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– **1. Exchangeability with known model parameters**

(a) $p(y_1 = \mathrm{b}, y_2 = \mathrm{w}) = p(y_1 = \mathrm{w}, y_2 = \mathrm{b}) = 1/4$. Observations y_1 and y_2 are exchangable. We put the balls back, ergo y_1 and y_2 are independent.

(b) y_1 and y_2 are exchangable, but not independent.

(c) y_1 and y_2 are exchangable, but not independent. However, there are 1 million balls for each color, and we only take twice. Hence, we can act as if two observations are independent.

– **5. Mixtures of independent distributions**

W.L.O.G, let's compute the covariance of θ_i and θ_j:

$$\begin{aligned}
\mathrm{cov}(\theta_i, \theta_j) &= \int_{\theta_i}\int_{\theta_j} p(\theta_i, \theta_j)(\theta_i - \mu_i)(\theta_j - \mu_j) d\theta_i d\theta_j \\
&= \int_{\alpha}\int_{\theta_i}\int_{\theta_j} p(\theta_i, \theta_j|\alpha)p(\alpha)(\theta_i - \mu_i)(\theta_j - \mu_j) d\alpha d\theta_i d\theta_j \\
&= \int_{\alpha}\int_{\theta_i}\int_{\theta_j} p(\theta_i|\alpha)(\theta_i - \mu_i) d\theta_i p(\theta_j|\alpha)(\theta_j - \mu_j) d\theta_j p(\alpha) d\alpha \\
&= \int_{\alpha}\left(\int_{\theta} p(\theta|\alpha)(\theta - \mu) d\theta\right)^2 p(\alpha) d\alpha \\
&\geq 0
\end{aligned}$$

– **Continuous mixture models**

(a) We have

$$\begin{aligned}
y|\theta &\sim \frac{1}{y!}\theta^y e^{-\theta} \\
\theta|\alpha, \beta &\sim \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}
\end{aligned}$$

Therefore, the prior predictive distribution of y:

$$\begin{aligned}
p(y|\alpha, \beta) &= \int_{\theta} p(y, \theta|\alpha, \beta) d\theta \\
&= \int_{\theta} p(y|\theta)p(\theta|\alpha, \beta) d\theta \\
&= \frac{\beta^\alpha}{y!\Gamma(\alpha)}\int \theta^{\alpha+y-1}e^{-(\beta+1)\theta} d\theta \\
&= \frac{\Gamma(\alpha+y)}{y!\Gamma(\alpha)}\frac{\beta^\alpha}{(\beta+1)^{\alpha+y}}\int \frac{(\beta+1)^{\alpha+y}}{\Gamma(\alpha+y)}\theta^{\alpha+y-1}e^{-(\beta+1)\theta} d\theta \\
&= \frac{\Gamma(\alpha+y)}{y!\Gamma(\alpha)}\left(\frac{\beta}{\beta+1}\right)^\alpha\left(\frac{1}{\beta+1}\right)^y
\end{aligned}$$

Hence, the negative binomial.

To calculate the mean and variance of $y|\alpha, \beta$, we have

$$\mathbb{E}(y) = \mathbb{E}\left(\mathbb{E}(y|\theta)\right) = \mathbb{E}(\theta) = \frac{\alpha}{\beta}$$

and

$$\mathbb{V}(y) = \mathbb{E}\left(\mathbb{V}(y|\theta)\right) + \mathbb{V}\left(\mathbb{E}(\theta|y)\right) = \mathbb{E}(\theta) + \mathbb{V}(\theta) = \frac{\alpha}{\beta^2}(\beta + 1)$$