

# Homework11

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## 4.5.4

(a)

**Denote** The  $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$  as  $B_v$

The  $i$ th column of  $[R]_{B_v}$  is  $[Rv_i]_{B_v}$ .

$$\begin{aligned} [Rv_1]_{B_v} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B_v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [Rv_2]_{B_v} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B_v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(b)

**Proof** By **Proposition 4.31** and **Definition of Orthogonal Matrix**, we know that if the columns are not **Othonormal**, then the matrix must not be orthogonal.

$$\langle A_1, A_2 \rangle = 2 \neq 0$$

Therefore,  $A$  is not orthogonal matrix.

(c)

**Proof** Because **Proposition 4.30** says, **Suppose  $B_v, B_w$  are orthonormal basis of  $V, W$ .**

But  $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$  itself is not an orthonormal basis.

## 4.5.6

**Proof** Since  $\mathbb{C}_{2\pi}(\mathbb{R})$  is equipped with innerproduct, so it is both an innerproduct space and a normed space.

Suppose  $\forall f(x) \in \mathbb{C}_{2\pi}(\mathbb{R})$ , which means  $f(x)$  is a continuous  $2\pi$  periodic function  $\Rightarrow \exists g(x) = f(x - t) \in \mathbb{C}_{2\pi}(\mathbb{R}), T(g)(x) = f(x + t)$ .  $T$  is a surjective linear map.

**Denote**

$$\begin{aligned} \int_a^b f(x) \overline{f(x)} dx &= F(b) - F(a) = \int_{a+2\pi}^{b+2\pi} f(x) \overline{f(x)} dx = F(b+2\pi) - F(a+2\pi) \\ \Rightarrow F(b+2\pi) - F(b) &= F(a+2\pi) - F(a) \end{aligned}$$

*So we get*

$$\begin{aligned} \|(Tf)(x)\| \int_0^{2\pi} (Tf)(x) \overline{(Tf)(x)} dx &= \int_0^{2\pi} f(x+t) \overline{f(x+t)} dx \\ &= \int_0^{2\pi} f(x+t) \overline{f(x+t)} d(x+t) \\ &= \int_t^{t+2\pi} f(x') \overline{f(x')} d(x') \\ &= F(t+2\pi) - F(t) \\ &= F(0+2\pi) - F(0) \\ &= \|(f)(x)\| \end{aligned}$$

*So, it is an isometry.*

**4.5.8**

**4.5.10**

**4.5.14**

(a)

**Proof** By **Proposition 4.30**, we know that if  $U$  is unitary, then all columns of  $U$  is orthonormal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonormal basis. By **Corrolary 4.30**, we Know that  $U$  is an isometry.

$$\|U\|_{op} = \max_{\|v\|=1, v \in \mathbb{C}^n} \|Uv\| = \|v\| = 1$$

(b)

By **Proposition 4.30**, we know that if  $U$  is unitary, then all columns of  $U$  is orthonormal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonormal basis. By **Corrolary 4.30**, we Know that  $U$  is an isometry.

$$\|U\|_F = \sqrt{\text{tr} U^* U} = \sqrt{\text{tr} I} = \sqrt{n}$$

**5.1.4**

**Denote**  $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$   $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$   $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

First use standard basis as  $(e_1, e_2, e_3)$ . Second use  $(e_2, e_3)$  as basis for output.

$$\begin{aligned} T(e_1) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 * e_2 \\ T(e_3) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 * e_3 \\ T(e_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 * e_2 \end{aligned}$$

Since  $\text{rank}T = 2$ , so the  $\sigma_1 = 1, \sigma_2 = 1$

The right vectors are  $(e_1, e_3, e_2)$ , left vectors are  $(e_2, e_3)$ , and  $\sigma_1 = 1, \sigma_2 = 1$ .

### 5.1.6

**Denote** The standard basis of  $\mathbb{V}$  is  $(e_1, e_2, \dots, e_n)$ , with  $\dim V = n$ . As well as  $P := P_U$  with  $U \subset V, \dim U = m, U = \text{span}(e_1, \dots, e_m)$ , and  $V = U \oplus U^\perp$ .

(a)

By **Theorem 5.3**, we know that the singular value are unique. So we just need to prove that we can find only 1, 0.

By **Theorem 4.16.2**, when  $i \leq m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = \langle e_i, e_{j=i} \rangle e_{j=i} = 1 * e_i$$

By **Theorem 4.16.2**, when  $i > m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = 0$$

So, the singular values are only 0, 1.

(b)

I choose  $(e_1, e_2, \dots, e_n)$  as both left and right singular vectors, and with the same sequence.

### 5.1.10

**Proof** Since  $T$  is invertible, it must be surjective, so  $n = \text{range}T = \dim W = \dim V$ . Since  $\forall i \in \{1, 2, \dots, n\}, \sigma_i > 0, \neq 0$ . By **Theorem 5.3**, we know the singular value is unique.

**Denote** The orthonormal basis of  $V$  is  $(e_1, e_2, \dots, e_n)$ , and the orthonormal basis of  $W$  is  $(f_1, f_2, \dots, f_n)$ .

**Consider** Construct  $T^{-1} \in \mathcal{L}(W, V)$ , with singular value  $\sigma'_1, \sigma'_2, \dots, \sigma'_n$

Since  $\forall i \in \{1, 2, \dots, n\}$

$$T(e_i) = \sigma_i f_i \Rightarrow T^{-1}(f_i) = \frac{1}{\sigma_i} e_i$$

$\forall i \geq j$

$$\sigma_i \geq \sigma_j \Rightarrow \frac{1}{\sigma_j} \geq \frac{1}{\sigma_i}$$

Therefore,  $\{\frac{1}{\sigma_n}, \frac{1}{\sigma_{n-1}}, \dots, \frac{1}{\sigma_1}\}$  are singular value of  $T^{-1}$ .

By **Key Ideas On Page 295 last point**, we know that the largest singular value is operator norm of that map.

$$\|T^{-1}\|_{op} = \frac{1}{\sigma_n} \Rightarrow \|T^{-1}\|_{op}^{-1} = \sigma_n$$

**Denote**  $v = \sum_{i=1}^{i=n} a_i e_i$ ,  $\|v\| = 1$

$$\begin{aligned} \|T(v)\| &= \|T(\sum_{i=1}^{i=n} a_i e_i)\| = \|\sum_{i=1}^{i=n} a_i T(e_i)\| = \|\sum_{i=1}^{i=n} a_i \sigma_i f_i\| \\ &= \sqrt{\sum_{i=1}^{i=n} |\sigma_i|^2 \|a_i f_i\|^2} \\ &\geq |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2 \|f_i\|^2} \\ &= |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2} = |\sigma_n| \\ &\Rightarrow \min_{\|v\|=1} \|Tv\| = \sigma_n \end{aligned}$$

#### 5.1.14

(a)

**Denote** The orthonormal basis of  $\mathbb{W}$  is  $(f_1, f_2, \dots, f_m)$ , with  $\dim \mathbb{W} = m$

$$\begin{aligned} \|Tv\| &= \|T(\sum_{i=j}^{i=p} a_i e_i + \sum_{i=p+1}^{i=n} a_i e_i)\| = \|\sum_{i=j}^{i=p} a_i T(e_i) + \sum_{i=p+1}^{i=n} a_i T(e_i)\| \\ &= \|\sum_{i=j}^{i=p} a_i \sigma_i f_i + \sum_{i=p+1}^{i=n} 0\| = \|\sum_{i=j}^{i=p} a_i \sigma_i f_i\| \\ &= \sqrt{\sum_{i=j}^{i=p} \|a_i \sigma_i f_i\|^2} \leq \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=p} \|a_i f_i\|^2} \\ &\leq \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=n} \|a_i f_i\|^2} = \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=n} \|a_i\|^2} \\ &= \sigma_j \|v\| \end{aligned}$$

(b)

**Denote** The orthonormal basis of  $\mathbb{W}$  is  $(f_1, f_2, \dots, f_m)$ , with  $\dim \mathbb{W} = m$

$$\begin{aligned}
\|Tv\| &= \|T(\sum_{i=1}^{i=j} a_i e_i)\| = \|\sum_{i=1}^{i=j} a_i T(e_i)\| \\
&= \|\sum_{i=1}^{i=j} a_i \sigma_i f_i\| = \sqrt{\sum_{i=1}^{i=j} \|a_i \sigma_i f_i\|^2} \\
&\geq \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=j} \|a_i f_i\|^2} = \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=j} \|a_i\|^2} \\
&= \sigma_j \|v\|
\end{aligned}$$

(c)

We will try to prove this by using contrapositive statement. **If**  $\forall v \in U$  **that**  $\|Tv\| \geq \sigma_j \|v\|$ , **then**  $U \in V, \dim U = n - j + 1$ .

$$\begin{aligned}
\sigma_j \|v\| &= \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=n} \|a_i\|^2} \leq \|Tv\| = \|\sum_{i=1}^{i=n} a_i T(v_i)\| = \|\sum_{i=1}^{i=p} a_i \sigma_i f_i\| \\
&\Rightarrow |\sigma_j|^2 \sum_{i=1}^{i=n} \|a_i\|^2 \leq \sum_{i=1}^{i=p} \|\sigma_i a_i\|^2
\end{aligned}$$