Homework11

Zhihao Wang

April 28, 2022

5.3.6

Denote *The adjoint* B *as matrix* $K \in M_n(\mathbb{C})$.

$\forall x, y \in \mathbb{C}^n$

$$\begin{aligned}
&\langle Bx, y\rangle_A = \langle ABx, Ay\rangle = \langle A^*ABx, y\rangle = \langle x, (A^*AB)^*y\rangle \\
&\langle x, Ky\rangle_A = \langle Ax, AKy\rangle = \langle x, A^*AKy\rangle \\
&\Rightarrow \langle x, ((A^*AB)^* - A^*AK)y\rangle = 0
\end{aligned}$$

Since A is invertible, to A^* is invertible, so well as A^*A. By **Proposition 4.2.4**

$$((A^*AB)^* - A^*AK)y = 0 \Rightarrow (A^*AB)^* - A^*AK = 0 \Rightarrow K = (A^*A)^{-1}B^*(A^*A)$$

5.3.8

(a)

$$\begin{aligned}
\langle D(p), q\rangle &= \int_{-\infty}^{\infty} p'(x)q(x)e^{-\frac{x^2}{2}}\, dx = \int_{-\infty}^{\infty} q(x)e^{-\frac{x^2}{2}}\, dp(x) \\
&= p(x)q(x)e^{-\frac{x^2}{2}}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} p(x)\, d(q(x)e^{-\frac{x^2}{2}}) \\
&= -\int_{-\infty}^{\infty} p(x)\, d(q(x)e^{-\frac{x^2}{2}}) \\
&= -\int_{-\infty}^{\infty} p(x)(q'(x) - xq(x))e^{-\frac{x^2}{2}}\, dx \\
&= \langle p, D^*(q)\rangle \\
&\Rightarrow D^*(q(x)) = -q'(x) + xq(x)
\end{aligned}$$

(b)

Denote *Basis* $(1, x, ..., x^{n-1})$ *is* B_v. *And Basis* $(1, x, ..., x^n)$ *is* B_w

The *ith* col of matrix of D with respect of B_v, B_w is

$$[D_i]_{B_w} = [D(x^{i-1})]_{B_w}$$

Therefore, the matrix is $n * (n+1)$

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 2 & 0 & ... & 0 \\ 0 & 0 & 0 & 3 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & n \end{bmatrix}$$

(c)

Denote *Basis* $(1, x, ..., x^n)$ *is* B_w*. And Basis* $(1, x, ..., x^{n-1})$ *is* B_v

The ith col of matrix of D with respect of B_w, B_v is

$$[D_i^*]_{B_v} = [D^*(x^{i-1})]_{B_v} = [-(i-1)x^{i-2} + x^i]_{B_v}$$

Therefore, the matrix is $(n+1) * n$

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 & ... & 0 \\ 1 & 0 & -2 & 0 & ... & 0 \\ 0 & 1 & 0 & -3 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1 \end{bmatrix}$$

(d)

Since the basis we use are not orthonomal basis. For example

$$\int_{-\infty}^{\infty} 1 * 1 * e^{-\frac{x^2}{2}} dx \neq 1$$

5.3.12

Since it is a self adjoint Map, $T^* = T$. We can rewrite v as following

$$v = T(v) + (v - T(v))$$

Now, consider $U = rangeT$, and we need to prove that $T(v)$ is orthogonal to $v - T(v)$

$$\langle T(v), v - T(v) \rangle = \langle v, T^*(v - T(v)) \rangle = \langle v, T(v - T(v)) \rangle = \langle v, T(v) - T^2(v) \rangle = 0$$

Therefore, $T(v)$ is orthogonal to $v - T(v)$, so we get T is the orthogonal projection onto $U = rangeT$.

5.3.18/19

Suppose *The random two distinct eignvalues are* λ_1, λ_2*, with two independent corrosponding eignvectors* v_1, v_2*.*

$$\begin{aligned}\langle v_1, v_1\rangle &= \frac{1}{\lambda_1}\langle T(v_1), v_1\rangle = \frac{1}{\lambda_1}\langle v_1, T^*(v_1)\rangle \\ &= -\frac{1}{\lambda_1}\langle v_1, T(v_1)\rangle = -\frac{\overline{\lambda_1}}{\lambda_1}\langle v_1, v_1\rangle\end{aligned}$$

So we get

$$\overline{\lambda_1} = -\lambda_1 \Rightarrow \overline{\lambda_1} + \lambda_1 = 0 = 2Re(\lambda_1)$$

Since the real part is zero, so it is purely imaginary. We can further prove that

$$\begin{aligned}\langle v_1, v_2\rangle &= \frac{1}{\lambda_1}\langle T(v_1), v_2\rangle = \frac{1}{\lambda_1}\langle v_1, T^*(v_2)\rangle \\ &= -\frac{1}{\lambda_1}\langle v_1, T(v_2)\rangle = -\frac{\overline{\lambda_2}}{\lambda_1}\langle v_1, v_2\rangle\end{aligned}$$

Denote $\lambda_1 = b_1 i, \lambda_2 = b_2 i$

$$-\frac{\overline{\lambda_2}}{\lambda_1} = 1 \Rightarrow b_2 i = b_1 i$$

Contradiction, so all of them are orthogonal.