– **9. Conjugate normal model**

Suppose y is an i.i.d sample of size n from the distribution $N(\mu, \sigma^2)$. Prior for (μ, σ^2) is

$$\textbf{N-Inv-}\chi^2(\mu, \sigma^2 | \mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$$

This is derived from

$$\begin{aligned} \mu | \sigma^2 &\sim N(\mu_0, \sigma^2/\kappa_0) \\ \sigma^2 &\sim \textbf{Inv-}\chi^2(\nu_0, \sigma_0^2) \end{aligned}$$

Notice the formula for Inv-$\chi^2(\nu, s^2)$ is

$$p(\theta) \propto s^\nu \theta^{-(\nu/2+1)} e^{1/(2\theta)}$$

and N-Inv-$\chi^2(\mu, \sigma^2 | \mu_0, \sigma_0^2/\kappa_0; \nu_0, \sigma_0^2)$

$$p(\mu, \sigma^2) \propto \sigma^{-1} (\sigma^2)^{-(\nu_0/2+1)} \exp\left(-\frac{1}{2\sigma^2}[\nu_0\sigma_0^2 + \kappa_0(\mu_0 - \mu)^2]\right)$$

Therefore, the posterior

$$\begin{aligned} p(\mu, \sigma^2 | y) \propto p(\mu, \sigma^2) \prod_{i=1}^{n} p(y_i | \mu, \sigma^2) \\ = \sigma^{-1} (\sigma^2)^{-(\nu_0/2+1)} \exp\left(-\frac{1}{2\sigma^2}[\nu_0\sigma_0^2 + \kappa_0(\mu_0 - \mu)^2]\right) \times \\ (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y} - \mu)^2]\right) \end{aligned}$$

The rest arithmetic is rather dull, but I guess it's good to do it at least once.
First, merge the terms outside the $\exp(\cdots)$:

$$\sigma^{-1}\sigma^{-((\nu_0+n)/2+1)}$$

Let's focus on the terms inside $\exp(\cdots)$, discarding the exponential:

$$\begin{aligned} & -\frac{1}{2\sigma^2}[\nu_0\sigma_0^2 + \kappa_0(\mu_0 - \mu)^2] - \frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y} - \mu)^2] \\ &= -\frac{1}{2\sigma^2}\left(\nu_0\sigma_0^2 + (n-1)s^2 + \kappa_0(\mu_0 - \mu)^2 + n(\bar{y} - \mu)^2\right) \\ &= -\frac{1}{2\sigma^2}\left(\nu_0\sigma_0^2 + (n-1)s^2 + \kappa_0\mu_0^2 + n\bar{y}^2 + (\kappa_0 + n)\mu^2 - 2(\kappa_0\mu_0 + n\bar{y})\mu\right) \\ &= -\frac{1}{2\sigma^2}\left(\nu_0\sigma_0^2 + (n-1)s^2 + \kappa_0\mu_0^2 + n\bar{y}^2 + (\kappa_0 + n)\left(\mu - \frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0 + n}\right)^2 - \frac{(\kappa_0\mu_0 + n\bar{y})^2}{\kappa_0 + n}\right) \\ &= -\frac{1}{2\sigma^2}\left(\nu_0\sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2 + (\kappa_0 + n)\left(\mu - \frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0 + n}\right)^2\right) \end{aligned}$$

Let

$$
\begin{aligned}
\mu_n &= \frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y} \\
\kappa_n &= \kappa_0 + n \\
\nu_n &= \nu_0 + n \\
\nu_n \sigma_n^2 &= \nu_0 \sigma_0^2 + (n-1)s^2 + \frac{\kappa_0 n}{\kappa_0 + n}(\bar{y} - \mu_0)^2
\end{aligned}
$$

, then

$$
p(\mu, \sigma^2 | y) = \textbf{N-Inv-}\chi^2(\mu, \sigma^2 | \mu_n, \sigma_n^2/\kappa_n; \nu_n, \sigma_n^2)
$$