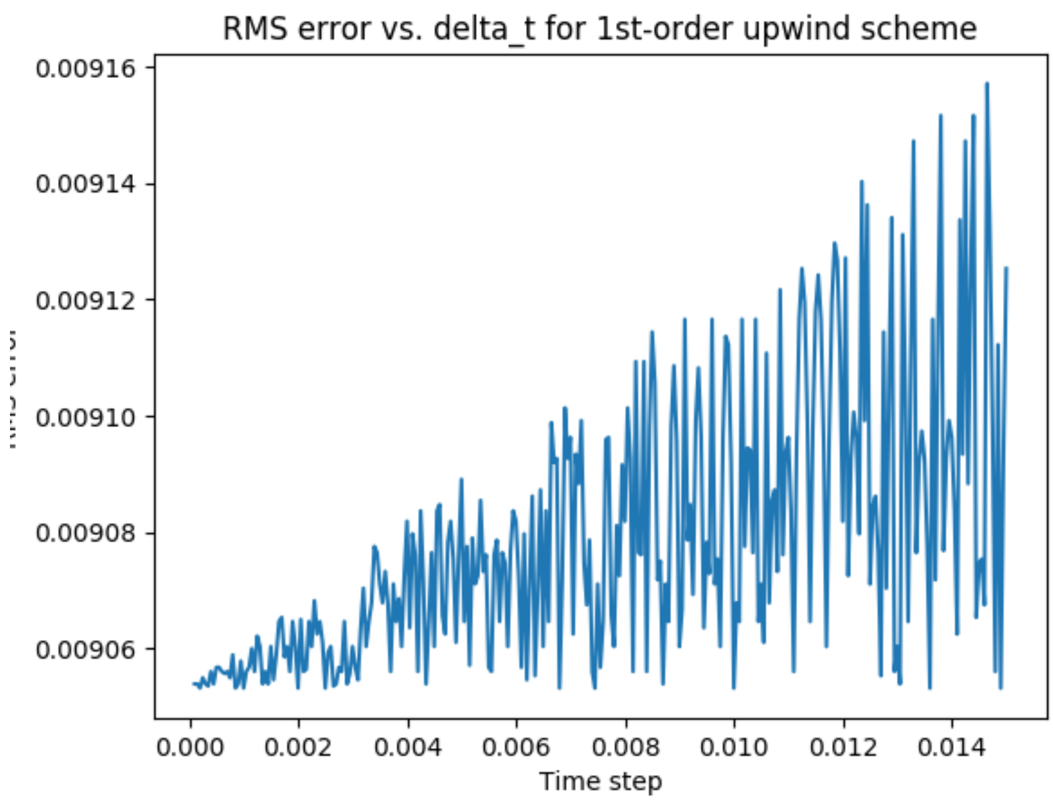



RMS error vs. grid points for 2nd-order central scheme
0.00190
0.00188
0.00186
0.00184
0.00182
0
2000
4000
6000
8000
10000
Grid point

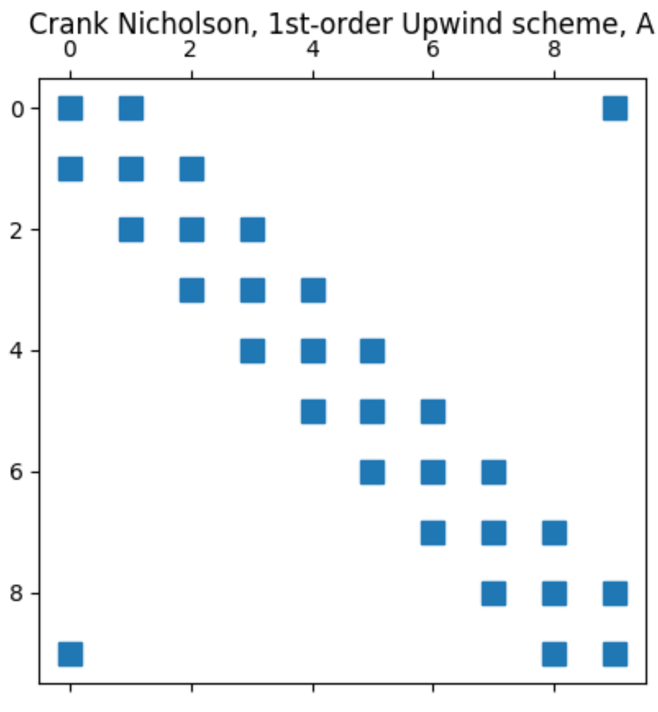

RMS error vs. grid points for 1st-order upwind scheme
0.010
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
RMS error
0
2000
4000
6000
8000
10000
Grid point

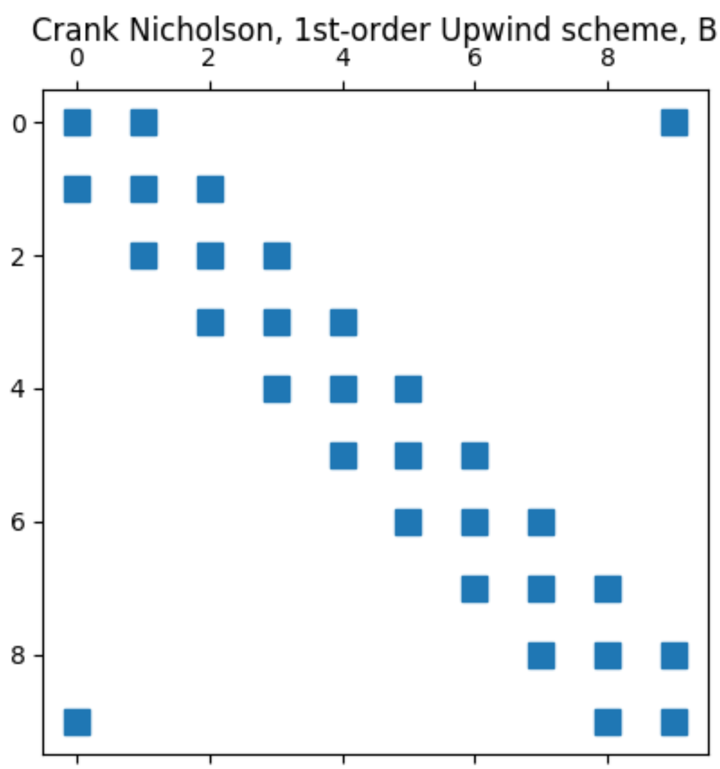

RMS error vs. delta_t for 2nd-order upwind scheme
0.000375
0.000350
0.000325
0.000300
0.000275
0.000250
0.000225
0.000200
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
Time step


RMS error vs. delta_t for 2nd-order central scheme
0.00010
0.00009
0.00008
0.00007
0.00006
0.00005
0.00004
0.00003
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
Time step

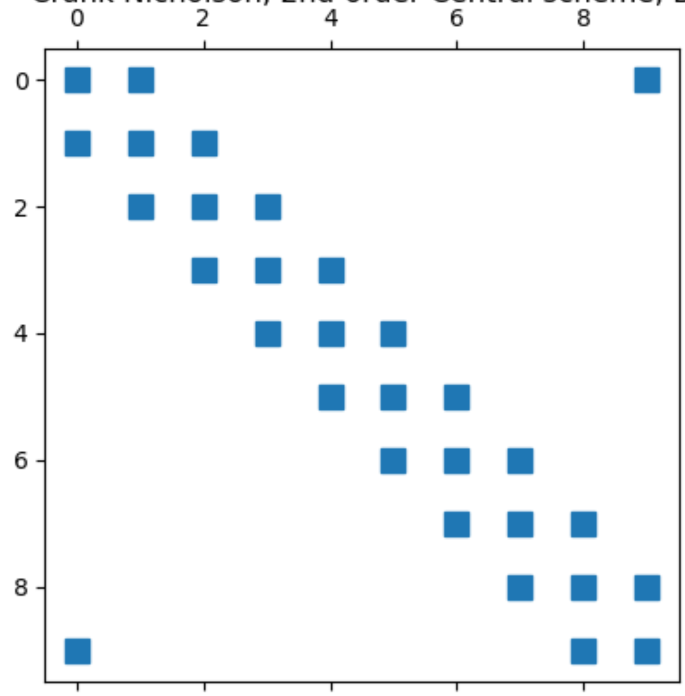

RMS error vs. delta_t for 1st-order upwind scheme
0.00916
0.00914
0.00912
0.00910
0.00908
0.00906
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.014
Time step

Problem 2 d

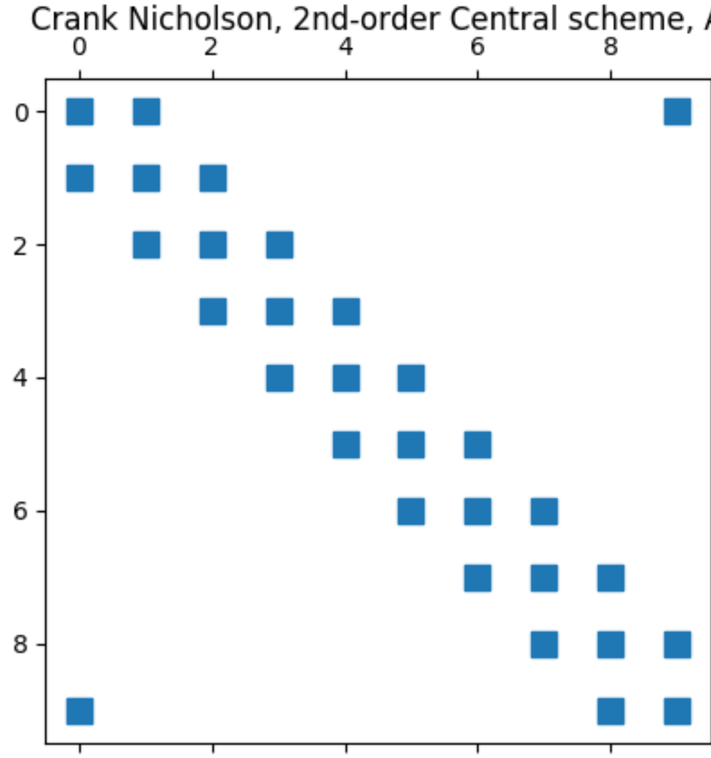

Crank Nicholson, 1st-order Upwind scheme, A
0
2
4
6
8
0
2
4
6
8

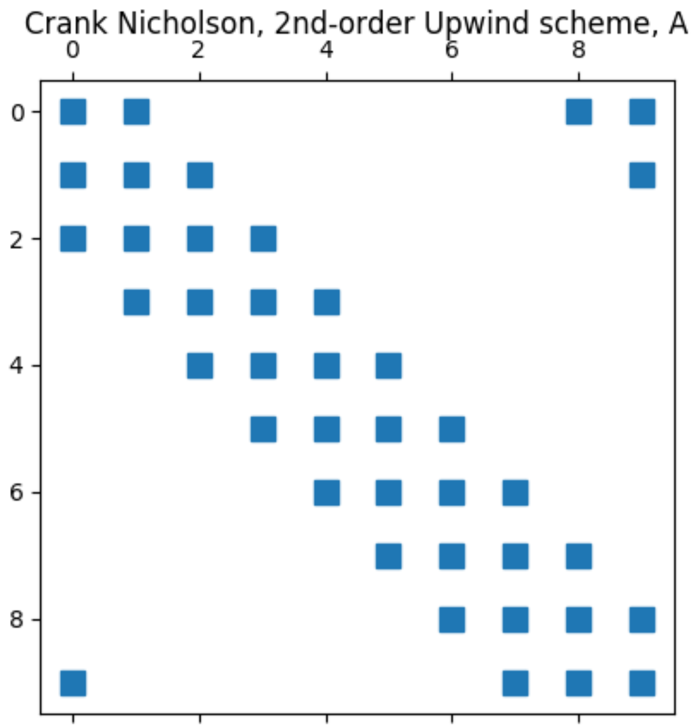

Crank Nicholson, 1st-order Upwind scheme, B
0
2
4
6
8
0
2
4
6
8

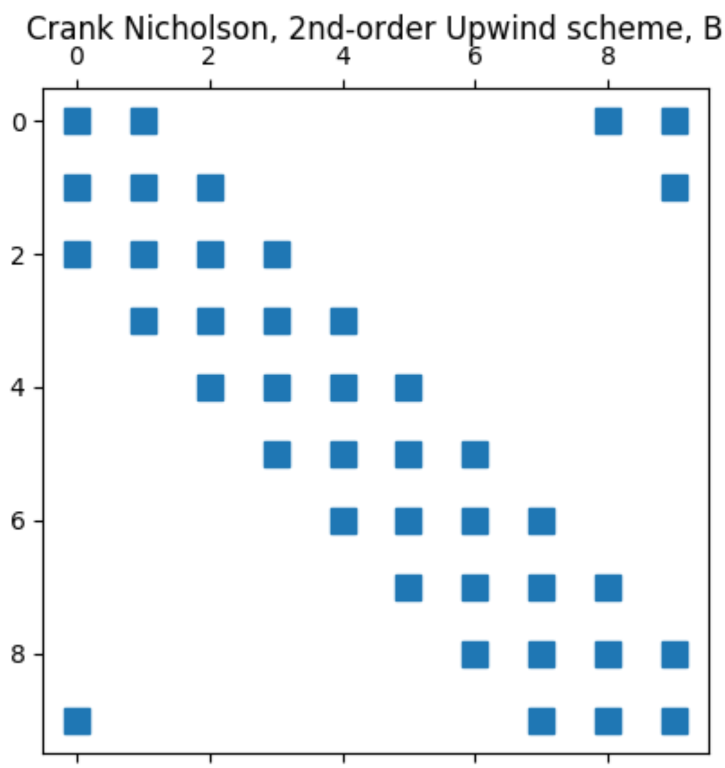
Crank Nicholson, 2nd-order Central scheme, B

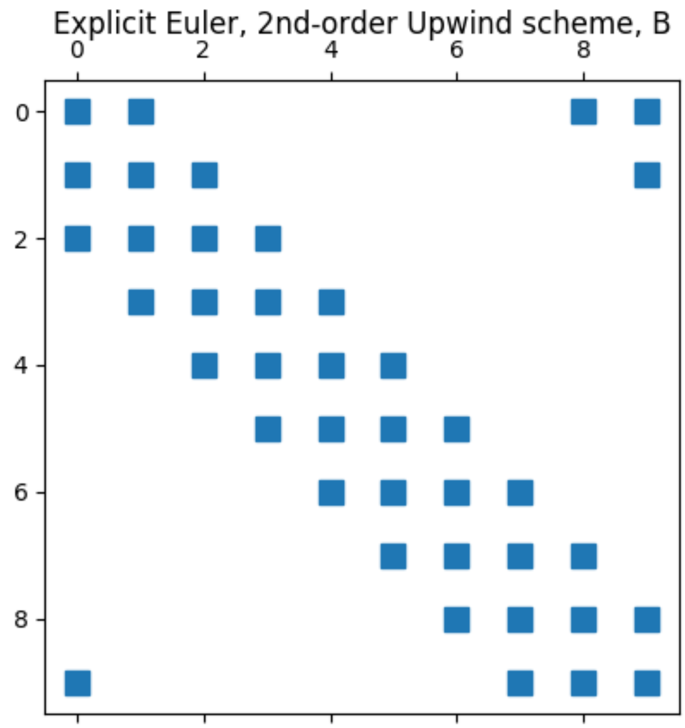


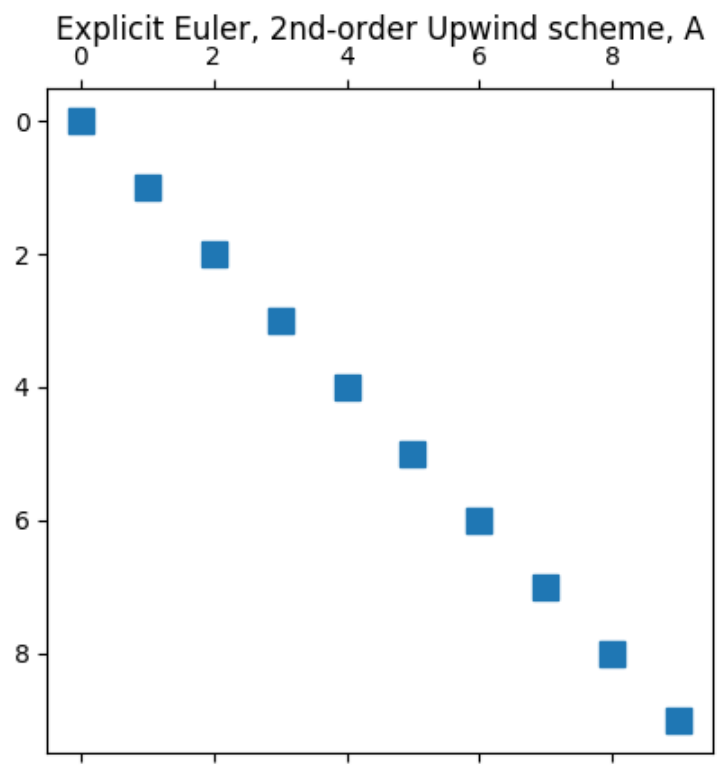

Crank Nicholson, 2nd-order Central scheme, A



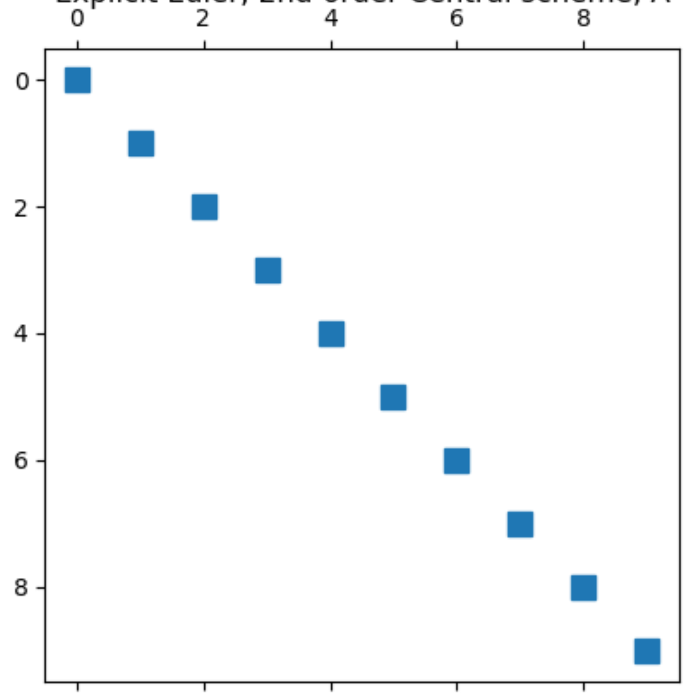

Crank Nicholson, 2nd-order Upwind scheme, A
0
2
4
6
8


Crank Nicholson, 2nd-order Upwind scheme, B
0
2
4
6
8

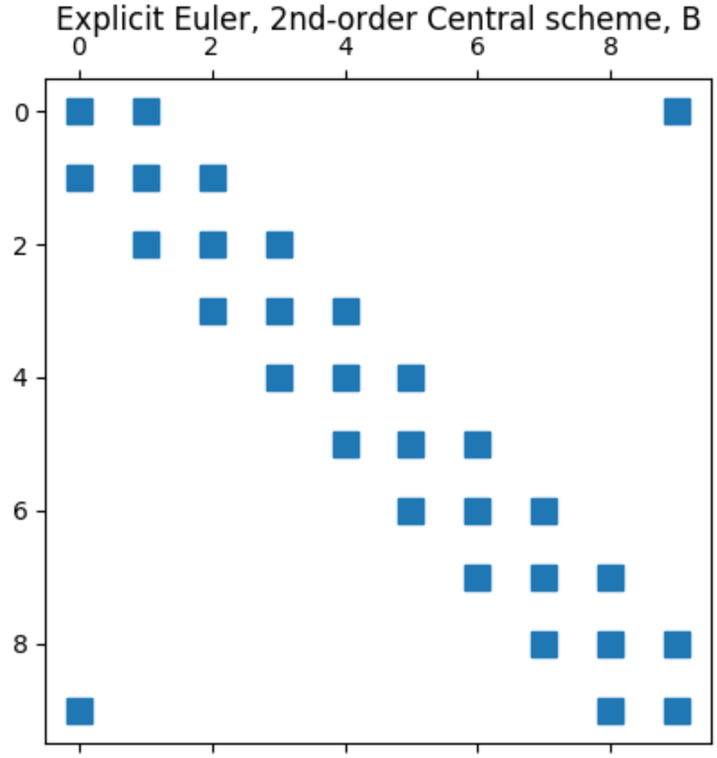

Explicit Euler, 2nd-order Upwind scheme, B

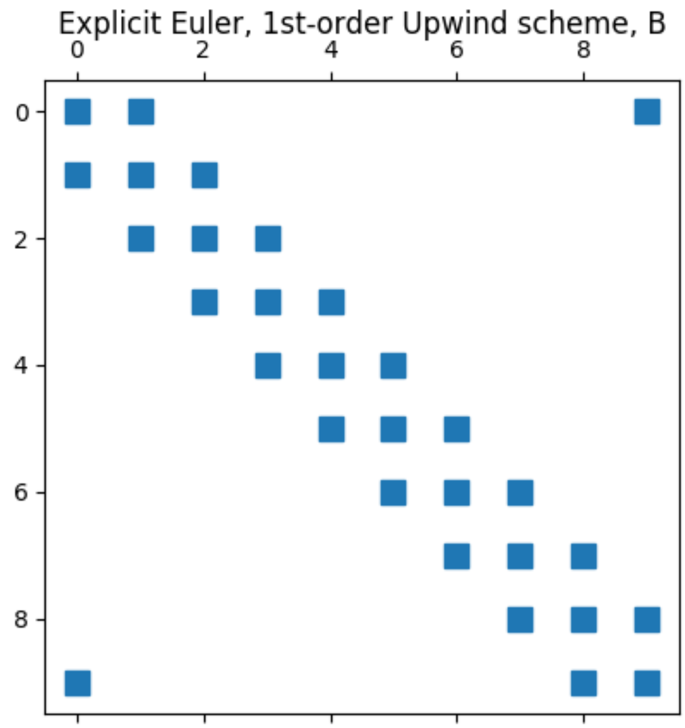

Explicit Euler, 2nd-order Upwind scheme, A

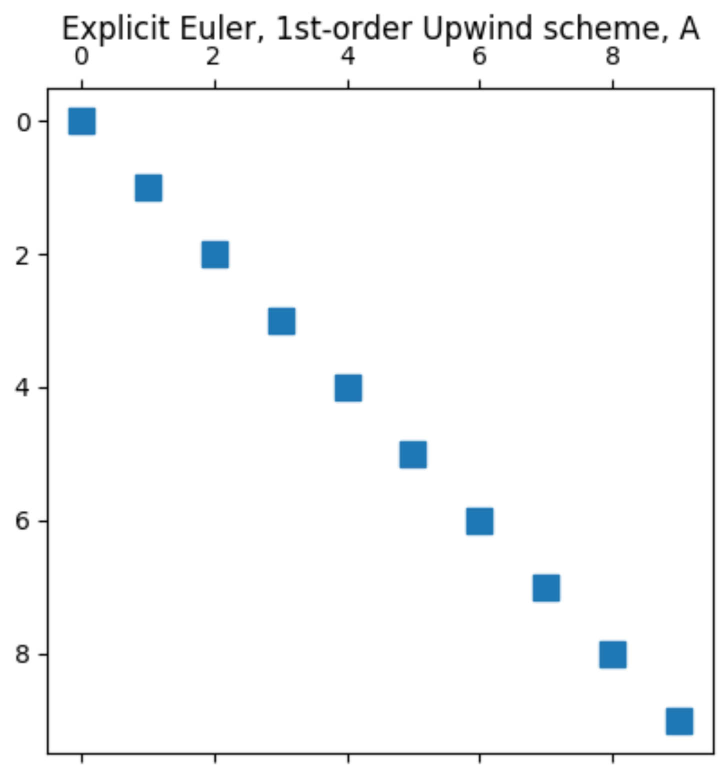
Explicit Euler, 2nd-order Central scheme, A




Explicit Euler, 2nd-order Central scheme, B




Explicit Euler, 1st-order Upwind scheme, B
0
2
4
6
8


Explicit Euler, 1st-order Upwind scheme, A
0
2
4
6
8