Determinants Linear Algebra X. Du

- $\det(A) = |A|$
- Applies only to square matrices
- For 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- All other determinants can be simplified into determinants of 2×2 matrices
- For 3×3 matrix A.
  - **Cofactor expansion (Laplace expansion)**:
    $$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
    - Generalize cofactor expansion to higher dimension matrices
    - Remember to alternate the signs
  - **Rule of Sarrus**: Copy the first two columns of A and place them on the right. Then add the product of the forward diagonals and subtract the product of the backwards diagonals. i.e.
    - $$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$
      $$\rightarrow a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$
- Represents signed area or volume inside matrix
  - Scalar representation of a vector space
- $\det(A) = \det(A^T)$
- If any row or column vector in matrix A is $\vec{0}$, then $\det(A) = 0$
- Alternate definition: $\det(A)$ is the sum of all signed elementary products of an $n\times n$ matrix A. An elementary product is a product of n entries from A, where no two of such entries share the same row or column. If the factors of the elementary product are ordered by their row indices, its sign is positive if the column indices follow an even permutation or negative if the column indices follow an odd permutation.
  - Relates determinants to combinatorics.
  - Explains why the number of products in computing $\det(A)$ is $n!$
- $\det(kA) = k^n \det(A)$
- Row operations:
  - Multiply one row by a constant, then your new determinant is $k\det(A)$
  - If you switch two rows or columns in A, your new determinant is $-\det(A)$
  - Add a multiple of one row to another row, your determinant stays the same.
- If one row or column is a linear combination of other rows or columns, then the determinant is 0.
- $\det(AB) = \det(A)\det(B)$