CMSC 25025 HW#5 Xinyu Wei

1. (a).

$$RHS = \sum_{w\in V}\sum_{c\in V}[NP(w,c)\log[\sigma(v_w^t v_c)] + kN P(w)\frac{NP(c)}{N(S^p)}\cdot\log[\sigma(-v_w^t v_c)]]$$

$$= \sum_{w\in V}\sum_{c\in V}(NP(w)\cdot NP(c|w)\cdot\log(\sigma(v_w^t v_c))) + \sum_{w\in V}\sum_{c\in V}(kNP(w)\cdot\frac{NP(c)}{N(S^p)}\log[\sigma(-v_w^t v_c)])$$

$$= \sum_{w\in V} NP(w)\cdot\left[\sum_{c\in V} N^p(c|w)\cdot\log(\sigma(v_w^t v_c)) + \sum_{c\in V} k\cdot\frac{NP(c)}{N(S^p)}\cdot\log[\sigma(-v_w^t v_c)]\right]$$

$$= \sum_{w\in V} NP(w)\cdot\left[\sum_{c\in S_w^p} N^p(c|w)\cdot\log[\sigma(v_w^t v_c)] + \sum_{c\in S_w^n} N^n(c|w)\cdot\log[\sigma(-v_w^t v_c)]\right]$$

since $N^n(c|w) = k\cdot\frac{N^p(c)}{N(S^p)}$

$$= l(\phi)$$

Thus, equation (1) holds.

(b).

$x = v_w^t v_c$

$$l(\phi) = \sum_{c\in V, w\in V} N^p(c,w)\cdot\log[\sigma(x)] + kN^p(w)\frac{N^p(c)}{N(S^p)}\log[\sigma(-x)]$$

Let $l_1(\phi) = \sum_{c\in V, w\in V} N^p(c,w)\cdot\log[\sigma(x)]$

$l_2(\phi) = \sum_{c\in V, w\in V} kN^p(w)\frac{N^p(c)}{N(S^p)}\cdot\log[\sigma(-x)]$.

$l(\phi) = l_1(\phi) + l_2(\phi)$

$$\frac{\partial l_1(\phi)}{\partial x} = \sum_{c\in V, w\in V} N^p(c,w)\cdot\frac{1}{\sigma(x)}\cdot\frac{\partial\sigma}{\partial x}$$

$$= \sum_{c\in V, w\in V} N^p(c,w)\cdot\frac{1}{e^x+1}$$

$$\frac{\partial l_2(\phi)}{\partial x} = \sum_{c\in V, w\in V} k\cdot N^p(w)\frac{N^p(c)}{N(S^p)}\cdot\frac{1}{\sigma(-x)}\cdot\frac{\partial\sigma(-x)}{\partial x}$$

$$= \sum_{c\in V, w\in V} k\cdot N^p(w)\cdot\frac{N^p(c)}{N(S^p)}\cdot\left(-\frac{e^x}{1+e^x}\right)$$

$\sigma(x) = \frac{1}{1+e^{-x}}$ $\quad\frac{\partial\sigma(x)}{\partial x} = (1+e^{-x})^{-2}\cdot e^{-x}$

$= \frac{e^{-x}}{(1+e^{-x})^2}$

$\frac{1}{\sigma(x)}\cdot\frac{\partial\sigma(x)}{\partial x} = \frac{e^{-x}}{1+e^{-x}} = \frac{1}{e^x+1}$

$\frac{1}{\sigma(-x)}\cdot\frac{\partial\sigma(-x)}{\partial x} = \left(\frac{1}{1+e^x}\right)^{-1}\left(-\frac{e^x}{(1+e^x)^2}\right)$

$= -\frac{e^x}{1+e^x}$

$$\frac{\partial l(\phi)}{\partial x} = \frac{\partial l_1(\phi)}{\partial x} + \frac{\partial l_2(\phi)}{\partial x}$$

$$= \sum_{c\in V, w\in V}\left(\underbrace{N^p(c,w)}_{A}\cdot\frac{1}{e^x+1} - \underbrace{kN^p(w)\frac{N^p(c)}{N(S^p)}}_{B}\cdot\frac{e^x}{1+e^x}\right)$$

$\frac{\partial l(\phi)}{\partial x} = 0 \iff A\cdot\frac{1}{e^x+1} - B\cdot\frac{e^x}{1+e^x} = 0$

$\iff A = B\cdot e^x$

$\iff e^x = \frac{A}{B}$ ①.

Equation (3) $\Leftrightarrow \quad e^{2x} - (\frac{A}{B} - 1)e^x - \frac{A}{B} = 0.$

$\Leftrightarrow \quad (e^x - \frac{A}{B})(e^x + 1) = 0$

$\Leftrightarrow \quad e^x - \frac{A}{B} = 0 \qquad$ since $e^x + 1 > 0.$

$\Leftrightarrow \quad e^x = \frac{A}{B}. \qquad$ ②.

① and ② are equivalent.

So $\frac{\partial l(\phi)}{\partial x} = 0$ is equivalent to solving equation (3).

(c). $\quad y = e^x$

Equation (3) $\Leftrightarrow \; y = \frac{A}{B}$

$y = e^x. \Rightarrow x = \log \frac{A}{B} = v_w^T v_c$

So $v_w^T v_c = \log \left(\frac{N^P(w,c)}{k \cdot N^P(w) \cdot \frac{N^P(c)}{N(S^P)}} \right)$

$= \log \left(\frac{N^P(w,c) \cdot N(S^P)}{N^P(w) \cdot N^P(c)} \right) - \log k$