The following example is taken from [Ler83].

Level 1

1 Theorem on canonical forms for Hermitian operators and matrices [Hal47]

Let V be a finite-dimensional complex inner-product space, and let T be a Hermitian operator on V. Then V has an orthonormal basis of characteristic vectors of T.

11 Let U be the subspace V spanned by all characteristic vectors of T. Then U = V, i.e.m the characteristic vectors of T span the whole space.

12 U has an orthonormal basis of characteristic vectors of T.

Proof. Cleary the two assertions 11 and 12 yield the conclusion of Theorem 1.

In the elevator

From here on the proof branches to two independent subproofs, rooted at 11 and 12. Note that while the presentation proceeds strictly top-down, it does not force a top-down *reading* of the proof. Some readers may prefer, for example, to read through the first branch (11) all the way to the bottom, then return to Level 2 and start on the second branch (12).

Level 2

11

Let U be the subspace V spanned by all characteristic vectors of T. Then U = V, i.e.m the characteristic vectors of T span the whole space.

111 U^{\perp} is T-invariant.

112 Every nonzero T-invariant subspace of V contains a characteristic vector of T.

Proof. To prove U=V we prove the equivalent statement $U^{\perp}=\{0\}$. This in turn will follow from 111 and 112. Since U^{\perp} cannot contain a characteristic vector of T (this would contradict $U \cap U^{\perp}=\{0\}$), we must have $U^{\perp}=\{0\}$, hence U=V.

12

U has an orthonormal basis of characteristic vectors of T.

121 Lemma Let T be a Hermitian operator. Then

- 1. All the (complex) characteristic values of T are real.
- 2. Characteristic vectors of T that belong to distinct characteristic values are orthogonal.

Proof. Let $\lambda_1, \ldots, \lambda_k$ be distinct characteristic values of T, and let U_1, \ldots, U_k be there respective characteristic spaces:

$$U_i = \{ v \in V \mid Tv = \lambda_i v \}.$$

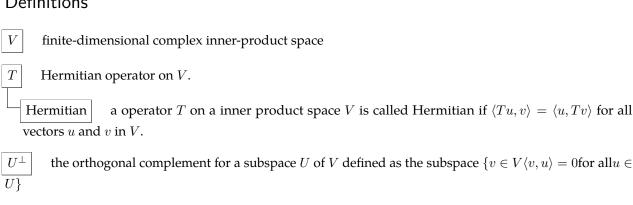
By the Gram-Schmidt orthogonalization process we can construct an orthonormal basis B_i to each subspace U_i , and it turns out that without any more car on our part, the set $B = B_1 \cup \cdots \cup B_k$ already forms an orthonormal basis for V. This is due to the important lemma proved in 121:

Characteristic vectors of T that belong to distinct characteristic values are orthogonal. Thus if $i \neq j$, B_i and B_j are orthogonal, hence B is indeed an orthonormal basis for U. Since B consists of characteristic vectors of T, it satisfies all the requirements of Level 12

In the elevator

We leave out the proofs of 111, 112 and 121, since Level 3 is the bottom level and the proofs appear in it in their standard (linear) form (see [HK]pp. 312-313). Note, however, that it is only in 111 and 121 that the main hypothesis of the theorem (namely that *T* is Hermitian) finally enters the proof. In contrast, 112 is true for operators on complex spaces in general.

Definitions



Bibliography

- [Hal47] Paul R Halmos. Finite dimensional vector spaces. Number 7. Princeton University Press, 1947.
 - [HK] Kenneth Hoffman and Ray Kunze. Linear algebra. 1971.
- [Ler83] U. Leron. Structuring mathematical proofs. *The American Mathematical Monthly*, 90(3):174–185, 1983.