Symplectic excision

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1 Introduction

A couple of years ago, Alan Weinstein circulated the following question, which he and Christian Blohmann put in writing in their paper [2, Question 11.2].

> Let M be a noncompact symplectic manifold, and let $R \subset M$ be the image of a proper embedding of the ray $[0, \infty)$. Is $M \setminus R$ symplectomorphic to M? Can a symplectomorphism between them be chosen to be the identity outside a prescribed neighbourhood U of R?

In his earlier paper [11], the second named author proved that such a symplectomorphism exists if the neighbourhood U is sufficiently big in the following sense: for some $\varepsilon > 0$, there exists a symplectic embedding of $B_0^{2n-2}(\varepsilon) \times \mathbb{R}^2$ into U that maps

$$R_0 \coloneqq \{0\} \times [0, \infty) \times \{0\}$$

onto R. Note that such a neighbourhood U has infinite volume. In this paper, we allow U to be arbitrary. This result is strictly stronger; for example, M can have finite volume.

Moreover, we extend this result to more general closed subsets Z of M. Our method is a novel variant of the symplectic isotopy extension theorem: we assume that there is a submanifold N of M that contains Z and a Hamiltonian flow $(\psi_t)_t$ of N that sends all the points of Z to infinity in time $\leqslant 1$ and whose time-1 flow is well defined on $N \setminus Z$, and we extend it to a Hamiltonian flow on M whose time-1 flow is well defined on $M \setminus Z$. The subtle point here is to ensure that the time-1 flow is well defined on $M \setminus Z$.

During our final preparations of this paper we learned of Bernd Stratmann's recent eprint [10], which achieves similar results. However, they excide the closed subset Z by time-dependent Hamiltonians instead of time-independent ones in the present paper and their result does not apply to the Cantor brush $C \times [0,\infty) \times \{0\}^{2n-2}$, where C is the Cantor set, in $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ as in Example 5.1.

Definition 1.1. Let (M,ω) be a symplectic manifold, and let Z be a closed subset of M. A *symplectic excision* of Z from M is a symplectomorphism $\varphi\colon M \setminus Z \to M$. The symplectic excision φ is *supported* in a neighbourhood U of Z if it restricts to the identity map on some neighbourhood of $M \setminus U$ in $M \setminus Z$. (Note that φ is not defined on all of U; it is defined only on $U \setminus Z$.) We similarly define *presymplectic excision* for a closed subset of a presymplectic manifold (meaning a manifold equipped with a closed two-form), *smooth excision* for a closed subset of a manifold, and *topological excision* for a closed subset of a topological manifold. Finally, a closed subset Z of a symplectic manifold M is *symplectically neighbourhood excisable* if for every neighbourhood U of Z in M there is a symplectic excision of Z from M that is supported in U. ◇

For a topological manifold M and a non-empty subset Z of M, if $M \setminus Z$ is homeomorphic to M, then Z must be closed in M and M must be noncompact.

2 Removing a ray

2.1 Removing a ray from $\mathbb{R}^{2n}$

In the present paper we use the convention $X_F \lrcorner\, \omega = \mathrm{d}F$. Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$, with the standard symplectic form $\omega_{\mathrm{can}} \coloneqq \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n$, and, in it, consider the ray

$$R_0 \coloneqq \{0\}^{2n-2} \times [0,\infty) \times \{0\}.$$

Theorem 2.1. *R_0 is symplectically neighbourhood excisable from $\mathbb{R}^{2n}$.*

Proof. Because there is a symplectomorphism from $(-1,1) \times \mathbb{R}$ to $\mathbb{R}^2$ that takes $[0,1) \times \{0\}$ to $[0,\infty) \times \{0\}$, (for example, take the cotangent lift of the diffeomorphism $t \mapsto t/(1-t)$ from $(-1,1)$ to $\mathbb{R}$), it is enough to show that

$$R_1 \coloneqq \{0\}^{2n-2} \times [0,1) \times \{0\}$$

is symplectically neighbourhood excisable from

$$M \coloneqq \mathbb{R}^{2n-2} \times (-1,1) \times \mathbb{R}.$$

We claim the following stronger result than is required. For any neighbourhood U of R_1 in M there exists a smooth function $F\colon M \to \mathbb{R}$, with Hamiltonian vector field X_F, whose flow domain $D \subset M \times \mathbb{R}$ is given by

$$D \cap (\{z\} \times \mathbb{R}) = \{z\} \times (S_z, T_z)$$

for all $z \in M$, such that

- F is supported in U;
- $T_z > 1$ in $M \setminus R_1$;
- $T_z \leqslant 1$ on R_1;
- $S_z < -1$ in M.

This claim completes the proof: the time-1 Hamiltonian flow of F is then a symplectomorphism from $M \setminus R_1$ to M.

We will now prove this claim. Write

$$p = (x_1, y_1, \dots, x_{n-1}, y_{n-1}).$$

Let U be an arbitrary neighbourhood of R_1 in M. Let $\varepsilon > 0$ and let

$$h\colon [-\varepsilon, 1) \to \mathbb{R}_{>0}$$

be a smooth strictly decreasing function that converges to 0 at 1, such that the set $\{x_n \in [-\varepsilon, 1),\ |p|^2 + y_n^2 \leqslant h(x_n)\}$ is contained in U. Let

$$U_1 := \Big\{x_n \in (-\varepsilon, 1),\ |p|^2 + y_n^2 < h(x_n)\Big\},$$

so that U_1 is contained in U, and

$$|p|^2 + y_n^2 \leqslant h(-\varepsilon) \quad \text{on} \quad U_1.$$

Fix a smooth function

$$\chi\colon M \to [0, 1]$$

that is supported in U_1 and is equal to 1 in a neighbourhood of R_1. Assume that $\mathrm{d}\chi = 0$ wherever $\chi = 0$; this can be achieved, for instance, by replacing χ by $\rho \circ \chi$ where $\rho(s) = 3s^2 - 2s^3$.

Let

$$F(z) := \frac{1 - x_n^2}{|p|^2 + 1 - x_n^2} \chi(z) y_n.$$

Then $F\colon M \to \mathbb{R}$ is supported in U.

We will calculate S_z and T_z explicitly for each $z \in M$.

Because the function F is the product of y_n with a function that takes values in $[0,1]$ and is supported in U_1, if $|F(z)| \geqslant c > 0$, then $z \in U_1$ and $|y_n| \geqslant c$. From the definition of U_1, this further implies that $x_n \in [-\varepsilon, 1)$ and $h(x_n) \geqslant c^2$. Because $h\colon [-\varepsilon, 1) \to \mathbb{R}_{>0}$ approaches 0, these inequalities imply that $x_n \in [-\varepsilon, b]$ for some $b \in [-\varepsilon, 1)$. Hence, for each $c > 0$, the set $\{|F| \geqslant c\}$ is compact, as it is closed in M and contained in $\{x_n \in [-\varepsilon, b],\ |p|^2 + y_n^2 \leqslant h(-\varepsilon)\}$. So

$$F|_{M \setminus F^{-1}(0)}\colon M \setminus F^{-1}(0) \to \mathbb{R} \setminus \{0\}$$

is a proper map. In particular, all the non-zero level sets of F are compact.

At each point z with $y_n \neq 0$, if $F(z) \neq 0$, then $F^{-1}(F(z))$ is compact, and if $F(z) = 0$, then ($\chi(z) = 0$; by the choice of χ also $\mathrm{d}\chi|_z = 0$; and so) $X_F(z) = 0$. We conclude that X_F is complete on $\{y_n \neq 0\}$. So for any point z with $y_n \neq 0$ we have $S_z = -\infty$ and $T_z = \infty$.

Compute X_F on $\{y_n = 0\}$ explicitly for each $z \in M$:

$$X_F(p, x_n, 0) = \frac{1 - x_n^2}{|p|^2 + 1 - x_n^2} \chi(p, x_n, 0) \partial_{x_n}.$$

Since X_F is proportional to ∂_{x_n} and vanishes for $x < -\varepsilon$, we have $S_z = -\infty$ for any point z with $y_n = 0$.

For a point z with $y_n = 0$ and $p \neq 0$, we have $|p|^2 > 0$. By the comparison

$$\frac{1 - x_n^2}{|p|^2 + 1 - x_n^2} \chi(p, x_n, 0) b \leqslant \frac{1 - x_n^2}{|p|^2 + 1 - x_n^2}$$

and the completeness of the vector field $\frac{1 - x_n^2}{b + 1 - x_n^2} \partial_{x_n}$ on $(-1, 1)$ for $b > 0$, we have $T_z = \infty$.

At each point with $y_n = 0$ and $p = 0$, which we write as $z = (0, x_n, 0)$, we have $X_F(0, x_n, 0) = \chi(0, x_n, 0) \partial_{x_n}$. Because $\chi(0, x_n, 0) = 1$ on the set $\{x_n \geqslant 0\}$, we have $T_z \leqslant 1$ when $x_n \geqslant 0$ and $T_z = 1$ when $x_n = 0$. Because the vector field is a positive multiple of ∂_{x_n}, the function $x_n \mapsto T_z$ is strictly decreasing, and so $T_z > 1$ whenever $x_n < 0$. So $T_z \leqslant 1$ if and only if $x_n \geqslant 0$.

We have now shown that $T_z \leqslant 1$ if and only if $z \in R_0$, and that $S_z = -\infty$ for all $z \in M$. This justifies our claim, and our proof is complete. $\square$

We contrast Theorem 2.1 with the analogous result of [11], which is weaker:

Theorem 2.2. *[11, Theorem 1.1] For any $\varepsilon > 0$, there is a symplectomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n} \setminus R_0$ that is the identity outside $\mathbb{B}_0^{2n-2}(\varepsilon) \times W(\varepsilon)$, where $W(\varepsilon) \coloneqq \mathbb{B}_0^2(\varepsilon) \cup \{(x, y) \in \mathbb{R}^2 \mid x > \frac{\sqrt{2}\varepsilon}{2},\ x|y| < \frac{\varepsilon^2}{2}\}$.*

2.2 Applications

Theorem 2.2 cannot be used to symplectically excise R from M if M has finite volume. In contrast, Theorem 2.1 allows us to remove a ray from *any* symplectic manifold.

Corollary 2.3. *Let (M, ω) be a symplectic manifold, and let $\gamma\colon [0, \infty) \to M$ be a proper embedding with image R. Then R is symplectically neighbourhood excisable in M.*

Proof. Let U be an open neighbourhood and $V \subset U$ be a closed neighbourhood of R in M. There exists an open neighbourhood U_0 of the standard ray R_0 in $\mathbb{R}^{2n}$ and a proper symplectic embedding $\psi\colon (U_0, \omega_{\mathrm{can}}) \to (M, \omega)$ such that

$$\psi(R_0) = R, \qquad W \coloneqq \psi(U_0) \subseteq U.$$

By Theorem 2.1, there is symplectic excision φ_0 of R_0 from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$ that is the identity outside of a closed neighbourhood $V_0 \subset U_0$ of R_0 in $\mathbb{R}^{2n}$. Then

$$\varphi \coloneqq \psi \circ \varphi_0 \circ \psi^{-1}\Big|_{W \setminus R}\colon (W \setminus R, \omega) \to (W, \omega)$$

is a symplectic excision of R from W that is the identity in $W \setminus \psi(V_0)$. Since ψ is a proper embedding, the set $\psi(V_0)$ is a closed neighbourhood of R in M. Extending φ by the identity we obtain a symplectic excision $\widetilde{\varphi}$ of R from W that is supported (in W, hence) in U. $\square$

Lemma 2.4. *Any exact 2-form on a smooth manifold has a primitive whose zero set is discrete.*

Proof. Let $\omega = \mathrm{d}\theta$ be an exact 2-form on M^m. Let $\phi = (x_1, \ldots, x_n)\colon M \to \mathbb{R}^n$ be a smooth embedding and then the function

$$F\colon M \times \mathbb{R}^n \to T^*M,$$
$$F(x, s) = \theta(x) + \sum_{i=1}^{n} s_i\, \mathrm{d}_x x_i,$$

where $s = (x_1, \ldots, x_n)$, is transversal to 0_{T^*M}, the zero section of T^*M. By Transversality Theorem, for instance, in [6], we deduce that $F(\cdot, s)$ is transversal to 0_{T^*M} for almost every $s \in \mathbb{R}^n$. Choose such an s and let $\rho = \sum_{i=1}^{n} s_i x_i$. Then the zero set of $\beta \coloneqq \theta + \mathrm{d}\rho = F(\cdot, s) \in \Omega^1(M)$ is a 0-submanifold of M, or equivalently, a discrete set of points in M, and we have $\mathrm{d}\beta = \mathrm{d}\theta = \omega$. $\square$

Theorem 2.5. *Any exact symplectic manifold has a nowhere vanishing primitive (a Liouville form) of its symplectic form.*

Proof. Let $(M, \omega = \mathrm{d}\theta)$ be an exact symplectic manifold. By Lemma 2.4 we choose a $\rho \in \Omega^1(M)$ such that the zeroes of $\beta = \theta + \mathrm{d}\rho$ are isolated. The existence of such a ρ is claimed in [2] and proven differently in [9].

We construct an exhaustion of M by a sequence of compact subsets $(K_j)_{j=1}^{\infty}$ such that for each $j \in \mathbb{N}$, any point in $K_{j+1} \setminus K_j$ is joint by a smooth path in $M \setminus K_j$ to a point in $M \setminus K_{j+1}$. The construction takes the unions of hypographs of an exhausion function for M with the bounded connected components of their complements, which was used in [4, 7, 8, 9].

Let $(z_i)_{i=1}^N$ where $N \in \mathbb{N} \cup \{\infty\}$ be the set of zeroes of β. We claim that there is a properly embedded ray R_i starting with z_i for each $i \in [1, N] \cap \mathbb{N}$ such that they are pairwise disjoint and their union is closed in M. Our construction is step-by-step inside of $M \setminus K_j$ for $j = 1, 2, \dots$. For any $j \in \mathbb{N}$, we draw a smooth path in $M \setminus K_j$ from each of z_i or endpoint of a previous path to a point to a point in $M \setminus K_{j+1}$, so that all the new paths are disjoint with each other and with previous paths, and the joint paths are smooth. In this way, we obtain pairwise disjoint rays $(R_i)_{i=1}^N$ with endpoints $(z_i)_{i=1}^N$. Moreover, $R = \bigcup_{i=1}^N R_i$ is closed in M since $R \cap K_j$ is compact for any $j \in \mathbb{N}$.

Let $(U_i)_{i=1}^N$ be pairwise disjoint open neighbourhoods of $(R_i)_{i=1}^N$ and let $U = \bigcup_{i=1}^N U_i$. By Corollary 2.3, for each $i \in [1, N] \cap \mathbb{N}$ there is a symplectic excision φ_i of R_i from M supported in U_i, and then let $\varphi \colon M \setminus R \to M$ be the composition of $(\varphi_i)_{i=1}^N$, which is supported in U. Let $\alpha = (\varphi^{-1})^* \beta|_{M \setminus R}$ and then we have

$$\mathrm{d}\alpha = \mathrm{d}((\varphi^{-1})^* \beta|_{M \setminus R}) = (\varphi^{-1})^* \omega|_{M \setminus R} = \omega.$$

Moreover, α has no zeroes since β has no zeroes in $M \setminus R$. $\square$

There are proofs of Corollary 2.3 and Theorem 2.5 in [9] by a more straightforward approach.

3 Extension of vector fields

We will construct symplectic vector fields on symplectic manifolds that extend presymplectic vector fields on constant rank submanifolds.

We follow the terminology of [2, 5].

Definition 3.1. A *presymplectic form* is a closed two-form. A *presymplectic manifold* is a manifold S equipped with a closed two-form ω_S. A *regular presymplectic* manifold is a presymplectic manifold (S, ω_S) in which ω_S has constant rank.

On a presymplectic manifold (S, ω_S), a *null vector field* is a vector field v that is everywhere in the null space $(TS)^{\omega_S}$ of the presymplectic form; a *presymplectic vector field* on S is a vector field v such that $\mathcal{L}_v \omega_S = 0$; a *Hamiltonian vector field* on S is a vector field v such that $v \lrcorner \omega_S$ is exact; a *Hamiltonian pair* is a pair (f, v) where v is a Hamiltonian vector field and f is a smooth function such that $df = v \lrcorner \omega_S$.

In a regular presymplectic manifold (S, ω_S), the null spaces $(TS)^{\omega_S}$ fit together into an involutive distribution that defines the *characteristic foliation* of (S, ω_S).

A *constant rank* submanifold of a symplectic manifold (M^{2n}, ω) is a submanifold S of M such that the pullback ω_S of ω to S has constant rank. (S, ω_S) is then a regular presymplectic manifold. Isotropic submanifolds, Lagrangian submanifolds, coisotropic submanifolds, and symplectic submanifolds are examples of constant rank submanifolds. ◇

Remark 3.1. Fix a presymplectic manifold (S, ω_S). If $\omega_S = 0$, then every vector field on S is presymplectic. In general, every null vector field on S is presymplectic. However, unless $\omega_S = 0$, presymplectic vector fields on S are generally not null vector fields on S.

Every Hamiltonian vector field is presymplectic.

The null vector fields are exactly those Hamiltonian vector fields that together with the zero function 0 form a Hamiltonian pair $(v, 0)$.

A function f that occurs in a Hamiltonian pair (v, f) is constant along the leaves of the Hamiltonian foliation. Thus, unless ω_S is symplectic, not every function belongs to a Hamiltonian pair.

Given a Hamiltonian pair (v, f) and a vector field u, the pair (u, f) is a Hamiltonian pair if and only if the difference $u - v$ is a null vector field. ◇

Proposition 3.1. *Let S be a subspace of $(\mathbb{R}^{2n}, \omega_{\text{can}})$ and let $\omega_S = \iota^*\omega$ where $\iota\colon S \to M$ is the embedding. A vector field v on (S, ω_S) is presymplectic if and only if there is a symplectic vector field $\tilde{v}$ on $(\mathbb{R}^{2n}, \omega_{\text{can}})$ that extends v.*

Proof. Let v be a vector field on S.

If $\tilde{v}$ is a symplectic vector field on $(\mathbb{R}^{2n}, \omega_{\text{can}})$ and $v = \tilde{v}|_S$, then

$$v \lrcorner \omega_S = \iota^*(\iota_* v \lrcorner \omega) = \iota^*(\tilde{v} \lrcorner \omega),$$

which implies that $\mathrm{d}(v \lrcorner \omega_S) = \iota^* \, \mathrm{d}(\tilde{v} \lrcorner \omega) = 0$ and v is presymplectic.

Conversely, let v be a presymplectic vector field on (S, ω_S). Let m be the dimension of S and let $2r$ be the rank of ω_S. Then we can find symplectic coordinates $z = (x_1, y_1, \ldots, x_n, y_n)$ of $\mathbb{R}^{2n}$, such that

$$x_1, y_1, \ldots, x_r, y_r, x_{r+1}, x_{r+2}, \ldots, x_{m-r}. \tag{3.1}$$

are coordinates on S and the remaining coordinates vanish on S. Let $\pi\colon M \to S$ be the projection to the coordinates in (3.1). Write v in these coordinates as

$$v = \sum_{i=1}^{m-r} a^i \partial_{x_i} + \sum_{i=1}^{r} b^i \partial_{y_i},$$

and let

$$g(z) \coloneqq \sum_{i=r+1}^{m-r} a^i(\pi(z)) y_i.$$

Now define an extension of v as

$$\tilde{v}(z) \coloneqq \sum_{i=1}^{r} \Big(a^i(\pi(z))\partial_{x_i} + b^i(\pi(z))\partial_{y_i} \Big) + X_g(z) = v(\pi(z)) + \sum_{i=r+1}^{m-r} y_i X_{a^i}(\pi(z)).$$

Note that $\omega = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$ and $\omega_S = \sum_{i=1}^{r} \mathrm{d}x_i \wedge \mathrm{d}y_i$. If v is presymplectic then

$$(\tilde{v} \lrcorner \omega)(z) = \sum_{i=1}^{r} \Big(a^i(\pi(z))\, \mathrm{d}y_i - b^i(\pi(z))\, \mathrm{d}x_i \Big) = (v \lrcorner \omega_S)(\pi(z))$$

is closed, which implies that $\tilde{v}$ is symplectic. □

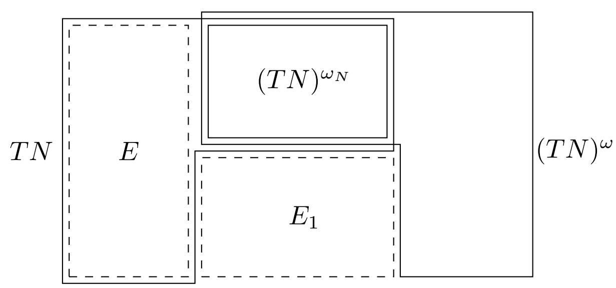

Figure 1: A splitting of the tangent bundle of M along N.

Recall that if $\iota\colon N \to M$ is the inclusion of the submanifold then a smooth map $\pi\colon W \subseteq M \to N$ is called a *smooth neighbourhood retraction* if $\pi \circ \iota = \mathrm{id}_N$ and W is a neighbourhood of N in M. If moreover (M, ω) is a symplectic manifold and the smooth neighbourhood retraction π satisfies that $(TN)^{\omega}$ is invariant under $\pi_*|_N\colon TM|_N \to TN$ then we call π a *symplectic neighbourhood retraction*.

Lemma 3.2. *Let (M, ω) be a symplectic manifold and (N, ω_N) be a constant rank submanifold of M. For any splitting $TN = (TN)^{\omega_N} \oplus E$ there is a symplectic neighbourhood retraction $\pi\colon W \subseteq M \to N$ with $E = (\ker \pi_*)^{\omega} \cap TN$. For any symplectic neighbourhood retraction $\pi\colon W \subseteq M \to N$, the tangent bundle of N has a splitting $TN = (TN)^{\omega_N} \oplus E$ where $E = (\ker \pi_*)^{\omega} \cap TN$.*

Proof. When we have a splitting $TN = (TN)^{\omega_N} \oplus E$ we choose another splitting $E^\omega = (TN)^\omega \oplus E_1$ and then they yields $TM|_N = TN \oplus (E \oplus E_1)^\omega$; see Figure 1. By identifying the normal bundle $TM|_N/TN$ with $(E \oplus E_1)^\omega$ we can construct a smooth neighbourhood retraction $\pi\colon W \subseteq M \to N$ that annihilates $(E \oplus E_1)^\omega \subseteq TM|_N$. Since $\pi_*|_N(TN \oplus E_1) = (TN)^{\omega_N} = TN \cap (TN)^\omega$, the map $\pi\colon W \to N$ a symplectic neighbourhood retraction and $(\ker \pi_*)^\omega \cap TN = (E \oplus E_1) \cap TN = E$.

Let $\pi\colon W \subseteq M \to N$ be a symplectic neighbourhood retraction. For each point $x \in N$ and vector $v \in T_xN$, let $v_2 \in T_xM$ be given by

$$\omega(v_2, u) = \omega(v, \pi_* u) \qquad \text{for all } u \in T_xM\,, \tag{3.2}$$

and let

$$v_1 \coloneqq v - v_2.$$

For any $u \in T_xN$, we have $\pi_* u = u$, and so

$$\omega(v_1, u) = \omega(v, u) - \omega(v_2, u) = \omega(v, u) - \omega(v, \pi_* u) = 0.$$

So $v_1 \in (T_xN)^\omega$. For any $u \in (T_xN)^\omega$, we have $\pi_* u \in (T_xN)^\omega$, and so

$$\omega(v_2, u) = \omega(v, \pi_* u) = 0.$$

So $v_2 \in T_xN$. Then also $v_1 = v - v_2 \in T_xN$, and so $v_1 \in (T_xN)^{\omega_N}$. For any $u \in \ker \pi_*|_x$, we also have $\omega(v_2, u) = 0$ since $\pi_* u = 0$. Therefore $v_2 \in E$. Since $v = v_1 + v_2$, this shows that the subspaces $(T_xN)^{\omega_N}$ and E_x span T_xN. To show that the intersection of these subspaces is trivial, let $v \in (T_xN)^{\omega_N} \cap E_x$. For any $u \in T_xM$,

$$\omega(v, u) = \omega(v, \underbrace{u - \pi_* u}_{\in \ker \pi_*}) + \omega_N(v, \underbrace{\pi_* u}_{\in TN}) = 0$$

because $v \in (\ker \pi_*)^\omega$ and $v \in (TN)^{\omega_N}$. So $v = 0$. Because $x \in N$ is arbitrary, we obtain a fibrewise splitting $TN = (TN)^{\omega_N} \oplus E$. Because N is a constant rank submanifold, the rank of $(TN)^{\omega_N}$, and hence of E, is constant. $\square$

Lemma 3.3. *Let M be a smooth manifold and $N \subset M$ a submanifold. Let α be a section of the conormal bundle $T^*_{M/N}$ to N in M; that is, α is a smooth section of $T^*M|_N$ that vanishes on TN. Then there is an open neighbourhood W of N in M and a smooth function $F\colon W \to \mathbb{R}$ that vanishes on N and such that $\mathrm{d}F|_N = \alpha$.*

Proof. As a section of $T^*_{M/N}$, we can view α as a function on the normal bundle $T_{M/N} \coloneqq TM|_N/TN$ that is linear on each fibre. By the tubular neighbourhood theorem, there exists a diffeomorphism ϵ from $T_{M/N}$ to an open neighbourhood W of N in M whose differential along the zero section induces the identity map on the normal bundle to the zero section. Let $F \coloneqq \alpha \circ \epsilon^{-1}\colon W \to \mathbb{R}$. Then $\mathrm{d}F|_N = \alpha$. $\square$

Lemma 3.4. *Any Hamiltonian pair on a constant rank submanifold of a symplectic manifold can be extended to a Hamiltonian pair in a neighbourhood of the submanifold.*

Proof. Let (M, ω) be a symplectic manifold with a constant rank submanifold (N, ω_N). By Lemma 3.2 we choose a splitting $TN = (TN)^{\omega_N} \oplus E$ and obtain a symplectic neighbourhood retraction $\pi\colon W_2 \subset M \to N$ such that $E = (\ker \pi_*)^\omega \cap TN$. Let (f, v) be a Hamiltonian pair on (N, ω_N) and we apply the splitting to v to obtain $v = v_1 + v_2$ with $v_1 \in \Gamma((TN)^{\omega_N})$ and $v_2 \in \Gamma(E)$. Since v_1 is null then $\alpha = v_1 \lrcorner\, \omega$ is a section of $T^*_{M/N}$ as it vanishes on sections of TN. By Lemma 3.3, there is an open neighbourhood W_1 of N in M and a smooth function $F_1\colon W_1 \to \mathbb{R}$ which vanishes on N and $\mathrm{d}F_1|_N = \alpha$. Note that (f, v_2) is also a Hamiltonian pair since $v_1 \lrcorner\, \omega_N = 0$. By the fact that π preserves both TN and $(TN)^\omega$, we have that $X_{f\circ\pi}|_N = v_2$. Let $W = W_1 \cap W_2$ and let $H = F_1 + f \circ \pi\colon W \to \mathbb{R}$. Then the Hamiltonian pair (H, X_H) extends (f, v). □

Recall that a function on a topological space *vanishes at infinity* if all of the positive epigraph of its absolute value is compact, which is equivalent to being proper away from the zero locus. Any smooth manifold admits a positive smooth function vanishing at infinity, for instance, as the reciprocal of a positive exhaustion function.

Theorem 3.5. *Let (M, ω) be a symplectic manifold with a hypersurface (N, ω_N). Suppose that v is a null vector field on (N, ω_N) whose support in N is closed in M. Let Z be a closed subset of N where v does not vanish. Then for any open neighbourhood U of Z in M there are smooth functions $\chi_N\colon N \to [0, 1]$ supported in $U \cap N$ and $F\colon M \to \mathbb{R}$ supported in U, such that $\chi_N = 1$ on Z, $X_F|_N = \chi_N v$, and any forward flow of X_F started in $M \setminus N$ is complete.*

Proof. Let $\pi\colon W \subseteq M \to N$ be a symplectic neighbourhood retraction and H be a smooth function on W constructed in Lemma 3.4, such that $X_H|_N = v$. Moreover, the construction identifies W with an open neighbourhood of the zero section of $T_{M/N}$ and then H is linear on fibers of $T_{M/N}$. In particular, for any $z \in M \setminus N$, $H(z) = 0$ if and only if $v(\pi(z)) = 0$, by the one-codimensionality of $N \subset M$. We shrink U if necessary so that v never vanishes in U, $U \subseteq W$, and $\pi(U) = U \cap N$.

Choose a closed neighbourhood V of N in W on which π is proper and let H_1 be a positive smooth function on M vanishing at infinity. Since H vanishes on N then $W_3 = \{z \in W \mid |H(z)| < H_1(z)\}$ is an open neighbourhood of N in W. Choose a smooth cut-off function $\chi\colon M \to [0, 1]$ supported in $U \cap V \cap W_3$ that equals 1 on Z. Moreover, suppose that $\mathrm{d}\chi = 0$ wherever $\chi = 0$. This can be achieved, for instance, by replacing χ by $\rho \circ \chi$ where $\rho(s) = 3s^2 - 2s^3$. Let $\chi_N = \chi|_N$.

The function $F \coloneqq \chi H \colon M \to \mathbb{R}$ satisfies $X_F|_N = (\chi X_H)|_N = \chi_N v$ (where the first equality is because H vanishes on N). We claim that $F|_{M \setminus N}$ satisfies $\mathrm{d}F|_{M\setminus N} = 0$ wherever $F|_{M\setminus N} = 0$. Suppose $H(z) = 0$ for some point $z \in W \setminus N$. Then $v(\pi(z)) = 0$, and then $\pi(z) \in N \setminus U$, and then $z \in \pi^{-1}(N \setminus U) \subseteq W \setminus U$. Therefore, for any $z \in M \setminus N$, $F(z) = 0$ implies that $z \notin U$; but in this case we have $\chi(z) = 0$ and then $\mathrm{d}\chi(z) = 0$, which implies that $\mathrm{d}F(z) = 0$. This justifies our claim. Since

$$|F| = |\chi H| \leqslant |\mathbb{1}_{W_3} H| \leqslant H_1,$$

we have that F vanishes at infinity, since H_1 does. Then F is proper away from the zero locus. In particular, all the non-zero level sets of F are compact. Then for any point $z \in M \setminus N$, either $F(z) \neq 0$ and $F^{-1}(F(z))$ is compact, or $F(z) = 0$ and $X_F(z)$ vanishes. This shows that the forward flow of X_F started at $z \in M \setminus N$ is complete. □

4 Presymplectic excision

We recall the solution theory of an automonous ordinary differential equation of first order. Let $I = (-1, 1)$. Consider a non-negative smooth function $v \colon I \to [0, \infty)$ and the ordinary differential equation on I

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v(x). \tag{4.1}$$

The solution $\gamma = \gamma_{t_0,x_0}$ to (4.1) with the initial condition (t_0, x_0) is given by Barrow's formula [1]

$$\begin{cases} t - t_0 = \displaystyle\int_{x_0}^{\gamma(t)} \frac{\mathrm{d}\xi}{v(\xi)} & \text{if } \ v(x_0) > 0\,; \\ \gamma(t) = x_0 & \text{if } \ v(x_0) = 0\,. \end{cases} \tag{4.2}$$

In general, the flow

$$\Phi_v \colon D_v \to I,$$
$$\Phi_v(t, x) = \gamma_{0,x}(t)$$

of v is only defined on an open subset D_v of $\mathbb{R} \times I$, called the *flow domain* of v. Let the *backward time function* $S_v \colon I \to [-\infty, 0)$ and the *forward time function* $T_v \colon I \to (0, \infty]$ of v be defined by

$$D_v = \{(t, x) \in \mathbb{R} \times I \mid S_v(x) < t < T_v(x)\}.$$

Suppose now that $v(x) = 0$ for x close to -1. Then, for any $x \in I$,

$$S_v(x) = -\infty \tag{4.3}$$

and

$$T_v(x) = \begin{cases} \displaystyle\int_x^1 \frac{\mathrm{d}\xi}{v(\xi)} \in (0, \infty] & \text{if } \ v(\xi) > 0 \ \text{ for all } \ \xi \in [x, 1)\,; \\ \infty & \text{if } \ v(\xi) = 0 \ \text{ for some } \ \xi \in [x, 1)\,. \end{cases} \tag{4.4}$$

Now we consider a model with parameters. Let $P = [0, 1] \times [-1, 1]$, recall that $I = (-1, 1)$, and define

$$u \colon I \times P \times I \to [0, \infty), \quad u(a, b, c; x) \coloneqq \chi_a(x)(1 - b)\frac{1 - x^2}{1 - x^2 + c} \tag{4.5}$$

where for $a \in I$,

$$\begin{aligned} &\chi_a \colon I \to [0, 1], \\ &\chi_a(x) \coloneqq \begin{cases} \frac{\exp(-\frac{1}{x-(a-1)/2})}{\exp(-\frac{1}{x-(a-1)/2})+\exp(-\frac{1}{a-x})}, & \frac{a-1}{2} \leqslant x \leqslant a; \\ 0, & -1 < x \leqslant \frac{a-1}{2}; \\ 1, & a \leqslant x < 1; \end{cases} \end{aligned} \tag{4.6}$$

is a non-decreasing smooth function.

Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}\gamma}{\mathrm{d}t} = u(a, b, c; \gamma), \\ \gamma(0) = x\,; \end{cases} \tag{4.7}$$

let $\gamma = \gamma_{a,b,c;x} \colon (S_u(a, b, c; x), T_u(a, b, c; x)) \to I$ be its maximal solution curve. The flow of u,

$$\Phi_u \colon D_u \to I, \quad \Phi_u(a, b, c; t, x) \coloneqq \gamma_{a,b,c;x}(t),$$

is defined on

$$D_u = \{(a, b, c, t, x) \in I \times P \times \mathbb{R} \times I \mid S_u(a, b, c; x) < t < T_u(a, b, c; x)\}.$$

We calculate D_u explicitly. The backward time $S_u(a, b, c; x)$ is always $-\infty$. For $x \leqslant \frac{a-1}{2}$, the forward time $T_u(a, b, c; x) = \infty$; otherwise, we have

$$T_u(a, b, c; x) = \int_x^1 \frac{\mathrm{d}\xi}{u(a, b, c; \xi)} = \int_x^1 \frac{1}{(1 - b)\chi_a(\xi)} \left(1 + \frac{c}{1 - \xi^2}\right) \mathrm{d}\xi.$$

If $x \geqslant a$ then

$$T_u(a,b,c;x) = \left[\frac{\xi}{1-b} + \frac{c}{2(1-b)}\ln\left|\frac{1+\xi}{1-\xi}\right|\right]_x^1 = \begin{cases} \frac{1-x}{1-b}, & c=0, b<1; \\ \infty, & c>0 \text{ or } b=1. \end{cases} \tag{4.8}$$

If $\frac{a-1}{2} < x < a$ then

$$T_u(a,b,c;x) = \begin{cases} \frac{1-a}{1-b} + \int_x^a \frac{1+\exp(\frac{1}{\xi-(a-1)/2} - \frac{1}{a-\xi})}{1-b}\,\mathrm{d}\xi, & c=0, b<1; \\ \infty, & c>0 \text{ or } b=1. \end{cases} \tag{4.9}$$

Define the function $\mu\colon I \times [-1,1) \to [-1,1)$ by

$$\int_{\mu(a,b)}^1 \frac{1}{(1-b)\chi_a(\xi)}\,\mathrm{d}\xi = 1.$$

By (4.8) and (4.9) we conclude that μ is a smooth function increasing in b such that $\mu(a,b) = b$ for $b \geqslant a$ and $\mu(a,b)$ is greater than both b and $\frac{a-1}{2}$ for $b < a$. Then for $a \in I$ define the set

$$\begin{aligned} Z_{u(a,\cdot)} &\coloneqq \{(b,c,x) \in P \times I \mid T_u(a,b,c;x) \leqslant 1\} \\ &= \{(b,0,x) \in P \times I \mid b \in [-1,1), x \geqslant \mu(a,b)\}. \end{aligned}$$

Lemma 4.1. *Let B be a smooth manifold and let $N = B \times I$. Let $\lambda\colon B \to (-1,1]$ be a function, smooth on the support of $1-\lambda$, such that $a \coloneqq \inf_B \lambda > -1$. Let $Z \subset N$ be the epigraph of λ. For any smooth $v\colon N \to [0,\infty)$ consider the initial value problem*

$$\begin{cases} \frac{\mathrm{d}\gamma}{\mathrm{d}t} = v(p;\gamma), \\ \gamma(0) = x \end{cases} \tag{4.10}$$

with solution $\gamma\colon (S_v(p;x), T_v(p;x)) \to I$. Then there is a choice of v such that $S_v = -\infty$ and $\{(p,x) \in N \mid T_v \leqslant 1\} = Z$.

Proof. Note that λ is smooth on the closed subset

$$C = \overline{\{p \in B \mid \lambda(p) < 1\}}.$$

Define $b\colon B \to [0,1]$ as a smooth extension of $\lambda|_C$ by Whitney extension theorem and define $c\colon B \to [0,1]$ as a smooth function whose zero locus is C by a theorem in [3]. Then we have

$$\begin{cases} b(p) = \lambda(p), c(p) = 0, & p \in C; \\ b(p) \in [-1,1], c(p) \in (0,1], & p \in B \setminus C. \end{cases}$$

If $\lambda \equiv 1$ then Z is empty and we set $v = 0$. Otherwise, let $a = \inf_B \lambda \in I$ and the map $b \times c \times \mathrm{id}_I \colon N = B \times I \to [-1,1] \times [0,1] \times I$ pulls back $u(a, \cdot)$ as in (4.5) to a function v on $B \times I$, in the sense that $v(p,x) = u(a, b(p), c(p); x)$. Then we have

$$\begin{aligned} Z_v &= (b \times c \times \mathrm{id}_I)^{-1}(Z_{u(a,\cdot)}) = \{z \in M \mid T_z \leqslant 1\} \\ &= \{(p,x) \in B \times I \mid c(p) = 0, x \geqslant b(p)\} = \{p \in C, x \in [\lambda(p), 1)\} = Z. \end{aligned}$$

□

In Lemma 4.1, when (B, ω_B) is a symplectic manifold and λ is smooth, we equip N with a presymplectic structure ω_N that is the pullback of ω_B by the projection $N = B \times I \to B$. Consider the flow $(\psi_t)_{t \in \mathbb{R}}$ generated by the vector field v on N. The domain and range of ψ_t are subsets of N, depending on S_v and T_v. To be precise,

$$\psi_t \colon \{(p,x) \in N \mid T_v(p;x) > t\} \to \{(p,x) \in N \mid S_v(p;x) < -t\}.$$

If we define subsets $K_t = \{S_v \geqslant -t\}$ and $Z_t = \{T_v \leqslant t\}$ of N then $\psi_t \colon (N \setminus Z_t, \omega_N) \to (N \setminus K_t, \omega_N)$ is a presymplectomorphism. In particular, when $K_t = \varnothing$, the map ψ_t is a presymplectic excision of Z_t from (N, ω_N).

5 Symplectic excision

5.1 Time-independent Hamiltonian flows

We prove a symplectic excision result for closed subsets of codimension at least one.

Theorem 5.1. *Let (M, ω) be a symplectic manifold with a properly embedded co-oriented hypersurface N. The co-oriented hypersurface N determines an orientation of its characteristic foliation ξ generated by $X_y \in \mathfrak{X}(N)$ where y is a defining function of N compatible with the co-orientation. Suppose that $\kappa \colon N \to N/\xi$ is a principle $\mathbb{R}$-bundle with the $\mathbb{R}$-action given by the flow of X_y. Then $B \coloneqq N/\xi$ is equipped with a symplectic form ω_B descended from ω_N by κ. Further suppose that all forward flow line of X_y goes to infinity in M.*

Choose a presymplectomorphism $\psi \colon (B \times I, \omega_B \oplus 0) \to (N, \omega_N)$ so that $\psi_* \partial_x$ is positively propotional to X_y, where $I = (-1,1)$ with coordinate x. Let $\lambda \colon C \to (-1,1]$ be a smooth function on a closed subset C of B and let $Z \subset N$ be the image of $\{(p,x) \in B \times I \mid x \geqslant \lambda(p)\}$ under ψ. Then Z is symplectically neighbourhood excisable from (M, ω).

Proof. As a closed subset of B, by a theorem in [3] , C can be realized as the zero locus of a smooth function $c\colon B \to [0,1]$. Let $b\colon B \to [0,1]$ be a smooth extension of λ. By Lemma 4.1, there is a smooth function $v\colon B \times I \to [0,\infty)$ so that $S_v = -\infty$ and $Z_v = Z$. By the abuse of notation we identify v with $v\partial_x$ where x is the coordinate on I in $B \times I$, and then v is a null vector field on $(B \times I, \omega_{B\times I})$. Note that v never vanishes on Z or otherwise at fixed points of v we would have $T_v = \infty$. Let U be an open neighbourhood of Z in M. By Theorem 3.5, there is a smooth $\chi_N\colon N \to [0,1]$ supported in $U \cap N$ which equals 1 on Z, and a smooth $F\colon M \to \mathbb{R}$ supported in U with Hamiltonian vector field $w = X_f$ such that $w|_N = \chi_N v$ and w is complete on $M \setminus N$. We now analyse the dynamical behavior of w. On $M \setminus N$, by completeness, $S_w = -\infty$ and $T_w = \infty$. On N, we have $S_w \leqslant S_v = -\infty$ and $T_w \geqslant T_v$; in particular, on Z, we have $T_w = T_v \leqslant 1$. On $N \setminus Z$, we have $T_w \geqslant T_v > 1$, and therefore $Z_w = Z_v = Z$. The time-1 map $\varphi = \Phi_w^1$ is a symplectic excision of Z from (M,ω) supported in U. □

It is interesting to note that we can excise subsets with complicated topology.

Example 5.1. The Cantor brush $C \times [0,\infty) \times \{0\}^{2n-2}$, where C is the Cantor set, is symplectically neighbourhood excisable from $(\mathbb{R}^{2n}, \omega_{\mathrm{can}})$; see Figure 2.

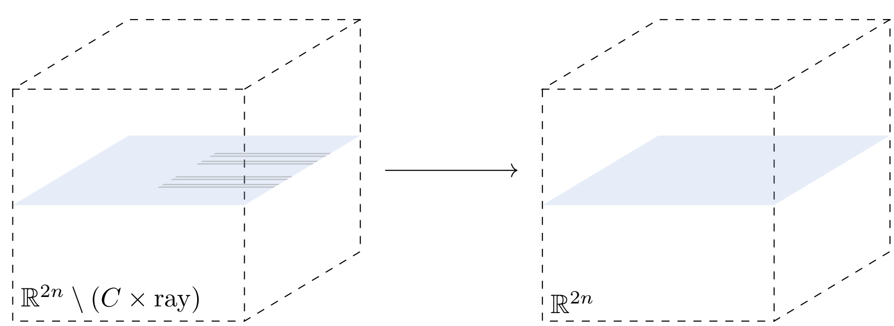

Figure 2: Removing the Cantor brush.

Theorem 2.1 is a special case of the results in this section. Namely, let $(M,\omega) = (\mathbb{R}^{2n}, \omega_{\mathrm{can}})$. Take $N \subset M$ to be given by the vanishing of y_n and co-oriented by y_n. Then $(B,\omega_B) = (\mathbb{R}^{2n-2}, \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_{n-1} \wedge \mathrm{d}y_{n-1})$ with coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ and we assume $\psi\colon B \times I \to N, \omega_N)$ maps $x = 0$ in I to $x_n = 0$ in N. Let $C = \{0\}^{2n-2} \subset B$, and $\lambda = 0$ on C. Then Theorem 2.1 follows from Theorem 5.1.

5.2 A remark on time-independent excisions

Throughout this paper, as in [11], in order to symplectically excise Z from (M, ω), we find a Hamiltonian on M, and we excise the set of points that escape to infinity in time $\leqslant 1$. Arranging the flow to be supported in an arbitrary neighbourhood of the set Z is non-trivial. The dynamics of the localized flow are complicated.

Lemma 5.2. *Let (M, ω) be a symplectic manifold with a smooth function $F\colon M \to \mathbb{R}$ supported in U. If the periodic orbits of X_F are not dense, then there is a symplectic embedding $\psi\colon (B^{2n-2}, \omega_{\mathrm{can}}) \times (\mathbb{R}^2, \omega_{\mathrm{can}}) \to (U, \omega)$.*

Proof. Since the periodic orbits of $w \coloneqq X_F$ is not dense, there is a ball $B^{2n}(\varepsilon)$ for some $\varepsilon > 0$ and a symplectic embedding

$$\psi_0\colon (B^{2n}(\varepsilon), \omega_{\mathrm{can}}) \hookrightarrow (U, \omega)$$

whose image intersects no periodic orbit of w. Let B be the intersection of $B^{2n}(\varepsilon)$ with a hyperplane S so that $\psi_0(B)$ is transversal to the flow of w. Denoting by ν a normal vector to S we extend ψ_0 to a symplectic embedding

$$\psi_1\colon (B + \mathbb{R}\nu, \omega_{\mathrm{can}}) \hookrightarrow (U, \omega)$$

in the way that ν and w are ψ_1-related. Combining some known facts in symplectic geometry, we obtain a chain of symplectic embeddings and symplectomorphisms

$$\psi\colon (B^{2n-2}, \omega_{\mathrm{can}}) \times (\mathbb{R}^2, \omega_{\mathrm{can}}) \hookrightarrow (B^{2n-1}(\varepsilon) \times \mathbb{R}, \omega_{\mathrm{can}}) \simeq (B + \mathbb{R}\nu, \omega_{\mathrm{can}}) \xhookrightarrow{\psi_1} (U, \omega).$$

$\square$

Corollary 5.3. *Let (M, ω) be a symplectic manifold and let $\gamma\colon [0, \infty) \to M$ be a proper embedding with image R. If U is an open neighbourhood of R in M that admits no symplectically embedded $(B^{2n-2}, \omega_{\mathrm{can}}) \times (\mathbb{R}^2, \omega_{\mathrm{can}})$, then there is no symplectic excision of R from (M, ω) supported in U realized as the time-1 map φ of the Hamiltonian flow of any $F\colon M \to \mathbb{R}$, unless the periodic orbits of X_F are dense.*

5.3 Excision in stages

Until now we only considered time-independent Hamiltonians. By iterating the excision procedure, that is, by comosing the flow maps of a sequence of such Hamiltonians, we can excise more complicated subsets. This gives us a glimps into what we could achieve with time-dependent Hamiltonians. Here we discuss two examples.

Recall that an *unrooted tree* is a connected directed graph with no cycles. We define a *half-open unrooted tree* as a tree with one of its node removed. A ray and a ray with two horns are examples of unrooted trees, while a singleton, a line segment, and a double Y are examples of half-open unrooted trees.

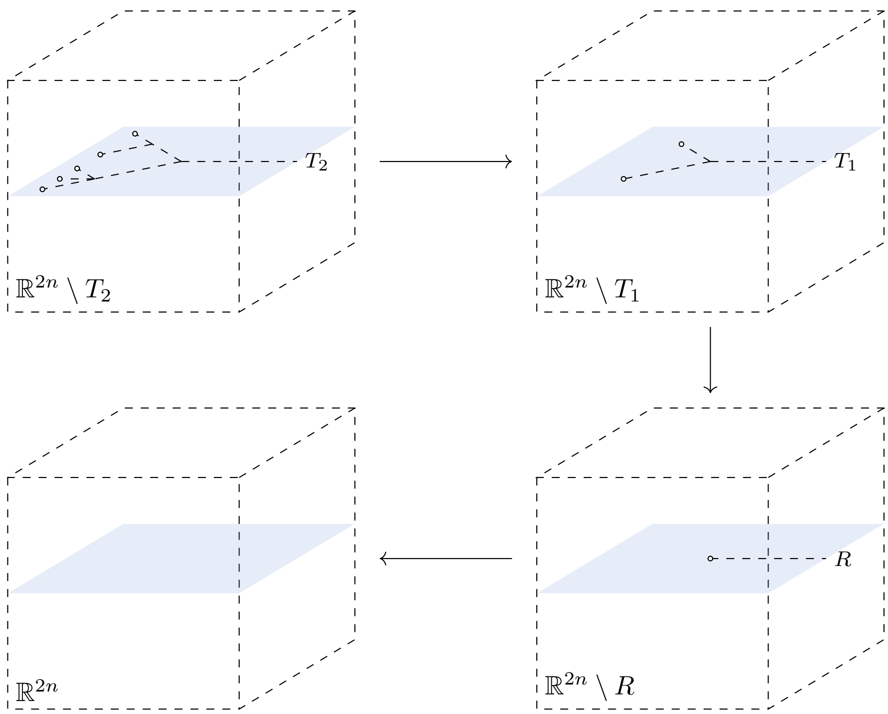

Figure 3: Removing a tree.

Example 5.2. Any properly embedded finite half-open unrooted tree T in a symplectic manifold (M, ω) is symplectically neighbourhood excisable from it.

Proof. We prove by induction on the size of the tree. Let U be an open neighbourhood of T in M. Fix a leaf z of T and let L_z be the branch connecting z that is closed at z and open at the other end. Let $U_z \subseteq U$ be an open neighbourhood of z in $M \setminus (T \setminus L_z)$. By Corollary 2.3 we obtain a symplectic excision $\varphi_z \colon M \setminus T \to M \setminus (T \setminus L_z)$ of L_z from $(M \setminus (T \setminus L_z), \omega)$ supported in U_z. If we know that $T \setminus L_z$ is symplectically neighbourhood excisable from (M, ω) by the induction hypothesis then so is T. $\square$

Another interesting question is that, given a compact subset K of a symplectic manifold (M, ω), is $(M \setminus K, \omega)$ symplectomorphic to M with several punctured points?

Example 5.3. Let (M, ω) be a symplectic manifold with a compact connected embedded unrooted tree T. Then for any open neighbourhood U of T in M there is a symplectomorphism $\varphi\colon (M \setminus T, \omega) \to (M \setminus \{z_0\}, \omega)$ supported in U for any $z_0 \in T$.

Proof. For any $z_0 \in T$, note that $T \setminus \{z_0\}$ is the disjoint union of finitely many half-open unrooted trees T_j, $j \in \{1, \dots, k\}$ for some $k \in \mathbb{N}$, each of which is properly embedded in $M \setminus \{z_0\}$. Applying Example 5.2 k times to each of T_i in $(M \setminus \{z_0\}, \omega)$ we obtain the symplectic neighbourhood excisability of $\bigsqcup_{j=1}^{k} T_j$ from $(M \setminus \{z_0\}, \omega)$. In other words, for any open neighbourhood U of T in M there is a symplectomorphism $\varphi\colon (M \setminus T, \omega) \to (M \setminus \{z_0\}, \omega)$ supported in U. □

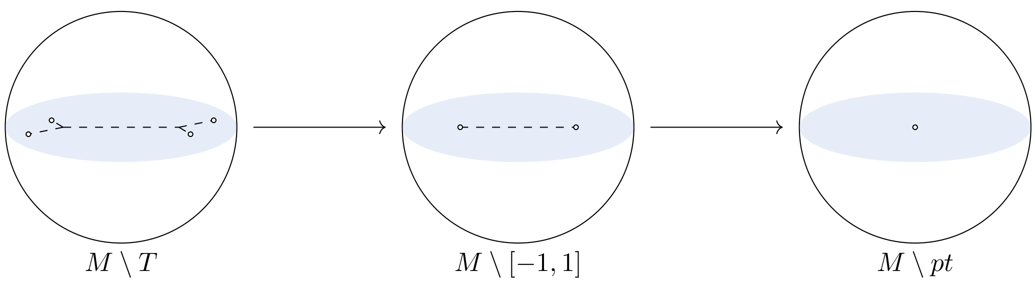

Figure 4: Retracting a hole.

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