Foundations of mathematics

en.wikipedia.org

March 20, 2022

Contents

1 Foundations of mathematics

Study of the basic mathematical concepts For the book by Hilbert and Bernays, see Grundlagen der Mathematik[1]. **Foundations of mathematics** is the study of the philosophical[2] and logical[1] and/or algorithmic[3] basis of mathematics[4], or, in a broader sense, the mathematical investigation of what underlies the philosophical theories concerning the nature of mathematics.[2] In this latter sense, the distinction between foundations of mathematics and philosophy of mathematics[5] turns out to be quite vague. Foundations of mathematics can be conceived as the study of the basic mathematical concepts (set, function, geometrical figure, number, etc.) and how they form hierarchies of more complex structures and concepts, especially the fundamentally important structures that form the language of mathematics[6] (formulas, theories and their models[7] giving a meaning to formulas, definitions, proofs, algorithms, etc.) also called metamathematical concepts[8], with an eye to the philosophical aspects and the unity of mathematics. The search for foundations of mathematics is a central question of the philosophy of mathematics; the abstract nature of mathematical objects presents special philosophical challenges.

The foundations of mathematics as a whole does not aim to contain the foundations of every mathematical topic. Generally, the *foundations* of a field of study refers to a more-or-less systematic analysis of its most basic or fundamental concepts, its conceptual unity and its natural ordering or hierarchy of concepts, which may help to connect it with the rest of human knowledge. The development, emergence[9], and clarification of the foundations can come late in the history of a field, and might not be viewed by everyone as its most interesting part.

Mathematics always played a special role in scientific thought, serving since ancient times as a model of truth and rigor for rational inquiry, and giving tools or even a foundation for other sciences (especially physics). Mathematics' many developments towards higher abstractions in the 19th century brought new challenges and paradoxes, urging for a deeper and more systematic examination of the nature and criteria of mathematical truth[10], as well as a unification of the diverse branches of mathematics into a coherent whole.

1 https://en.wikipedia.org/wiki/Grundlagen_der_Mathematik
2 https://en.wikipedia.org/wiki/Philosophy
3 https://en.wikipedia.org/wiki/Algorithm
4 https://en.wikipedia.org/wiki/Mathematics
5 https://en.wikipedia.org/wiki/Philosophy_of_mathematics
6 https://en.wikipedia.org/wiki/Language_of_mathematics
7 https://en.wikipedia.org/wiki/Model_theory
8 https://en.wikipedia.org/wiki/Metamathematics
9 https://en.wikipedia.org/wiki/Emergence
10 https://en.wikipedia.org/wiki/Mathematical_truth

The systematic search for the foundations of mathematics started at the end of the 19th century and formed a new mathematical discipline called mathematical logic[11], which later had strong links to theoretical computer science[12]. It went through a series of crises with paradoxical results, until the discoveries stabilized during the 20th century as a large and coherent body of mathematical knowledge with several aspects or components (set theory[13], model theory[14], proof theory[15], etc.), whose detailed properties and possible variants are still an active research field. Its high level of technical sophistication inspired many philosophers to conjecture that it can serve as a model or pattern for the foundations of other sciences.

1.1 Historical context

See also: History of logic[16] and History of mathematics[17]

1.1.1 Ancient Greek mathematics

Further information: Ancient Greek mathematics[18] While the practice of mathematics had previously developed in other civilizations, special interest in its theoretical and foundational aspects was clearly evident in the work of the Ancient Greeks.

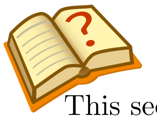
Early Greek philosophers disputed as to which is more basic, arithmetic or geometry. Zeno of Elea[19] (490 − c. 430 BC) produced four paradoxes that seem to show the impossibility of change. The Pythagorean school of mathematics[20] originally insisted that only natural and rational numbers exist. The discovery of the irrationality[21] of $\sqrt{2}$, the ratio of the diagonal of a square to its side (around 5th century BC), was a shock to them which they only reluctantly accepted. The discrepancy between rationals and reals was finally resolved by Eudoxus of Cnidus[22] (408–355 BC), a student of Plato[23], who reduced the comparison of two irrational ratios to comparisons of multiples of the magnitudes involved. His method anticipated that of the Dedekind cut[24] in the modern definition of real numbers by Richard Dedekind[25] (1831−1916).[3]

In the *Posterior Analytics*[26], Aristotle[27] (384–322 BC) laid down the axiomatic method[28] for organizing a field of knowledge logically by means of primitive concepts, axioms, postulates,

11 https://en.wikipedia.org/wiki/Mathematical_logic
12 https://en.wikipedia.org/wiki/Theoretical_computer_science
13 https://en.wikipedia.org/wiki/Set_theory
14 https://en.wikipedia.org/wiki/Model_theory
15 https://en.wikipedia.org/wiki/Proof_theory
16 https://en.wikipedia.org/wiki/History_of_logic
17 https://en.wikipedia.org/wiki/History_of_mathematics
18 https://en.wikipedia.org/wiki/Ancient_Greek_mathematics
19 https://en.wikipedia.org/wiki/Zeno_of_Elea
20 https://en.wikipedia.org/wiki/Pythagoreanism
21 https://en.wikipedia.org/wiki/Irrational_number
22 https://en.wikipedia.org/wiki/Eudoxus_of_Cnidus
23 https://en.wikipedia.org/wiki/Plato
24 https://en.wikipedia.org/wiki/Dedekind_cut
25 https://en.wikipedia.org/wiki/Richard_Dedekind
26 https://en.wikipedia.org/wiki/Posterior_Analytics
27 https://en.wikipedia.org/wiki/Aristotle
28 https://en.wikipedia.org/wiki/Axiomatic_method

definitions, and theorems. Aristotle took a majority of his examples for this from arithmetic and from geometry. This method reached its high point with Euclid[29]'s *Elements*[30] (300 BC), a treatise on mathematics structured with very high standards of rigor: Euclid justifies each proposition by a demonstration in the form of chains of syllogisms[31] (though they do not always conform strictly to Aristotelian templates). Aristotle's syllogistic logic, together with the axiomatic method exemplified by Euclid's *Elements*, are recognized as scientific achievements of ancient Greece.

1.1.2 Platonism as a traditional philosophy of mathematics



This section **needs additional citations for verification[32]**. Please help improve this article[33] by adding citations to reliable sources[34]. Unsourced material may be challenged and removed. *(October 2014)(Learn how and when to remove this template message[35])*

Further information: Platonism (mathematics)[36] Starting from the end of the 19th century, a Platonist view of mathematics became common among practicing mathematicians.

The *concepts* or, as Platonists would have it, the *objects* of mathematics are abstract and remote from everyday perceptual experience: geometrical figures are conceived as idealities to be distinguished from effective drawings and shapes of objects, and numbers are not confused with the counting of concrete objects. Their existence and nature present special philosophical challenges: How do mathematical objects differ from their concrete representation? Are they located in their representation, or in our minds, or somewhere else? How can we know them?

The ancient Greek philosophers took such questions very seriously. Indeed, many of their general philosophical discussions were carried on with extensive reference to geometry and arithmetic. Plato[37] (424/423 BC – 348/347 BC) insisted that mathematical objects, like other platonic *Ideas* (forms or essences), must be perfectly abstract and have a separate, non-material kind of existence, in a world of mathematical objects independent of humans. He believed that the truths about these objects also exist independently of the human mind, but is *discovered* by humans. In the *Meno*[38] Plato's teacher Socrates asserts that it is possible to come to know this truth by a process akin to memory retrieval.

Above the gateway to Plato's academy appeared a famous inscription: ”Let no one who is ignorant of geometry enter here”. In this way Plato indicated his high opinion of geometry.

29 https://en.wikipedia.org/wiki/Euclid
30 https://en.wikipedia.org/wiki/Euclid%27s_Elements
31 https://en.wikipedia.org/wiki/Syllogisms
32 https://en.wikipedia.org/wiki/Wikipedia:Verifiability
33 https://en.wikipedia.org/w/index.php?title=Foundations_of_mathematics&action=edit
34 https://en.wikipedia.org/wiki/Help:Referencing_for_beginners
35 https://en.wikipedia.org/wiki/Help:Maintenance_template_removal
36 https://en.wikipedia.org/wiki/Platonism_(mathematics)
37 https://en.wikipedia.org/wiki/Plato
38 https://en.wikipedia.org/wiki/Meno

He regarded geometry as "the first essential in the training of philosophers", because of its abstract character.

This philosophy of *Platonist mathematical realism*[39] is shared by many mathematicians. It can be argued that Platonism somehow comes as a necessary assumption underlying any mathematical work.[4]

In this view, the laws of nature and the laws of mathematics have a similar status, and the effectiveness[40] ceases to be unreasonable. Not our axioms, but the very real world of mathematical objects forms the foundation.

Aristotle dissected and rejected this view in his Metaphysics[41]. These questions provide much fuel for philosophical analysis and debate.

1.1.3 Middle Ages and Renaissance

For over 2,000 years, Euclid's Elements stood as a perfectly solid foundation for mathematics, as its methodology of rational exploration guided mathematicians, philosophers, and scientists well into the 19th century.

The Middle Ages saw a dispute over the ontological status of the universals (platonic Ideas): Realism[42] asserted their existence independently of perception; conceptualism[43] asserted their existence within the mind only; nominalism[44] denied either, only seeing universals as names of collections of individual objects (following older speculations that they are words, "*logoi*").

René Descartes[45] published *La Géométrie*[46] (1637), aimed at reducing geometry to algebra by means of coordinate systems, giving algebra a more foundational role (while the Greeks embedded arithmetic into geometry by identifying whole numbers with evenly spaced points on a line). Descartes' book became famous after 1649 and paved the way to infinitesimal calculus.

Isaac Newton[47] (1642–1727) in England and Leibniz[48] (1646–1716) in Germany independently developed the infinitesimal calculus[49] based on heuristic methods greatly efficient, but direly lacking rigorous justifications. Leibniz even went on to explicitly describe infinitesimals as actual infinitely small numbers (close to zero). Leibniz also worked on formal logic but most of his writings on it remained unpublished until 1903.

39 https://en.wikipedia.org/wiki/Platonism_(mathematics)
40 https://en.wikipedia.org/wiki/The_Unreasonable_Effectiveness_of_Mathematics_in_the_Natural_Sciences
41 https://en.wikipedia.org/wiki/Metaphysics_(Aristotle)
42 https://en.wikipedia.org/wiki/Philosophical_realism
43 https://en.wikipedia.org/wiki/Conceptualism
44 https://en.wikipedia.org/wiki/Nominalism
45 https://en.wikipedia.org/wiki/Ren%C3%A9_Descartes
46 https://en.wikipedia.org/wiki/La_G%C3%A9om%C3%A9trie
47 https://en.wikipedia.org/wiki/Isaac_Newton
48 https://en.wikipedia.org/wiki/Gottfried_Wilhelm_Leibniz
49 https://en.wikipedia.org/wiki/Infinitesimal_calculus

The Protestant philosopher George Berkeley[50] (1685–1753), in his campaign against the religious implications of Newtonian mechanics, wrote a pamphlet on the lack of rational justifications of infinitesimal calculus:[5] ”They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?”

Then mathematics developed very rapidly and successfully in physical applications, but with little attention to logical foundations.

1.1.4 19th century

In the 19th century[51], mathematics became increasingly abstract. Concerns about logical gaps and inconsistencies in different fields led to the development of axiomatic systems.

Real analysis

See also: Mathematical analysis § History[52] Cauchy[53] (1789–1857) started the project of formulating and proving the theorems of infinitesimal calculus[54] in a rigorous manner, rejecting the heuristic principle of the generality of algebra[55] exploited by earlier authors. In his 1821 work *Cours d'Analyse*[56] he defines infinitely small quantities[57] in terms of decreasing sequences that converge to 0, which he then used to define continuity. But he did not formalize his notion of convergence.

The modern (ε, δ)-definition of limit[58] and continuous functions[59] was first developed by Bolzano[60] in 1817, but remained relatively unknown. It gives a rigorous foundation of infinitesimal calculus based on the set of real numbers, arguably resolving the Zeno paradoxes and Berkeley's arguments.

Mathematicians such as Karl Weierstrass[61] (1815–1897) discovered pathological functions such as continuous, nowhere-differentiable functions[62]. Previous conceptions of a function as a rule for computation, or a smooth graph, were no longer adequate. Weierstrass began to advocate the arithmetization of analysis[63], to axiomatize analysis using properties of the natural numbers.

In 1858, Dedekind[64] proposed a definition of the real numbers as cuts[65] of rational numbers. This reduction of real numbers and continuous functions in terms of rational numbers, and

50 https://en.wikipedia.org/wiki/George_Berkeley
51 https://en.wikipedia.org/wiki/History_of_mathematics#19th_century
52 https://en.wikipedia.org/wiki/Mathematical_analysis#History
53 https://en.wikipedia.org/wiki/Cauchy
54 https://en.wikipedia.org/wiki/Infinitesimal_calculus
55 https://en.wikipedia.org/wiki/Generality_of_algebra
56 https://en.wikipedia.org/wiki/Cours_d%27Analyse
57 https://en.wikipedia.org/wiki/Infinitesimal
58 https://en.wikipedia.org/wiki/(%CE%B5,_%CE%B4)-definition_of_limit
59 https://en.wikipedia.org/wiki/Continuous_functions
60 https://en.wikipedia.org/wiki/Bernard_Bolzano
61 https://en.wikipedia.org/wiki/Karl_Weierstrass
62 https://en.wikipedia.org/wiki/Weierstrass_function
63 https://en.wikipedia.org/wiki/Arithmetization_of_analysis
64 https://en.wikipedia.org/wiki/Richard_Dedekind
65 https://en.wikipedia.org/wiki/Dedekind_cuts

thus of natural numbers, was later integrated by Cantor[66] in his set theory, and axiomatized in terms of second order arithmetic[67] by Hilbert and Bernays.

Group theory

See also: History of group theory[68] For the first time, the limits of mathematics were explored. Niels Henrik Abel[69] (1802–1829), a Norwegian, and Évariste Galois[70], (1811–1832) a Frenchman, investigated the solutions of various polynomial equations, and proved that there is no general algebraic solution to equations of degree greater than four (Abel–Ruffini theorem[71]). With these concepts, Pierre Wantzel[72] (1837) proved that straightedge and compass alone cannot trisect an arbitrary angle[73] nor double a cube[74]. In 1882, Lindemann[75] building on the work of Hermite[76] showed that a straightedge and compass quadrature of the circle[77] (construction of a square equal in area to a given circle) was also impossible by proving that π[78] is a transcendental number[79]. Mathematicians had attempted to solve all of these problems in vain since the time of the ancient Greeks.

Abel and Galois's works opened the way for the developments of group theory[80] (which would later be used to study symmetry[81] in physics and other fields), and abstract algebra[82]. Concepts of vector spaces[83] emerged from the conception of barycentric coordinates[84] by Möbius[85] in 1827, to the modern definition of vector spaces and linear maps by Peano in 1888. Geometry was no more limited to three dimensions. These concepts did not generalize numbers but combined notions of functions and sets which were not yet formalized, breaking away from familiar mathematical objects.

Non-Euclidean geometries

See also: Non-Euclidean geometry § History[86] After many failed attempts to derive the parallel postulate[87] from other axioms, the study of the still hypothetical hyperbolic geometry[88]

66 https://en.wikipedia.org/wiki/Georg_Cantor
67 https://en.wikipedia.org/wiki/Second_order_arithmetic
68 https://en.wikipedia.org/wiki/History_of_group_theory
69 https://en.wikipedia.org/wiki/Niels_Henrik_Abel
70 https://en.wikipedia.org/wiki/%C3%89variste_Galois
71 https://en.wikipedia.org/wiki/Abel%E2%80%93Ruffini_theorem
72 https://en.wikipedia.org/wiki/Pierre_Wantzel
73 https://en.wikipedia.org/wiki/Trisect_an_arbitrary_angle
74 https://en.wikipedia.org/wiki/Doubling_the_cube
75 https://en.wikipedia.org/wiki/Ferdinand_von_Lindemann
76 https://en.wikipedia.org/wiki/Charles_Hermite
77 https://en.wikipedia.org/wiki/Quadrature_of_the_circle
78 https://en.wikipedia.org/wiki/Pi
79 https://en.wikipedia.org/wiki/Transcendental_number
80 https://en.wikipedia.org/wiki/Group_theory
81 https://en.wikipedia.org/wiki/Symmetry
82 https://en.wikipedia.org/wiki/Abstract_algebra
83 https://en.wikipedia.org/wiki/Vector_spaces
84 https://en.wikipedia.org/wiki/Barycentric_coordinates_(mathematics)
85 https://en.wikipedia.org/wiki/August_Ferdinand_M%C3%B6bius
86 https://en.wikipedia.org/wiki/Non-Euclidean_geometry#History
87 https://en.wikipedia.org/wiki/Parallel_postulate
88 https://en.wikipedia.org/wiki/Hyperbolic_geometry

by Johann Heinrich Lambert[89] (1728–1777) led him to introduce the hyperbolic functions[90] and compute the area of a hyperbolic triangle[91] (where the sum of angles is less than 180°). Then the Russian mathematician Nikolai Lobachevsky[92] (1792–1856) established in 1826 (and published in 1829) the coherence of this geometry (thus the independence of the parallel postulate[93]), in parallel with the Hungarian mathematician János Bolyai[94] (1802–1860) in 1832, and with Gauss[95]. Later in the 19th century, the German mathematician Bernhard Riemann[96] developed Elliptic geometry[97], another non-Euclidean geometry[98] where no parallel can be found and the sum of angles in a triangle is more than 180°. It was proved consistent by defining point to mean a pair of antipodal points on a fixed sphere and line to mean a great circle[99] on the sphere. At that time, the main method for proving the consistency of a set of axioms was to provide a model[100] for it.

Projective geometry

One of the traps in a deductive system[101] is circular reasoning[102], a problem that seemed to befall projective geometry[103] until it was resolved by Karl von Staudt[104]. As explained by Russian historians:[6]

> In the mid-nineteenth century there was an acrimonious controversy between the proponents of synthetic and analytic methods in projective geometry, the two sides accusing each other of mixing projective and metric concepts. Indeed the basic concept that is applied in the synthetic presentation of projective geometry, the cross-ratio[105] of four points of a line, was introduced through consideration of the lengths of intervals.

The purely geometric approach of von Staudt was based on the complete quadrilateral[106] to express the relation of projective harmonic conjugates[107]. Then he created a means of expressing the familiar numeric properties with his Algebra of Throws[108]. English language versions of this process of deducing the properties of a field[109] can be found in either the

89 https://en.wikipedia.org/wiki/Johann_Heinrich_Lambert
90 https://en.wikipedia.org/wiki/Hyperbolic_functions
91 https://en.wikipedia.org/wiki/Hyperbolic_triangle
92 https://en.wikipedia.org/wiki/Nikolai_Lobachevsky
93 https://en.wikipedia.org/wiki/Parallel_postulate
94 https://en.wikipedia.org/wiki/J%C3%A1nos_Bolyai
95 https://en.wikipedia.org/wiki/Gauss
96 https://en.wikipedia.org/wiki/Bernhard_Riemann
97 https://en.wikipedia.org/wiki/Elliptic_geometry
98 https://en.wikipedia.org/wiki/Non-Euclidean_geometry
99 https://en.wikipedia.org/wiki/Great_circle
100 https://en.wikipedia.org/wiki/Model_(mathematical_logic)
101 https://en.wikipedia.org/wiki/Deductive_system
102 https://en.wikipedia.org/wiki/Circular_reasoning
103 https://en.wikipedia.org/wiki/Projective_geometry
104 https://en.wikipedia.org/wiki/Karl_von_Staudt
105 https://en.wikipedia.org/wiki/Cross-ratio
106 https://en.wikipedia.org/wiki/Complete_quadrilateral
107 https://en.wikipedia.org/wiki/Projective_harmonic_conjugate
108 https://en.wikipedia.org/wiki/Karl_von_Staudt#Algebra_of_throws
109 https://en.wikipedia.org/wiki/Field_(mathematics)

book by Oswald Veblen[110] and John Young, *Projective Geometry* (1938), or more recently in John Stillwell[111]'s *Four Pillars of Geometry* (2005). Stillwell writes on page 120

> ... projective geometry is *simpler* than algebra in a certain sense, because we use only five geometric axioms to derive the nine field axioms.

The algebra of throws is commonly seen as a feature of cross-ratios since students ordinarily rely upon numbers[112] without worry about their basis. However, cross-ratio calculations use metric[113] features of geometry, features not admitted by purists. For instance, in 1961 Coxeter[114] wrote *Introduction to Geometry* without mention of cross-ratio.

Boolean algebra and logic

Attempts of formal treatment of mathematics had started with Leibniz and Lambert[115] (1728–1777), and continued with works by algebraists such as George Peacock[116] (1791–1858). Systematic mathematical treatments of logic came with the British mathematician George Boole[117] (1847) who devised an algebra that soon evolved into what is now called Boolean algebra[118], in which the only numbers were 0 and 1 and logical combinations (conjunction, disjunction, implication and negation) are operations similar to the addition and multiplication of integers. Additionally, De Morgan[119] published his laws[120] in 1847. Logic thus became a branch of mathematics. Boolean algebra is the starting point of mathematical logic and has important applications in computer science[121].

Charles Sanders Peirce[122] built upon the work of Boole to develop a logical system for relations[123] and quantifiers[124], which he published in several papers from 1870 to 1885.

The German mathematician Gottlob Frege[125] (1848–1925) presented an independent development of logic with quantifiers in his Begriffsschrift[126] (formula language) published in 1879, a work generally considered as marking a turning point in the history of logic. He exposed deficiencies in Aristotle's *Logic*, and pointed out the three expected properties of a mathematical theory[*citation needed*[127]]

1. Consistency[128]: impossibility of proving contradictory statements.

110 https://en.wikipedia.org/wiki/Oswald_Veblen
111 https://en.wikipedia.org/wiki/John_Stillwell
112 https://en.wikipedia.org/wiki/Number
113 https://en.wikipedia.org/wiki/Metric_(mathematics)
114 https://en.wikipedia.org/wiki/Coxeter
115 https://en.wikipedia.org/wiki/Johann_Heinrich_Lambert
116 https://en.wikipedia.org/wiki/George_Peacock
117 https://en.wikipedia.org/wiki/George_Boole
118 https://en.wikipedia.org/wiki/Boolean_algebra
119 https://en.wikipedia.org/wiki/Augustus_De_Morgan
120 https://en.wikipedia.org/wiki/De_Morgan%27s_laws
121 https://en.wikipedia.org/wiki/Computer_science
122 https://en.wikipedia.org/wiki/Charles_Sanders_Peirce
123 https://en.wikipedia.org/wiki/Relation_(logic)
124 https://en.wikipedia.org/wiki/Quantifier_(logic)
125 https://en.wikipedia.org/wiki/Gottlob_Frege
126 https://en.wikipedia.org/wiki/Begriffsschrift
128 https://en.wikipedia.org/wiki/Consistency

2. Completeness[129]: any statement is either provable or refutable (i.e. its negation is provable).
3. Decidability[130]: there is a decision procedure to test any statement in the theory.

He then showed in *Grundgesetze der Arithmetik (Basic Laws of Arithmetic)* how arithmetic could be formalised in his new logic.

Frege's work was popularized by Bertrand Russell[131] near the turn of the century. But Frege's two-dimensional notation had no success. Popular notations were (x) for universal and (∃x) for existential quantifiers, coming from Giuseppe Peano[132] and William Ernest Johnson[133] until the ∀symbol was introduced by Gerhard Gentzen[134] in 1935 and became canonical in the 1960s.

From 1890 to 1905, Ernst Schröder[135] published *Vorlesungen über die Algebra der Logik* in three volumes. This work summarized and extended the work of Boole, De Morgan, and Peirce, and was a comprehensive reference to symbolic logic[136] as it was understood at the end of the 19th century.

Peano arithmetic

Main article: Peano axioms[137] The formalization of arithmetic[138] (the theory of natural numbers[139]) as an axiomatic theory started with Peirce in 1881 and continued with Richard Dedekind[140] and Giuseppe Peano[141] in 1888. This was still a second-order[142] axiomatization (expressing induction in terms of arbitrary subsets, thus with an implicit use of set theory[143]) as concerns for expressing theories in first-order logic[144] were not yet understood. In Dedekind's work, this approach appears as completely characterizing natural numbers and providing recursive definitions of addition and multiplication from the successor function[145] and mathematical induction[146].

129 https://en.wikipedia.org/wiki/Completeness_(logic)
130 https://en.wikipedia.org/wiki/Decidability_(logic)
131 https://en.wikipedia.org/wiki/Bertrand_Russell
132 https://en.wikipedia.org/wiki/Giuseppe_Peano
133 https://en.wikipedia.org/wiki/William_Ernest_Johnson
134 https://en.wikipedia.org/wiki/Gerhard_Gentzen
135 https://en.wikipedia.org/wiki/Ernst_Schr%C3%B6der_(mathematician)
136 https://en.wikipedia.org/wiki/Mathematical_logic#Symbolic_logic
137 https://en.wikipedia.org/wiki/Peano_axioms
138 https://en.wikipedia.org/wiki/Arithmetic
139 https://en.wikipedia.org/wiki/Natural_numbers
140 https://en.wikipedia.org/wiki/Richard_Dedekind
141 https://en.wikipedia.org/wiki/Giuseppe_Peano
142 https://en.wikipedia.org/wiki/Second-order_logic
143 https://en.wikipedia.org/wiki/Set_theory
144 https://en.wikipedia.org/wiki/First-order_logic
145 https://en.wikipedia.org/wiki/Successor_function
146 https://en.wikipedia.org/wiki/Mathematical_induction

1.2 Foundational crisis

The **foundational crisis of mathematics** (in German[147] *Grundlagenkrise der Mathematik*) was the early 20th century's term for the search for proper foundations for mathematics.

Several schools of the philosophy of mathematics[148] ran into difficulties one after the other in the 20th century, as the assumption that mathematics had any foundation that could be consistently[149] stated within mathematics itself was heavily challenged by the discovery of various paradoxes[150] (such as Russell's paradox[151]).

The name *"paradox"* should not be confused with *contradiction*. A contradiction[152] in a formal theory is a formal proof of an absurdity inside the theory (such as $2 + 2 = 5$), showing that this theory is inconsistent[153] and must be rejected. But a paradox may be either a surprising but true result in a given formal theory, or an informal argument leading to a contradiction, so that a candidate theory, if it is to be formalized, must disallow at least one of its steps; in this case the problem is to find a satisfying theory without contradiction. Both meanings may apply if the formalized version of the argument forms the proof of a surprising truth. For instance, Russell's paradox may be expressed as "there is no set of all sets" (except in some marginal axiomatic set theories).

Various schools of thought opposed each other. The leading school was that of the formalists[154], of which David Hilbert[155] was the foremost proponent, culminating in what is known as Hilbert's program[156], intending to ground mathematics on a small basis of a logical system proved sound by metamathematical[157] finitistic[158] means. The main opponent of the formalist school was the intuitionist[159] school, led by L. E. J. Brouwer[160], which resolutely discarded formalism as a meaningless game with symbols.[7] The fight was acrimonious. In 1920 Hilbert succeeded in having Brouwer, whom he considered a threat to mathematics, removed from the editorial board of *Mathematische Annalen*[161], the leading mathematical journal of the time.

1.2.1 Philosophical views

Main article: Philosophy of mathematics[162] At the beginning of the 20th century, three schools of philosophy of mathematics opposed each other: Formalism, Intuitionism and

147 https://en.wikipedia.org/wiki/German_language
148 https://en.wikipedia.org/wiki/Philosophy_of_mathematics
149 https://en.wikipedia.org/wiki/Consistent
150 https://en.wikipedia.org/wiki/Paradox
151 https://en.wikipedia.org/wiki/Russell%27s_paradox
152 https://en.wikipedia.org/wiki/Contradiction
153 https://en.wikipedia.org/wiki/Inconsistent
154 https://en.wikipedia.org/wiki/Formalism_(mathematics)
155 https://en.wikipedia.org/wiki/David_Hilbert
156 https://en.wikipedia.org/wiki/Hilbert%27s_program
157 https://en.wikipedia.org/wiki/Metamathematics
158 https://en.wikipedia.org/wiki/Finitism
159 https://en.wikipedia.org/wiki/Intuitionism
160 https://en.wikipedia.org/wiki/L._E._J._Brouwer
161 https://en.wikipedia.org/wiki/Mathematische_Annalen
162 https://en.wikipedia.org/wiki/Philosophy_of_mathematics

Logicism. The Second Conference on the Epistemology of the Exact Sciences [163] held in Königsberg[164] in 1930 gave space to these three schools.

Formalism

Main article: Formalism (mathematics)[165] It has been claimed that formalists, such as David Hilbert[166] (1862−1943), hold that mathematics is only a language and a series of games. Indeed, he used the words "formula game" in his 1927 response to L. E. J. Brouwer[167]'s criticisms:

> And to what extent has the formula game thus made possible been successful? This formula game enables us to express the entire thought-content of the science of mathematics in a uniform manner and develop it in such a way that, at the same time, the interconnections between the individual propositions and facts become clear ... The formula game that Brouwer so deprecates has, besides its mathematical value, an important general philosophical significance. For this formula game is carried out according to certain definite rules, in which the *technique of our thinking* is expressed. These rules form a closed system that can be discovered and definitively stated.[8]

Thus Hilbert is insisting that mathematics is not an *arbitrary* game with *arbitrary* rules; rather it must agree with how our thinking, and then our speaking and writing, proceeds.[8]

> We are not speaking here of arbitrariness in any sense. Mathematics is not like a game whose tasks are determined by arbitrarily stipulated rules. Rather, it is a conceptual system possessing internal necessity that can only be so and by no means otherwise.[9]

The foundational philosophy of formalism, as exemplified by David Hilbert[168], is a response to the paradoxes of set theory[169], and is based on formal logic[170]. Virtually all mathematical theorems[171] today can be formulated as theorems of set theory. The truth of a mathematical statement, in this view, is represented by the fact that the statement can be derived from the axioms of set theory[172] using the rules of formal logic.

Merely the use of formalism alone does not explain several issues: why we should use the axioms we do and not some others, why we should employ the logical rules we do and not some others, why do "true" mathematical statements (e.g., the laws of arithmetic[173]) appear to be true, and so on. Hermann Weyl[174] would ask these very questions of Hilbert:

> What "truth" or objectivity can be ascribed to this theoretic construction of the world, which presses far beyond the given, is a profound philosophical problem. It is closely

163 https://en.wikipedia.org/wiki/Second_Conference_on_the_Epistemology_of_the_Exact_Sciences
164 https://en.wikipedia.org/wiki/K%C3%B6nigsberg
165 https://en.wikipedia.org/wiki/Formalism_(mathematics)
166 https://en.wikipedia.org/wiki/David_Hilbert
167 https://en.wikipedia.org/wiki/L._E._J._Brouwer
168 https://en.wikipedia.org/wiki/David_Hilbert
169 https://en.wikipedia.org/wiki/Set_theory
170 https://en.wikipedia.org/wiki/Formal_logic
171 https://en.wikipedia.org/wiki/Theorem
172 https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory
173 https://en.wikipedia.org/wiki/Peano_axioms
174 https://en.wikipedia.org/wiki/Hermann_Weyl

> connected with the further question: what impels us to take as a basis precisely the particular axiom system developed by Hilbert? Consistency is indeed a necessary but not a sufficient condition. For the time being we probably cannot answer this question ...[10]

In some cases these questions may be sufficiently answered through the study of formal theories, in disciplines such as reverse mathematics[175] and computational complexity theory[176]. As noted by Weyl, formal logical systems[177] also run the risk of inconsistency[178]; in Peano arithmetic[179], this arguably has already been settled with several proofs of consistency[180], but there is debate over whether or not they are sufficiently finitary[181] to be meaningful. Gödel's second incompleteness theorem[182] establishes that logical systems of arithmetic can never contain a valid proof of their own consistency[183]. What Hilbert wanted to do was prove a logical system S was consistent, based on principles P that only made up a small part of S. But Gödel proved that the principles P could not even prove P to be consistent, let alone S.

Intuitionism

Main articles: Intuitionism[184] and Constructivism (mathematics)[185] Intuitionists, such as L. E. J. Brouwer[186] (1882–1966), hold that mathematics is a creation of the human mind. Numbers, like fairy tale characters, are merely mental entities, which would not exist if there were never any human minds to think about them.

The foundational philosophy of *intuitionism*[187] or *constructivism*[188], as exemplified in the extreme by Brouwer[189] and Stephen Kleene[190], requires proofs to be "constructive" in nature – the existence of an object must be demonstrated rather than inferred from a demonstration of the impossibility of its non-existence. For example, as a consequence of this the form of proof known as reductio ad absurdum[191] is suspect.

Some modern theories[192] in the philosophy of mathematics deny the existence of foundations in the original sense. Some theories tend to focus on mathematical practice[193], and aim to

175 https://en.wikipedia.org/wiki/Reverse_mathematics
176 https://en.wikipedia.org/wiki/Computational_complexity_theory
177 https://en.wikipedia.org/wiki/Formal_logical_system
178 https://en.wikipedia.org/wiki/Consistency_proof
179 https://en.wikipedia.org/wiki/Peano_axioms
180 https://en.wikipedia.org/wiki/Consistency_proof
181 https://en.wikipedia.org/wiki/Finitism
182 https://en.wikipedia.org/wiki/G%C3%B6del%27s_incompleteness_theorem
183 https://en.wikipedia.org/wiki/Consistency_proof
184 https://en.wikipedia.org/wiki/Intuitionism
185 https://en.wikipedia.org/wiki/Constructivism_(mathematics)
186 https://en.wikipedia.org/wiki/L._E._J._Brouwer
187 https://en.wikipedia.org/wiki/Intuitionism
188 https://en.wikipedia.org/wiki/Constructivism_(mathematics)
189 https://en.wikipedia.org/wiki/Luitzen_Egbertus_Jan_Brouwer
190 https://en.wikipedia.org/wiki/Stephen_Kleene
191 https://en.wikipedia.org/wiki/Reductio_ad_absurdum
192 https://en.wikipedia.org/wiki/Theory
193 https://en.wikipedia.org/wiki/Mathematical_practice

describe and analyze the actual working of mathematicians as a social group[194]. Others try to create a cognitive science of mathematics[195], focusing on human cognition as the origin of the reliability of mathematics when applied to the real world. These theories would propose to find foundations only in human thought, not in any objective outside construct. The matter remains controversial.

Logicism

Main article: Logicism[196] Logicism[197] is a school of thought, and research programme, in the philosophy of mathematics, based on the thesis that mathematics is an extension of a logic or that some or all mathematics may be derived in a suitable formal system whose axioms and rules of inference are 'logical' in nature. Bertrand Russell[198] and Alfred North Whitehead[199] championed this theory initiated by Gottlob Frege[200] and influenced by Richard Dedekind[201].

Set-theoretic Platonism

Main article: Set-theoretic Platonism[202] Many researchers in axiomatic set theory[203] have subscribed to what is known as set-theoretic Platonism[204], exemplified by Kurt Gödel[205].

Several set theorists followed this approach and actively searched for axioms that may be considered as true for heuristic reasons and that would decide the continuum hypothesis[206]. Many large cardinal[207] axioms were studied, but the hypothesis always remained independent[208] from them and it is now considered unlikely that CH can be resolved by a new large cardinal axiom. Other types of axioms were considered, but none of them has reached consensus on the continuum hypothesis yet. Recent work by Hamkins[209] proposes a more flexible alternative: a set-theoretic multiverse[210] allowing free passage between set-theoretic universes that satisfy the continuum hypothesis and other universes that do not.

194 https://en.wikipedia.org/wiki/Social_group
195 https://en.wikipedia.org/wiki/Cognitive_science_of_mathematics
196 https://en.wikipedia.org/wiki/Logicism
197 https://en.wikipedia.org/wiki/Logicism
198 https://en.wikipedia.org/wiki/Bertrand_Russell
199 https://en.wikipedia.org/wiki/Alfred_North_Whitehead
200 https://en.wikipedia.org/wiki/Gottlob_Frege
201 https://en.wikipedia.org/wiki/Richard_Dedekind
202 https://en.wikipedia.org/wiki/Set-theoretic_Platonism
203 https://en.wikipedia.org/wiki/Axiomatic_set_theory
204 https://en.wikipedia.org/wiki/Platonism#Modern_Platonism
205 https://en.wikipedia.org/wiki/Kurt_G%C3%B6del
206 https://en.wikipedia.org/wiki/Continuum_hypothesis
207 https://en.wikipedia.org/wiki/Large_cardinal
208 https://en.wikipedia.org/wiki/Independence_(mathematical_logic)
209 https://en.wikipedia.org/wiki/Joel_David_Hamkins
210 https://en.wikipedia.org/wiki/Multiverse

Indispensability argument for realism

Main article: Quine–Putnam indispensability argument[211] This argument[212] by Willard Quine[213] and Hilary Putnam[214] says (in Putnam's shorter words),

> ... quantification over mathematical entities is indispensable for science ... therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question.

However, Putnam was not a Platonist.

Rough-and-ready realism

Few mathematicians are typically concerned on a daily, working basis over logicism, formalism or any other philosophical position. Instead, their primary concern is that the mathematical enterprise as a whole always remains productive. Typically, they see this as ensured by remaining open-minded, practical and busy; as potentially threatened by becoming overly-ideological, fanatically reductionistic or lazy.

Such a view has also been expressed by some well-known physicists.

For example, the Physics Nobel Prize laureate Richard Feynman[215] said

> People say to me, "Are you looking for the ultimate laws of physics?" No, I'm not ... If it turns out there is a simple ultimate law which explains everything, so be it – that would be very nice to discover. If it turns out it's like an onion with millions of layers ... then that's the way it is. But either way there's Nature and she's going to come out the way She is. So therefore when we go to investigate we shouldn't predecide what it is we're looking for only to find out more about it.[11]

And Steven Weinberg[216]:[12]

> The insights of philosophers have occasionally benefited physicists, but generally in a negative fashion – by protecting them from the preconceptions of other philosophers. ... without some guidance from our preconceptions one could do nothing at all. It is just that philosophical principles have not generally provided us with the right preconceptions.

Weinberg believed that any undecidability in mathematics, such as the continuum hypothesis, could be potentially resolved despite the incompleteness theorem, by finding suitable further axioms to add to set theory.

211 https://en.wikipedia.org/wiki/Quine%E2%80%93Putnam_indispensability_argument
212 https://en.wikipedia.org/wiki/Quine%E2%80%93Putnam_indispensability_thesis
213 https://en.wikipedia.org/wiki/Willard_Quine
214 https://en.wikipedia.org/wiki/Hilary_Putnam
215 https://en.wikipedia.org/wiki/Richard_Feynman
216 https://en.wikipedia.org/wiki/Steven_Weinberg

Philosophical consequences of Gödel's completeness theorem

Main article: Gödel's completeness theorem[217] Gödel's completeness theorem establishes an equivalence in first-order logic between the formal provability of a formula and its truth in all possible models. Precisely, for any consistent first-order theory it gives an ”explicit construction” of a model described by the theory; this model will be countable if the language of the theory is countable. However this ”explicit construction” is not algorithmic. It is based on an iterative process of completion of the theory, where each step of the iteration consists in adding a formula to the axioms if it keeps the theory consistent; but this consistency question is only semi-decidable (an algorithm is available to find any contradiction but if there is none this consistency fact can remain unprovable).

This can be seen as giving a sort of justification to the Platonist view that the objects of our mathematical theories are real. More precisely, it shows that the mere assumption of the existence of the set of natural numbers as a totality (an actual infinity) suffices to imply the existence of a model (a world of objects) of any consistent theory. However several difficulties remain:

- For any consistent theory this usually does not give just one world of objects, but an infinity of possible worlds that the theory might equally describe, with a possible diversity of truths between them.
- In the case of set theory, none of the models obtained by this construction resemble the intended model, as they are countable while set theory intends to describe uncountable infinities. Similar remarks can be made in many other cases. For example, with theories that include arithmetic, such constructions generally give models that include non-standard numbers, unless the construction method was specifically designed to avoid them.
- As it gives models to all consistent theories without distinction, it gives no reason to accept or reject any axiom as long as the theory remains consistent, but regards all consistent axiomatic theories as referring to equally existing worlds. It gives no indication on which axiomatic system should be preferred as a foundation of mathematics.
- As claims of consistency are usually unprovable, they remain a matter of belief or non-rigorous kinds of justifications. Hence the existence of models as given by the completeness theorem needs in fact two philosophical assumptions: the actual infinity of natural numbers and the consistency of the theory.

Another consequence of the completeness theorem is that it justifies the conception of infinitesimals as actual infinitely small nonzero quantities, based on the existence of non-standard models as equally legitimate to standard ones. This idea was formalized by Abraham Robinson[218] into the theory of nonstandard analysis[219].

217 https://en.wikipedia.org/wiki/G%C3%B6del%27s_completeness_theorem
218 https://en.wikipedia.org/wiki/Abraham_Robinson
219 https://en.wikipedia.org/wiki/Nonstandard_analysis

1.2.2 More paradoxes

See also: List of statements independent of ZFC[220] and List of paradoxes[221] The following lists some notable results in metamathematics. Zermelo–Fraenkel set theory[222] is the most widely studied axiomatization of set theory. It is abbreviated **ZFC** when it includes the axiom of choice[223] and **ZF** when the axiom of choice is excluded.

- 1920: Thoralf Skolem[224] corrected Leopold Löwenheim[225]'s proof of what is now called the downward Löwenheim–Skolem theorem[226], leading to Skolem's paradox[227] discussed in 1922, namely the existence of countable models of ZF, making infinite cardinalities a relative property.
- 1922: Proof by Abraham Fraenkel[228] that the axiom of choice cannot be proved from the axioms of Zermelo set theory[229] with urelements[230].
- 1931: Publication of Gödel's incompleteness theorems[231], showing that essential aspects of Hilbert's program could not be attained. It showed how to construct, for any sufficiently powerful and consistent recursively axiomatizable system – such as necessary to axiomatize the elementary theory of arithmetic[232] on the (infinite) set of natural numbers – a statement that formally expresses its own unprovability, which he then proved equivalent to the claim of consistency of the theory; so that (assuming the consistency as true), the system is not powerful enough for proving its own consistency, let alone that a simpler system could do the job. It thus became clear that the notion of mathematical truth can not be completely determined and reduced to a purely formal system[233] as envisaged in Hilbert's program. This dealt a final blow to the heart of Hilbert's program, the hope that consistency could be established by finitistic means (it was never made clear exactly what axioms were the "finitistic" ones, but whatever axiomatic system was being referred to, it was a 'weaker' system than the system whose consistency it was supposed to prove).
- 1936: Alfred Tarski[234] proved his truth undefinability theorem[235].
- 1936: Alan Turing[236] proved that a general algorithm to solve the halting problem[237] for all possible program-input pairs cannot exist.
- 1938: Gödel proved the consistency of the axiom of choice and of the generalized continuum hypothesis[238].

220 https://en.wikipedia.org/wiki/List_of_statements_independent_of_ZFC
221 https://en.wikipedia.org/wiki/List_of_paradoxes
222 https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory
223 https://en.wikipedia.org/wiki/Axiom_of_choice
224 https://en.wikipedia.org/wiki/Thoralf_Skolem
225 https://en.wikipedia.org/wiki/Leopold_L%C3%B6wenheim
226 https://en.wikipedia.org/wiki/Downward_L%C3%B6wenheim%E2%80%93Skolem_theorem
227 https://en.wikipedia.org/wiki/Skolem%27s_paradox
228 https://en.wikipedia.org/wiki/Abraham_Fraenkel
229 https://en.wikipedia.org/wiki/Zermelo_set_theory
230 https://en.wikipedia.org/wiki/Urelement
231 https://en.wikipedia.org/wiki/G%C3%B6del%27s_incompleteness_theorems
232 https://en.wikipedia.org/wiki/Arithmetic
233 https://en.wikipedia.org/wiki/Formal_system
234 https://en.wikipedia.org/wiki/Alfred_Tarski
235 https://en.wikipedia.org/wiki/Tarski%27s_undefinability_theorem
236 https://en.wikipedia.org/wiki/Alan_Turing
237 https://en.wikipedia.org/wiki/Halting_problem
238 https://en.wikipedia.org/wiki/Constructible_universe

- 1936–1937: Alonzo Church[239] and Alan Turing[240], respectively, published independent papers showing that a general solution to the Entscheidungsproblem[241] is impossible: the universal validity of statements in first-order logic is not decidable (it is only semi-decidable[242] as given by the completeness theorem[243]).
- 1955: Pyotr Novikov[244] showed that there exists a finitely presented group G such that the word problem for G is undecidable.
- 1963: Paul Cohen[245] showed that the Continuum Hypothesis is unprovable from ZFC[246]. Cohen's proof developed the method of forcing[247], which is now an important tool for establishing independence[248] results in set theory.
- 1964: Inspired by the fundamental randomness in physics, Gregory Chaitin[249] starts publishing results on algorithmic information theory[250] (measuring incompleteness and randomness in mathematics).[13]
- 1966: Paul Cohen showed that the axiom of choice is unprovable in ZF even without urelements[251].
- 1970: Hilbert's tenth problem[252] is proven unsolvable: there is no recursive solution to decide whether a Diophantine equation[253] (multivariable polynomial equation) has a solution in integers.
- 1971: Suslin's problem[254] is proven to be independent from ZFC.

1.3 Toward resolution of the crisis

Starting in 1935, the Bourbaki[255] group of French mathematicians started publishing a series of books to formalize many areas of mathematics on the new foundation of set theory.

The intuitionistic school did not attract many adherents, and it was not until Bishop[256]'s work in 1967 that constructive mathematics[257] was placed on a sounder footing.[14]

One may consider that Hilbert's program has been partially completed[258], so that the crisis is essentially resolved, satisfying ourselves with lower requirements than Hilbert's original

239 https://en.wikipedia.org/wiki/Alonzo_Church
240 https://en.wikipedia.org/wiki/Alan_Turing
241 https://en.wikipedia.org/wiki/Entscheidungsproblem
242 https://en.wikipedia.org/wiki/Semi-decidable
243 https://en.wikipedia.org/wiki/Completeness_theorem
244 https://en.wikipedia.org/wiki/Pyotr_Novikov
245 https://en.wikipedia.org/wiki/Paul_Cohen_(mathematician)
246 https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory
247 https://en.wikipedia.org/wiki/Forcing_(mathematics)
248 https://en.wikipedia.org/wiki/Independence_(mathematical_logic)
249 https://en.wikipedia.org/wiki/Gregory_Chaitin
250 https://en.wikipedia.org/wiki/Algorithmic_information_theory
251 https://en.wikipedia.org/wiki/Urelements
252 https://en.wikipedia.org/wiki/Hilbert%27s_tenth_problem
253 https://en.wikipedia.org/wiki/Diophantine_equation
254 https://en.wikipedia.org/wiki/Suslin%27s_problem
255 https://en.wikipedia.org/wiki/Nicolas_Bourbaki
256 https://en.wikipedia.org/wiki/Errett_Bishop
257 https://en.wikipedia.org/wiki/Constructivism_(mathematics)
258 https://en.wikipedia.org/wiki/Hilbert%27s_program#Hilbert's_program_after_G%C3%B6del

ambitions. His ambitions were expressed in a time when nothing was clear: it was not clear whether mathematics could have a rigorous foundation at all.

There are many possible variants of set theory, which differ in consistency strength, where stronger versions (postulating higher types of infinities) contain formal proofs of the consistency of weaker versions, but none contains a formal proof of its own consistency. Thus the only thing we don't have is a formal proof of consistency of whatever version of set theory we may prefer, such as ZF.

In practice, most mathematicians either do not work from axiomatic systems, or if they do, do not doubt the consistency of ZFC[259], generally their preferred axiomatic system. In most of mathematics as it is practiced, the incompleteness and paradoxes of the underlying formal theories never played a role anyway, and in those branches in which they do or whose formalization attempts would run the risk of forming inconsistent theories (such as logic and category theory), they may be treated carefully.

The development of category theory[260] in the middle of the 20th century showed the usefulness of set theories guaranteeing the existence of larger classes than does ZFC, such as Von Neumann–Bernays–Gödel set theory[261] or Tarski–Grothendieck set theory[262], albeit that in very many cases the use of large cardinal axioms or Grothendieck universes is formally eliminable.

One goal of the reverse mathematics[263] program is to identify whether there are areas of "core mathematics" in which foundational issues may again provoke a crisis.

1.4 See also

- Mathematics portal[264]
- Philosophy portal[265]

- Aristotelian realist philosophy of mathematics[266]
- Mathematical logic[267]
- Brouwer–Hilbert controversy[268]
- Church–Turing thesis[269]
- Controversy over Cantor's theory[270]
- Epistemology[271]

259 https://en.wikipedia.org/wiki/ZFC
260 https://en.wikipedia.org/wiki/Category_theory
261 https://en.wikipedia.org/wiki/Von_Neumann%E2%80%93Bernays%E2%80%93G%C3%B6del_set_theory
262 https://en.wikipedia.org/wiki/Tarski%E2%80%93Grothendieck_set_theory
263 https://en.wikipedia.org/wiki/Reverse_Mathematics
264 https://en.wikipedia.org/wiki/Portal:Mathematics
265 https://en.wikipedia.org/wiki/Portal:Philosophy
266 https://en.wikipedia.org/wiki/Aristotelian_realist_philosophy_of_mathematics
267 https://en.wikipedia.org/wiki/Mathematical_logic
268 https://en.wikipedia.org/wiki/Brouwer%E2%80%93Hilbert_controversy
269 https://en.wikipedia.org/wiki/Church%E2%80%93Turing_thesis
270 https://en.wikipedia.org/wiki/Controversy_over_Cantor%27s_theory
271 https://en.wikipedia.org/wiki/Epistemology

- *Euclid's Elements*[272]
- Hilbert's problems[273]
- Implementation of mathematics in set theory[274]
- Liar paradox[275]
- New Foundations[276]
- Philosophy of mathematics[277]
- *Principia Mathematica*[278]
- Quasi-empiricism in mathematics[279]
- Mathematical thought of Charles Peirce[280]

1.5 Notes

1. Joachim Lambek (2007), ”Foundations of mathematics”, *Encyc. Britannica*[281]
2. Leon Horsten (2007, rev. 2012), ”Philosophy of Mathematics” *SEP*[282]
3. *The thirteen books of Euclid's Elements, edited by Sir Thomas Heath*[283]. Vol. 2 (Book V). Translated by Heiberg. New York: Dover Publications[284]. 1956. pp. 124–126. ISBN[285] 0-486-60089-0[286].
4. Karlis Podnieks, Platonism, intuition and the nature of mathematics: 1. Platonism - the Philosophy of Working Mathematicians[287]
5. *The Analyst*[288]*, A Discourse Addressed to an Infidel Mathematician*
6. Laptev, B.L. & B.A. Rozenfel'd (1996) *Mathematics of the 19th Century: Geometry*, page 40, Birkhäuser[289] ISBN[290] 3-7643-5048-2[291]
7. van Dalen D. (2008), ”Brouwer, Luitzen Egbertus Jan (1881–1966)”, in Biografisch Woordenboek van Nederland. URL:[292] [2008-03-13]
8. Hilbert 1927 *The Foundations of Mathematics* in van Heijenoort 1967:475
9. p. 14 in Hilbert, D. (1919–20), Natur und Mathematisches Erkennen: Vorlesungen, gehalten 1919–1920 in Göttingen. Nach der Ausarbeitung von Paul Bernays (Edited and with an English introduction by David E. Rowe), Basel, Birkhauser (1992).

272 https://en.wikipedia.org/wiki/Euclid%27s_Elements
273 https://en.wikipedia.org/wiki/Hilbert%27s_problems
274 https://en.wikipedia.org/wiki/Implementation_of_mathematics_in_set_theory
275 https://en.wikipedia.org/wiki/Liar_paradox
276 https://en.wikipedia.org/wiki/New_Foundations
277 https://en.wikipedia.org/wiki/Philosophy_of_mathematics
278 https://en.wikipedia.org/wiki/Principia_Mathematica
279 https://en.wikipedia.org/wiki/Quasi-empiricism_in_mathematics
280 https://en.wikipedia.org/wiki/Charles_Sanders_Peirce#Mathematics
281 http://www.britannica.com/EBchecked/topic/369221/foundations-of-mathematics
282 http://plato.stanford.edu/entries/philosophy-mathematics/#MatLogFouMat
283 https://archive.org/details/thirteenbookseu00heibgoog/page/n136/mode/2up
284 https://en.wikipedia.org/wiki/Dover_Publications
285 https://en.wikipedia.org/wiki/ISBN_(identifier)
286 https://en.wikipedia.org/wiki/Special:BookSources/0-486-60089-0
287 http://www.ltn.lv/~podnieks/gt1.html#BM1_1
288 https://en.wikipedia.org/wiki/The_Analyst
289 https://en.wikipedia.org/wiki/Springer_Science%2BBusiness_Media
290 https://en.wikipedia.org/wiki/ISBN_(identifier)
291 https://en.wikipedia.org/wiki/Special:BookSources/3-7643-5048-2
292 http://www.inghist.nl/Onderzoek/Projecten/BWN/lemmata/bwn2/brouwerle

10. Weyl 1927 *Comments on Hilbert's second lecture on the foundations of mathematics* in van Heijenoort 1967:484. Although Weyl the intuitionist believed that "Hilbert's view" would ultimately prevail, this would come with a significant loss to philosophy: "*I see in this a decisive defeat of the philosophical attitude of pure phenomenology* , which thus proves to be insufficient for the understanding of creative science even in the area of cognition that is most primal and most readily open to evidence – mathematics" (ibid).
11. Richard Feynman, *The Pleasure of Finding Things Out* p. 23
12. Steven Weinberg, chapter *Against Philosophy*[293] wrote, in *Dreams of a final theory*
13. Chaitin, Gregory[294] (2006), "The Limits Of Reason"[295] (PDF), *Scientific American*, **294** (3): 74–81, Bibcode[296]:2006SciAm.294c..74C[297], doi[298]:10.1038/scientificamerican0306-74[299], PMID[300] 16502614[301], archived from the original[302] (PDF) on 2016-03-04, retrieved 2016-02-22
14. Andrej Bauer (2017), "Five stages of accepting constructive mathematics", *Bull. Amer. Math. Soc.*, **54** (3): 485, doi[303]:10.1090/bull/1556[304]

1.6 References

- Avigad, Jeremy[305] (2003) *Number theory and elementary arithmetic*[306], Philosophia Mathematica Vol. 11, pp. 257–284
- Eves, Howard[307] (1990), *Foundations and Fundamental Concepts of Mathematics Third Edition*, Dover Publications, INC, Mineola NY, ISBN[308] 0-486-69609-X[309] (pbk.) cf §9.5 Philosophies of Mathematics pp. 266–271. Eves lists the three with short descriptions prefaced by a brief introduction.
- Goodman, N.D.[310] (1979), "Mathematics as an Objective Science[311]", in Tymoczko (ed., 1986).
- Hart, W.D.[312] (ed., 1996), *The Philosophy of Mathematics*, Oxford University Press, Oxford, UK.

293 http://libcom.org/library/unexpected-uselessness-philosophy
294 https://en.wikipedia.org/wiki/Gregory_Chaitin
295 https://web.archive.org/web/20160304192140/https://www.cs.auckland.ac.nz/~chaitin/sciamer3.pdf
296 https://en.wikipedia.org/wiki/Bibcode_(identifier)
297 https://ui.adsabs.harvard.edu/abs/2006SciAm.294c..74C
298 https://en.wikipedia.org/wiki/Doi_(identifier)
299 https://doi.org/10.1038%2Fscientificamerican0306-74
300 https://en.wikipedia.org/wiki/PMID_(identifier)
301 http://pubmed.ncbi.nlm.nih.gov/16502614
302 https://www.cs.auckland.ac.nz/~chaitin/sciamer3.pdf
303 https://en.wikipedia.org/wiki/Doi_(identifier)
304 https://doi.org/10.1090%2Fbull%2F1556
305 https://en.wikipedia.org/wiki/Jeremy_Avigad
306 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.105.6509&rep=rep1&type=pdf
307 https://en.wikipedia.org/wiki/Howard_Eves
308 https://en.wikipedia.org/wiki/ISBN_(identifier)
309 https://en.wikipedia.org/wiki/Special:BookSources/0-486-69609-X
310 https://en.wikipedia.org/w/index.php?title=Nicholas_D._Goodman&action=edit&redlink=1
311 http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.461.6921&rep=rep1&type=pdf
312 https://en.wikipedia.org/wiki/Wilbur_Dyre_Hart

- Hersh, R.[313] (1979), "Some Proposals for Reviving the Philosophy of Mathematics", in (Tymoczko 1986).
- Hilbert, D.[314] (1922), "Neubegründung der Mathematik. Erste Mitteilung", *Hamburger Mathematische Seminarabhandlungen* 1, 157–177. Translated, "The New Grounding of Mathematics. First Report", in (Mancosu 1998).
- Katz, Robert (1964), *Axiomatic Analysis*, D. C. Heath and Company.
- Kleene, Stephen C.[315] (1991) [1952]. *Introduction to Meta-Mathematics* (Tenth impression 1991 ed.). Amsterdam NY: North-Holland Pub. Co. ISBN[316] 0-7204-2103-9[317].

In Chapter III *A Critique of Mathematic Reasoning, §11. The paradoxes*, Kleene discusses Intuitionism[318] and Formalism[319] in depth. Throughout the rest of the book he treats, and compares, both Formalist (classical) and Intuitionist logics with an emphasis on the former. Extraordinary writing by an extraordinary mathematician.

- Mancosu, P. (ed., 1998), *From Hilbert to Brouwer. The Debate on the Foundations of Mathematics in the 1920s*, Oxford University Press, Oxford, UK.
- Putnam, Hilary[320] (1967), "Mathematics Without Foundations", *Journal of Philosophy* 64/1, 5–22. Reprinted, pp. 168–184 in W.D. Hart (ed., 1996).
- —, "What is Mathematical Truth?", in Tymoczko (ed., 1986).
- Sudac, Olivier (Apr 2001). "The prime number theorem is PRA-provable"[321]. *Theoretical Computer Science.* **257** (1–2): 185–239. doi[322]:10.1016/S0304-3975(00)00116-X[323].
- Troelstra, A. S.[324] (no date but later than 1990), "A History of Constructivism in the 20th Century"[325], A detailed survey for specialists: §1 Introduction, §2 Finitism & §2.2 Actualism, §3 Predicativism and Semi-Intuitionism, §4 Brouwerian Intuitionism, §5 Intuitionistic Logic and Arithmetic, §6 Intuitionistic Analysis and Stronger Theories, §7 Constructive Recursive Mathematics, §8 Bishop's Constructivism, §9 Concluding Remarks. Approximately 80 references.
- Tymoczko, T.[326] (1986), "Challenging Foundations", in Tymoczko (ed., 1986).
- —,(ed., 1986), *New Directions in the Philosophy of Mathematics*[327], 1986. Revised edition, 1998.

313 https://en.wikipedia.org/wiki/Reuben_Hersh
314 https://en.wikipedia.org/wiki/David_Hilbert
315 https://en.wikipedia.org/wiki/Stephen_Kleene
316 https://en.wikipedia.org/wiki/ISBN_(identifier)
317 https://en.wikipedia.org/wiki/Special:BookSources/0-7204-2103-9
318 https://en.wikipedia.org/wiki/Intuitionism
319 https://en.wikipedia.org/wiki/Formalism_(mathematics)
320 https://en.wikipedia.org/wiki/Hilary_Putnam
321 https://doi.org/10.1016%2FS0304-3975%2800%2900116-X
322 https://en.wikipedia.org/wiki/Doi_(identifier)
323 https://doi.org/10.1016%2FS0304-3975%2800%2900116-X
324 https://en.wikipedia.org/wiki/Anne_Sjerp_Troelstra
325 https://web.archive.org/web/20060209210015/http://staff.science.uva.nl/~anne/hhhist.pdf
326 https://en.wikipedia.org/wiki/Thomas_Tymoczko
327 https://books.google.com/books?hl=en&lr=&id=a-M9DwAAQBAJ&oi=fnd&pg=PR9&dq=%22New+Directions+in+the+Philosophy+of+Mathematics%22&ots=atf_Ne8rLn&sig=Vf0ZioQ2KmW7UOOpZPiC4fNmEWA#v=onepage&q=%22New%20Directions%20in%20the%20Philosophy%20of%20Mathematics%22&f=false

- van Dalen D. (2008), "Brouwer, Luitzen Egbertus Jan (1881–1966)", in Biografisch Woordenboek van Nederland. URL:[328] [2008-03-13]
- Weyl, H.[329] (1921), "Über die neue Grundlagenkrise der Mathematik", *Mathematische Zeitschrift* 10, 39–79. Translated, "On the New Foundational Crisis of Mathematics", in (Mancosu 1998).
- Wilder, Raymond L.[330] (1952), *Introduction to the Foundations of Mathematics*, John Wiley and Sons, New York, NY.

1.7 External links

Wikiquote has quotations related to: ***Foundations of mathematics***[331]

- Media related to Foundations of mathematics[332] at Wikimedia Commons
- "Philosophy of mathematics"[333]. *Internet Encyclopedia of Philosophy*[334].
- Logic and Mathematics[335]
- Foundations of Mathematics: past, present, and future[336], May 31, 2000, 8 pages.
- A Century of Controversy over the Foundations of Mathematics[337] by Gregory Chaitin. arXiv[338]:chao-dyn/9909001[339]

Portals[340]: Mathematics[341] History of science[342] Philosophy[343] Contents/Mathematics and logic[344] Judaism[345]

Metalogic and metamathematics

328 http://www.inghist.nl/Onderzoek/Projecten/BWN/lemmata/bwn2/brouwerle
329 https://en.wikipedia.org/wiki/Hermann_Weyl
330 https://en.wikipedia.org/wiki/Raymond_Louis_Wilder
331 https://en.wikiquote.org/wiki/Special:Search/Foundations_of_mathematics
332 https://commons.wikimedia.org/wiki/Category:Foundations_of_mathematics
333 http://www.iep.utm.edu/category/s-l-m/math/
334 https://en.wikipedia.org/wiki/Internet_Encyclopedia_of_Philosophy
335 http://www.personal.psu.edu/t20/papers/philmath/
336 https://cpb-us-w2.wpmucdn.com/u.osu.edu/dist/1/1952/files/2014/01/fom12pt5.31.00-1jkl4df.pdf
337 https://web.archive.org/web/20130727184333/http://www.cs.auckland.ac.nz/~chaitin/lowell.html
338 https://en.wikipedia.org/wiki/ArXiv_(identifier)
339 https://arxiv.org/abs/chao-dyn/9909001
340 https://en.wikipedia.org/wiki/Wikipedia:Contents/Portals
341 https://en.wikipedia.org/wiki/Portal:Mathematics
342 https://en.wikipedia.org/wiki/Portal:History_of_science
343 https://en.wikipedia.org/wiki/Portal:Philosophy
344 https://en.wikipedia.org/wiki/Portal:Contents/Mathematics_and_logic
345 https://en.wikipedia.org/wiki/Portal:Judaism

Mathematical logic

Mathematics (areas of mathematics)

Major topics in Foundations of Mathematics

346 http://en.wikipedia.org/wiki/Wikipedia:Text_of_Creative_Commons_Attribution-ShareAlike_3.0_Unported_License
347 http://creativecommons.org/licenses/by-sa/3.0/
348 http://foundation.wikimedia.org/wiki/Terms_of_Use
349 http://foundation.wikimedia.org/wiki/Privacy_policy
350 http://www.wikimediafoundation.org/

2 Contributors

Edits	User
1	"alyosha"[1]
1	Adam Chlipala[2]
1	Addbot[3]
1	Ahecht[4]
1	AioftheStorm[5]
1	Ajithhillemane[6]
1	Alduin2000[7]
3	Aliotra[8]
1	Allforrous[9]
5	Ancheta Wis[10]
1	Annette Maon[11]
6	AnomieBOT[12]
2	Anonymous Dissident[13]
3	Ant1lg[14]
1	Apparition11[15]
1	Arjayay[16]
1	Aubreybardo[17]
1	Augustus Leonhardus Cartesius[18]
1	Aurora2698[19]
3	AxelBoldt[20]

1 https://en.wikipedia.org/wiki/User:%2522alyosha%2522
2 https://en.wikipedia.org/wiki/User:Adam_Chlipala
3 https://en.wikipedia.org/wiki/User:Addbot
4 https://en.wikipedia.org/wiki/User:Ahecht
5 https://en.wikipedia.org/wiki/User:AioftheStorm
6 https://en.wikipedia.org/w/index.php%3ftitle=User:Ajithhillemane&action=edit&redlink=1
7 https://en.wikipedia.org/wiki/User:Alduin2000
8 https://en.wikipedia.org/wiki/User:Aliotra
9 https://en.wikipedia.org/wiki/User:Allforrous
10 https://en.wikipedia.org/wiki/User:Ancheta_Wis
11 https://en.wikipedia.org/wiki/User:Annette_Maon
12 https://en.wikipedia.org/wiki/User:AnomieBOT
13 https://en.wikipedia.org/wiki/User:Anonymous_Dissident
14 https://en.wikipedia.org/w/index.php%3ftitle=User:Ant1lg&action=edit&redlink=1
15 https://en.wikipedia.org/wiki/User:Apparition11
16 https://en.wikipedia.org/wiki/User:Arjayay
17 https://en.wikipedia.org/w/index.php%3ftitle=User:Aubreybardo&action=edit&redlink=1
18 https://en.wikipedia.org/wiki/User:Augustus_Leonhardus_Cartesius
19 https://en.wikipedia.org/wiki/User:Aurora2698
20 https://en.wikipedia.org/wiki/User:AxelBoldt

1 BD2412[21]
2 BG19bot[22]
1 BHGbot[23]
31 Baking Soda[24]
1 BattyBot[25]
2 Beland[26]
4 Bender235[27]
1 BethNaught[28]
1 Bfinn[29]
1 BillyPreset[30]
1 Bjankuloski06en~enwiki[31]
1 Blaisorblade[32]
1 Blue Mist 1[33]
1 Borislav[34]
6 BrownHairedGirl[35]
1 C17GMaster[36]
1 CRGreathouse[37]
1 Caseyfloyd6[38]
1 Charles Matthews[39]
1 Chicdat[40]
1 Chobot[41]
5 Citation bot[42]
3 ClueBot NG[43]
1 Comp.arch[44]
1 Curpsbot-unicodify[45]

21 https://en.wikipedia.org/wiki/User:BD2412
22 https://en.wikipedia.org/wiki/User:BG19bot
23 https://en.wikipedia.org/wiki/User:BHGbot
24 https://en.wikipedia.org/wiki/User:Baking_Soda
25 https://en.wikipedia.org/wiki/User:BattyBot
26 https://en.wikipedia.org/wiki/User:Beland
27 https://en.wikipedia.org/wiki/User:Bender235
28 https://en.wikipedia.org/wiki/User:BethNaught
29 https://en.wikipedia.org/wiki/User:Bfinn
30 https://en.wikipedia.org/w/index.php%3ftitle=User:BillyPreset&action=edit&redlink=1
31 https://en.wikipedia.org/wiki/User:Bjankuloski06en~enwiki
32 https://en.wikipedia.org/wiki/User:Blaisorblade
33 https://en.wikipedia.org/wiki/User:Blue_Mist_1
34 https://en.wikipedia.org/wiki/User:Borislav
35 https://en.wikipedia.org/wiki/User:BrownHairedGirl
36 https://en.wikipedia.org/wiki/User:C17GMaster
37 https://en.wikipedia.org/wiki/User:CRGreathouse
38 https://en.wikipedia.org/w/index.php%3ftitle=User:Caseyfloyd6&action=edit&redlink=1
39 https://en.wikipedia.org/wiki/User:Charles_Matthews
40 https://en.wikipedia.org/wiki/User:Chicdat
41 https://en.wikipedia.org/wiki/User:Chobot
42 https://en.wikipedia.org/wiki/User:Citation_bot
43 https://en.wikipedia.org/wiki/User:ClueBot_NG
44 https://en.wikipedia.org/wiki/User:Comp.arch
45 https://en.wikipedia.org/wiki/User:Curpsbot-unicodify

1 Cyan[46]
3 D.Lazard[47]
1 DYLAN LENNON~enwiki[48]
1 DannyS712[49]
3 DaveApter[50]
1 Deacon Vorbis[51]
7 Degsy88[52]
1 Denisarona[53]
3 Down time[54]
1 Drakide[55]
1 Dukkani[56]
2 ELApro[57]
1 Eelworm[58]
2 Elwikipedista~enwiki[59]
3 EmausBot[60]
2 Epicgenius[61]
1 Ernest lk lam[62]
1 Everyking[63]
1 Faizan[64]
3 Firebirth[65]
1 FlaBot[66]
1 Frietjes[67]
1 Fuckboi666[68]
1 GTBacchus[69]
1 Galileo9[70]

46 https://en.wikipedia.org/wiki/User:Cyan
47 https://en.wikipedia.org/wiki/User:D.Lazard
48 https://en.wikipedia.org/wiki/User:DYLAN_LENNON~enwiki
49 https://en.wikipedia.org/wiki/User:DannyS712
50 https://en.wikipedia.org/wiki/User:DaveApter
51 https://en.wikipedia.org/wiki/User:Deacon_Vorbis
52 https://en.wikipedia.org/wiki/User:Degsy88
53 https://en.wikipedia.org/wiki/User:Denisarona
54 https://en.wikipedia.org/wiki/User:Down_time
55 https://en.wikipedia.org/w/index.php%3ftitle=User:Drakide&action=edit&redlink=1
56 https://en.wikipedia.org/w/index.php%3ftitle=User:Dukkani&action=edit&redlink=1
57 https://en.wikipedia.org/wiki/User:ELApro
58 https://en.wikipedia.org/wiki/User:Eelworm
59 https://en.wikipedia.org/wiki/User:Elwikipedista~enwiki
60 https://en.wikipedia.org/wiki/User:EmausBot
61 https://en.wikipedia.org/wiki/User:Epicgenius
62 https://en.wikipedia.org/wiki/User:Ernest_lk_lam
63 https://en.wikipedia.org/wiki/User:Everyking
64 https://en.wikipedia.org/wiki/User:Faizan
65 https://en.wikipedia.org/wiki/User:Firebirth
66 https://en.wikipedia.org/wiki/User:FlaBot
67 https://en.wikipedia.org/wiki/User:Frietjes
68 https://en.wikipedia.org/w/index.php%3ftitle=User:Fuckboi666&action=edit&redlink=1
69 https://en.wikipedia.org/wiki/User:GTBacchus
70 https://en.wikipedia.org/w/index.php%3ftitle=User:Galileo9&action=edit&redlink=1

1 Gene.arboit[71]
3 Giftlite[72]
1 Gonzalcg[73]
1 GreenC bot[74]
1 Greenrd[75]
3 Gregbard[76]
5 Hairy Dude[77]
1 Helpful Pixie Bot[78]
1 Hyacinth[79]
1 I dream of horses[80]
2 I.hidekazu[81]
2 Iain.dalton[82]
1 IamFuturePr00f[83]
1 Incnis Mrsi[84]
3 InternetArchiveBot[85]
1 Intrnlerr[86]
1 Inwind[87]
1 Iosef[88]
1 JAnDbot[89]
2 JJL[90]
3 JP.Martin-Flatin[91]
6 JRSpriggs[92]
1 James oluwafumilayo[93]
1 Jandalhandler[94]
3 Jarble[95]

71 https://en.wikipedia.org/wiki/User:Gene.arboit
72 https://en.wikipedia.org/wiki/User:Giftlite
73 https://en.wikipedia.org/wiki/User:Gonzalcg
74 https://en.wikipedia.org/wiki/User:GreenC_bot
75 https://en.wikipedia.org/wiki/User:Greenrd
76 https://en.wikipedia.org/wiki/User:Gregbard
77 https://en.wikipedia.org/wiki/User:Hairy_Dude
78 https://en.wikipedia.org/wiki/User:Helpful_Pixie_Bot
79 https://en.wikipedia.org/wiki/User:Hyacinth
80 https://en.wikipedia.org/wiki/User:I_dream_of_horses
81 https://en.wikipedia.org/wiki/User:I.hidekazu
82 https://en.wikipedia.org/w/index.php%3ftitle=User:Iain.dalton&action=edit&redlink=1
83 https://en.wikipedia.org/w/index.php%3ftitle=User:IamFuturePr00f&action=edit&redlink=1
84 https://en.wikipedia.org/wiki/User:Incnis_Mrsi
85 https://en.wikipedia.org/wiki/User:InternetArchiveBot
86 https://en.wikipedia.org/wiki/User:Intrnlerr
87 https://en.wikipedia.org/wiki/User:Inwind
88 https://en.wikipedia.org/wiki/User:Iosef
89 https://en.wikipedia.org/wiki/User:JAnDbot
90 https://en.wikipedia.org/wiki/User:JJL
91 https://en.wikipedia.org/wiki/User:JP.Martin-Flatin
92 https://en.wikipedia.org/wiki/User:JRSpriggs
93 https://en.wikipedia.org/w/index.php%3ftitle=User:James_oluwafumilayo&action=edit&redlink=1
94 https://en.wikipedia.org/wiki/User:Jandalhandler
95 https://en.wikipedia.org/wiki/User:Jarble

1 Jaredscribe[96]
1 John of Reading[97]
9 Jon Awbrey[98]
2 Julesd[99]
1 JustAGal[100]
1 KamielC~enwiki[101]
1 KamikazeBot[102]
1 Karl-Henner[103]
1 KasparBot[104]
3 Katzmik[105]
1 Kerowyn[106]
1 Kinewma[107]
1 Kingdon[108]
1 KolbertBot[109]
2 LaForge[110]
25 Lambiam[111]
1 Lefabaru[112]
2 Leslie.Hetherington[113]
2 Leutha[114]
1 Lightbot[115]
3 LilHelpa[116]
1 LilyKitty[117]
1 LinkTiger[118]
1 Llamaa2[119]
1 Lofty abyss[120]

96 https://en.wikipedia.org/wiki/User:Jaredscribe
97 https://en.wikipedia.org/wiki/User:John_of_Reading
98 https://en.wikipedia.org/wiki/User:Jon_Awbrey
99 https://en.wikipedia.org/wiki/User:Julesd
100 https://en.wikipedia.org/wiki/User:JustAGal
101 https://en.wikipedia.org/wiki/User:KamielC~enwiki
102 https://en.wikipedia.org/wiki/User:KamikazeBot
103 https://en.wikipedia.org/wiki/User:Karl-Henner
104 https://en.wikipedia.org/wiki/User:KasparBot
105 https://en.wikipedia.org/wiki/User:Katzmik
106 https://en.wikipedia.org/wiki/User:Kerowyn
107 https://en.wikipedia.org/wiki/User:Kinewma
108 https://en.wikipedia.org/wiki/User:Kingdon
109 https://en.wikipedia.org/wiki/User:KolbertBot
110 https://en.wikipedia.org/w/index.php%3ftitle=User:LaForge&action=edit&redlink=1
111 https://en.wikipedia.org/wiki/User:Lambiam
112 https://en.wikipedia.org/w/index.php%3ftitle=User:Lefabaru&action=edit&redlink=1
113 https://en.wikipedia.org/wiki/User:Leslie.Hetherington
114 https://en.wikipedia.org/wiki/User:Leutha
115 https://en.wikipedia.org/wiki/User:Lightbot
116 https://en.wikipedia.org/wiki/User:LilHelpa
117 https://en.wikipedia.org/wiki/User:LilyKitty
118 https://en.wikipedia.org/wiki/User:LinkTiger
119 https://en.wikipedia.org/w/index.php%3ftitle=User:Llamaa2&action=edit&redlink=1
120 https://en.wikipedia.org/wiki/User:Lofty_abyss

2 Ludvikus[121]
4 M a s[122]
1 Macmat1[123]
1 Magic links bot[124]
8 Manifesto50[125]
2 Markhurd[126]
3 Materialscientist[127]
1 Mathematron[128]
2 Maurice Carbonaro[129]
1 Maximillion Pegasus[130]
1 Mdd[131]
1 Mets501[132]
1 Mexistential Crisis[133]
4 Michael Hardy[134]
2 Monkbot[135]
1 Msh210[136]
1 NONIS STEFANO[137]
1 Neelix[138]
3 NeilOnWiki[139]
2 Newbyguesses[140]
2 Newone[141]
1 Nixdorf[142]
2 OAbot[143]
1 Oleg Alexandrov[144]
3 Omnipaedista[145]

121 https://en.wikipedia.org/wiki/User:Ludvikus
122 https://en.wikipedia.org/wiki/User:M_a_s
123 https://en.wikipedia.org/w/index.php%3ftitle=User:Macmat1&action=edit&redlink=1
124 https://en.wikipedia.org/wiki/User:Magic_links_bot
125 https://en.wikipedia.org/w/index.php%3ftitle=User:Manifesto50&action=edit&redlink=1
126 https://en.wikipedia.org/wiki/User:Markhurd
127 https://en.wikipedia.org/wiki/User:Materialscientist
128 https://en.wikipedia.org/w/index.php%3ftitle=User:Mathematron&action=edit&redlink=1
129 https://en.wikipedia.org/wiki/User:Maurice_Carbonaro
130 https://en.wikipedia.org/wiki/User:Maximillion_Pegasus
131 https://en.wikipedia.org/wiki/User:Mdd
132 https://en.wikipedia.org/wiki/User:Mets501
133 https://en.wikipedia.org/w/index.php%3ftitle=User:Mexistential_Crisis&action=edit&redlink=1
134 https://en.wikipedia.org/wiki/User:Michael_Hardy
135 https://en.wikipedia.org/wiki/User:Monkbot
136 https://en.wikipedia.org/wiki/User:Msh210
137 https://en.wikipedia.org/w/index.php%3ftitle=User:NONIS_STEFANO&action=edit&redlink=1
138 https://en.wikipedia.org/wiki/User:Neelix
139 https://en.wikipedia.org/wiki/User:NeilOnWiki
140 https://en.wikipedia.org/wiki/User:Newbyguesses
141 https://en.wikipedia.org/wiki/User:Newone
142 https://en.wikipedia.org/wiki/User:Nixdorf
143 https://en.wikipedia.org/wiki/User:OAbot
144 https://en.wikipedia.org/wiki/User:Oleg_Alexandrov
145 https://en.wikipedia.org/wiki/User:Omnipaedista

1 Ozob[146]
3 Paolo Lipparini[147]
1 Paradoctor[148]
1 Patrickwooldridge[149]
1 Pcap[150]
1 Pgr94[151]
7 Porcher[152]
1 Portalian[153]
12 Quixotex[154]
1 R.e.b.[155]
1 RekishiEJ[156]
7 Rgdboer[157]
1 RibotBOT[158]
1 Rich Farmbrough[159]
3 Rick Norwood[160]
1 Rjwilmsi[161]
2 Robert Merkel[162]
1 Rtbittaker[163]
1 Rubinbot[164]
1 Sabrejo[165]
1 Salgueiro~enwiki[166]
1 Sardinastrale[167]
1 Sbmehta[168]
1 Schlafly[169]
1 ScratchMonkey[170]

146 https://en.wikipedia.org/wiki/User:Ozob
147 https://en.wikipedia.org/wiki/User:Paolo_Lipparini
148 https://en.wikipedia.org/wiki/User:Paradoctor
149 https://en.wikipedia.org/wiki/User:Patrickwooldridge
150 https://en.wikipedia.org/wiki/User:Pcap
151 https://en.wikipedia.org/wiki/User:Pgr94
152 https://en.wikipedia.org/wiki/User:Porcher
153 https://en.wikipedia.org/wiki/User:Portalian
154 https://en.wikipedia.org/w/index.php%3ftitle=User:Quixotex&action=edit&redlink=1
155 https://en.wikipedia.org/wiki/User:R.e.b.
156 https://en.wikipedia.org/wiki/User:RekishiEJ
157 https://en.wikipedia.org/wiki/User:Rgdboer
158 https://en.wikipedia.org/wiki/User:RibotBOT
159 https://en.wikipedia.org/wiki/User:Rich_Farmbrough
160 https://en.wikipedia.org/wiki/User:Rick_Norwood
161 https://en.wikipedia.org/wiki/User:Rjwilmsi
162 https://en.wikipedia.org/wiki/User:Robert_Merkel
163 https://en.wikipedia.org/wiki/User:Rtbittaker
164 https://en.wikipedia.org/wiki/User:Rubinbot
165 https://en.wikipedia.org/w/index.php%3ftitle=User:Sabrejo&action=edit&redlink=1
166 https://en.wikipedia.org/w/index.php%3ftitle=User:Salgueiro~enwiki&action=edit&redlink=1
167 https://en.wikipedia.org/w/index.php%3ftitle=User:Sardinastrale&action=edit&redlink=1
168 https://en.wikipedia.org/wiki/User:Sbmehta
169 https://en.wikipedia.org/wiki/User:Schlafly
170 https://en.wikipedia.org/w/index.php%3ftitle=User:ScratchMonkey&action=edit&redlink=1

1 Sgutkind[171]
1 Silver hr[172]
1 SkyGazer 512[173]
1 Sligocki[174]
1 Smasongarrison[175]
1 Smithpith[176]
1 Somejan[177]
1 Soobtoob[178]
2 Soulparadox[179]
5 Spoirier~enwiki[180]
1 Srnec[181]
1 Stevertigo[182]
1 TaBOT-zerem[183]
7 Tea2min[184]
3 The Tetrast[185]
1 The Transhumanist[186]
1 The.megapode[187]
4 Thenub314[188]
1 Thriley[189]
1 Tigercompanion25[190]
1 Tim Retout[191]
14 Tkuvho[192]
2 Tompw[193]
1 Trackteur[194]
14 Trovatore[195]

171 https://en.wikipedia.org/w/index.php%3ftitle=User:Sgutkind&action=edit&redlink=1
172 https://en.wikipedia.org/w/index.php%3ftitle=User:Silver_hr&action=edit&redlink=1
173 https://en.wikipedia.org/wiki/User:SkyGazer_512
174 https://en.wikipedia.org/wiki/User:Sligocki
175 https://en.wikipedia.org/wiki/User:Smasongarrison
176 https://en.wikipedia.org/wiki/User:Smithpith
177 https://en.wikipedia.org/w/index.php%3ftitle=User:Somejan&action=edit&redlink=1
178 https://en.wikipedia.org/w/index.php%3ftitle=User:Soobtoob&action=edit&redlink=1
179 https://en.wikipedia.org/wiki/User:Soulparadox
180 https://en.wikipedia.org/wiki/User:Spoirier~enwiki
181 https://en.wikipedia.org/wiki/User:Srnec
182 https://en.wikipedia.org/wiki/User:Stevertigo
183 https://en.wikipedia.org/wiki/User:TaBOT-zerem
184 https://en.wikipedia.org/wiki/User:Tea2min
185 https://en.wikipedia.org/wiki/User:The_Tetrast
186 https://en.wikipedia.org/wiki/User:The_Transhumanist
187 https://en.wikipedia.org/wiki/User:The.megapode
188 https://en.wikipedia.org/wiki/User:Thenub314
189 https://en.wikipedia.org/wiki/User:Thriley
190 https://en.wikipedia.org/wiki/User:Tigercompanion25
191 https://en.wikipedia.org/wiki/User:Tim_Retout
192 https://en.wikipedia.org/wiki/User:Tkuvho
193 https://en.wikipedia.org/wiki/User:Tompw
194 https://en.wikipedia.org/wiki/User:Trackteur
195 https://en.wikipedia.org/wiki/User:Trovatore

1 Tweisbach[196]
1 Victorgrigas[197]
1 Vieque[198]
1 Vina-iwbot~enwiki[199]
2 Vladimirias[200]
2 VolkovBot[201]
1 Wahrmund[202]
2 Wanavehv.adt[203]
13 Wcherowi[204]
1 Welsh[205]
1 WikiCleanerBot[206]
2 Woohookitty[207]
2 Workstern[208]
21 Wvbailey[209]
1 XJaM[210]
1 Xqbot[211]
1 Y-S.Ko[212]
3 Yobot[213]
1 YurikBot[214]
1 ZX81[215]
1 Zbisem[216]
1 Zeke, the Mad Horrorist[217]
2 Zlk199011[218]
1 ZéroBot[219]

196 https://en.wikipedia.org/wiki/User:Tweisbach
197 https://en.wikipedia.org/wiki/User:Victorgrigas
198 https://en.wikipedia.org/wiki/User:Vieque
199 https://en.wikipedia.org/wiki/User:Vina-iwbot~enwiki
200 https://en.wikipedia.org/wiki/User:Vladimirias
201 https://en.wikipedia.org/wiki/User:VolkovBot
202 https://en.wikipedia.org/wiki/User:Wahrmund
203 https://en.wikipedia.org/w/index.php%3ftitle=User:Wanavehv.adt&action=edit&redlink=1
204 https://en.wikipedia.org/wiki/User:Wcherowi
205 https://en.wikipedia.org/wiki/User:Welsh
206 https://en.wikipedia.org/wiki/User:WikiCleanerBot
207 https://en.wikipedia.org/wiki/User:Woohookitty
208 https://en.wikipedia.org/w/index.php%3ftitle=User:Workstern&action=edit&redlink=1
209 https://en.wikipedia.org/wiki/User:Wvbailey
210 https://en.wikipedia.org/wiki/User:XJaM
211 https://en.wikipedia.org/wiki/User:Xqbot
212 https://en.wikipedia.org/wiki/User:Y-S.Ko
213 https://en.wikipedia.org/wiki/User:Yobot
214 https://en.wikipedia.org/wiki/User:YurikBot
215 https://en.wikipedia.org/wiki/User:ZX81
216 https://en.wikipedia.org/w/index.php%3ftitle=User:Zbisem&action=edit&redlink=1
217 https://en.wikipedia.org/wiki/User:Zeke,_the_Mad_Horrorist
218 https://en.wikipedia.org/w/index.php%3ftitle=User:Zlk199011&action=edit&redlink=1
219 https://en.wikipedia.org/wiki/User:Z%25C3%25A9roBot

List of Figures

- GFDL: Gnu Free Documentation License. http://www.gnu.org/licenses/fdl.html
- cc-by-sa-3.0: Creative Commons Attribution ShareAlike 3.0 License. http://creativecommons.org/licenses/by-sa/3.0/
- cc-by-sa-2.5: Creative Commons Attribution ShareAlike 2.5 License. http://creativecommons.org/licenses/by-sa/2.5/
- cc-by-sa-2.0: Creative Commons Attribution ShareAlike 2.0 License. http://creativecommons.org/licenses/by-sa/2.0/
- cc-by-sa-1.0: Creative Commons Attribution ShareAlike 1.0 License. http://creativecommons.org/licenses/by-sa/1.0/
- cc-by-2.0: Creative Commons Attribution 2.0 License. http://creativecommons.org/licenses/by/2.0/
- cc-by-2.0: Creative Commons Attribution 2.0 License. http://creativecommons.org/licenses/by/2.0/deed.en
- cc-by-2.5: Creative Commons Attribution 2.5 License. http://creativecommons.org/licenses/by/2.5/deed.en
- cc-by-3.0: Creative Commons Attribution 3.0 License. http://creativecommons.org/licenses/by/3.0/deed.en
- GPL: GNU General Public License. http://www.gnu.org/licenses/gpl-2.0.txt
- LGPL: GNU Lesser General Public License. http://www.gnu.org/licenses/lgpl.html
- PD: This image is in the public domain.
- ATTR: The copyright holder of this file allows anyone to use it for any purpose, provided that the copyright holder is properly attributed. Redistribution, derivative work, commercial use, and all other use is permitted.
- EURO: This is the common (reverse) face of a euro coin. The copyright on the design of the common face of the euro coins belongs to the European Commission. Authorised is reproduction in a format without relief (drawings, paintings, films) provided they are not detrimental to the image of the euro.
- LFK: Lizenz Freie Kunst. http://artlibre.org/licence/lal/de
- CFR: Copyright free use.

- EPL: Eclipse Public License. `http://www.eclipse.org/org/documents/epl-v10.php`

Copies of the GPL, the LGPL as well as a GFDL are included in chapter Licenses[220]. Please note that images in the public domain do not require attribution. You may click on the image numbers in the following table to open the webpage of the images in your webbrower.

220 Chapter 3 on page 41

1	en:User:Saranphat.cha[221], Anomie, J.delanoy, JJMC89, Jo-Jo Eumerus, John M Wolfson, Kimchi.sg, Luk, MSGJ, PeterSymonds, Redrose64, Salvidrim!, ToBeFree, Topbanana	
2	David Vignoni[222] (original icon); Flamurai[223] (SVG convertion); bayo[224] (color), David Vignoni[225] (original icon); Flamurai[226] (SVG convertion); bayo[227] (color)	
3	Reidab[228], Grunt[229], 3247[230], Pumbaa[231], Anomie, Callanecc, CambridgeBayWeather, JJMC89, Jo-Jo Eumerus, RHaworth	
4	David Vignoni[232] (original icon); Flamurai[233] (SVG convertion); bayo[234] (color), David Vignoni[235] (original icon); Flamurai[236] (SVG convertion); bayo[237] (color)	
5	David Brewster	
6	The original uploader was Magnus Manske[238] at English Wikipedia[239]. Later versions were uploaded by Optimager[240] at en.wikipedia[241].	
7	User:Eubulides[242], Anomie, JJMC89, Jo-Jo Eumerus, Topbanana	
8	Zscout370[243], Zscout370[244]	

221 https://en.wikipedia.org/wiki/User:Saranphat.cha
222 https://en.wikipedia.org/wiki/David_Vignoni
223 http://commons.wikimedia.org/wiki/User:Flamurai
224 http://commons.wikimedia.org/wiki/User:Bayo
225 https://en.wikipedia.org/wiki/David_Vignoni
226 https://commons.wikimedia.org/wiki/User:Flamurai
227 https://commons.wikimedia.org/wiki/User:Bayo
228 https://meta.wikimedia.org/wiki/User:Reidab
229 https://commons.wikimedia.org/w/index.php?title=User:Grunt&action=edit&redlink=1
230 https://commons.wikimedia.org/wiki/User:3247
231 https://commons.wikimedia.org/wiki/User:Pumbaa80
232 https://en.wikipedia.org/wiki/David_Vignoni
233 http://commons.wikimedia.org/wiki/User:Flamurai
234 http://commons.wikimedia.org/wiki/User:Bayo
235 https://en.wikipedia.org/wiki/David_Vignoni
236 https://commons.wikimedia.org/wiki/User:Flamurai
237 https://commons.wikimedia.org/wiki/User:Bayo
238 https://en.wikipedia.org/wiki/User:Magnus_Manske
239 https://en.wikipedia.org/wiki/
240 https://en.wikipedia.org/wiki/User:Optimager
241 https://en.wikipedia.org
242 https://commons.wikimedia.org/wiki/User:Eubulides
243 http://commons.wikimedia.org/wiki/User:Zscout370
244 https://commons.wikimedia.org/wiki/User:Zscout370

3 Licenses

3.1 GNU GENERAL PUBLIC LICENSE

Version 3, 29 June 2007

Copyright © 2007 Free Software Foundation, Inc. <http://fsf.org/>

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed. Preamble

The GNU General Public License is a free, copyleft license for software and other kinds of works.

The licenses for most software and other practical works are designed to take away your freedom to share and change the works. By contrast, the GNU General Public License is intended to guarantee your freedom to share and change all versions of a program–to make sure it remains free software for all its users. We, the Free Software Foundation, use the GNU General Public License for most of our software; it applies also to any other work released this way by its authors. You can apply it to your programs, too.

When we speak of free software, we are referring to freedom, not price. Our General Public Licenses are designed to make sure that you have the freedom to distribute copies of free software (and charge for them if you wish), that you receive source code or can get it if you want it, that you can change the software or use pieces of it in new free programs, and that you know you can do these things.

To protect your rights, we need to prevent others from denying you these rights or asking you to surrender the rights. Therefore, you have certain responsibilities if you distribute copies of the software, or if you modify it: responsibilities to respect the freedom of others.

For example, if you distribute copies of such a program, whether gratis or for a fee, you must pass on to the recipients the same freedoms that you received. You must make sure that they, too, receive or can get the source code. And you must show them these terms so they know their rights.

Developers that use the GNU GPL protect your rights with two steps: (1) assert copyright on the software, and (2) offer you this License giving you legal permission to copy, distribute and/or modify it.

For the developers' and authors' protection, the GPL clearly explains that there is no warranty for this free software. For both users' and authors' sake, the GPL requires that modified versions be marked as changed, so that their problems will not be attributed erroneously to authors of previous versions.

Some devices are designed to deny users access to install or run modified versions of the software inside them, although the manufacturer can do so. This is fundamentally incompatible with the aim of protecting users' freedom to change the software. The systematic pattern of such abuse occurs in the area of products for individuals to use, which is precisely where it is most unacceptable. Therefore, we have designed this version of the GPL to prohibit the practice for those products. If such problems arise substantially in other domains, we stand ready to extend this provision to those domains in future versions of the GPL, as needed to protect the freedom of users.

Finally, every program is threatened constantly by software patents. States should not allow patents to restrict development and use of software on general-purpose computers, but in those that do, we wish to avoid the special danger that patents applied to a free program could make it effectively proprietary. To prevent this, the GPL assures that patents cannot be used to render the program non-free.

The precise terms and conditions for copying, distribution and modification follow. TERMS AND CONDITIONS 0. Definitions.

“This License” refers to version 3 of the GNU General Public License.

“Copyright” also means copyright-like laws that apply to other kinds of works, such as semiconductor masks.

“The Program” refers to any copyrightable work licensed under this License. Each licensee is addressed as “you”. “Licensees” and “recipients” may be individuals or organizations.

To “modify” a work means to copy from or adapt all or part of the work in a fashion requiring copyright permission, other than the making of an exact copy. The resulting work is called a “modified version” of the earlier work or a work “based on” the earlier work.

A “covered work” means either the unmodified Program or a work based on the Program.

To “propagate” a work means to do anything with it that, without permission, would make you directly or secondarily liable for infringement under applicable copyright law, except executing it on a computer or modifying a private copy. Propagation includes copying, distribution (with or without modification), making available to the public, and in some countries other activities as well.

To “convey” a work means any kind of propagation that enables other parties to make or receive copies. Mere interaction with a user through a computer network, with no transfer of a copy, is not conveying.

An interactive user interface displays “Appropriate Legal Notices” to the extent that it includes a convenient and prominently visible feature that (1) displays an appropriate copyright notice, and (2) tells the user that there is no warranty for the work (except to the extent that warranties are provided), that licensees may convey the work under this License, and how to view a copy of this License. If the interface presents a list of user commands or options, such as a menu, a prominent item in the list meets this criterion. 1. Source Code.

The “source code” for a work means the preferred form of the work for making modifications to it. “Object code” means any non-source form of a work.

A “Standard Interface” means an interface that either is an official standard defined by a recognized standards body, or, in the case of interfaces specified for a particular programming language, one that is widely used among developers working in that language.

The “System Libraries” of an executable work include anything, other than the work as a whole, that (a) is included in the normal form of packaging a Major Component, but which is not part of that Major Component, and (b) serves only to enable use of the work with that Major Component, or to implement a Standard Interface for which an implementation is available to the public in source code form. A “Major Component”, in this context, means a major essential component (kernel, window system, and so on) of the specific operating system (if any) on which the executable work runs, or a compiler used to produce the work, or an object code interpreter used to run it.

The “Corresponding Source” for a work in object code form means all the source code needed to generate, install, and (for an executable work) run the object code and to modify the work, including scripts to control those activities. However, it does not include the work's System Libraries, or general-purpose tools or generally available free programs which are used unmodified in performing those activities but which are not part of the work. For example, Corresponding Source includes interface definition files associated with source files for the work, and the source code for shared libraries and dynamically linked subprograms that the work is specifically designed to require, such as by intimate data communication or control flow between those subprograms and other parts of the work.

The Corresponding Source need not include anything that users can regenerate automatically from other parts of the Corresponding Source.

The Corresponding Source for a work in source code form is that same work. 2. Basic Permissions.

All rights granted under this License are granted for the term of copyright on the Program, and are irrevocable provided the stated conditions are met. This License explicitly affirms your unlimited permission to run the unmodified Program. The output from running a covered work is covered by this License only if the output, given its content, constitutes a covered work. This License acknowledges your rights of fair use or other equivalent, as provided by copyright law.

You may make, run and propagate covered works that you do not convey, without conditions so long as your license otherwise remains in force. You may convey covered works to others for the sole purpose of having them make modifications exclusively for you, or provide you with facilities for running those works, provided that you comply with the terms of this License in conveying all material for which you do not control copyright. Those thus making or running the covered works for you must do so exclusively on your behalf, under your direction and control, on terms that prohibit them from making any copies of your copyrighted material outside their relationship with you.

Conveying under any other circumstances is permitted solely under the conditions stated below. Sublicensing is not allowed; section 10 makes it unnecessary. 3. Protecting Users' Legal Rights From Anti-Circumvention Law.

No covered work shall be deemed part of an effective technological measure under any applicable law fulfilling obligations under article 11 of the WIPO copyright treaty adopted on 20 December 1996, or similar laws prohibiting or restricting circumvention of such measures.

When you convey a covered work, you waive any legal power to forbid circumvention of technological measures to the extent such circumvention is effected by exercising rights under this License with respect to the covered work, and you disclaim any intention to limit operation or modification of the work as a means of enforcing, against the work's users, your or third parties' legal rights to forbid circumvention of technological measures. 4. Conveying Verbatim Copies.

You may convey verbatim copies of the Program's source code as you receive it, in any medium, provided that you conspicuously and appropriately publish on each copy an appropriate copyright notice; keep intact all notices stating that this License and any non-permissive terms added in accord with section 7 apply to the code; keep intact all notices of the absence of any warranty; and give all recipients a copy of this License along with the Program.

You may charge any price or no price for each copy that you convey, and you may offer support or warranty protection for a fee. 5. Conveying Modified Source Versions.

You may convey a work based on the Program, or the modifications to produce it from the Program, in the form of source code under the terms of section 4, provided that you also meet all of these conditions:

* a) The work must carry prominent notices stating that you modified it, and giving a relevant date. * b) The work must carry prominent notices stating that it is released under this License and any conditions added under section 7. This requirement modifies the requirement in section 4 to “keep intact all notices”. * c) You must license the entire work, as a whole, under this License to anyone who comes into possession of a copy. This License will therefore apply, along with any applicable section 7 additional terms, to the whole of the work, and all its parts, regardless of how they are packaged. This License gives no permission to license the work in any other way, but it does not invalidate such permission if you have separately received it. * d) If the work has interactive user interfaces, each must display Appropriate Legal Notices; however, if the Program has interactive interfaces that do not display Appropriate Legal Notices, your work need not make them do so.

A compilation of a covered work with other separate and independent works, which are not by their nature extensions of the covered work, and which are not combined with it such as to form a larger program, in or on a volume of a storage or distribution medium, is called an “aggregate” if the compilation and its resulting copyright are not used to limit the access or legal rights of the compilation's users beyond what the individual works permit. Inclusion of a covered work in an aggregate does not cause this License to apply to the other parts of the aggregate. 6. Conveying Non-Source Forms.

You may convey a covered work in object code form under the terms of sections 4 and 5, provided that you also convey the machine-readable Corresponding Source under the terms of this License, in one of these ways:

* a) Convey the object code in, or embodied in, a physical product (including a physical distribution medium), accompanied by the Corresponding Source fixed on a durable physical medium customarily used for software interchange. * b) Convey the object code in, or embodied in, a physical product (including a physical distribution medium), accompanied by a written offer, valid for at least three years and valid for as long as you offer spare parts or customer support for that product model, to give anyone who possesses the object code either (1) a copy of the Corresponding Source for all the software in the product that is covered by this License, on a durable physical medium customarily used for software interchange, for a price no more than your reasonable cost of physically performing this conveying of source, or (2) access to copy the Corresponding Source from a network server at no charge. * c) Convey individual copies of the object code with a copy of the written offer to provide the Corresponding Source. This alternative is allowed only occasionally and noncommercially, and only if you received the object code with such an offer, in accord with subsection 6b. * d) Convey the object code by offering access from a designated place (gratis or for a charge), and offer equivalent access to the Corresponding Source in the same way through the same place at no further charge. You need not require recipients to copy the Corresponding Source along with the object code. If the place to copy the object code is a network server, the Corresponding Source may be on a different server (operated by you or a third party) that supports equivalent copying facilities, provided you maintain clear directions next to the object code saying where to find the Corresponding Source. Regardless of what server hosts the Corresponding Source, you remain obligated to ensure that it is available for as long as needed to satisfy these requirements. * e) Convey the object code using peer-to-peer transmission, provided you inform other peers where the object code and Corresponding Source of the work are being offered to the general public at no charge under subsection 6d.

A separable portion of the object code, whose source code is excluded from the Corresponding Source as a System Library, need not be included in conveying the object code work.

A “User Product” is either (1) a “consumer product”, which means any tangible personal property which is normally used for personal, family, or household purposes, or (2) anything designed or sold for incorporation into a dwelling. In determining whether a product is a consumer product, doubtful cases shall be resolved in favor of coverage. For a particular product received by a particular user, “normally used” refers to a typical or common use of that class of product, regardless of the status of the particular user or of the way in which the particular user actually uses, or expects or is expected to use, the product. A product is a consumer product regardless of whether the product has substantial commercial, industrial or non-consumer uses, unless such uses represent the only significant mode of use of the product.

“Installation Information” for a User Product means any methods, procedures, authorization keys, or other information required to install and execute modified versions of a covered work in that User Product from a modified version of its Corresponding Source. The information must suffice to ensure that the continued functioning of the modified object code is in no case prevented or interfered with solely because modification has been made.

If you convey an object code work under this section in, or with, or specifically for use in, a User Product, and the conveying occurs as part of a transaction in which the right of possession and use of the User Product is transferred to the recipient in perpetuity or for a fixed term (regardless of how the transaction is characterized), the Corresponding Source conveyed under this section must be accompanied by the Installation Information. But this requirement does not apply if neither you nor any third party retains the ability to install modified object code on the User Product (for example, the work has been installed in ROM).

The requirement to provide Installation Information does not include a requirement to continue to provide support service, warranty, or updates for a work that has been modified or installed by the recipient, or for the User Product in which it has been modified or installed. Access to a network may be denied when the modification itself materially and adversely affects the operation of the network or violates the rules and protocols for communication across the network.

Corresponding Source conveyed, and Installation Information provided, in accord with this section must be in a format that is publicly documented (and with an implementation available to the public in source code form), and must require no special password or key for unpacking, reading or copying. 7. Additional Terms.

“Additional permissions” are terms that supplement the terms of this License by making exceptions from one or more of its conditions. Additional permissions that are applicable to the entire Program shall be treated as though they were included in this License, to the extent that they are valid under applicable law. If additional permissions apply only to part of the Program, that part may be used separately under those permissions, but the entire Program remains governed by this License without regard to the additional permissions.

When you convey a copy of a covered work, you may at your option remove any additional permissions from that copy, or from any part of it. (Additional permissions may be written to require their own removal in certain cases when you modify the work.) You may place additional permissions on material, added by you to a covered work, for which you have or can give appropriate copyright permission.

Notwithstanding any other provision of this License, for material you add to a covered work, you may (if authorized by the copyright holders of that material) supplement the terms of this License with terms:

* a) Disclaiming warranty or limiting liability differently from the terms of sections 15 and 16 of this License; or * b) Requiring preservation of specified reasonable legal notices or author attributions in that material or in the Appropriate Legal Notices displayed by works containing it; or * c) Prohibiting misrepresentation of the origin of that material, or requiring that modified versions of such material be marked in reasonable ways as different from the original version; or * d) Limiting the use for publicity purposes of names of licensors or authors of the material; or * e) Declining to grant rights under trademark law for use of some trade names, trademarks, or service marks; or * f) Requiring indemnification of licensors and authors of that material by anyone who conveys the material (or modified versions of it) with contractual assumptions of liability to the recipient, for any liability that these contractual assumptions directly impose on those licensors and authors.

All other non-permissive additional terms are considered “further restrictions” within the meaning of section 10. If the Program as you received it, or any part of it, contains a notice stating that it is governed by this License along with a term that is a further restriction, you may remove that term. If a license document contains a further restriction but permits relicensing or conveying under this License, you may add to a covered work material governed by the terms of that license document, provided that the further restriction does not survive such relicensing or conveying.

If you add terms to a covered work in accord with this section, you must place, in the relevant source files, a statement of the additional terms that apply to those files, or a notice indicating where to find the applicable terms.

Additional terms, permissive or non-permissive, may be stated in the form of a separately written license, or stated as exceptions; the above requirements apply either way. 8. Termination.

You may not propagate or modify a covered work except as expressly provided under this License. Any attempt otherwise to propagate or modify it is void, and will automatically terminate your rights under this License (including any patent licenses granted under the third paragraph of section 11).

However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60 days after the cessation.

Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.

Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, you do not qualify to receive new licenses for the same material under section 10. 9. Acceptance Not Required for Having Copies.

You are not required to accept this License in order to receive or run a copy of the Program. Ancillary propagation of a covered work occurring solely as a consequence of using peer-to-peer transmission to receive a copy likewise does not require acceptance. However, nothing other than this License grants you permission to propagate or modify any covered work. These actions infringe copyright if you do not accept this License. Therefore, by modifying or propagating a covered work, you indicate your acceptance of this License to do so. 10. Automatic Licensing of Downstream Recipients.

Each time you convey a covered work, the recipient automatically receives a license from the original licensors, to run, modify and propagate that work, subject to this License. You are not responsible for enforcing compliance by third parties with this License.

An “entity transaction” is a transaction transferring control of an organization, or substantially all assets of one, or subdividing an organization, or merging organizations. If propagation of a covered work results from an entity transaction, each party to that transaction who receives a copy of the work also receives whatever licenses to the work the party's predecessor in interest had or could give under the previous paragraph, plus a right to possession of the Corresponding Source of the work from the predecessor in interest, if the predecessor has it or can get it with reasonable efforts.

You may not impose any further restrictions on the exercise of the rights granted or affirmed under this License. For example, you may not impose a license fee, royalty, or other charge for exercise of rights granted under this License, and you may not initiate litigation (including a cross-claim or counterclaim in a lawsuit) alleging that any patent claim is infringed by making, using, selling, offering for sale, or importing the Program or any portion of it. 11. Patents.

A “contributor” is a copyright holder who authorizes use under this License of the Program or a work on which the Program is based. The work thus licensed is called the contributor's “contributor version”.

A contributor's “essential patent claims” are all patent claims owned or controlled by the contributor, whether already acquired or hereafter acquired, that would be infringed by some manner, permitted by this License, of making, using, or selling its contributor version, but do not include claims that would be infringed only as a consequence of further modification of the contributor version. For purposes of this definition, “control” includes the right to grant patent sublicenses in a manner consistent with the requirements of this License.

Each contributor grants you a non-exclusive, worldwide, royalty-free patent license under the contributor's essential patent claims, to make, use, sell, offer for sale, import and otherwise run, modify and propagate the contents of its contributor version.

In the following three paragraphs, a “patent license” is any express agreement or commitment, however denominated, not to enforce a patent (such as an express permission to practice a patent or covenant not to sue for patent infringement). To “grant” such a patent license to a party means to make such an agreement or commitment not to enforce a patent against the party.

If you convey a covered work, knowingly relying on a patent license, and the Corresponding Source of the work is not available for anyone to copy, free of charge and under the terms of this License, through a publicly available network server or other readily accessible means, then you must either (1) cause the Corresponding Source to be so available, or (2) arrange to deprive yourself of the benefit of the patent license for this particular work, or (3) arrange, in a manner consistent with the requirements of this License, to extend the patent license to downstream recipients. “Knowingly relying” means you have actual knowledge that, but for the patent license, your conveying the covered work in a country, or your recipient's use of the covered work in a country, would infringe one or more identifiable patents in that country that you have reason to believe are valid.

If, pursuant to or in connection with a single transaction or arrangement, you convey, or propagate by procuring conveyance of, a covered work, and grant a patent license to some of the parties receiving the covered work authorizing them to use, propagate, modify or convey a specific copy of the covered work, then the patent license you grant is automatically extended to all recipients of the covered work and works based on it.

A patent license is “discriminatory” if it does not include within the scope of its coverage, prohibits the exercise of, or is conditioned on the non-exercise of one or more of the rights that are specifically granted under this License. You may not convey a covered work if you are a party to an arrangement with a third party that is in the business of distributing software, under which you make payment to the third party based on the extent of your activity of conveying the work, and under which the third party grants, to any of the parties who would receive the covered work from you, a discriminatory patent license (a) in connection with copies of the covered work conveyed by you (or copies made from those copies), or (b) primarily for and in connection with specific products or compilations that contain the covered work, unless you entered into that arrangement, or that patent license was granted, prior to 28 March 2007.

Nothing in this License shall be construed as excluding or limiting any implied license or other defenses to infringement that may otherwise be available to you under applicable patent law. 12. No Surrender of Others' Freedom.

If conditions are imposed on you (whether by court order, agreement or otherwise) that contradict the conditions of this License, they do not excuse you from the conditions of this License. If you cannot convey a covered work so as to satisfy simultaneously your obligations under this License and any other pertinent obligations, then as a consequence you may not convey it at all. For example, if you agree to terms that obligate you to collect a royalty for further conveying from those to whom you convey the Program, the only way you could satisfy both those terms and this License would be to refrain entirely from conveying the Program. 13. Use with the GNU Affero General Public License.

Notwithstanding any other provision of this License, you have permission to link or combine any covered work with a work licensed under version 3 of the GNU Affero General Public License into a single combined work, and to convey the resulting work. The terms of this License will continue to apply to the part which is the covered work, but the special requirements of the GNU Affero General Public License, section 13, concerning interaction through a network will apply to the combination as such. 14. Revised Versions of this License.

The Free Software Foundation may publish revised and/or new versions of the GNU General Public License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns.

Each version is given a distinguishing version number. If the Program specifies that a certain numbered version of the GNU General Public License “or any later version” applies to it, you have the option of following the terms and conditions either of that numbered version or of any later version published by the Free Software Foundation. If the Program does not specify a version number of the GNU General Public License, you may choose any version ever published by the Free Software Foundation.

If the Program specifies that a proxy can decide which future versions of the GNU General Public License can be used, that proxy's public statement of acceptance of a version permanently authorizes you to choose that version for the Program.

Later license versions may give you additional or different permissions. However, no additional obligations are imposed on any author or copyright holder as a result of your choosing to follow a later version. 15. Disclaimer of Warranty.

THERE IS NO WARRANTY FOR THE PROGRAM, TO THE EXTENT PERMITTED BY APPLICABLE LAW. EXCEPT WHEN OTHERWISE STATED IN WRITING THE COPYRIGHT HOLDERS AND/OR OTHER PARTIES PROVIDE THE PROGRAM “AS IS” WITHOUT WARRANTY OF ANY KIND, EITHER EXPRESSED OR IMPLIED, INCLUDING, BUT NOT LIMITED TO, THE IMPLIED WARRANTIES OF MERCHANTABILITY AND FITNESS FOR A PARTICULAR PURPOSE. THE ENTIRE RISK AS TO THE QUALITY AND PERFORMANCE OF THE PROGRAM IS WITH YOU. SHOULD THE PROGRAM PROVE DEFECTIVE, YOU ASSUME THE COST OF ALL NECESSARY SERVICING, REPAIR OR CORRECTION. 16. Limitation of Liability.

IN NO EVENT UNLESS REQUIRED BY APPLICABLE LAW OR AGREED TO IN WRITING WILL ANY COPYRIGHT HOLDER, OR ANY OTHER PARTY WHO MODIFIES AND/OR CONVEYS THE PROGRAM AS PERMITTED ABOVE, BE LIABLE TO YOU FOR DAMAGES, INCLUDING ANY GENERAL, SPECIAL, INCIDENTAL OR CONSEQUENTIAL DAMAGES ARISING OUT OF THE USE OR INABILITY TO USE THE PROGRAM (INCLUDING BUT NOT LIMITED TO LOSS OF DATA OR DATA BEING RENDERED INACCURATE OR LOSSES SUSTAINED BY YOU OR THIRD PARTIES OR A FAILURE OF THE PROGRAM TO OPERATE WITH ANY OTHER PROGRAMS), EVEN IF SUCH HOLDER OR OTHER PARTY HAS BEEN ADVISED OF THE POSSIBILITY OF SUCH DAMAGES. 17. Interpretation of Sections 15 and 16.

If the disclaimer of warranty and limitation of liability provided above cannot be given local legal effect according to their terms, reviewing courts shall apply local law that most closely approximates an absolute waiver of all civil liability in connection with the Program, unless a warranty or assumption of liability accompanies a copy of the Program in return for a fee.

END OF TERMS AND CONDITIONS How to Apply These Terms to Your New Programs

If you develop a new program, and you want it to be of the greatest possible use to the public, the best way to achieve this is to make it free software which everyone can redistribute and change under these terms.

To do so, attach the following notices to the program. It is safest to attach them to the start of each source file to most effectively state the exclusion of warranty; and each file should have at least the “copyright” line and a pointer to where the full notice is found.

<one line to give the program's name and a brief idea of what it does.> Copyright (C) <year> <name of author>

This program is free software: you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation, either version 3 of the License, or (at your option) any later version.

This program is distributed in the hope that it will be useful, but WITHOUT ANY WARRANTY; without even the implied warranty of MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the GNU General Public License for more details.

You should have received a copy of the GNU General Public License along with this program. If not, see <http://www.gnu.org/licenses/>.

Also add information on how to contact you by electronic and paper mail.

If the program does terminal interaction, make it output a short notice like this when it starts in an interactive mode:

<program> Copyright (C) <year> <name of author> This program comes with ABSOLUTELY NO WARRANTY; for details type ‘show w’. This is free software, and you are welcome to redistribute it under certain conditions; type ‘show c’ for details.

The hypothetical commands ‘show w’ and ‘show c’ should show the appropriate parts of the General Public License. Of course, your program's commands might be different; for a GUI interface, you would use an “about box”.

You should also get your employer (if you work as a programmer) or school, if any, to sign a “copyright disclaimer” for the program, if necessary. For more information on this, and how to apply and follow the GNU GPL, see <http://www.gnu.org/licenses/>.

The GNU General Public License does not permit incorporating your program into proprietary programs. If your program is a subroutine library, you may consider it more useful to permit linking proprietary applications with the library. If this is what you want to do, use the GNU Lesser General Public License instead of this License. But first, please read <http://www.gnu.org/philosophy/why-not-lgpl.html>.

3.2 GNU Free Documentation License

Version 1.3, 3 November 2008

Copyright © 2000, 2001, 2002, 2007, 2008 Free Software Foundation, Inc. <http://fsf.org/>

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed. 0. PREAMBLE

The purpose of this License is to make a manual, textbook, or other functional and useful document "free" in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondarily, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of "copyleft", which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference. 1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The "Document", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "you". You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A "Modified Version" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A "Secondary Section" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The "Invariant Sections" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The "Cover Texts" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A "Transparent" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not "Transparent" is called "Opaque".

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or PDF produced by some word processors for output purposes only.

The "Title Page" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work's title, preceding the beginning of the body of the text.

The "publisher" means any person or entity that distributes copies of the Document to the public.

A section "Entitled XYZ" means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as "Acknowledgements", "Dedications", "Endorsements", or "History".) To "Preserve the Title" of such a section when you modify the Document means that it remains a section "Entitled XYZ" according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties: any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License. 2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies. 3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.

If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a computer-network location from which the general network-using public has access to download using public-standard network protocols a complete Transparent copy of the Document, free of added material. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.

It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document. 4. MODIFICATIONS

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

* A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission. * B. List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement. * C. State on the Title page the name of the publisher of the Modified Version, as the publisher. * D. Preserve all the copyright notices of the Document. * E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices. * F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below. * G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice. * H. Include an unaltered copy of this License. * I. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence. * J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission. * K. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein. * L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles. * M. Delete any section Entitled "Endorsements". Such a section may not be included in the Modified Version. * N. Do not retitle any existing section to be Entitled "Endorsements" or to conflict in title with any Invariant Section. * O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section Entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version. 5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled "History" in the various original documents, forming one section Entitled "History"; likewise combine any sections Entitled "Acknowledgements", and any sections Entitled "Dedications". You must delete all sections Entitled "Endorsements". 6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document. 7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an "aggregate" if the copyright resulting from the compilation is not used to limit the legal rights of the compilation's users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document's Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate. 8. TRANSLATION

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled "Acknowledgements", "Dedications", or "History", the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title. 9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense, or distribute it is void, and will automatically terminate your rights under this License.

However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60 days after the cessation.

Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.

Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, receipt of a copy of some or all of the same material does not give you any rights to use it. 10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See http://www.gnu.org/copyleft/.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation. If the Document specifies that a proxy can decide which future versions of this License can be used, that proxy's public statement of acceptance of a version permanently authorizes you to choose that version for the Document. 11. RELICENSING

"Massive Multiauthor Collaboration Site" (or "MMC Site") means any World Wide Web server that publishes copyrightable works and also provides prominent facilities for anybody to edit those works. A public wiki that anybody can edit is an example of such a server. A "Massive Multiauthor Collaboration" (or "MMC") contained in the site means any set of copyrightable works thus published on the MMC site.

"CC-BY-SA" means the Creative Commons Attribution-Share Alike 3.0 license published by Creative Commons Corporation, a not-for-profit corporation with a principal place of business in San Francisco, California, as well as future copyleft versions of that license published by that same organization.

"Incorporate" means to publish or republish a Document, in whole or in part, as part of another Document.

An MMC is "eligible for relicensing" if it is licensed under this License, and if all works that were first published under this License somewhere other than this MMC, and subsequently incorporated in whole or in part into the MMC, (1) had no cover texts or invariant sections, and (2) were thus incorporated prior to November 1, 2008.

The operator of an MMC Site may republish an MMC contained in the site under CC-BY-SA on the same site at any time before August 1, 2009, provided the MMC is eligible for relicensing. ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright (C) YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.3 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the "with ... Texts." line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.

3.3 GNU Lesser General Public License

GNU LESSER GENERAL PUBLIC LICENSE

Version 3, 29 June 2007

Copyright © 2007 Free Software Foundation, Inc. <http://fsf.org/>

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

This version of the GNU Lesser General Public License incorporates the terms and conditions of version 3 of the GNU General Public License, supplemented by the additional permissions listed below. 0. Additional Definitions.

As used herein, "this License" refers to version 3 of the GNU Lesser General Public License, and the "GNU GPL" refers to version 3 of the GNU General Public License.

"The Library" refers to a covered work governed by this License, other than an Application or a Combined Work as defined below.

An "Application" is any work that makes use of an interface provided by the Library, but which is not otherwise based on the Library. Defining a subclass of a class defined by the Library is deemed a mode of using an interface provided by the Library.

A "Combined Work" is a work produced by combining or linking an Application with the Library. The particular version of the Library with which the Combined Work was made is also called the "Linked Version".

The "Minimal Corresponding Source" for a Combined Work means the Corresponding Source for the Combined Work, excluding any source code for portions of the Combined Work that, considered in isolation, are based on the Application, and not on the Linked Version.

The "Corresponding Application Code" for a Combined Work means the object code and/or source code for the Application, including any data and utility programs needed for reproducing the Combined Work from the Application, but excluding the System Libraries of the Combined Work. 1. Exception to Section 3 of the GNU GPL.

You may convey a covered work under sections 3 and 4 of this License without being bound by section 3 of the GNU GPL. 2. Conveying Modified Versions.

If you modify a copy of the Library, and, in your modifications, a facility refers to a function or data to be supplied by an Application that uses the facility (other than as an argument passed when the facility is invoked), then you may convey a copy of the modified version:

* a) under this License, provided that you make a good faith effort to ensure that, in the event an Application does not supply the function or data, the facility still operates, and performs whatever part of its purpose remains meaningful, or * b) under the GNU GPL, with none of the additional permissions of this License applicable to that copy.

3. Object Code Incorporating Material from Library Header Files.

The object code form of an Application may incorporate material from a header file that is part of the Library. You may convey such object code under terms of your choice, provided that, if the incorporated material is not limited to numerical parameters, data structure layouts and accessors, or small macros, inline functions and templates (ten or fewer lines in length), you do both of the following:

* a) Give prominent notice with each copy of the object code that the Library is used in it and that the Library and its use are covered by this License. * b) Accompany the object code with a copy of the GNU GPL and this license document.

4. Combined Works.

You may convey a Combined Work under terms of your choice that, taken together, effectively do not restrict modification of the portions of the Library contained in the Combined Work and reverse engineering for debugging such modifications, if you also do each of the following:

* a) Give prominent notice with each copy of the Combined Work that the Library is used in it and that the Library and its use are covered by this License. * b) Accompany the Combined Work with a copy of the GNU GPL and this license document. * c) For a Combined Work that displays copyright notices during execution, include the copyright notice for the Library among these notices, as well as a reference directing the user to the copies of the GNU GPL and this license document. * d) Do one of the following: o 0) Convey the Minimal Corresponding Source under the terms of this License, and the Corresponding Application Code in a form suitable for, and under terms that permit, the user to recombine or relink the Application with a modified version of the Linked Version to produce a modified Combined Work, in the manner specified by section 6 of the GNU GPL for conveying Corresponding Source. o 1) Use a suitable shared library mechanism for linking with the Library. A suitable mechanism is one that (a) uses at run time a copy of the Library already present on the user's computer system, and (b) will operate properly with a modified version of the Library that is interface-compatible with the Linked Version. * e) Provide Installation Information, but only if you would otherwise be required to provide such information under section 6 of the GNU GPL, and only to the extent that such information is necessary to install and execute a modified version of the Combined Work produced by recombining or relinking the Application with a modified version of the Linked Version. (If you use option 4d0, the Installation Information must accompany the Minimal Corresponding Source and Corresponding Application Code. If you use option 4d1, you must provide the Installation Information in the manner specified by section 6 of the GNU GPL for conveying Corresponding Source.)

5. Combined Libraries.

You may place library facilities that are a work based on the Library side by side in a single library together with other library facilities that are not Applications and are not covered by this License, and convey such a combined library under terms of your choice, if you do both of the following:

* a) Accompany the combined library with a copy of the same work based on the Library, uncombined with any other library facilities, conveyed under the terms of this License. * b) Give prominent notice with the combined library that part of it is a work based on the Library, and explaining where to find the accompanying uncombined form of the same work.

6. Revised Versions of the GNU Lesser General Public License.

The Free Software Foundation may publish revised and/or new versions of the GNU Lesser General Public License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns.

Each version is given a distinguishing version number. If the Library as you received it specifies that a certain numbered version of the GNU Lesser General Public License "or any later version" applies to it, you have the option of following the terms and conditions either of that published version or of any later version published by the Free Software Foundation. If the Library as you received it does not specify a version number of the GNU Lesser General Public License, you may choose any version of the GNU Lesser General Public License ever published by the Free Software Foundation.

If the Library as you received it specifies that a proxy can decide whether future versions of the GNU Lesser General Public License shall apply, that proxy's public statement of acceptance of any version is permanent authorization for you to choose that version for the Library.