The following example is taken from [Ler83].

Level 1

1 Theorem on canonical forms for Hermitian operators and matrices [Hal47]
Let V be a finite-dimensional complex inner-product space, and let T be a Hermitian operator on V. Then V has an orthonormal basis of charateristic vectors of T.

11 *Let U be the subspace V spanned by all characteristic vectors of T. Then $U = V$, i.e.m the characteristic vectors of T span the whole space.*

12 *U has an orthonormal basis of characteristic vectors of T.*

Proof. Cleary the two assertions 11 and 12 yield the conclusion of Theorem 1. □

In the elevator

From here on the proof branches to two independent subproofs, rooted at 11 and 12. Note that while the presentation proceeds strictly top-down, it does not force a top-down *reading* of the proof. Some readers may prefer, for example, to read through the first branch (11) all the way to the bottom, then return to Level 2 and start on the second branch (12).

Level 2

11
Let U be the subspace V spanned by all characteristic vectors of T. Then $U = V$, i.e.m the characteristic vectors of T span the whole space.

111 *$U^\perp$ is T-invariant.*

112 *Every nonzero T-invariant subspace of V contains a characteristic vector of T.*

Proof. To prove $U = V$ we prove the equivalent statement $U^\perp = \{0\}$. This in turn will follow from 111 and 112. Since $U^\perp$ cannot contain a characteristic vector of T (this would contradict $U \cap U^\perp = \{0\}$), we must have $U^\perp = \{0\}$, hence $U = V$. □

12
U has an orthonormal basis of characteristic vectors of T.

121 Lemma *Let T be a Hermitian operator. Then*

1. *All the (complex) characteristic values of T are real.*
2. *Characteristic vectors of T that belong to distinct characteristic values are orthogonal.*

Proof. Let $\lambda_1, \ldots \lambda_k$ be distinct characteristic values of T, and let $U_1, \ldots, U_k$ be there respective characteristic spaces:

$$U_i = \{v \in V \mid Tv = \lambda_i v\}\,.$$

By the Gram-Schmidt orthogonalization process we can construct an orthonormal basis B_i to each subspace U_i, and it turns out that without any more car on our part, the set $B = B_1 \cup \cdots \cup B_k$ already forms an orthonormal basis for V. This is due to the important lemma proved in 121:

Characteristic vectors of T that belong to distinct characteristic values are orthogonal. Thus if $i \neq j$, B_i and B_j are orthogonal, hence B is indeed an orthonormal basis for U. Since B consists of characteristic vectors of T, it satisfies all the requirements of Level 12 $\square$

In the elevator

We leave out the proofs of 111, 112 and 121, since Level 3 is the bottom level and the proofs appear in it in their standard (linear) form (see [HK]pp. 312-313). Note, however, that it is only in 111 and 121 that the main hypothesis of the theorem (namely that T is Hermitian) finally enters the proof. In contrast, 112 is true for operators on complex spaces in general.

Definitions

$\boxed{V}$ finite-dimensional complex inner-product space

$\boxed{T}$ Hermitian operator on V.

$\boxed{\text{Hermitian}}$ a operator T on a inner product space V is called Hermitian if $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all vectors u and v in V.

$\boxed{U^\perp}$ the orthogonal complement for a subspace U of V defined as the subspace $\{v \in V \langle v, u\rangle = 0 \text{ for all } u \in U\}$

Bibliography

[Hal47] Paul R Halmos. *Finite dimensional vector spaces*. Number 7. Princeton University Press, 1947.

[HK] Kenneth Hoffman and Ray Kunze. Linear algebra. 1971.

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