Homework9

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3.7.4

(a)

Since this matrix is uppertriangular matrix $\rightarrow$ The eignvalue of the matrix is its diagonal. There are two different eignvalues. By **Theorem 3.8**, there are at least two independant eignvectors. By **Theorm 3.28**, they are basis of $\mathbb{C}$. By **Corollary 3.16**, these two independant eignvectors can form an invertible matrix S. So it is diagonalizable.

(b)

Since this matrix is uppertriangular matrix $\rightarrow$ The eignvalue of the matrix is its diagonal. There are only one eignvalue which is 1. We know that $Eig_\lambda T = Ker(T - \lambda I)$.

$$A - \lambda I = A - I \rightarrow \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

If we do **RREF** to this matrix: $\begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Apparently, there are only two pivots in this matrix. By **Theorem 3.35**, $Rank(T-\lambda I) = 2 \Rightarrow Null(T-\lambda I) = 1$ there are only one independnt eignvectors of matrix. By **Theorm 3.28**, It can not form an invertible matrix S, therefore, it is not diagonolizable.

(c)

Since this matrix is uppertriangular matrix $\rightarrow$ The eignvalue of the matrix is its diagonal. There are only two eignvalue which are $1, 2$. We know that $Eig_\lambda T = Ker(T - \lambda I)$.

$$A - \lambda_1 I = A - I \rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} A - \lambda_2 I = A - 3I \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

Apparently, there are two pivots in the first matrix. By **Theorem 3.35**, $Rank(T - \lambda I) = 2 \Rightarrow Null(T - \lambda I) = 1$ there is only one independnt eignvector of matrix. There is only one pivot in the second matrix. By **Theorem 3.35**, $Rank(T - \lambda I) = 1 \Rightarrow Null(T - \lambda I) = 2$ there are two independnt eignvectors of matrix. By **Theorem 3.8**, these three eignvectos are independant. By **Theorm 3.28**, It can form an invertible matrix S, therefore, it is diagonolizable.

(d)

Since this matrix is uppertriangular matrix $\rightarrow$ The eignvalue of the matrix is its diagonal. There are only two eignvalue which are $1, 3$. We know that $Eig_\lambda T = Ker(T - \lambda I)$.

$$A - \lambda_1 I = A - I \rightarrow \begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix} A - \lambda_2 I = A - 2I \rightarrow \begin{bmatrix} 0 & 0 & 7 \\ 0 & -1 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

If we do **RREF** to the second matrix: $\begin{bmatrix} 0 & 0 & 7 \\ 0 & -1 & 8 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 & 8 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$. Apparently, there are two pivots in the first matrix. By **Theorem 3.35**, $Rank(T - \lambda I) = 2 \Rightarrow Null(T - \lambda I) = 1$ there is only one independnt eignvectors of matrix. There are two pivots in the second matrix. By **Theorem 3.35**, $Rank(T - \lambda I) = 2 \Rightarrow Null(T - \lambda I) = 1$ there is only one independnt eignvector of matrix. By **Theorem 3.8**, these two eignvectos are independant. By **Theorm 3.28**, It can not form an invertible matrix S, therefore, it is not diagonolizable.

3.7.10

(a)

Proof *Prove it by doing arithmetic calculation*

$$\begin{aligned} p(A)x &= (a_0 I + a_1 A + a_2 A^2 + ... + a_n A^n)x = (a_0 x + a_1 \lambda x + a_2 \lambda A x + ... + a_n \lambda A^{n-1} x) \\ &= (a_0 x + a_1 \lambda x + a_2 \lambda^2 x + ... + a_n \lambda^2 A^{n-2} x) = (a_0 x + a_1 \lambda x + a_2 \lambda^2 x + ... + a_n \lambda^3 A^{n-3} x) \\ &= ...... \\ &= (a_0 + a_1 \lambda + a_2 \lambda^2 x + ... + a_n \lambda^n)x = p(\lambda)x \end{aligned}$$

Since x is eignvector of A, x is not $\{0\} \rightarrow x$ is an eignvector of $p(A)$ with eignvalue $p(\lambda)$.

(b)

Proof *From part **(a)**: we know that $p(A)x = p(\lambda)x$. Therefore, $p(A)x = 0 \Rightarrow p(\lambda)x = 0$. Since $x \neq 0$, $p(\lambda) = 0$.*

(c)

Proof *From part **(b)**: we know that if $p(A)x = 0$, then $p(\lambda) = 0$.*
If $p(A) = 0 \rightarrow p(A)x = 0 \rightarrow p(\lambda) = 0$

3.7.12

Proof Denote *The basis of F^n as $(e_1, e_2, ..., e_n)$.*

*We choose a random vector e_j from $(e_1, e_2, ..., e_n)$. Since A is an uppertriangular matrix, by **Lemma 3.68**, $Ae_j = \sum_{i=1}^{i=j} a_i e_i$. Since it is strictly uppertriangular matrix, $a_j = 0$, we get that*

$Ae_j = \sum_{i=1}^{i=j-1} a_i e_i$. (Ae_j) *is jth column of A.* $(A^k e_j)$ *is jth column of* A^k.

$$A^n e_j = A^{n-1}(\sum_{i=1}^{i=j-1} a_i e_i) \Rightarrow A^{n-2}(AAe_j) = A^{n-2}(\sum_{i=1}^{i=j-1} Aa_i e_i)$$
$$= A^{n-2}(\sum_{i=1}^{i=j-1} a_i Ae_i) = A^{n-2}(\sum_{i=1}^{i=j-1} a_i \sum_{k=1}^{k=i-1} b_k e_k)$$

$\sum_{i=1}^{i=j-1} a_i \sum_{k=1}^{k=i-1} b_k e_k \in Span(e_1, e_2, ..., e_{j-2})$. *So we can conclude that* $A^n e_j = A^{n-2} \sum_{i=1}^{i=j-2} c_i e_i$.

$$\begin{aligned} A^n e_j &= A^{n-2} \sum_{i=1}^{i=j-2} c_i e_i \\ &= A^{n-3}(\sum_{i=1}^{i=j-3} d_i e_i) \\ &= ... \\ &= A^{n-j}(\sum_{i=1}^{i=0} x_i e_i) \\ &= 0 \end{aligned}$$

It is true $\forall j \in \{1, ..., n\}$. *So all columns of* A^n *are zero.*

3.7.14

Proof Suppose *The* $dim V = n$. $T^0 = I$.

If we want to find a non-zero polynomial, which is equivalent to find a list of dependent vectors. By ***Theorem 2.5****, we now that* $\mathcal{L}(\mathcal{V})$ *itself is a vector space. By* ***Corollary 3.43****, we know* $dim\mathcal{T} = dimV * dimV = n^2$.

Consider *Construct a List* $(I, T^1, T^2, ..., T^{n^2})$. *By* ***Proposition 3.21****, we know that the number of independent vectors in a list is at most* $(\dim V)^2 = n^2$ *. Since there are* $n^2 + 1 > n^2$ *elements in* $(I, T^1, T^2, ..., T^{n^2})$*, so it can not be independant, therefore it is a list of dependent vectors. By* ***Definition Of Linearly Dependent****, there must exist a list of coefficients* $(a_0, a_1, ..., a_{n^2})$ *which is not all zero. It makes* $\sum_{i=0}^{i=n^2} a_i T^i = 0$. *Therefore, there exists a nonzero polynomial* $p(x) = \sum_{i=0}^{i=n^2} aix^i$.

4.1.2

(a)

Since the field is $\mathbb{R}$, therefore, $\langle v, w\rangle = \langle w, v\rangle$.

$$\begin{aligned} 3 = \|v - w\|^2 &= \langle v - w, v - w\rangle \\ &= \langle v, v\rangle + \langle w, w\rangle - 2\langle v, w\rangle \\ 7 = \|v + w\|^2 &= \langle v + w, v + w\rangle \\ &= \langle v, v\rangle + \langle w, w\rangle + 2\langle v, w\rangle \end{aligned}$$

Subtract those two equations, we will get $4 = 4\langle v, w\rangle \Rightarrow \langle v, w\rangle = 1$

(b)

Use the answer from **(a)**

$$7 = \|v + w\|^2 = \langle v, v\rangle + \langle w, w\rangle + 2$$
$$\Rightarrow \|v\|^2 + \|w\|^2 = 5$$

4.1.6

Denote *The ith entry of a_j, b_k are a_{ij}, b_{ik}.*

Proof

$$[A^*B]_{jk} = \sum_{i=1}^{i=m} a^*_{ji} b_{ik} = \sum_{i=1}^{i=m} \overline{a_{ij}} b_{ik} = \sum_{i=1}^{i=m} b_{ik}\overline{a_{ij}} = \langle b_k, a_j\rangle$$

4.1.10

Proof *Inorder to prove this will defines an inner product on $\mathbb{W}$. We need to follow the **Definition** on page 226, 227. Since T is an isomorphism.*

Denote $\forall w_1, w_2, w_3 \in \mathbb{W}, \exists v_1 = T^{-1}(w_1), v_2 = T^{-1}(w_2), v_3 = T^{-1}(w_3)$, *where* $v_1, v_2, v_3 \in \mathbb{V}, \forall a \in \mathbb{F}$.

Inner product is scalar: *Since $\mathbb{V}$ is a innerproduct vectorspace, $\langle v_1, v_2\rangle = k \in (F)$. $\forall w_1, w_2 \in \mathbb{W}$, $\langle w_1, w_2\rangle = \langle T^{-1}(w_1), T^{-1}(w_2)\rangle = \langle v_1, v_2\rangle = k \in (F)$.*
Distributive Law: *Since $\mathbb{V}$ is a innerproduct vectorspace, $\langle v_1 + v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$. Therefore, $\langle w_1 + w_2, w_3\rangle = \langle T^{-1}(w_1) + T^{-1}(w_2), T^{-1}(w_3)\rangle = \langle v_1 + v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle = \langle T^{-1}(w_1), T^{-1}(w_3)\rangle + \langle T^{-1}(w_2), T^{-1}(w_3)\rangle = \langle w_1, w_3\rangle + \langle w_2, w_3\rangle$.*
Homogenity: *Since $\mathbb{V}$ is a innerproduct vectorspace, $\langle av_1, v_2\rangle = a\langle v_1, v_2\rangle$. Therefore, $\langle aw_1, w_2\rangle = \langle T^{-1}(aw_1), T^{-1}(w_2)\rangle = \langle aT^{-1}(w_1), T^{-1}(w_2)\rangle = \langle av_1, v_2\rangle = a\langle v_1, v_2\rangle = a\langle T^{-1}(w_1), T^{-1}(w_2)\rangle = a\langle w_1, w_2\rangle$.*
Symmetry: *Since $\mathbb{V}$ is a innerproduct vectorspace, $\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}$. Therefore, $\langle w_1, w_2\rangle = \langle T^{-1}(w_1), T^{-1}(w_2)\rangle = \langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle} = \overline{\langle T^{-1}(w_2), T^{-1}(w_1)\rangle} = \overline{\langle w_2, w_1\rangle}$.*
Nonenegtive: *Since $\mathbb{V}$ is a innerproduct vectorspace, $\langle v_1, v_1\rangle \geq 0$. Therefore, $\langle w_1, w_1\rangle = \langle T^{-1}(w_1), T^{-1}(w_1)\rangle = \langle v_1, v_1\rangle \geq 0$.*
Definiteness: *Since $\mathbb{V}$ is a innerproduct vectorspace, if $\langle v_1, v_1\rangle = 0 \to v_1 = 0$. Therefore, if $0 = \langle w_1, w_1\rangle = \langle T^{-1}(w_1), T^{-1}(w_1)\rangle = \langle v_1, v_1\rangle \to v_1 = 0 \to w_1 = T(v_1) = T(0) = 0$.*

4.1.12

(a)

Proof $\Rightarrow$ *If $v = 0$, by **Theorem 4.2(3)***

$$\langle v, w\rangle = \langle 0, w\rangle = 0, \forall w \in \mathbb{V}$$

$\Leftarrow$ *If $\langle v, w\rangle = 0$, $\forall w \in \mathbb{V}$, by **Theorem 4.2(4)***

$$v = 0$$

(b)

Proof $\Rightarrow$ *If* $v = w$

$$\langle v, u\rangle = \langle w, u\rangle = \langle w, u\rangle, \forall u \in \mathbb{V}.$$

$\Leftarrow$ *If* $\langle v, u\rangle = \langle w, u\rangle$, $\forall u \in \mathbb{V}$. *Using result from* ***(a)***

$$\langle v, u\rangle = \langle w, u\rangle \to \langle v - w, u\rangle = 0 \to v - w = 0 \to v = w$$

(c)

Proof $\Rightarrow$ *If* $S = T$.

$$\langle Sv_1, v_2\rangle = \langle Tv_1, v_2\rangle, \forall v_1, v_2 \in \mathbb{V}$$

$\Leftarrow$ *If* $\langle Sv_1, v_2\rangle = \langle Tv_1, v_2\rangle, \forall v_1, v_2 \in \mathbb{V}$. *Using conclusion from* ***(a), (b)***

$$\begin{aligned}
\langle Sv_1, v_2\rangle = \langle Tv_1, v_2\rangle &\to \langle Sv_1 - T(v_1), v_2\rangle = 0 \\
&\to \forall v_2, \langle (S-T)(v_1), v_2\rangle = 0 \\
&\to (S-T)(v_1) = 0 \\
&\to \forall v_1, S(v_1) = T(v_1) \\
&S = T
\end{aligned}$$