Homework10

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4.1.4

Denote *The jth col of* A, B *is* A_j, B_j. *And* $A_j, B_j \in \mathbb{C}^m$. *So that,* $\|A_j\|^2 = \langle A_j, A_j \rangle$, *as well as for* B_j.

Proof *By the definition of inner product* $\sum_{i=1}^{i=m}[A]_{ij}[B]_{ij} = \langle A_j, B_j \rangle$. *For the inner product of two matrices:* $\langle A, B \rangle_F = \sum_{j=1}^{j=n}\sum_{i=1}^{i=m}[A]_{ij}[B]_{ij} = \sum_{j=1}^{j=n}\langle A_j, B_j \rangle$. *By the definition of the norm:* $\|A\|^2{}_F = \langle A, A \rangle_F = \sum_{j=1}^{j=n}\langle A_j, A_j \rangle = \sum_{j=1}^{j=n}\|A_j\|^2$.

4.1.8

Suppose there exists $w \in V$. $\langle v - w, v - w \rangle = \langle w - v, w - v \rangle \Rightarrow \|v - w\| = \|w - v\|$

$$\|v\| = \|v - u + u\| \leq \|v - u\| + \|u\| \Rightarrow \|v - u\| \geq \|v\| - \|u\| = 11 - 2 = 9$$
$$\|v - u\| \leq \|v - w\| + \|w - u\| = \|v - w\| + \|u - w\| = 8$$

Contradiction.

4.1.14

(a)

$$\frac{1}{4}(\|v + w\|^2 - \|v - w\|^2) = \frac{1}{4}(\langle v + w, v + w \rangle - \langle v - w, v - w \rangle) = \frac{1}{4}(2\langle v, w \rangle - (-2)\langle v, w \rangle) = \langle v, w \rangle$$

(b)

$$\begin{aligned}
&\frac{1}{4}(\|v + w\|^2 - \|v - w\|^2 + i\|v + iw\|^2 - i\|v - iw\|^2) \\
&= \frac{1}{4}(\langle v + w, v + w \rangle - \langle v - w, v - w \rangle + i(\langle v + iw, v + iw \rangle - \langle v - iw, v - iw \rangle)) \\
&= \frac{1}{4}(2\langle w, v \rangle + 2\langle v, w \rangle + i(2i\langle w, v \rangle - 2i\langle v, w \rangle)) = \langle v, w \rangle
\end{aligned}$$

4.1.14

Proof $\forall v, w \in \mathbb{V}$. *Use* ***Definition*** *and* ***Proposition 4.2.2***.

$$\langle 0, v \rangle = 0 * \langle w, v \rangle = 0$$
$$\langle v, 0 \rangle = \overline{0} * \langle v, w \rangle = 0$$

4.2.4

This Exercise we need to ouse **Theorem 4.9**

(a)

$$\begin{aligned}\left\langle \begin{bmatrix}1\\2\\-3\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\rangle = 2\sqrt{2} \\ \left\langle \begin{bmatrix}1\\2\\-3\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}\right\rangle = -\sqrt{6}\end{aligned} \Rightarrow \begin{bmatrix}2\sqrt{2}\\-\sqrt{6}\end{bmatrix}$$

(b)

Refer to **Exercise 4.1.4**

$$\begin{aligned}\left\langle \begin{bmatrix}5&7\\7&2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}\right\rangle = \frac{21}{2} \\ \left\langle \begin{bmatrix}5&7\\7&2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\right\rangle = \frac{3}{2} \\ \left\langle \begin{bmatrix}5&7\\7&2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1&-1\\1&-1\end{bmatrix}\right\rangle = \frac{3}{2} \\ \left\langle \begin{bmatrix}5&7\\7&2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\right\rangle = \frac{-7}{2}\end{aligned} \Rightarrow \begin{bmatrix}\frac{21}{2}\\\frac{3}{2}\\\frac{3}{2}\\\frac{-7}{2}\end{bmatrix}$$

(c)

$$\begin{aligned}\left\langle \begin{bmatrix}-3\\0\\1\\2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\rangle = 0 \\ \left\langle \begin{bmatrix}-3\\0\\1\\2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\i\\-1\\-i\end{bmatrix}\right\rangle = -2+i \\ \left\langle \begin{bmatrix}-3\\0\\1\\2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right\rangle = -2 \\ \left\langle \begin{bmatrix}-3\\0\\1\\2\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-i\\-1\\i\end{bmatrix}\right\rangle = -2-i\end{aligned} \Rightarrow \begin{bmatrix}0\\-2+i\\-2\\-2-i\end{bmatrix}$$

4.2.6

This question we will use **Theorem 4.9**. Assume the *ith* row and *jth* column of the matrix is the following equation:

$$[[T]_{B_\mathbb{V}B_\mathbb{V}}]_{ij} = \langle T(e_j), e_i \rangle$$

(a)

$$\left.\begin{aligned}
[[T]_{B_\mathbb{V}}]_{11} &= \langle T(e_1), e_1 \rangle = \langle \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rangle = -\frac{1}{2} \\
[[T]_{B_\mathbb{V}}]_{21} &= \langle T(e_1), e_2 \rangle = \langle \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rangle = \frac{\sqrt{3}}{2} \\
[[T]_{B_\mathbb{V}}]_{12} &= \langle T(e_2), e_1 \rangle = \langle \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rangle = -\frac{\sqrt{3}}{2} \\
[[T]_{B_\mathbb{V}}]_{22} &= \langle T(e_2), e_2 \rangle = \langle \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rangle = -\frac{1}{2}
\end{aligned}\right. \Rightarrow \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

(b)

$$\begin{bmatrix} \langle T(e_1), e_1 \rangle & \langle T(e_2), e_1 \rangle & \langle T(e_3), e_1 \rangle & \langle T(e_4), e_1 \rangle \\ \langle T(e_1), e_2 \rangle & \langle T(e_2), e_2 \rangle & \langle T(e_3), e_2 \rangle & \langle T(e_4), e_2 \rangle \\ \langle T(e_1), e_3 \rangle & \langle T(e_2), e_3 \rangle & \langle T(e_3), e_3 \rangle & \langle T(e_4), e_3 \rangle \\ \langle T(e_1), e_4 \rangle & \langle T(e_2), e_4 \rangle & \langle T(e_3), e_4 \rangle & \langle T(e_4), e_4 \rangle \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} \langle T(e_1), e_1 \rangle & \langle T(e_2), e_1 \rangle & \langle T(e_3), e_1 \rangle & \langle T(e_4), e_1 \rangle \\ \langle T(e_1), e_2 \rangle & \langle T(e_2), e_2 \rangle & \langle T(e_3), e_2 \rangle & \langle T(e_4), e_2 \rangle \\ \langle T(e_1), e_3 \rangle & \langle T(e_2), e_3 \rangle & \langle T(e_3), e_3 \rangle & \langle T(e_4), e_3 \rangle \\ \langle T(e_1), e_4 \rangle & \langle T(e_2), e_4 \rangle & \langle T(e_3), e_4 \rangle & \langle T(e_4), e_4 \rangle \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$$

4.2.8

The main idea is use orthonormal basis to help use calculate the innerproduct.

Proof *Because we are using* $\mathbb{R}$ *for this question, so* $|\langle v,w\rangle|^2 = \langle v,w\rangle * \overline{\langle v,w\rangle} = \langle v,w\rangle^2$.
Use the ***Theorem 4.10****, and according to the example on* ***Page 240****:*

Denote *The othonomal basis of triangonometric polynomial space is* $B_v = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}sin(ix), \frac{1}{\sqrt{\pi}}cos(jx)|i = 1,...n \in \mathbb{N}, j = 1,...m \in \mathbb{N}\}$*. The ith entry of* $[f]_{B_v}$*, which is under the triangonometric polynomial space is* $t_i = \langle f, e_i\rangle$.

$$\begin{aligned}
\|f\|^2 &= \sum_{i=1}^{i=1+m+n} |\langle v,e_i\rangle|^2 = \sum_{i=1}^{i=1+m+n} \langle v,e_i\rangle^2 = \sum_{i=1}^{i=1+m+n} t_i^2 \\
&= b_0^2 2\pi + \sum_{k=1}^{k=n} a_k^2\pi + \sum_{l=1}^{l=m} b_l^2\pi \\
&\Rightarrow \|f\| = \sqrt{b_0^2 2\pi + \sum_{k=1}^{k=n} a_k^2\pi + \sum_{l=1}^{l=m} b_l^2\pi}
\end{aligned}$$

4.2.10

Because we are using $\mathbb{R}$ for this question, so $\forall k \in \mathbb{F}, |k|^2 = k * \overline{k} = k^2$.

(a)

Proof

$$\begin{aligned}
\|3-2x+x^2\|^2 = \langle 3-2x+x^2, 3-2x+x^2\rangle &= \int_0^1 (3-2x+x^2)\overline{3-2x+x^2}\,dx \\
= \int_0^1 (3-2x+x^2)^2\,dx &= [\frac{(x-1)^5}{5} + \frac{4(x-1)^3}{3} + 4x - 4 + C]_0^1 = \frac{83}{15}
\end{aligned}$$

(b)

Proof *By refering to the conclusion on* ***Page 240.6****, we know that* $\{1, x, x^2\}$ *is not even orthogonal. Therefore, they can not bt othonomal basis. So it will not follow tht* ***Theorem 4.10***

4.2.18

(a)

Denote A^* *is the conjugate transpose of A.* $[A]_{ij} = \overline{[A^*]_{ji}}$.

Proof *Since the basis we are using is orthonomal basis.*
By ***Theorem 4.10****:*

$$\langle x,y\rangle_A = \langle Ax, Ay\rangle \Rightarrow \langle e_j, e_k\rangle_A = \langle Ae_j, Ae_k\rangle = \sum_{i=1}^{i=n}\langle Ae_j, e_i\rangle\overline{\langle Ae_k, e_i\rangle} = \sum_{i=1}^{i=n}[A]_{ij}[A^*]_{ki} = [A^*A]_{kj}$$

(b)

Proof *Use the formula we get from **(a)**, and refer to the **Definition Of Orthonormal Basis**:*

$$\left.\begin{array}{l} k \neq j \rightarrow 0 = \langle e_j, e_k \rangle_A = [A^*A]_{kj} = 0 = [I]_{kj} = [A^{-1}A]_{kj} \\ k = j \rightarrow 1 = \langle e_j, e_j \rangle_A = [A^*A]_{jj} = 1 = [I]_{jj} = [A^{-1}A]_{jj} \end{array}\right. \Rightarrow A^* = A^{-1}$$