Receptive field MuReNN_AP

Definition of the problem

The structure of Murenn_AP:

$$x \xrightarrow{wt} A \xrightarrow{conv1d} B \xrightarrow{iwt} y \tag{1}$$

With:

- x: the input 1d signal
- wt: wavelet transform
- conv1d: random filters with i.i.d. gaussian distributed weights
- iwt: inverse wavelet transform

Assume all the operations are continuous. Denote the input signal at time t as x_t, the center point of the output signal as y_0, our goal is to calculate the gradient $\frac{\partial y_0}{\partial x_t}, \forall t$.

Key points to note:

1. Since the conv1d filters are random, the result is a random vector.
2. The system is linear, so the result is independent of the values of $\{x_t\}$.

Conclusion

Splitting the system into three operations

1. $x \xrightarrow{wt} A$:

$$\begin{gathered} A_{su} = a(u,s) = \int_{-\infty}^{+\infty} x(t)\overline{\psi_s}(t-u)dt \\ \Rightarrow \frac{\partial a(u,s)}{\partial x(t)} = \overline{\psi_s}(t-u) \end{gathered} \tag{2}$$

2. $A \xrightarrow{conv1d} B$:

$$\begin{gathered} B_{su'} = b(u',s) = \int_{-\infty}^{+\infty} a(u,s)\omega_s(u'-u)du \\ \Rightarrow \frac{\partial b(u',s)}{\partial a(u,s)} = \omega_s(u'-u) \end{gathered} \tag{3}$$

3. $B \xrightarrow{iwt} y$:

$$
\begin{aligned}
y(u'') &= \frac{1}{C_\psi}\int_0^{s_0}\int_{-\infty}^{+\infty} b(u',s)\psi_s(u''-u')du'\frac{ds}{s^2} \\
y(0) &= \frac{1}{C_\psi}\int_0^{s_0}\int_{-\infty}^{+\infty} b(u',s)\psi_s(0-u')du'\frac{ds}{s^2} \\
&\Rightarrow \frac{\partial y(0)}{\partial b(u',s)} = \frac{\psi_s(-u')}{s^2}
\end{aligned} \tag{4}
$$

According to the chain rule,

$$
\frac{\partial y_0}{\partial x_t} = \frac{1}{C_\psi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^{+\infty}\frac{1}{s^2}\psi_s(-u')\omega_s(u'-u)\overline{\psi_s}(t-u)dudu'ds \tag{5}
$$

By changing the variable: $\tau = u - u' \Rightarrow d\tau = du, t - u = t - \tau - u'$

$$
\Rightarrow \frac{\partial y_0}{\partial x_t} = \frac{1}{C_\psi}\int_0^{+\infty}\frac{1}{s^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\psi_s(-u')\overline{\psi_s}(t-\tau-u')\omega_s(-\tau)du'd\tau ds \tag{6}
$$

$$
= \frac{1}{C_\psi}\int_0^{+\infty}\frac{1}{s^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\psi_s(-u')\psi_s^*(u'+\tau-t)\omega_s(-\tau)du'd\tau ds \tag{7}
$$

$$
= \frac{1}{C_\psi}\int_0^{+\infty}\frac{1}{s^2}\int_{-\infty}^{+\infty}(\psi_s * \psi_s^*(\tau-t))\omega_s(-\tau)d\tau ds \tag{8}
$$

$$
= \frac{1}{C_\psi}\int_0^{+\infty}\frac{1}{s^2}\psi_s * \psi_s^* * \omega_s(t)ds \tag{9}
$$