

Homework11

Zhihao Wang

04/22/2022

4.5.4

(a)

Denote The $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ as B_v

The i th column of $[R]_{B_v}$ is $[Rv_i]_{B_v}$.

$$\begin{aligned} [Rv_1]_{B_v} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{B_v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ [Rv_2]_{B_v} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{B_v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(b)

Proof By **Proposition 4.31** and **Definition of Orthogonal Matrix**, we know that if the columns are not **Othonormal**, then the matrix must not be orthogonal.

$$\langle A_1, A_2 \rangle = 2 \neq 0$$

Therefore, A is not orthogonal matrix.

(c)

Proof Because **Proposition 4.30** says, **Suppose B_v, B_w are orthonormal basis of V, W .**

But $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ itself is not an orthonormal basis.

4.5.6

Proof Since $\mathbb{C}_{2\pi}(\mathbb{R})$ is equipped with innerproduct, so it is both an innerproduct space and a normed space.

Suppose $\forall f(x) \in \mathbb{C}_{2\pi}(\mathbb{R})$, which means $f(x)$ is a continuous 2π periodic function $\Rightarrow \exists g(x) = f(x - t) \in \mathbb{C}_{2\pi}(\mathbb{R}), T(g)(x) = f(x + t)$. T is a surjective linear map.

Denote

$$\begin{aligned} \int_a^b f(x) \overline{f(x)} dx &= F(b) - F(a) = \int_{a+2\pi}^{b+2\pi} f(x) \overline{f(x)} dx = F(b+2\pi) - F(a+2\pi) \\ \Rightarrow F(b+2\pi) - F(b) &= F(a+2\pi) - F(a) \end{aligned}$$

So we get

$$\begin{aligned} \|(Tf)(x)\| \int_0^{2\pi} (Tf)(x) \overline{(Tf)(x)} dx &= \int_0^{2\pi} f(x+t) \overline{f(x+t)} dx \\ &= \int_0^{2\pi} f(x+t) \overline{f(x+t)} d(x+t) \\ &= \int_t^{t+2\pi} f(x') \overline{f(x')} d(x') \\ &= F(t+2\pi) - F(t) \\ &= F(0+2\pi) - F(0) \\ &= \|(f)(x)\| \end{aligned}$$

So, it is an isometry.

4.5.8

4.5.10

4.5.14

(a)

Proof By **Proposition 4.30**, we know that if U is unitary, then all columns of U is orthonormal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonormal basis. By **Corrolary 4.30**, we Know that U is an isometry.

$$\|U\|_{op} = \max_{\|v\|=1, v \in \mathbb{C}^n} \|Uv\| = \|v\| = 1$$

(b)

By **Proposition 4.30**, we know that if U is unitary, then all columns of U is orthonormal. By **Theorem 4.3**, we know that all columns are independent. By **Theorem 3.28**, we know that the columns are orthonormal basis. By **Corrolary 4.30**, we Know that U is an isometry.

$$\|U\|_F = \sqrt{\text{tr} U^* U} = \sqrt{\text{tr} I} = \sqrt{n}$$

5.1.4

Denote $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

First use standard basis as (e_1, e_2, e_3) . Second use (e_2, e_3) as basis for output.

$$\begin{aligned} T(e_1) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1 * e_2 \\ T(e_3) &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 * e_3 \\ T(e_2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 * e_2 \end{aligned}$$

Since $\text{rank} T = 2$, so the $\sigma_1 = 1, \sigma_2 = 1$

The right vectors are (e_1, e_3, e_2) , left vectors are (e_2, e_3) , and $\sigma_1 = 1, \sigma_2 = 1$.

5.1.6

Denote The standard basis of \mathbb{V} is (e_1, e_2, \dots, e_n) , with $\dim V = n$. As well as $P := P_U$ with $U \subset V, \dim U = m, U = \text{span}(e_1, \dots, e_m)$, and $V = U \oplus U^\perp$.

(a)

By **Theorem 5.3**, we know that the singular value are unique. So we just need to prove that we can find only 1, 0.

By **Theorem 4.16.2**, when $i \leq m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = \langle e_i, e_{j=i} \rangle e_{j=i} = 1 * e_i$$

By **Theorem 4.16.2**, when $i > m$

$$P_U(e_i) = \sum_{j=1}^{j=m} \langle e_i, e_j \rangle e_j = 0$$

So, the singular values are only 0, 1.

(b)

I choose (e_1, e_2, \dots, e_n) as both left and right singular vectors, and with the same sequence.

5.1.10

Proof Since T is invertible, so $n = \dim V = \dim W$. Since $\forall i \in \{1, 2, \dots, n\}, \sigma_i > 0, \neq 0$. By **Theorem 5.3**, we know the singular value is unique.

Denote The orthonormal basis of V is (e_1, e_2, \dots, e_n) , and the orthonormal basis of W is (f_1, f_2, \dots, f_n) .

Consider Construct $T^{-1} \in \mathcal{L}(W, V)$, with singular value $\sigma'_1, \sigma'_2, \dots, \sigma'_n$

Since $\forall i \in \{1, 2, \dots, n\}$

$$T(e_i) = \sigma_i f_i \Rightarrow T^{-1}(f_i) = \frac{1}{\sigma_i} e_i$$

$\forall i \geq j$

$$\sigma_i \geq \sigma_j \Rightarrow \frac{1}{\sigma_j} \geq \frac{1}{\sigma_i}$$

Therefore, $\{\frac{1}{\sigma_n}, \frac{1}{\sigma_{n-1}}, \dots, \frac{1}{\sigma_1}\}$ are singular value of T^{-1} .

By **Key Ideas On Page 295 last point**, we know that the largest singular value is operator norm of that map.

$$\|T^{-1}\|_{op} = \frac{1}{\sigma_n} \Rightarrow \|T^{-1}\|_{op}^{-1} = \sigma_n$$

Denote $v = \sum_{i=1}^{i=n} a_i e_i$, $\|v\| = 1$

$$\begin{aligned} \|T(v)\| &= \|T(\sum_{i=1}^{i=n} a_i e_i)\| = \|\sum_{i=1}^{i=n} a_i T(e_i)\| = \|\sum_{i=1}^{i=n} a_i \sigma_i f_i\| \\ &\geq \|\sigma_n \sum_{i=1}^{i=n} a_i f_i\| = |\sigma_n| \|\sum_{i=1}^{i=n} a_i f_i\| \\ &= |\sigma_n| \sqrt{\sum_{i=1}^{i=n} \|a_i f_i\|^2} = |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2 \|f_i\|^2} \\ &= |\sigma_n| \sqrt{\sum_{i=1}^{i=n} |a_i|^2} = |\sigma_n| \\ &\Rightarrow \min_{\|v\|=1} \|Tv\| = \sigma_n \end{aligned}$$