TAIPEI TECH 國立臺北科技大學 National Taipei University of Technology

誠,樸,精,勤

Matlab HW8: Maximum Likelihood Estimator

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Matlab HW8

1. Maximum Likelihood Estimator
   - Algorithm 9.4: Estimation of time delay

Estimation of time delay

1. **Problem**:
   - Estimate the time delay of a known deterministic signal in WGN.
2. **Application area example**:
   - Laser range finders called Light Detection and Ranging (LIDAR) are utilized to determine the range of an object.
   - They do so by transmitting a signal and measuring the time it takes to propagate to the object and return to the receiver.
   - Then the range is given by $R = c\tau_0/2$, where τ_0 is the round trip propagation time, i.e., the *delay time*, and c is the speed of propagation.

Estimation of time delay

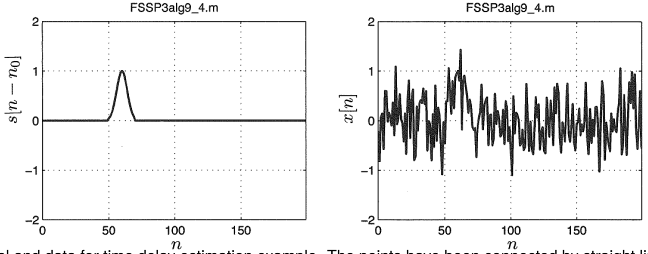

Figure 1: Signal and data for time delay estimation example. The points have been connected by straight lines for easier viewing.

3. **Data model/assumptions**:

$$x[n] = s[n - n_0] + w[n] \quad n = 0, 1, ..., N - 1$$

- $s[n]$ is the known signal, M samples in length, and is nonzero for $n = 0, 1, ...M - 1$
- n_0 is the delay to be estimated
- $w[n]$ is WGN with variance σ^2 (need not be known to implement estimator)
- For all possible values of n_0, the delayed signal $s[n - n_0]$ is contained in the observation interval $0 \leq n \leq N - 1$

Estimation of time delay

4. **Estimator**: Derive the maximum likelihood estimator (MLE) as follows:

- Original: $x(t) = s(t - \tau_0) + w(t)$
- Sampled signal: $x[n\triangle] = s(n\triangle - n_0) + w[n\triangle]$, where $n_0 = \tau_0/\triangle$ delay in samples

$$x[n] = \begin{cases} w[n] & 0 \leq n \leq n_0 - 1 \\ s[n-n_0] + w[n] & n_0 \leq n \leq n_0 + M - 1 \\ w[n] & n_0 + M \leq n \leq N - 1 \end{cases}$$

$$\begin{aligned} p(\mathbf{x}; n_0) = & \prod_{n=0}^{n_0-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2} x^2[n]\right] \\ & \cdot \prod_{n=n_0}^{n_0+M-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2} (x[n] - s[n-n_0])^2\right] \\ & \cdot \prod_{n=n_0+M}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2} x^2[n]\right] \end{aligned}$$

Estimation of time delay

$$p(\mathbf{x}; n_0) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1} x^2[n]\right]$$

$$\cdot \exp\left[-\frac{1}{2\sigma^2}\sum_{n=n_0}^{n_0+M-1}\left(-2x[n]s[n-n_0] + s^2[n-n_0]\right)\right]$$

The MLE is found by maximizing the likelihood function

$$\exp\left[-\frac{1}{2\sigma^2}\sum_{n=n_0}^{n_0+M-1}\left(-2x[n]s[n-n_0] + s^2[n-n_0]\right)\right]$$

or equivalently by minimizing

$$\sum_{n=n_0}^{n_0+M-1}\left(-2x[n]s[n-n_0] + s^2[n-n_0]\right)$$

The MLE of n_0 is found by maximizing

$$\sum_{n=n_0}^{n_0+M-1} x[n]s[n-n_0]$$

Estimation of time delay

4. **Estimator**:

- The estimator is a **running correlator**, which correlates each possible received signal with the data as

$$J(n_0) = \sum_{n=n_0}^{n_0+M-1} x[n]s[n-n_0] \quad 0 \leq n_0 \leq N-M \qquad (1)$$

  and chooses the value of n_0 that produces a maximum

- The estimate of the delay in second is $\tau_0 = n_0 \triangle$, where $\triangle$ is the time interval between samples

Estimation of time delay

5. **Example**:
   - Consider a signal that is a Gaussian pulse whose time delay is $n_0 = 50$ as shown in Fig. 1(a)
   - WGN with variance $\sigma^2 = 0.25$ results in Fig. 1(b)
   - After implementing the running correlator, the function $J(n_0)$ to be maximized is shown in Fig. 2
     - The maximum occurs exactly at the correct delay of $n_0 = 50$

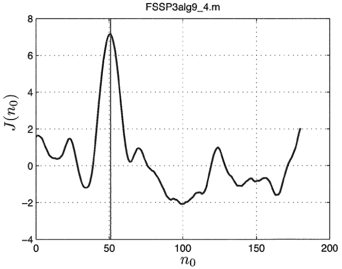

Figure 2: Function to be maximized to obtain MLE of time delay in samples

Estimation of time delay

6. **Explanation**:
   - Since $x[n] = s[n - n_{0_{true}}] + w[n]$, the cost function $J(n_0)$ in (1) becomes

$$J(n_0) = \sum_{n=n_0}^{n_0+M-1} (s[n - n_{0_{true}}]s[n - n_0]) + \sum_{n=n_0}^{n_0+M-1} w[n]s[n - n_0]$$

   so that when $n_0 = n_{0_{true}}$ the contribution from the first sum, which is an autocorrelation of the signal, becomes

$$x[n] = \sum_{n=n_{0_{true}}}^{n_{0_{true}}+M-1} s^2[n - n_{0_{true}}] = \mathcal{E}$$

   which is the total energy
   - The autocorrelation sequence is known to peak at zero lag (when the two signals are aligned) and to be smaller otherwise
   - Note that “on the average” the valuer of the peak is the energy

Estimation of time delay

Example 1 (Estimation of time delay)

Try to plot Figs. 1 and 2 using the program **FSSP3alg9_4.m** and understand how to implement MLE for the time delay estimation problem

Question & Answer