Homework10

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4.1.4

Denote *The jth col of* A, B *is* A_j, B_j*. And* $A_j, B_j \in \mathbb{C}^m$*. So that,* $\|A_j\|^2 = \langle A_j, A_j\rangle$*, as well as for* B_j*.*

Proof *By the definition of inner product* $\sum_{i=1}^{i=m}[A]_{ij}[B]_{ij} = \langle A_j, B_j\rangle$*. For the inner product of two matrices:* $\langle A, B\rangle_F = \sum_{j=1}^{j=n}\sum_{i=1}^{i=m}[A]_{ij}[B]_{ij} = \sum_{j=1}^{j=n}\langle A_j, B_j\rangle$*. By the definition of the norm:* $\|A\|^2{}_F = \langle A, A\rangle_F = \sum_{j=1}^{j=n}\langle A_j, A_j\rangle = \sum_{j=1}^{j=n}\|A_j\|^2$*.*

4.1.8

Suppose there exists $w \in V$. $\langle v-w, v-w\rangle = \langle w-v, w-v\rangle \Rightarrow \|v-w\| = \|w-v\|$

$$\|v\| = \|v-u+u\| \le \|v-u\| + \|u\| \Rightarrow \|v-u\| \ge \|v\| - \|u\| = 11 - 2 = 9$$
$$\|v-u\| \le \|v-w\| + \|w-u\| = \|v-w\| + \|u-w\| = 8$$

Contradiction.

4.1.14

(a)

$$\frac{1}{4}(\|v+w\|^2 - \|v-w\|^2) = \frac{1}{4}(\langle v+w, v+w\rangle - \langle v-w, v-w\rangle) = \frac{1}{4}(2\langle v,w\rangle - (-2)\langle v,w\rangle) = \langle v,w\rangle$$

(b)

$$\begin{aligned}
&\frac{1}{4}(\|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2) \\
&= \frac{1}{4}(\langle v+w, v+w\rangle - \langle v-w, v-w\rangle + i(\langle v+iw, v+iw\rangle - \langle v-iw, v-iw\rangle)) \\
&= \frac{1}{4}(2\langle w,v\rangle + 2\langle v,w\rangle + i(2i\langle w,v\rangle - 2i\langle v,w\rangle)) = \langle v,w\rangle
\end{aligned}$$

4.1.20

Proof $\forall v, w \in \mathbb{V}$. *Use **Definition** and **Proposition 4.2.2**.*

$$\langle 0, v\rangle = 0 * \langle w, v\rangle = 0$$
$$\langle v, 0\rangle = \overline{0} * \langle v, w\rangle = 0$$

4.2.4

This Exercise we need to ouse **Theorem 4.9**

(a)

$$\begin{aligned}
&\langle \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rangle = 2\sqrt{2} \\
&\langle \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rangle = -\sqrt{6}
\end{aligned} \Rightarrow \begin{bmatrix} 2\sqrt{2} \\ -\sqrt{6} \end{bmatrix}$$

(b)

Refer to **Exercise 4.1.4**

$$\begin{aligned}
&\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rangle = \frac{21}{2} \\
&\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \rangle = \frac{3}{2} \\
&\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rangle = \frac{3}{2} \\
&\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rangle = \frac{-7}{2}
\end{aligned} \Rightarrow \begin{bmatrix} \frac{21}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ \frac{-7}{2} \end{bmatrix}$$

(c)

$$\begin{aligned}
&\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rangle = 0 \\
&\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \rangle = -2+i \\
&\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \rangle = -2 \\
&\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} \rangle = -2-i
\end{aligned} \Rightarrow \begin{bmatrix} 0 \\ -2+i \\ -2 \\ -2-i \end{bmatrix}$$

4.2.6

This question we will use **Theorem 4.9**. Assume the *ith* row and *jth* column of the matrix is the following equation:

$$[[T]_{B_{\mathbb{V}}B_{\mathbb{V}}}]_{ij} = \langle T(e_j), e_i\rangle$$

(a)

$$\begin{aligned}
[[T]_{B_{\mathbb{V}}}]_{11} &= \langle T(e_1), e_1\rangle = \langle \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}\rangle = -\frac{1}{2}\\
[[T]_{B_{\mathbb{V}}}]_{21} &= \langle T(e_1), e_2\rangle = \langle \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}\rangle = \frac{\sqrt{3}}{2}\\
[[T]_{B_{\mathbb{V}}}]_{12} &= \langle T(e_2), e_1\rangle = \langle \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}\rangle = -\frac{\sqrt{3}}{2}\\
[[T]_{B_{\mathbb{V}}}]_{22} &= \langle T(e_2), e_2\rangle = \langle \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}\rangle = -\frac{1}{2}
\end{aligned}
\Rightarrow \begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$$

(b)

$$\begin{bmatrix}
\langle T(e_1), e_1\rangle & \langle T(e_2), e_1\rangle & \langle T(e_3), e_1\rangle & \langle T(e_4), e_1\rangle\\
\langle T(e_1), e_2\rangle & \langle T(e_2), e_2\rangle & \langle T(e_3), e_2\rangle & \langle T(e_4), e_2\rangle\\
\langle T(e_1), e_3\rangle & \langle T(e_2), e_3\rangle & \langle T(e_3), e_3\rangle & \langle T(e_4), e_3\rangle\\
\langle T(e_1), e_4\rangle & \langle T(e_2), e_4\rangle & \langle T(e_3), e_4\rangle & \langle T(e_4), e_4\rangle
\end{bmatrix}
= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

(c)

$$\begin{bmatrix}
\langle T(e_1), e_1\rangle & \langle T(e_2), e_1\rangle & \langle T(e_3), e_1\rangle & \langle T(e_4), e_1\rangle\\
\langle T(e_1), e_2\rangle & \langle T(e_2), e_2\rangle & \langle T(e_3), e_2\rangle & \langle T(e_4), e_2\rangle\\
\langle T(e_1), e_3\rangle & \langle T(e_2), e_3\rangle & \langle T(e_3), e_3\rangle & \langle T(e_4), e_3\rangle\\
\langle T(e_1), e_4\rangle & \langle T(e_2), e_4\rangle & \langle T(e_3), e_4\rangle & \langle T(e_4), e_4\rangle
\end{bmatrix}
= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & -1 & 0\\ 0 & i & 0 & 0\end{bmatrix}$$

4.2.8

The main idea is use orthonormal basis to help use calculate the innerproduct.

Proof *Because we are using* $\mathbb{R}$ *for this question, so* $|\langle v, w\rangle|^2 = \langle v, w\rangle * \overline{\langle v, w\rangle} = \langle v, w\rangle^2$.
Use the ***Theorem 4.10****, and according to the example on* ***Page 240****:*

Denote *The othonomal basis of triangonometric polynomial space is* $B_v = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}sin(kx), \frac{1}{\sqrt{\pi}}cos(lx)|k = 1, ...n \in \mathbb{N}, l = 1, ...m \in \mathbb{N}\}$. *The ith entry of* $[f]_{B_v}$*, which is under the triangonometric polynomial space is* $t_i = \langle f, e_i\rangle$.

Since the element of the function is constitute by the orthonoml basis. So when any part of the function, $b_0, a_k sin(k\theta), b_l cos(l\theta)$*, we can tret them as constant times of one of the orthonomal basis.*

$$
\begin{aligned}
\|f\|^2 &= \sum_{i=1}^{i=1+m+n} |\langle f, e_i\rangle|^2 = \sum_{i=1}^{i=1+m+n} \langle f, e_i\rangle^2 = \sum_{i=1}^{i=1+m+n} t_i^2 \\
&= \sum_{i=1}^{i=1+m+n} (|\langle\sum_{k=1}^{k=n} a_k sin(k\theta), e_i\rangle|^2 + |\langle\sum_{l=1}^{l=m} b_l cos(l\theta), e_i\rangle|^2 + |\langle b_0, e_i\rangle|^2) \\
&= \sum_{i=1}^{i=1+m+n} (|\langle\sqrt{\pi}\sum_{k=1}^{k=n} \frac{1}{\sqrt{\pi}} a_k sin(k\theta), e_i\rangle|^2 + |\langle\sqrt{\pi}\sum_{l=1}^{l=m} \frac{1}{\sqrt{\pi}} b_l cos(l\theta), e_i\rangle|^2 + |\sqrt{2\pi} b_0 \langle\frac{1}{\sqrt{2\pi}}, e_i\rangle|^2) \\
&= b_0^2 2\pi + \sum_{k=1}^{k=n} a_k^2\pi + \sum_{l=1}^{l=m} b_l^2\pi \\
&\Rightarrow \|f\| = \sqrt{b_0^2 2\pi + \sum_{k=1}^{k=n} a_k^2\pi + \sum_{l=1}^{l=m} b_l^2\pi}
\end{aligned}
$$

4.2.10

Because we are using $\mathbb{R}$ for this question, so $\forall k \in \mathbb{F}, |k|^2 = k * \overline{k} = k^2$.

(a)

Proof

$$
\begin{aligned}
\|3 - 2x + x^2\|^2 = \langle 3 - 2x + x^2, 3 - 2x + x^2\rangle &= \int_0^1 (3 - 2x + x^2)\overline{3 - 2x + x^2}\, dx \\
&= \int_0^1 (3 - 2x + x^2)^2\, dx = [\frac{(x-1)^5}{5} + \frac{4(x-1)^3}{3} + 4x - 4 + C]_0^1 = \frac{83}{15}
\end{aligned}
$$

(b)

Proof *By refering to the conclusion on* ***Page 240.6****, we know that* $\{1, x, x^2\}$ *is not even orthogonal. Therefore, they can not bt othonomal basis. So it will not follow tht* ***Theorem 4.10***

4.2.18

(a)

Denote *A^* is the conjugate transpose of A. $[A]_{ij} = \overline{[A^*]_{ji}}$.*

Proof *Since the basis we are using is orthonomal basis.*
*By **Theorem 4.10**:*

$$\langle x,y\rangle_A = \langle Ax,Ay\rangle \Rightarrow \langle e_j,e_k\rangle_A = \langle Ae_j,Ae_k\rangle = \sum_{i=1}^{i=n}\langle Ae_j,e_i\rangle\overline{\langle Ae_k,e_i\rangle} = \sum_{i=1}^{i=n}[A]_{ij}[A^*]_{ki} = [A^*A]_{kj}$$

(b)

Proof *Use the formula we get from **(a)**, and refer to the **Definition Of Orthonormal Basis**:*

$$\begin{matrix} k\neq j \rightarrow 0 = \langle e_j,e_k\rangle_A = [A^*A]_{kj} = 0 = [I]_{kj} = [A^{-1}A]_{kj} \\ k = j \rightarrow 1 = \langle e_j,e_j\rangle_A = [A^*A]_{jj} = 1 = [I]_{jj} = [A^{-1}A]_{jj} \end{matrix} \Rightarrow A^* = A^{-1}$$

4.3.2

(a)

Refer to **Proposition 4.18**

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix})^{-1}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix}$$

(b)

Refer to **Proposition 4.18**

$$A = \begin{bmatrix} 1 & 0 \\ i & 1 \\ 0 & i \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 0 \\ i & 1 \\ 0 & i \end{bmatrix}(\begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & -i \end{bmatrix}\begin{bmatrix} 1 & 0 \\ i & 1 \\ 0 & i \end{bmatrix})^{-1}\begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & -i \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -i & 1 \\ i & 2 & -i \\ 1 & i & 2 \end{bmatrix}$$

4.3.4

(a)

Refer to the **Theorem 4.16.8**

$$A = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} (\begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix})^{-1} \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 9 & 3 & 12 \\ 3 & 1 & 4 \\ 12 & 4 & 16 \end{bmatrix}$$

$$[P_{U^\perp}]_\xi = I - [P_U]_\xi = \frac{1}{26}(\begin{bmatrix} 26 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 26 \end{bmatrix} - \begin{bmatrix} 9 & 3 & 12 \\ 3 & 1 & 4 \\ 12 & 4 & 16 \end{bmatrix}) = \frac{1}{26} \begin{bmatrix} 17 & -3 & -12 \\ -3 & 25 & -4 \\ -12 & -4 & 10 \end{bmatrix}$$

(b)

It is easy to find out that we need to find $\langle \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \rangle^\perp$.

$$A = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} (\begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix})^{-1} \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 9 & 6 & 3 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$[P_{U^\perp}]_\xi = I - [P_U]_\xi = \frac{1}{14}(\begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 3 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}) = \frac{1}{14} \begin{bmatrix} 5 & -6 & -3 \\ -6 & 10 & -2 \\ -3 & -2 & 13 \end{bmatrix}$$

(c)

It is easy to find out that we need to find $\langle \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix} \rangle^\perp$.

$$A = \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix} (\begin{bmatrix} 1 & i & -i \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ i \end{bmatrix})^{-1} \begin{bmatrix} 1 & i & -i \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & i & -i \\ -i & 1 & -1 \\ i & -1 & 1 \end{bmatrix}$$

$$[P_{U^\perp}]_\xi = I - [P_U]_\xi = \frac{1}{3}(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & i & -i \\ -i & 1 & -1 \\ i & -1 & 1 \end{bmatrix}) = \frac{1}{3} \begin{bmatrix} 2 & -i & i \\ i & 2 & 1 \\ -i & 1 & 2 \end{bmatrix}$$

(d)

Do REF to the given matrix, we get $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Which means that we only get first and second cols as basis.

$A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$

$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} (\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix})^{-1} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} = \frac{1}{83} \begin{bmatrix} 14 & -11 & 2 & 29 \\ -11 & 62 & 34 & -5 \\ 2 & 34 & 24 & 16 \\ 29 & -5 & 16 & 66 \end{bmatrix}$

4.3.14

(a)

Proof *Refer to* ***Theorem 1.10*** *Define* 0 *is zero matrix in* $\mathbb{V}, \mathbb{W}$. $\forall v, v_1, v_2, w, w_1, w_2 \in \mathbb{V}, \mathbb{W}$, $c \in \mathbb{F}$*we have*

$$\begin{aligned} &0 + v = v, 0^T = 0 \Rightarrow 0 \in \mathbb{V} \\ &0 + w = w, 0^T = -0 \Rightarrow 0 \in \mathbb{W} \\ &v_1 + cv_2 = (v_1 + cv_2)^T \Rightarrow v_1 + cv_2 \in \mathbb{V} \\ &w_1 + cw_2 = (-w_1 - cw_2)^T = -(w_1 + cw_2)^T \Rightarrow w_1 + cw_2 \in \mathbb{W} \end{aligned}$$

So $\mathbb{V}, \mathbb{W}$ *is subspace of* $M_n(\mathbb{R})$.

(b)

Refer to **Frobenius Inner Product**: $\forall X, Y \in M_n(\mathbb{R}), X^T = X^*, \langle X, Y \rangle = tr(XY^*) = tr(XY^T)$, since the field we are using here is $\mathbb{R}$.

Claim *Rewrite* V *as* $V = \{\frac{X+X^T}{2} | X \in M_n(\mathbb{R})\}$, W *as* $W = \{\frac{X-X^T}{2} | X \in M_n(\mathbb{R})\}$. $(\frac{X+X^T}{2})^T = \frac{X^T+X}{2} = \frac{X+X^T}{2}$, *and* $(\frac{X-X^T}{2})^T = \frac{X^T-X}{2} = -\frac{X-X^T}{2}$

By the **Definition Of Orthogonal Projection**, I can build a function $P_V : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ and $X = \frac{X+X^T}{2} + \frac{X-X^T}{2}, X \in M_n(\mathbb{R}), \frac{X+X^T}{2} \in \mathbb{V}, \frac{X-X^T}{2} \in \mathbb{W}, P_V(X) = \frac{X+X^T}{2}$.

Proof *By the* ***Proposition 3.60*** $\forall v \in \mathbb{V}, \forall w \in \mathbb{W}$, $tr(vw^T) = -tr(vw) = -tr(v^Tw) = -tr(wv^T) = -tr(wv^T)^T = -tr(vw^T) \Rightarrow tr(vw^T) = 0$. *Therfore the function I build is valid.* $\mathbb{V}^\perp = \mathbb{W}$.

(c)

Refer to the conclusion from **(b)**: $P_V(X) = \frac{X+X^T}{2} = \frac{X+X^*}{2} = Re(A)$, $P_W(X)(= P_{V^\perp}(X) = I(X) - P_V(X)) = \frac{X-X^T}{2} = \frac{X-X^*}{2} = i\frac{X-X^*}{2i} = iIm(A)$

4.3.22

(a)

Proof $\forall i \in \{1,2,...,n\}$, *the ith col of* $[P_U]_B$ *is* $\begin{bmatrix} \langle P_U(e_i), e_1\rangle \\ \langle P_U(e_i), e_2\rangle \\ ... \\ \langle P_U(e_i), e_n\rangle \end{bmatrix}$.

When $i \le m$, $\begin{bmatrix} \langle P_U(e_i), e_1\rangle \\ \langle P_U(e_i), e_2\rangle \\ ... \\ \langle P_U(e_i), e_n\rangle \end{bmatrix} = \begin{bmatrix} \langle \sum_{k=1}^{k=m} \langle e_i, e_k\rangle e_k, e_1\rangle \\ \langle \sum_{k=1}^{k=m} \langle e_i, e_k\rangle e_k, e_2\rangle \\ ... \\ \langle \sum_{k=1}^{k=m} \langle e_i, e_k\rangle e_k, e_n\rangle \end{bmatrix} = \begin{bmatrix} \langle e_i, e_1\rangle \\ \langle e_i, e_2\rangle \\ ... \\ \langle e_i, e_n\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ... \\ 1 \\ ... \\ 0 \end{bmatrix}$.

The ith row will be one which is $[[P_U]_B]_{ii} = 1$.

When $i > m$, $\begin{bmatrix} \langle P_U(e_i), e_1\rangle \\ \langle P_U(e_i), e_2\rangle \\ ... \\ \langle P_U(e_i), e_n\rangle \end{bmatrix} = \begin{bmatrix} \langle \sum_{k=1}^{k=m} \langle e_i, e_k\rangle e_k, e_1\rangle \\ \langle \sum_{k=1}^{k=m} \langle e_i, e_k\rangle e_k, e_2\rangle \\ ... \\ \langle \sum_{k=1}^{k=m} \langle e_i, e_k\rangle e_k, e_n\rangle \end{bmatrix} = \begin{bmatrix} \langle 0, e_1\rangle \\ \langle 0, e_2\rangle \\ ... \\ \langle 0, e_n\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \\ ... \\ 0 \end{bmatrix}$.

$\Rightarrow [P_U]_B = diag(1,...1,0,...,0)$.

(b)

Proof *Refer to the* ***Definition Of Orthogonal Projection****:* $\forall u \in rangeP_U, u \in U$
$\Rightarrow RangeP_U \subset U$.
By the conclusion of ***part2****, we know* $\forall u \in U, u = \sum_{j=1}^{j=m} a_j e_j, P_U(u) = \sum_{i=1}^{i=m} \langle u, e_i\rangle e_i = \sum_{i=1}^{i=m} \langle \sum_{j=1}^{j=m} a_j e_j, e_i\rangle e_i = \sum_{i=1}^{i=m}\sum_{j=1}^{j=m} a_j \langle e_j, e_i\rangle e_i = \sum_{i=1}^{i=m} a_i e_i = u \Rightarrow \forall u \in U, P_U(u) = u$.
I can conclude that $\forall u \in U, \exists u \in U \subset V, u = P_U(u) \in RangeP_U \Rightarrow U \subset RangeP_U$.
$\Rightarrow U = RangeP_U$

(c)

Proof *Refer to* ***Definition Of Orthogonal Projection****:* $\forall v \in V, u \in U, w \in U^{\perp}, v = u + w, P_U(v) = u$.
$\forall v \in KerP_U, P_U(v) = 0 \Rightarrow v = 0 + v \Rightarrow v \in U^{\perp} \Rightarrow KerP_U \subset U^{\perp}$
$\forall v \in U^{\perp} \subset V, v = 0 + v \Rightarrow P_U(v) = 0 \Rightarrow v \in KerU^{\perp} \Rightarrow U^{\perp} \subset KerP_U$
$\Rightarrow U^{\perp} = KerP_U$

(d)

Proof *Suppose* $\forall v \in V, v = u + w, u \in U, w \in U^{\perp}$

$$(I - P_U)v = v - P_U(v) = v - u = w = P_{U^{\perp}}(v) \Rightarrow I - P_U = P_{U^{\perp}}$$

(e)

Proof *Refer to the conclusion from* ***(b)***

$$\forall v \in V, u \in U, w \in U^{\perp}, v = u + w, P_U^2(v) = P_U(u) = u = P_U(v) \Rightarrow P_U^2 = P_U$$