Assumption:

1. ω a conv1d filter with support $[-L, L]$ and weights i.i.d normal variables:

$$\omega \sim \mathcal{N}(0, \sigma^2 \mathrm{I}_{2L+1}) \tag{1}$$

2. To have $||\omega||^2 = 1$, $\sigma^2$and L should satisfy:

$$\sigma^2(2L+1) = 1 \tag{2}$$

Conv1D's case:

$$\begin{aligned} \mathbb{E}[\sum_{t=-L}^{L} t\omega(t)^2] &= \sum_{t=-L}^{L} t\mathbb{E}[\omega(t)^2] \\ &= \sum_{t=-L}^{L} t\sigma^2 \\ &= 0 \end{aligned} \tag{3}$$

$$\begin{aligned} \mathbb{E}[\sum_{t=-L}^{L} t^2\omega(t)^2] &= \sum_{t=-L}^{L} t^2\mathbb{E}[\omega(t)^2] \\ &= \sigma^2 \sum_{t=-L}^{L} t^2 \\ &= \sigma^2 \frac{L(L+1)(2L+1)}{3} \\ &= \frac{L(L+1)}{3} \end{aligned} \tag{4}$$

Dilated Conv1D's case:

The filter ω_d is defined by:

$$\omega_d(n) = \sum_{t=-L}^{L} \omega(t)\delta(n - 2^j t) \tag{5}$$

Obviously $\mathbb{E}[\sum_{t=-L}^{L} t\omega_d(t)^2] = 0$, the time spread of the filter:

$$\mathbb{E}[\sum_{t=-2^jL}^{2^jL} t^2\omega_d(t)^2] \tag{6}$$

By using the change of variable $t = 2^j\tau$, we have :

$$
\begin{aligned}
\mathbb{E}[\sum_{t=-2^jL}^{2^jL} t^2\omega_d(t)^2] &= \mathbb{E}[\sum_{\tau=-L}^{L}(2^j\tau)^2\omega_d(2^j\tau)^2] \\
&= \mathbb{E}[\sum_{\tau=-L}^{L}(2^j\tau)^2\omega(\tau)^2] \\
&= \mathbb{E}[\omega(\tau)^2]\sum_{\tau=-L}^{L}(2^j\tau)^2 \\
&= 4^j\sigma^2\frac{L(L+1)(2L+1)}{3} \\
&= 4^j\frac{L(L+1)}{3}
\end{aligned}
\tag{7}
$$

All Pass Dilated Conv1D's case:

Assumption:

1. ω_j the Conv1D Gaussian filter of level j
2. $J(2L+1)\sigma^2 = 1$

Definition of the filter:

$$
\omega_{ad}(n) = \sum_{j=1}^{J}\sum_{t=-L}^{L}\omega_j(t)\delta(n-2^jt) \tag{8}
$$

Time spread of the filter:

$$
\begin{aligned}
&\mathbb{E}(\sum_{n=-2^JL}^{2^JL} n^2\omega_{ad}(n)^2) \\
&= \sum_{n=-2^JL}^{2^JL} n^2\mathbb{E}[\sum_{j=1}^{J}\sum_{j'=1}^{J}\sum_{t=-L}^{L}\sum_{t'=-L}^{L}\omega_j(t)\omega_{j'}(t')\delta(n-2^jt)\delta(n-2^{j'}t')] \\
&= \sum_{n=-2^jL}^{2^jL} n^2\sum_{j=1}^{J}\sum_{t=-L}^{L}\sigma^2\delta(n-2^jt) \\
&= \sum_{j=1}^{J}\sum_{t=-L}^{L}(2^jt)^2\sigma^2 \\
&= \sigma^2\sum_{j=1}^{J}4^j\sum_{t=-L}^{L}t^2 \\
&= \frac{4(4^J-1)L(L+1)}{9J}
\end{aligned}
\tag{9}
$$

Comparison with Conv1d :

The support of the Conv1D filter has to be $[-2^JL, 2^JL]$ to have the same receptive field with ω_d, the time spread becomes:

$$\mathbb{E}[\sum_{t=-2^J L}^{2^J L} t^2 \omega(t)^2] = 4^J \frac{L(L+2^{-J})}{3} \tag{10}$$

In this case,

$$\begin{aligned} \frac{TL(\omega_{ad})}{TL(\omega)} &= \frac{4(4^J-1)L(L+1)}{3J4^J(L(L+2^{-J}L))} \\ &\approx \frac{4}{3J} \end{aligned} \tag{11}$$