

# Homework11

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## 4.5.4

(a)

**Denote** The  $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$  as  $B_v$

The  $i$ th column of  $[R]_{B_v}$  is  $[Rv_i]_{B_v}$ .

$$\begin{aligned} [Rv_1]_{B_v} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B_v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [Rv_2]_{B_v} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B_v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(b)

**Proof** By **Proposition 4.31** and **Definition of Orthogonal Matrix**, we know that if the columns are not **Othonormal**, then the matrix must not be orthogonal.

$$\langle A_1, A_2 \rangle = 2 \neq 0$$

Therefore,  $A$  is not orthogonal matrix.

(c)

**Proof** Because **Proposition 4.30** says, **Suppose  $B_v, B_w$  are orthonormal basis of  $V, W$ .**

But  $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$  itself is not an orthonormal basis.

## 4.5.6

**Proof** Since  $\mathbb{C}_{2\pi}(\mathbb{R})$  is equipped with innerproduct, so it is both an innerproduct space and a normed space.

Suppose  $\forall f(x) \in \mathbb{C}_{2\pi}(\mathbb{R})$ , which means  $f(x)$  is a continuous  $2\pi$  periodic function  $\Rightarrow \exists g(x) = f(x - t) \in \mathbb{C}_{2\pi}(\mathbb{R}), T(g)(x) = f(x + t)$ .  $T$  is a surjective linear map.

**Denote**

$$\begin{aligned}\int_a^b f(x) \overline{f(x)} dx &= F(b) - F(a) = \int_{a+2\pi}^{b+2\pi} f(x) \overline{f(x)} dx = F(b+2\pi) - F(a+2\pi) \\ \Rightarrow F(b+2\pi) - F(b) &= F(a+2\pi) - F(a)\end{aligned}$$

*So we get*

$$\begin{aligned}\|(Tf)(x)\| \int_0^{2\pi} (Tf)(x) \overline{(Tf)(x)} dx &= \int_0^{2\pi} f(x+t) \overline{f(x+t)} dx \\ &= \int_0^{2\pi} f(x+t) \overline{f(x+t)} d(x+t) \\ &= \int_t^{t+2\pi} f(x') \overline{f(x')} d(x') \\ &= F(t+2\pi) - F(t) \\ &= F(0+2\pi) - F(0) \\ &= \|(f)(x)\|\end{aligned}$$

*So, it is an isometry.*

#### 4.5.8

**On Hand Writing**

#### 4.5.10

**On Hand Writing**

#### 4.5.14

(a)

**Proof** By **Proposition 4.30**, we know that if  $U$  is unitary, then all columns of  $U$  is orthonormal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonormal basis. By **Corrolary 4.30**, we Know that  $U$  is an isometry.

$$\|U\|_{op} = \max_{\|v\|=1, v \in \mathbb{C}^n} \|Uv\| = \|v\| = 1$$

(b)

By **Proposition 4.30**, we know that if  $U$  is unitary, then all columns of  $U$  is orthonormal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonormal basis. By **Corrolary 4.30**, we Know that  $U$  is an isometry.

$$\|U\|_F = \sqrt{\text{tr} U^* U} = \sqrt{\text{tr} I} = \sqrt{n}$$

#### 5.1.4

**Denote**  $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$   $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$   $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

First use standard basis as  $(e_1, e_2, e_3)$ . Second use  $(e_2, e_3)$  as basis for output.

$$\begin{aligned} T(e_1) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 * e_2 \\ T(e_3) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 * e_3 \\ T(e_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 * e_2 \end{aligned}$$

Since  $\text{rank}T = 2$ , so the  $\sigma_1 = 1, \sigma_2 = 1$

The right vectors are  $(e_1, e_3, e_2)$ , left vectors are  $(e_2, e_3)$ , and  $\sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 0$ .

#### 5.1.6

**Denote** The standard basis of  $\mathbb{V}$  is  $(e_1, e_2, \dots, e_n)$ , with  $\dim V = n$ . As well as  $P := P_U$  with  $U \subset V, \dim U = m, U = \text{span}(e_1, \dots, e_m)$ , and  $V = U \oplus U^\perp$ .

(a)

By **Theorem 5.3**, we know that the singular value are unique. So we just need to prove that we can find only 1, 0.

By **Theorem 4.16.2**, when  $i \leq m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = \langle e_i, e_{j=i} \rangle e_{j=i} = 1 * e_i$$

By **Theorem 4.16.2**, when  $i > m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = 0$$

So, the singular values are only 0, 1.

(b)

I choose  $(e_1, e_2, \dots, e_n)$  as both left and right singular vectors, and with the same sequence.

#### 5.1.10

**Proof** Since  $T$  is invertible, it must be surjective, so  $n = \text{range}T = \dim W = \dim V$ . Since  $\forall i \in \{1, 2, \dots, n\}, \sigma_i > 0, \neq 0$ . By **Theorem 5.3**, we know the singular value is unique.

**Denote** The orthonormal basis of  $V$  is  $(e_1, e_2, \dots, e_n)$ , and the orthonormal basis of  $W$  is  $(f_1, f_2, \dots, f_n)$ .

**Consider** Construct  $T^{-1} \in \mathfrak{L}(W, V)$ , with singular value  $\sigma'_1, \sigma'_2, \dots, \sigma'_n$

Since  $\forall i \in \{1, 2, \dots, n\}$

$$T(e_i) = \sigma_i f_i \Rightarrow T^{-1}(f_i) = \frac{1}{\sigma_i} e_i$$

$\forall i \geq j$

$$\sigma_i \geq \sigma_j \Rightarrow \frac{1}{\sigma_j} \geq \frac{1}{\sigma_i}$$

Therefore,  $\{\frac{1}{\sigma_n}, \frac{1}{\sigma_{n-1}}, \dots, \frac{1}{\sigma_1}\}$  are singular value of  $T^{-1}$ .

By **Key Ideas On Page 295 last point**, we know that the largest singular value is operator norm of that map.

$$\|T^{-1}\|_{op} = \frac{1}{\sigma_n} \Rightarrow \|T^{-1}\|_{op}^{-1} = \sigma_n$$

**Denote**  $v = \sum_{i=1}^{i=n} a_i e_i$ ,  $\|v\| = 1$

$$\begin{aligned} \|T(v)\| &= \|T(\sum_{i=1}^{i=n} a_i e_i)\| = \|\sum_{i=1}^{i=n} a_i T(e_i)\| = \|\sum_{i=1}^{i=n} a_i \sigma_i f_i\| \\ &= \sqrt{\sum_{i=1}^{i=n} |\sigma_i|^2 \|a_i f_i\|^2} \\ &\geq |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2 \|f_i\|^2} \\ &= |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2} = |\sigma_n| \\ &\Rightarrow \min_{\|v\|=1} \|Tv\| = \sigma_n \end{aligned}$$

#### 5.1.14

(a)

**Denote** The orthonormal basis of  $\mathbb{W}$  is  $(f_1, f_2, \dots, f_m)$ , with  $\dim \mathbb{W} = m$

$$\begin{aligned}
\|Tv\| &= \|T(\sum_{i=j}^{i=p} a_i e_i + \sum_{i=p+1}^{i=n} a_i e_i)\| = \|\sum_{i=j}^{i=p} a_i T(e_i) + \sum_{i=p+1}^{i=n} a_i T(e_i)\| \\
&= \|\sum_{i=j}^{i=p} a_i \sigma_i f_i + \sum_{i=p+1}^{i=n} 0\| = \|\sum_{i=j}^{i=p} a_i \sigma_i f_i\| \\
&= \sqrt{\sum_{i=j}^{i=p} \|a_i \sigma_i f_i\|^2} \leq \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=p} \|a_i f_i\|^2} \\
&\leq \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=n} \|a_i f_i\|^2} = \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=n} \|a_i\|^2} \\
&= \sigma_j \|v\|
\end{aligned}$$

(b)

**Denote** The orthonormal basis of  $\mathbb{W}$  is  $(f_1, f_2, \dots, f_m)$ , with  $\dim \mathbb{W} = m$

$$\begin{aligned}
\|Tv\| &= \|T(\sum_{i=1}^{i=j} a_i e_i)\| = \|\sum_{i=1}^{i=j} a_i T(e_i)\| \\
&= \|\sum_{i=1}^{i=j} a_i \sigma_i f_i\| = \sqrt{\sum_{i=1}^{i=j} \|a_i \sigma_i f_i\|^2} \\
&\geq \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=j} \|a_i f_i\|^2} = \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=j} \|a_i\|^2} \\
&= \sigma_j \|v\|
\end{aligned}$$

(c)

First I could build a vector space  $V = \langle e_1, e_2, \dots, e_j \rangle$ ,  $\dim V = j$ . By **Lemma 3.22**,

$$\dim U + \dim V = n - j + 1 + j \geq n \Rightarrow U \cap V \neq 0$$

By conclusion from **Part (b)**

$$\exists v_0 \in U, v_0 \in V, \|Tv_0\| \geq \sigma_j v_0$$

(d)

Use the conclusion from **Part (c)**, we know that, if  $\dim U = n - j + 1 \Rightarrow \exists v_0 \in U, \|Tv\| \geq \sigma_j \|v_0\|$  This is equivalent to

$$\dim U = n - j + 1 \Rightarrow \max_{v \in U} \|Tv\| \geq \sigma_j \|v\|$$

When  $\forall v \in U, \|v\| = 1$ , we get

$$\dim U = n - j + 1 \Rightarrow \max_{v \in U, \|v\|=1} \|Tv\| \geq \sigma_j$$

Since it is true  $\forall U \subset V, \dim U = n - j + 1$ , the statement is equivalent to

$$\min_{\dim U = n-j+1} \max_{v \in U, \|v\|=1} \|Tv\| \geq \sigma_j \quad (1)$$

Now we need to prove another side. Now we build  $U' = \langle e_j, e_{j+1}, \dots, e_n \rangle, \dim U' = n - j + 1$ . From conclusion from **Part (a)**, we get

$$\max_{v \in U', \|v\|=1} \|Tv\| \leq \sigma_j \quad (2)$$

From **Equation (1), and (2)**, we get

$$\min_{\dim U = n-j+1} \max_{v \in U, \|v\|=1} \|Tv\| \leq \max_{v \in U', \|v\|=1} \|Tv\| \leq \sigma_j \quad (3)$$

From **Equation (1), and (3)**, I prove

$$\min_{\dim U = n-j+1} \max_{v \in U, \|v\|=1} \|Tv\| = \sigma_j$$

### 5.2.2

(a)

The singular value of matrix  $A$  is the sqrt root of eigenvalue of  $AA^*$  or  $AA^*$ .

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 5. And it gets two independent eigenvectors

So the singular value is  $\sigma_1 = \sigma_2 = \sqrt{5}$ .

(b)

The singular value of matrix  $A$  is the sqrt root of eigenvalue of  $AA^*$  or  $AA^*$ .

$$\begin{bmatrix} 4 & -3 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 25, 100. And it gets two independent eigenvectors

So the singular value is  $\sigma_1 = 10, \sigma_2 = 5$ .

(c)

The singular value of matrix  $A$  is the sqrt root of eigenvalue of  $AA^*$  or  $AA^*$ .

$$\begin{bmatrix} i & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 2, 1.

So the singular value is  $\sigma_1 = \sqrt{2}, \sigma_2 = 1$ .

(d)

The singular value of matrix  $A$  is the sqrt root of eigenvalue of  $AA^*$  or  $AA^*$ .

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 6, 3.  
So the singular value is  $\sigma_1 = \sqrt{6}, \sigma_2 = \sqrt{3}$ .

(e)

The singular value of matrix  $A$  is the sqrt root of eigenvalue of  $AA^*$  or  $AA^*$ .

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 9, 4, 1  
So the singular value is  $\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1$ .

### 5.2.8

By **Proposition 5.5**, we know the singular value of  $A$  is positive sqrt root of eigenvalue of  $A^*A$  or  $AA^*$ .

$$A^*A = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \overline{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & \overline{\lambda_2} & 0 & \dots & 0 \\ 0 & 0 & \overline{\lambda_3} & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \overline{\lambda_n} \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 & 0 & 0 & \dots & 0 \\ 0 & |\lambda_2|^2 & 0 & \dots & 0 \\ 0 & 0 & |\lambda_3|^2 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & |\lambda_n|^2 \end{bmatrix}$$

Since it is a diagonal matrix, so the eigenvalue of  $A^*A$  is its diagonal. Singular value of  $A$  is sqrt root of its eigenvalue  $\Rightarrow$  The singular value of  $A$  is  $|\lambda_1|, |\lambda_2|, |\lambda_3|, \dots, |\lambda_n|$

### 5.2.16

(a)

$$AA^\dagger b = b \Rightarrow \exists x, x = A^\dagger b, Ax = b$$

From the conclusion **5.2.15(d)**

$$\exists x, Ax = b \Rightarrow AA^\dagger Ax = Ax = b \Rightarrow AA^\dagger b = b$$

(b)

First show that:

$$\Sigma^\dagger \Sigma = \text{diag}(1, \dots, 1, 0, \dots, 0)$$

Since the system is consistent

$$A^\dagger Ax = A^\dagger b$$

So we get,  $U, V, U^*, V^*$  are isometry.

$$\begin{aligned} \|A^\dagger b\| &= \|A^\dagger Ax\| = \|V\Sigma^\dagger U^* U \Sigma V^* x\| = \|V\Sigma^\dagger \Sigma V^* x\| \\ &= \|\Sigma^\dagger \Sigma V^* x\| \leq \|IV^* x\| = \|V^* x\| = \|x\| \end{aligned}$$

(c)

$$\forall v_0 \in \ker A, Av_0 = 0 \Rightarrow (I_n - A^\dagger A)v_0 = v_0 - 0 = v_0 \Rightarrow v_0 \in C(I - A^\dagger A)$$

So we get

$$\ker A \subset C(I - A^\dagger A)$$

For another side

$$\begin{aligned} \forall x \in C(I - A^\dagger A), x &= (I - A^\dagger A)v_0 \Rightarrow Ax = A(I - A^\dagger A)v_0 = (A - AA^\dagger A)v_0 = 0 \\ \Rightarrow C(I - A^\dagger A) &\subset \ker A \end{aligned}$$

We get

$$C(I - A^\dagger A) = \ker A$$

Since  $Ax = b$  is consistent, so  $AA^\dagger b = b$ ,  $A^\dagger b$  is one of the solution. By **Proposition 2.42**

$$\{x \in F^n | Ax = b\} = \{A^\dagger b + k | k \in \ker A\} = \{A^\dagger b + [I_n - A^\dagger A]w | w \in F^n\}$$

### 5.2.20

At each  $i$ th position:

$$\Sigma = \sum_{i=1}^{i=r} \text{diag}(0, 0, \dots, \sigma_i, 0, \dots, 0)$$



So we get

$$\begin{aligned}
A &= U\Sigma V^* = U\left(\sum_{i=1}^{i=r} \text{diag}(0, 0, \dots, \sigma_i, 0, \dots, 0)\right)V^* \\
&= U\sum_{i=1}^{i=r} \text{diag}(0, 0, \dots, \sigma_i, 0, \dots, 0)V^* \\
&= U\left(\sum_{i=1}^{i=r} \begin{bmatrix} 0^* \\ 0^* \\ \dots \\ \sigma_i(V_i)^* \\ 0^* \\ \dots \\ 0^* \end{bmatrix}\right) = \sum_{i=1}^{i=r} U\begin{bmatrix} 0^* \\ 0^* \\ \dots \\ \sigma_i(V_i)^* \\ 0^* \\ \dots \\ 0^* \end{bmatrix} \\
&= \sum_{i=1}^{i=r} u_i \sigma_i(v_i)^* = \sum_{i=1}^{i=r} \sigma_i u_i(v_i)^*
\end{aligned}$$

$$\|T(v)\|^2 = \left\| \sum_{i=1}^{i=\dim V} a_i T(e_i) \right\|^2 = \left\| \sum_{i=1}^{i=p} a_i \sigma_i f_i \right\|^2 \geq \sigma_p \sum_{i=1}^{i=p} \|a_i\|^2$$