CS224n HW1

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1 Softmax

(a)

Take a look at any element i,

$$\text{softmax}(\boldsymbol{x}+c)_i = \frac{e^{\boldsymbol{x}_i+c}}{\sum_j e^{\boldsymbol{x}_j+c}} = \frac{e^c \cdot e^{\boldsymbol{x}_i}}{e^c \cdot \sum_j e^{\boldsymbol{x}_j}} = \text{softmax}(\boldsymbol{x})_i$$

Therefore, we have $\text{softmax}(\boldsymbol{x}+c) = \text{softmax}(\boldsymbol{x})$

(b)

Note the first case illustrate the broadcasting principle (dimension match) in numpy:

```
if len(x.shape) > 1:
    #matrix
    x = x - np.max(x, axis=1, keepdims=True)
    x = np.exp(x) / np.sum(np.exp(x), axis=1, keepdims=True)
else:
    # vector
    x = x - x.max() # normalize
    x = np.exp(x)/np.sum(np.exp(x))
```

2 Neural Network Basics

(a)

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1-e^{-x})^2} = \sigma(x)(1-\sigma(x))$$

(b)

First, we have

$$\hat{y}_i = \frac{e^{\theta_i}}{\sum_j e^{\theta_j}}$$

For one-hot encoding, only k-th element in $\boldsymbol{y}$ is *one*, so we have

$$\begin{aligned} CE(\boldsymbol{y}, \hat{\boldsymbol{y}}) &= -\log \hat{y}_k = -\log \frac{e^{\theta_k}}{\sum_j e^{\theta_j}} \\ &= -\theta_k + \log \sum_j e^{\theta_j} \end{aligned}$$

Therefore,

$$\frac{\partial CE(\boldsymbol{y}, \hat{\boldsymbol{y}})}{\partial \theta_k} = -1 + \frac{e^{\theta_k}}{\sum_j e^{\theta_j}}$$
$$\frac{\partial CE(\boldsymbol{y}, \hat{\boldsymbol{y}})}{\partial \theta_i} = \frac{e^{\theta_i}}{\sum_j e^{\theta_j}}, \forall i \neq k$$

Put them altogether,

$$\frac{\partial CE(\boldsymbol{y}, \hat{\boldsymbol{y}})}{\partial \boldsymbol{\theta}} = -\boldsymbol{y} + \mathrm{softmax}(\boldsymbol{\theta}) = \hat{\boldsymbol{y}} - \boldsymbol{y}$$

(c)

Denote $\boldsymbol{a}_1 = \boldsymbol{x}\boldsymbol{W}_1 + \boldsymbol{b}_1$ and $\boldsymbol{a}_2 = \boldsymbol{h}\boldsymbol{W}_2 + \boldsymbol{b}_2$. Therefore, we have

$$\boldsymbol{z}_1 = \frac{\partial CE}{\partial \boldsymbol{a}_2} = \hat{\boldsymbol{y}} - \boldsymbol{y}$$
$$\boldsymbol{z}_2 = \frac{\partial CE}{\partial \boldsymbol{h}} = \frac{\partial CE}{\partial \boldsymbol{a}_2} \times \frac{\partial \boldsymbol{a}_2}{\partial \boldsymbol{h}} = \boldsymbol{z}_1 \boldsymbol{W}_2$$
$$\boldsymbol{z}_3 = \frac{\partial CE}{\partial \boldsymbol{x}} = \frac{\partial CE}{\partial \boldsymbol{h}} \times \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} = \boldsymbol{z}_2 \sigma'(\boldsymbol{x}\boldsymbol{W}_1 + \boldsymbol{b}_1)\boldsymbol{W}_1$$