Symplectic excision

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1 Introduction

A couple of years ago, Alan Weinstein circulated the following question, which appeared in his paper with Christian Blohmann [2, Question 11.2]:

> Let M be a noncompact symplectic manifold, and let $R \subset M$ be the image of a proper embedding of the ray $[0, \infty)$. Is $M \setminus R$ symplectomorphic to M? Can a symplectomorphism between them be chosen to be the identity outside a prescribed neighbourhood U of R?

In his earlier paper [10], the second named author proved that such a symplectomorphism exists if the neighbourhood U is sufficiently big in the following sense: for some $\varepsilon > 0$, there exists a symplectic embedding of $B_0^{2n-2}(\varepsilon) \times \mathbb{R}^2$ into U that maps

$$R_0 \coloneqq \{0\} \times [0, \infty) \times \{0\}$$

onto R. Note that such a neighbourhood U has infinite volume. In this paper, we allow U to be arbitrary. This result is strictly stronger; for example, M can have finite volume. Moreover, we extend this result to more general closed subsets Z of symplectic manifolds M.

During our final preparations of this paper we learned of Bernd Stratmann's recent eprint [9]. Stratmann uses time-dependent flows to excise what he calls "parametrized rays": he excises a closed submanifold of the form $S = [0, \infty) \times Q$ on which the two-form ω_S comes from a two-form on Q. We use time-*independent* flows to excise similar sets. Moreover, we use time-independent flows to excise more general sets; for example, we excise from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ the Cantor brush $\{0\}^{2n-2} \times C \times [0,\infty)$ (see Example 5.2). Furthermore, using time-*dependent* flows, we excise even more general subsets, for example, a "ray with two horns", or, more generally, a "tree with a root at infinity" (see Example 5.5).

Definition 1.1. A subset Z of a symplectic manifold (M,ω) is *Hamiltonian excisable* if there is a Hamiltonian flow on M whose forward-trajectories that start in Z stay in Z as long as they're defined and whose time-one flow is a symplectomorphism of $M \setminus Z$ with M. It is *time-independent Hamiltonian excisable* if the Hamiltonian flow can be chosen to be time-independent. $\diamond$

Remark 1.2. If Z is Hamiltonian excisable from (M,ω) and is non-empty, then M is non-compact and Z is closed in M. More generally, for a topological manifold M and a non-empty subset Z of M, if $M \setminus Z$ is homeomorphic to M, then Z must be closed in M and M must be noncompact. $\diamond$

Remark 1.3. If Z is Hamiltonian excisable from (M,ω), then for any neighbourhood U of Z in M we can choose a Hamiltonian flow as in Definition 1.1 that restricts to the identity map on some neighbourhood of $M \setminus U$ in $M \setminus Z$. $\diamond$

We denote by X_F the Hamiltonian vector field X_F of a smooth function $F \colon M \to \mathbb{R}$ on a symplectic manifold (M,ω). We use the convention $X_F \lrcorner \omega = \mathrm{d}F$.

2 Removing a ray

2.1 Removing a ray from $\mathbb{R}^{2n}$

Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$, with the standard symplectic form $\omega_{\text{can}} \coloneqq \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n$, and, in it, consider the ray

$$R_0 \coloneqq \{0\}^{2n-2} \times [0, \infty) \times \{0\}.$$

Theorem 2.1. *R_0 is time-independent Hamiltonian excisable from $\mathbb{R}^{2n}$.*

Proof. Because there is a symplectomorphism from $(-1,1) \times \mathbb{R}$ to $\mathbb{R}^2$ that takes $[0,1) \times \{0\}$ to $[0,\infty) \times \{0\}$, (for example, take the cotangent lift of the diffeomorphism $t \mapsto t/(1-t)$ from $(-1,1)$ to $\mathbb{R}$,) it is enough to show that

$$R_1 \coloneqq \{0\}^{2n-2} \times [0,1) \times \{0\}$$

is time-independent Hamiltonian excisable from

$$M \coloneqq \mathbb{R}^{2n-2} \times (-1,1) \times \mathbb{R}.$$

We claim the following stronger result than is required. For any neighbourhood U of R_1 in M there exists a smooth function $F \colon M \to \mathbb{R}$, with Hamiltonian vector field X_F, whose flow domain $D \subset M \times \mathbb{R}$ is given by

$$D \cap (\{z\} \times \mathbb{R}) = \{z\} \times (S(z), T(z))$$

for all $z \in M$, such that

- F is supported in U;
- $T > 1$ on $M \setminus R_1$;
- $T \leqslant 1$ on R_1;
- $S < -1$ everywhere on M.

This claim completes the proof: the time-1 Hamiltonian flow of F is then a symplectomorphism from $M \setminus R_1$ to M.

We will now prove this claim. We write points of M as

$$z = (p; x_n, y_n) \quad \text{with} \quad p = (x_1, y_1, \ldots, x_{n-1}, y_{n-1}).$$

Let U be an arbitrary neighbourhood of R_1 in M. Let $\varepsilon > 0$ and let

$$h\colon [-\varepsilon, 1) \to \mathbb{R}_{>0}$$

be a smooth strictly decreasing function that converges to 0 at 1, such that the set $\{x_n \in [-\varepsilon, 1),\ |p|^2 + y_n^2 \leqslant h(x_n)\}$ is contained in U. Let

$$U_1 := \Big\{x_n \in (-\varepsilon, 1),\ |p|^2 + y_n^2 < h(x_n)\Big\}.$$

Then U_1 is contained in U, and

$$|p|^2 + y_n^2 \leqslant h(-\varepsilon) \quad \text{ on } \quad U_1.$$

Fix a smooth function

$$\chi\colon M \to [0, 1]$$

that is supported in U_1 and is equal to 1 in a neighbourhood of R_1. Assume that $\mathrm{d}\chi = 0$ wherever $\chi = 0$; this can be achieved, for instance, by replacing χ by $\rho \circ \chi$ where $\rho(s) = 3s^2 - 2s^3$.

Let

$$F(z) := \frac{1 - x_n^2}{|p|^2 + 1 - x_n^2}\chi(z) y_n.$$

Then $F\colon M \to \mathbb{R}$ is supported in U.

Fix $c > 0$. Because the function F is the product of y_n with a function that takes values in $[0, 1]$ and is supported in U_1, if $|F(z)| \geqslant c > 0$, then $z \in U_1$ and $|y_n| \geqslant c$. From the definition of U_1, this further implies that $x_n \in [-\varepsilon, 1)$ and $h(x_n) \geqslant c^2$. Because $h\colon [-\varepsilon, 1) \to \mathbb{R}_{>0}$ approaches 0, these inequalities imply that $x_n \in [-\varepsilon, b]$ for some $b \in [-\varepsilon, 1)$. Hence, the set $\{|F| \geqslant c\}$ is compact, as it is closed in M and contained in $\{x_n \in [-\varepsilon, b],\ |p|^2 + y_n^2 \leqslant h(-\varepsilon)\}$.

So

$$F|_{M \setminus F^{-1}(0)}\colon M \setminus F^{-1}(0) \to \mathbb{R} \setminus \{0\}$$

is a proper map. In particular, all the non-zero level sets of F are compact.

We will now calculate $S(z)$ and $T(z)$ explicitly for each $z \in M$.

On $\{y_n \neq 0\}$: At each point z with $y_n \neq 0$, if $F(z) \neq 0$, then $F^{-1}(F(z))$ is compact, and if $F(z) = 0$, then ($\chi(z) = 0$; by the choice of χ also $\mathrm{d}\chi|_z = 0$; and so) $X_F(z) = 0$. We conclude that X_F is complete on $\{y_n \neq 0\}$. So for any point z with $y_n \neq 0$ we have $S(z) = -\infty$ and $T(z) = \infty$.

On $\{y_n = 0\}$*:* At each point z with $y_n = 0$, we have

$$X_F(p, x_n, 0) = \frac{1 - x_n^2}{|p|^2 + 1 - x_n^2}\chi(p, x_n, 0)\partial_{x_n}.$$

Since X_F is proportional to ∂_{x_n} and vanishes for $x < -\varepsilon$, we have $S(z) = -\infty$ for any point z with $y_n = 0$. It remains to calculate $T(z)$ for such z.

On $\{y_n = 0\} \cap \{p \neq 0\}$*:* At each point z with $y_n = 0$ and $p \neq 0$, we have $|p|^2 > 0$. By the comparison

$$\frac{1 - x_n^2}{|p|^2 + 1 - x_n^2}\chi(p, x_n, 0) \leqslant \frac{1 - x_n^2}{|p|^2 + 1 - x_n^2}$$

and the completeness of the vector field $\frac{1-x_n^2}{b+1-x_n^2}\partial_{x_n}$ on $(-1, 1)$ for $b > 0$, we have $T(z) = \infty$.

On $\{y_n = 0\} \cap \{p = 0\}$*:* At each point with $y_n = 0$ and $p = 0$, which we write as $z = (0, x_n, 0)$, we have $X_F(0, x_n, 0) = \chi(0, x_n, 0)\partial_{x_n}$. Because $\chi(0, x_n, 0) = 1$ on the set $\{x_n \geqslant 0\}$, we have $T(z) \leqslant 1$ when $x_n \geqslant 0$ and $T(z) = 1$ when $x_n = 0$. Because the vector field is a positive multiple of ∂_{x_n}, the function $x_n \mapsto T(z)$ is strictly decreasing, and so $T(z) > 1$ whenever $x_n < 0$. So $T(z) \leqslant 1$ if and only if $x_n \geqslant 0$.

We have now shown that $T(z) \leqslant 1$ if and only if $z \in R_0$, and that $S(z) = -\infty$ for all $z \in M$. This justifies our claim, and our proof is complete. □

In contrast to Theorem 2.1, the analogous result of [10] only guarantees that, for any $\varepsilon > 0$, there is a symplectomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n} \setminus R_0$ that is the identity outside $\mathbb{B}_0^{2n-2}(\varepsilon) \times W(\varepsilon)$, where $W(\varepsilon) := \mathbb{B}_0^2(\varepsilon) \cup \{(x, y) \in \mathbb{R}^2 \mid x > \frac{\sqrt{2}\varepsilon}{2},\ x|y| < \frac{\varepsilon^2}{2}\}$. (This implies the result that we quoted in the introduction.) This result cannot be used to symplectically excise R from M if M has finite volume. In contrast, Theorem 2.1 allows us to remove a ray from *any* symplectic manifold, as we now show.

2.2 Applications

Corollary 2.2. *Let* (M, ω) *be a symplectic manifold, and let* $\gamma\colon [0, \infty) \to M$ *be a proper embedding with image* R*. Then* R *is time-independent Hamiltonian excisable from* M*.*

Proof. By Theorem 2.1, there exists a smooth function $F^0\colon \mathbb{R}^{2n} \to \mathbb{R}$ with Hamiltonian flow whose forward-trajectories that start in R_0 stay in R_0 and whose time-one flow is a symplectomorphism of $\mathbb{R}^{2n} \setminus R_0$ with $\mathbb{R}^{2n}$. By the Weinstein symplectic tubular neighbourhood theorem, there exists a symplectic embedding $\psi\colon (U_0, \omega_{\mathrm{can}}) \to (M, \omega)$ from an open neighbourhood U_0 of R_0 to an open neighbourhood U of R in M, sending R_0 to R. Let $\chi\colon M \to [0,1]$ be a smooth function supported in U that equals 1 in a neighbourhood of R in M. Then the function $F \coloneqq \chi \cdot (F^0 \circ \psi^{-1})\colon M \to \mathbb{R}$ is smooth, and it generates a Hamiltonian flow whose forward-trajectories that start in R stay in R and whose time-one map is a symplectomorphism of $M \setminus R$ with M. $\square$

Lemma 2.3. *Any exact* 2*-form on a smooth manifold has a primitive whose zero set is discrete.*

Proof. Let $\omega = \mathrm{d}\theta$ be an exact 2-form on M^m. Let $\phi = (x_1, \ldots, x_n)\colon M \to \mathbb{R}^n$ be a smooth embedding. The function

$$F\colon M \times \mathbb{R}^n \to T^*M,$$
$$F(x,s) = \theta(x) + \sum_{i=1}^{n} s_i \, \mathrm{d}_x x_i,$$

where $s = (x_1, \ldots, x_n)$, is transverse to the zero section 0_{T^*M} of T^*M. By the Transversality Theorem (see, for instance, [5]), we deduce that $F(\cdot, s)\colon M \to T^*M$ is transverse to 0_{T^*M} for almost every $s \in \mathbb{R}^n$. Fix such an s, and let $\rho \coloneqq \sum_{i=1}^{n} s_i x_i\colon M \to \mathbb{R}$. Then the zero set of $\beta \coloneqq \theta + \mathrm{d}\rho = F(\cdot, s) \in \Omega^1(M)$ is a discrete set of points in M, and we have $\mathrm{d}\beta = \mathrm{d}\theta = \omega$. $\square$

Theorem 2.4. *Any exact symplectic manifold has a nowhere vanishing primitive (a Liouville form) of its symplectic form.*

Proof. Let $(M, \omega = \mathrm{d}\theta)$ be an exact symplectic manifold. By Lemma 2.3 we choose a $\rho \in \Omega^1(M)$ such that the zeroes of $\beta = \theta + \mathrm{d}\rho$ are isolated. The existence of such a ρ is claimed in [2] and proven differently in [8].

We construct an exhaustion of M by a sequence of compact subsets $(K_j)_{j=1}^{\infty}$ such that for each $j \in \mathbb{N}$, any point in $K_{j+1} \setminus K_j$ can be joined by a smooth path in $M \setminus K_j$ to a point in $M \setminus K_{j+1}$. This can be achieved by taking the unions of regular superlevelsets of an exhausion function for M with the bounded connected components of their complements, as was done in [4, 6, 7, 8].

Let $(z_i)_{i \geqslant 1}$, be the (finite or countable) set of zeroes of β. We claim that for each i there is a properly embedded ray R_i starting with z_i such that the rays are pairwise disjoint and each point of M has a neighbourhood that meets only finitely many of the rays. Our construction is step-by-step inside $M \setminus K_j$ for $j = 1, 2, \dots$. For any $j \in \mathbb{N}$, we draw a smooth path in $M \setminus K_j$ from each z_i or endpoint of a previous path that is contained in $K_{j+1} \setminus K_j$ to a point in $M \setminus K_{j+1}$, such that all the new paths are disjoint from each other, from previous paths, and from all the points z_i. Moreover, we arrange the path to have non-zero velocity, and whenever we extend an earlier path, we arrange that the concatenated path will be smooth.

In this way, we obtain propely embedded pairwise disjoint rays R_i emanating from the points z_i, such that each point in M has a neighbourhood that meets only finitely may of the rays.

Let $(U_i)_{i \geqslant 1}$ be pairwise disjoint open neighbourhoods of $(R_i)_{i \geqslant 1}$, and let $U = \bigcup_{i \geqslant 1} U_i$. By Corollary 2.2, for each i, there is a symplectic excision φ_i of R_i from M supported in U_i. Let $\varphi \colon M \setminus R \to M$ be the composition of $(\varphi_i)_{i \geqslant 1}$, which is supported in U. Let $\alpha := (\varphi^{-1})^*(\beta|_{M \setminus R})$. Then we have

$$\mathrm{d}\alpha = \mathrm{d}((\varphi^{-1})^*(\beta|_{M \setminus R})) = (\varphi^{-1})^*(\omega|_{M \setminus R}) = \omega.$$

Moreover, α has no zeroes, because β has no zeroes in $M \setminus R$. $\square$

Stratmann [8] has a more direct proof of Theorem 2.4.

3 Extension of null vector fields

In this section, we construct symplectic vector fields on symplectic manifolds that extend null vector fields on submanifolds.

Definition 3.1. A *presymplectic form* is a closed two-form. A *presymplectic manifold* is a manifold S equipped with a closed two-form ω_S. On a presymplectic manifold (S, ω_S), a *null vector field* is a vector field v that is everywhere in the null space $(TS)^{\omega_S}$ of the presymplectic form. $\diamond$

On a presymplectic manifold (S, ω_S), every null vector field v is *presymplectic*, in the sense that $\mathcal{L}_v \omega_S = 0$; moreover, it is *Hamiltonian* in the sense that $v \lrcorner \omega_S$ is exact.

A *smooth neighbourhood retraction* for a submanifold N of a manifold M is a smooth map $\pi \colon W \to N$ from a neighbourhood W of N that restricts to the identity map on N.

A function on a topological space *vanishes at infinity* if all of the positive epigraphs of its absolute value are compact. This is equivalent to being proper away from the zero locus, as a map to $\mathbb{R} \setminus \{0\}$. Any smooth manifold admits a positive smooth function vanishing at infinity, for instance, as the reciprocal of a positive exhaustion function.

Theorem 3.2. *Let (M, ω) be a symplectic manifold with a codimension one submanifold (N, ω_N). Let Z be a subset of N that is closed in M. Let v be a null vector field on (N, ω_N) that is non-vanishing on Z. Then there are smooth functions $\chi \colon M \to [0, 1]$ and $F \colon M \to \mathbb{R}$ such that $\chi = 1$ on Z, $X_F|_N = \chi v$, and every trajectory of X_F that starts on $M \setminus N$ is defined for all times.*

Proof. Because v is a null vector field, $\alpha := v \lrcorner \omega$ is a section of the conormal bundle $T^*_{M/N}$. We view α as a function on the normal bundle $T_{M/N} := TM|_N/TN$ that is linear on each fibre. By the tubular neighbourhood theorem, there exist an open neighbourhood W of N in M and a diffeomorphism

$$\epsilon \colon T_{M/N} \to W$$

that identifies the zero section with N and whose differential along the zero section induces the identity map on the normal bundle. Let $H := \alpha \circ \epsilon^{-1} \colon W \to \mathbb{R}$. Then $\mathrm{d}H|_N = \alpha$, and $X_H|_N = v$.

Let $\pi\colon W \to N$ be the composition of $\epsilon^{-1}\colon W \to T_{M/N}$ with projection map $T_{M/N} \to N$. By the construction of H, and since N has codimension 1,

$$\text{for any } z \in W \setminus N, \quad H(z) = 0 \text{ if and only if } v(\pi(z)) = 0. \tag{3.1}$$

Let H_1 be a positive smooth function on M vanishing at infinity. Because v is non-vanishing on Z and H vanishes on N, the set

$$U_1 := \{z \in W \mid v(\pi(z)) \neq 0 \text{ and } H(z) < H_1(z)\}$$

is an open neighbourhood of Z in M. Let $\chi\colon M \to [0,1]$ be a smooth function that is supported in U_1 and is equal to 1 on Z. Moreover, suppose that $\mathrm{d}\chi = 0$ wherever $\chi = 0$; this can be achieved, for instance, by replacing χ by $\rho \circ \chi$ where $\rho(s) = 3s^2 - 2s^3$.

The function

$$F \coloneqq \chi H\colon M \to \mathbb{R}$$

satisfies $X_F|_N = \chi X_H|_N = \chi v$ (where the first equality is because H vanishes on N). We will now show that, on the subset $M \setminus N$, we have $dF = 0$ whenever $F = 0$.

Suppose that $z \in M \setminus N$ and $F(z) = 0$. We claim that $\chi(z) = 0$. Indeed, if $z \notin U_1$, then $\chi(z) = 0$ because the support of χ is contained in U_1. And if $z \in U_1$, then $z \in W \setminus N$ and $v(\pi(z)) \neq 0$ by the choice of U_1; by (3.1) we have $H(z) \neq 0$; since $(\chi H)(z) = F(z) = 0$, we have $\chi(z) = 0$. This proves our claim that $\chi(z) = 0$. By the choice of χ, we then have that $d\chi|_z = 0$, and so $dF|_z = 0$.

Since

$$|F| = |\chi H| \leqslant |\mathbb{1}_{U_1} H| < H_1,$$

the function F vanishes at infinity, since H_1 does. Then F is proper away from the zero locus, i.e., $F|_{\{F\neq 0\}}\colon \{F \neq 0\} \to \mathbb{R} \setminus \{0\}$ is proper. In particular, all the non-zero level sets of F are compact. Then for any $z \in M \setminus N$, either $F(z) \neq 0$ and $F^{-1}(F(z))$ is compact, or $F(z) = 0$ and $X_F(z)$ vanishes. This shows that the restriction of X_F to $M \setminus N$ is complete. $\square$

4 Hamiltonian excision along null vector fields

Throughout this section, let

$$I := (-1, 1).$$

We recall the solution theory of an automonous ordinary differential equation of first order on I. Consider a non-negative smooth function $v\colon I \to [0, \infty)$ and the ordinary differential equation on I

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v(x). \tag{4.1}$$

Let $\gamma = \gamma_{t_0,x_0}$ be the maximal solution to (4.1) with the initial condition (t_0, x_0). By Barrow's formula [1],

$$\begin{cases} t - t_0 = \displaystyle\int_{x_0}^{\gamma(t)} \frac{\mathrm{d}\xi}{v(\xi)} & \text{if } \ v(x_0) > 0\,; \\ \gamma(t) = x_0 & \text{if } \ v(x_0) = 0\,. \end{cases} \tag{4.2}$$

The flow of v,

$$\Phi_v\colon D_v \to I, \quad \text{given by} \quad \Phi_v(t, x) := \gamma_{0,x}(t),$$

is defined on the *flow domain* of v, which is an open subset D_v of $\mathbb{R} \times I$ of the form

$$D_v = \{(t, x) \in \mathbb{R} \times I \mid S_v(x) < t < T_v(x)\}$$

for some upper semi-continuous function

$$S_v\colon I \to [-\infty, 0)$$

and lower semi-continuous function

$$T_v\colon I \to (0, \infty];$$

we call these functions, respectively, the *backward time function* and the *forward time function*.

Suppose now that $v(x) = 0$ for x close to the left endpoint -1 of I. Then, for any $x \in I$,

$$S_v(x) = -\infty \tag{4.3}$$

and

$$T_v(x) = \begin{cases} \displaystyle\int_x^1 \frac{\mathrm{d}\xi}{v(\xi)} \in (0, \infty] & \text{if } \ v(\xi) > 0 \ \text{ for all } \ \xi \in [x, 1)\,; \\ \infty & \text{if } \ v(\xi) = 0 \ \text{ for some } \ \xi \in [x, 1)\,. \end{cases} \tag{4.4}$$

Now we consider a model with parameters.

Recall that $I = (-1, 1)$, and define $u\colon I \times [-1, 1] \times [0, 1] \times I \to [0, \infty)$ by

$$u(a, b, c; x) \coloneqq \chi_a(x)(1 - b)\frac{1 - x^2}{1 - x^2 + c}, \tag{4.5}$$

where $\chi_a\colon I \to [0, 1]$, for $a \in I$, is the non-decreasing smooth function

$$\chi_a(x) \coloneqq \begin{cases} 0 & \text{if } \ -1 < x \leqslant \dfrac{a-1}{2}; \\ \dfrac{\exp(-\frac{1}{x-(a-1)/2})}{\exp(-\frac{1}{x-(a-1)/2}) + \exp(-\frac{1}{a-x})} & \text{if } \ \dfrac{a-1}{2} \leqslant x \leqslant a; \\ 1 & \text{if } \ a \leqslant x < 1. \end{cases} \tag{4.6}$$

Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}\gamma}{\mathrm{d}t} = u(a, b, c; \gamma), \\ \gamma(0) = x\,; \end{cases} \tag{4.7}$$

let $\gamma = \gamma_{a,b,c;x}\colon (S_u(a, b, c; x), T_u(a, b, c; x)) \to I$ be its maximal solution curve. The flow of $u\frac{\partial}{\partial x}$,

$$\Phi_u\colon D_u \to I, \quad \Phi_u(a, b, c; t, x) \coloneqq \gamma_{a,b,c;x}(t),$$

is defined on

$$D_u = \{(a, b, c, t, x) \in I \times [-1, 1] \times [0, 1] \times \mathbb{R} \times I \mid S_u(a, b, c; x) < t < T_u(a, b, c; x)\}.$$

We calculate D_u explicitly. The backward time is always $-\infty$:

$$S_u(a, b, c; x) = -\infty \text{ for all } (a, b, c; x). \tag{4.8}$$

If $x \leqslant \frac{a-1}{2}$ or $b = 1$, the forward time $T_u(a, b, c; x)$ is ∞. Otherwise, $\frac{a-1}{2} < x < 1$ and $b < 1$, and we have

$$T_u(a, b, c; x) = \int_x^1 \frac{\mathrm{d}\xi}{u(a, b, c; \xi)} = \int_x^1 \frac{1}{(1 - b)\chi_a(\xi)}\left(1 + \frac{c}{1 - \xi^2}\right)\mathrm{d}\xi.$$

If $a \leqslant x < 1$, then

$$T_u(a, b, c; x) = \left[\frac{\xi}{1 - b} + \frac{c}{2(1 - b)}\ln\left|\frac{1 + \xi}{1 - \xi}\right|\right]_x^1 = \begin{cases} \dfrac{1 - x}{1 - b}, & c = 0; \\ \infty, & c > 0. \end{cases} \tag{4.9}$$

If $\frac{a-1}{2} < x < a$ then

$$T_u(a,b,c;x) = \begin{cases} \dfrac{1-a}{1-b} + \displaystyle\int_x^a \frac{1+\exp(\frac{1}{\xi-(a-1)/2} - \frac{1}{a-\xi})}{1-b}\,\mathrm{d}\xi, & c=0; \\ \infty, & c>0. \end{cases} \tag{4.10}$$

Define the function $\mu\colon I \times [-1,1) \to [-1,1)$ by

$$\int_{\mu(a,b)}^1 \frac{1}{(1-b)\chi_a(\xi)}\,\mathrm{d}\xi = 1.$$

By (4.9) and (4.10) we conclude that μ is a smooth function increasing in b such that $\mu(a,b) = b$ for $b \geqslant a$ and $\mu(a,b)$ is greater than both b and $\frac{a-1}{2}$ for $b < a$. Then, for all $(a,b,c;x) \in I \times [-1,1] \times [0,1] \times I$,

$$T_u(a,b,c;x) \leqslant 1 \quad \text{if and only if} \quad b<1,\ c=0, \ \text{ and } \mu(a,b) \leqslant x. \tag{4.11}$$

Lemma 4.1. *Let B be a smooth manifold, let $C \subseteq B$ be a closed subset, and let $\lambda\colon C \to (-1,1]$ be a smooth function. Then there exists a vector field on $B \times I$, of the form $v(p,x)\frac{\partial}{\partial x}$ with $v(p,x) \geqslant 0$, whose forward time function T_v and backward time function S_v satisfy*

$$T_v(p,x) \leqslant 1 \quad \textit{if and only if} \quad p \in C \textit{ and } \lambda(p) \leqslant x < 1$$

and

$$S_v(p,x) = -\infty \quad \textit{for all} \quad (p,x) \in B \times I.$$

Proof. By the definition of a smooth function on a subset of a manifold, and because the subset C of B is closed, λ is the restriction to C of a smooth function $b\colon B \to (-1,1]$. Let $a\colon B \to I$ be a smooth function such that $a(p) \leqslant b(p)$ for all p; for example, we can take $a(p) = \frac{-1+b(p)}{2}$. Because C is closed, it is the zero locus of a smooth function $c\colon B \to [0,1]$ (see for instance [3]). We set

$$v(p;x) := u(a(p), b(p), c(p); x)$$

with the function $u(a,b,c;x)$ of (4.5). Then for all $(p,x) \in B \times I$, we have $T_v(p,x) = T_u(a(p),b(p),c(p);x)$ and $S_v(p,x) = S_u(a(p),b(p),c(p);x)$. By (4.8), $S_v(p,x) = -\infty$ for all (p,x). By (4.11), for all $(p,x) \in B \times I$,

$$T_v(p,x) \leqslant 1 \quad \text{if and only if} \quad b(p) < 1,\ c(p) = 0, \ \text{ and } \ \mu(a(p),b(p)) \leqslant x.$$

Because $a(p) \leqslant b(p)$, we have $\mu(a(p), b(p)) = b(p)$; the condition $c(p) = 0$ means that $p \in C$, which implies that $b(p) = \lambda(p)$; because $x \in I$, we have $x < 1$; so

$$T_v(p, x) \leqslant 1 \quad \text{if and only if} \quad p \in C \ \text{ and } \ \lambda(p) \leqslant x < 1.$$

□

Corollary 4.2. *Let (B, ω_B) be a presymplectic manifold, and equip $N := B \times I$ with the presymplectic structure $\omega_N := \omega_B \oplus 0$, namely, the pullback of ω_B by the projection to B. Let $C \subseteq B$ be a closed subset, let $\lambda\colon C \to (-1, 1]$ be a smooth function, and let Z be the epigraph of λ in N:*

$$Z := \{(p, x) \in B \times I \mid p \in C \text{ and } x \geqslant \lambda(p)\}.$$

Then the vector field of Lemma 4.1 is a null vector field, and the time-1 map of the flow that it generates is a presymplectomorphism

$$N \setminus Z \to N.$$

Proof. Let v be the vector field of Lemma 4.1, and let $(\psi_t)_{t \in \mathbb{R}}$ be the flow that v generates. Each ψ_t is a diffeomorphism between open subsets of N. The domain of ψ_t is $\{z \in N \mid T_v(z) > t\}$, and the range of ψ_t is $\{z \in N \mid S_v(z) < -t\}$. Because $S_v(z) = -\infty$ for all z, the range of ψ_t is all of N. Because $T_v(z) \leqslant 1$ iff $z \in Z$, the domain of ψ_1 is $N \setminus Z$. The vector field v is a null vector field because it is a multiple of $\frac{\partial}{\partial x}$ (where we write coordinates on $B \times I$ as (p, x)) and ω_N is the pullback of a two-form on B. It follows that each ψ_t is a presymplectomorphism. So ψ_1 is a presymplectomorphism from $N \setminus Z$ to N. □

5 Symplectic excision

5.1 Time-independent Hamiltonian flows

We prove a symplectic excision result for closed subsets of codimension at least one. As before, let $I = (-1, 1)$.

Theorem 5.1. *Let (M, ω) be a $2n$ dimensional symplectic manifold, let (B, ω_B) be a $2n-2$ dimensional symplectic manifold, and let $\psi\colon B \times I \to M$ an embedding such that $\psi^*\omega = \omega_B \oplus 0$. Let $\lambda\colon C \to (-1, 1]$ be a smooth function on a closed subset C of B. Let Z be the image under ψ of the epigraph $\{(p, x) \in B \times I \mid x \in C \text{ and } x \geqslant \lambda(p)\}$. Suppose that Z is closed in M. Then Z is time-independent Hamiltonian excisable from (M, ω).*

Proof. We use ψ to identify $N := B \times I$ with its image in M; we write ω_N for the pullback of ω to N, which is equal to $\omega_B \oplus 0$.

Let $v(p, x)$ be as in Lemma 4.1, so that the forward time function T_v and backward time function S_v of the vector field $v(p, x)\frac{\partial}{\partial x}$ on N satisfy

$$T_v(z) \leqslant 1 \quad \text{if and only if} \quad z \in Z$$

and

$$S_v(z) = -\infty \quad \text{for all} \quad z \in N.$$

Note that v is a null vector field on (N, ω_N) that is non-vanishing on Z. By Theorem 3.2, there are smooth functions $\chi\colon M \to [0, 1]$ and $F\colon M \to \mathbb{R}$, such that $\chi = 1$ on Z, $X_F|_N = \chi v$, and every trajectory of X_F that starts on $M \setminus N$ is defined for all times.

We now analyse the dynamical behavior of X_F. Let S_F and T_F be its forward and backward time functions. By the choice of F, for all $z \in M \setminus N$ we have $S_M(z) = -\infty$ and $T_M(z) = \infty$. Because the restriction of X_F to N is a multiple of v by a function with values in $[0, 1]$, for all $z \in N$ we have $S_F(z) \leqslant S_v(z)$ and $T_F(z) \geqslant T_v(z)$. Finally, for all $z \in Z$ we have $T_v(z) = T_F(z)$, because $X_F|_Z = v$ and because Z is invariant under the forward flow of v and is closed in M. We conclude that

$$T_F(z) \leqslant 1 \quad \text{if and only if} \quad z \in Z$$

and

$$S_F(z) = -\infty \quad \text{for all} \quad z \in M.$$

The time-1 map of the flow generated by X_F is then a symplectomorphism from $M \setminus Z$ to M. $\square$

Theorem 2.1, about excising the ray $\{0\}^{2n-2} \times [0,\infty) \times \{0\}$ from $(M,\omega) = (\mathbb{R}^{2n}, dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n)$, follows from Theorem 5.1. To see this, take $N \subset M$ to be given by the vanishing of the y_n coordinate, take $(B, \omega_B) = (\mathbb{R}^{2n-2}, dx_1 \wedge dy_1 + \ldots + dx_{n-1} \wedge dy_{n-1})$, identify $B \times I$ with N through a diffeomorphism $I \to \mathbb{R}$ in the x_n coordinate, and take $C = \{0\}^{2n-2}$ and $\lambda = 0$.

It is interesting to note that we can excise subsets with complicated topology:

Example 5.2. The Cantor brush $Z := \{0\}^{2n-2} \times C \times [0,\infty)$, where C is the Cantor set, is Hamiltonian excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$; see Figure 5.1. To see this, note that there is a closed subset C' of $\mathbb{R}$ and a symplectomorphism of $\mathbb{R} \times (-1,1)$ with $\mathbb{R}^2$ that takes $C' \times [0,1)$ to $C \times [0,\infty)$, and apply Theorem 5.1 with the function $\lambda = 0$ on C'.

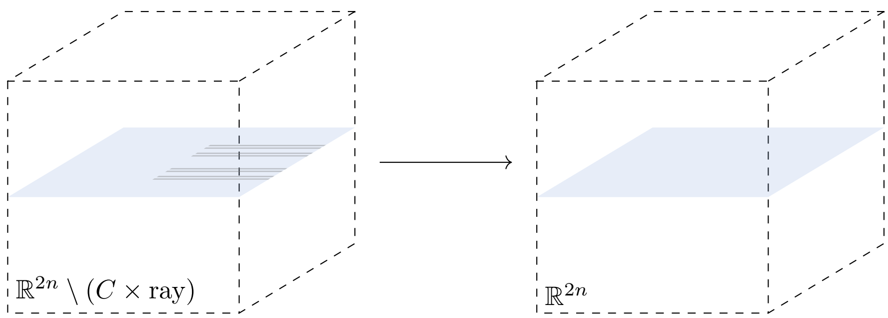

Figure 5.1: Removing the Cantor brush.

5.2 A remark on time-independent excisions

Throughout this paper, as in [10], in order to symplectically excise Z from (M,ω), we find a Hamiltonian on M, and we excise the set of points that escape to infinity in time $\leqslant 1$. Arranging the flow to be supported in an arbitrary neighbourhood of the set Z is non-trivial. The dynamics of the localized flow are complicated:

Lemma 5.3. *Let (M,ω) be a symplectic manifold with a smooth function $F\colon M \to \mathbb{R}$ supported in U. If the periodic orbits of X_F are not dense, then there is a symplectic embedding $\psi\colon (B^{2n-2}, \omega_{\mathrm{can}}) \times (\mathbb{R}^2, \omega_{\mathrm{can}}) \to (U, \omega)$.*

Proof. Since the periodic orbits of $w := X_F$ is not dense, there is a ball $B^{2n}(\varepsilon)$ for some $\varepsilon > 0$ and a symplectic embedding

$$\psi_0\colon (B^{2n}(\varepsilon), \omega_{\mathrm{can}}) \hookrightarrow (U, \omega)$$

whose image intersects no periodic orbit of w. Let B be the intersection of $B^{2n}(\varepsilon)$ with a hyperplane S so that $\psi_0(B)$ is transversal to the flow of w. Denoting by ν a normal vector to S we extend ψ_0 to a symplectic embedding

$$\psi_1\colon (B + \mathbb{R}\nu, \omega_{\text{can}}) \hookrightarrow (U, \omega)$$

in the way that ν and w are ψ_1-related. Combining some known facts in symplectic geometry, we obtain a chain of symplectic embeddings and symplectomorphisms

$$\psi\colon (B^{2n-2}, \omega_{\text{can}}) \times (\mathbb{R}^2, \omega_{\text{can}}) \hookrightarrow (B^{2n-1}(\varepsilon) \times \mathbb{R}, \omega_{\text{can}}) \simeq (B + \mathbb{R}\nu, \omega_{\text{can}}) \xhookrightarrow{\psi_1} (U, \omega).$$

□

Corollary 5.4. *Let (M, ω) be a symplectic manifold and let $\gamma\colon [0, \infty) \to M$ be a proper embedding with image R. If U is an open neighbourhood of R in M that admits no symplectically embedded $(B^{2n-2}, \omega_{\text{can}}) \times (\mathbb{R}^2, \omega_{\text{can}})$, then there is no symplectic excision of R from (M, ω) supported in U realized as the time-1 map φ of the Hamiltonian flow of any $F\colon M \to \mathbb{R}$, unless the periodic orbits of X_F are dense.*

5.3 Excision in stages

Until now we only considered time-independent Hamiltonians. By iterating the excision procedure, that is, by comosing the flow maps of a sequence of such Hamiltonians, we can excise more complicated subsets. This gives us a glimps into what we could achieve with time-dependent Hamiltonians. Here we discuss two examples.

Recall that an *unrooted tree* is a connected directed graph with no cycles. We define a *half-open unrooted tree* as a tree with one of its node removed. A ray and a ray with two horns are examples of unrooted trees, while a singleton, a line segment, and a double Y are examples of half-open unrooted trees.

Example 5.5. Any properly embedded finite half-open unrooted tree T in a symplectic manifold (M, ω) is symplectically neighbourhood excisable from it.

Proof. We prove by induction on the size of the tree. Let U be an open neighbourhood of T in M. Fix a leaf z of T and let L_z be the branch connecting z that is closed at z and open at the other end. Let $U_z \subseteq U$ be an open neighbourhood of z in $M \setminus (T \setminus L_z)$. By Corollary 2.2 we obtain a symplectic excision

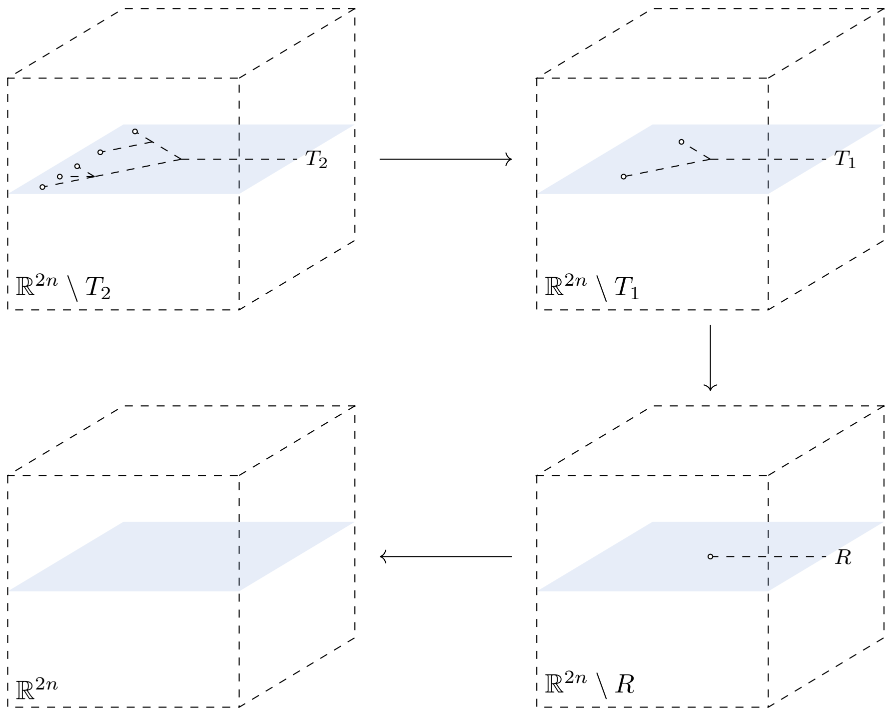

Figure 5.2: Removing a tree.

$\varphi_z \colon M \setminus T \to M \setminus (T \setminus L_z)$ of L_z from $(M \setminus (T \setminus L_z), \omega)$ supported in U_z. If we know that $T \setminus L_z$ is symplectically neighbourhood excisable from (M, ω) by the induction hypothesis then so is T. □

Another interesting question is that, given a compact subset K of a symplectic manifold (M, ω), is $(M \setminus K, \omega)$ symplectomorphic to M with several punctured points?

Example 5.6. Let (M, ω) be a symplectic manifold with a compact connected embedded unrooted tree T. Then for any open neighbourhood U of T in M there is a symplectomorphism $\varphi \colon (M \setminus T, \omega) \to (M \setminus \{z_0\}, \omega)$ supported in U for any $z_0 \in T$.

Proof. For any $z_0 \in T$, note that $T \setminus \{z_0\}$ is the disjoint union of finitely many half-open unrooted trees T_j, $j \in \{1, \ldots, k\}$ for some $k \in \mathbb{N}$, each of which is properly embedded in $M \setminus \{z_0\}$. Applying Example 5.5 k times to each of T_i in $(M \setminus \{z_0\}, \omega)$ we obtain the symplectic neighbourhood excisability of $\bigsqcup_{j=1}^{k} T_j$ from

$(M \setminus \{z_0\}, \omega)$. In other words, for any open neighbourhood U of T in M there is a symplectomorphism $\varphi\colon (M \setminus T, \omega) \to (M \setminus \{z_0\}, \omega)$ supported in U. $\square$

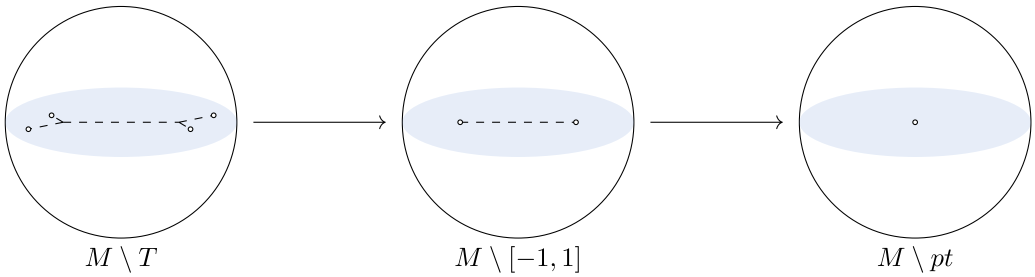

Figure 5.3: Retracting a hole.

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