Homework10

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4.1.4

Denote *The jth col of* A, B *is* A_j, B_j*. And* $A_j, B_j \in \mathbb{C}^m$*. So that,* $\|A_j\|^2 = \langle A_j, A_j\rangle$*, as well as for* B_j*.*

Proof *By the definition of inner product* $\sum_{i=1}^{i=m}[A]_{ij}[B]_{ij} = \langle A_j, B_j\rangle$*. For the inner product of two matrices:* $\langle A, B\rangle_F = \sum_{j=1}^{j=n}\sum_{i=1}^{i=m}[A]_{ij}[B]_{ij} = \sum_{j=1}^{j=n}\langle A_j, B_j\rangle$*. By the definition of the norm:* $\|A\|^2{}_F = \langle A, A\rangle_F = \sum_{j=1}^{j=n}\langle A_j, A_j\rangle = \sum_{j=1}^{j=n}\|A_j\|^2$*.*

4.1.8

Suppose there exists $w \in V$. $\langle v-w, v-w\rangle = \langle w-v, w-v\rangle \Rightarrow \|v-w\| = \|w-v\|$

$$\|v\| = \|v-u+u\| \le \|v-u\| + \|u\| \Rightarrow \|v-u\| \ge \|v\| - \|u\| = 11-2 = 9$$
$$\|v-u\| \le \|v-w\| + \|w-u\| = \|v-w\| + \|u-w\| = 8$$

Contradiction.

4.1.14

(a)

$$\frac{1}{4}(\|v+w\|^2 - \|v-w\|^2) = \frac{1}{4}(\langle v+w, v+w\rangle - \langle v-w, v-w\rangle) = \frac{1}{4}(2\langle v,w\rangle - (-2)\langle v,w\rangle) = \langle v,w\rangle$$

(b)

$$\begin{aligned}
&\frac{1}{4}(\|v+w\|^2 - \|v-w\|^2 + i\|v+iw\|^2 - i\|v-iw\|^2) \\
&= \frac{1}{4}(\langle v+w, v+w\rangle - \langle v-w, v-w\rangle + i(\langle v+iw, v+iw\rangle - \langle v-iw, v-iw\rangle)) \\
&= \frac{1}{4}(2\langle w,v\rangle + 2\langle v,w\rangle + i(2i\langle w,v\rangle - 2i\langle v,w\rangle)) = \langle v,w\rangle
\end{aligned}$$

4.1.14

Proof $\forall v, w \in \mathbb{V}$. *Use **Definition** and **Proposition 4.2.2**.*

$$\langle 0, v\rangle = 0 * \langle w, v\rangle = 0$$
$$\langle v, 0\rangle = \overline{0} * \langle v, w\rangle = 0$$

4.2.4

This Exercise we need to ouse **Theorem 4.9**

(a)

$$\langle \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rangle = 2\sqrt{2}$$

$$\langle \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rangle = -\sqrt{6}$$

$$\Rightarrow \begin{bmatrix} 2\sqrt{2} \\ -\sqrt{6} \end{bmatrix}$$

(b)

Refer to **Exercise 4.1.4**

$$\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rangle = \frac{21}{2}$$

$$\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \rangle = \frac{3}{2}$$

$$\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rangle = \frac{3}{2}$$

$$\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rangle = \frac{-7}{2}$$

$$\Rightarrow \begin{bmatrix} \frac{21}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ \frac{-7}{2} \end{bmatrix}$$

(c)

$$\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rangle = 0$$

$$\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \rangle = -2 + i$$

$$\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \rangle = -2$$

$$\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} \rangle = -2 - i$$

$$\Rightarrow \begin{bmatrix} 0 \\ -2 + i \\ -2 \\ -2 - i \end{bmatrix}$$

4.2.6

This question we will use **Theorem 4.9**. Assume the *ith* row and *jth* column of the matrix is the following equation:

$$[[T]_{B_\mathbb{V} B_\mathbb{V}}]_{ij} = \langle T(e_j), e_i \rangle$$

(a)

$$\begin{aligned}
[[T]_{B_\mathbb{V}}]_{11} &= \langle T(e_1), e_1 \rangle = \langle \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rangle = -\frac{1}{2} \\
[[T]_{B_\mathbb{V}}]_{21} &= \langle T(e_1), e_2 \rangle = \langle \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rangle = \frac{\sqrt{3}}{2} \\
[[T]_{B_\mathbb{V}}]_{12} &= \langle T(e_2), e_1 \rangle = \langle \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rangle = -\frac{\sqrt{3}}{2} \\
[[T]_{B_\mathbb{V}}]_{22} &= \langle T(e_2), e_2 \rangle = \langle \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \rangle = -\frac{1}{2}
\end{aligned}
\Rightarrow \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

(b)

$$\begin{bmatrix} \langle T(e_1), e_1 \rangle & \langle T(e_2), e_1 \rangle & \langle T(e_3), e_1 \rangle & \langle T(e_4), e_1 \rangle \\ \langle T(e_1), e_2 \rangle & \langle T(e_2), e_2 \rangle & \langle T(e_3), e_2 \rangle & \langle T(e_4), e_2 \rangle \\ \langle T(e_1), e_3 \rangle & \langle T(e_2), e_3 \rangle & \langle T(e_3), e_3 \rangle & \langle T(e_4), e_3 \rangle \\ \langle T(e_1), e_4 \rangle & \langle T(e_2), e_4 \rangle & \langle T(e_3), e_4 \rangle & \langle T(e_4), e_4 \rangle \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} \langle T(e_1), e_1 \rangle & \langle T(e_2), e_1 \rangle & \langle T(e_3), e_1 \rangle & \langle T(e_4), e_1 \rangle \\ \langle T(e_1), e_2 \rangle & \langle T(e_2), e_2 \rangle & \langle T(e_3), e_2 \rangle & \langle T(e_4), e_2 \rangle \\ \langle T(e_1), e_3 \rangle & \langle T(e_2), e_3 \rangle & \langle T(e_3), e_3 \rangle & \langle T(e_4), e_3 \rangle \\ \langle T(e_1), e_4 \rangle & \langle T(e_2), e_4 \rangle & \langle T(e_3), e_4 \rangle & \langle T(e_4), e_4 \rangle \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \end{bmatrix}$$

4.2.8