Singular Value Decomposition Linear Algebra X. Du

- Motivation: we know that we can decompose a symmetric matrix into $A = QDQ^T$ using eigenvalue and eigenvector analysis.
- As it turns out, we can also decompose any $m \times n$ matrix $A = U\Sigma V^T$
    - $A = U\Sigma V^T$ is the **singular value decomposition**.
    - U is an $m \times m$ orthogonal matrix and V is an $n \times n$ orthogonal matrix.
    - Σ is an $m \times n$ block matrix with the form $\Sigma = \begin{pmatrix} D & O \\ O & O \end{pmatrix}$, where D is a diagonal.
    - The column vectors of U are the **left-singular vectors** and the column vectors of V are the **right-singular vectors**.
- For any $m \times n$ matrix A, consider $A^T A$.
    - $A^T A$ is symmetric $\quad (A^T A)^T = A^T A^{T^T} = A^T A$
    - The **singular values** of A are the eigenvalues of $A^T A$. All eigenvalues of $A^T A$ are real and non-negative.
    - Note the utility of $A^T A$. It can be used to determine whether or not a matrix has linearly independent column vectors.
- Finding the singular value decomposition.
    - Note that $A^T A = V\Sigma^T U^T U\Sigma V^T = V\Sigma\Sigma^T V^T$. $\Sigma\Sigma^T$ is just an $m \times m$ diagonal matrix with the diagonal entries as the singular values squared.
    - Note that $AV = U\Sigma$, so $A\vec{v}_i = \vec{u}_i \sigma_i$.
    - Step 1: Compute $A^T A$.
    - Step 2: Compute the singular values of A by taking the square roots of the eigenvalues of $A^T A$.
    - Step 3: Compute the right-singular vectors in V by finding the normalized eigenvectors of $A^T A$.
    - Step 4: Compute the left-singular vectors in U by using $\vec{u}_i = \dfrac{1}{\sigma_i} A\vec{v}_i$.
        - Alternatively, you can find the normalized eigenvectors of AA^T.