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- **1. Normal Approximations**

(a) Log posterior density:

$$\begin{aligned}\log p(\theta|y_1, \cdots, y_5) &= \log p(\theta) \log(y1, \cdots, y_5|\theta) \\ &= \sum_{i=1}^{5} \log p(y_i|\theta) \propto \sum_{i=1}^{5} \log \frac{1}{1+(y_i-\theta)^2} \\ &= -\sum_{i=1}^{5} \log\left(1+(y_i-\theta)^2\right)\end{aligned}$$

Hence, first derivative:

$$\frac{dp(\theta|y)}{d\theta} = 2\sum_{i=1}^{5} \frac{y_i-\theta}{1+(y_i-\theta)^2}$$

Second derivative:

$$\frac{d^2p(\theta|y)}{d\theta^2} = 2\sum_{i=1}^{5} \frac{(y_i-\theta)^2-1}{(y_i-\theta)^2+1}$$

(b) The posterior mode $\hat{\theta} = -0.125$

(c) The posterior normal approximation:

$$\begin{aligned}\log p(\theta|y) &\approx p(\hat{\theta}|y) + \frac{1}{2}(\theta-\hat{\theta})^2 \times \left[\frac{d^2p(\theta|y)}{d\theta^2}\right]_{\theta=\hat{\theta}} \\ &= -5.45 + \frac{1}{2} \times 1.30 \times (\theta+0.125)^2\end{aligned}$$

Therefore, the approximated posterior distribution is $\theta|y \sim N(-0.125, 0.877^2)$.

- **2. Normal Approximation**

Note: *Trivial arithmetic*

In bioassay example, we have the posterior

$$p(\alpha,\beta|y) = \prod_{i=1}^{4} \textbf{logit}^{-1}(\alpha+\beta x_i)^{y_i}\left(1-\textbf{logit}^{-1}(\alpha+\beta x_i)\right)^{n_i-y_i}$$

Denote $\text{logit}^{-1}(x)$ to be $f(x)$; hence, $f(x) = 1/\left(1+e^{-x}\right)$, the derivative $df(x)/dx = (1-f)f$. Using $f(x)$, the log likelihood:

$$\log p(\alpha,\beta|y) = \sum_{i=1}^{4} y_i \log f(\alpha+\beta x_i) + (n_i-y_i)\log\left(1-f(\alpha+\beta x_i)\right)$$

First compute mode:

$$\begin{aligned}\frac{\partial \log p(\alpha,\beta|y)}{\partial\alpha} &= \sum_{i=1}^{4} y_i - n_i f(\alpha+\beta x_i) \\ \frac{\partial \log p(\alpha,\beta|y)}{\partial\beta} &= \sum_{i=1}^{4} x_i(y_i - n_i f(\alpha+\beta x_i))\end{aligned}$$

Second derivative:

$$\frac{\partial^2 \log p(\alpha, \beta|y)}{\partial \alpha^2} = -\sum_{i=1}^{4} n_i f(\alpha + \beta x_i)\,(1 - f(\alpha + \beta x_i)) \tag{1}$$

$$\frac{\partial^2 \log p(\alpha, \beta|y)}{\partial \alpha \partial \beta} = -\sum_{i=1}^{4} n_i x_i f(\alpha + \beta x_i)(1 - f(\alpha + \beta x_i)) \tag{2}$$

$$\frac{\partial^2 \log p(\alpha, \beta|y)}{\partial \alpha \partial \beta} = -\sum_{i=1}^{4} n_i x_i^2 f(\alpha + \beta x_i)(1 - f(\alpha + \beta x_i)) \tag{3}$$

The information matrix

$$I(\alpha, \beta) = \begin{bmatrix} \sum_{i=1}^{4} n_i f(\alpha + \beta x_i)\,(1 - f(\alpha + \beta x_i)) & \sum_{i=1}^{4} n_i x_i f(\alpha + \beta x_i)(1 - f(\alpha + \beta x_i)) \\ \sum_{i=1}^{4} n_i x_i f(\alpha + \beta x_i)(1 - f(\alpha + \beta x_i)) & \sum_{i=1}^{4} n_i x_i^2 f(\alpha + \beta x_i)(1 - f(\alpha + \beta x_i)) \end{bmatrix}$$