All 1511 students should take a shot at the first two problems. 2110 students, and CS 1511 students shooting for an A, should take a shot at the last two problems.

5. (2 points) Let $K(X)$ be the Kolmogorov complexity of binary string X.
   (a) Let X and Y be a binary strings. Show $K(XY) \leq K(X) + K(Y) + O(1)$. Here XY is the string X concatenated the string Y.
   (b) Let X be a binary string. Show $K(XX) \leq K(X) + O(1)$. Here XX is the string X concatenated with itself.
   (c) Are there finitely or infinitely many binary strings X such that $K(X) = \lg |X| + O(1)$? Here $|X|$ is the number of bits in X. Justify your answer.
   (d) Are there finitely or infinitely many binary strings X such that $K(X) = \lg \lg |X| + O(1)$? Justify your answer.
   (e) Are there finitely or infinitely many binary strings X such that $K(X) = O(1)$? Justify your answer.

6. (4 points) Show that there are only finitely many incompressible strings that have the property that there are no bits equal to 1 in the string (so the string only contains bits that are equal to 0). Recall that a string is incompressible if its Kolmogorov complexity is at least its length.

7. (6 points) Show that there are only finitely many incompressible strings that have the property that the number of bits that are 0 in the string is equal to the number of bits that are 1 in the string.

8. (8 points) The chain rule for entropy (`https://en.wikipedia.org/wiki/Conditional_entropy#Chain_rule`) states that $H(X, Y) = H(X) + H(Y|X)$, which means the amount of information in two (possibly correlated) random variables is the amount of information in the first variable, plus the amount of information in the second variable given that one already knew the value of the first variable. Further since $H(Y|X) \leq H(Y)$, this implies $H(X, Y) \leq H(X) + H(Y)$. So the information in two random variables is at most the sum of the information in each random variable.

   The purpose of this problem is to show that an analogous chain rule does not hold for Komogorov complexity. Let $K(x, y)$ be the information in the string x and the string y together. So here the strings are not interchangeable. So the information is not just what the two strings are, but also which string is x and which string is y. So in particular, $K(x, y)$ may not equal $K(y, x)$.

   (a) Give a reasonable formal definition of $K(x, y)$. Use this definition for the following subproblems. If any of the statements in the following subproblems is false with your definition, then the most likely explanation is that your definition is not reasonable.
   (b) Let x and y be strings such $|x| + |y| = n$. Show that t $K(x, y) \leq K(x) + K(y) + O(\log n)$.

(c) Let z be an n bit string. Explain why there are $n+1$ different pairs of strings x and y such that $xy = z$. Here we are counting the empty string as a possible string.

(d) Explain why for all n there exists an n bit string z, and strings x and y with $z = xy$ such that $K(x, y) \geq n + \Omega(\log n))$.

(e) Explain why for all n there exists an n bit string z, and strings x and y with $z = xy$ such that $K(x, y) \geq K(x) + K(y) + \Omega(\log n)$.

(f) Explain why for all n there exists an n bit string z, and strings x and y with $z = xy$ such that $K(x, y) \geq K(xy) + \Omega(\log n)$.

(g) Intuitively why can x, y contain a logarithmically additive amount of more information than xy, but not more than an additive logarithmic amount of information.