汎関数の計算

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定義域が関数族であるような関数を物理では**汎関数**functional と呼ぶ. 例えば, $F : \{A \to B\} \to C$ など. このとき, $\varphi : A \to B$ に対して $F[\varphi(x)] \in C$ と書くことが多い. ただし表記中 $x \in A$ は「ダミー」であって, 汎関数の定義中で用いられる変数を明示しているだけに過ぎない. 単に $F[\varphi]$ と書かれることもある. この文章中では, ダミー変数を添字にした $F_x[\varphi]$, $F_{x\in A}[\varphi]$ も用いる[*1]. $F[\varphi(x)]$ が汎関数であるとき, 通常の関数 $g : C \to D$ を合成した $g(F[\varphi(x)])$ もまた汎関数である.

物理では, 汎関数といえば専ら積分である. この文章では頻出する汎関数の基本的な計算方法についてまとめる. 数学的な厳密性は一切考慮していない. 高校微積分程度の習得を目指している[*2]. また, 勝手な解釈も多く含んでいるので, 気持ち程度に読んでほしい.

0.1 汎関数の考え方

例として関数 $\varphi : [a, b] \to \mathbb{R}$ の汎関数 $F[\varphi(x)]$ を考える. I の分割 $a = x_0 < \cdots < x_N = b$ に対し, 関数値を $\varphi_n := \varphi(x_n)$ として, 汎関数 $F[\varphi(x)]$ はある関数 $f_N(\varphi_0, \dots, \varphi_N)$ の分割数 N を極限まで増やしたものと見做すことができる. たとえば積分 $F[\varphi(x)] = \int_a^b \mathrm{d}x\, \varphi(x)$ では, 分割幅を $\Delta x_n := x_n - x_{n-1}$ として, Riemann 積分の考え方を用いれば[*3],

$$
\begin{aligned}
f_N(\varphi_0, \dots, \varphi_N) &= \sum_{n=1}^{N} \Delta x_n \times \varphi(x_n) \\
\xrightarrow{N\to\infty} \quad F[\varphi(x)] &= \int_a^b \mathrm{d}x\, \varphi(x).
\end{aligned}
$$

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*1 $F[\varphi(x)]$ という表記法は誤解を生む. たとえば, 十分に小さい x の関数 $\varepsilon : A \to A$ に対して $F[\varphi(x + \varepsilon(x))]$ を考える. このとき,

$$\varphi(x + \varepsilon(x)) = \varphi(x) + \varphi'(x)\varepsilon(x)$$

であるが,

$$F[\varphi(x + \varepsilon(x))] \neq F[\varphi(x) + \varphi'(x)\varepsilon(x)]$$

である. ダミー変数を添字にした $F_x[\varphi]$ という表記法を用いれば, 不等号の理由は明らかであろう:

$$F_{x+\varepsilon(x)}[\varphi] \neq F_x[\varphi + \varphi'\varepsilon].$$

*2 それすら怪しいかもしれない. 気付いたことがあれば随時更新する.

*3 これは Riemann 積分ではなく「区分求積法」である. Riemann 和を用いるならば $\varphi_n = \varphi(x_n)$ ではなく, 代表点 $x_{n-1} \le \xi_n \le x_n$ を用いて $\varphi_n := \varphi(\xi_n)$ とするべき. しかし, ここでは計算を主目的としているので, 細かいことは気にしない.

または, 等間隔な分割 $x_n := a + \dfrac{n(b-a)}{N}$, 分割幅 $\Delta x := \dfrac{b-a}{N}$ に対し, 例えば $\varphi(x) := x^2$ とすると,

$$\begin{aligned}
f_N(x_0^2, \dots, x_N^2) &= \sum_{n=1}^{N} \Delta x \times x_n^2 \\
\xrightarrow{N\to\infty} \quad F[x^2] &= \int_a^b \mathrm{d}x\, x^2.
\end{aligned}$$

このような汎関数の離散的な表現を考えることも重要である. 特に, 汎関数積分の計算においては離散表現が必須である.

0.1.1　汎関数の例

以下は汎関数である:

1. 積分

$$\begin{aligned}
i_N(\varphi_0, \dots, \varphi_N) &= \sum_{n=1}^{N} \Delta x \times g(\varphi_n) \\
\xrightarrow{N\to\infty} \quad I[\varphi(x)] &= \int \mathrm{d}x\, g(\varphi(x)).
\end{aligned}$$

2. 代入

$$\begin{aligned}
s(\varphi_0, \dots, \varphi_N; x_m = x') &= \sum_{n=1}^{N} \Delta x \times \varphi_n \frac{\delta_{nm}}{\Delta x} = \varphi_m \\
\xrightarrow{N\to\infty} \quad S[\varphi(x)](x') &= \int \mathrm{d}x\, \varphi(x)\delta(x-x') = \varphi(x').
\end{aligned}$$

汎関数中のデルタ関数 $\delta(x-x')$ は, 離散表現の $\dfrac{\delta_{nm}}{\Delta x}$ に対応している.

3. Fourier 変換

$$\begin{aligned}
f_N(\varphi_0, \dots, \varphi_N; k_m) &= \sum_{n=1}^{N} \frac{\Delta x}{\sqrt{2\pi}} \times \varphi_n e^{-ik_m x_n} \\
\xrightarrow{N\to\infty} \quad \mathcal{F}[\varphi(x)](k) &= \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \varphi(x) e^{-ikx}.
\end{aligned}$$

4. Fourier 逆変換

$$\begin{aligned}
f_N^{\text{“}-1\text{”}}(\tilde{\varphi}_0, \dots, \tilde{\varphi}_N; x_n) &= \sum_{m=1}^{N} \frac{\Delta k}{\sqrt{2\pi}} \times \tilde{\varphi}_m e^{ik_m x_n} \\
\xrightarrow{N\to\infty} \quad \mathcal{F}^{-1}[\tilde{\varphi}(k)](x) &= \int \frac{\mathrm{d}k}{\sqrt{2\pi}} \tilde{\varphi}(k) e^{ikx};
\end{aligned}$$

実際, $\mathcal{F}^{-1}[\mathcal{F}[\varphi(\tilde{x})](k)](x) = \varphi(x)$.

5. 汎関数のダミー変数を関数にしたもの

$$\begin{aligned}
g_N(x_0, \dots, x_N) &= f_N(\varphi_0, \dots, \varphi_N) \\
\xrightarrow{N\to\infty} \quad G_t[x] &:= F_{x(t)}[\varphi].
\end{aligned}$$

ただし, $x_n = x(t_n)$. 例えば $F_x[\varphi] := \displaystyle\int \mathrm{d}x\, \varphi(x)$ に対して,

$$\begin{aligned}
g_N(x_0, \dots, x_N) = f_N(\varphi_0, \dots, \varphi_N) &= \sum_{n=1}^{N} \Delta x \times \varphi_n = \sum_{n=1}^{N} \Delta t \times \frac{\Delta x}{\Delta t} \varphi(x_n) \\
\xrightarrow{N\to\infty} \quad G_t[x] = F_{x(t)}[\varphi] &= \int \mathrm{d}x(t)\, \varphi(x(t)) = \int \mathrm{d}t\, \frac{\mathrm{d}x}{\mathrm{d}t} \varphi(x(t)).
\end{aligned}$$

0.2　汎関数微分

汎関数 $F[\varphi(x)]$ の点 y における**汎関数微分**functional derivative は, 以下で定義される:

$$\frac{\delta F[\varphi(x)]}{\delta\varphi(y)} := \lim_{h\to 0}\frac{F[\varphi(x)+h\delta(x-y)]-F[\varphi(x)]}{h}.$$

物理では汎関数微分を変分とも呼び, 単に $\dfrac{\delta F[\varphi]}{\delta\varphi}$ とも略記される.

汎関数微分の離散的な表現は, $y = x_m$ として, 定義から

$$\begin{aligned}
&\lim_{h\to 0}\frac{f_N\left(\varphi_1+h\frac{\delta_{1m}}{\Delta x},\dots,\varphi_N+h\frac{\delta_{Nm}}{\Delta x}\right)-f_N(\varphi_1,\dots,\varphi_N)}{h}\\
&=\frac{1}{\Delta x}\lim_{h\to 0}\frac{f_N(\varphi_1,\dots,\varphi_m+h/\Delta x,\dots,\varphi_N)-f_N(\varphi_1,\dots,\varphi_N)}{h/\Delta x}\\
&=\frac{1}{\Delta x}\frac{\partial f_N}{\partial\varphi_m}.
\end{aligned}$$

したがって, 汎関数微分演算子 $\dfrac{\delta}{\delta\varphi(y)}$ に対応する離散表現は $\dfrac{1}{\Delta x}\dfrac{\partial}{\partial\varphi_m}$ である.

汎関数微分は線形性

$$\frac{\delta}{\delta\varphi(y)}\Big\{aF[\varphi(x)]+bG[\varphi(x)]\Big\} = a\frac{\delta F[\varphi(x)]}{\delta\varphi(y)}+b\frac{\delta G[\varphi(x)]}{\delta\varphi(y)}$$

や Leibniz 則

$$\frac{\delta}{\delta\varphi(y)}\Big\{F[\varphi(x)]G[\varphi(x)]\Big\} = \frac{\delta F[\varphi(x)]}{\delta\varphi(y)}G[\varphi(x)]+F[\varphi(x)]\frac{\delta G[\varphi(x)]}{\delta\varphi(y)}$$

を満たす.

0.2.1　汎関数微分の計算例

以下の汎関数 $F[\varphi(x)]$ について汎関数微分 $\dfrac{\delta F[\varphi(x)]}{\delta\varphi(y)}$ を計算する:

1. $F[\varphi(x)] = \displaystyle\int \mathrm{d}x\, g(x)\varphi(x)$:

$$\begin{aligned}
&\frac{\delta}{\delta\varphi(y)}\int \mathrm{d}x\, g(x)\varphi(x)\\
&=\lim_{h\to 0}\frac{1}{h}\left[\int \mathrm{d}x\, g(x)(\varphi(x)+h\delta(x-y))-\int \mathrm{d}x\, g(x)\varphi(x)\right]\\
&=\lim_{h\to 0}\frac{1}{h}\int \mathrm{d}x\, g(x)h\delta(x-y)\\
&=\int \mathrm{d}x\, g(x)\delta(x-y)=g(y).
\end{aligned}$$

離散表現では, $y = x_m$ として,

$$\frac{1}{\Delta x}\frac{\partial}{\partial\varphi_m}\sum_{n=1}^{N}\Delta x\times g(x_n)\varphi_n = g(x_m).$$

2. $F[\varphi(x)] = \varphi(x')$:

$$\frac{\delta\varphi(x')}{\delta\varphi(y)} = \frac{\delta}{\delta\varphi(y)}\int \mathrm{d}z\,\varphi(z)\delta(x'-z) = \delta(x'-y).$$

離散表現では, $y = x_m$, $x' = x_k$ として,

$$\frac{1}{\Delta x}\frac{\partial}{\partial\varphi_m}\sum_{n=1}^{N}\Delta x \times \varphi_n \frac{\delta_{nk}}{\Delta x} = \frac{\delta_{mk}}{\Delta x}.$$

3. $F[\varphi(x)] = \displaystyle\int \mathrm{d}x\, g(\varphi(x))$:

$$\begin{aligned}
&\frac{\delta}{\delta\varphi(y)}\int \mathrm{d}x\, g(\varphi(x)) \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int \mathrm{d}x\, g(\varphi(x) + h\delta(x-y)) - \int \mathrm{d}x\, g(\varphi(x))\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\int \mathrm{d}x\left[h\frac{\mathrm{d}g(\varphi(x))}{\mathrm{d}\varphi(x)}\delta(x-y) + O(h^2)\right]\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\left[h\frac{\mathrm{d}g(\varphi(y))}{\mathrm{d}\varphi(y)} + O(h^2)\right] \\
&= \frac{\mathrm{d}g(\varphi(y))}{\mathrm{d}\varphi(y)}.
\end{aligned}$$

離散表現では, $y = x_m$ として,

$$\frac{1}{\Delta x}\frac{\partial}{\partial\varphi_m}\sum_{n=1}^{N}\Delta x \times g(\varphi_n) = \frac{\mathrm{d}g(\varphi_m)}{\mathrm{d}\varphi_m}.$$

4. $F[\varphi(x)] = \displaystyle\int \mathrm{d}x\, g(\varphi'(x))$:

$$\begin{aligned}
&\frac{\delta}{\delta\varphi(y)}\int \mathrm{d}x\, g(\varphi'(x)) \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int \mathrm{d}x\, g\left(\frac{\mathrm{d}\{\varphi(x) + h\delta(x-y)\}}{\mathrm{d}x}\right) - \int \mathrm{d}x\, g\left(\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\int \mathrm{d}x\, g\left(\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} + h\frac{\mathrm{d}\delta(x-y)}{\mathrm{d}x}\right) - \int \mathrm{d}x\, g\left(\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\int \mathrm{d}x\left[h\frac{\mathrm{d}g(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}{\mathrm{d}(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}\frac{\mathrm{d}\delta(x-y)}{\mathrm{d}x} + O(h^2)\right]\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\int \mathrm{d}x\left[-h\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}g(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}{\mathrm{d}(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}\delta(x-y) + h\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}g(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}{\mathrm{d}(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}\delta(x-y)\right) + O(h^2)\right]\right\} \\
&\quad (\because\text{部分積分}) \\
&= \lim_{h\to 0}\frac{1}{h}\left[-h\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}g(\mathrm{d}\varphi(y)\,/\,\mathrm{d}y)}{\mathrm{d}(\mathrm{d}\varphi(y)\,/\,\mathrm{d}y)} + h\int \mathrm{d}\left(\frac{\mathrm{d}g(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}{\mathrm{d}(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}\delta(x-y)\right) + O(h^2)\right] \\
&= -\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}g(\mathrm{d}\varphi(y)\,/\,\mathrm{d}y)}{\mathrm{d}(\mathrm{d}\varphi(y)\,/\,\mathrm{d}y)} + \int \mathrm{d}\left(\frac{\mathrm{d}g(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}{\mathrm{d}(\mathrm{d}\varphi(x)\,/\,\mathrm{d}x)}\delta(x-y)\right) \\
&= -\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}g(\varphi'(y))}{\mathrm{d}\varphi'(y)} + \int \mathrm{d}\left(\frac{\mathrm{d}g(\varphi'(x))}{\mathrm{d}\varphi'(x)}\delta(x-y)\right).
\end{aligned}$$

特に y が積分範囲の内部にあるとき, 発散項を消すことができて,

$$\frac{\delta}{\delta\varphi(y)}\int \mathrm{d}x\, g(\varphi'(x)) = -\frac{\mathrm{d}}{\mathrm{d}y}\frac{\mathrm{d}g(\varphi'(y))}{\mathrm{d}\varphi'(y)}.$$

離散表現では, $y = x_m$ として,

$$\frac{1}{\Delta x}\frac{\partial}{\partial\varphi_m}\sum_{n=1}^{N}\Delta x \times g\Big(\frac{\varphi_n - \varphi_{n-1}}{\Delta x}\Big) = -\frac{g'\Big(\frac{\varphi_{m+1}-\varphi_m}{\Delta x}\Big) - g'\Big(\frac{\varphi_m - \varphi_{m-1}}{\Delta x}\Big)}{\Delta x}.$$

5. $F[\varphi(x)] = \int \mathrm{d}x\, g(\varphi(x), \varphi'(x))$:
上の例を繰り返し使うことで,

$$\frac{\delta}{\delta\varphi(y)}\int \mathrm{d}x\, g(\varphi(x), \varphi'(x)) = \frac{\partial g}{\partial\varphi(y)} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial g}{\partial\varphi'(y)} + \int \mathrm{d}\Big(\frac{\partial g}{\partial\varphi'(x)}\delta(x-y)\Big),$$

あるいは, y が積分範囲の内部にあるとき,

$$\frac{\delta}{\delta\varphi(y)}\int \mathrm{d}x\, g(\varphi(x), \varphi'(x)) = \frac{\partial g}{\partial\varphi(y)} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial g}{\partial\varphi'(y)}.$$

0.3 汎関数冪級数

連続な汎関数は Tayler 級数に相当する以下の冪級数に展開することができる. これを **Volterra 級数** Volterra series という: 微小な関数 $\eta(x)$ を用いて,

$$\begin{aligned}F[\varphi(x)+\eta(x)] = F[\varphi(x)] &+ \int \mathrm{d}y\, \frac{\delta F[\varphi(x)]}{\delta\varphi(y)}\eta(y) \\ &+ \frac{1}{2}\int \mathrm{d}y_1 \int \mathrm{d}y_2\, \frac{\delta^2 F[\varphi(x)]}{\delta\varphi(y_1)\delta\varphi(y_2)}\eta(y_1)\eta(y_2) + \cdots \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}y_1 \cdots \int \mathrm{d}y_n\, \frac{\delta^n F[\varphi(x)]}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}\eta(y_1)\cdots\eta(y_n).\end{aligned}$$

特に, $\varphi = 0$ まわりの冪展開は,

$$\begin{aligned}F[\varphi(x)] &= F[0] + \int \mathrm{d}y\, \frac{\delta F[\varphi(x)]}{\delta\varphi(y)}\bigg|_{\varphi=0}\varphi(y) + \frac{1}{2}\int \mathrm{d}y_1 \int \mathrm{d}y_2\, \frac{\delta^2 F[\varphi(x)]}{\delta\varphi(y_1)\delta\varphi(y_2)}\bigg|_{\varphi=0}\varphi(y_1)\varphi(y_2) + \cdots \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}y_1 \cdots \int \mathrm{d}y_n\, \frac{\delta^n F[\varphi(x)]}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}\bigg|_{\varphi=0}\varphi(y_1)\cdots\varphi(y_n).\end{aligned}$$

汎関数冪級数の離散表現は,

$$\begin{aligned}&f_N(\varphi_0+\eta_0, \dots, \varphi_N+\eta_N) \\ &= f_N(\varphi_0, \dots, \varphi_N) + \sum_{m=0}^{N}\Delta x \frac{1}{\Delta x}\frac{\partial f_N}{\partial\varphi_m}\eta_m + \frac{1}{2}\sum_{m_1=0}^{N}\Delta x \sum_{m_2=0}^{N}\Delta x \frac{1}{(\Delta x)^2}\frac{\partial^2 f_N}{\partial\varphi_{m_1}\partial\varphi_{m_2}}\eta_{m_1}\eta_{m_2} + \cdots \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{m_1=0}^{N}\Delta x \cdots \sum_{m_n=0}^{N}\Delta x \frac{1}{(\Delta x)^n}\frac{\partial^n f_N(\varphi_0, \dots, \varphi_N)}{\partial\varphi_{m_1}\cdots\partial\varphi_{m_n}}\eta_{m_1}\cdots\eta_{m_n}.\end{aligned}$$

この表現は関数 $f_N(\varphi_0+\eta_0, \dots, \varphi_N+\eta_N)$ の $(\varphi_0, \dots, \varphi_N)$ まわりでの Taylor 展開になっている.

n 階汎関数微分 $\frac{\delta^n F[\varphi(x)]}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}$ が $y_1, \dots, y_n$ について対称であると仮定して, $\frac{\delta^n F}{\delta\varphi^n}$ と略記する. また,

$$\frac{\delta^n F}{\delta\varphi^n} * \eta^n := \int \mathrm{d}y_1 \cdots \int \mathrm{d}y_n\, \frac{\delta^n F[\varphi(x)]}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}\eta(y_1)\cdots\eta(y_n)$$

とすると, Volterra 級数は以下のように書き直せる:

$$F[\varphi(x)+\eta(x)] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\delta^n F}{\delta\varphi^n} * \eta^n.$$

0.3.1 冪級数を用いた計算例

1. $\dfrac{\delta^n F}{\delta\varphi^n} * \eta^n$ の $\eta(y)$ による汎関数微分:

$$\begin{aligned}
&\frac{\delta}{\delta\eta(y)}\left(\frac{\delta^n F}{\delta\varphi^n} * \eta^n\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left[\int \mathrm{d}y_1 \cdots \int \mathrm{d}y_n \, \frac{\delta^n F[\varphi(x)]}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}[\eta(y_1)+h\delta(y_1-y)]\cdots[\eta(y_n)+h\delta(y_n-y)]\right. \\
&\qquad\qquad \left. - \int \mathrm{d}y_1 \cdots \int \mathrm{d}y_n \, \frac{\delta^n F[\varphi(x)]}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}\eta(y_1)\cdots\eta(y_n)\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\sum_{i=0}^{n} \int \mathrm{d}y_1 \cdots \int \mathrm{d}y_n \, \frac{\delta^n F[\varphi(x)]}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}\eta(y_1)\cdots\widehat{\eta(y_i)}\cdots\eta(y_n)h\delta(y_i-y)+O(h^2)\right] \\
&= \sum_{i=0}^{n} \int \mathrm{d}y_1 \cdots \int \mathrm{d}y_n \, \frac{\delta^n F[\varphi(x)]}{\delta\varphi(y_1)\cdots\delta\varphi(y_n)}\eta(y_1)\cdots\widehat{\eta(y_i)}\cdots\eta(y_n)\delta(y_i-y) \\
&= n\int \mathrm{d}y_1 \cdots \int \mathrm{d}y_{n-1} \, \frac{\delta^n F[\varphi(x)]}{\delta\varphi(y)\delta\varphi(y_1)\cdots\delta\varphi(y_{n-1})}\eta(y_1)\cdots\eta(y_{n-1}) \\
&= n\frac{\delta}{\delta\varphi(y)}\left(\frac{\delta^{n-1}F}{\delta\varphi^{n-1}}\right) * \eta^{n-1} \quad \left(=: n\frac{\delta^n F}{\delta\varphi^n} * \eta^{n-1} \text{ とも書く}\right).
\end{aligned}$$

2. $g(F[\varphi(x)])$ の汎関数微分:

$$\begin{aligned}
&\frac{\delta g(F[\varphi(x)])}{\delta\varphi(y)} \\
&= \lim_{h\to 0}\frac{1}{h}[g(F[\varphi(x)+h\delta(x-y)])-g(F[\varphi(x)])] \\
&= \lim_{h\to 0}\frac{1}{h}\left[g\left(F[\varphi(x)]+\int \mathrm{d}z\,\frac{\delta F[\varphi(x)]}{\delta\varphi(z)}h\delta(z-y)+O(h^2)\right)-g(F[\varphi(x)])\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[g\left(F[\varphi(x)]+h\frac{\delta F[\varphi(x)]}{\delta\varphi(y)}+O(h^2)\right)-g(F[\varphi(x)])\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[h\frac{\mathrm{d}g(F[\varphi(x)])}{\mathrm{d}F[\varphi(x)]}\frac{\delta F[\varphi(x)]}{\delta\varphi(y)}+O(h^2)\right] \\
&= \frac{\mathrm{d}g(F[\varphi(x)])}{\mathrm{d}F[\varphi(x)]}\frac{\delta F[\varphi(x)]}{\delta\varphi(y)}.
\end{aligned}$$

3. x の積分で定義される汎関数 $F[\varphi(x,t)]$ に対し, 微分 $\dfrac{\mathrm{d}}{\mathrm{d}t}F[\varphi(x,t)]$:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}F[\varphi(x,t)] \\
&= \lim_{h\to 0}\frac{F[\varphi(x,t+h)]-F[\varphi(x,t)]}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{F\left[\varphi(x,t)+h\frac{\partial\varphi(x,t)}{\partial t}+O(h^2)\right]-F[\varphi(x,t)]\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{F[\varphi(x,t)]+h\int \mathrm{d}y\,\frac{\delta F[\varphi(x)]}{\delta\varphi(y)}\frac{\partial\varphi(y,t)}{\partial t}+O(h^2)-F[\varphi(x,t)]\right\} \\
&= \int \mathrm{d}y\,\frac{\delta F[\varphi(x,t)]}{\delta\varphi(y,t)}\frac{\partial\varphi(y,t)}{\partial t}.
\end{aligned}$$

4. 微小変換 $x(t) \mapsto x'(t) = x(t) + \delta x(t)$ に対し $\varphi(x(t)) \mapsto \varphi'(x'(t)) = \varphi(x(t)) + \delta\varphi(x(t))$ と変換されるとき, 汎関数 $F_{x'(t)}[\varphi']$ を 1 次まで展開することを考える. 汎関数 $F_{x(t)}[\varphi']$ をパラメータ $x(t)$ に関する汎関数 $G_t[x] := F_{x(t)}[\varphi']$ と見れば $\delta x(t)$ の 1 次で展開することができ,

$$
\begin{aligned}
&F_{x'(t)}[\varphi'] \\
&= F_{x(t)+\delta x(t)}[\varphi'] \\
&\left(= G_t[x+\delta x] = G_t[x] + \int \mathrm{d}x_0 \, \frac{\delta G_t[x]}{\delta x(t_0)} \delta x(t_0)\right) \\
&= F_{x(t)}[\varphi'] + \int \mathrm{d}t_0 \, \frac{\delta F_{x(t)}[\varphi']}{\delta x(t_0)} \delta x(t_0) \\
&= F_{x(t)}[\varphi + \delta^L \varphi] + \int \mathrm{d}t_0 \, \frac{\delta F_{x(t)}[\varphi + \delta^L \varphi]}{\delta x(t_0)} \delta x(t_0) \\
&= F_{x(t)}[\varphi + \delta^L \varphi] + \int \mathrm{d}t_0 \, \frac{\delta F_{x(t)}[\varphi]}{\delta x(t_0)} \delta x(t_0).
\end{aligned}
$$

ただし, $\delta^L \varphi(x(t))$ は Lie 微分である:

$$
\delta^L \varphi(x(t)) := \varphi'(x(t)) - \varphi(x(t)) = \delta\varphi(x(t)) - \frac{\mathrm{d}\varphi(x(t))}{\mathrm{d}x(t)} \delta x(t).
$$

次に $F_{x(t)}[\varphi']$ を 1 次で展開して,

$$
\begin{aligned}
&F_{x(t)}[\varphi + \delta^L \varphi] \\
&= F_{x(t)}[\varphi] + \int \mathrm{d}x(t_0) \, \frac{\delta F_{x(t)}[\varphi]}{\delta\varphi(x(t_0))} \delta^L \varphi(x(t_0)) \\
&= F_{x(t)}[\varphi] + \int \mathrm{d}x(t_0) \, \frac{\delta F_{x(t)}[\varphi]}{\delta\varphi(x(t_0))} \delta\varphi(x(t_0)) - \int \mathrm{d}x(t_0) \, \frac{\delta F_{x(t)}[\varphi]}{\delta\varphi(x(t_0))} \frac{\mathrm{d}\varphi(x(t_0))}{\mathrm{d}x(t_0)} \delta x(t_0) \\
&= F_{x(t)}[\varphi] + \int \mathrm{d}x(t_0) \, \frac{\delta F_{x(t)}[\varphi]}{\delta\varphi(x(t_0))} \delta\varphi(x(t_0)) - \int \mathrm{d}t_0 \, \frac{\delta F_{x(t)}[\varphi]}{\delta\varphi(x(t_0))} \frac{\mathrm{d}\varphi(x(t_0))}{\mathrm{d}t_0} \delta x(t_0).
\end{aligned}
$$

これを前の式に代入すれば, $F_{x'(t)}[\varphi']$ の 1 次の展開が得られる:

$$
\begin{aligned}
F_{x'(t)}[\varphi'] &= F_{x(t)}[\varphi] + \int \mathrm{d}x(t_0) \, \frac{\delta F_{x(t)}[\varphi]}{\delta\varphi(x(t_0))} \delta^L \varphi(x(t_0)) + \int \mathrm{d}t_0 \, \frac{\delta F_{x(t)}[\varphi]}{\delta x(t_0)} \delta x(t_0) \\
&= F_{x(t)}[\varphi] + \int \mathrm{d}x(t_0) \, \frac{\delta F_{x(t)}[\varphi]}{\delta\varphi(x(t_0))} \delta\varphi(x(t_0)) \\
&\qquad + \int \mathrm{d}t_0 \left[\frac{\delta F_{x(t)}[\varphi]}{\delta x(t_0)} - \frac{\delta F_{x(t)}[\varphi]}{\delta\varphi(x(t_0))} \frac{\mathrm{d}\varphi(x(t_0))}{\mathrm{d}t_0}\right] \delta x(t_0).
\end{aligned}
$$

5. 一般の汎関数微分:

$$
\begin{aligned}
(DF)[\varphi(x)][\eta(x)] &:= \left.\frac{\mathrm{d}F[\varphi(x) + \lambda\eta(x)]}{\mathrm{d}\lambda}\right|_{\lambda=0} \\
&= \lim_{h \to 0} \frac{F[\varphi(x) + h\eta(x)] - F[\varphi(x)]}{h}.
\end{aligned}
$$

先に定義した汎関数微分は

$$
\frac{\delta F[\varphi(x)]}{\delta\varphi(y)} = (DF)[\varphi(x)][\delta(x-y)]
$$

と書ける. また, $F[\varphi(x)+h\eta(x)]$ を冪展開すると

$$F[\varphi(x)+h\eta(x)] = F[\varphi(x)] + h\int \mathrm{d}y\,\frac{\delta F[\varphi(x)]}{\delta\varphi(y)}\eta(y) + O(h^2)$$

だから, 定義式に代入すれば, 一般の汎関数微分の表示が得られる*4:

$$(DF)[\varphi(x)][\eta(x)] = \int \mathrm{d}x\,\frac{\delta F[\varphi(x)]}{\delta\varphi(x)}\eta(x) = \frac{\delta F}{\delta\varphi} * \eta.$$

また, この表示を汎関数冪級数に代入すれば, 一般の汎関数微分に関する冪級数展開が得られる:

$$F[\varphi(x)+\eta(x)] = \sum_{n=0}^{\infty}\frac{1}{n!}(D^nF)[\varphi(x)]\underbrace{[\eta(x)]\cdots[\eta(x)]}_{n}.$$

0.4 汎関数積分

$x\in[a,b]$ の関数上で定義される $F[\varphi(x)]$ の**汎関数積分** functional integration は, 以下で定義される:

$$\begin{aligned}\int \mathcal{D}\varphi(x)F[\varphi(x)] &:= \frac{1}{\theta}\left(\prod_{x\in[a,b]}\int \mathrm{d}\varphi(x)\right)F[\varphi(x)] \\ &:= \lim_{N\to\infty}\frac{1}{\theta(N)}\int \mathrm{d}\varphi_0\cdots\int \mathrm{d}\varphi_N\, f_N(\varphi_0,\dots,\varphi_N).\end{aligned}$$

ただし, θ は有限値に収束させるための正規化因子, $f_N(\varphi_0,\dots,\varphi_N)$ は $F[\varphi(x)]$ の離散表現である. 単に $\int \mathcal{D}\varphi F[\varphi]$ とも略記される.

$\varphi(x)$ の端を固定した汎関数積分も重要である:

$$\begin{aligned}\int_{\varphi_0}^{\varphi}\mathcal{D}\varphi(x)F[\varphi(x)] &:= \frac{1}{\theta}\left(\prod_{x\in(a,b)}\int \mathrm{d}\varphi(x)\right)F[\varphi(x)]\Bigg|_{\varphi(a)=\varphi_0}^{\varphi(b)=\varphi} \\ &:= \lim_{N\to\infty}\frac{1}{\theta(N)}\int \mathrm{d}\varphi_1\cdots\int \mathrm{d}\varphi_{N-1}\, f_N(\varphi_0,\varphi_1,\dots,\varphi_{N-1},\varphi).\end{aligned}$$

これは, 端点を固定した経路の経路上各点について積分した積になっていることから, **経路積分**とも呼ばれる. 経路積分の表記法については別記事を参照.

0.4.1 汎関数積分の計算例

1. 自由粒子型:

$$I(\varphi) = \int_{\varphi_0}^{\varphi}\mathcal{D}\varphi(x)\exp\left[i\int_a^b \mathrm{d}x\,\frac{A}{2}\{\varphi'(x)\}^2\right],$$

*4 $\delta F[\varphi(x)]/\delta\varphi(y)$ が y の寄与に対してだけデルタ関数を足した微分であったことを思い出せば, $(DF)[\varphi(x)][\eta(x)]$ は $\eta(x)$ で特徴付けられる方向に沿った微分と考えることができる. これはちょうど偏微分 $\partial f(\boldsymbol{x})/\partial x_i$ と方向微分 $\boldsymbol{v}\cdot\nabla f(\boldsymbol{x})$ の関係に対応している.

ただし $\displaystyle\int \mathrm{d}\varphi\, I(\varphi)=1$ として正規化する. $F[\varphi(x)]=\exp\left[i\int_a^b \mathrm{d}x\,\frac{A}{2}\{\varphi'(x)\}^2\right]$ の離散表現は,

$$f_N(\varphi_0,\varphi_1,\dots,\varphi_{N-1},\varphi)=\exp\left[i\sum_{n=1}^{N}\Delta x\times\frac{A}{2}\Big(\frac{\varphi_n-\varphi_{n-1}}{\Delta x}\Big)^2\right]_{\varphi_0=\varphi_0}^{\varphi_N=\varphi}.$$

ただし, 分割幅を $\Delta x := (b-a)/N$ とした. したがって $F[\varphi(x)]$ の汎関数積分は,

$$\begin{aligned}
I(\varphi)&=\int_{\varphi(a)=\varphi_0}^{\varphi(b)=\varphi}\mathcal{D}\varphi(x)\exp\left[i\int_a^b\mathrm{d}x\,\frac{A}{2}\{\varphi'(x)\}^2\right]\\
&=\lim_{N\to\infty}\frac{1}{\theta(N)}\int\mathrm{d}\varphi_1\cdots\int\mathrm{d}\varphi_{N-1}\exp\left[i\sum_{n=1}^{N}\Delta x\times\frac{A}{2}\Big(\frac{\varphi_n-\varphi_{n-1}}{\Delta x}\Big)^2\right]_{\varphi_0=\varphi_0}^{\varphi_N=\varphi}\\
&=\lim_{N\to\infty}\frac{1}{\theta(N)}\int\mathrm{d}\varphi_1\cdots\int\mathrm{d}\varphi_{N-1}\exp\left[\frac{iA}{2\Delta x}\sum_{n=1}^{N}(\varphi_n-\varphi_{n-1})^2\right]_{\varphi_0=\varphi_0}^{\varphi_N=\varphi}\\
&=\lim_{N\to\infty}\frac{1}{\theta(N)}\int\mathrm{d}\varphi_1\cdots\int\mathrm{d}\varphi_{N-1}\exp\left\{\frac{iA}{2\Delta x}\left[(\varphi-\varphi_{N-1})^2+\sum_{k=1}^{N-1}(\varphi_{N-k}-\varphi_{N-(k+1)})^2\right]\right\}_{\varphi_0=\varphi_0}.
\end{aligned}$$

ここで φ_{N-k} の積分について考えると,

$$\begin{aligned}
&\int\mathrm{d}\varphi_{N-k}\exp\left\{\frac{iA}{2\Delta x}\left[\frac{1}{k}(\varphi-\varphi_{N-k})^2+(\varphi_{N-k}-\varphi_{N-(k+1)})^2\right]\right\}\\
=&\int\mathrm{d}\varphi_{N-k}\exp\left\{\frac{iA}{2\Delta x}\left[\frac{k+1}{k}\varphi_{N-k}^2-2\Big(\frac{1}{k}\varphi+\varphi_{N-(k+1)}\Big)\varphi_{N-k}+\Big(\frac{1}{k}\varphi^2+\varphi_{N-(k+1)}^2\Big)\right]\right\}\\
=&\int\mathrm{d}\varphi_{N-k}\exp\left[\frac{iA}{2\Delta x}\frac{k+1}{k}\varphi_{N-k}^2-\frac{iA}{2\Delta x}2\Big(\frac{1}{k}\varphi+\varphi_{N-(k+1)}\Big)\varphi_{N-k}+\frac{iA}{2\Delta x}\Big(\frac{1}{k}\varphi^2+\varphi_{N-(k+1)}^2\Big)\right]\\
&\quad\left(\int\mathrm{d}x\exp\left(-iax^2+ibx\right)=\sqrt{\frac{\pi}{ia}}\exp\left(\frac{ib^2}{4a}\right)\right)\\
=&\sqrt{\frac{k}{k+1}}\sqrt{\frac{2\pi i\Delta x}{A}}\exp\left[-\frac{iA}{2\Delta x}\frac{k}{k+1}(\varphi+\varphi_{N-(k+1)})^2+\frac{iA}{2\Delta x}\Big(\frac{1}{k}\varphi^2+\varphi_{N-(k+1)}^2\Big)\right]\\
=&\sqrt{\frac{k}{k+1}}\sqrt{\frac{2\pi i\Delta x}{A}}\exp\left[\frac{iA}{2\Delta x}\frac{1}{k+1}\big(\varphi-\varphi_{N-(k+1)}\big)^2\right]
\end{aligned}$$

より, $k=1,\dots,N-1$ で順に積分することで,

$$\begin{aligned}
I(\varphi)&=\lim_{N\to\infty}\frac{1}{\theta(N)}\sqrt{\frac{1}{2}}\sqrt{\frac{2}{3}}\cdots\sqrt{\frac{N-1}{N}}\left(\sqrt{\frac{2\pi i\Delta x}{A}}\right)^{N-1}\exp\left[\frac{iA}{2N\Delta x}(\varphi-\varphi_0)^2\right]\\
&=\lim_{N\to\infty}\frac{1}{\theta(N)}\frac{1}{\sqrt{N}}\Big(\frac{2\pi i\Delta x}{A}\Big)^{(N-1)/2}\exp\left[\frac{iA}{2N\Delta x}(\varphi-\varphi_0)^2\right].
\end{aligned}$$

ここで, 定数 C を用いて $\theta(N)=\dfrac{1}{C}\Big(\dfrac{2\pi i\Delta x}{A}\Big)^{N/2}$ とすれば,

$$\begin{aligned}
I(\varphi)&=\lim_{N\to\infty}C\Big(\frac{A}{2\pi i\Delta x}\Big)^{N/2}\frac{1}{\sqrt{N}}\Big(\frac{2\pi i\Delta x}{A}\Big)^{(N-1)/2}\exp\left[\frac{iA}{2N\Delta x}(\varphi-\varphi_0)^2\right]\\
&=\lim_{N\to\infty}C\sqrt{\frac{a}{2\pi iN\Delta x}}\exp\left[\frac{iA}{2N\Delta x}(\varphi-\varphi_0)^2\right]\\
&=C\sqrt{\frac{A}{2\pi i(b-a)}}\exp\left[i\frac{A}{2}\frac{(\varphi-\varphi_0)^2}{b-a}\right].
\end{aligned}$$

正規化条件より定数 C を決定すると,

$$1 = \int \mathrm{d}\varphi\, I(\varphi) = C\int \mathrm{d}\varphi \sqrt{\frac{A}{2\pi i(b-a)}}\exp\left[i\frac{A}{2}\frac{(\varphi-\varphi_0)^2}{b-a}\right] = C.$$

したがって,

$$I(\varphi) = \int_{\varphi(a)=\varphi_0}^{\varphi(b)=\varphi} \mathcal{D}\varphi(x)\exp\left[i\int_a^b \mathrm{d}x\,\frac{A}{2}\{\varphi'(x)\}^2\right] = \sqrt{\frac{A}{2\pi i(b-a)}}\exp\left[i\frac{A}{2}\frac{(\varphi-\varphi_0)^2}{b-a}\right].$$

2. 汎関数積分の連結:

$x_3 > x_2 > x_1$ に対し, $x \in [x_3, x_1]$ の関数上で定義される汎関数 $F[\varphi(x)]$ について,

$$\int_{\varphi_1}^{\varphi_2} \mathcal{D}\varphi(x)\int \mathrm{d}\varphi_2 \int_{\varphi_2}^{\varphi_3} \mathcal{D}\varphi(x)F[\varphi(x)] = \int_{\varphi_1}^{\varphi_3} \mathcal{D}\varphi(x)F[\varphi(x)].$$

実際,

$$\begin{aligned}
&\int_{\varphi_1}^{\varphi_2} \mathcal{D}\varphi(x)\int \mathrm{d}\varphi_2 \int_{\varphi_2}^{\varphi_3} \mathcal{D}\varphi(x)g(\varphi_2)F[\varphi(x)] \\
&= \frac{1}{\theta}\left(\prod_{x\in(t_1,t_2)}\int \mathrm{d}\varphi(x)\right)\int \mathrm{d}\varphi(x_2)\left(\prod_{x\in(t_2,t_3)}\int \mathrm{d}\varphi(x)\right)F[\varphi(x)] \\
&= \frac{1}{\theta}\left(\prod_{x\in(t_1,t_3)}\int \mathrm{d}\varphi(x)\right)F[\varphi(x)] \quad (\because (t_1,t_2)\cup\{t_2\}\cup(t_2,t_3) = (t_1,t_3)) \\
&= \int_{\varphi_1}^{\varphi_3} \mathcal{D}\varphi(x)F[\varphi(x)].
\end{aligned}$$

特に, 指数法則 $F_{x\in A}[\varphi]F_{x\in B}[\varphi] = F_{x\in A\cup B}[\varphi]$ を満たす汎関数 (例えば $F_{x\in[a,b]}[\varphi] = \exp\left[\int_a^b \mathrm{d}x\,\varphi(x)\right]$) に対しては,

$$\begin{aligned}
&\int \mathrm{d}\varphi_2\, g(\varphi_2)\left(\int_{\varphi_1}^{\varphi_2} \mathcal{D}\varphi(x)F_{x\in[x_1,x_2]}[\varphi]\right)\left(\int_{\varphi_2}^{\varphi_3} \mathcal{D}\varphi(x)F_{x\in[x_2,x_3]}[\varphi]\right) \\
&= \int_{\varphi_1}^{\varphi_3} \mathcal{D}\varphi(x)F_{x\in[x_1,x_3]}[\varphi]g(\varphi(x_2)).
\end{aligned}$$

実際,

$$\begin{aligned}
&\int \mathrm{d}\varphi_2\, g(\varphi_2)\left(\int_{\varphi_1}^{\varphi_2} \mathcal{D}\varphi(x)F_{x\in[x_1,x_2]}[\varphi]\right)\left(\int_{\varphi_2}^{\varphi_3} \mathcal{D}\varphi(x)F_{x\in[x_2,x_3]}[\varphi]\right) \\
&= \int_{\varphi_1}^{\varphi_2} \mathcal{D}\varphi(x)\int \mathrm{d}\varphi_2 \int_{\varphi_2}^{\varphi_3} \mathcal{D}\varphi'(x)g(\varphi_2)F_{x\in[x_1,x_2]}[\varphi]F_{x\in[x_2,x_3]}[\varphi'] \\
&= \int_{\varphi_1}^{\varphi_3} \mathcal{D}\varphi(x)g(\varphi(x_2))F_{x\in[x_1,x_3]}[\varphi].
\end{aligned}$$

3. デルタ汎関数 $\Delta[\varphi(x)]$:

汎関数積分で

$$\Delta[\varphi(x)] := \int \mathcal{D}\xi(x)\exp\left[i\int \mathrm{d}x\,\varphi(x)\xi(x)\right]$$

と定義される. 離散表現は

$$
\begin{aligned}
\delta_N(\{\varphi_n\}) &\equiv \frac{1}{\theta(N)}\left(\prod_n \int \mathrm{d}\xi_n\right)\exp\left[i\sum_n \Delta x \times \varphi_n \xi_n\right] \\
&= \frac{1}{\theta(N)}\prod_n \int \mathrm{d}\xi_n \exp\Big(i\Delta x \times \varphi_n \xi_n\Big) \\
&= \frac{1}{\theta(N)}\prod_n 2\pi\delta(\varphi_n \Delta x) \\
&= \frac{1}{\theta(N)}\left(\frac{2\pi}{\Delta x}\right)^N \prod_n \delta(\varphi_n)
\end{aligned}
$$

となって, 正規化因子を $\theta(N) = (2\pi/\Delta x)^N$ と置けば

$$
\delta_N(\{\varphi_n\}) = \prod_n \delta(\varphi_n)
$$

であるから, $N \to \infty$ の極限で, デルタ汎関数は

$$
\Delta[\varphi(x)] = \prod_x \delta(\varphi(x))
$$

と書ける. さて, 汎関数デルタ関数は,

$$
\int \mathcal{D}\varphi(x) F[\varphi(x)]\Delta[\varphi(x) - \tilde{\varphi}(x)] = F[\tilde{\varphi}(x)],
$$

$$
\int \mathcal{D}\varphi(x)\Delta[\varphi(x)] = 1
$$

の性質を満たす. 実際,

$$
\begin{aligned}
&\int \mathcal{D}\varphi(x) F[\varphi(x)]\Delta[\varphi(x) - \tilde{\varphi}(x)] \\
&= \frac{1}{\theta}\left(\prod_x \int \mathrm{d}\varphi(x)\right) F[\varphi(x)] \left(\prod_x \delta(\varphi(x) - \tilde{\varphi}(x))\right) \\
&= \frac{1}{\theta}\left(\prod_x \int \mathrm{d}\varphi(x)\, \delta(\varphi(x) - \tilde{\varphi}(x))\right) F[\varphi(x)] \\
&= F[\tilde{\varphi}(x)]
\end{aligned}
$$

であって, 恒等的に $F[\varphi(x)] = 1$, $\tilde{\varphi}(x) = 1$ とすれば第二式が得られる.

0.5 汎関数 Fourier 変換

汎関数 $F[\varphi(x)]$ に対する汎関数 Fourier 変換 $\mathcal{F}$ は

$$\mathcal{F}\{F[\varphi(x)]\}[\xi(x)] := \int \mathcal{D}\varphi(x) F[\varphi(x)] \exp\left[-i \int \mathrm{d}x\, \varphi(x)\xi(x)\right],$$

また, $\tilde{F}[\xi(x)]$ に対する逆変換 $\mathcal{F}^{-1}$ は

$$\mathcal{F}^{-1}\{\tilde{F}[\xi(x)]\}[\varphi(x)] := \int \mathcal{D}\xi(x) \tilde{F}[\xi(x)] \exp\left[i \int \mathrm{d}x\, \varphi(x)\xi(x)\right]$$

で定義され,

$$\mathcal{F}^{-1}\{\mathcal{F}\{F[\tilde{\varphi}(x)]\}[\xi(x)]\}[\varphi(x)] = F[\varphi(x)]$$

を満たす.

離散表現で汎関数 Fourier 変換の標識を導出する. 汎関数の離散表現 $f_N(\{\varphi_n\})$ と $\tilde{f}_N(\{\xi_n\})$ に対し, $g_N(\{\sqrt{\Delta x}\varphi_n\}) \equiv f_N(\{\varphi_n\})$ と $\tilde{g}_N(\{\sqrt{\Delta x}\xi_n\}) \equiv \tilde{f}_N(\{\xi_n\})$ を定義して, g_N と $\tilde{g}_N$ の間の多変数 Fourier 変換を考えると,

$$\begin{aligned}
\tilde{g}_N(\{\sqrt{\Delta x}\xi_n\}) &= \left(\prod_n \int \frac{\sqrt{\Delta x}\, \mathrm{d}\varphi_n}{\sqrt{2\pi}}\right) g_N(\{\sqrt{\Delta x}\varphi_n\}) \exp\left[-i \sum_n \sqrt{\Delta x}\varphi_n \times \sqrt{\Delta x}\xi_n\right] \\
&= \left(\frac{\Delta x}{2\pi}\right)^{N/2} \left(\prod_n \int \mathrm{d}\varphi_n\right) g_N(\{\sqrt{\Delta x}\varphi_n\}) \exp\left[-i \sum_n \Delta x \times \varphi_n \xi_n\right]
\end{aligned}$$

となるから, 結局 f_N と $\tilde{f}_N$ の関係は

$$\tilde{f}_N(\{\xi_n\}) = \left(\frac{\Delta x}{2\pi}\right)^{N/2} \left(\prod_n \int \mathrm{d}\varphi_n\right) f_N(\{\varphi_n\}) \exp\left[-i \sum_n \Delta x \times \varphi_n \xi_n\right]$$

となって, $N \to \infty$ の極限で汎関数 Fourier 変換が得られる. 逆変換も同様.

汎関数デルタ関数 $\Delta[\varphi(x)]$ を用いれば, 容易に逆変換であることがわかる:

$$\begin{aligned}
&\mathcal{F}^{-1}\{\mathcal{F}\{F[\tilde{\varphi}(x)]\}[\xi(x)]\}[\varphi(x)] \\
&= \int \mathcal{D}\xi(x) \mathcal{F}\{F[\tilde{\varphi}(x)]\}[\xi(x)] \exp\left[i \int \mathrm{d}x\, \varphi(x)\xi(x)\right] \\
&= \int \mathcal{D}\tilde{\varphi}(x) F[\tilde{\varphi}(x)] \int \mathcal{D}\xi(x) \exp\left\{i \int \mathrm{d}x \left[\varphi(x) - \tilde{\varphi}(x)\right]\xi(x)\right\} \\
&= \int \mathcal{D}\tilde{\varphi}(x) F[\tilde{\varphi}(x)] \Delta[\varphi(x) - \tilde{\varphi}(x)] \\
&= F[\varphi(x)].
\end{aligned}$$

0.5.1 汎関数 Fourier 変換の計算例

規格化定数は都合の良いように取る.

1. 1 (恒等的に 1 である汎関数) の汎関数 Fourier 変換:

$$\mathcal{F}\{1\}[\xi(x)] = \int \mathcal{D}\varphi(x) \exp\left[-i \int \mathrm{d}x\, \varphi(x)\xi(x)\right] = \Delta[\xi(x)].$$

2. デルタ汎関数 $\Delta[\varphi(x)]$ の汎関数 Fourier 変換:

$$\begin{aligned}\mathcal{F}\{\Delta[\varphi(x)]\}[\xi(x)] &= \int \mathcal{D}\varphi(x)\Delta[\varphi(x)]\exp\left[-i\int \mathrm{d}x\,\varphi(x)\xi(x)\right] \\ &= \exp\left[-i\int \mathrm{d}x\,0\times\xi(x)\right] \\ &= 1\end{aligned}$$

0.6 参考文献

- Stevens, C. F. The Six Core Theories of Modern Physics (United Kingdom, MIT Press, 1995)