

Homework10

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4.1.4

Denote The j th col of A, B is A_j, B_j . And $A_j, B_j \in \mathbb{C}^m$. So that, $\|A_j\|^2 = \langle A_j, A_j \rangle$, as well as for B_j .

Proof By the definition of inner product $\sum_{i=1}^{i=m} [A]_{ij} [B]_{ij} = \langle A_j, B_j \rangle$. For the inner product of two matrices: $\langle A, B \rangle_F = \sum_{j=1}^{j=n} \sum_{i=1}^{i=m} [A]_{ij} [B]_{ij} = \sum_{j=1}^{j=n} \langle A_j, B_j \rangle$. By the definition of the norm: $\|A\|_F^2 = \langle A, A \rangle_F = \sum_{j=1}^{j=n} \langle A_j, A_j \rangle = \sum_{j=1}^{j=n} \|A_j\|^2$.

4.1.8

Suppose there exists $w \in V$. $\langle v - w, v - w \rangle = \langle w - v, w - v \rangle \Rightarrow \|v - w\| = \|w - v\|$

$$\|v\| = \|v - u + u\| \leq \|v - u\| + \|u\| \Rightarrow \|v - u\| \geq \|v\| - \|u\| = 11 - 2 = 9$$

$$\|v - u\| \leq \|v - w\| + \|w - u\| = \|v - w\| + \|u - w\| = 8$$

Contradiction.

4.1.14

(a)

$$\frac{1}{4}(\|v + w\|^2 - \|v - w\|^2) = \frac{1}{4}(\langle v + w, v + w \rangle - \langle v - w, v - w \rangle) = \frac{1}{4}(2\langle v, w \rangle - (-2)\langle v, w \rangle) = \langle v, w \rangle$$

(b)

$$\begin{aligned} & \frac{1}{4}(\|v + w\|^2 - \|v - w\|^2 + i\|v + iw\|^2 - i\|v - iw\|^2) \\ &= \frac{1}{4}(\langle v + w, v + w \rangle - \langle v - w, v - w \rangle + i(\langle v + iw, v + iw \rangle - \langle v - iw, v - iw \rangle)) \\ &= \frac{1}{4}(2\langle w, v \rangle + 2\langle v, w \rangle + i(2i\langle w, v \rangle - 2i\langle v, w \rangle)) = \langle v, w \rangle \end{aligned}$$

4.1.20

Proof $\forall v, w \in \mathbb{V}$. Use **Definition** and **Proposition 4.2.2**.

$$\langle 0, v \rangle = 0 * \langle w, v \rangle = 0$$

$$\langle v, 0 \rangle = \bar{0} * \langle v, w \rangle = 0$$

4.2.4

This Exercise we need to use **Theorem 4.9**

(a)

$$\begin{aligned} \left\langle \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle &= 2\sqrt{2} \\ \left\langle \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle &= -\sqrt{6} \end{aligned} \Rightarrow \begin{bmatrix} 2\sqrt{2} \\ -\sqrt{6} \end{bmatrix}$$

(b)

Refer to **Exercise 4.1.4**

$$\begin{aligned} \left\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle &= \frac{21}{2} \\ \left\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right\rangle &= \frac{3}{2} \\ \left\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right\rangle &= \frac{3}{2} \\ \left\langle \begin{bmatrix} 5 & 7 \\ 7 & 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\rangle &= \frac{-7}{2} \end{aligned} \Rightarrow \begin{bmatrix} \frac{21}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ \frac{-7}{2} \end{bmatrix}$$

(c)

$$\begin{aligned} \left\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle &= 0 \\ \left\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \right\rangle &= -2 + i \\ \left\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right\rangle &= -2 \\ \left\langle \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} \right\rangle &= -2 - i \end{aligned} \Rightarrow \begin{bmatrix} 0 \\ -2 + i \\ -2 \\ -2 - i \end{bmatrix}$$

4.2.6

This question we will use **Theorem 4.9**. Assume the i th row and j th column of the matrix is the following equation:

$$[[T]_{B_V B_V}]_{ij} = \langle T(e_j), e_i \rangle$$

(a)

$$\begin{aligned} [[T]_{B_V}]_{11} &= \langle T(e_1), e_1 \rangle = \left\langle \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle = -\frac{1}{2} \\ [[T]_{B_V}]_{21} &= \langle T(e_1), e_2 \rangle = \left\langle \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle = \frac{\sqrt{3}}{2} \\ [[T]_{B_V}]_{12} &= \langle T(e_2), e_1 \rangle = \left\langle \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle = -\frac{\sqrt{3}}{2} \\ [[T]_{B_V}]_{22} &= \langle T(e_2), e_2 \rangle = \left\langle \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle = -\frac{1}{2} \end{aligned} \Rightarrow \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

(b)

$$\begin{aligned} &\begin{bmatrix} \langle T(e_1), e_1 \rangle & \langle T(e_2), e_1 \rangle & \langle T(e_3), e_1 \rangle & \langle T(e_4), e_1 \rangle \\ \langle T(e_1), e_2 \rangle & \langle T(e_2), e_2 \rangle & \langle T(e_3), e_2 \rangle & \langle T(e_4), e_2 \rangle \\ \langle T(e_1), e_3 \rangle & \langle T(e_2), e_3 \rangle & \langle T(e_3), e_3 \rangle & \langle T(e_4), e_3 \rangle \\ \langle T(e_1), e_4 \rangle & \langle T(e_2), e_4 \rangle & \langle T(e_3), e_4 \rangle & \langle T(e_4), e_4 \rangle \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(c)

$$\begin{aligned} &\begin{bmatrix} \langle T(e_1), e_1 \rangle & \langle T(e_2), e_1 \rangle & \langle T(e_3), e_1 \rangle & \langle T(e_4), e_1 \rangle \\ \langle T(e_1), e_2 \rangle & \langle T(e_2), e_2 \rangle & \langle T(e_3), e_2 \rangle & \langle T(e_4), e_2 \rangle \\ \langle T(e_1), e_3 \rangle & \langle T(e_2), e_3 \rangle & \langle T(e_3), e_3 \rangle & \langle T(e_4), e_3 \rangle \\ \langle T(e_1), e_4 \rangle & \langle T(e_2), e_4 \rangle & \langle T(e_3), e_4 \rangle & \langle T(e_4), e_4 \rangle \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -1 & 0 \\ 0 & i & 0 & 0 \end{bmatrix} \end{aligned}$$

4.2.8

The main idea is use orthonormal basis to help use calculate the innerproduct.

Proof Because we are using \mathbb{R} for this question, so $|\langle v, w \rangle|^2 = \langle v, w \rangle * \overline{\langle v, w \rangle} = \langle v, w \rangle^2$.
Use the **Theorem 4.10**, and according to the example on **Page 240**:

Denote The orthonormal basis of triangonometric polynomial space is $B_v = \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin(ix), \frac{1}{\sqrt{\pi}} \cos(jx) | i = 1, \dots, n \in \mathbb{N}, j = 1, \dots, m \in \mathbb{N}\}$. The i th entry of $[f]_{B_v}$, which is under the triangonometric polynomial space is $t_i = \langle f, e_i \rangle$.

$$\begin{aligned} \|f\|^2 &= \sum_{i=1}^{i=1+m+n} |\langle v, e_i \rangle|^2 = \sum_{i=1}^{i=1+m+n} \langle v, e_i \rangle^2 = \sum_{i=1}^{i=1+m+n} t_i^2 \\ &= b_0^2 2\pi + \sum_{k=1}^{k=n} a_k^2 \pi + \sum_{l=1}^{l=m} b_l^2 \pi \\ \Rightarrow \|f\| &= \sqrt{b_0^2 2\pi + \sum_{k=1}^{k=n} a_k^2 \pi + \sum_{l=1}^{l=m} b_l^2 \pi} \end{aligned}$$

4.2.10

Because we are using \mathbb{R} for this question, so $\forall k \in \mathbb{F}, |k|^2 = k * \bar{k} = k^2$.

(a)

Proof

$$\begin{aligned} \|3 - 2x + x^2\|^2 &= \langle 3 - 2x + x^2, 3 - 2x + x^2 \rangle = \int_0^1 (3 - 2x + x^2) \overline{3 - 2x + x^2} dx \\ &= \int_0^1 (3 - 2x + x^2)^2 dx = \left[\frac{(x-1)^5}{5} + \frac{4(x-1)^3}{3} + 4x - 4 + C \right]_0^1 = \frac{83}{15} \end{aligned}$$

(b)

Proof By refering to the conclusion on **Page 240.6**, we know that $\{1, x, x^2\}$ is not even orthogonal. Therefore, they can not be orthonormal basis. So it will not follow the **Theorem 4.10**

4.2.18

(a)

Denote A^* is the conjugate transpose of A . $[A]_{ij} = \overline{[A^*]_{ji}}$.

Proof Since the basis we are using is orthonormal basis.

By **Theorem 4.10**:

$$\langle x, y \rangle_A = \langle Ax, Ay \rangle \Rightarrow \langle e_j, e_k \rangle_A = \langle Ae_j, Ae_k \rangle = \sum_{i=1}^{i=n} \langle Ae_j, e_i \rangle \overline{\langle Ae_k, e_i \rangle} = \sum_{i=1}^{i=n} [A]_{ij} \overline{[A^*]_{ki}} = [A^* A]_{kj}$$

(b)

Proof Use the formula we get from (a), and refer to the *Definition Of Orthonormal Basis*:

$$\begin{aligned} k \neq j &\rightarrow 0 = \langle e_j, e_k \rangle_A = [A^*A]_{kj} = 0 = [I]_{kj} = [A^{-1}A]_{kj} \Rightarrow A^* = A^{-1} \\ k = j &\rightarrow 1 = \langle e_j, e_j \rangle_A = [A^*A]_{jj} = 1 = [I]_{jj} = [A^{-1}A]_{jj} \end{aligned}$$

4.3.2

(a)

Refer to **Proposition 4.18**

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 7 & 4 & 1 & -2 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -2 & 1 & 4 & 7 \end{bmatrix}$$

(b)

Refer to **Proposition 4.18**

$$A = \begin{bmatrix} 1 & 0 \\ i & 1 \\ 0 & i \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 0 \\ i & 1 \\ 0 & i \end{bmatrix} \left(\begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & -i \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -i & 0 \\ 0 & 1 & -i \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -i & 1 \\ -i & 2 & -i \\ 1 & i & 2 \end{bmatrix}$$

4.3.4

(a)

Refer to the **Theorem 4.16.8**

$$A = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \left(\begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 9 & 3 & 12 \\ 3 & 1 & 4 \\ 12 & 4 & 16 \end{bmatrix}$$

$$[P_{U^\perp}]_\xi = I - [P_U]_\xi = \frac{1}{26} \left(\begin{bmatrix} 26 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 26 \end{bmatrix} - \begin{bmatrix} 9 & 3 & 12 \\ 3 & 1 & 4 \\ 12 & 4 & 16 \end{bmatrix} \right) = \frac{1}{26} \begin{bmatrix} 17 & -3 & -12 \\ -3 & 25 & -4 \\ -12 & -4 & 10 \end{bmatrix}$$

(b)

It is easy to find out that we need to find $\langle \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \rangle^\perp$.

$$A = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} ([3 \ 2 \ 1] \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix})^{-1} [3 \ 2 \ 1] = \frac{1}{14} \begin{bmatrix} 9 & 6 & 3 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$[P_{U^\perp}]_\xi = I - [P_U]_\xi = \frac{1}{14} \left(\begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 3 \\ 6 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix} \right) = \frac{1}{14} \begin{bmatrix} 5 & -6 & -3 \\ -6 & 10 & -2 \\ -3 & -2 & 13 \end{bmatrix}$$

(c)

It is easy to find out that we need to find $\langle \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix} \rangle^\perp$.

$$A = \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix} ([1 \ -i \ i] \begin{bmatrix} 1 \\ i \\ -i \end{bmatrix})^{-1} [1 \ -i \ i] = \frac{1}{3} \begin{bmatrix} 1 & -i & i \\ i & 1 & -1 \\ -i & -1 & 1 \end{bmatrix}$$

$$[P_{U^\perp}]_\xi = I - [P_U]_\xi = \frac{1}{3} \left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -i & i \\ i & 1 & -1 \\ -i & -1 & 1 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} 2 & i & -i \\ -i & 2 & 1 \\ i & 1 & 2 \end{bmatrix}$$

(d)

Do REF to the given matrix, we get $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Which means that we only get first and

second cols as basis.

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$

$$[P_U]_\xi = A(A^*A)^{-1}A^* = \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 3 \\ 0 & 2 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} = \frac{1}{83} \begin{bmatrix} 14 & -11 & 2 & 29 \\ -11 & 62 & 34 & -5 \\ 2 & 34 & 24 & 16 \\ 29 & -5 & 16 & 66 \end{bmatrix}$$

4.3.14

(a)

Proof Refer to **Theorem 1.10** Define 0 is zero matrix in \mathbb{V}, \mathbb{W} . $\forall v, v_1, v_2, w, w_1, w_2 \in \mathbb{V}, \mathbb{W}$, $c \in \mathbb{F}$ we have

$$0 + v = v, 0^T = 0 \Rightarrow 0 \in \mathbb{V}$$

$$0 + w = w, 0^T = -0 \Rightarrow 0 \in \mathbb{W}$$

$$v_1 + cv_2 = (v_1 + cv_2)^T \Rightarrow v_1 + cv_2 \in \mathbb{V}$$

$$w_1 + cw_2 = (-w_1 - cw_2)^T = -(w_1 + cw_2)^T \Rightarrow w_1 + cw_2 \in \mathbb{W}$$

So \mathbb{V}, \mathbb{W} is subspace of $M_n(\mathbb{R})$.

(b)

Refer to **Frobenius Inner Product**: $\forall X, Y \in M_n(\mathbb{R}), X^T = X^*, \langle X, Y \rangle = \text{tr}(XY^*) = \text{tr}(XY^T)$, since the field we are using here is \mathbb{R} .

Claim Rewrite V as $V = \left\{ \frac{X+X^T}{2} \mid X \in M_n(\mathbb{R}) \right\}$, W as $W = \left\{ \frac{X-X^T}{2} \mid X \in M_n(\mathbb{R}) \right\}$. $\left(\frac{X+X^T}{2} \right)^T = \frac{X^T+X}{2} = \frac{X+X^T}{2}$, and $\left(\frac{X-X^T}{2} \right)^T = \frac{X^T-X}{2} = -\frac{X-X^T}{2}$

By the **Definition Of Orthogonal Projection**, I can build a function $P_V : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ and $X = \frac{X+X^T}{2} + \frac{X-X^T}{2}$, $X \in M_n(\mathbb{R})$, $\frac{X+X^T}{2} \in \mathbb{V}$, $\frac{X-X^T}{2} \in \mathbb{W}$, $P_V(X) = \frac{X+X^T}{2}$.

Proof By the **Proposition 3.60** $\forall v \in \mathbb{V}, \forall w \in \mathbb{W}$, $\text{tr}(vw^T) = -\text{tr}(vw) = -\text{tr}(v^T w) = -\text{tr}(wv^T) = -\text{tr}(wv^T)^T = -\text{tr}(vw^T) \Rightarrow \text{tr}(vw^T) = 0$. Therefore the function I build is valid. $\mathbb{V}^\perp = \mathbb{W}$.

(c)

Refer to the conclusion from (b): $P_V(X) = \frac{X+X^T}{2} = \frac{X+X^*}{2} = \text{Re}(A)$, $P_W(X) (= P_{V^\perp}(X) = I(X) - P_V(X)) = \frac{X-X^T}{2} = \frac{X-X^*}{2} = i\frac{X-X^*}{2i} = i\text{Im}(A)$

4.3.22

(a)

Proof $\forall i \in \{1, 2, \dots, n\}$, the i th col of $[P_U]_B$ is $\begin{bmatrix} \langle P_U(e_i), e_1 \rangle \\ \langle P_U(e_i), e_2 \rangle \\ \dots \\ \langle P_U(e_i), e_n \rangle \end{bmatrix}$.

$$\text{When } i \leq m, \begin{bmatrix} \langle P_U(e_i), e_1 \rangle \\ \langle P_U(e_i), e_2 \rangle \\ \dots \\ \langle P_U(e_i), e_n \rangle \end{bmatrix} = \begin{bmatrix} \langle \sum_{k=1}^{k=m} \langle e_i, e_k \rangle e_k, e_1 \rangle \\ \langle \sum_{k=1}^{k=m} \langle e_i, e_k \rangle e_k, e_2 \rangle \\ \dots \\ \langle \sum_{k=1}^{k=m} \langle e_i, e_k \rangle e_k, e_n \rangle \end{bmatrix} = \begin{bmatrix} \langle e_i, e_1 \rangle \\ \langle e_i, e_2 \rangle \\ \dots \\ \langle e_i, e_n \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{bmatrix}.$$

The i th row will be one which is $[[P_U]_B]_{ii} = 1$.

$$\text{When } i > m, \begin{bmatrix} \langle P_U(e_i), e_1 \rangle \\ \langle P_U(e_i), e_2 \rangle \\ \dots \\ \langle P_U(e_i), e_n \rangle \end{bmatrix} = \begin{bmatrix} \langle \sum_{k=1}^{k=m} \langle e_i, e_k \rangle e_k, e_1 \rangle \\ \langle \sum_{k=1}^{k=m} \langle e_i, e_k \rangle e_k, e_2 \rangle \\ \dots \\ \langle \sum_{k=1}^{k=m} \langle e_i, e_k \rangle e_k, e_n \rangle \end{bmatrix} = \begin{bmatrix} \langle 0, e_1 \rangle \\ \langle 0, e_2 \rangle \\ \dots \\ \langle 0, e_n \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

$$\Rightarrow [P_U]_B = \text{diag}(1, \dots, 1, 0, \dots, 0).$$

(b)

Proof Refer to the **Definition Of Orthogonal Projection**: $\forall u \in \text{range } P_U, u \in U$

$\Rightarrow \text{Range } P_U \subset U$.

By the conclusion of **part2**, we know $\forall u \in U, u = \sum_{j=1}^{j=m} a_j e_j, P_U(u) = \sum_{i=1}^{i=m} \langle u, e_i \rangle e_i = \sum_{i=1}^{i=m} \langle \sum_{j=1}^{j=m} a_j e_j, e_i \rangle e_i = \sum_{i=1}^{i=m} \sum_{j=1}^{j=m} a_j \langle e_j, e_i \rangle e_i = \sum_{i=1}^{i=m} a_i e_i = u \Rightarrow \forall u \in U, P_U(u) = u$.
I can conclude that $\forall u \in U, \exists u \in U \subset V, u = P_U(u) \in \text{Range } P_U \Rightarrow U \subset \text{Range } P_U$.

$\Rightarrow U = \text{Range } P_U$

(c)

Proof Refer to **Definition Of Orthogonal Projection**: $\forall v \in V, u \in U, w \in U^\perp, v = u + w, P_U(v) = u$.

$\forall v \in \text{Ker } P_U, P_U(v) = 0 \Rightarrow v = 0 + v \Rightarrow v \in U^\perp \Rightarrow \text{Ker } P_U \subset U^\perp$

$\forall v \in U^\perp \subset V, v = 0 + v \Rightarrow P_U(v) = 0 \Rightarrow v \in \text{Ker } P_U \Rightarrow U^\perp \subset \text{Ker } P_U$

$\Rightarrow U^\perp = \text{Ker } P_U$

(d)

Proof Suppose $\forall v \in V, v = u + w, u \in U, w \in U^\perp$

$$(I - P_U)v = v - P_U(v) = v - u = w = P_{U^\perp}(v) \Rightarrow I - P_U = P_{U^\perp}$$

(e)

Proof Refer to the conclusion from (b)

$$\forall v \in V, u \in U, w \in U^\perp, v = u + w, P_U^2(v) = P_U(u) = u = P_U(v) \Rightarrow P_U^2 = P_U$$