

Homework11

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4.5.4

(a)

Denote The $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ as B_v

The i th column of $[R]_{B_v}$ is $[Rv_i]_{B_v}$.

$$\begin{aligned} [Rv_1]_{B_v} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B_v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [Rv_2]_{B_v} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B_v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(b)

Proof By **Proposition 4.31** and **Definition of Orthogonal Matrix**, we know that if the columns are not **Othonormal**, then the matrix must not be orthogonal.

$$\langle A_1, A_2 \rangle = 2 \neq 0$$

Therefore, A is not orthogonal matrix.

(c)

Proof Because **Proposition 4.30** says, **Suppose B_v, B_w are orthonormal basis of V, W .**

But $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ itself is not an orthonormal basis.

4.5.6

Proof Since $\mathbb{C}_{2\pi}(\mathbb{R})$ is equipped with innerproduct, so it is both an innerproduct space and a normed space.

Suppose $\forall f(x) \in \mathbb{C}_{2\pi}(\mathbb{R})$, which means $f(x)$ is a continuous 2π periodic function $\Rightarrow \exists g(x) = f(x - t) \in \mathbb{C}_{2\pi}(\mathbb{R}), T(g)(x) = f(x + t)$. T is a surjective linear map.

Denote

$$\begin{aligned}\int_a^b f(x) \overline{f(x)} dx &= F(b) - F(a) = \int_{a+2\pi}^{b+2\pi} f(x) \overline{f(x)} dx = F(b+2\pi) - F(a+2\pi) \\ \Rightarrow F(b+2\pi) - F(b) &= F(a+2\pi) - F(a)\end{aligned}$$

So we get

$$\begin{aligned}\|(Tf)(x)\| \int_0^{2\pi} (Tf)(x) \overline{(Tf)(x)} dx &= \int_0^{2\pi} f(x+t) \overline{f(x+t)} dx \\ &= \int_0^{2\pi} f(x+t) \overline{f(x+t)} d(x+t) \\ &= \int_t^{t+2\pi} f(x') \overline{f(x')} d(x') \\ &= F(t+2\pi) - F(t) \\ &= F(0+2\pi) - F(0) \\ &= \|(f)(x)\|\end{aligned}$$

So, it is an isometry.

4.5.14

(a)

Proof By **Proposition 4.30**, we know that if U is unitary, then all columns of U is orthonormal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonormal basis. By **Corrolary 4.30**, we Know that U is an isometry.

$$\|U\|_{op} = \max_{\|v\|=1, v \in \mathbb{C}^n} \|Uv\| = \|v\| = 1$$

(b)

By **Proposition 4.30**, we know that if U is unitary, then all columns of U is orthonormal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonormal basis. By **Corrolary 4.30**, we Know that U is an isometry.

$$\|U\|_F = \sqrt{\text{tr} U^* U} = \sqrt{\text{tr} I} = \sqrt{n}$$

5.1.4

Denote $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

First use standard basis as (e_1, e_2, e_3) . Second use (e_2, e_3) as basis for output.

$$\begin{aligned} T(e_1) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 * e_2 \\ T(e_3) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 * e_3 \\ T(e_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 * e_2 \end{aligned}$$

Since $\text{rank}T = 2$, so the $\sigma_1 = 1, \sigma_2 = 1$

The right vectors are (e_1, e_3, e_2) , left vectors are (e_2, e_3) , and $\sigma_1 = 1, \sigma_2 = 1, \sigma_3 = 0$.

5.1.6

Denote The standard basis of \mathbb{V} is (e_1, e_2, \dots, e_n) , with $\dim V = n$. As well as $P := P_U$ with $U \subset V, \dim U = m, U = \text{span}(e_1, \dots, e_m)$, and $V = U \oplus U^\perp$.

(a)

By **Theorem 5.3**, we know that the singular value are unique. So we just need to prove that we can find only 1, 0.

By **Theorem 4.16.2**, when $i \leq m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = \langle e_i, e_{j=i} \rangle e_{j=i} = 1 * e_i$$

By **Theorem 4.16.2**, when $i > m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = 0$$

So, the singular values are only 0, 1.

(b)

I choose (e_1, e_2, \dots, e_n) as both left and right singular vectors, and with the same sequence.

5.1.10

Proof Since T is invertible, it must be surjective, so $n = \text{range}T = \dim W = \dim V$. Since $\forall i \in \{1, 2, \dots, n\}, \sigma_i > 0, \neq 0$. By **Theorem 5.3**, we know the singular value is unique.

Denote The orthonormal basis of V is (e_1, e_2, \dots, e_n) , and the orthonormal basis of W is (f_1, f_2, \dots, f_n) .

Consider Construct $T^{-1} \in \mathcal{L}(W, V)$, with singular value $\sigma'_1, \sigma'_2, \dots, \sigma'_n$

Since $\forall i \in \{1, 2, \dots, n\}$

$$T(e_i) = \sigma_i f_i \Rightarrow T^{-1}(f_i) = \frac{1}{\sigma_i} e_i$$

$\forall i \geq j$

$$\sigma_i \geq \sigma_j \Rightarrow \frac{1}{\sigma_j} \geq \frac{1}{\sigma_i}$$

Therefore, $\{\frac{1}{\sigma_n}, \frac{1}{\sigma_{n-1}}, \dots, \frac{1}{\sigma_1}\}$ are singular value of T^{-1} .

By **Key Ideas On Page 295 last point**, we know that the largest singular value is operator norm of that map.

$$\|T^{-1}\|_{op} = \frac{1}{\sigma_n} \Rightarrow \|T^{-1}\|_{op}^{-1} = \sigma_n$$

Denote $v = \sum_{i=1}^{i=n} a_i e_i$, $\|v\| = 1$

$$\begin{aligned} \|T(v)\| &= \|T(\sum_{i=1}^{i=n} a_i e_i)\| = \|\sum_{i=1}^{i=n} a_i T(e_i)\| = \|\sum_{i=1}^{i=n} a_i \sigma_i f_i\| \\ &= \sqrt{\sum_{i=1}^{i=n} |\sigma_i|^2 \|a_i f_i\|^2} \\ &\geq |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2 \|f_i\|^2} \\ &= |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2} = |\sigma_n| \\ &\Rightarrow \min_{\|v\|=1} \|Tv\| = \sigma_n \end{aligned}$$

5.1.14

(a)

Denote The orthonormal basis of \mathbb{W} is (f_1, f_2, \dots, f_m) , with $\dim \mathbb{W} = m$

$$\begin{aligned} \|Tv\| &= \|T(\sum_{i=j}^{i=p} a_i e_i + \sum_{i=p+1}^{i=n} a_i e_i)\| = \|\sum_{i=j}^{i=p} a_i T(e_i) + \sum_{i=p+1}^{i=n} a_i T(e_i)\| \\ &= \|\sum_{i=j}^{i=p} a_i \sigma_i f_i + \sum_{i=p+1}^{i=n} 0\| = \|\sum_{i=j}^{i=p} a_i \sigma_i f_i\| \\ &= \sqrt{\sum_{i=j}^{i=p} \|a_i \sigma_i f_i\|^2} \leq \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=p} \|a_i f_i\|^2} \\ &\leq \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=n} \|a_i f_i\|^2} = \sqrt{|\sigma_j|^2 \sum_{i=j}^{i=n} \|a_i\|^2} \\ &= \sigma_j \|v\| \end{aligned}$$

(b)

Denote The orthonormal basis of \mathbb{W} is (f_1, f_2, \dots, f_m) , with $\dim \mathbb{W} = m$

$$\begin{aligned}
\|Tv\| &= \|T(\sum_{i=1}^{i=j} a_i e_i)\| = \|\sum_{i=1}^{i=j} a_i T(e_i)\| \\
&= \|\sum_{i=1}^{i=j} a_i \sigma_i f_i\| = \sqrt{\sum_{i=1}^{i=j} \|a_i \sigma_i f_i\|^2} \\
&\geq \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=j} \|a_i f_i\|^2} = \sqrt{|\sigma_j|^2 \sum_{i=1}^{i=j} \|a_i\|^2} \\
&= \sigma_j \|v\|
\end{aligned}$$

(c)

First I could build a vector space $V = \langle e_1, e_2, \dots, e_j \rangle$, $\dim V = j$. By **Lemma 3.22**,

$$\dim U + \dim V = n - j + 1 + j \geq n \Rightarrow U \cap V \neq 0$$

By conclusion from **Part (b)**

$$\exists v_0 \in U, v_0 \in V, \|Tv_0\| \geq \sigma_j v_0$$

(d)

Use the conclusion from **Part (c)**, we know that, if $\dim U = n - j + 1 \Rightarrow \exists v_0 \in U, \|Tv\| \geq \sigma_j \|v_0\|$ This is equivalent to

$$\dim U = n - j + 1 \Rightarrow \max_{v \in U} \|Tv\| \geq \sigma_j \|v\|$$

When $\forall v \in U, \|v\| = 1$, we get

$$\dim U = n - j + 1 \Rightarrow \max_{v \in U, \|v\|=1} \|Tv\| \geq \sigma_j$$

Since it is true $\forall U \subset V, \dim U = n - j + 1$, the statement is equivalent to

$$\min_{\dim U = n - j + 1} \max_{v \in U, \|v\|=1} \|Tv\| \geq \sigma_j \quad (1)$$

Now we need to prove another side. Now we build $U' = \langle e_j, e_{j+1}, \dots, e_n \rangle$, $\dim U' = n - j + 1$. From conclusion from **Part (a)**, we get

$$\max_{v \in U', \|v\|=1} \|Tv\| \leq \sigma_j \quad (2)$$

From **Equation (1), and (2)**, we get

$$\min_{\dim U = n - j + 1} \max_{v \in U, \|v\|=1} \|Tv\| \leq \max_{v \in U', \|v\|=1} \|Tv\| \leq \sigma_j \quad (3)$$

From **Equation (1), and (3)**, I prove

$$\min_{\dim U = n - j + 1} \max_{v \in U, \|v\|=1} \|Tv\| = \sigma_j$$

5.2.2

(a)

The singular value of matrix A is the sqrt root of eigenvalue of AA^* or AA^* .

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 5. And it gets two independent eigenvectors

So the singular value is $\sigma_1 = \sigma_2 = \sqrt{5}$.

(b)

The singular value of matrix A is the sqrt root of eigenvalue of AA^* or AA^* .

$$\begin{bmatrix} 4 & -3 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 100 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 25, 100. And it gets two independent eigenvectors

So the singular value is $\sigma_1 = 10, \sigma_2 = 5$.

(c)

The singular value of matrix A is the sqrt root of eigenvalue of AA^* or AA^* .

$$\begin{bmatrix} i & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 2, 1.

So the singular value is $\sigma_1 = \sqrt{2}, \sigma_2 = 1$.

(d)

The singular value of matrix A is the sqrt root of eigenvalue of AA^* or AA^* .

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 6, 3.

So the singular value is $\sigma_1 = \sqrt{6}, \sigma_2 = \sqrt{3}$.

(e)

The singular value of matrix A is the sqrt root of eigenvalue of AA^* or AA^* .

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Since it is an uppertriangular matrix, so the eigenvalue is its diagonal, which is 9, 4, 1

So the singular value is $\sigma_1 = 3, \sigma_2 = 2, \sigma_3 = 1$.

5.2.8

By **Proposition 5.5**, we know the singular value of A is positive sqrt root of eigenvalue of A^*A or AA^* .

$$A^*A = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \overline{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & \overline{\lambda_2} & 0 & \dots & 0 \\ 0 & 0 & \overline{\lambda_3} & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & \overline{\lambda_n} \end{bmatrix} = \begin{bmatrix} |\lambda_1|^2 & 0 & 0 & \dots & 0 \\ 0 & |\lambda_2|^2 & 0 & \dots & 0 \\ 0 & 0 & |\lambda_3|^2 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & |\lambda_n|^2 \end{bmatrix}$$

Since it is a diagonal matrix, so the eigenvalue of A^*A is its diagonal. Singular value of A is sqrt root of its eigenvalue \Rightarrow The singular value of A is $|\lambda_1|, |\lambda_2|, |\lambda_3|, \dots, |\lambda_n|$

5.2.16

(a)

$$AA^\dagger b = b \Rightarrow \exists x, x = A^\dagger b, Ax = b$$

From the conclusion **5.2.15(d)**

$$\exists x, Ax = b \Rightarrow AA^\dagger Ax = Ax = b \Rightarrow AA^\dagger b = b$$

(b)

First show that:

$$\Sigma^\dagger \Sigma = \text{diag}(1, \dots, 1, 0, \dots, 0)$$

Since the system is consistent

$$A^\dagger Ax = A^\dagger b$$

So we get, U, V, U^*, V^* are isometry.

$$\begin{aligned} \|A^\dagger b\| &= \|A^\dagger Ax\| = \|V\Sigma^\dagger U^* U \Sigma V^* x\| = \|V\Sigma^\dagger \Sigma V^* x\| \\ &= \|\Sigma^\dagger \Sigma V^* x\| \leq \|IV^* x\| = \|V^* x\| = \|x\| \end{aligned}$$

(c)

$$\forall v_0 \in \ker A, Av_0 = 0 \Rightarrow (I_n - A^\dagger A)v_0 = v_0 - 0 = v_0 \Rightarrow v_0 \in C(I - A^\dagger A)$$

So we get

$$\ker A \subset C(I - A^\dagger A)$$

For another side

$$\begin{aligned} \forall x \in C(I - A^\dagger A), x = (I - A^\dagger A)v_0 \Rightarrow Ax = A(I - A^\dagger A)v_0 = (A - AA^\dagger A)v_0 = 0 \\ \Rightarrow C(I - A^\dagger A) \subset \ker A \end{aligned}$$

We get

$$C(I - A^\dagger A) = \ker A$$

Since $Ax = b$ is consistent, so $AA^\dagger b = b$, $A^\dagger b$ is one of the solution. By **Proposition 2.42**

$$\{x \in F^n | Ax = b\} = \{A^\dagger b + k | k \in \ker A\} = \{A^\dagger b + [I_n - A^\dagger A]w | w \in F^n\}$$

5.2.20

At each i th position:

$$\Sigma = \sum_{i=1}^{i=r} \text{diag}(0, 0, \dots, \sigma_i, 0, \dots, 0)$$

So we get

$$\begin{aligned} A &= U\Sigma V^* = U\left(\sum_{i=1}^{i=r} \text{diag}(0, 0, \dots, \sigma_i, 0, \dots, 0)\right)V^* \\ &= U\sum_{i=1}^{i=r} \text{diag}(0, 0, \dots, \sigma_i, 0, \dots, 0)V^* \\ &= U\left(\sum_{i=1}^{i=r} \begin{bmatrix} 0^* \\ 0^* \\ \dots \\ \sigma_i(V_i)^* \\ 0^* \\ \dots \\ 0^* \end{bmatrix}\right) = \sum_{i=1}^{i=r} U \begin{bmatrix} 0^* \\ 0^* \\ \dots \\ \sigma_i(V_i)^* \\ 0^* \\ \dots \\ 0^* \end{bmatrix} \\ &= \sum_{i=1}^{i=r} u_i \sigma_i(v_i)^* = \sum_{i=1}^{i=r} \sigma_i u_i(v_i)^* \end{aligned}$$