Homework11

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5.3.6

Denote *The adjoint B as matrix $K \in M_n(\mathbb{C})$.*

$\forall x, y \in \mathbb{C}^n$

$$\begin{aligned}
&\langle Bx, y\rangle_A = \langle ABx, Ay\rangle = \langle A^*ABx, y\rangle = \langle x, (A^*AB)^*y\rangle \\
&\langle x, Ky\rangle_A = \langle Ax, AKy\rangle = \langle x, A^*AKy\rangle \\
&\Rightarrow \langle x, ((A^*AB)^* - A^*AK)y\rangle = 0
\end{aligned}$$

Since A is invertible, to A^* is invertible, so well as A^*A. By **Proposition 4.2.4**

$$((A^*AB)^* - A^*AK)y = 0 \Rightarrow (A^*AB)^* - A^*AK = 0 \Rightarrow K = (A^*A)^{-1}B^*(A^*A)$$

5.3.8

(a)

$$\begin{aligned}
\langle D(p), q\rangle &= \int_{-\infty}^{\infty} p'(x)q(x)e^{-\frac{x^2}{2}}\,dx = \int_{-\infty}^{\infty} q(x)e^{-\frac{x^2}{2}}\,dp(x) \\
&= p(x)q(x)e^{-\frac{x^2}{2}}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} p(x)\,d(q(x)e^{-\frac{x^2}{2}}) \\
&= -\int_{-\infty}^{\infty} p(x)\,d(q(x)e^{-\frac{x^2}{2}}) \\
&= -\int_{-\infty}^{\infty} p(x)(q'(x) - xq(x))e^{-\frac{x^2}{2}}\,dx \\
&= \langle p, D^*(q)\rangle \\
&\Rightarrow D^*(q(x)) = -q'(x) + xq(x)
\end{aligned}$$

(b)

Denote *Basis $(1, x, ..., x^{n-1})$ is B_v. And Basis $(1, x, ..., x^n)$ is B_w*

The *ith* col of matrix of D with respect of B_v, B_w is

$$[D_i]_{B_w} = [D(x^{i-1})]_{B_w}$$

Therefore, the matrix is $n*(n+1)$

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 2 & 0 & ... & 0 \\ 0 & 0 & 0 & 3 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & n \end{bmatrix}$$

(c)

Denote *Basis* $(1, x, ..., x^n)$ *is* B_w*. And Basis* $(1, x, ..., x^{n-1})$ *is* B_v

The ith col of matrix of D with respect of B_w, B_v is

$$[D_i^*]_{B_v} = [D^*(x^{i-1})]_{B_v} = [-(i-1)x^{i-2} + x^i]_{B_v}$$

Therefore, the matrix is $(n+1)*n$

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 & ... & 0 \\ 1 & 0 & -2 & 0 & ... & 0 \\ 0 & 1 & 0 & -3 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

(d)

Since the basis we use are not orthonomal basis. For example

$$\int_{-\infty}^{\infty} 1*1*e^{-\frac{x^2}{2}}dx \neq 1$$

5.3.12

Since it is a self adjoint Map, $T^* = T$. We can rewrite v as following

$$v = T(v) + (v - T(v))$$

Now, consider $U = rangeT$, and we need to prove that $T(v)$ is orthogonal to $v - T(v)$

$$\langle T(v), v - T(v)\rangle = \langle v, T^*(v - T(v))\rangle = \langle v, T(v - T(v))\rangle = \langle v, T(v) - T^2(v)\rangle = 0$$

Therefore, $T(v)$ is orthogonal to $v - T(v)$, so we get T is the orthogonal projection onto $U = rangeT$.

5.3.18/19

Suppose *The random two distinct eignvalues are* λ_1, λ_2*, with two independent corrosponding eignvectors* v_1, v_2*.*

$$\begin{aligned}\langle v_1, v_1\rangle &= \frac{1}{\lambda_1}\langle T(v_1), v_1\rangle = \frac{1}{\lambda_1}\langle v_1, T^*(v_1)\rangle \\ &= -\frac{1}{\lambda_1}\langle v_1, T(v_1)\rangle = -\frac{\overline{\lambda_1}}{\lambda_1}\langle v_1, v_1\rangle\end{aligned}$$

So we get

$$\overline{\lambda_1} = -\lambda_1 \Rightarrow \overline{\lambda_1} + \lambda_1 = 0 = 2Re(\lambda_1)$$

Since the real part is zero, so it is purely imaginary. We can further prove that

$$\begin{aligned}\langle v_1, v_2\rangle &= \frac{1}{\lambda_1}\langle T(v_1), v_2\rangle = \frac{1}{\lambda_1}\langle v_1, T^*(v_2)\rangle \\ &= -\frac{1}{\lambda_1}\langle v_1, T(v_2)\rangle = -\frac{\overline{\lambda_2}}{\lambda_1}\langle v_1, v_2\rangle\end{aligned}$$

Denote $\lambda_1 = b_1 i, \lambda_2 = b_2 i$

$$-\frac{\overline{\lambda_2}}{\lambda_1} = 1 \Rightarrow b_2 i = b_1 i$$

Contradiction, so all of them are orthogonal.

5.4.2

(a)

First check whether it is hermitian matrix.

$$\begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix}^* = \begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix}$$

So it is a hermitian. So all the eignvalues are real.

$$\begin{bmatrix} 7-\lambda & 6 \\ 6 & -2-\lambda \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 7-\lambda & 6 \\ 0 & \lambda^2 - 5\lambda - 50 \end{bmatrix} \Rightarrow \lambda_1 = 10, \lambda_2 = -5$$

So we get two eignvalues, with corrosponding eignvectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

We can get the corrosponding othonormal basis

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Finally,

$$\begin{bmatrix} 7 & 6 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}$$

(b)

We find it is a normal matrix. Now we need to find

$$\begin{bmatrix} 1-\lambda & i \\ i & 1-\lambda \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1-\lambda & i \\ 0 & \lambda^2 - 2\lambda + 2 \end{bmatrix} \Rightarrow \lambda_1 = 1+i, \lambda_2 = 1-i$$

So the corrosponding two eignvectors are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Finally, we get

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

(c)

We find it is a normal matrix. Now we need to find

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & \lambda^2 - 4 \end{bmatrix}$$
$$\Rightarrow \lambda_1 = 1, \lambda_1 = 2, \lambda_1 = -2$$

So the corrosponding eignvectors are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Finally we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

5.4.6

Since A is hermitian matrix, and is invertible, so all eignvalue of A are nonzero.

Suppose $A = Udiag(\lambda_1, ..., \lambda_n)U^*, \forall i \in 1, ..., n, \lambda_i \neq 0$

Refer to the equation on **Page 324**

$$f(A) = Udiag(f(\lambda_1), ..., f(\lambda_n))U^* \Rightarrow \frac{1}{A} = Udiag(\frac{1}{\lambda_1}, ..., \frac{1}{\lambda_n})U^*$$

For A^{-1}, we get

$$A^{-1} = (Udiag(\lambda_1, ..., \lambda_n)U^*)^{-1} = Udiag(\frac{1}{\lambda_1}, ..., \frac{1}{\lambda_n})U^*$$

Therefore, we get

$$\frac{1}{A} = A^{-1}$$

5.4.10

(a)

We need to prove **5.4.9** first.

Suppose $B = Udiag(\lambda_1, ..., \lambda_n)U^*, \forall i \in 1, ..., n, \lambda_i \geq 0$

If every eignvalue of Hermitian matrix is nonnegative.

$$\langle Bx, x\rangle = \langle Udiag(\lambda_1, ..., \lambda_n)U^*x, x\rangle = \sum_{i=1}^{i=n} \lambda_i |U^*x|^2 \geq 0$$

if $\langle Bx, x\rangle \geq 0, \forall x \in V$, when we set x as its eignvectors,

$$\langle Bx, x\rangle \geq 0 \Rightarrow \lambda_i \|x_i\|^2 \geq 0 \Rightarrow \lambda_i \geq 0$$

Now for **5.4.10**, we can prove that A^*A is hermitian.

$$(A^*A)^* = A^*A$$

So we can get

$$\langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0$$

So A^*A is positive semidefinite.

(b)

FIRST METHOD: Since $rankA = n$

$$nullA = n - rankA = 0$$

Therefore, A is injective.

$$Ax = 0 \iff x = 0$$

If, $y \in kerA^*A$, by **Proposition 5.16**

$$A^*Ay = 0 \iff Ay = 0, or\ Ay \in kerA^* \iff Ay = 0, or\ Ay \in (rangeA)^\perp$$

Because , $A \neq 0$ So we get

$$\langle Ay, Ay\rangle = 0 \iff \|Ay\|^2 = 0 \iff y = 0$$

So 0 can not be eignvalue of A, all the eignvalue is positive. By **Conclusion from (a)** it is positive definite.

SECOND METHOD: Since $rankA = n$

$$nullA = n - rankA = 0$$

Therefore, A is injective.

$$Ax = 0 \iff x = 0$$

So we can conclude that

$$A^*Ay = 0 \Rightarrow \langle A^*Ay, y\rangle = 0 \iff \|Ay\|^2 = 0 \iff y = 0$$

So 0 can not be eignvalue of A, all the eignvalue is positive. By **Conclusion from (a)** it is positive definite.

5.4.22

Suppose *The random two distinct eignvaluesof A are* λ_1, λ_2*, with two independent corresponding eignvectors* x_1, x_2*, with coordinate* $v_1, ..., v_n$ *and* $w_1, ..., w_n$*. Write* $e_1, ..., e_n$ *as standard basis, which is also an orthonomal basis.*

Lemma *Suppose*

$$Dx_1 = \sum_{i=1}^{i=n} v_i\lambda_i e_i = \lambda_1 x_1 = \sum_{i=1}^{i=n} v_i\lambda_1 e_i$$
$$\Rightarrow \sum_{i=1}^{i=n} v_i(\lambda_i - \lambda_1)e_i = 0$$

Since $e_1, ..., e_n$ *are independent, so that* $v_i = 0$ *when* $\lambda_i \neq \lambda_1$*. These statement also holds true for* x_2*. So we get*

$$x_1 = \sum^{\lambda_i=\lambda_1} v_i e_i, x_2 = \sum^{\lambda_i=\lambda_2} w_i e_i$$

Since $\{e_i|\lambda_i = \lambda_1\} \cap \{e_i|\lambda_i = \lambda_2\} = \emptyset$*. We get*

$$\langle x_1, x_2\rangle = \langle \sum^{\lambda_i=\lambda_1} v_i e_i, \sum^{\lambda_i=\lambda_2} 2_i e_i\rangle = 0$$

Therefore, eignvectors with different eignvalues of diagonal matrix is orthogonal.

Since A is a noormal matrix, by **Theorem 5.23**we can rewrite it as $A = UDU^*$, with U, U^* is an unitary matrix.

$$Ax_1 = UDU^*x_1 = \lambda_1 x_1 \Rightarrow D(U^*x_1) = \lambda_1(U^*x_1)$$
$$Ax_2 = UDU^*x_2 = \lambda_2 x_2 \Rightarrow D(U^*x_2) = \lambda_2(U^*x_2)$$

By **Theorem 4.25**

$$\langle x_1, x_2\rangle = \langle U^*(x_1), U^*(x_2)\rangle$$

Consider $U^*(x_1), U^*(x_2)$ as eignvectors of D with different eignvalues λ_1, λ_2. By the **Lemma**, we know that eignvec with diff eignvalues are orthogonal.

$$\langle x_1, x_2\rangle = \langle U^*(x_1), U^*(x_2)\rangle = 0$$

So, eignvectors with different eignvalues of normal matrix is orthogonal