Gambling for Redemption or Ripoff, and the Impact of Superpriority*

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ABSTRACT

New U.S. bankruptcy laws exempt repos and derivatives from the automatic stay and clawbacks, giving them "superpriority" over claims resolved in bankruptcy. Motivated by the new laws, we study gambling by firms. We find that gambling can be understood from two polar cases. "Gambling for redemption" preserves firms' continuation value more often without hurting bondholders, while "gambling for ripoff" destroys firms' continuation value most of the time, benefiting owners at the expense of bondholders. If anticipated, the ability to gamble at large scale (possibly due to superpriority) impairs borrowing capacity and reduces the equity value of the firm.

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1 Introduction

In the early days of Federal Express, the company's cash once dwindled to $5,000, too little to cover the $24,000 jet fuel bill due the following Monday. With the firm hanging on the edge, the founder Frederick Smith flew to Las Vegas over the weekend and played blackjack to convert the $5,000 into $32,000, enough to keep the company afloat for another week.[1] This gambling was obviously beneficial to the firm's owners since it avoided bankruptcy with positive probability, but it was probably also beneficial to the other claimants including the fuel company, who were unlikely to receive much in bankruptcy. Gambling by a firm can also benefit owners at the expense of creditors because they receive most of the upside of large gambles but most of the downside is borne by the creditors. In this paper, we study pure gambling by the firm. We can understand the impact of this gambling through two polar cases. *Gambling for redemption*, which means gambling just enough to meet the required payments as in the Federal Express example, is good for the owners, the creditors and for overall efficiency, while *gambling for ripoff*, which is at a larger scale, benefits the owners at the expense of the creditors and overall economic efficiency. Our results show that when gambling at large scale is possible, the anticipation of gambling for ripoff makes it hard for firms to borrow in the first place and reduces the equilibrium value of equity.

Gambling for ripoff is of special current interest because of recent legislation that exempts repos and other derivative securities from important provisions of bankruptcy, including the automatic stay and clawbacks, causing some people to call them superpriority claims.[2][3] One important aspect of superpriority is that it enables the firm to gamble away assets, which may also be possible due to poor specification or enforcement of property rights and bankruptcy law in under-developed countries. In the United States, it has traditionally been difficult to redeploy assets for gambling. While common law allows for asset seizure in satisfaction of debts, seizure or sales in violation of bond covenants can be clawed back in bankruptcy, typically for a transfer within 90 days before bankruptcy.[4] Traditionally, bonds contain covenants to prevent asset sales, typically placing the firm in default on the loan if the covenant is violated. Together with cross-default clauses in bonds saying that a default on one bond places the company in default on all its bonds, this would normally result in the firm entering bankruptcy. According to the baseline bankruptcy law, an asset

[1] Frock (2006)

[2] Roe (2020)

[3] Superpriority protects the contractual right of derivatives counterparties to "terminate, liquidate, or accelerate" a derivatives contract before the commencement of the case. See 11 U.S. Code §362(b)(6), §546(e). Stockholders who are members of Securities Investor Protection Corporation (SIPC) are liquidated under SIPA with similar rules, instead of the Code, see 15 U.S. Code §78eee(b)(2)(C).

[4] The clawback extends back one year for a preferential transfer to an insider, or up to two years for a fraudulent conveyance.

sale in satisfaction of a particular claim within 90 days before bankruptcy is considered preferential if the firm cannot satisfy all claimants and can be clawed back (reversed by the court), which tends to make asset seizure or sale pointless. Consequently, any promise by the firm to transfer assets when it fails the gamble would be unreliable unless the gambling counterparties are sure that the firm will not be pushed into bankruptcy. However, the new exemption from bankruptcy law for "superpriority" claims sidesteps these laws. The gambling counterparties are now granted immunity from the clawback and can redeem their gains without being stayed in the firm's estate in bankruptcy. Thus, it makes possible for a financially shaky firm on a path to bankruptcy to gamble assets, and derivative securities makes it easier to shape the exact distribution of the gamble a firm chooses.

Following Myers (1977), we know that limited gambling, in the form of "asset substitution" to an inefficient but noisier production technology, can benefit firm owners at the expense of bondholders and overall efficiency. Gambling with derivatives permits gambling with negligible efficiency loss, and is a sharper tool for gambling just what is needed (for example, a firm can buy a digital option paying off exactly this period's required debt payment). This is consistent with Federal Express's gambling, but gambling using superpriority can also operate at a much larger scale and in the presence of accounting controls.

We start our analysis with a single-period model which has some debt coming due now along with some cash flow coming in. Managers' incentives are assumed to be aligned with the owners' to maximize equity value.[5] Bankruptcy is triggered when the firm's cash flow cannot cover its maturing debt and is modeled in a very simple way with two costs: the loss of the continuation value, and an administrative cost paid out of the surviving assets. In the model, the loss of continuation value is borne primarily by the owners and the administrative cost borne primarily by the bondholders. With cash shortfalls, firms can either declare bankruptcy or gamble. If the firm gambles for redemption, just to the level needed to repay the debt, the gambling benefits both the owners and the bondholders by minimizing both costs. Absent the cost paid out of the surviving assets, the bondholders would be just indifferent about gambling for redemption since we stay on the linear part of their payoff (receiving 50 cents on the dollar 100% of the time versus 100 cents on the dollar 50% of the time have the same expected value), and gambling for redemption allows the bondholders to avoid paying the administrative cost. However, whenever the face value of debt is larger than the value lost in bankruptcy, the firm owners prefer for the firm not to continue since the value lost is trumped by not having to pay off the debt. The firm owners may simply "take

[5] Our firms are more like proprietorships than corporations in order to focus on the role of gambling. Sometimes we use "the firm" to refer to the firm owners, and whenever we have "firm owners" in the model we simply mean the single entity who makes decisions as a whole. Similarly, "bondholders" are also considered as a whole. Future work could study gambling more generally, for example by a rogue trader or a CEO who does not maximize on behalf of shareholders.

the money and run", and gambling provides a legal way of doing it. With gambling for ripoff, the bondholders, instead of getting some of the remaining assets, are ripped off since they may only receive full repayment .1% of the time and zero 99.9% of the time. If it is possible to gamble some of the assets as well (for example, due to superpriority), gambling for ripoff becomes more attractive because it also transfers part or all of the asset value (which should be subject to a stay or a clawback absent superpriority) to the firm and reduces the value loss in bankruptcy for the firm owners.

In the single-period model, the face value of debt is exogenous. This might be a good assumption at the time of the superpriority legislation, if the legislation was a surprise to the bondholders with debt in place. It is perhaps more interesting to think about the impact of the law once it is understood by bondholders and is priced out in the lending decision. For this, we have a multi-period model that endogenizes the level of borrowing. The multiperiod model also endogenizes the continuation value and permits the firm to borrow to repay debt. Our main result from the multiperiod model is that if there is significant liquidation value (for example due to superpriority), being able to gamble against assets reduces the maximum amount the firm can borrow at first place and reduces the market value of equity.

(talk about some economics)

How much can be gambled in a firm is crucial for our analysis and depends on legal considerations. As a consequence of superpriority law, the bondholders cannot rely on protections in bankruptcy through negative pledge covenants which preclude asset sales, but security interests if perfected are still honored under UCC Article 9. In a parallel provision, the liquidation of stockbrokers who are members of Securities Investor Protection Corporation (SIPC) is governed by SIPA[6] rather than the Bankruptcy Code. SIPA generally provides similar superpriority protections to qualified financial contracts, except that a foreclosure of related securities collateral may still subject to a stay.[7]

The "superpriority" claims we are talking about obtained their exemption from bankruptcy in a series of laws passed between 1978 and 2006. See Schwarcz and Sharon (2014) for a detailed history of the law. The game changer seems to be the 2005 amendment to bankruptcy code (BAPCA), which extends the exemption, which started with some commodity futures and previously extended to repos and swaps, to all derivative securities. Taken together, these laws exempt qualified contracts from the automatic stay of bankruptcy (which stops collection efforts), clawback of preferential treatment and also provide favorable terms for netting of claims.[8]

[6]The Securities Investor Protection Act in 1970.

[7]15 U.S. Code §78eee.

[8]Supepriority also favors derivatives by exempting clawback of constructive (but not actual) fraudulent transfers under them. See Vasser (2005). However, the exemption from avoidance of fraudulent transfer may not apply in the context of gambling, since the transfer is in satisfaction of an existing claim and reflects a fair value.

The exemption has drawn a lot of attention since the 2008 financial crisis. Roe (2010) suggests that superpriority played a central role in accelerating the crisis by encouraging firms to shift away from traditional financing and lowering the incentives of derivatives counterparties to monitor the firm. While this cheaper way of financing facilitates more liquidity that otherwise would not occur, it would worsen the "too big to fail" problem if used by the systemically important firms. Besides the costs, Duffie and Skeel (2012) also provides "benefits" of the safe harbor exemption on QFCs, such as increasing reliance of firms using critical hedges and reducing self-fulfilling security runs. Existence of superpriority claims may cause other damages. It may push firms towards inefficient overuse of collateral, such as that described by Donaldson et al. (2019, 2020).[9] Previous economics literature also focused on the repo market fire sales, which dilutes the collateral value for the secured creditors.[10] Our paper suggests another angle to understand the impact of the law by incorporating firms' gambling decisions and also provides meaningful economic insights to understand gambling.

In normal times (when it is beneficial for the firm owners to continue the firm), gambling would not be a problem because a liquid firm would not gamble; whereas if delinquent, the firm will gamble for redemption to maximize the stake of staying in business and consequently minimize bankruptcy costs for both the firm and the bondholders. Gambling is even more beneficial for both counterparties with future borrowing to repay the debt, because the firm can further reduce the gambling odds in order to stay (and hence increase the chance of survival) and the chance of retaining continuation value is greater with fair gambling.

Gambling becomes a greater problem when the firm's continuation value is small compared to debt, which can happen when bad news hit the industry or the firm, and the result is a drop in asset value. When the drop is large enough, the firm owners will gamble for ripoff which loots the value that should have been collected by the bondholders. Interestingly, if the owners benefit from bankruptcy, they favor such extreme gambling regardless having enough cash to cover debt or not. In fact, gambling becomes socially damaging if there is enough cash: without gambling, the continuation value should have been maintained for sure, but is dissipated when the owners gamble for ripoff.

When the economy goes south, superpriority law can accelerate the process of the firm falling into the plight of gambling for ripoff and make failure more possible. Therefore, providing liquidity in a downturn may help to keep the firm temporarily, but may not be useful enough to change the risk taking by the firm and help preserve continuation value if the longer term perspective of the firm is not enhanced. Rather, policies that increase firms' going concern surplus (may or may

[9] Overuse of collateral can be inefficient because it damages the flexibility of assets redeployment and may result in underinvestment in good projects in the future as in Donaldson et al. (2020), and also undermines the benefit of using covenants as a weak protection as in Donaldson et al. (2019).

[10] See Infante (2013), Oehmke (2014), Antinolfi et al. (2015) and Auh et al. (2018).

not include cash flow) or block superpriority gambling may be more socially efficient.

The analysis of gambling can further extend to other scenarios. For instance, it may encourage gambling for ripoff in Chapter 11 resolution after Congress enacted section 1114 in 1988 to give special priority to retiree medical benefits, because increases of debt obligation will reduce the going concern surplus of the firm. Consequently, gambling for ripoff wipes out most of the assets and the assets dilution effect may be larger than purely having additional debt. This highly risky behavior may make it harder for the firm to raise funds, and bondholders may only be willing to lend if the firm promise to file for Chapter 7 liquidation to evade the legislation with underfunded retiree insurance benefits.[11]

The paper is organized as follows. Section 2 focuses on the two polar cases of *"gambling for redemption"* and *"gambling for ripoff"* by examining a stripped-down single-period model. Section 3 presents a multi-period model using the building block in Section 2 and with endogenous decision making to study the ex ante effect of gambling with and without superpriority. Section 4 characterizes equilibrium properties of the model and provides numerical examples to illustrate the results, and Section 5 concludes.

2 Optimal gambling: the single-period model

We start with a stripped-down model in which we show there can be two different forms of gambling: *"gambling for redemption"* and *"gambling for ripoff."* *Gambling for redemption* occurs when the firm cannot pay the debt immediately, and the owners would suffer a net loss from bankruptcy. In this case, bankruptcy is bad for firm owners, so they will minimize the probability of bankruptcy by gambling just to what is needed to pay the debt, and the firm owners and the bondholders are all better off. By contrast, *gambling for ripoff* occurs when the owners would gather a net gain from bankruptcy. In this case, bankruptcy is good for the owners, so they will maximize the probability of bankruptcy to evade debt obligations but still collect the firm's value by gambling to a large payoff, benefiting the owners at the expense of the bondholders. Superpriority claims reduce the net loss to owners because they allow them to collect part of the firm's asset value directly without paying the bondholders, which makes gambling for ripoff more appealing to the firm owners.

To illustrate these main ideas, this section assumes that everything except for the firm's gambling decision is exogenous. Absent gambling, if the current cash flow π exceeds the face value of the debt F, the debt is paid off and the value to owners is $\pi - F + C$, where C is the continuation value. However, if $\pi < F$ and there is no gambling, then the firm fails, the bondholders collects

[11] We thank Dan Keating for providing the example of underfunded retiree insurance benefits. For further reading of this issue, see Keating (1990) and Keating (1991).

$(1-c)(\pi+L)$ and the owners get nothing, where $L \in [0,C)$ is the liquidation value of the firm's assets, and $(1-c) \in [0,1]$ is the fraction of remaining value that the bondholders can collect in bankruptcy. Therefore, the total cost of bankruptcy to all financial claimants is $C+(\pi+L)c$.

With gambling, the owner can gamble π plus the part of the liquidation value that is available for gambling. Superpriority increases the proportion of L available for gambling, other things equal, and therefore allows the owners to make large gambles. In this section, for simplicity that superpriority increases the amount available for gambling from 0 to the entire liquidation value L. Gambling is not unlimited. We require it to be "fair" and "feasible," meaning that the firm can purchase mean-zero market claims with any risk distribution subject to the limitation of what is available to pay. If $\mathbf{p}$ is a stochastic gambling payoff, the two requirements are

$$\begin{aligned} &\text{gambling being fair:} && \mathrm{E}[\mathbf{p}] = \pi, \quad \text{and} \\ &\text{gambling being feasible:} && -\mathbb{1}_s L \leq \mathbf{p} \leq \bar{\pi}, \end{aligned}$$

where $\bar{\pi}$ is the upper bound of gambling and can be infinite, and $\mathbb{1}_s$ is the indicator whether gambling assets is feasible (due to superpriority):

$$\mathbb{1}_s \equiv \begin{cases} 1, & \text{with superpriority} \\ 0, & \text{absent superpriority.} \end{cases}$$

"Fair gambling" assumes the efficiency of gambling, because using derivatives largely reduces the cost of gambling (for example, using a digital option which pays x with probability $1/x$ and pays 0 otherwise). In this paper, everything is under risk-neutral probabilities or market valuation, and we ignore transaction costs of participating in gambling, i.e., the costs in searching for gambling counterparties, writing gambling contracts, transportation, etc. The conditions of gambling for redemption or gambling for ripoff may vary with risk-averse agents and/or costly gambling, but readers can use the similar method and intuition as we describe in the following content, and our conclusion is drawn without loss of generality.

Superpriority makes feasible larger scales of gambling. Absent superpriority, the owners at most lose all the cash flow π, because the gambling counterparties know that the firm cannot reliably promise more than the sure cash. However, superpriority makes available other assets in the firm to be pledged as collateral and then the owners can gamble down to $-L$. We show an example to illustrate the assumptions: if the firm has \$100 in cash and uses it to gamble, a fair gamble should be worth exactly \$100. Without being able to pledge assets (as in the case absent superpriority), the largest value the firm can lose is \$100. Knowing the firm's limit, the gambling counterparty would not gamble with the firm if the firm promises to pay \$200, unless the firm can pledge its other assets with market value of \$100 (if superpriority makes it feasible). Specifically, not to gamble ($\mathbf{p} \equiv \pi$) can be regarded as a special case of gambling.

The purpose of gambling – to get quick cash when the firm is likely at the edge of bankruptcy– also suggests that gambling, unlike hedging by firms, is more likely to be short-term. Being short-term also indicates a hidden assumption of independence for gambling and randomness in the firm. It also makes sense for the firms' gambling counterparties who are more certain about firm's condition in the short run and thus are more willing to participate in gambling.

The firm owner maximizes the expected payoff to equity, and the firm owners' problem is formally stated as follows.

Firm's problem

Given the cash flow π, the continuation value C, the liquidation value L, and the face value of the debt F, the firm's problem is to choose a fair gamble $\mathbf{p}(\tilde{x})$, where $\tilde{x}$ is the underlying randomness: $\tilde{x} \sim_d U(0,1)$, maximizing[12]

$$\mathrm{E}\big[\big(\mathbf{p}(\tilde{x}) - F\big)^+ + \big(\mathbf{p}(\tilde{x}) \geq F\big)C\big], \tag{1}$$

subject to the gamble being fair,

$$\mathrm{E}[\mathbf{p}(\tilde{x})] = \pi, \tag{2}$$

and the constraint of feasible gambling outcome

$$-\mathbb{1}_s L \leq \mathbf{p}(\tilde{x}) \leq \bar{\pi}\ (\to +\infty). \tag{3}$$

Given any choice of gambling $\mathbf{p}$, we have

$$\textit{bond value} = \mathrm{E}\Big[(\mathbf{p}(\tilde{x}) \geq F)F + (\mathbf{p}(\tilde{x}) < F)\Big(\big((1-c)(\mathbf{p}(\tilde{x}) + L)\big) \wedge F\Big)\Big].$$

Available liquidation value L may be small because of the nature of the firm and its capital, but we are mostly interested in its availability for gambling. In our simple model, we think of L being the liquidation value of the firm and also the available liquidation value for gambling. In

[12]The notations we use are defined as follows:

$$(aRb) \equiv \begin{cases} 1, & \text{for } aRb \\ 0, & \text{otherwise} \end{cases}$$

$$a \wedge b \equiv \min\{a, b\},$$

$$(A)^+ \equiv \begin{cases} A, & \text{for } A \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

our examples, we focus on gambling only cash (without superpriority) and gambling with all the liquidation value (with superpriority).

Example 1: cannot gamble much (no superpriority)

In general, "gambling for redemption" happens when continuing the firm has net benefit to the owners: the face value of the debt to be repaid for the firm to stay in business is less than the continuation value lost in bankruptcy. We show in the simplest form that *gambling for redemption* minimizes bankruptcy costs for the owners and bondholders and makes everyone better off, while *gambling for ripoff* maximizes both costs with a transfer from bondholders to the owners.

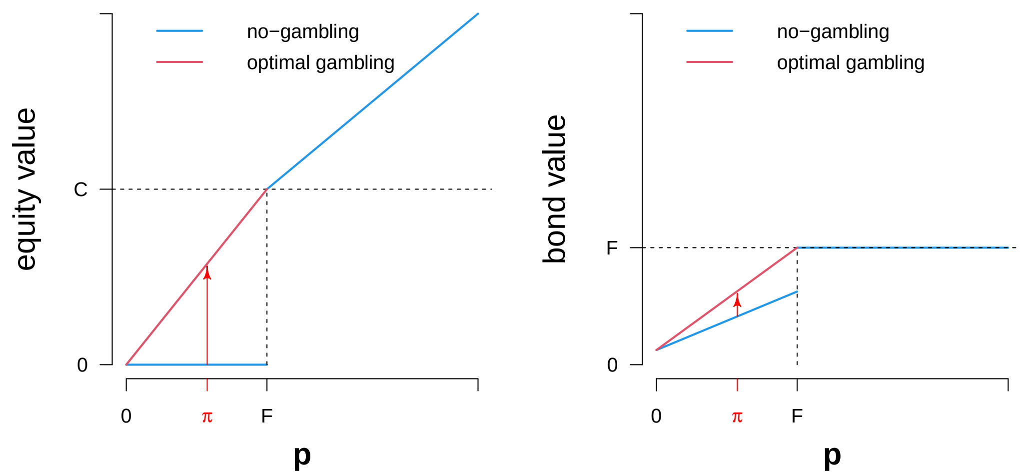


Figure 1: ***When $F < C$ (Gambling for redemption.) Dependence of equity value and bond value on cash flow* p.** *Any gambling should reach the convex combination of the values, and the optimal strategy "concavifies" the utility function of the firm owner. The red arrow shows the increment of the expected equity value through optimal gambling given cash flow π.*

Figure 1 demonstrates the equity value and bond value as functions of cash flow π when $F < C$. In each sub-figure, the blue lines depict the values without gambling: equity value increases linearly when the firm has enough cash, but becomes zero as the cash flow π is below F. In the latter case the bondholders grab $(1-c)(\pi+L)$ and the remaining $c(\pi+L)$ is lost.

Fair gambling in expectation achieves convex combinations of the gambling outcomes. To reach the maximal expected equity value, an optimal gambling strategy, depicted by the red lines, should "concavify" the blue curves. Shown in the left graph of Figure 1, when $\pi \geq F$, the sound firm will only gamble along the 45 degree segment, which is equivalent to no gambling; yet, when $\pi < F$ the equity is out of money, the firm is inclined to be in the money since $C > F$. An optimal gambling should retain the continuation value as often as possible, and hence the gambling randomizes payoffs between F, with probability $\frac{\pi}{F}$, and 0, with probability $1-\frac{\pi}{F}$. The bondholders will obtain full repayment of debt F with probability $\frac{\pi}{F}$ and $(1-c)L$ with probability $1-\frac{\pi}{F}$. In effect, the firm owners achieve positive expected equity value instead of zero, and the bondholders collect

$$\frac{\pi}{F}\times F+(1-\frac{\pi}{F})\times(1-c)L=(1-c)(\pi+L)+\frac{\pi}{F}(F-(1-c)(L+F)),$$

which is bigger than the amount $(1-c)(\pi+L)$ if $F>(1-c)(L+F)$ due to costly bankruptcy. To conclude, gambling for redemption adds value to the firm owners and the bondholders due to less frequent value loss in bankruptcy.

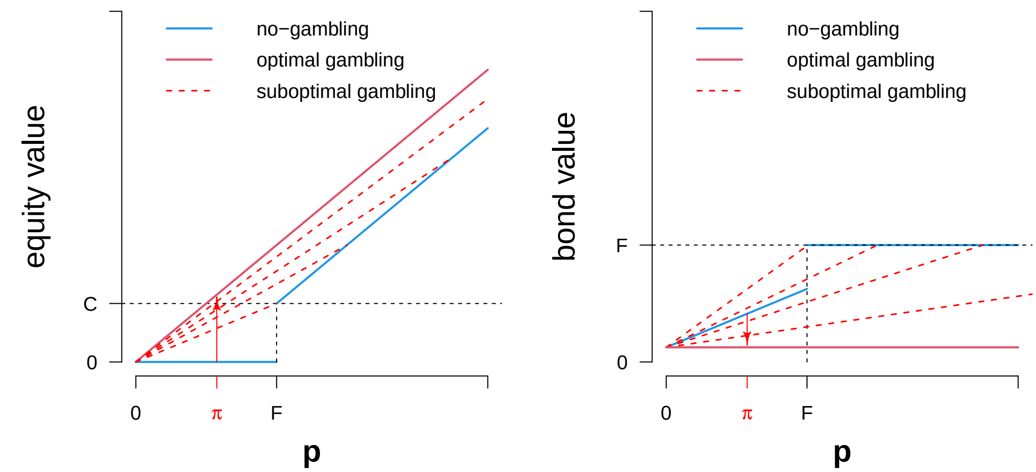


Figure 2: ***When*** $F>C$ ***(Gambling for ripoff.) Dependence of equity value and bond value on cash flow* p.** *The concavification as optimal gambling of the utility function is a linear segment shown by the solid red line. In this case, equity value increases when the firm take bigger gambles and the bond value decreases accordingly. It happens even if* $\pi>F$.

However, when the face value of the debt F is greater than the value lost in bankruptcy C, *"gambling for redemption"* is no longer optimal. The dashed red lines in Figure 2 give the payoffs of fair Bernoulli gambles

$$\mathbf{p}=\begin{cases}\bar{\pi} & \text{with probability } \pi/\bar{\pi}\\ 0 & \text{with probability } 1-\pi/\bar{\pi}\end{cases}$$

As the payoff $\bar{\pi}$ increases, the probability of winning declines but the owners benefit more because not paying F is more important to them than not receiving C. The maximum to owners is achieved in the limit as $\bar{\pi}$ approaches infinity. In the limit, gambling will allow the firm obtain $\bar{\pi}$ with probability $\frac{\pi}{\bar{\pi}}$, and 0 with probability $1-\frac{\pi}{\bar{\pi}}$ which in the firm value is $\lim_{\bar{\pi}\uparrow\infty}\left(\frac{\pi}{\bar{\pi}}(\bar{\pi}-F+C)\right)=\pi$. In the limit, the bond value is $\lim_{\bar{\pi}\uparrow\infty}\left(\frac{\pi}{\bar{\pi}}F+(1-\frac{\pi}{\bar{\pi}})(1-c)L\right)=(1-c)L$, i.e., the bondholders almost always only receive part of the liquidation value. Interestingly, gambling for ripoff is optimal in this example even if the cash flow is enough to repay the debt. If the cash flow does not cover the face value of the debt, *gambling for redemption* would increase the total value of bond and equity because the continuation value would be preserved as often as possible, but the owners would rather choose a larger gamble to transfer value from the bondholders. Even worse, with cash surplus the firm would have survived and preserved continuation value for sure if gambling were not allowed, but now the continuation value is lost almost surely.

Example 2: can gamble a lot (superpriority)

Positive available liquidation value to gamble will change the shape of gambling if $C > F > C - L$, as illustrated in Figure 3. Superpriority makes the liquidation value available for gambling, allowing firm owners to gamble down to $-L$ instead of 0. In Figure 3, the continuation value C is

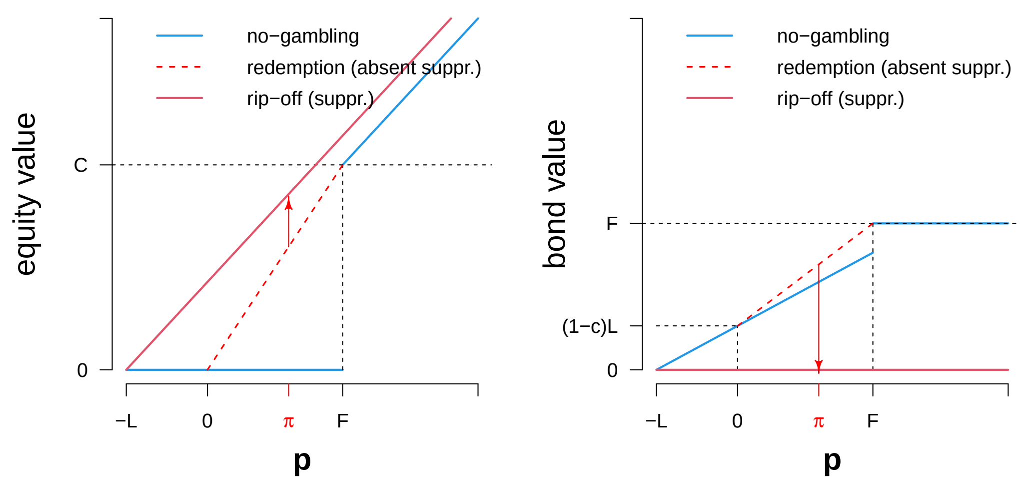


Figure 3: ***(Can gamble a lot)*** *when $C > F > C - L$, firms "gambles for redemption" absent superpriority (dashed red line), but will "gamble for ripoff" with superpriority (solid red line). The red arrow represents the increase in equity value.*

greater than the face value of the debt F, so that absent superpriority the firm will gamble its cash flow for redemption of the face value and obtain $\frac{\pi}{F}C$, depicted by the dashed red line. By allowing gambling with superpriority claims, firms can gamble down to $-L$, and superpriority gambling yields greater benefits when the firm *"gambles for ripoff,"* as shown by the solid red line in the figure. It is straightforward in the graph that the determinant of gambling for redemption or ripoff falls to the comparison of $L + F$ and C, or F and $C - L$, where $C - L$ is the value lost in bankruptcy.

These graphic observations are formally stated by the following propositions:

PROPOSITION 2.1 *when $F < C - \mathbb{1}_s L$ (the face value of the debt is less than the value lost in bankruptcy), it is optimal to* ***gamble for redemption****. Under this parameter restriction, gambling strictly increases the value of both bond and equity when $\pi < F$, and leaves both unchanged when $\pi \geq F$. Specifically,*

(1) If $\pi \geq F$, $\mathbf{p}^*(\tilde{x}) \equiv \pi$ is one solution. Under linear utility, adding noise such that $\mathbf{p}^*(\tilde{x}) \geq F$ with probability one is also optimal. Otherwise,

(2) If $\pi < F$, the optimal gamble is

$$\mathbf{p}^*(\tilde{x}) = \begin{cases} F, & \text{for } 0 < x \leq \frac{\pi + \mathbb{1}_s L}{F + \mathbb{1}_s L} \\ -\mathbb{1}_s L, & \text{for } \frac{\pi + \mathbb{1}_s L}{F + \mathbb{1}_s L} < x < 1 \end{cases} \tag{4}$$

(3) The payoffs are

$$\text{equity value} = \begin{cases} \pi - F + C, & \text{for } \pi \geq F \\ \frac{\pi + \mathbb{1}_s L}{F + \mathbb{1}_s L} C, & \text{for } \pi < F \end{cases} \qquad \text{bond value} = \begin{cases} F, & \text{for } \pi \geq F \\ \frac{\pi + \mathbb{1}_s L}{F + \mathbb{1}_s L} F, & \text{for } \pi < F \end{cases}$$

$$\text{bond+equity} = \begin{cases} \pi + C, & \text{for } \pi \geq F \\ \frac{\pi + \mathbb{1}_s L}{F + \mathbb{1}_s L}(C + F), & \text{for } \pi < F \end{cases}$$

Proof. See Appendix A. ■

The gambling outcome is a function of a continuum of states and does not have to be binary. However, any optimal gambling can have a binary equivalence because otherwise we can always find an improvement.

In Proposition 2.1(1), if we believe that gambling is costly, or if we are not using risk neutral probabilities (the owners are risk averse), then $\mathbf{p}^*(\tilde{x}) \equiv \pi$ (no gambling) should be the unique solution.

PROPOSITION 2.2 *when $F > C - \mathbb{1}_s L$, it is optimal for the firm to* ***gamble for ripoff.*** *Under this parameter restriction, gambling purely transfers value from bondholders to firm owners when $\pi < F$, and also destroys continuation value when $\pi \geq F$. Specifically,*

(1) The optimal gambling is

$$\mathbf{p}^*(\tilde{x}) = \begin{cases} \bar{\pi}, & \text{for } 0 < x \leq \frac{\pi + \mathbb{1}_s L}{\bar{\pi} + \mathbb{1}_s L} \\ -\mathbb{1}_s L, & \text{for } \frac{\pi + \mathbb{1}_s L}{\bar{\pi} + \mathbb{1}_s L} < x < 1 \end{cases} \tag{5}$$

(2) *The payoff of the firm is* $\frac{\pi+\mathbb{1}_s L}{\bar{\pi}+\mathbb{1}_s L}(\bar{\pi}-F+C)$, *which increases to* $\pi+\mathbb{1}_s L$ *as* $\bar{\pi}\to\infty$. *The value of the debt is* $\frac{\pi+\mathbb{1}_s L}{\bar{\pi}+\mathbb{1}_s L}F$, *and declines to* 0 *as* $\bar{\pi}\to\infty$. *For any* $\pi>0$, *the total value of bond and equity is always* $\pi+\mathbb{1}_s L$ *when* $\bar{\pi}\to\infty$.

Proof. See Appendix A. ■

There is also a knife-edge case when $F=C-\mathbb{1}_s L$. Then any fair gamble with outcomes distributed long the 45-degree linear segment would yield the same expected value. That is to say, *gambling for redemption* and *gambling for ripoff* give the same outcome for the owners, and anything in between the two polar cases is also optimal, so that there are uncountable many solutions. Though these optimal gambles generate different values for the bondholders (for example, we still have *gambling for redemption* makes the bondholders better-off and *gambling for redemption* worse-off), we don't want to go into the details of what the equilibrium (equilibria) is (are) because it is reasonable to believe that $F=C-\mathbb{1}_s L$ almost never happen in application.

These results suggest that having maturing debt larger than the firm value lost in bankruptcy is bad because the firm would be more likely to evade liability by taking on high risk gambling. Even if the firm has enough liquidity, it does not necessarily save the firm from going bankrupt. On the contrary, the worst situation happens when the the cash is enough to cover the debt: if not for gambling, the continuation value would not be lost. Superpriority worsen the situation by making bankruptcy more appealing to the firm owners and results in gambling for ripoff.

In the trade-offs between *gambling for redemption* and *gambling for ripoff*, superpriority also plays an important role. It transfers owners the liquidation value which should be grabbed by bondholders, making *gambling for ripoff* more appealing to the owners. With more "ripoff" cases, continuation value can be more easily destroyed.

With new borrowing (pending)

We next discuss an extension allowing borrowing against the continuation value to show how robust our results are. Without a structure of pricing the new debt in this simple model, we assume borrowing is deducted from the continuation value of the firm without loss of generality, and subject to an exogenous borrowing constraint. Generally, more borrowing means higher bankruptcy risk and hence increasing marginal cost, so we should expect more curvature in the equity value when the firm is approaching the edge of bankruptcy. This is exactly the feature of borrowing in the multi-period model. However, adding curvature here does not qualitatively change the results.

With such simplified assumption of new borrowing, when the firm's cash flow is slightly less than the face value, the firm can borrow to repay the debt. The blue curves in Figure 4 show the adjusted equity value. Then, following the same method that optimal gambling concavifies the function, we shall observe that when $F < C$, (or $F < C - L$ with gambling assets,) the firm still gambles for redemption. The red solid line on the left shows that the firm is strictly better off with borrowing which also benefits the bondholders because they have higher probability to get repaid. When $F > C$, (or $F > C - L$ with gambling assets,) gambling for ripoff still prevails. After all, the owners have an incentive to evade debt, then why bother to borrow?

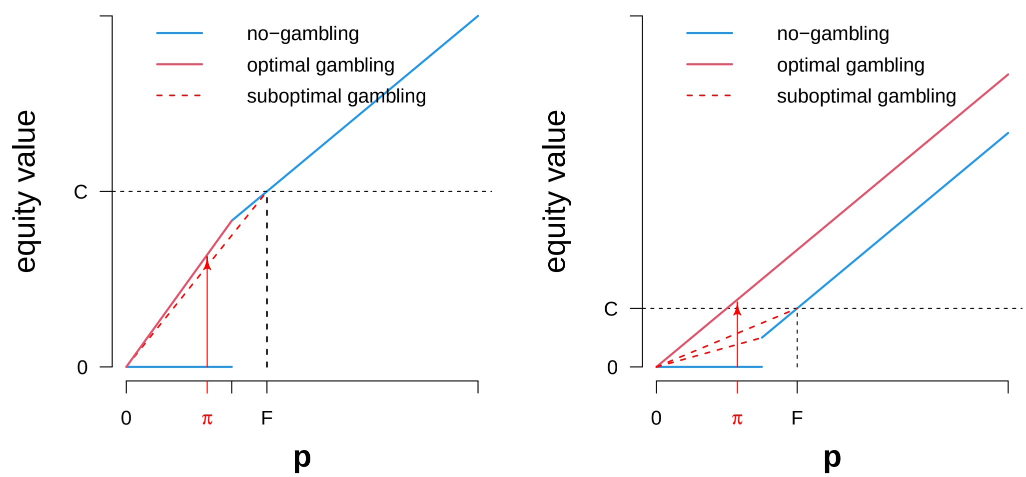


Figure 4: ***(With new borrowing)*** *When $F < C$, the firm gambles for redemption, which benefits all the firm owners and the bondholders; when $F > C$, the firm gambles for ripoff and borrowing has no effect.*

Thus far, this section has examined the conditions for *"gambling for redemption"* and *"gambling for ripoff"*, depending on the net loss for the owners in bankruptcy. In the stripped-down framework, gambling for redemption and gambling for ripoff can both happen depending on exogenous values. It is useful to do ex post analysis when there is a shock due to changes in economic environment, but it does not tell to what extent gambling for redemption and gambling for ripoff are to occur in equilibrium. Is gambling for redemption always efficient enough so that it always tends to increase the firm value, or does the possibility of gambling for ripoff destroy more value? What is the impact of superpriority law on the firm's ability of fundraising, if anticipated? Starting from the next section, we study the general multi-period model and welfare of gambling with superpriority.

3 The Dynamic Model with Liquidation

We consider a setting in which both the firm and the potential bondholders understand the game and know the possible payoffs of gambling. In each period t, the firm enters with capital K_t and a maturing debt F_t. The capital does not depreciate. It pays a cash flow $vK_t > 0$ and is subject to an i.i.d. shock $\tilde{\delta} > 0$ with $\mathrm{E}[\tilde{\delta}] = 1$, so capital is $\widehat{K}_t = \tilde{\delta} K_t$ after the shock.[13] This assumption reflects the fluctuation of firm's asset value under economic uncertainty: in good times, the asset value increases and so does the firm's going concern value; when the economy goes south, the negative shock of asset value may lower the firm's going concern value down below the face value of the maturing debt. The shock can also be industry or firm-specific. Good news such as a change of corporate tax legislation or regulation that favors the industry may increase firm value, but a financial market crash, a war or a pandemic that hampers the business for some firms but thrives other firms can impose different shocks to firms. In either case the firm can gamble, liquidate, borrow and invest.

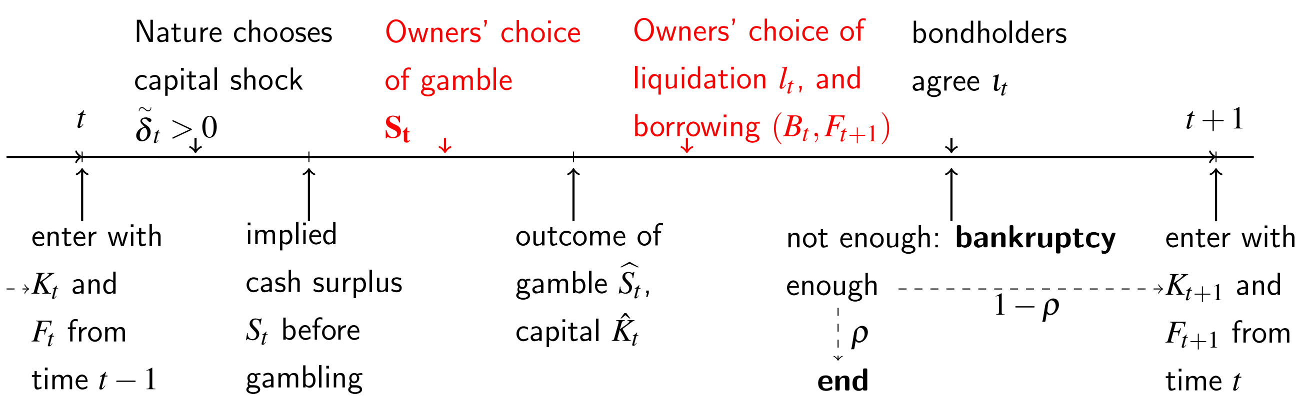
Gambling is similar to the single-period model. Before the adoption of the superpriority law, the firm can purchase fair claims to gamble in a frictionless competitive market using only the cash flow, but can deploy cash flow plus liquidation value if there is superpriority. The liquidation value is θ per unit of capital. With a fraction $l_t \in [0,1]$ liquidated, the firm acquires $\theta l_t \widehat{K}_t$ in cash and remains productive capital $(1-l_t)\widehat{K}_t$. The firm's new borrowing has a face value F_{t+1} and amount B_t priced as the expected value determined in equilibrium, where $B_t < 0$ is interpreted as risk-free lending or savings. If there is a cash surplus after clearing all the debt, the firm may also increase the capital at a growth rate capped by g per period, and we require $(1+g)(1-\rho) < 1$ to avoid

[13] We assume i.i.d. distribution for simplicity. It may be interesting to model correlated shocks or temporary shocks to better mirror the reality, but the simplest case is i.i.d. In our model, cash flow in this period is not subject to the capital shock immediately, but will adjust in the later period. Of course, we can also assume that the shock affects the cash flow, or the cash flow has a shock correlated with the capital shock. These changes do not alter the basic results.

Ponzi scheme.[14] If the firm's shortfall is cover by asset liquidation and new borrowing, the firm is saved.

The firm owners care about the expected net payoff at firm's termination without discount, i.e., the shadow risk-free interest rate is zero. The life of the firm can be determined in one of the two ways: (1) if the state of nature is unfavorable, the owners can abandon the firm at any time and deserts all assets, or (2) the game ends with an exogenous probability ρ (the hazard ratio), in which case the payoff to the owners is $K_{T+1} - B_T$, where K_{T+1} is the capital at the end of the firm's life. The timeline is shown below:

The timeline[15]



The implied cash surplus $S_t \equiv vK_t - F_t$ is the net cash before gambling. Choice of gambling $\mathbf{S_t}$ randomizes the cash surplus S_t and is fair. Similar to the single period model, superpriority allows gambling also the liquidation value $\theta \widehat{K}_t$, so that

$$\begin{aligned} \mathrm{E}\left[\mathbf{S_t}\right] &= S_t, \text{ and} \\ \mathbf{S_t} &\geq -\mathbb{1}_s \theta \widehat{K}_t - F_t. \end{aligned} \tag{*}$$

The outcome of gambling $\widehat{S}_t$ and capital after the shock $\hat{K}_t$ are the state variables. The case absent gambling can be seen as a special case in which $\widehat{S}_t = \mathbf{S_t} = S_t$. Gambling in this model has short duration. We think that it is optimal for the owners to use short-maturity derivative since they have to acquire current information about the various positions at the time the derivative matures.[16]

[14] It is a reasonable assumption because acquisitions are usually time consuming, and firms usually have limited capacity to expand within a period of time. We also need this assumption to rule out infinite borrowing.

[15] We formally describe the information sets and the trajectory of events in Appendix B for reference.

[16] Firms' motivations for holding derivatives (either to gamble or to hedge) might be distinguished by duration

After the realization of the gamble, the firm chooses a fraction of liquidation $l_t \in [0,1]$ and a debt contract (B_t, F_{t+1}) given $\widehat{S}_t$ and $\hat{K}_t$, and the bondholders decide a probability $\iota_t \in [0,1]$ of granting the debt. We can assume $\iota_t \equiv 1$ in equilibrium if the bondholders always accept a fairly priced debt.[17] If liquidation and borrowing cannot cover the shortfall, the firm files for bankruptcy and all the assets are liquidated to repay the debt. If enough, the firm continues with probability $1-\rho$, the new capital K_{t+1} is expressed as the result of liquidation, borrowing, debt repayment and new investment:

$$K_{t+1} = ((1-l_t)\widehat{K}_t + l_t\theta\widehat{K}_t + \widehat{S}_t + B_t)(l_t\theta\widehat{K}_t + \widehat{S}_t + B_t \geq 0).$$

In bankruptcy (when $l_t\theta\widehat{K}_t + \widehat{S}_t + B_t < 0$), bondholders obtains $(1-c)(\widehat{S}_t + F_t + \theta\widehat{K}_t)$, a fraction of the remaining assets value in the firm, and a fraction c is lost in bankruptcy.

The firm's problem in a form of Bellman equation is stated below.

Bellman equation

Given capital $\widehat{K}_0$ and cash surplus after gambling $\widehat{S}_0$, the firm chooses adapted liquidation fraction $\{l_t\}_{t=0}^T \in [0,1]$, debt contracts $\{(B_t, F_{t+1})\}_{t=0}^T \geq 0$, and gambling $\{\mathbf{S_{t+1}}(\widetilde{x})\}_{t=0}^T$, to maximize expected net capital at T, $\mathrm{E}_0\left[K_{T+1} - B_T\right]$. The value function at time t satisfies[18]

$$C_t(\widehat{K}_t, \widehat{S}_t) = \left\{\max_{l_t, B_t, F_{t+1}, \mathbf{S_{t+1}}(\widetilde{x})} \mathrm{E}_t\left[\rho(K_{t+1} - B_t) + (1-\rho)C_{t+1}(\widehat{K}_{t+1}, \mathbf{S_{t+1}}(\widetilde{x}))\right]\right\}^+,$$

of derivatives. Long-dated derivatives are probably for hedging, while short-dated derivatives are more likely for gambling. We thank Harold Zhang for raising this discussion.

[17] If the firm does not borrow, ι_t is out of the path. In equilibrium, if the firm needs borrowing to stay alive, it must be true that the loan is fairly priced and is granted. Otherwise the firm defaults, any debt contract is equivalent to offering $(0,0)$ and is accepted by the bondholders. In either case, assuming $\iota_t = 1$ does not change value.

[18] We assume the hazard ratio ρ is constant and independent of other variables. Then the firm's value function can be written as the following:

$$\begin{aligned}
C_0(\widehat{K}_0, \widehat{S}_0) &= \max_{\{l_t, B_t, F_{t+1}, \mathbf{S_{t+1}}(\widetilde{x})\}_{t=0}^T} \mathrm{E}_0\left[K_{T+1} - B_T\right] \\
&= \max_{\{l_t, B_t, F_{t+1}, \mathbf{S_{t+1}}(\widetilde{x})\}_{t=0}^T} \mathrm{E}_0\left[\rho\sum_{t=0}^{\infty}(1-\rho)^t(K_{t+1} - B_t)\right] \\
C_t(\widehat{K}_t, \widehat{S}_t) &= \max_{\{l_t, B_t, F_{t+1}, \mathbf{S_{t+1}}(\widetilde{x})\}} \mathrm{E}_t\left[\rho(K_{t+1} - B_t) + (1-\rho)C_{t+1}(\widehat{K}_{t+1}, \mathbf{S_{t+1}}(\widetilde{x}))\right]
\end{aligned}$$

If the value is negative, the owners can simply run away and the value is bounded below by 0. We write $C_t(\widehat{K}_t, \widehat{S}_t)$ with $\{\cdot\}^+$ indicating running away as a plausible choice.

subject to the constraints of fair gambling

$$\mathrm{E}\big[\mathbf{S_{t+1}}(\tilde{x})\big] = S_{t+1}, \quad \mathbf{S_{t+1}}(\tilde{x}) \geq -\mathbb{1}_s \theta \widehat{K}_{t+1} - F_{t+1},$$

and the bondholders' willingness to lend[19]

$$\mathrm{E}_t\left[\Big(l_{t+1}\theta\widehat{K}_{t+1} + \mathbf{S_{t+1}}(\tilde{x}) + B_{t+1} < 0\Big)\Big((1-c)\Big(\mathbf{S_{t+1}}(\tilde{x}) + \theta\widehat{K}_{t+1} - cF_{t+1}\Big)\Big)^{-} + F_{t+1}\right] \geq B_t, \quad (6)$$

and the capital growth rate not exceeding g

$$K_{t+1} \leq \widehat{K}_t(1+g), \tag{7}$$

where

$$\begin{aligned} K_{t+1} &= ((1-l_t)\widehat{K}_t + l_t\theta\widehat{K}_t + \widehat{S}_t + B_t)(l_t\theta\widehat{K}_t + \widehat{S}_t + B_t \geq 0), \\ \widehat{K}_{t+1} &= \tilde{\delta}_{t+1}K_{t+1}, \text{ and} \\ S_{t+1} &= vK_{t+1} - F_{t+1}. \end{aligned} \tag{8}$$

Again, the firm's problem without gambling is a special case when $\widehat{S}_t = \mathbf{S_t}(\tilde{x}) = S_t$, for every $t \geq 0$. We next simplify the firm's problem to only one state variable by dividing $\widehat{K}_t$, so everything can be expressed as value per unit of capital.

Normalization

The firm's problem is homogeneous in $\widehat{K}_t$, so the problem and the solution can be expressed in terms of per unit of capital by dividing $\widehat{K}_t$.

Given $l_t\theta\widehat{K}_t + \widehat{S}_t + B_t \geq 0$, we define

$$\begin{gathered} s \equiv \frac{\widehat{S}_t}{\widehat{K}_t},\ b \equiv \frac{B_t}{\widehat{K}_t};\quad s' \equiv \frac{\widehat{S}_{t+1}}{\widehat{K}_{t+1}},\ b' \equiv \frac{B_{t+1}}{\widehat{K}_{t+1}}; \\ \beta \equiv \frac{B_t}{K_{t+1}},\ \phi' \equiv \frac{F_{t+1}}{K_{t+1}};\quad l \equiv l_t,\ l' \equiv l_{t+1}. \end{gathered}$$

[19]Bond value given gambling outcome $\widehat{S}_{t+1}$ is

$$\begin{aligned} &\Big(l_t\theta\widehat{K}_{t+1} + \widehat{S}_{t+1} + B_{t+1} < 0\Big)\Big[\Big((1-c)(\widehat{S}_{t+1} + F_{t+1} + \theta\widehat{K}_{t+1})\Big) \wedge F_{t+1}\Big] + \Big(l_t\theta\widehat{K}_{t+1} + \widehat{S}_{t+1} + B_{t+1} \geq 0\Big)F_{t+1} \\ =&\Big(l_t\theta\widehat{K}_{t+1} + \widehat{S}_{t+1} + B_{t+1} < 0\Big)\Big((1-c)(\widehat{S}_{t+1} + F_{t+1} + \theta\widehat{K}_{t+1}) - F_{t+1}\Big)^{-} + F_{t+1} \end{aligned}$$

The bond price is composed of the face value and the value lost in the case of bankruptcy, which in equilibrium happens when $l_{t+1}\theta\widehat{K}_{t+1} + S_{t+1} + B_{t+1} < 0$.

The "prime" on any variable refers to the variable at $t+1$. For example, s is the cash surplus per unit of capital after gambling in this period, while s' is cash surplus per capital after gambling in the next period. The choice variables are l and (β, ϕ'), representing liquidation fraction and debt per unit of capital. We can compute other variables accordingly.[20]

Now we restate the firm's problem:

Firm's problem: *Given an s (cash surplus per unit of capital after gambling), the owners choose adapted liquidation fraction $l \in [0,1)$, debt contract (β, ϕ') and gambling for the next period $\mathbf{s}(\tilde{x}, \phi', \tilde{\delta}') \in [-\frac{\phi'}{\tilde{\delta}'}, +\infty)$, (where $\tilde{x} \sim_d U(0,1)$ is the underlying randomness,) to maximize expected net capital. The value function has the similar form*

$$C(s) = \left\{ \max_{\mathbf{s}(\tilde{x},\phi',\tilde{\delta}'),l,\beta,\phi'} \left(\rho \cdot (1+s-l(1-\theta)) + (1-\rho) \times \frac{1+s-l(1-\theta)}{1-\beta} \mathrm{E}\left[\tilde{\delta}' \cdot C(\mathbf{s}(\tilde{x}, \phi', \tilde{\delta}'))\right] \right) \right\}^+, \tag{9}$$

subject to the constraints of fair gambling

$$\begin{aligned} &\mathbf{s}(\tilde{x}, \phi', \tilde{\delta}') \in [-\mathbb{1}_s \theta - \frac{\phi'}{\tilde{\delta}'}, +\infty), \text{ and} \\ &\mathrm{E}[\mathbf{s}(\tilde{x}, \phi', \tilde{\delta}') | \tilde{\delta}'] = \frac{v - \phi'}{\tilde{\delta}'}, \end{aligned} \tag{10}$$

and the bondholders' willingness to lend

$$\mathrm{E}\left[\left(\mathbf{s}(\tilde{x}, \phi', \tilde{\delta}') < \underline{s}\right)\left((1-c)(\mathbf{s}(\tilde{x}, \phi', \tilde{\delta}') + \theta)\tilde{\delta}' - c\phi'\right)^-\right] + \phi' \geq \beta, \tag{11}$$

and the constraints of borrowing

$$-\frac{s + l\theta}{1-l} \leq \beta \leq \frac{g - s + l(1-\theta)}{1+g} \tag{12}$$

In (11), $\underline{s}$ is a presumed threshold of cash flow, and below which the firm defaults and declares

[20] We have

$$\frac{\widehat{K}_{t+1}}{\widehat{K}_t} = \tilde{\delta}'(1+s+b-l(1-\theta)), \ \frac{K_{t+1}}{\widehat{K}_t} = 1+s+b-l(1-\theta);$$

$$\beta = \begin{cases} 1, \text{ if } 1+s-l(1-\theta) = 0, \\ \frac{b}{1+s+b-l(1-\theta)}, \text{ otherwise.} \end{cases} \quad b = \frac{(1+s-l(1-\theta))\beta}{1-\beta}, (\beta \neq 1).$$

bankruptcy. In equilibrium, there will be a set $\mathscr{S}$ of values of s such that the firm will borrow enough to avoid bankruptcy. We conjecture that $\mathscr{S}$ is of the form $\mathscr{S} = \{s | s \geq \underline{s}\}$.

Without gambling, $s = \frac{v-\phi}{\delta}$ and $\mathbf{s}(\tilde{x}, \phi', \tilde{\delta}') \equiv \frac{v-\phi'}{\tilde{\delta}'}$ is a special case of the above firm's problem. The difference between with and without superpriority is also implicit in the constraints of borrower's willingness to lend. Without superpriority, the bondholders can at least redeem the liquidation value which would be taken by gambling counterparties with superpriority.

4 Equilibrium and Graphic Illustration

The firm's problem does not have a closed form solution because of the inter-reliance of gambling and (continuation) value function, but the equilibrium properties and the numerical results provide useful implications for understanding gambling by firm in a dynamic setting. We begin with propositions which are immediate derivations from the model setting.

PROPOSITION 4.1 *In all three cases, the firm will not liquidate capital and grow capital at the same time. Then, conditional on $l\theta + s + b \geq 0$,*

$$\frac{K_{t+1}}{\widehat{K}_t} = \begin{cases} \left(0 \vee \left(\frac{\theta+s}{\theta-\beta} \wedge \frac{1+s}{1-\beta}\right)\right), & \textit{for } \beta \neq \theta \\ 1-l, & \textit{if } \beta = \theta \textit{ and } s = -\theta \\ \left(0 \vee \frac{1+s}{1-\theta}\right), & \textit{if } \beta = \theta \textit{ and } s \neq -\theta \end{cases} \tag{13}$$

$$l(s,\beta) = \begin{cases} \left(\left(0 \vee \frac{\beta+s}{\beta-\theta}\right) \wedge 1\right), & \textit{for } \beta \neq \theta \\ l \in [0,1], & \textit{for } \beta = \theta \textit{ and } s = -\theta \\ 0, & \textit{for } \beta = \theta \textit{ and } s \neq -\theta \end{cases} \tag{14}$$

Proof. The firm will not liquidate and grow capital at the same time, otherwise the firm bears loss in liquidation, which is avoidable if the firm keeps the capital.

If the firm liquidates, $l\theta + s + b = 0 \Rightarrow l = -\frac{s+b}{\theta}$. Then

$$\begin{aligned} \frac{K_{t+1}}{\widehat{K}_t} &= 1 + s + b - l(1-\theta) = 1 - l \\ &= 1 + \frac{s+b}{\theta} = \begin{cases} 1-l, & \text{if } \beta = \theta \text{ and } s = -\theta \\ (\text{cannot have liquidation}), & \text{if } \beta = \theta \text{ and } s \neq -\theta \\ \frac{\theta+s}{\theta-\beta}, & \text{otherwise.} \end{cases} \end{aligned}$$

If the firm grows capital, $\frac{K_{t+1}}{\widehat{K}_t} = \frac{1+s}{1-\beta}$. ■

This proposition claims that ϕ' and l are functions of β and s in equilibrium, and the firm's choice variables can be reduced to only β. This alternative is useful to simplify the problem when computing the numerical solutions.

PROPOSITION 4.2 *In all three cases, given the growth rate g, cash flow v per unit of capital, and ending of the firm at a rate ρ, the firm's value per unit of capital satisfies $C(s) \leq 1+s+\frac{(1-\rho)(1+g)}{\rho-g+\rho g}v$, and $\lim_{s\to\infty} C(s) = 1+s+\frac{(1-\rho)(1+g)}{\rho-g+\rho g}v$.*

Proof. Since firm's growth is bounded by g in each period, the value is capped by growing at maximum in each period perpetually, equivalently $\frac{(1-\rho)(1+g)}{\rho-g+\rho g}v$. When the firm's cash surplus s is approaching infinity, the firm achieves (or tends to achieve) the maximum perpetual growth, and an increment of s raises firm's value at a one-for-one rate. ■

Bond pricing

For further analysis we assume $\tilde{\delta}'$ follows a uniform distribution in $(\bar{\delta}, \underline{\delta})$, without loss of generality.

The firm maximizing the form owners' wealth should always offer a fair bond price in equilibrium that equals to the face value subtracts the loss from bankruptcy. Without gambling, it is explicitly a function of face value ϕ':

$$\beta(\phi') = \phi' + \mathrm{E}\big[\big(\frac{v-\phi'}{\tilde{\delta}'} < \underline{s}\big)\big((1-c)(v+\theta\tilde{\delta}') - \phi'\big)^{-}\big] \tag{15}$$

Figure 5 shows two possibilities: risky borrowing and risk free borrowing (only). The curves represent the balance between borrowing and risk of bankruptcy. When the face value (compared to capital level) is small, borrowing can be less risky and the bondholders obtain full repayment more often and is demonstrated by the linear (or close to linear) part of the curve. As the face value becomes greater, the risk of default becomes bigger and bond value has smaller increment and can be decreasing when the face value is high enough. The black solid upward curves are efficient borrowing frontiers which represent the set of the bond contracts that the firm would choose in equilibrium. The firm can never write a contract beyond the maximum borrowing and the upper bound is an endogenous borrowing constraint. Comparing the two graphs, when capital shocks are relatively volatile (Figure 5 left graph), firms may choose risky debt; when capital is "stable,"

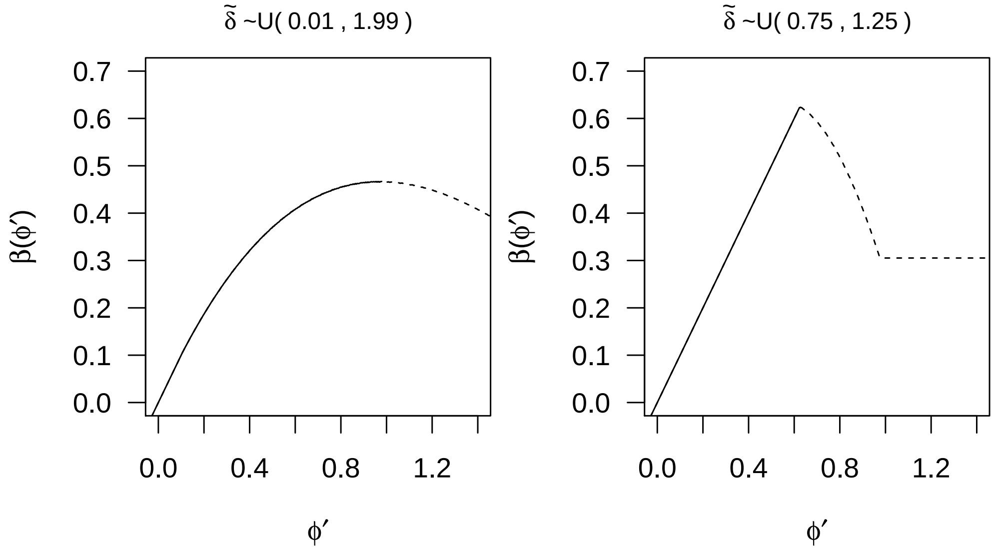


Figure 5: ***Bond pricing without gambling*** *(*$\theta = 0.5$*,* $c = 0.5$*,* $v = 0.1$*,* $\theta = 0.5$*,* $g = 0.05$*,* $\rho = 0.25$*,* $\underline{s} = -0.7$*) bond value as a function of face value. Feasible bond pricing (solid curves) represents the rational choice set of the bond contracts that satisfy the pricing function* (15).

borrowing is always safe.[21]

Bond pricing with gambling requires explicit functional expression of gambling and the continuation value. Figure 6 provides an example. We can observe some sensible features: for each curve the upward linear segment implies a risk-free bond absent bankruptcy risk when face value is small. As face value is getting bigger, the growing possibility of bankruptcy generates higher costs which offset the promise of higher repayment.

As can be seen, bond pricing has a feature of credit rationing even with the presence of perfect capital market: with greater promise of repayment, bankruptcy costs are also greater. with gambling, when face value is getting bigger, "gambling for ripoff" is more often when the firm can gamble assets, which value is depleted eventually.

Equity value

Table 1 shows the parameter values we use for the numerical exercise in this section:[22] We fix g, ρ and $\tilde{\delta}$ and vary v and θ. The result is shown in Figure 7.

We can observe the effects of gambling and superpriority separately. Gambling and superprior-

[21]Safe borrowing is probably not an interesting case, therefore we mainly focus on the situations in which capital volatility is large.

[22]The hazard ratio ρ is set large because we want the curves to converge quicker. The capital shock is also set with a large range because we are more interested in the case with risky borrowing.

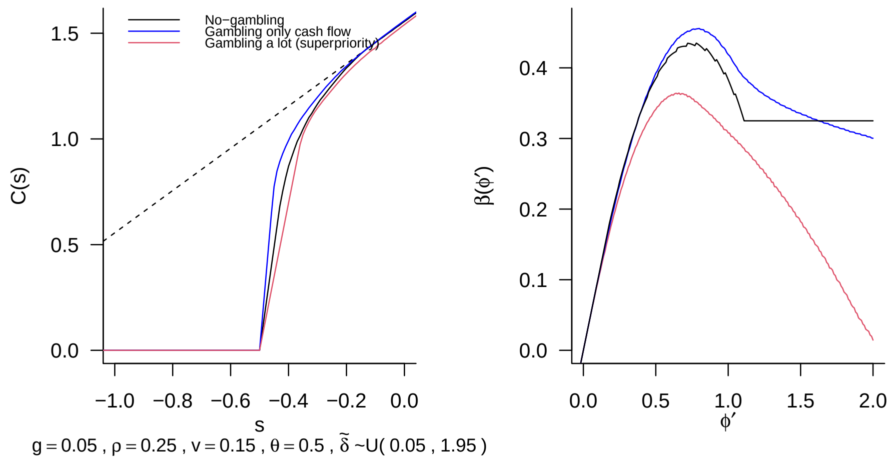


Figure 6: ***Firm value (left) and Bond value as a function of face value(right).***

g	0.05
ρ	0.25
c	0.5
$\tilde{\delta}$	$U(0.05, 1.95)$
v	$\{0.05, 0.1, 0.2\}$
θ	$\{0, 0.5, 0.7\}$

Table 1: Parameter values.

ity matter when the firm is in trouble, i.e., when the firm has low level of cash surplus. Comparing the black curves and the blue curves, we can find that the pure effect of gambling without gambling assets can be either beneficial or damaging depending on liquidation value per unit of capital. The first row shows cases in which the firm's assets have no market value and superpriority plays no role. Then we can observe that gambling is actually value enhancing probably because gambling for redemption dominates. The bondholders are also more willing to lend and the borrowing constraints are relaxed. All the value of gambling is reflected in the equity value (since bond is fair). The benefit of gambling is enlarged as the cash flow is greater - we assume cash flow is fixed for simplicity, but the idea is that higher cash flow increases the chance of winning a fair gamble and thus further alleviates borrowing and makes the firm better off. This result goes against previous work concluding that extra risk taking by firm is bad. For bigger liquidation value, the comparison between the black and blue curves suggests that the advantage of gambling tends to be smaller as borrowing may rely more on asset value instead of gaining from gambling, probably because borrowing grows faster than continuation value, resulting in the fewer contingencies of "gambling for redemption." Firm value is largely impaired when gambling assets is plausible. Compared to gambling without superpriority, two effects may contribute to the dissipation: first, the assets are diluted because the ability of asset redeployment by the firm owners; second, "gambling for ripoff" would be more likely to prevail in a lot of the cases.

The effect of superpriority can be shown by comparing the blue solid curves and red dashed curves. Both curves coincide in the first row, because $\theta = 0$ makes gambling problems of the owners exactly the same with and without superpriority. These graphs suggest that. When $\theta > 0$, equity value tends to be smaller when the firm can use the liquidation value to gamble. Of course, when the cash surplus s is very negative (for example in some graphs $s < -0.5$), a firm has no value with and without superpriority. With larger cash surplus (for example in some graphs $s > 0$), the blue curves and red curves tend to converge since there is very small probability that the firm has to gamble. This is also shown by the convergence to the black (no gambling) curves as well as the dashed black lines, which represents the maximum firm values.

Optimal gambling

The optimal gambling can be found following the same logic as in the single-period model, that is, by "concavifying" the value function $C(s)$ for each contingency. Notice that s' is the pre-gambling cash surplus in the next period (before gambling), which can be used to target where can the firm gamble to. For example, given any s' and $\frac{\phi'}{\delta'}$, the firm can use s' to gamble down to $-\frac{\phi'}{\delta'}$ without gambling assets (or to $-\frac{\phi'}{\delta'} - \theta$ with gambling assets) and up to a tangent point of $C(s)$ and the linear line going through point $(-\frac{\phi'}{\delta'}, 0)$ (or $(-\frac{\phi'}{\delta'} - \theta, 0)$ with gambling assets). By using the optimal gambling and the value function, we can compute the pre-gambling value functions of the firm and the bond, denoted as $\widehat{C}(s', \frac{\phi'}{\delta'})$ and $\widehat{\phi}(s', \frac{\phi'}{\delta'})$, respectively.

We first define

$$s_0 \equiv -1 - \frac{(1-\rho)(1+g)}{\rho - g + \rho g} v, \quad s_1 \equiv \underline{s} - \frac{C(\underline{s})}{C'(\underline{s})},$$

where s_0 is the interception of the upper bound of value function (see Prop. 4.2) and the x axis, and s_1 is the smallest s on the x axis through which the tangent point on the value function $C(s)$ is the "kink." (see Figure 8.) Also define

$$i(-\frac{\phi'}{\delta'}) \equiv -\frac{\phi'}{\delta'} - \mathbb{1}_s \theta$$

and

$$I(-\frac{\phi'}{\delta'}) \equiv \begin{cases} +\infty, & \text{if } i(-\frac{\phi'}{\delta'}) \leq s_0 \\ \arg\max_s \frac{C(s)}{s - i(-\phi'/\delta')}, & \text{if } s_0 < i(-\frac{\phi'}{\delta'}) \end{cases}$$

where $i(-\frac{\phi'}{\delta'})$ is the minimum value that the firm can gamble. Absent superpriority, the firm can only gamble down to $-\frac{\phi'}{\delta'}$, while the firm can gamble down to $-\frac{\phi'}{\delta'} - \theta$ if there is superpriority.

$I(-\frac{\phi'}{\delta'})$ is the tangent point of $C(s)$ and the straight line going through $i(-\frac{\phi'}{\delta'})$, if not infinite.

If $i(-\frac{\phi'}{\delta'})$ falls on the left of s_0, we cannot find a tangent point of $i(-\frac{\phi'}{\delta'})$ on the value function $C(s)$, thence the firm will gambling for "ripoff." If $i(-\frac{\phi'}{\delta'})$ falls between s_1 and $\underline{s}$, then the tangent point is exactly the "kink," and the firm gambles for redemption. Going through any point in between s_0 and s_1 we can always find tangent point(s) on $C(s)$, and the tangent point(s) should be the odds that the firm gamble towards. Proposition 4.3 formally states the gambling feature:

PROPOSITION 4.3 ***(Optimal gambling)*** *Given ϕ', δ',*

1. *the optimal gambling for the firm is*

$$\mathbf{s}^*(\tilde{x}, \phi', \delta') = \begin{cases} I(-\frac{\phi'}{\delta'}) \vee s', & \text{if } 0 < x < w(s', \frac{\phi'}{\delta'}) \\ i(-\frac{\phi'}{\delta'}), & \text{if } w(s', \frac{\phi'}{\delta'}) \le x < 1 \end{cases}$$

where $w(s', \frac{\phi'}{\delta'}) \equiv \frac{s' - i(-\phi'/\delta')}{I(-\phi'/\delta') \vee s' - i(-\phi'/\delta')}$ is the probability/weight of the firm gets up in value. This result indicates that if $I(-\frac{\phi'}{\delta'}) \le s'$, then $\mathbf{s}^*(\tilde{x}, \phi', \delta') \equiv s'$ (i.e., the firm does not choose to gamble); otherwise, if $I(-\frac{\phi'}{\delta'}) > s'$, the firm gambles s' up to $I(-\frac{\phi'}{\delta'})$, and down to $i(-\frac{\phi'}{\delta'})$.

2. *for $I(-\frac{\phi'}{\delta'}) > s'$, define*

$$\sigma(-\frac{\phi'}{\delta'}) \equiv \sup_{s \ge \underline{s}} \frac{C(s)}{s - i(-\phi'/\delta')}, \quad \gamma(-\frac{\phi'}{\delta'}) \equiv \begin{cases} 0, & \text{if } i(-\frac{\phi'}{\delta'}) \le s_0, \\ \frac{\phi'}{I(-\phi'/\delta') - i(-\phi'/\delta')}, & \text{if } s_0 < i(-\frac{\phi'}{\delta'}). \end{cases}$$

Then the value functions of the firm and the bond before gambling are

$$\textbf{\textit{firm:}}\ \widehat{C}(s', \frac{\phi'}{\delta'}) = \begin{cases} \sigma(-\frac{\phi'}{\delta'})(s' - i(-\frac{\phi'}{\delta'})), & I(-\frac{\phi'}{\delta'}) > s', \\ C(s'), & I(-\frac{\phi'}{\delta'}) \le s'. \end{cases} \tag{16}$$

$$\textbf{\textit{bond:}}\ \widehat{\phi}(s', \frac{\phi'}{\delta'}) = \begin{cases} \gamma(-\frac{\phi'}{\delta'})(s' - i(-\frac{\phi'}{\delta'})), & I(-\frac{\phi'}{\delta'}) > s', \\ \phi', & I(-\frac{\phi'}{\delta'}) \le s'. \end{cases} \tag{17}$$

or, equivalently,

$$\textbf{\textit{firm:}}\ \widehat{C}(s', \frac{\phi'}{\delta'}) = w(s', \frac{\phi'}{\delta'}) \cdot C\left(I(-\frac{\phi'}{\delta'}) \vee s'\right), \tag{18}$$

$$\textbf{\textit{bond:}}\ \widehat{\phi}(s', \frac{\phi'}{\delta'}) = w(s', \frac{\phi'}{\delta'}) \cdot \phi'. \tag{19}$$

The optimal gambling has a feature different from the existing literature: the firm does not always choose extreme risks. Rather, gambling has a mixed feature of gambling for redemption and gambling for ripoff and is continuous rather than jumping between extremes.

The way of how gambling works can be easily applied to other settings. For example, if the firm value has a decreased margin when cash flow increases, gambling for extreme ripoff may not happen since we can find a finite tangent point on the flatter segment of the function. Yet the conclusion that gambling is bigger and more damaging with higher level of liability and liquidation value is still true.

5 Conclusions

We provided a simple framework to analyze gambling by firms: "gambling for redemption" is a Pareto improvement and occurs when the firm owners are willing to maintain the firm, whereas "gambling for ripoff" can be socially costly and occurs when continuing a firm is no longer beneficial. By making gambling some of the assets possible, superpriority law lowers the value lost in bankruptcy and increases the incentives for the firm owners to gamble for ripoff. In the more realistic intertemporal model with endogenous borrowing and endogenous continuation value, the firm's choices of gambling are mixtures of gambling for redemption and ripoff. We find that superpriority increases the scale of gambling taken by the firm and makes funding more difficult (through traditional way of financing). Our results suggest an interesting empirical question for which we do not have a good answer: how do we distinguish "gambling for redemption" and "gambling for ripoff" ex post since they look the same in the case of failure?

One possible implication of superpriority law will be the adoption of financing that reduces the scale of superpriority gambling. One possibility is the adoption by bond issuers of more defensive measures that protect against superpriority claims. For example, it may be more common to protect bonds to specific perfected collateral instead of passive covenants claiming the preclusion of asset sales and security transfers. It may also incentivize the bondholders to issue short-term bonds which have less exposure to a stay in bankruptcy, or even use repos which are also protected against bankruptcy. The substitution away from traditional financing to repo financing can cause an asset grab race which undermines the purposes of bankruptcy law to facilitate an orderly liquidation (or reorganization) and to give a breathing space for the firm owners to resolve financial difficulties.

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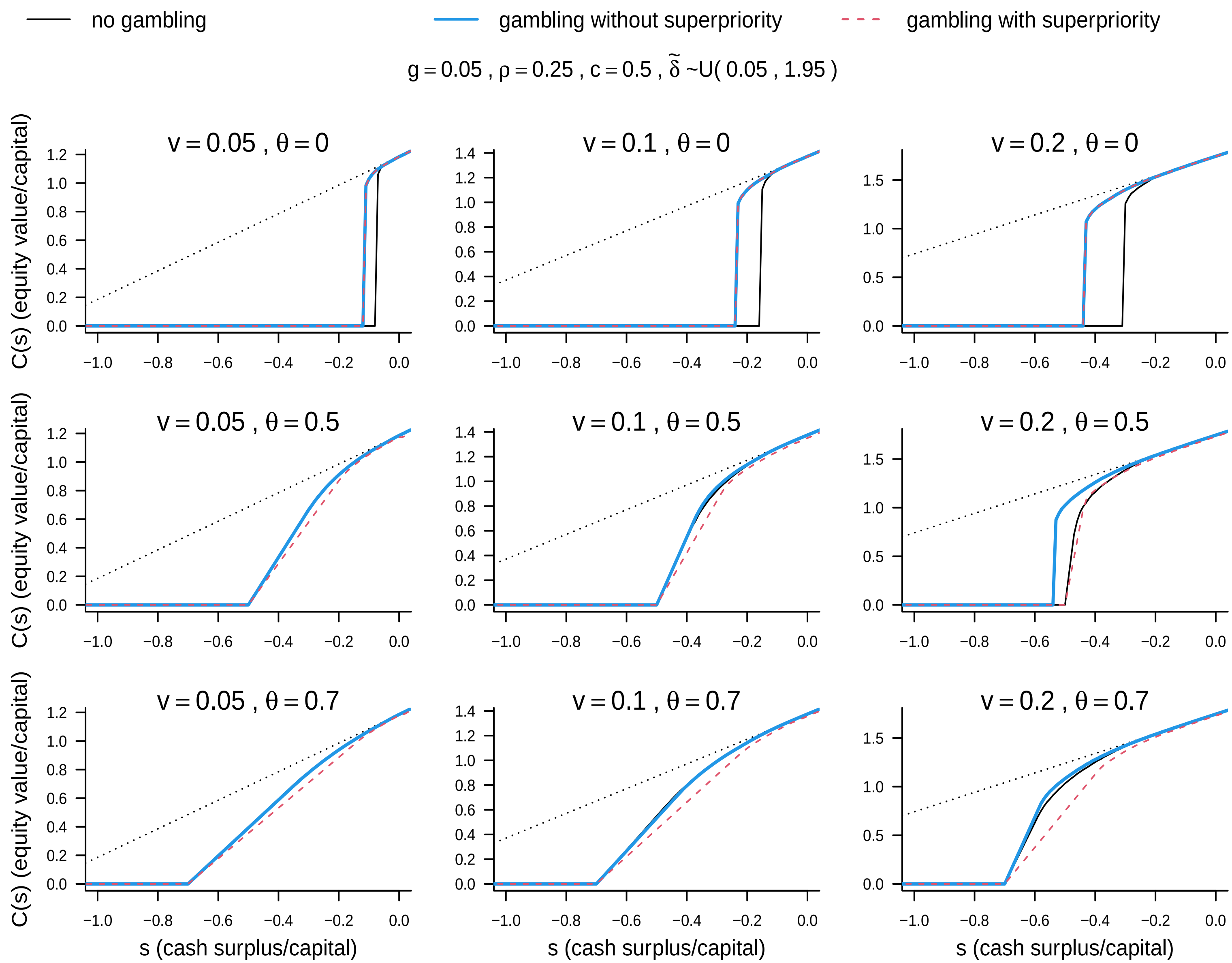

Figure 7: ***(equity value/capital as a function of cash surplus/capital)***
v : cash flow/capital; θ : liquidation value available/capital. If $\theta = 0$, superpriority is irrelevant, and increasing θ implies increasing damage from superpriority, especially when v is large.

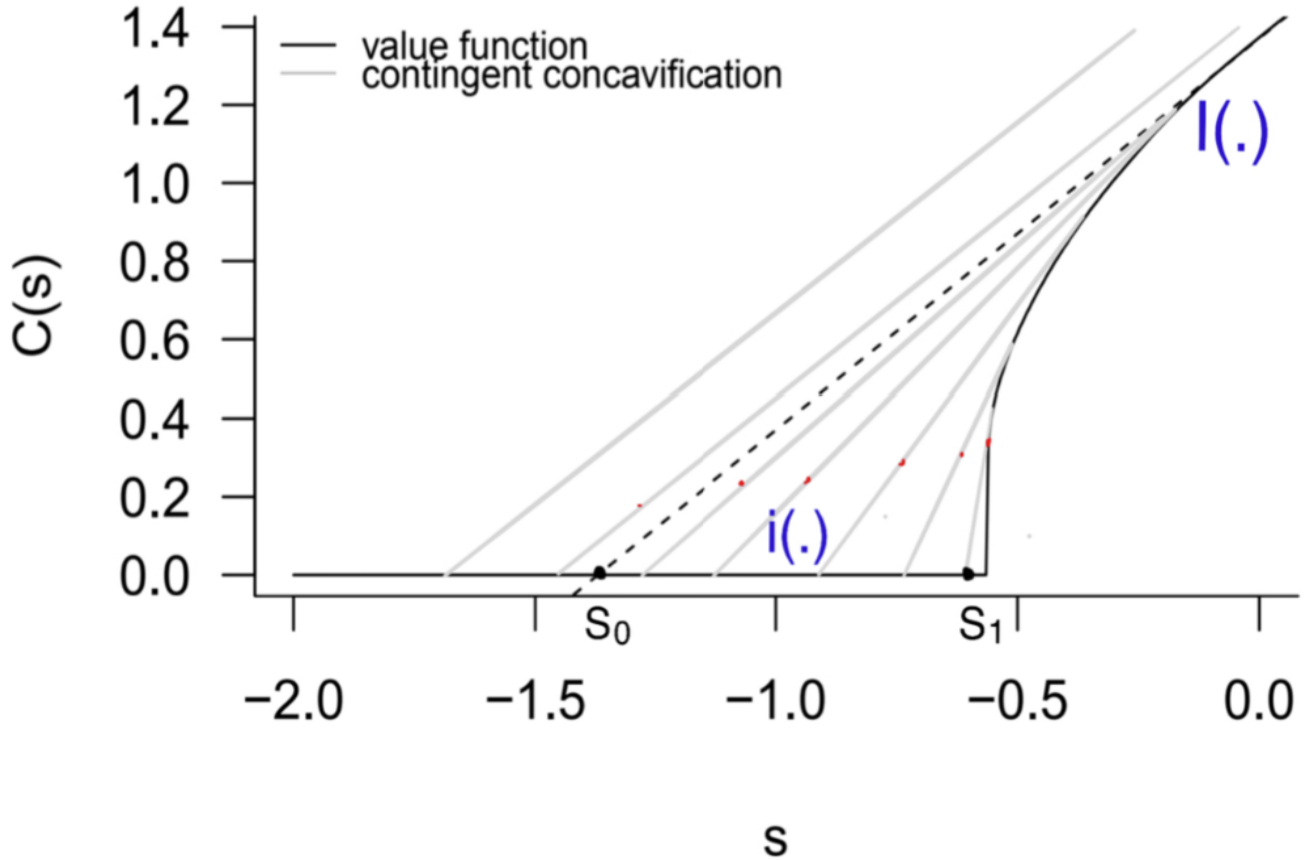

Figure 8: ***An example of gambling*** *gambling has a mixed feature of "gambling for redemption" and "gambling for ripoff" and is continuous rather than jumping between extremes.*

Appendix A.

Notice that this proof works for the superpriority case when liquidation value L is used for gambling, but to prove the no-superpriority case, we can simply assume $L = 0$. Given constants $F, C, \pi \in \mathbb{R}_{++}$, and $L \in \mathbb{R}_{+}$, the question becomes

$$\max_{\mathbf{p}(x)} \int_0^1 \left\{ (\mathbf{p}(x) \geq F)(\mathbf{p}(x) - F + C) \right\} dx$$
$$\text{s.t.} \int_0^1 \mathbf{p}(x) dx = 1, \text{ and } -L \leq \mathbf{p}(x) \leq \bar{\pi}$$

To get the necessary conditions for the solution, we first concavify the function $(\mathbf{p}(\tilde{x}) \geq F)(\mathbf{p}(\tilde{x}) - F + C)$ to make it continuous.

Gambling for redemption: When $C > F + L$, define the concavified function

$$G(\mathbf{p}(x)) \equiv (\mathbf{p}(x) < F) \frac{C}{F+L} (\mathbf{p}(x) + L) + (\mathbf{p}(x) \geq F)(\mathbf{p}(x) - F + C)$$

Assume that $q(x) = \int_0^x \mathbf{p}(t) dt$, then $q'(x) = \mathbf{p}(x)$. For short, we use $\mathbf{p}$ to represent $\mathbf{p}(x)$. Rewrite the concavified problem

$$\max_{\mathbf{p}} \int_0^1 G(\mathbf{p}) dx$$
$$\text{s.t. } q(0) = 0, q(1) = \pi,$$
$$q' = \mathbf{p}, \text{ and } -L \leq \mathbf{p} \leq \bar{\pi}.$$

$G(\mathbf{p})$ is continuous together with its partial derivative $\frac{\partial G(\mathbf{p})}{\partial q} \equiv 0$, and is piece-wise smooth in $\mathbf{p}$. We can assume that $\mathbf{p}(\tilde{x})$ is piece-wise smooth in x, and $q(x)$ is continuous and piece-wise smooth. Thus, we have the necessary conditions for the solution.

Since $G(\mathbf{p}(x)) \geq (\mathbf{p}(\tilde{x}) \geq F)(\mathbf{p}(\tilde{x}) - F + C)$, then any solution of the concavified problem falling into the domain of the original function $(\mathbf{p}(\tilde{x}) \geq F)(\mathbf{p}(\tilde{x}) - F + C)$ is also the solution of the original problem.

Then $\mathbf{p}(\tilde{x})$ is to be chosen at each x to maximize the Hamiltonian

$$\mathscr{H}(\lambda, \mathbf{p}) = G(\mathbf{p}) + \lambda \mathbf{p}, \text{ s.t. } -L \leq \mathbf{p} \leq \bar{\pi}.$$

The Lagrangian, with the multipliers w_1 and w_2, of the new problem, is

$$\mathscr{L}(\lambda, w_1, w_2, \mathbf{p}) = G(\mathbf{p}) + \lambda \mathbf{p} + w_1(\mathbf{p} + L) + w_2(\bar{\pi} - \mathbf{p})$$

When $C > F$, necessary conditions for $\mathbf{p}$ to be maximizing are

$$0 = \mathscr{L}_p(\lambda, w_1, w_2, \mathbf{p}) = \begin{cases} [\frac{C}{F+L} + \lambda + w_1 - w_2, +\infty), & \text{for } \mathbf{p} = -L \\ \frac{C}{F+L} + \lambda + w_1 - w_2, & \text{for } -L < \mathbf{p} < F \\ [1 + \lambda + w_1 - w_2, \frac{C}{F+L} + \lambda + w_1 - w_2], & \text{for } \mathbf{p} = F \\ 1 + \lambda + w_1 - w_2, & \text{for } F < \mathbf{p} < \bar{\pi} \\ [0, 1 + \lambda + w_1 - w_2], & \text{for } \mathbf{p} = \bar{\pi} \end{cases}$$

$$w_1 \geq 0, \quad w_1(\mathbf{p} + L) = 0, \text{ and}$$
$$w_2 \geq 0, \quad w_2(\bar{\pi} - \mathbf{p}) = 0.$$

Further,

$$\lambda' = -\partial \mathscr{H} / \partial q = -\partial (G(\mathbf{p}) + \lambda \mathbf{p}) / \partial q = 0,$$

so that $\lambda(x)$ is a constant, for all x. Then,

$$\lambda(x) = \begin{cases} (-\infty, -w_1 - \frac{C}{F+L}], & \text{for } \mathbf{p} = -L, (w_2 = 0) \\ -\frac{C}{F+L}, & \text{for } -L < \mathbf{p} < F, (w_1 = w_2 = 0) \\ [-\frac{C}{F+L}, -1], & \text{for } \mathbf{p} = F, (w_1 = w_2 = 0) \\ -1, & \text{for } F < \mathbf{p} < \bar{\pi}, (w_1 = w_2 = 0) \\ [-1 + w_2, +\infty), & \text{for } \mathbf{p} = \bar{\pi}, (w_1 = 0) \end{cases}$$

Since $-w_1 - \frac{C}{F+L} \leq -\frac{C}{F+L} < -1 \leq -1 + w_2$, we cannot have $\mathbf{p} < F$ and $\mathbf{p} > F$ at the same time as a solution of $\mathbf{p}$. Then the solution should be either $\mathbf{p}(x) \leq F$, $\forall x$, or $\mathbf{p}(x) \geq F$, $\forall x$.

- If $\pi < F$, we cannot have $\mathbf{p}(x) \geq F$, $\forall x$, otherwise $\int_0^1 \mathbf{p}(x)dx \geq F > \pi$, which does not satisfy the constraint. Therefore, $\mathbf{p}(x) \leq F$, $\forall x$. If we further constrain the value $\mathbf{p}(x) \in \{0\} \cup [F, \bar{\pi}]$, we have $\lambda = -\frac{C}{F+L}$ and the unique solution (we assume decreasing $\mathbf{p}(x)$ with respect to x)

  $$\mathbf{p}^*(\tilde{x}) = \begin{cases} F, \text{ if } x \leq \frac{\pi+L}{F+L} \\ -L, \text{ otherwise} \end{cases}$$

  This applies to the original problem when $L = 0$. Proposition 2.1(2) is proved.

- If $\pi > F$, we cannot have $\mathbf{p}(x) \leq F$ for the same reason. Then any randomization of $\mathbf{p}(\tilde{x})$ above or equal to F would be optimal. Of course, this include $\mathbf{p}(\tilde{x}) \equiv \pi$, which is not to gamble at all. If $\pi = F$, then the only possible solution is $\mathbf{p}(\tilde{x}) = F$, for all x. This proves

Proposition 2.1(1).

Gambling for rip-off: when $C < F + L$, the concavified function is $G(\mathbf{p}) = \frac{\bar{\pi}-F+C}{\bar{\pi}+L}(\mathbf{p}+L)$. The Lagrangian is

$$\mathscr{L}(\lambda, w_1, w_2, \mathbf{p}) = \frac{\bar{\pi} - F + C}{\bar{\pi} + L}(\mathbf{p}+L) + \lambda \mathbf{p} + w_1(\mathbf{p}+L) + w_2(\bar{\pi} - \mathbf{p}).$$

The necessary conditions are

$$\begin{aligned}
&\lambda' = 0,\\
&0 = \mathscr{L}_p(\lambda, w_1, w_2, \mathbf{p}) = \begin{cases} [1 - \frac{L+F-C}{\bar{\pi}+L} + \lambda + w_1 - w_2, +\infty), & \text{for } \mathbf{p} = -L \\ 1 - \frac{L+F-C}{\bar{\pi}+L} + \lambda + w_1 - w_2, & \text{for } -L < \mathbf{p} < \bar{\pi} \\ [0, 1 - \frac{L+F-C}{\bar{\pi}+L} + \lambda + w_1 - w_2], & \text{for } \mathbf{p} = \bar{\pi} \end{cases}\\
&w_1 \geq 0, \quad w_1(\mathbf{p}+L) = 0,\\
&w_2 \geq 0, \quad w_2(\bar{\pi} - \mathbf{p}) = 0.
\end{aligned}$$

Similarly,

$$\lambda(x) = \begin{cases} (-\infty, -w_1 - 1 + \frac{L+F-C}{\bar{\pi}+L}], & \text{for } \mathbf{p} = -L, (w_2 = 0) \\ -1 + \frac{L+F-C}{\bar{\pi}+L}, & \text{for } -L < \mathbf{p} < \bar{\pi}, (w_1 = w_2 = 0) \\ [-1 + \frac{L+F-C}{\bar{\pi}+L} + w_2, +\infty), & \text{for } \mathbf{p} = \bar{\pi}, (w_1 = 0) \end{cases}$$

When $w_1 = w_2 = 0$, any randomization along the line is the optimal solution. Specifically, if we constrain the solution on $\mathbf{p}(\tilde{x}) \in \{-L, \bar{\pi}\}$, which is the set falling into the domain of the original problem. We have $\lambda = -1 + \frac{L+F-C}{\bar{\pi}+L}$ and the unique solution

$$\mathbf{p}^*(\tilde{x}) = \begin{cases} \bar{\pi}, \text{ if } x \leq \frac{\pi+L}{\bar{\pi}+L} \\ -L, \text{ otherwise} \end{cases}$$

This proves Proposition 2.2.

Appendix B. The trajectory and history

This section provides a rigorous description of the information sets and timing of events for reference.

A trajectory represented as $\mathscr{P}$ is a possible path of events known to both the firm and the prospective lender:

$$\mathscr{P} = (F_0, K_0, \delta_0, S_0, \mathbf{S_0}, \widehat{S}_0, \widehat{K}_0, l_0, B_0, F_1, K_1, \delta_1, S_1, \mathbf{S_1}, \widehat{S}_1, \widehat{K}_1, l_1, B_1, F_2, K_2, \delta_2, S_2, \mathbf{S_2}, \widehat{S}_2, \widehat{K}_2, l_2, B_2, \ldots).$$

where $S_t \equiv vK_t - F_t$ is the implied cash surplus defined as the excessive cash after repaying the maturing debt in full (or more properly called the "shortfall" if smaller than zero). Gambling choice $\mathbf{S_t}$ randomizes the cash surplus S_t, and the realization is $\widehat{S}_t$. Therefore, the case without gambling is a special case in which $\mathbf{S_t} = \widehat{S}_t = S_t$.

The history at time t contains the first $7+9t$ elements in the relevant trajectory, denoted

$$\mathscr{H}^{7+9t} \equiv \mathscr{P}_{1:(7+9t)} = (F_0, K_0, \delta_0, \ldots, F_t, K_t, \delta_t, S_t, \mathbf{S_t}, \widehat{S}_t, \widehat{K}_t).$$

Along the path, we have

$\widehat{S}_t$: the cash surplus after gambling in period t *(state variable)*

$\widehat{K}_t$: capital after the shock in period t *(state variable)*

$l_t = l^*(\mathscr{H}^{7+9t})$: the fraction of liquidation *(choice variable)*

$B_t = B^*(\mathscr{H}^{7+9t})$: new borrowing *(choice variable)*

$F_{t+1} = F^*(\mathscr{H}^{7+9t})$: face value of the debt *(choice variable)*

$K_{t+1} = ((1-l_t)\widehat{K}_t + l_t\theta\widehat{K}_t + \widehat{S}_t + B_t)(l_t\theta\widehat{K}_t + \widehat{S}_t + B_t \geq 0)$

δ_{t+1} : the realized capital shock at $t+1$

$\widehat{K}_{t+1} = \delta_{t+1}K_{t+1}$: capital after the shock at $t+1$

$\mathbf{S_{t+1}}$: gambling choice subject to $\mathrm{E}\left[\mathbf{S_{t+1}}\right] = S_{t+1}$, and $\mathbf{S_{t+1}} \geq -F_{t+1}$ without superpriority, or $\mathbf{S_{t+1}} \geq -\theta\widehat{K}_{t+1} - F_{t+1}$ with superpriority *(choice variable)*

We intentionally place $\widehat{S}_t$ and $\widehat{K}_t$ at the beginning of the cycle and the choice of gambling in the end, so that the timeline coincides with the firm owners' problem. There are different time points in a period at which we can look at the equity value, but we choose the moment after the realization of gambling. The purpose is threefold: firstly, to have only two state variables in the firm's Bellman equation instead of three before gambling; secondly, to have a comparable value function to our benchmark case; thirdly, to mirror the continuation value in the single-period model to firm owners' value function, upon which the gambling strategy should depend.