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Matlab HW9: MMSE and Bayesian Estimation

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Estimation of Random Signal in Noise

1. **Problem**:
   - Extract a random signal from noise. The signal is assumed to be the outcome of the random process.
2. **Application area example**:
   - A typical use is to enhance images corrupted by noise, which requires a straightforward extension to a two-dimensional signal. An example is in high resolution electron microscope images.

Estimation of Random Signal in Noise

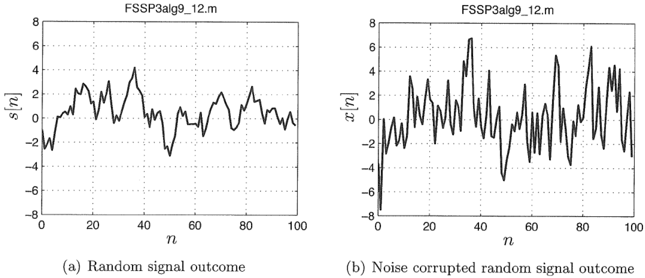

Figure 1: Outcome of random signal and observed noisy data. The points have been connected by straight lines for easier viewing.

3. **Data model/assumptions**:

The Gaussian signal to be extracted $s[n]$ is assumed to be embedded in WGN with known variance σ^2,

$$x[n] = s[n] + w[n] \quad n = 0, 1, ..., N-1$$

- $s[n]$ is assumed to be the outcome of a zero mean Gaussian random process with a known $N \times N$ covariance matrix $\mathbf{C}_s$.

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4. **Estimator**: The MMSE estimator is given as

$$\hat{\mathbf{s}} = \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{x} \tag{1}$$

- $\hat{\mathbf{s}} = [\hat{s}[0]\ \hat{s}[1]\ ...\ \hat{s}[N-1]]^T$
- If the signal is from a stationary random process, then the covariance matrix becomes an autocorrelation matrix $\mathbf{R}_s$ given by

$$\mathbf{C}_s = \mathbf{R}_s = \begin{bmatrix} r_s[0] & r_s[1] & \dots & r_s[N-1] \\ r_s[1] & r_s[0] & \dots & r_s[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_s[(N-1)] & r_s[N-2] & \dots & r_s[0] \end{bmatrix}$$

where $r_s[k]$ is the autocorrelation sequence for $s[n]$.

Estimation of Random Signal in Noise

Theorem 1 (MMSE estimator)

$$\mathbf{x} = \mathbf{H}\theta + \mathbf{w}$$

where $\mathbf{x} = [x[0]\ x[1]\ \ldots\ x[N-1]]^T$. *The Bayesian linear model the posterior PDF is given by*

$$p(\boldsymbol{\theta}|\mathbf{x}) = \frac{1}{(2\pi)^{N/2} det^{1/2}(\mathbf{C}_{\theta|x})} exp\left[-\frac{1}{2}(\boldsymbol{\theta}-E[\boldsymbol{\theta}|\mathbf{x}])^T \mathbf{C}_{\boldsymbol{\theta}|\mathbf{x}}^{-1}(\theta-E[\boldsymbol{\theta}|\mathbf{x}])\right]$$

where the mean is

$$\begin{aligned} E[\boldsymbol{\theta}|\mathbf{x}] &= \boldsymbol{\mu}_\theta + \left(\mathbf{C}_\theta^{-1} + \frac{1}{\sigma^2}\mathbf{H}^T\mathbf{H}\right)^{-1}\frac{\mathbf{H}^T}{\sigma^2}(\mathbf{x} - \mathbf{H}\boldsymbol{\mu}_\theta) \\ &= \boldsymbol{\mu}_\theta + \mathbf{C}_\theta\mathbf{H}^T\left(\mathbf{H}\mathbf{C}_\theta\mathbf{H}^T + \mathbf{C}_w\right)^{-1}(\mathbf{x} - \mathbf{H}\boldsymbol{\mu}_\theta) \end{aligned}$$

and the covariance matrix is

$$\mathbf{C}_{\boldsymbol{\theta}|\mathbf{x}} = \mathbf{C}_{\boldsymbol{\theta}} - \mathbf{C}_{\boldsymbol{\theta}}\mathbf{H}^T(\mathbf{H}\mathbf{C}_{\boldsymbol{\theta}}\mathbf{H}^T + \mathbf{C}_w)^{-1}\mathbf{H}\mathbf{C}_{\boldsymbol{\theta}}$$

The MMSE estimator is $\boldsymbol{\theta} = E[\boldsymbol{\theta}|\mathbf{x}]$.

Estimation of Random Signal in Noise

Back to the problem of estimation of random signal in noise

$$\mathbf{x} = \mathbf{s} + \mathbf{w}$$

Compare to the $\mathbf{x} = \mathbf{H}\theta + \mathbf{w}$, the MMSE estimator is

$$E[\boldsymbol{\theta}|\mathbf{x}] = \boldsymbol{\mu}_\theta + \mathbf{C}_\theta \mathbf{H}^T \left(\mathbf{H}\mathbf{C}_\theta \mathbf{H}^T + \mathbf{C}_w \right)^{-1} (\mathbf{x} - \mathbf{H}\boldsymbol{\mu}_\theta)$$

Therefore, we let

$$\mathbf{H} = \mathbf{I}$$

$$\boldsymbol{\mu}_\theta = [0\ 0\ \ldots\ 0]^T = \mathbf{0}$$

$$\mathbf{C}_\theta = \mathbf{C}_s$$

$$\mathbf{C}_w = \sigma^2 \mathbf{I}$$

Then,

$$\hat{\mathbf{s}} = E[\boldsymbol{\theta}|\mathbf{x}] = \mathbf{C}_s (\mathbf{C}_s + \sigma^2 \mathbf{I})^{-1} \mathbf{x}$$

Estimation of Random Signal in Noise

- For this case, if the PSD is given by $P_s(f)$, then an approximate estimate (which avoids the inversion of the $N \times N$ autocorrelation matrix) is to filter the data $x[n]$ with the noncausual filter with frequency response

$$H(f) = \frac{P_s(f)}{P_s(f) + \sigma^2}$$

- It is called a Wiener smoothing filter.

Estimation of Random Signal in Noise

5. **Performance**:
   - The MMSE estimator is optimal in that minimizes the Bayesian mean square error of the estimator, assuming the signal and noise are Gaussian
   - The minimum mean square error for the estimator $\hat{s}[n]$ in (1) for $n = 0, 1, ..., N-1$ is the $[n, n]$ diagonal element of the $N \times N$ mean square error matrix

$$\mathbf{M}_{\hat{s}} = \mathbf{C}_s - \mathbf{C}_s(\mathbf{C}_s + \sigma^2\mathbf{I})^{-1}\mathbf{C}_s$$

Estimation of Random Signal in Noise

6. **Example**:
   - Consider a random signal that is an outcome of a Gaussian AR random process of order one

$$s[n] = -a[1]s[n-1] + u[n]$$

   - where $a[1] = -0.9$ and $\sigma_u^2 = 1$
   - It is embedded in WGN $w[n]$ with variance $\sigma^2 = 5$

$$x[n] = s[n] + w[n] = (-a[1]s[n-1] + u[n]) + w[n]$$

   - The signal and the noise corrupted signal are shown as Fig. 1
   - The signal is an outcome of a stationary random process so that we can estimate it using the (1) with the covariance matrix replaced by the autocorrelation matrix. The autocorrelation sequence is given by

$$r_s[k] = \frac{\sigma_u^2}{1 - a^2[1]}(-a[1])^{|k|}$$

Estimation of Random Signal in Noise

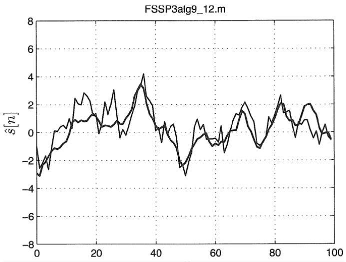

Figure 2: Estimated signal using the matrix Wiener smoother (heavy line) and the true signal (light signal). The points have been connected by straight lines for easier viewing

- The Wiener smoothed signal along with the true signal is shown in Fig. 2
- It is closer to the true signal than noise corrupted signal but exhibits some smoothing

Estimation of Random Signal in Noise

6. **Explanation**:
   - The Wiener smoother cannot separate out the signal from the noise perfectly since the PSDs overlap in frequency.
   - It attempts to reduce the WGN as much as possible in an effort to reduce the overall Bayesian mean square error.
   - This result in reducing the power in some of the high frequency signal bands, resulting in a smoothed estimate.

Estimation of Random Signal in Noise

Example 2 (Estimation of time delay)

Try to plot Fig. 2 using the program **FSSP3alg9_12.m** and understand how to implement MMSE for the random signal estimation problem. Try to use different noise variance and compare the result:

- $\sigma^2 = 5$ as in Fig. 2
- A higher SNR case: $\sigma^2 = 1$

Question & Answer