Tensor Singular Value Decomposition and Composition

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Overview

1. Tensor Singular Value Decomposition (t-SVD)
2. Tensor Composition

Notation

$$\mathbf{A}^{(j)} \equiv \mathcal{A}(:,:,j), \qquad \hat{\mathcal{A}} = \mathtt{fft}(\mathcal{A},[],3)$$

$$\mathtt{bcirc}(\mathcal{A}) = \begin{bmatrix} \mathbf{A}^{(1)} & \mathbf{A}^{(n)} & \mathbf{A}^{(n-1)} & \cdots & \mathbf{A}^{(2)} \\ \mathbf{A}^{(2)} & \mathbf{A}^{(1)} & \mathbf{A}^{(n)} & \cdots & \mathbf{A}^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}^{(n)} & \mathbf{A}^{(n-1)} & \cdots & \mathbf{A}^{(2)} & \mathbf{A}^{(1)} \end{bmatrix}$$

$$\mathtt{unfold}(\mathcal{A}) = \begin{bmatrix} \mathbf{A}^{(1)} \\ \mathbf{A}^{(2)} \\ \vdots \\ \mathbf{A}^{(n)} \end{bmatrix}, \qquad \mathtt{fold}(\mathtt{unfold}(\mathcal{A})) = \mathcal{A}$$

t-Product

Definition. t-product

Let $\mathcal{A}$ be $n_1 \times n_2 \times n_3$ and $\mathcal{B}$ be $n_2 \times n_4 \times n_3$. Then the t-product $\mathcal{A} * \mathcal{B}$ is the $n_1 \times n_4 \times n_3$ tensor

$$\mathcal{A} * \mathcal{B} = \texttt{fold}(\texttt{bcirc}(\mathcal{A}) \cdot \texttt{unfold}(\mathcal{B})).$$

Example 1. Let $\mathcal{A} \in \mathbb{R}^{3\times 2\times 2}$ with frontal faces

$$\mathbf{A}^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{(2)} = \begin{bmatrix} -2 & 1 \\ -2 & 7 \\ 0 & -1 \end{bmatrix},$$

and let $\mathcal{B} \in \mathbb{R}^{2\times 1\times 2}$ with frontal faces

$$\mathbf{B}^{(1)} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^{(2)} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

t-Product

$$\mathcal{A} * \mathcal{B} = \texttt{fold}\left(\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 2 & -2 & 7 \\ -1 & 3 & 0 & -1 \\ \hline -2 & 1 & 1 & 0 \\ -2 & 7 & 0 & 2 \\ 0 & -1 & -1 & 3 \end{array}\right] \left[\begin{array}{c} 3 \\ -1 \\ \hline -2 \\ -3 \end{array}\right] \right)$$

$$= \texttt{fold}\left(\left[\begin{array}{c} 4 \\ -19 \\ -3 \\ \hline -9 \\ -19 \\ -6 \end{array}\right] \right) \in \mathbb{R}^{3\times 1\times 2}$$

is a $3 \times 1 \times 2$ tensor. In other words, in this example, $\mathcal{C} := \mathcal{A} * \mathcal{B}$ is a 3×2 matrix, oriented as a lateral slice of a third-order tensor.

t-SVD

Theorem. t-SVD

For $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, the t-SVD of $\mathcal{A}$ is given by

$$\mathcal{A} = \mathcal{U} * \mathcal{S} * \mathcal{V}^T,$$

where $\mathcal{U}$ and $\mathcal{V}$ are orthogonal tensors of size $n_1 \times n_1 \times n_3$ and $n_2 \times n_2 \times n_3$ respectively. $\mathcal{S}$ is a rectangular f-diagonal tensor of size $n_1 \times n_2 \times n_3$, and $*$ denotes the t-product.

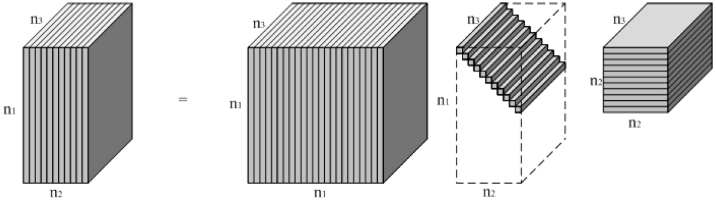

Figure: The t-SVD of an $n_1 \times n_2 \times n_3$ tensor.

t-SVD

Algorithm 1 t-SVD

Require: Tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$;

Ensure: $\mathcal{U} \in \mathbb{R}^{n_1 \times n_1 \times n_3}$, $\mathcal{S} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathcal{V} \in \mathbb{R}^{n_2 \times n_2 \times n_3}$;

$\hat{\mathcal{A}} := \texttt{fft}(\mathcal{A}, [], 3)$;

for $i = 1$ to n_3 **do**

  $[U, S, V] = \texttt{SVD}(\hat{\mathcal{A}}(:, :, i))$;

  $\hat{\mathcal{U}}(:, :, i) := U$;

  $\hat{\mathcal{S}}(:, :, i) := S$;

  $\hat{\mathcal{V}}(:, :, i) := V$;

end for

$\mathcal{U} := \texttt{ifft}(\hat{\mathcal{U}}, [], 3)$;

$\mathcal{S} := \texttt{ifft}(\hat{\mathcal{S}}, [], 3)$;

$\mathcal{V} := \texttt{ifft}(\hat{\mathcal{V}}, [], 3)$;

Example 2. Take the tensor $\mathcal{A} \in \mathbb{R}^{3\times 2\times 2}$ same in Example 1. We have the Fourier transformation of $\mathcal{A}$

$$\hat{\mathcal{A}} = \texttt{fold}\left(\left[\begin{array}{cc} -1 & 1 \\ -2 & 9 \\ -1 & 2 \\ \hline 3 & -1 \\ 2 & -5 \\ -1 & 4 \end{array}\right]\right).$$

Now compute the $i = 1$-th frontal face's SVD using $[U, S, V] = \texttt{SVD}(\hat{\mathcal{A}}(:, :, 1))$.

$$\hat{\mathcal{U}}(:, :, 1) := U = \begin{bmatrix} -0.1268 & 0.8066 & -0.5774 \\ -0.9653 & -0.2343 & -0.1155 \\ -0.2284 & 0.5427 & 0.8083 \end{bmatrix},$$

t-SVD

$$\hat{\mathcal{S}}(:,:,1) := S = \begin{bmatrix} 9.5487 & 0 \\ 0 & 0.9070 \\ 0 & 0 \end{bmatrix},$$

$$\hat{\mathcal{V}}(:,:,1) := V = \begin{bmatrix} 0.2394 & -0.9709 \\ -0.9709 & -0.2394 \end{bmatrix}.$$

Next, we can use the same method to get $\hat{\mathcal{U}}(:,:,2)$, $\hat{\mathcal{S}}(:,:,2)$ and $\hat{\mathcal{V}}(:,:,2)$. Last, compute the inverse Fourier transformation of $\hat{\mathcal{U}}$, $\hat{\mathcal{S}}$ and $\hat{\mathcal{V}}$, obtaining $\mathcal{U}$, $\mathcal{S}$ and $\mathcal{V}$.

t-SVD

$$\mathcal{U} = \texttt{ifft}(\hat{\mathcal{U}}, [], 3)$$

$$= \texttt{ifft}\left(\texttt{fold}\left(\left[\begin{array}{ccc} -0.1268 & 0.8066 & -0.5774 \\ -0.9653 & -0.2343 & -0.1155 \\ -0.2284 & 0.5427 & 0.8083 \\ \hline -0.3089 & 0.9351 & -0.1735 \\ -0.7600 & -0.1330 & 0.6361 \\ 0.5718 & 0.3284 & 0.7518 \end{array}\right]\right), [], 3\right)$$

$$= \texttt{fold}\left(\left[\begin{array}{ccc} -0.2178 & 0.8709 & -0.3754 \\ -0.8626 & -0.1837 & 0.2603 \\ 0.1717 & 0.4355 & 0.7800 \\ \hline 0.0911 & -0.0643 & -0.2019 \\ -0.1026 & -0.0507 & -0.3758 \\ -0.4001 & 0.1072 & 0.0282 \end{array}\right]\right).$$

t-SVD

$$
\begin{aligned}
\mathcal{S} &= \texttt{ifft}(\hat{\mathcal{S}}, [], 3) \\
&= \texttt{ifft}\left(\texttt{fold}\left(\begin{bmatrix} 9.5487 & 0 \\ 0 & 0.9070 \\ 0 & 0 \\ \hline 7.0727 & 0 \\ 0 & 2.4448 \\ 0 & 0 \end{bmatrix}\right), [], 3\right) \\
&= \texttt{fold}\left(\begin{bmatrix} 8.3107 & 0 \\ 0 & 1.6759 \\ 0 & 0 \\ \hline 1.2380 & 0 \\ 0 & -0.7689 \\ 0 & 0 \end{bmatrix}\right).
\end{aligned}
$$

t-SVD

$$
\begin{aligned}
\mathcal{V} &= \texttt{ifft}(\hat{\mathcal{V}}, [], 3) \\
&= \texttt{ifft}\left(\texttt{fold}\left(\left[\begin{array}{cc} 0.2394 & -0.9709 \\ -0.9709 & -0.2394 \\ \hline -0.4268 & 0.9044 \\ 0.9044 & 0.4268 \end{array}\right]\right), [], 3\right) \\
&= \texttt{fold}\left(\left[\begin{array}{cc} -0.0937 & -0.0333 \\ -0.0333 & 0.0937 \\ \hline 0.3331 & -0.9376 \\ -0.9376 & -0.3331 \end{array}\right]\right).
\end{aligned}
$$

Tensor Composition

Algorithm 2 Tensor Composition

Require: $\mathcal{U} \in \mathbb{R}^{n_1 \times n_1 \times n_3}$, $\mathcal{S} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ and $\mathcal{V} \in \mathbb{R}^{n_2 \times n_2 \times n_3}$;

Ensure: $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$;

```
Û := fft(U, [], 3);
Ŝ := fft(S, [], 3);
V̂ := fft(V, [], 3);
for i = 1 to n3 do
    U := Û(:, :, i);
    S := Ŝ(:, :, i);
    V := V̂(:, :, i);
    Â(:, :, i) = U S V^T;
end for
A := ifft(Â, [], 3);
```

$\hat{\mathcal{U}} := \mathtt{fft}(\mathcal{U}, [], 3)$;
$\hat{\mathcal{S}} := \mathtt{fft}(\mathcal{S}, [], 3)$;
$\hat{\mathcal{V}} := \mathtt{fft}(\mathcal{V}, [], 3)$;
for $i = 1$ to n_3 **do**
  $U := \hat{\mathcal{U}}(:, :, i)$;
  $S := \hat{\mathcal{S}}(:, :, i)$;
  $V := \hat{\mathcal{V}}(:, :, i)$;
  $\hat{\mathcal{A}}(:, :, i) = USV^T$;
end for
$\mathcal{A} := \mathtt{ifft}(\hat{\mathcal{A}}, [], 3)$;

The End