Homework11

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4.5.4

(a)

Denote *The* $\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$ *as* B_v

The ith column of $[R]_{B_v}$ is $[Rv_i]_{B_v}$.

$$[Rv_1]_{B_v} = \begin{bmatrix}1\\0\end{bmatrix}_{B_v} = \begin{bmatrix}1\\0\end{bmatrix}$$
$$[Rv_2]_{B_v} = \begin{bmatrix}1\\-1\end{bmatrix}_{B_v} = \begin{bmatrix}2\\-1\end{bmatrix}$$
$$\Rightarrow \begin{bmatrix}1 & 2\\0 & -1\end{bmatrix}$$

(b)

Proof *By **Proposition 4.31** and **Definition of Orthogonal Matrix**, we know that if the columns are not **Othonomal**, then the matrix must not be orthogonal.*

$$\langle A_1, A_2\rangle = 2 \neq 0$$

Therefore, A is not orthogonal matrix.

(c)

Proof *Because **Proposition 4.30** says, **Suppose** B_v, B_w **are orthonormal basis of** $\mathbb{V}, \mathbb{W}$. But $\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right)$ itself is not an orthonomal basis.*

4.5.6

Proof *Since $\mathbb{C}_{2\pi}(\mathbb{R})$ is equipped with innerporduct, so it is both an innerproduct space and a normed space.*
Suppose $\forall f(x) \in \mathbb{C}_{2\pi}(\mathbb{R})$, which means $f(x)$ is a continuous 2π periodic function $\Rightarrow \exists g(x) = f(x-t) \in \mathbb{C}_{2\pi}(\mathbb{R}), T(g)(x) = f(x+t)$. T is a surjective lineaer map.

Denote

$$\int_a^b f(x)\overline{f(x)}\,d(x) = F(b) - F(a) = \int_{a+2\pi}^{b+2\pi} f(x)\overline{f(x)}\,d(x) = F(b+2\pi) - F(a+2\pi)$$
$$\Rightarrow F(b+2\pi) - F(b) = F(a+2\pi) - F(a)$$

So we get

$$\begin{aligned}
\|(Tf)(x)\| \int_0^{2\pi} (Tf)(x)\overline{(Tf)(x)}\,dx &= \int_0^{2\pi} f(x+t)\overline{f(x+t)}\,dx \\
&= \int_0^{2\pi} f(x+t)\overline{f(x+t)}\,d(x+t) \\
&= \int_t^{t+2\pi} f(x')\overline{f(x')}\,d(x') \\
&= F(t+2\pi) - F(t) \\
&= F(0+2\pi) - F(0) \\
&= \|(f)(x)\|
\end{aligned}$$

So, it is an isometry.

4.5.8

On Hand Writing

4.5.10

On Hand Writing

4.5.14

(a)

Proof *By **Proposition 4.30**, we know that if U is unitary, then all columns of U is orthonomal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonomal basis. By **Corrolary 4.30**, we Know that U is an isometry.*

$$\|U\|_{op} = \max_{\|v\|=1, v\in\mathbb{C}^n} \|Uv\| = \|v\| = 1$$

(b)

By **Proposition 4.30**, we know that if U is unitary, then all columns of U is orthonomal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonomal basis. By **Corrolary 4.30**, we Know that U is an isometry.

$$\|U\|_F = \sqrt{trU^*U} = \sqrt{trI} = \sqrt{n}$$

5.1.4

Denote $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

First use standard basis as (e_1, e_2, e_3). Second use (e_2, e_3) as basis for output.

$$T(e_1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 * e_2$$

$$T(e_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 * e_3$$

$$T(e_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 * e_2$$

Since $rankT = 2$, so the $\sigma_1 = 1, \sigma_2 = 1$
The right vectors are (e_1, e_3, e_2), left vectors are (e_2, e_3), and $\sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 0$.

5.1.6

Denote *The standard basis of* $\mathbb{V}$ *is* $(e_1, e_2, ..., e_n)$, *with* $dimV = n$. *As well as* $P := P_U$ *with* $U \subset V, dimU = m, U = span(e_1, ..., e_m)$, *and* $V = U \oplus U^{\perp}$.

(a)

By **Theorem 5.3**, we know that the singular value are unique. So we just need to prove that we can find only $1, 0$.
By **Theorem 4.16.2**, when $i \leq m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = \langle e_i, e_{j=i} \rangle e_{j=i} = 1 * e_i$$

By **Theorem 4.16.2**, when $i > m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = 0$$

So, the singular values are only $0, 1$.

(b)

I choose $(e_1, e_2, ..., e_n)$ as both left and right singular vectors, and with the same sequence.

5.1.10

Proof *Since* T *is invertible, it must be surjective, so* $n = rangeT = dimW = dimV$. *Since* $\forall i \in \{1, 2, ..., n\}, \sigma_i > 0, \neq 0$. *By* ***Theorem5.3***, *we know the singular value is unique.*

Denote *The orthonomal basis of* V *is* $(e_1, e_2, ..., e_n)$, *and the orthonomal basis of* W *is* $(f_1, f_2, ..., f_n)$.

Consider *Construct* $T^{-1} \in \mathfrak{L}(W, V)$, *with singular value* $\sigma_1', \sigma_2', ..., \sigma_n'$

Since $\forall i \in \{1, 2, ..., n\}$

$$T(e_i) = \sigma_i f_i \Rightarrow T^{-1}(f_i) = \frac{1}{\sigma_i} e_i$$

$\forall i \geq j$

$$\sigma_i \geq \sigma_j \Rightarrow \frac{1}{\sigma_j} \geq \frac{1}{\sigma_i}$$

Therefore, $\{\frac{1}{\sigma_n}, \frac{1}{\sigma_{n-1}}, ..., \frac{1}{\sigma_1}\}$ *are singular value of* T^{-1}.
By ***Key Ideas On Page295 last point****, we know that the largest singular value is operator norm of that map.*

$$\|T^{-1}\|_{op} = \frac{1}{\sigma_n} \Rightarrow \|T^{-1}\|_{op}^{-1} = \sigma_n$$

Denote $v = \sum_{i=1}^{i=n} a_i e_i$, $\|v\| = 1$

$$\begin{aligned}
\|T(v)\| &= \|T(\sum_{i=1}^{i=n} a_i e_i)\| = \|\sum_{i=1}^{i=n} a_i T(e_i)\| = \|\sum_{i=1}^{i=n} a_i \sigma_i f_i\| \\
&= \sqrt{\sum_{i=1}^{i=n} |\sigma_i|^2 \|a_i f_i\|^2} \\
&\geq |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2 \|f_i\|^2} \\
&= |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2} = |\sigma_n| \\
&\Rightarrow \underset{\|v\|=1}{min} \|Tv\| = \sigma_n
\end{aligned}$$

5.1.14

(a)

Denote *The orthonomal basis of* $\mathbb{W}$ *is* $(f_1, f_2, ..., f_m)$, *with* $dim\mathbb{W} = m$

$$
\begin{aligned}
\|Tv\| &= \|T(\sum_{i=j}^{i=p} a_i e_i + \sum_{i=p+1}^{i=n} a_i e_i)\| = \|\sum_{i=j}^{i=p} a_i T(e_i) + \sum_{i=p+1}^{i=n} a_i T(e_i)\| \\
&= \|\sum_{i=j}^{i=p} a_i \sigma_i f_i + \sum_{i=p+1}^{i=n} 0\| = \|\sum_{i=j}^{i=p} a_i \sigma_i f_i\| \\
&= \sqrt{\sum_{i=j}^{i=p} \|a_i \sigma_i f_i\|^2} \leq \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=p} \|a_i f_i\|^2} \\
&\leq \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=n} \|a_i f_i\|^2} = \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=n} \|a_i\|^2} \\
&= \sigma_j \|v\|
\end{aligned}
$$

(b)

Denote *The orthonomal basis of* $\mathbb{W}$ *is* $(f_1, f_2, ..., f_m)$*, with* $dim\mathbb{W} = m$

$$
\begin{aligned}
\|Tv\| &= \|T(\sum_{i=1}^{i=j} a_i e_i)\| = \|\sum_{i=1}^{i=j} a_i T(e_i)\| \\
&= \|\sum_{i=1}^{i=j} a_i \sigma_i f_i\| = \sqrt{\sum_{i=1}^{i=j} \|a_i \sigma_i f_i\|^2} \\
&\geq \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=j} \|a_i f_i\|^2} = \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=j} \|a_i\|^2} \\
&= \sigma_j \|v\|
\end{aligned}
$$

(c)

First I could build a vector space $V = \langle e_1, e_2, ..., e_j \rangle, dimV = j$. By **Lemma 3.22**,

$$dimU + dimV = n - j + 1 + j \geq n \Rightarrow U \cap V \neq 0$$

By conclusion from **Part (b)**

$$\exists v_0 \in U, v_0 \in V, \|Tv_0\| \geq \sigma_j v_0$$

(d)

Use the conclusion from **Part (c)**, we know that, if $dimU = n - j + 1 \Rightarrow \exists v_0 \in U, \|Tv\| \geq \sigma_j \|v_0\|$ This is equivalent to

$$dimU = n - j + 1 \Rightarrow \underset{v \in U}{max} \|Tv\| \geq \sigma_j \|v\|$$

When $\forall v \in U, \|v\| = 1$, we get

$$dimU = n - j + 1 \Rightarrow \underset{v \in U, \|v\|=1}{max} \|Tv\| \geq \sigma_j$$

Since it is true $\forall U \subset V, dimU = n-j+1$, the statement is equivalent to

$$\underset{dimU=n-j+1}{min} \underset{v \in U, \|v\|=1}{max} \|Tv\| \geq \sigma_j \tag{1}$$

Now we need to prove another side. Now we build $U' = \langle e_j, e_{j+1}, ..., e_n \rangle, dimU' = n-j+1$. From conclusion from **Part (a)**, we get

$$\underset{v \in U', \|v\|=1}{max} \|Tv\| \leq \sigma_j \tag{2}$$

From **Equation (1), and (2)**, we get

$$\underset{dimU=n-j+1}{min} \underset{v \in U, \|v\|=1}{max} \|Tv\| \leq \underset{v \in U', \|v\|=1}{max} \|Tv\| \leq \sigma_j \tag{3}$$

From **Eqution (1), and (3)**, I prove

$$\underset{dimU=n-j+1}{min} \underset{v \in U, \|v\|=1}{max} \|Tv\| = \sigma_j$$

5.2.2

(a)

The singualr value of matrix A is the sqrt root of eignvealue of AA^* or AA^*.

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Since it is an uppertraingular matrix, so the eignvalue is its diagonal, which is 5. And it gets two independent eignvectors
So the singualr value is $\sigma_1 = \sigma_2 = \sqrt{5}$.

(b)

The singualr value of matrix A is the sqrt root of eignvealue of AA^* or AA^*.

$$\begin{bmatrix} 4 & -3 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix}$$

Since it is an uppertraingular matrix, so the eignvalue is its diagonal, which is $25, 100$. And it gets two independent eignvectors
So the singualr value is $\sigma_1 = 10, \sigma_2 = 5$.

(c)

The singualr value of matrix A is the sqrt root of eignvealue of AA^* or AA^*.

$$\begin{bmatrix} i & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Since it is an uppertraingular matrix, so the eignvalue is its diagonal, which is $2, 1$.
So the singualr value is $\sigma_1 = \sqrt{2}, \sigma_2 = 1$.

(d)

The singualr value of matrix A is the sqrt root of eignvealue of AA^* or AA^*.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$$

Since it is an uppertraingular matrix, so the eignvalue is its diagonal, which is $6, 3$.
So the singualr value is $\sigma_1 = \sqrt{6}, \sigma_2 = \sqrt{3}$.

(e)

The singualr value of matrix A is the sqrt root of eignvealue of AA^* or AA^*.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Since it is an uppertraingular matrix, so the eignvalue is its diagonal, which is$9, 4, 1$
So the singualr value is $\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1$.

5.2.8

By **Proposition 5.5**, we know the singular valeu of A is positive sqrt root of eignvalue of A^*A or AA^*.

$$A^*A = \begin{bmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ 0 & 0 & \lambda_3 & ... & 0 \\ ... \\ 0 & 0 & 0 & ... & \lambda_n \end{bmatrix} \begin{bmatrix} \overline{\lambda_1} & 0 & 0 & ... & 0 \\ 0 & \overline{\lambda_2} & 0 & ... & 0 \\ 0 & 0 & \overline{\lambda_3} & ... & 0 \\ ... \\ 0 & 0 & 0 & ... & \overline{\lambda_n} \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 & 0 & 0 & ... & 0 \\ 0 & |\lambda_2|^2 & 0 & ... & 0 \\ 0 & 0 & |\lambda_3|^2 & ... & 0 \\ ... \\ 0 & 0 & 0 & ... & |\lambda_n|^2 \end{bmatrix}$$

Since it is a diagonal matrix, so the eignvalue of A^*A is its diagonal. Singualr value of A is sqrt root of its eignvalue $\Rightarrow$ The singular value of A is $|\lambda_1|, |\lambda_2|, |\lambda_3|, ..., |\lambda_n|$

5.2.16

(a)

$$AA^\dagger b = b \Rightarrow \exists x, x = A^\dagger b, Ax = b$$

From the conclusion **5.2.15(d)**

$$\exists x, Ax = b \Rightarrow AA^\dagger Ax = Ax = b \Rightarrow AA^\dagger b = b$$

(b)

First showo that:

$$\Sigma^\dagger \Sigma = diag(1, ..., 1, 0, .., 0)$$

Since the system is consistent

$$A^\dagger A x = A^\dagger b$$

So we get, U, V, U^*, V^* are isometry.

$$\begin{aligned} \|A^\dagger b\| &= \|A^\dagger A x\| = \|V\Sigma^\dagger U^* U \Sigma V^* x\| = \|V\Sigma^\dagger \Sigma V^* x\| \\ &= \|\Sigma^\dagger \Sigma V^* x\| \leq \|IV^* x\| = \|V^* x\| = \|x\| \end{aligned}$$

(c)

$$\forall v_0 \in kerA, Av_0 = 0 \Rightarrow (I_n - A^\dagger A)v_0 = v_0 - 0 = v_0 \Rightarrow v_0 \in C(I - A^\dagger A)$$

So we get

$$kerA \subset C(I - A^\dagger A)$$

For another side

$$\begin{aligned} &\forall x \in C(I - A^\dagger A), x = (I - A^\dagger A)v_0 \Rightarrow Ax = A(I - A^\dagger A)v_0 = (A - AA^\dagger A)v_0 = 0 \\ \Rightarrow &C(I - A^\dagger A) \subset kerA \end{aligned}$$

We get

$$C(I - A^\dagger A) = kerA$$

Since $Ax = b$ is consistent, so $AA^\dagger b = b$, $A^\dagger b$ is one of the solution. By **Proposition 2.42**

$$\{x \in F^n | Ax = b\} = \{A^\dagger b + k | k \in KerA\} = \{A^\dagger b + [I_n - A^\dagger A]w | w \in F^n\}$$

5.2.20

At each ith position:

$$\Sigma = \sum_{i=1}^{i=r} diag(0, 0, ..., \sigma_i, 0..., 0)$$

So we get

$$
\begin{aligned}
A = U\Sigma V^* &= U(\sum_{i=1}^{i=r} diag(0,0,...,\sigma_i,0...,0))V^* \\
&= U\sum_{i=1}^{i=r} diag(0,0,...,\sigma_i,0...,0))V^* \\
&= U(\sum_{i=1}^{i=r} \begin{bmatrix} 0^* \\ 0^* \\ ... \\ \sigma_i(V_i)^* \\ 0^* \\ ... \\ 0^* \end{bmatrix}) = \sum_{i=1}^{i=r} U \begin{bmatrix} 0^* \\ 0^* \\ ... \\ \sigma_i(V_i)^* \\ 0^* \\ ... \\ 0^* \end{bmatrix} \\
&= \sum_{i=1}^{i=r} u_i\sigma_i(v_i)^* = \sum_{i=1}^{i=r} \sigma_i u_i(v_i)^*
\end{aligned}
$$