Compressed Distributed Gradient Descent: Communication-Efficient Consensus over Networks

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Network Consensus Optimization

Wireless Networks

Distributed Machine Learning

Tactical Multi-Agent Swarm

Smart Power Grid

Network Consensus Optimization

- First Order Method:
  - Decentralized Gradient Descent ([Nedic, Ozdaglar'09],[Shi et al.'15], [Yuan et al.'16], [Zeng, Yin'18])
  - Decentralized Stochastic Gradient Descent ([Lian et al.'17], [Jiang et al.'17], [Pu, Nedic'18])
- Second Order Method:
  - Decentralized Newton Method ([Mansoori, Wei'18])
  - Decentralized Quasi Newton Method ([Eisen et al.'17])
- Dual-Primal Method:
  - Decentralized ADMM ([Shi et al.'14], [Ling, Ribeiro'14])

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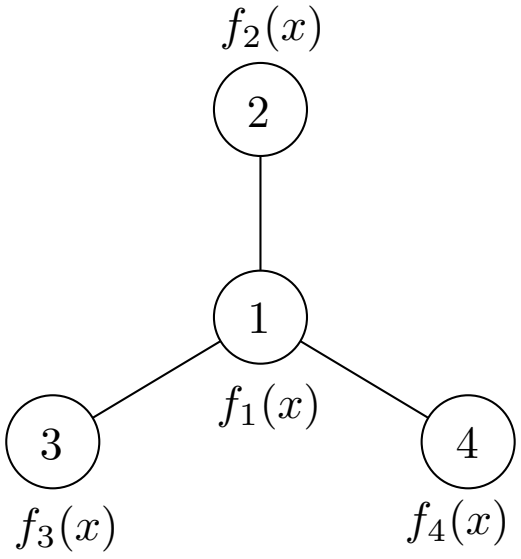
Mainly focus on Decentralized Gradient Descent

Network Consensus Optimization: The Basics

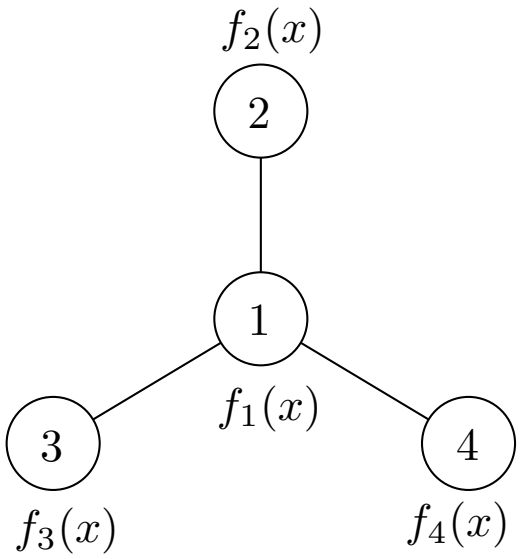
- A network $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, with $|\mathcal{N}| = N$, $|\mathcal{L}| = L$
- Let $x \in \mathbb{R}^P$ be a global state/decision variable
- Each node i can only access some local objective function $f_i(x)$
- Global objective function is: $\sum_{i=1}^{N} f_i(x)$



Network Consensus Optimization: The Basics

- Goal: To solve the following optimization problem distributively & collaboratively

$$\min_{x \in \mathbb{R}^P} f(x) = \min_{x \in \mathbb{R}^P} \sum_{i=1}^{N} f_i(x)$$



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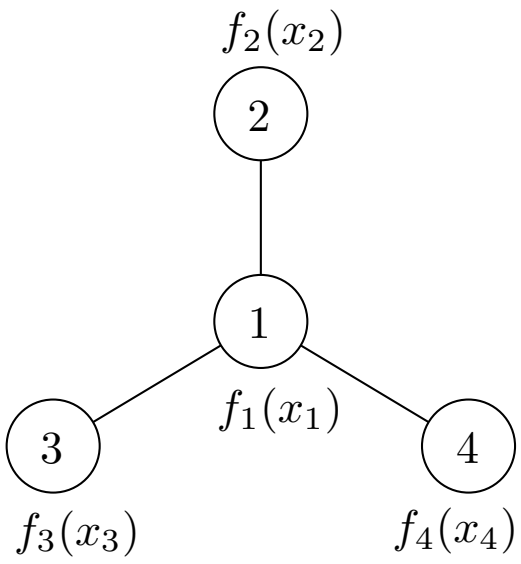
- Clearly, this problem can be rewritten in a consensus form:

$$\min_{x_i \in \mathbb{R}^P, \forall i} \left\{ \sum_{i=1}^{N} f_i(x_i) \middle| x_i = x_j, \forall (i,j) \in \mathcal{L} \right\}$$



Decentralized Gradient Descent Method (DGD)

- It has been shown in [Nedic, Ozdaglar'09] that

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} \quad & x_i = x_j, \qquad \forall (i,j) \in \mathcal{L}. \end{aligned}$$

can be rewritten as:

$$\begin{aligned} \text{Minimize} \quad & \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} \quad & (\mathbf{W} \otimes \mathbf{I}_P)\mathbf{x} = \mathbf{x}, \end{aligned} \tag{1}$$

where $\mathbf{x} \triangleq [x_1^\top, \ldots, x_n^\top]^\top \in \mathbb{R}^{NP}$, $\mathbf{I}_P$ is the P-dimensional identity matrix, and the operator $\otimes$ is the Kronecker product.

Decentralized Gradient Descent Method (DGD)

In Problem (1), $\mathbf{W} \in \mathbb{R}^{N \times N}$ is called the consensus matrix and satisfies:

- $\mathbf{W}$ is doubly stochastic: $\sum_{i=1}^{N} [\mathbf{W}]_{ij} = \sum_{j=1}^{N} [\mathbf{W}]_{ij} = 1$.

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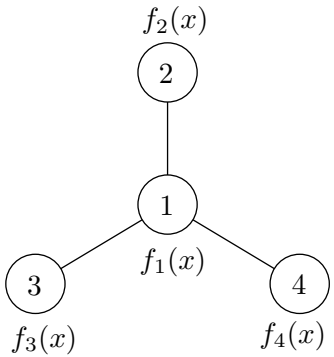

$$\mathbf{W} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 1/4 & 0 & 0 & 3/4 \end{bmatrix}$$

Decentralized Gradient Descent [Nedic, Ozdaglar'09]

The DGD algorithm:

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3. Upon reception of all local copies from its neighbors, each node updates its local copy:

$$x_{i,k+1} = \underbrace{\sum\nolimits_{j\in\mathcal{N}_i} [\mathbf{W}]_{ij} x_{j,k}}_{\text{Consensus step}} - \underbrace{\alpha_k \nabla f_i(x_{i,k})}_{\text{Gradient step}},$$

where $\mathcal{N}_i \triangleq \{j \in \mathcal{N} : (i,j) \in \mathcal{L}\}$.

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Cons:

- Performance not satisfactory for solving high-dimensional consensus optimization problems over network with low communication speed
  - Large amount of information exchange
  - Communication bottlenecks
  - Sharing info between nodes is time-consuming

Related Work on Communication-Efficient Methods

- Second-Order Approach (using second-order Hessian information)
  - [Jordan, Lee, Yang'16], [Wang, Kolar, Srebro, Zhang'16]
  - Key Idea: Treating local hessian as global hessian;
  - Pros: Faster convergence (hence less info exchanges);
  - Cons: High per-iteration complexity (matrix inversion, $\Omega(d^2 \log d)$ for d-dimensional problems);

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All existing works suffer high complexity

A Natural Idea for Reducing Communication Cost

- Message Conpression
  - Idea: Instead of exact message(e.g. gradient, iterates), transmitting some imprecise but 'simple' message;
  - Method: Unbiased Stochastic Compression;
  - Related Work:
    - Ternary compression operator [Wen et al. NIPS '17]
    - Quantized compression operator [Alistarh et al. NIPS'17]
    - Sparsification operator [Wangni et al.NIPS'18]
  - Most of the works are based on Master/Slave system, could this technique improve network optimization algorithm?

Unbiased Stochastic Compression Operator

Definition 1 (Unbiased Stochastic Compression Operator)

A stochastic compression operator $C(\cdot)$ is unbiased if it satisfies $C(z) = z + \epsilon_z$, with $\mathbb{E}[\epsilon_z] = 0$ and $\mathbb{E}[\epsilon_z^2] \leq \sigma^2, \forall\, z$.

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- Quantized compression operator [Alistarh et al. NIPS'17]

$$[C(\mathbf{z})]_k = \begin{cases} \lfloor z_k \rfloor + 1, \text{with probability } p_k, \\ \lfloor z_k \rfloor, \text{with probability } (1 - p_k). \end{cases}$$

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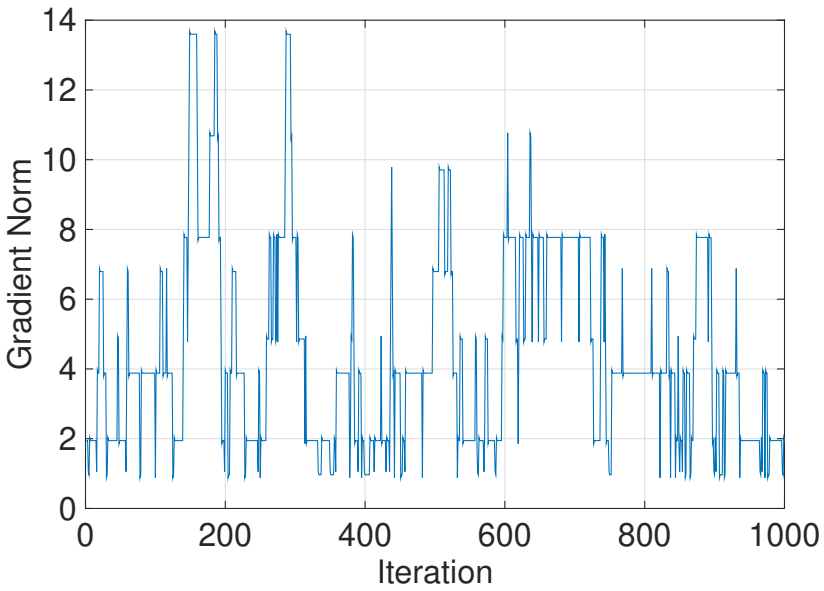
  - $\mathbb{E}[C(\mathbf{z})]_k = (\lfloor z_k \rfloor + 1) \times p_k + \lfloor z_k \rfloor \times (1 - p_k) = \lfloor z_k \rfloor + p_k = z_k$
  - $\text{Var}[C(\mathbf{z})]_k = (\lfloor z_k \rfloor + 1 - z_k)^2 \times p_k + (\lfloor z_k \rfloor - z_k)^2 \times (1 - p_k) \leq \frac{1}{4}$
  - stochastically mapping precise floating point number to inexact integer number;

Fundamental Questions

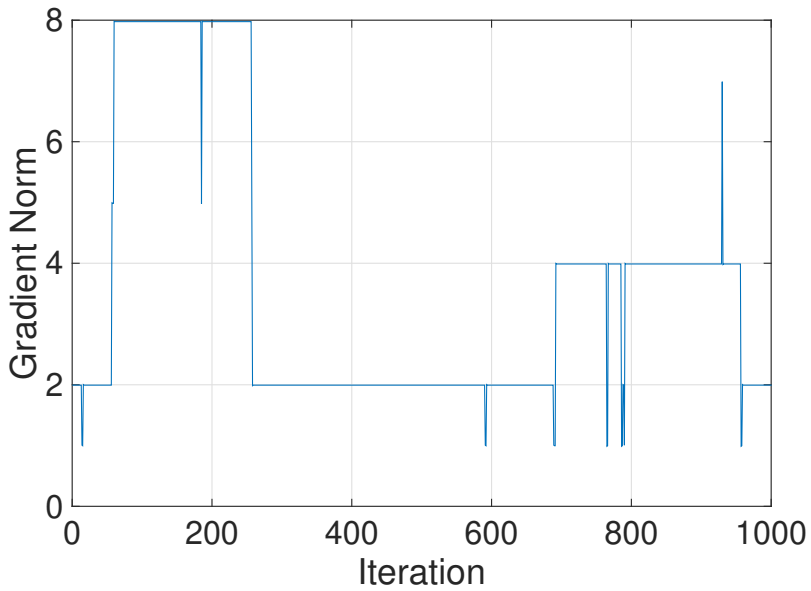
1. Will DGD with straightforward compressed info still converge?
2. If answer to (1) is "No," could we fix and make it work w/ compressed info?
3. If answer to (2) Is "Yes," how fast does the new algorithm converge?

DGD w/ Straightforward Compressed Info Fails

- The node sends compressed message $C(x_i)$ to its neighbors.



(a) $\alpha = 0.001$.



(b) $\alpha = 0.001/\sqrt{k}$.

Figure 1: Simulation results for DGD with quantized compression operator for a 2-node network, for which DGD fails to converge after 1000 iterations.

DGD w/ Straightforward Compressed Info Fails

$$
\begin{aligned}
x_{i,k+1} &= \sum\nolimits_{j\in\mathcal{N}_i} [\mathbf{W}]_{ij} C(x_{j,k}) - \alpha_k \nabla f_i(x_{i,k}) \\
&= \underbrace{\sum\nolimits_{j=\mathcal{N}_i} [\mathbf{W}]_{ij} x_{j,k} - \alpha_k \nabla f_i(x_{i,k})}_{\text{Exact DGD}} + \underbrace{\sum\nolimits_{j\in\mathcal{N}_i} [\mathbf{W}]_{ij} \epsilon_{x_{j,k}}}_{\text{Accumulated noise term}}
\end{aligned}
$$

- The accumulated noise term is similar to the noise term in SGD;
- In SGD, diminishing step-size/variance reduction to guarantee convergence;
- Variance reduction strategy might be helpful.

Open Fundamental Questions

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Our Solution: Amplified-Differential Compression DGD

Amplified-Differential Compression DGD (ADC-DGD)

The ADC-DGD algorithm:

1. Initialization: Let $k = 1$. Let $x_{i,0} = \tilde{x}_{i,0} = 0, \forall i$. Choose initial step-size α_1 and an amplification exponent γ. Let $x_{i,1} = y_{i,1} = -\alpha_1 \nabla f_i(x_{i,0})$.

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2. In k-th iteration: Each node sends compressed amplified-differential $d_{i,k} = C(k^\gamma y_{i,k})$ to its neighbors, where k^γ is the amplification factor, local differential $y_{i,k} = x_{i,k} - \tilde{x}_{i,k-1}$ and $C(\cdot)$ is some unbiased stochastic compression operator.

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3. Upon reception of all local copies from its neighbors, each node estimates neighbors' values: $\tilde{x}_{j,k} = \tilde{x}_{j,k-1} + d_{j,k}/k^\gamma$. Then, each node updates its local value:

$$x_{i,k+1} = \underbrace{\sum_{j \in \mathcal{N}_i} [\mathbf{W}]_{ij} \tilde{x}_{j,k}}_{\text{Compressed consensus}} - \underbrace{\alpha_k \nabla f_i(x_{i,k})}_{\text{gradient step}}.$$

Each node updates local differential: $y_{i,k+1} = x_{i,k+1} - \tilde{x}_{i,k}$.

Intuition Behind ADC-DGD

- A closer look at ADC-DGD update reveals that:

$$\begin{aligned}\tilde{x}_{j,k} =& \tilde{x}_{j,k-1} + C(k^\gamma y_{j,k})/k^\gamma \\ =& \tilde{x}_{j,k-1} + (k^\gamma y_{j,k} + \epsilon_{k^\gamma y_{j,k}})/k^\gamma \\ =& \tilde{x}_{j,k-1} + y_{j,k} + (\epsilon_{k^\gamma y_{j,k}})/k^\gamma \\ =& x_{j,k} + \underbrace{(\epsilon_{k^\gamma y_{j,k}})/k^\gamma}_{\text{Vanishing noise term}} \quad .\end{aligned}$$

- Thanks to properties of unbiased stochastic operator, the noise term in the last step of has zero mean and a vanishing variance $\frac{\sigma^2}{k^{2\gamma}}$ as $k \to \infty$.

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Indeed, proofs for main theorems are based on this intuition

Key Features of ADC-DGD

- Compared to DGD, requires additional memory to store (imprecise) neighbors' values: $\{\tilde{x}_{j,k-1} : (i,j) \in \mathcal{L}\}$
  - Allows to transmit only the difference between successive iterations $y_{i,k} = x_{i,k} - \tilde{x}_{i,k-1}$ rather than $x_{i,k}$ directly
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- With $\tilde{x}_{j,k}$, $j \in \mathcal{N}_i$, the update follows the same structure as in DGD
  - Also contains a consensus step and a local gradient step.
  - Complexity of ADC-DGD is almost the same to the original DGD

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Main Theoretical Results

Assumption 1

The local objective functions $f_i(\cdot)$ satisfy:

- *(Lower boundedness) $\exists$ an optimal x^* with $\|x^*\| < \infty$ such that $\forall x \neq x^*$, $\sum_{i=1}^N f_i(x) \geq \sum_{i=1}^N f_i(x^*)$*
- *(Lipschitz continuous gradient) there exists a constant $L > 0$ such that $\forall x, y$, $\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|$, $\forall i$.*

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Assumption 2 (Growth Rate at Infinity)

$\exists$ a constant $M > 0$ such that:

$$\lim_{\|\mathbf{x}\| \to \infty} \frac{\|\mathbf{x}\|}{f(\mathbf{x})} = \lim_{\sum_{i=1}^n \|x_i\| \to \infty} \frac{\sum_{i=1}^N \|x_i\|}{\sum_{i=1}^N f_i(x_i)} \leq M.$$

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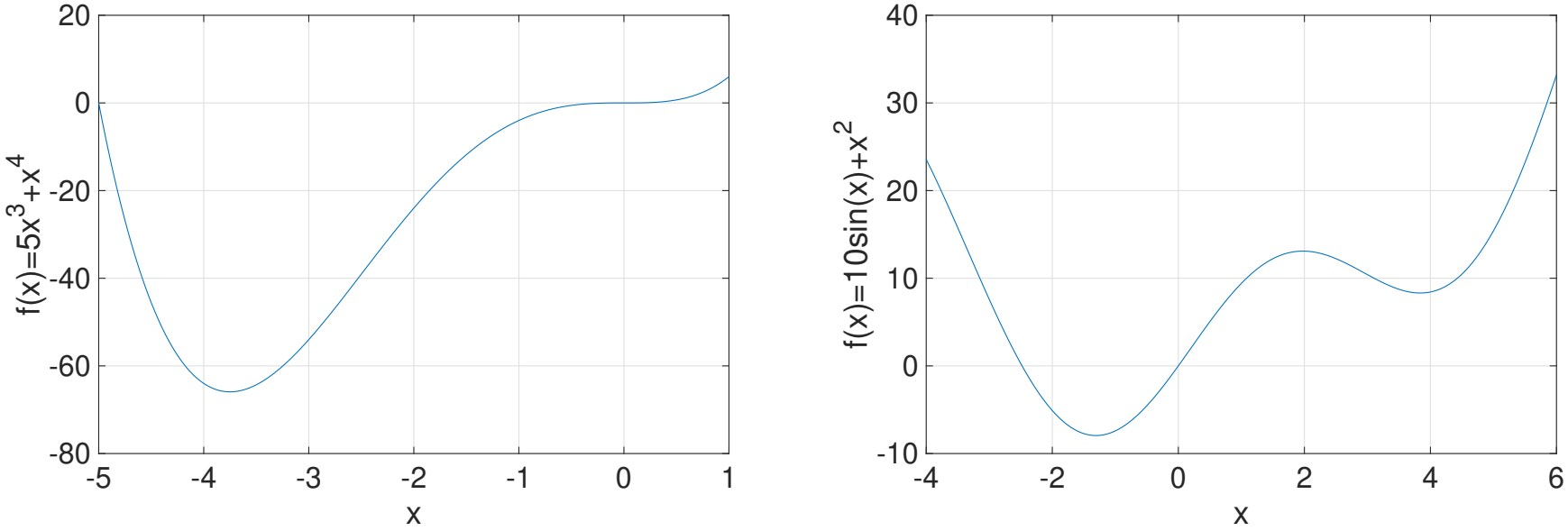

Figure 2: Examples of non-convex functions satisfying Assumption 2.

Main Theoretical Results

Theorem 3 (Convergence to Mean Vector)

Let the mean vector at the k-th iteration be $\bar{\mathbf{x}}^k = \mathbf{1} \otimes \bar{x}^k \in \mathbb{R}^{NP}$, *where* $\bar{x}^k = \frac{1}{N}\sum_{i=1}^{N} x_{i,k}$. *Under Assumptions 1, if* $\mathbb{E}[\|\nabla \mathbf{f}(\mathbf{x}^i)\|] \leq D$ *and the amplifying exponent is* γ, *then:*

- *For constant step-size* $\alpha_i = \alpha, \forall i$, $\mathbb{E}[\|\mathbf{x}^k - \bar{\mathbf{x}}^k\|] \leq \frac{\alpha D}{1-\beta} + O(\frac{\sqrt{NP}\sigma}{k^\gamma})$;
- *For diminishing step-size* $\alpha_i = O(\frac{1}{k^\eta})$ *with some constant* $\eta \in (0, 1]$, $\mathbb{E}[\|\mathbf{x}^k - \bar{\mathbf{x}}^k\|] = O(\frac{1}{k^{\min(\eta,\gamma)}})$.

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Remark

- *Local copies converge to mean vector with a diminishing step-size*
- *Local copies stay within a bounded error ball of the mean vector with a constant step-size.*

Main Theoretical Results

Theorem 4 (Constant Step-Size)

With constant step-size $\alpha_k = \alpha < \frac{1+\lambda_N(\mathbf{W})}{L}$, $\forall k$, *Under Assump. 1–2, if the amplified exponent* $\gamma > \frac{1}{2}$, *then it holds that:*

$$\min_k \left\{ \mathbb{E}\left[\left\| \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(\bar{x}^k) \right\|^2 \right] \right\} \leq \frac{C_1 \alpha^2}{N(1-\beta)^2} + \frac{C_2}{\alpha k} + O\left(\frac{2L\sqrt{P}\sigma}{\sqrt{N} k^{\gamma}} \right),$$

where $C_1 \triangleq L^4 (B + \|\mathbf{x}^*\|)^2$ *and* $C_2 \triangleq 2[\frac{1}{N}\sum_{i=1}^{N} f_i(0) - \frac{1}{N}\sum_{i=1}^{N} f_i(x^*)]$.

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Remark

Under the conditions of Theorem 4, ADC-DGD has:

- *A convergence rate* $O\left(\frac{1}{k^{\min(1,\gamma)}}\right)$ *until reaching an error ball* $O\left(\left(\frac{\alpha}{1-\beta}\right)^2\right)$
- *The fastest convergence rate of ADC-DGD is* $O\left(\frac{1}{k}\right)$. *(Same as GD)*

Main Theoretical Results

Theorem 5 (Diminishing Step-Sizes)

Under Assump. 1–2, if $\exists$ *a constant* $D > 0$ *such that* $\|\nabla f_i(x)\| \leq D$, $\forall x$, *and* $\gamma > \frac{1}{2}$, $\eta \geq \frac{1}{2}$, *then with diminishing step-size* $\alpha_k = O(\frac{1}{k^\eta})$, *it holds that*

$$\left\| \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(\bar{x}^k) \right\|^2 = o\left(\frac{1}{k^{1-\eta}} \right) \text{ almost surely.}$$

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Remark

- *The exponent* $\eta \geq \frac{1}{2}$. *Thus, the best convergence rate for ADC-DGD is* $o(1/\sqrt{k})$, *faster than all existing first-order approaches.*
- *Convergence is in "Small-O", stronger than conventional "Big-O" results.*

Understanding the Role of the Amplifying Exponent γ

- The amplifying exponent γ is a key component to adjust the communication rate.
  - within $(1/2, 1]$, the larger γ means the faster convergence;
  - a larger γ leads to a larger $k^\gamma \mathbf{y}^k$, which may lead to overflow error for the transmitted value $C(k^\gamma \mathbf{y}^k)$
  - for example, type 'int8' in Matlab could only present data within $(-128, 127)$.

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Proposition 1

Under Assumptions 1-2, with $\gamma > \frac{1}{2}$, the transmitted value $k^\gamma \mathbf{y}^k$ satisfies $\mathbb{E}[\|k^\gamma \mathbf{y}^k\|] = o(k^{\gamma - \frac{1}{2}})$.

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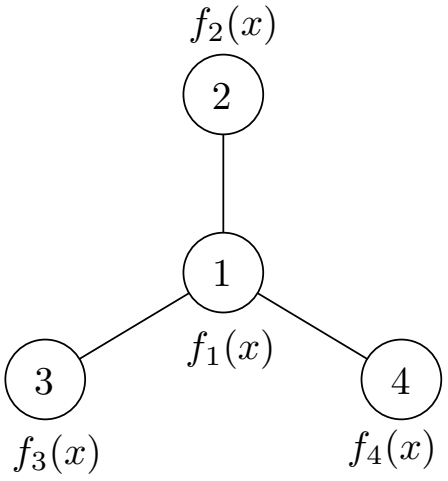
- with $\gamma \in (\frac{1}{2}, 1]$, the growth speed for the transmitted value is slower than $o(\sqrt{k})$, which is not very fast.

Numerical Results

- Effect of Compression:



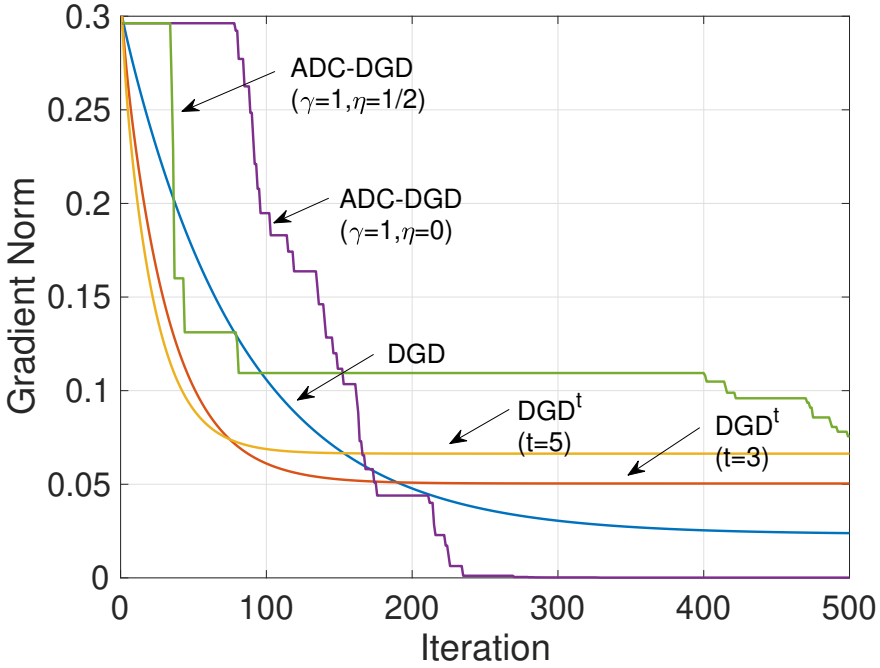
A four-node star network.

$$\mathbf{W} = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 3/4 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 1/4 & 0 & 0 & 3/4 \end{bmatrix}$$

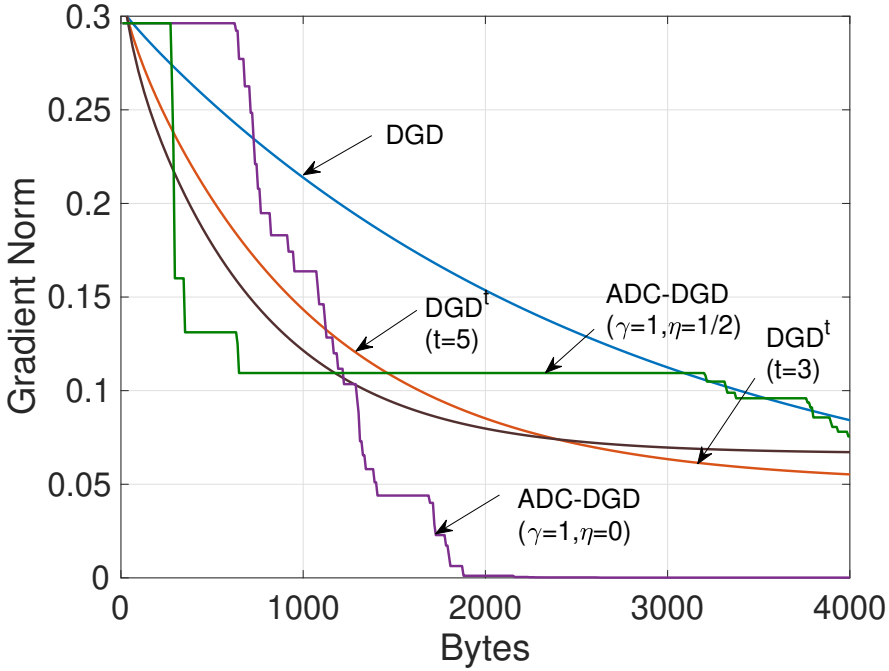
The consensus matrix.

- $\min_x f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x)$, with $f_1(x) = -4x^2$, $f_2(x) = 2(x-0.2)^2$, $f_3(x) = 2(x+0.3)^2$, and $f_4(x) = 5(x-0.1)^2$.

Numerical Results



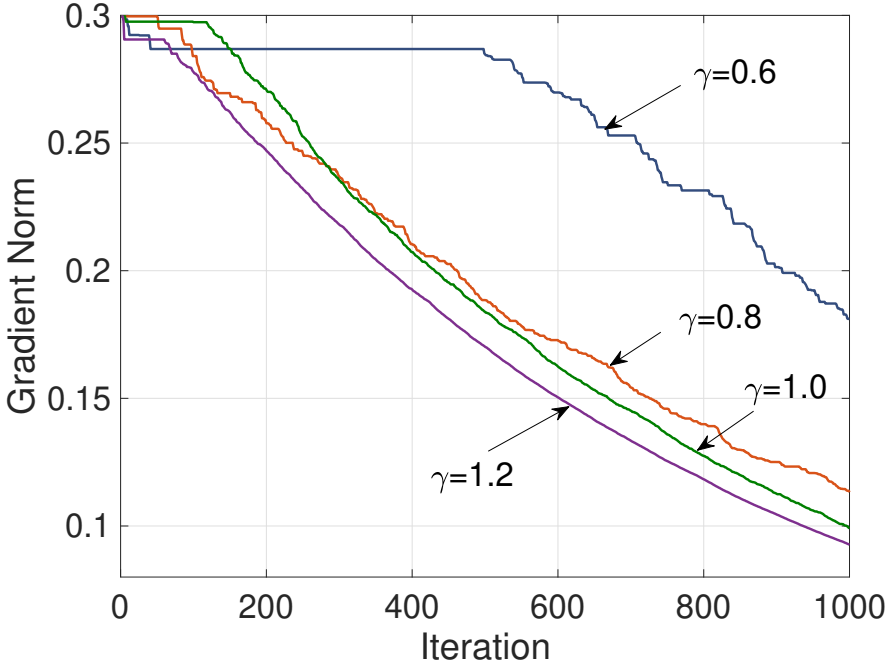
Convergence comparisons between ADC-DGD, DGD, and DGD^t.



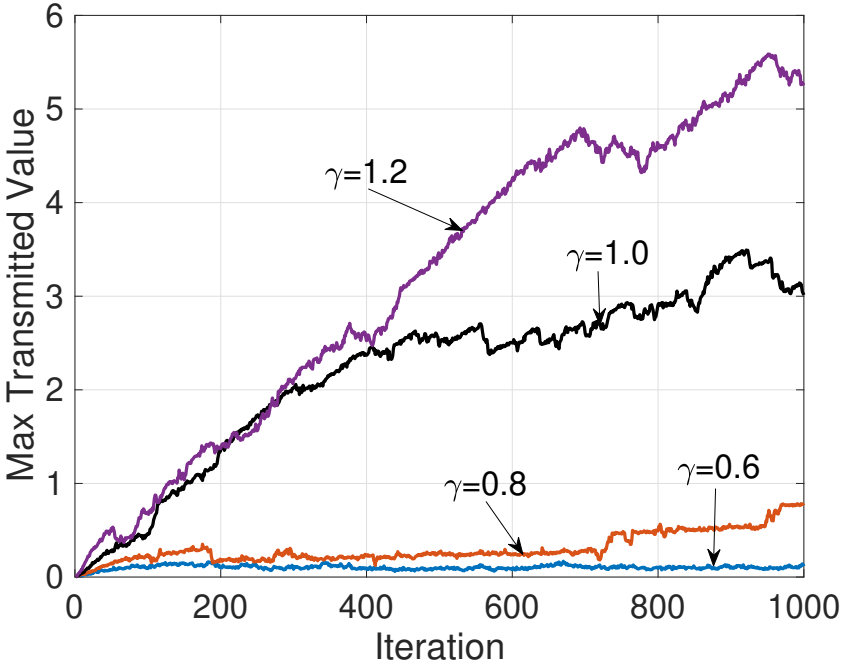
Amount of exchanged information (bytes) vs gradient norm.

Numerical Results

- Effect of Amplifying Exponent: Choose $\gamma \in \{0.6, 0.8, 1.0, 1.2\}$



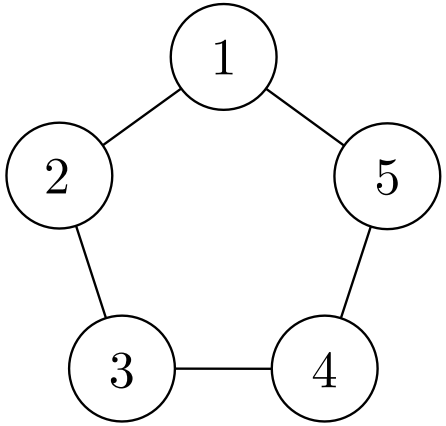
Convergence behaviors under different γ's.



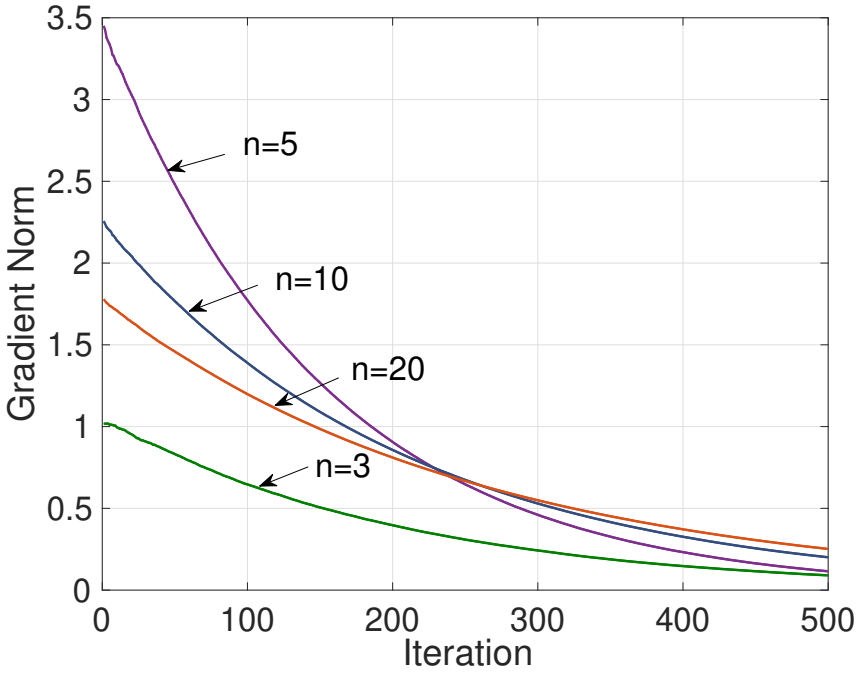
Growth of transmitted values ($k^{\gamma y}$) vs. number of iterations.

Numerical Results

- Effect of Network Size: Choose $N \in \{3, 5, 10, 20\}$



The 5-node circle topology.



The effect of network size.

Conclusion

Developed a new algorithm called ADC-DGD based on compression to reduce communication costs:

1. Convergence results with constant and diminished step-sizes
2. Understanding of compression in network consensus optimization

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1. Convergence results with constant and diminished step-sizes
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Future work:

- We claim that the memory requirement is modest and store the additional values, but the network are sensors with limited memory;
- The large values of the amplifying exponent γ may lead to overflow errors, and a solution for setting γ in a distributed manner is necessary;
- This work is mainly based on the quantized compression operator, which might not be the most efficient. Designing more efficient unbiased stochastic compression operator would be helpful.

Thank You!