Definiteness of a Matrix X. Du

- Definitions:
  - **Positive definite**: A symmetric matrix A is positive definite iff $\vec{x}^T A \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$.
  - **Positive semi-definite**: A symmetric matrix A is positive semi-definite iff $\vec{x}^T A \vec{x} \geq 0$ for all $\vec{x}$.
  - Likewise for **negative definite** and **negative semi-definite**
- Eigenvalues:
  - A is positive definite iff all eigenvalues λ_i of A are positive, i.e. $\lambda_i > 0$.
  - A is positive semi-definite iff all eigenvalues λ_i of A are non-negative, i.e. $\lambda_i \geq 0$.
- Gram matrix: A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite iff it is Gram matrix of n linearly independent vectors. Similarly, a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite iff it is a Gram matrix for some set of n vectors.
  - The Gram matrix A of a set of vectors $x_1, ..., x_n \in \mathbb{R}^m$ is $A_{ij} = \langle \vec{x}_i, \vec{x}_j \rangle$.
  - Similarly, A has linearly independent columns iff $A^T A$ is positive definite. $A^T A$ is positive semi-definite for any A.
- **Sylvester's criterion**: A symmetric matrix A is positive definite iff all leading principle minors of A are positive.
  - The k **leading principle minor** is the $k \times k$ submatrix of A anchored at the top-left of A.
- A positive definite matrix is always invertible, and its inverse is also positive definite.
- A matrix A is positive semi-definite iff there is a positive semi-definite B such that $BB = A$, where B is unique. In other words, A has a unique square root.
- Probability: A matrix is positive semi-definite if it is the covariance matrix of a multivariate distribution.