Vector Projections and the Gram-Schmidt Process Linear Algebra X. Du

- **Scalar projection**
  - $\text{comp}_{\vec{a}}\vec{b}$ (the component of $\vec{b}$ on $\vec{a}$).
  - $\text{comp}_{\vec{a}}\vec{b} = \vec{b}\cos\theta$ This is obvious geometrically.
  - $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ Derive this easily using $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
- **Vector projection**
  - $\text{proj}_{\vec{a}}\vec{b}$ (the projection of $\vec{b}$ onto $\vec{a}$).
  - $\text{proj}_{\vec{a}}\vec{b}$ is the vector component of $\vec{b}$ in the direction of $\vec{a}$
  - $\text{proj}_{\vec{a}}\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\left(\dfrac{\vec{a}}{|\vec{a}|}\right)$
    - $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)$ gives magnitude since $\left|\text{proj}_{\vec{a}}\vec{b}\right| = \text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$
    - $\hat{a} = \left(\dfrac{\vec{a}}{|\vec{a}|}\right)$ gives direction
  - $\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$
- **Vector rejection**
  - $\text{proj}_{\vec{a}}^{\perp}\vec{b}$ (the orthogonal projection of $\vec{b}$ onto $\vec{a}$).
  - $\text{proj}_{\vec{a}}^{\perp}\vec{b}$ is the vector component of $\vec{b}$ orthogonal to $\vec{a}$
  - $\left|\text{proj}_{\vec{a}}^{\perp}\vec{b}\right| = \vec{b}\sin\theta$ This is obvious geometrically.
  - $\text{proj}_{\vec{a}}\vec{b} + \text{proj}_{\vec{a}}^{\perp}\vec{b} = \vec{b}$ Draw a right triangle. This is also geometrically obvious.
  - $\text{proj}_{\vec{a}}^{\perp}\vec{b} = \vec{b} - \text{proj}_{\vec{a}}\vec{b}$
- **Gram-Schmidt Process**
  - Produce an orthogonal basis from any basis.
  - Steps:
    - Let $S = \{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ be any basis for V.
    - Let $\vec{u}_1 = \vec{v}_1$.
    - $\vec{u}_2 = \text{proj}_{\vec{v}_1}^{\perp}\vec{v}_2 = \vec{v}_2 - \text{proj}_{\vec{v}_1}\vec{v}_2$
    - $\vec{u}_k = \vec{v}_k - (\text{proj}_{\vec{v}_1}\vec{v}_k + \text{proj}_{\vec{v}_2}\vec{v}_k + ... + \text{proj}_{\vec{v}_{k-1}}\vec{v}_k)$
    - Then $S' = \{\vec{u}_1, \vec{u}_2, ..., \vec{u}_n\}$ is an orthogonal basis.