Tougaloo College
Spring 2020

Final Exam Time: 9:00 - 11:00 am
Final Exam Date: Monday, Apr. 27

MATH 221 Cal I Final Exam Review and Practice

1. Find the value of the limit, and, when applicable, indicate the limit theorems being used.

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} \\ &= \lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} \cdot \frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}} \end{aligned}$$

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{1-\cos x}{\sin x} \\ &= \lim_{x\to 0} \frac{1-\cos x}{\sin x} \cdot \frac{x}{x} \\ &= \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{x}{\sin x} \end{aligned}$$

$$\begin{aligned} f'(2) &= \lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{2}{2x}-\frac{x}{2x}}{x-2} \\ &= \lim_{x\to 2} \frac{\frac{2-x}{2x}}{x-2} \end{aligned}$$

$$\begin{aligned} f'(2^-) &= \lim_{x\to 2^-} \frac{x^3+8}{x^2-4} \\ &= \lim_{x\to 2^-} \frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)} \\ &= \lim_{x\to 2^-} \frac{x^2-2x+4}{x-2} \\ &= \frac{(2^-)^2-2(2^-)+4}{(2^-)-2} \\ &= \frac{4}{0^-} = -\infty \end{aligned}$$

2. Let

$$f(x) = \frac{x^2-9}{|x-3|} = \frac{(x+3)(x-3)}{|x-3|} = (x+3)\frac{x-3}{|x-3|}$$

Find each limit (if it exists).

$$\begin{aligned} &\lim_{x\to 3^-} f(x) \\ &= \lim_{x\to 3^-} (x+3)\frac{x-3}{|x-3|} \\ &= \lim_{x\to 3^-} (x+3) \cdot \lim_{x\to 3^-} \frac{x-3}{|x-3|} \\ &= (3+3)\cdot(-1) = -6 \end{aligned}$$

$$\begin{aligned} &\lim_{x\to 3^+} f(x) \\ &= \lim_{x\to 3^+} (x+3)\frac{x-3}{|x-3|} \\ &= \lim_{x\to 3^+} (x+3) \cdot \lim_{x\to 3^+} \frac{x-3}{|x-3|} \\ &= (3+3)\cdot(1) = -6 \end{aligned}$$

$$\lim_{x\to 3} f(x) = DNE$$

3. Let

$$f(x) = \begin{cases} x+2, & x \le -1 \\ x^2-1, & -1 \le x < 2 \\ \sqrt{x+1}, & x > 2 \end{cases}$$

Find each limit (if it exists).

$$\lim_{x \to -1^-} f(x) = (-1) + 2 = 1$$

$$\lim_{x \to -1^+} f(x) = (-1)^2 - 1 = 0$$

$$\lim_{x \to -1} f(x) = \text{DNE}$$

$$\lim_{x \to 2^-} f(x) = (2)^2 - 1 = 3$$

$$\lim_{x \to 2^+} f(x) = \sqrt{2+1} = \sqrt{3}$$

$$\lim_{x \to 2} f(x) = \text{DNE}$$

$$\lim_{x \to 3} f(x) = \sqrt{3}$$

4. Suppose that

$$\lim_{x \to 3} (4f(x) - 5g(x)) = 1$$

and $\lim_{x \to 3} 2g(x) = 6$. Find the value of $\lim_{x \to 3} f(x)$

5. Suppose that

$$f(x) = \begin{cases} \frac{(x+5)^2-25}{2x}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

If $f(x)$ is continuous at all real numbers, what is the value of c?
When $x \neq 0$, $f(x)$ is $\frac{0}{0}$ Form. There exist common factor between Numerator and Denominator. So the first step is to cancel common factor

$$\begin{aligned} f(x) &= \frac{(x+5)^2 - 25}{2x} \\ &= \frac{(x)^2 + 10x + 25 - 25}{2x} \\ &= \frac{(x)^2 + 10x}{2x} \\ &= \frac{x+10}{2} \end{aligned}$$

Step 2: Set $f(x) = c$ at $x = 0$, that is $\frac{x+10}{2} = c$ at $x = 0$. Then we have $c = 5$

6. Find the following derivatives.

(a) $f(x) = 2x^5 + \frac{1}{\sqrt[3]{x^2}} - \pi$

(b) $h(x) = \frac{2x^3+4}{x^2-4x+1}$

7. Let $g(x) = \frac{1}{x^2}$, use the **LimitDefinition** of the Derivative to find $g'(x)$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^2}{x^2(x+h)^2} - \frac{(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^2-(x^2+2hx+h^2)}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-2hx-h^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{-h(2x+h)}{x^2(x+h)^2} \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{-(2x+h)}{x^2(x+h)^2} \\
&= \frac{-(2x+0)}{x^2(x+0)^2} \\
&= -\frac{2}{x^3}
\end{aligned}$$

8. Let

$$f(x) = 2\cos x - (\sqrt{2})x$$

(a) Find the $x-$coordinates of all points at which the tangent line is horizontal.

(b) Find an equation of tangent line at $x = \frac{\pi}{4}$

9. Let $f(x) = (x^2 - 4)^3$. (10 pt)

(a) find $f'(x)$

(b) Find the points on the graph of $y = f(x)$ at which the tangent line is horizontal.

10. Show that the curve $x^2 + y^2 = 8$ and the line $y = x$ are orthogonal.

11. Consider the given equation $4x^2 - xy^2 - y^3 = 17$. Assume that it determines an implicit differentiable function f such that $y = f(x)$.

  (a) find $\frac{dy}{dx}$ (You may use the y' notation, instead). Use implicit differentiation. (10 points)

  step 1: Take derivative on both sides of equation,

$$\begin{aligned}
[4x^2]' - [xy^2]' - [y^3]' &= [17]' \\
8x - [y^2 + 2xyy'] - 3y^2y' &= 0 \\
8x - y^2 - 2xyy' - 3y^2y' &= 0 \\
y'[-2xy - 3y^2] &= y^2 - 8x \\
y' &= \frac{y^2 - 8x}{-2xy - 3y^2}
\end{aligned}$$

  (b) Use part a) to find the slope of the tangent line to the graph of $4x^2 - xy^2 - y^3 = 17$ at the point $(2, 1)$, which lies on the graph. (5 points)

$$\begin{aligned}
m = y'|_{(2,1)} &= \left.\frac{y^2 - 8x}{-2xy - 3y^2}\right|_{(2,1)} \\
&= \frac{(1)^2 - 8(2)}{-2(2)(1) - 3(1)^2} \\
&= \frac{15}{7}
\end{aligned}$$

12. Consider $f(\theta) = \sin^2\theta + \cos\theta$. Find and box in all critical number(s) of $f(x)$.

  1. Derivative: $f'(\theta) = 2\sin(\theta)\cos(\theta) - \sin(\theta)$
  2. Critical Number: Set $f'(\theta) = 0$ and solve for θ.

$$\begin{aligned}
f'(\theta) &= 0 \\
2\sin(\theta)\cos(\theta) - \sin(\theta) &= 0 \\
\sin(\theta)[2\cos(\theta) - 1] &= 0
\end{aligned}$$

$$\sin(\theta) = 0 \text{ or } 2\cos(\theta) - 1 = 0$$

$$\boxed{\theta = 0, \pi} \text{ or } \cos(\theta) = \frac{1}{2}$$

$$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}}$$

13. Given the function

$$y = \frac{(x+1)^4}{(x-3)^8}$$

Find the derivative of y using the quotient rule combined with the Chain Rule.

14. Find the absolute maximum and absolute minimum values of the following functions with given closed interval

(a) $f(x) = x^3 - 12x + 1$ over $[-3, 3]$

1. Derivative: $f'(x) = 3x^2 - 12$
2. Critical Number: Set $f'(x) = 0$ and solve for x.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 12 &= 0 \\ 3(x-2)(x+2) &= 0 \\ x &= -2 \text{ or } x = 2 \end{aligned}$$

3. Find A.Max/A.Min:

x	$f(x)$	A. Max/Min
-3	10	
-2	17	A.Max
2	-15	A.Min
3	-8	

(b) $f(x) = \sqrt[3]{x}(8 - x)$ over $[0, 8]$

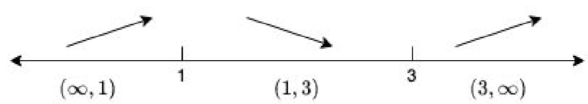
15. For the function $f(x) = x^3 - 6x^2 + 9x + 13$, find the critical points. Determine the intervals for which $f(x)$ is increasing and decreasing. Determine any local minimums or maximums

    1. Derivative: $f'(x) = 3x^2 - 12x + 9$
    2. Critical Number: Set $f'(x) = 0$ and solve for x.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 12x + 9 &= 0 \\ 3(x-1)(x-3) &= 0 \\ x &= 1 \text{ or } x = 3 \end{aligned}$$

    3. Number Line

$$\begin{aligned} f'(0) = 3(0)^2 - 12(0) + 9 = 9 > 0, &\quad \text{Increasing} \\ f'(2) = 3(2)^2 - 12(2) + 9 = -3 < 0, &\quad \text{decreasing} \\ f'(4) = 3(4)^2 - 12(4) + 9 = 9 > 0, &\quad \text{Increasing} \end{aligned}$$



16. For the function $f(x) = x^4 - 8x^3 - 72x^2 + 31x + 111$, find the inflection points. Determine the intervals for which $f(x)$ is concave upwards and concave downward.

    1. Derivatives:

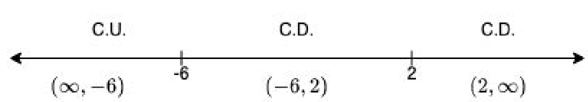
$$\begin{aligned} f'(x) &= 4x^3 - 24x^2 - 144x + 31 \\ f''(x) &= 12x^2 - 48x - 144 \end{aligned}$$

    2. Critical Number: Set $f''(x) = 0$ and solve for x.

$$\begin{aligned} f''(x) &= 0 \\ 12x^2 - 48x - 144 &= 0 \\ 12(x-6)(x=2) &= 0 \\ x &= -2 \text{ or } x = 6 \end{aligned}$$

    3. Number Line

$$\begin{aligned} f''(-7) = 12(-7)^2 - 48(-7) - 144 = 780 > 0, &\quad \text{Concave Up} \\ f''(0) = 12(0)^2 - 48(0) - 144 = -144 > 0, &\quad \text{Concave Down} \\ f''(3) = 12(3)^2 - 48(3) - 144 = -180 > 0, &\quad \text{Concave Down} \end{aligned}$$



17. Sketch the graph of $y = f(x)$, where $f(x) = x^3 - 12x^2 + 36x$ in the usual xy-plane.

(a) Find the domain, x-intercept and y-intercept of f.

1. Domain: the function is polynomial, thus the domain is $(-\infty, \infty)$
2. x-intercept: Set $f(x) = 0$ and solve for x.

$$\begin{aligned} f(x) &= 0 \\ x^3 - 12x^2 + 36x &= 0 \\ x(x-6)^2 &= 0 \\ x &= 0 \text{ or } x = 6 \end{aligned}$$

3. y-intercept: Find $(0, f(0))$. that is $(0, 0)$

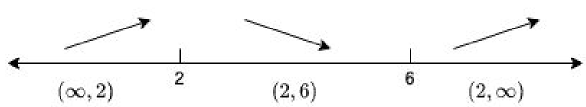
(b) Calculate the derivative $f'(x)$ and find the critical numbers of f. Make a number line and determine the intervals for which f is increasing, decreasing.

1. Derivative: $f'(x) = 3x^2 - 24x + 36$
2. Critical Number: Set $f'(x) = 0$ and solve for x.

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 - 24x + 36 &= 0 \\ 3(x-2)(x-6) &= 0 \\ x &= 2 \text{ or } x = 6 \end{aligned}$$

3. Number Line

$$\begin{aligned} f'(0) &= 3(0)^2 - 24(0) + 36 = 36 > 0, \quad \text{Increasing} \\ f'(3) &= 3(3)^2 - 24(3) + 36 = -9 < 0, \quad \text{decreasing} \\ f'(7) &= 3(7)^2 - 24(7) + 36 = 15 > 0, \quad \text{Increasing} \end{aligned}$$



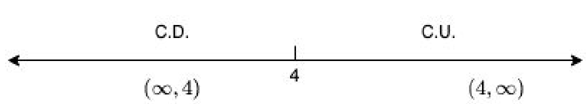
(c) Calculate the second derivative $f''(x)$ and find the possible inflection point of f. Make a number line and determine the intervals for which f is concave up and concave down.

1. Derivative: $f'(x) = 6x - 24x$
2. Critical Number: Set $f'(x) = 0$ and solve for x.

$$\begin{aligned} f'(x) &= 0 \\ 6x - 24 &= 0 \\ 6(x-4) &= 0 \\ x &= 4 \end{aligned}$$

3. Number Line

$$\begin{aligned} f''(0) &= 6(0) - 24 = -24 < 0, \quad \text{Concave Down} \\ f''(7) &= 6(7) - 24 = 18 > 0, \quad \text{Concave Up} \end{aligned}$$



(d) Use your information to determine the local extrema and the inflection points of f. Calculate the $y - values$ of each local extrema and inflection point.

1. Loacal maximum: based on First Derivative Test and information in (b), Function arrive at local maximum at $x = 6$. i.e. piint $(6, 0)$.

2. Loacal minimum: based on First Derivative Test and information in (b), Function arrive at local maximum at $x = 2$. i.e. piint $(2, 32)$.

3. Inflection Point: based on Second Derivative and information in (c), Function have a inflection point at $x = 4$. i.e. piint $(4, 16)$.

(e) Use your information to sketch a graph of $f(x)$.