12

U has an orthonormal basis of characteristic vectors of T.

121 Lemma *Let T be a Hermitian operator. Then*

1. *All the (complex) characteristic values of T are real.*
2. *Characteristic vectors of T that belong to distinct characteristic values are orthogonal.*

Proof. Let $\lambda_1, \ldots \lambda_k$ be distinct characteristic values of T, and let $U_1, \ldots, U_k$ be there respective characteristic spaces:

$$U_i = \{v \in V \mid Tv = \lambda_i v\}.$$

By the Gram-Schmidt orthogonalization process we can construct an orthonormal basis B_i to each subspace U_i, and it turns out that without any more car on our part, the set $B = B_1 \cup \cdots \cup B_k$ already forms an orthonormal basis for V. This is due to the important lemma proved in 121:

Characteristic vectors of T that belong to distinct characteristic values are orthogonal. Thus if $i \neq j$, B_i and B_j are orthogonal, hence B is indeed an orthonormal basis for U. Since B consists of characteristic vectors of T, it satisfies all the requirements of Level 12 $\square$

In the elevator

We leave out the proofs of 111, 112 and 121, since Level 3 is the bottom level and the proofs appear in it in their standard (linear) form (see [HK]pp. 312-313). Note, however, that it is only in 111 and 121 that the main hypothesis of the theorem (namely that T is Hermitian) finally enters the proof. In contrast, 112 is true for operators on complex spaces in general.