1 9/19 Introduction

1.1 What is the Putnam?

The Putnam is hosted the first Saturday of every December. It is the main collegiate olympiad math competition. The competition consists of two 3-hours sessions (one in the morning, one in the afternoon). Each session will contain 6 problems of roughly increasing difficulty, and each problem will require you to write out a full proof.

A correct proof is 10 points. The median score is usually no points at all! Aim to correctly solve about 3 problems per session on the real exam – if you can do that perfectly, you're comfortably getting an Honorable Mention at least!

1.2 Logistics of this Seminar

Today, we will be going over 3 sample problems to provide some intuition on how to approach Putnam proofs and writing them efficiently. Future sessions will be more seminar-esque with people collecting and presenting interesting problems – or writing them on your own!

The primary text of this lecture is "Putnam and Beyond" by Razvan Gelca and Titu Andreescu. We are not assuming any prior mathematical experience. Experience with proofs and olympiad math is welcome, and please help each other out when you can! Since this is an introductory seminar, we will be going over the more fundamental concepts in "Putnam and Beyond." In the spring, we will aim to finish the textbook based on how much we have covered this semester, so please attend as much as you can to get a good foundation!

Each week, we will cover a core concept in the Putnam. Someone will be presenting a section of the textbook, along with an example problem, a problem they will have written, and two Putnam problems in the subject. Please sign up on the sheet provided for the week and topic you would like to present for! This week, we will begin with some warm-up problems to get a taste of how to approach olympiad math.

1.3 Problems

1. Let $\times$ represent the cross product in $\mathbb{R}^3$. For what positive integers n does there exist a set $S \subset \mathbb{R}^3$ with exactly n elements such that
$$S = \{v \times w : v, w \in S\}?$$
(2022 Putnam B2)

2. Find the smallest positive integer j such that for every polynomial $p(x)$ with integer coefficients and for every integer k, the integer
$$p^{(j)}(k) = \left.\frac{d^j}{dx^j}p(x)\right|_{x=k}$$
(the j-th derivative of $p(x)$ at k) is divisible by 2016. (2016 Putnam A1)

3. Determine all ordered pairs of real numbers (a, b) such that the line $y = ax+b$ intersects the curve $y = \ln(1+x^2)$ in exactly one point. (2022 Putnam A1)

1.4 Solutions

1. **2022 Putnam B2** Let's begin by analyzing this problem a bit. What the problem is really asking is to look, in $\mathbb{R}^3$, at which sets S have the property all the elements in it are cross products of some other two elements within it. We should begin by constructing some basic example to work with.

   But looking at at a bunch of 3-D vectors is hard and annoying! Don't be tempted into generalizing vectors into their components with variables, start with something simple, like a unit vector along one of the coordinate axes, or in an even simpler (and thus, better) example, the zero vector!

When we let $S = \{0\}$, we find that this handily solves our problem. Alternatively, we can get to this point by trying out values of n starting at 1 (again, the simplest possible approach).

Now that we have some rudimentary example, we can start building on it! Notice that if we have a vector $v \in S$, we necessarily need to have $0 \in S$ because $v \times v = 0$, so $n = 1$ is a valid solution. This is encouraging stuff, what kind of weird behavior do we get if we toss in a $w \in S$?

Now, it gets tempting to try to bash this out with coordinates v_1, v_2, v_3 and throwing in some w_1, w_2, w_3 for good measure, but resist the urge! Very few Putnam problems can really be bashed out, and there's no good restriction on the set S – or at least, some concrete property – that we can use to narrow down our bashing. Instead, let's look at some simple combinations of v and w that we can elaborate on.

Drawing from our above knowledge that $0, v \in S$, we can start thinking of ways that $v \times w$ can also lead to the 0-vector to narrow down our possible candidates for w. From this, we can realize that if w is collinear to v, then this is indeed the case. So we must look at w that are not collinear with v.

Another interesting combination of vectors are orthogonal ones. Perhaps there is something to be said about the behavior of S when v and w are orthogonal?

Indeed, if v and w are orthogonal, then we produce a vector u that is also orthogonal! This is very exciting, because 3 orthogonal vectors in $\mathbb{R}^3$ form a basis to work with – think the 3 coordinate axes. So we can add $u = v \times w$ to S while we explore.

Ah, but there is an issue! Knowing that all our vectors are orthonormal, if either $||v||$ and $||w||$ are greater than one, then $||u||$ will be ever so slightly even bigger. And when we do $u' = u \times v$, we will have some result that has an even bigger magnitude. Even worse, u' will be collinear to w! Conversely, if either $||v||$ and $||w||$ are less reverse, this same effect will happen in reverse! This is a disaster!

Now, we can wrangle this numerically by trying to come up with some scheme that allows for slight increases and decreases in magnitude so long as they compensate one another, or else to hand-wave entirely and say "well the number of elements doesn't change so we don't care." But these are not rigorous ways to bound the sets you're interested in (sets with 3 orthogonal vectors)! Fortunately there is a way out of this predicament.

We set all the orthogonal vectors to magnitude 1, just like the orthonormal coordinate vectors in $\mathbb{R}^3$. And just like those vectors, these too become orthonormal! This is very exciting, because it allows us to tightly control the magnitude of the resulting $v \times w$ vectors into a limited set. Indeed, working out all the cross-products of this set, we get $S = \{0, \pm v, \pm w, \pm u\}$, where v, w, u are an orthonormal basis of $\mathbb{R}^3$. Thus, $n = 7$ is also a valid answer.

Now that we have an orthonormal basis, we recognize a key quality of any valid S: it must consist of orthonormal vectors! If some v and w are not orthogonal, then the result w would be orthogonal to both, and create an infinite set of vectors from the resulting cross-products. Similarly, if some v and w are orthogonal but have non-1 cross-product values, we get a similar infinite cascade of vectors. With these wishy-washy intuition in mind, we can put it into more concrete mathematical terms.

First, we must prove the rigorous bound that all nonzero elements in S must be unit vectors. If we choose an initial u that is collinear, we falter because S could then only contain 0, as there is no cross-product that produces a collinear vector. Given some nonzero $v \in S$, let $w \in S$ be a vector not collinear with v with magnitude unequal to 1. S must contain the nonzero vector $u_1 = v \times w$. It must also contain any u_k defined by

$$u_l = v \times u_{k-1} \tag{1}$$

u_k is orthogonal to v by definition, we have $||u_k|| = ||v||||u_{k-1}||$. The sequence $||u_k||$ consists of distinct numbers and thus S is infinite, so we are uninterested in it. Note that this also rules out collinear vectors, because we demonstrate that all u must have magnitude 1.

Next, we must demonstrate that all nonzero elements in S must be orthogonal. Since $v \times w$ produces a vector of magnitude 1, the same magnitude as both v and w, then v and w must be orthogonal. Thus, we have defined the orthonormal basis we sketched out early.

With an orthonormal basis, we cannot add more elements to this set in $\mathbb{R}^3$, so there are no more possible finite sets S. Thus, $n = 1, 7$, QED.

2. **2016 Putnam A1** Test writers like to scare you! At first glance... there's no way that this could be an A1 problem! Let's break this down into something more manageable.

First, let's get some intuition into what it's asking. Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \tag{2}$$

where $n \geq 0$ can be arbitrarily large (provided it is an integer), and the coefficients a_i are also integers. Now, we need to find the j-th derivative of $p(x)$

$$\begin{aligned} p^{(j)}(x) &= \frac{d^j}{dx^j}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0) && (3)\\ &= \frac{d^{j-1}}{dx^{j-1}}(na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1) && (4)\\ &= \cdots && (5)\\ &= n(n-1)\cdots(n-j+1)a_n x^{n-j} + (n-1)\cdots(n-j)a_{n-1}x^{n-j-1} + \cdots + j(j-1)\cdots(2)a_j && (6) \end{aligned}$$

Keep in mind that for polynomials of degree $< j$, $p^{(j)}(x) = 0$. Now that we have some bound of which polynomials for a given j always work, we want to try to set a stricter bound.

But how do we go about finding a bound here? Well, in algebra problems, we like constants because they give us something concrete to work with, and help with getting a real sense of what's going on. More variables means more properties to keep track of, which makes problems hard, instead, let's see how we can work with the final term of the Right Hand Side (RHS).

With this train of logic, for some j, consider polynomial of degree j.

$$p^{(j)}(x) = j(j-1)\cdots(2)(1) = j! \tag{7}$$

This is very convenient, because now, however we choose k (in other words, however we choose a value of x), we will always have the same number ($j!$). With this in mind, we are looking for values of j where $j!$ is divisible by 2016.

Now, going into the year 2023, we want to be familiar with the prime factorization of 2023. This year it's weird ($2023 = 7 * 17^2$). After the Putnam, you can promptly forget this and start remembering the many factors of 2024. If you were taking this exam in 2016, you should already be familiar with the prime factorization of 2016.

$$2016 = 2^5 * 3^2 * 7 \tag{8}$$

We want the smallest factorial that contains this! After checking 7! (since we know we need a 7), we realize that doesn't work because

$$7! = 2^4 * 3^2 * 5 * 7 \tag{9}$$

Conveniently, we are simply missing a 2. Thus, we know that $j = 8$ is the minimal answer to the solution.

However, we are now at an impasse. Just because $j!$ is divisible by 2016 doesn't mean all the other coefficients of $p^{(j)}(x)$ are.... or does it?

Let's revisit the form of $p^{(j)}(x)$. One observation that we can make is that the coefficients for each x^i term look a bit like a combinatorial term. Well, if we wrangle out a $j!$ somehow, then we can prove 2016 divides all the individual terms in $p^{(j)}(x)$, and we should be happy, right? To test this, let's rewrite $p^{(j)}(x)$ as a summation.

$$\begin{aligned} p^{(j)}(x) &= \sum_{m=j}^{n} j(j-1)\cdots(m-j+1)a_m x^{m-j} \\ &= \sum_{m=j}^{n} \binom{m}{j} j! a_m x^{m-8} \end{aligned}$$

Thus, for any given j, for any given polynomial of degree n, we can guarantee that $j!$ divides each of the terms of $p^{(j)}(x)$. If $j \geq 8$, then we can guarantee that 2016 divides $p^{(j)}(k)$ for any integer k.

Are we done? It feels like we are done... But we are not! We must prove that $j = 7$ does not work!

Thankfully, above we noted that 2016 does not divide any $j!$ for $j \leq 7$. Thus, choosing $j \leq 7$ means polynomials of degree j fail the divisibility test. Thus, not only is $j \geq 8$ a viable solution, it is the set of only viable solutions. The smallest viable value for j is 8. Thus, QED.

Remark. *We can also find this solution by testing out higher values of n after the step where we find that $j = 8$ is a viable solution for polynomials of degree j. For polynomials of degree $n > 8$ where $j = 8$, we get*

$$\begin{aligned} p^{(8)}(x) &= \sum_{m=8}^{n} m(m-1)\cdots(m-7)a_m x^{m-8} \\ &= \sum_{m=8}^{n} \binom{m}{8} 8! a_m x^{m-8} \end{aligned}$$

We thus can guarantee that $j = 8$ is a viable solution to the problem by similar logic as above.

3. **2022 Putnam A1** Let's begin by characterizing what we're actually looking at. For convenience – because there are too many y's out there – let $f(x) = \ln(1+x^2)$. First question: what in the world does $f(x)$ look like? Without Desmos, it feels like we are hopelessly lost....

   Let's find some easy values – 0 and 1. Solving for this in $f(x)$, we get that the points $(0,0)$ and $(\pm\sqrt{e-1}, 1)$. Solving for this in x, we also get $(\pm 1, \ln 2 \approx 0.7$. We also know that functions of the form $g(t) = \ln(t)$ have a wiggly, loggish form, assuming that t doesn't overwhelm the ln.

   Again, this is very hand-wavy, but it will provide us the opportunity to sketch out a sense of what we're dealing with. We know that $f(x)$ is symmetrical about the y-axis, never dips below 0, and – after plotting some more x at 2, e, 10, etc. – looks loggish at large values. Now, since we know it's not exactly loggish, we should investigate for small numbers.

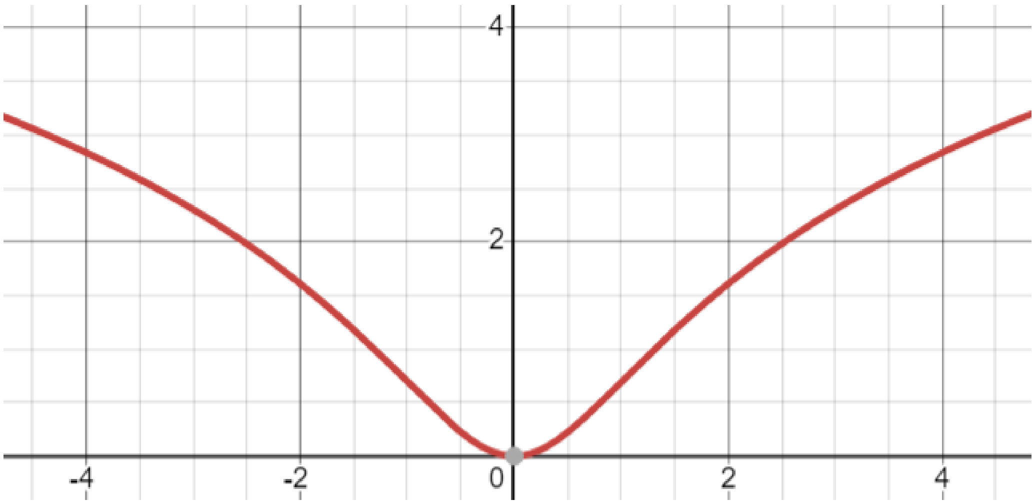
   But small numbers are super scary... fractions get super messy... and graphing out different points between 0 and 1 is really hard... Instead, we realize that as long as we know where the *weird* stuff happens, we can get a *sense* of what is going on! In this case, we want to know when the x^2-ness of $f(x)$ dominates over its ln-ness. In other words, we want to know when this functions behavior changes.... perhaps a point at which it changes over to a different state...

   Let's take the derivative! x^2 is concave up, ln is concave down, so comparing when $f(x)$ switches could help! In a more intuitive sense, we know that x^2 wants to make the function bigger at a bigger rate, while ln wants to make that increase smaller and smaller over time. When one starts to dominate, their *effect* starts to dominate, so once we learn where this happens, it should be good enough to use to analyze the function (for now, if something comes up we can deal with it later)!

$$f'(x) = \frac{2x}{1+x^2} \tag{10}$$

$$f''(x) = -\frac{2(x^2-1)}{(x^2+1)^2} \tag{11}$$

This allows us to confirm that there's a divot at $x = 0$, since the slope of the tangent line is 0 at that point. It also demonstrates that for $|x| > 1$, ln-ish behavior dominates, since the slope starts to decrease there forever (i.e. it is an inflection point) Note that because $f(x)$ is symmetrical, we can just look at positive values of x for now. With this information, we can finally get a decent graph of what $f(x)$ looks like.



Okay... so what? For many similar problems, the process of getting a good sketch of the function is also key to solving the problem itself because you get the fundamental properties that make the function tick. For example, knowing the symmetricness of the function around 0, it suffices to consider the case $a \geq 0$. This is because $y = ax + b$ and $y = -ax + b$ intersect $f(x)$ the same number of times, so we can simplify our search there. To get rid of that annoying equality, we can set $a = 0$, we find that $b = 0$ since the lowest point in $f(x)$ must be 0 (and it's a vertex point!).

At this point, we want to try exploiting our bounds on a more. We can notice that $f'(x) \leq 1$, and $f'(x)$ achieves each value in $(0, 1)$ exactly twice on $[0, \infty)$. This is useful because we have an upper bound on the slope of the tangent line to $f(x)$, which is the key property that interacts with the choice of a. Toying around with large values of $a \geq 1$, we soon find that any choice of b would work! But we should probably do something more rigorous, utilizing known bounds on $f'(x)$...

Thus, we can use the Intermediate Value Theorem! For some arbitrarily small $x < 0$, $ax + b < 0 < f(x)$. For some arbitrarily large $x > 0$, we can compute the limit $\lim_{x\to\infty} \frac{\ln(1+x^2)}{x} = 0$. Thus, $ax + b > f(x)$ for very large x.

Then, we note from above that for $a > 1$, $f'(x) < a$, and thus by the Mean Value Theorem y and $f(x)$ cannot intersect more than once. Dealing with $a = 1$, we can set up a small proof by contradiction.

Suppose they intersect at two distinct points (x_0, y_0) and (x_1, y_1). Then

$$1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\int_{x_0}^{x_1} f'(x)\,dx}{x_1 - x_0} < 1 \tag{12}$$

Since $f'(x)$ is continuous and $f'(x) \leq 1$ with equality only at one point, the inequality holds because the numerator must be less than or equal to the denominator at every corresponding point, with at most one point achieving equality in a range of infinitesimals. Thus, for any choice of $a \geq 1$, any choice of b will satisfy the requirements.

Finally, we have a very tight bound on the remaining possible values of a. We will look at $a \in (0, 1)$. We are interested in where $f'(x) = a$.

Solving this, we get $x' = \pm\frac{1\pm\sqrt{1-a^2}}{a}$. In other words, these are the two points where the slopes of y and the tangent line to $f(x)$ line up. We need to choose the correct b such that exactly one of these points actually exists on y. To reformulate this in a way that's tractable, we are interested in the value(s) of b that allow us to move y up or down such that these tangent points strike exactly once.

We can characterize this movement by looking at the gap between y and $f(x)$. Let us define $g(x) = f(x) - y = f(x) - ax - b$. Same as before, taking the derivative will let us look at the behavior of $g(x)$ better.

$$g'(x) = f'(x) - a \tag{13}$$
$$g''(x) = f''(x) \tag{14}$$

From this, we can find that $\lim_{x \to -\infty} g(x) = \infty$ while $\lim_{x \to \infty} g(x) = -\infty$. Visually, we can also intuit that g' is strictly decreasing on $(-\infty, x'_-)$, strictly increasing on (x'_-, x'_+), and strictly decreasing on (x'_+, ∞).

It follows that $g(x) = b$ has exactly one solution for $b < g(x'_-)$ or $b > g(x'_+)$, exactly three solutions for $g(x'_-) < b < g(x'_+)$, and exactly two solutions for $b = g(x'_\pm)$. So we are done! QED.