

Homework11

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5.3.6

Denote The adjoint B as matrix $K \in M_n(\mathbb{C})$.

$\forall x, y \in \mathbb{C}^n$

$$\begin{aligned}\langle Bx, y \rangle_A &= \langle ABx, Ay \rangle = \langle A^*ABx, y \rangle = \langle x, (A^*AB)^*y \rangle \\ \langle x, Ky \rangle_A &= \langle Ax, AKy \rangle = \langle x, A^*AKy \rangle \\ &\Rightarrow \langle x, ((A^*AB)^* - A^*AK)y \rangle = 0\end{aligned}$$

Since A is invertible, to A^* is invertible, so well as A^*A . By **Proposition 4.2.4**

$$((A^*AB)^* - A^*AK)y = 0 \Rightarrow (A^*AB)^* - A^*AK = 0 \Rightarrow K = (A^*A)^{-1}B^*(A^*A)$$

5.3.8

(a)

$$\begin{aligned}\langle D(p), q \rangle &= \int_{-\infty}^{\infty} p'(x)q(x)e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} q(x)e^{-\frac{x^2}{2}} dp(x) \\ &= p(x)q(x)e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} p(x) d(q(x)e^{-\frac{x^2}{2}}) \\ &= - \int_{-\infty}^{\infty} p(x) d(q(x)e^{-\frac{x^2}{2}}) \\ &= - \int_{-\infty}^{\infty} p(x)(q'(x) - xq(x))e^{-\frac{x^2}{2}} dx \\ &= \langle p, D^*(q) \rangle \\ &\Rightarrow D^*(q(x)) = -q'(x) + xq(x)\end{aligned}$$

(b)

Denote Basis $(1, x, \dots, x^{n-1})$ is B_v . And Basis $(1, x, \dots, x^n)$ is B_w

The i th col of matrix of D with respect of B_v, B_w is

$$[D_i]_{B_w} = [D(x^{i-1})]_{B_w}$$

Therefore, the matrix is $n * (n + 1)$

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & n \end{bmatrix}$$

(c)

Denote Basis $(1, x, \dots, x^n)$ is B_w . And Basis $(1, x, \dots, x^{n-1})$ is B_v

The i th col of matrix of D with respect of B_w, B_v is

$$[D_i^*]_{B_v} = [D^*(x^{i-1})]_{B_v} = [-(i-1)x^{i-2} + x^i]_{B_v}$$

Therefore, the matrix is $(n + 1) * n$

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -2 & 0 & \dots & 0 \\ 0 & 1 & 0 & -3 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

(d)

Since the basis we use are not orthonormal basis. For example

$$\int_{-\infty}^{\infty} 1 * 1 * e^{-\frac{x^2}{2}} dx \neq 1$$

5.3.12

Since it is a self adjoint Map, $T^* = T$. We can rewrite v as following

$$v = T(v) + (v - T(v))$$

Now, consider $U = \text{range}T$, and we need to prove that $T(v)$ is orthogonal to $v - T(v)$

$$\langle T(v), v - T(v) \rangle = \langle v, T^*(v - T(v)) \rangle = \langle v, T(v - T(v)) \rangle = \langle v, T(v) - T^2(v) \rangle = 0$$

Therefore, $T(v)$ is orthogonal to $v - T(v)$, so we get T is the orthogonal projection onto $U = \text{range}T$.

5.3.18/19

Suppose The random two distinct eigenvalues are λ_1, λ_2 , with two independent corresponding eigenvectors v_1, v_2 .

$$\begin{aligned}\langle v_1, v_1 \rangle &= \frac{1}{\lambda_1} \langle T(v_1), v_1 \rangle = \frac{1}{\lambda_1} \langle v_1, T^*(v_1) \rangle \\ &= -\frac{1}{\lambda_1} \langle v_1, T(v_1) \rangle = -\frac{\overline{\lambda_1}}{\lambda_1} \langle v_1, v_1 \rangle\end{aligned}$$

So we get

$$\overline{\lambda_1} = -\lambda_1 \Rightarrow \overline{\lambda_1} + \lambda_1 = 0 = 2\operatorname{Re}(\lambda_1)$$

Since the real part is zero, so it is purely imaginary. We can further prove that

$$\begin{aligned}\langle v_1, v_2 \rangle &= \frac{1}{\lambda_1} \langle T(v_1), v_2 \rangle = \frac{1}{\lambda_1} \langle v_1, T^*(v_2) \rangle \\ &= -\frac{1}{\lambda_1} \langle v_1, T(v_2) \rangle = -\frac{\overline{\lambda_2}}{\lambda_1} \langle v_1, v_2 \rangle\end{aligned}$$

Denote $\lambda_1 = b_1 i, \lambda_2 = b_2 i$

$$-\frac{\overline{\lambda_2}}{\lambda_1} = 1 \Rightarrow b_2 i = b_1 i$$

Contradiction, so all of them are orthogonal.

5.4.2

(a)

First check whether it is hermitian matrix.

$$\begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix}^* = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix}$$

So it is a hermitian. So all the eigenvalues are real.

$$\begin{bmatrix} 7-\lambda & 6 \\ 6 & -2-\lambda \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 7-\lambda & 6 \\ 0 & \lambda^2 - 5\lambda - 50 \end{bmatrix} \Rightarrow \lambda_1 = 10, \lambda_2 = -5$$

So we get two eigenvalues, with corresponding eigenvectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We can get the corresponding orthonormal basis

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Finally,

$$\begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

(b)

We find it is a normal matrix. Now we need to find

$$\begin{bmatrix} 1-\lambda & i \\ i & 1-\lambda \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1-\lambda & i \\ 0 & \lambda^2 - 2\lambda + 2 \end{bmatrix} \Rightarrow \lambda_1 = 1+i, \lambda_2 = 1-i$$

So the corresponding two eigenvectors are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Finally, we get

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

(c)

We find it is a normal matrix. Now we need to find

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & \lambda^2 - 4 \end{bmatrix} \\ \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -2$$

So the corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Finally we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

5.4.6

Since A is hermitian matrix, and is invertible, so all eigenvalue of A are nonzero.

Suppose $A = U \text{diag}(\lambda_1, \dots, \lambda_n) U^*, \forall i \in 1, \dots, n, \lambda_i \neq 0$

Refer to the equation on **Page 324**

$$f(A) = U \text{diag}(f(\lambda_1), \dots, f(\lambda_n)) U^* \Rightarrow \frac{1}{A} = U \text{diag}\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}\right) U^*$$

For A^{-1} , we get

$$A^{-1} = (U \text{diag}(\lambda_1, \dots, \lambda_n) U^*)^{-1} = U \text{diag}\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}\right) U^*$$

Therefore, we get

$$\frac{1}{A} = A^{-1}$$

5.4.10

(a)

We need to prove **5.4.9** first.

Suppose $B = U \text{diag}(\lambda_1, \dots, \lambda_n) U^*, \forall i \in 1, \dots, n, \lambda_i \geq 0$

If every eigenvalue of Hermitian matrix is nonnegative.

$$\langle Bx, x \rangle = \langle U \text{diag}(\lambda_1, \dots, \lambda_n) U^* x, x \rangle = \sum_{i=1}^n \lambda_i |U^* x|^2 \geq 0$$

if $\langle Bx, x \rangle \geq 0, \forall x \in V$, when we set x as its eigenvectors,

$$\langle Bx, x \rangle \geq 0 \Rightarrow \lambda_i \|x_i\|^2 \geq 0 \Rightarrow \lambda_i \geq 0$$

Now for **5.4.10**, we can prove that A^*A is hermitian.

$$(A^*A)^* = A^*A$$

So we can get

$$\langle A^*Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \geq 0$$

So A^*A is positive semidefinite.

(b)

FIRST METHOD: Since $\text{rank} A = n$

$$\text{null} A = n - \text{rank} A = 0$$

Therefore, A is injective.

$$Ax = 0 \iff x = 0$$

If, $y \in \ker A^*A$, by **Proposition 5.16**

$$A^*Ay = 0 \iff Ay = 0, \text{ or } Ay \in \ker A^* \iff Ay = 0, \text{ or } Ay \in (\text{range} A)^\perp$$

Because, $A \neq 0$ So we get

$$\langle Ay, Ay \rangle = 0 \iff \|Ay\|^2 = 0 \iff y = 0$$

So 0 can not be eigenvalue of A , all the eigenvalue is positive. By **Conclusion from (a)** it is positive definite.

SECOND METHOD: Since $\text{rank} A = n$

$$\text{null} A = n - \text{rank} A = 0$$

Therefore, A is injective.

$$Ax = 0 \iff x = 0$$

So we can conclude that

$$A^*Ay = 0 \Rightarrow \langle A^*Ay, y \rangle = 0 \iff \|Ay\|^2 = 0 \iff y = 0$$

So 0 can not be eigenvalue of A , all the eigenvalue is positive. By **Conclusion from (a)** it is positive definite.

5.4.22

Suppose The random two distinct eigenvalues of A are λ_1, λ_2 , with two independent corresponding eigenvectors x_1, x_2 , with coordinate v_1, \dots, v_n and w_1, \dots, w_n . Write e_1, \dots, e_n as standard basis, which is also an orthonormal basis.

Lemma Suppose

$$\begin{aligned} Dx_1 &= \sum_{i=1}^{i=n} v_i \lambda_i e_i = \lambda_1 x_1 = \sum_{i=1}^{i=n} v_i \lambda_1 e_i \\ &\Rightarrow \sum_{i=1}^{i=n} v_i (\lambda_i - \lambda_1) e_i = 0 \end{aligned}$$

Since e_1, \dots, e_n are independent, so that $v_i = 0$ when $\lambda_i \neq \lambda_1$. These statement also holds true for x_2 . So we get

$$x_1 = \sum_{\lambda_i=\lambda_1} v_i e_i, x_2 = \sum_{\lambda_i=\lambda_2} w_i e_i$$

Since $\{e_i | \lambda_i = \lambda_1\} \cap \{e_i | \lambda_i = \lambda_2\} = \emptyset$. We get

$$\langle x_1, x_2 \rangle = \left\langle \sum_{\lambda_i=\lambda_1} v_i e_i, \sum_{\lambda_i=\lambda_2} w_i e_i \right\rangle = 0$$

Therefore, eigenvectors with different eigenvalues of diagonal matrix is orthogonal.

Since A is a normal matrix, by **Theorem 5.23** we can rewrite it as $A = UDU^*$, with U, U^* is an unitary matrix.

$$\begin{aligned} Ax_1 &= UDU^*x_1 = \lambda_1 x_1 \Rightarrow D(U^*x_1) = \lambda_1(U^*x_1) \\ Ax_2 &= UDU^*x_2 = \lambda_2 x_2 \Rightarrow D(U^*x_2) = \lambda_2(U^*x_2) \end{aligned}$$

By **Theorem 4.25**

$$\langle x_1, x_2 \rangle = \langle U^*(x_1), U^*(x_2) \rangle$$

Consider $U^*(x_1), U^*(x_2)$ as eigenvectors of D with different eigenvalues λ_1, λ_2 . By the **Lemma**, we know that eigenvectors with different eigenvalues are orthogonal.

$$\langle x_1, x_2 \rangle = \langle U^*(x_1), U^*(x_2) \rangle = 0$$

So, eigenvectors with different eigenvalues of normal matrix is orthogonal