- **Linear Approximation** - first-degree multivariable Taylor polynomial
  - **Tangent Planes**: Approximates surfaces
    - $z = f(x, y)$ at $(x_o, y_o)$
      - $z = f(x_o, y_o) + f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)$
    - $f(x, y, z) = k$ at point $(x_o, y_o, z_o)$ such that $f(x_o, y_o, z_o) = k$.
      - $f_x(x - x_o) + f_y(y - y_o) + f_z(z - z_o) = 0$.
  - **Tangent Hyperplanes**: Approximates any *n*-manifold ($n > 1$)
    - $y = f(x_1, x_2, ...x_n)$ at $(a_1, a_2, ...a_n)$
      - $y = f(a_1, a_2, ...a_n) + f_{x_1}(x_1 - a_1) + f_{x_2}(x_2 - a_2) + ... + f_{x_n}(x_n - a_n)$
    - $f(x_1, x_2, ...x_n) = k$ at point $(a_1, a_2, ...a_n)$ such that $f(x_1, x_2, ...x_n) = k$
      - $\nabla f \cdot \langle (x_1 - a_1), (x_2 - a_2), ...(x_n - a_n) \rangle = 0$
  - **Differentials**:
    - Example: Let $z = f(x, y)$. Then $dz = f_x dx + f_y dy$. Extend to *n* variables.
    - Let $y = f(x_1, x_2, ...x_n)$. Then $dy = f_{x_1} dx_1 + f_{x_2} dx_2 + ... + f_{x_n} dx_n$
- **Taylor Polynomials**
  - Very conceptual to extend to *n* variables, but very tedious and often inefficient.
  - Examples:
  - First-degree polynomial of $f(x, y)$ on a disk about $(a, b)$:

    $P_1(x, y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$
  - Second-degree polynomial of $f(x, y)$ on a disk about $(a, b)$:

    $$P_2(x, y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b) + \frac{f_{xx}(a,b)}{2}(x - a)^2$$

    $$+ f_{xy}(a,b)(x - a)(y - b) + \frac{f_{yy}(a,b)}{2}(y - b)^2$$
  - Use Pascal's triangle (i.e. Leibniz's Rule) to help you deal with an *n*-degree Taylor polynomial of two variables.
  - Use Hessian matrices to help you deal with second-degree Taylor polynomials of $f(x_1, x_2, ...x_n)$
  - First-degree polynomial of $f(x, y, z)$ on a sphere about $(a, b, c)$:

    $P_1(x, y, z) = f(a,b,c) + f_x(a,b,c)(x - a) + f_y(a,b,c)(y - b) + f_z(a,b,c)(z - c)$
  - In general, the first-degree Taylor polynomial of $f(\vec{x}) = f(x_1, x_2, ..., x_n)$ centered about $\vec{a} = (a_1, a_2, ...a_n)$ is $P_1(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$.
  - Extend these concepts to *m*-degree Taylor polynomials of functions of *n* variables - it gets ugly!