All 1511 students should take a shot at the first two problems. 2110 students, and CS 1511 students shooting for an A, should also take a shot at the last two problems.

9. (2 points) We are in a setting where models are collections of bit strings, and each axiom specifies a string that must be in the model. So for example, one possible model in this setting is

$$\{0, 0000, 1010, 111, 010101\}$$

We consider the following axiomatic system. The axioms are $\{00, 11\}$. So the first axiom states that the string 00 is in the model, and the second axiom states that the string 11 is in the model.

The proof rule is: If we have deduced that the string x is in the model and we have deduced the string y is in the model, then we can deduce the string xy is in the model. Here xy is the concatenation of x and y. So if $x = 0101010$ and $y = 111$ then we can conclude 0101010111.

   (a) Give two different proofs that the string 00110011 is in the model. Recall that a proof is a sequence of strings, such that each string in the proof is either an axiom, or follows by the proof rule applied to two prior strings in the sequence.

   (b) Explain why there is not a proof that the string 1010 is in the model.

   (c) Give an example of a model that satisfies these axioms that does not contain the string 1010.

   (d) Give an example of a model that satisfies these axioms that does contain the string 1010.

10. (4 points) We are again in a setting where models are collections of bit strings, and each axiom specifies a string that must be in the model. And again the proof rule we will consider is: If we have deduced the string x and we have deduced the string y, then we can deduce the string xy, which is the concatenation of x and y.

We will consider various models. For each of the models, you should either give a complete and sound finite axiomatization, and explain why it is complete and sound, or explain why there isn't a complete and sound finite axiomatization. Recall that a finite axiomatization is a finite collection of strings. For the purposes of this homework, the empty string (which contains no letters) does not count as a string. Recall that an axiomatization is complete if there is a proof for every true string. Recall that an axiomatization is sound if every proof ends with a true string.

Let us consider an example. The model M_0 contains the binary strings where there are no 1's. A finite axiomatization exists, in particular the set $A = \{0\}$ consisting of a single string 0 is a complete and sound axiomatization. This axiomatization is finite because there are only a finite number of strings in A. This axiomatization is sound because each axiom is true in the model, and if neither of the strings x and y contain a 1 then neither does the string xy. This axiomatization is complete because a proof of the string 0^i that consists of i 0's is:

$$0, 00, 000, \ldots, 0^{i-1}, 0^i$$

The first string in this proof is an axiom, and the string j follows from applying the proof rule to the first string and string $j-1$. So a proof of 0^6 is

$$0, 00, 000, 0000, 00000, 000000$$

Another proof of 0^6 is

$$0, 00, 0000, 000000$$

Here the first string is an axiom, the second string follows from the proof rule when both x and y are the first string, the third string follows from the proof rule when both x and y are the second string, and finally the last string follows by applying the proof rule when x is the second string and y is the third string.

(a) In model M_1 the true statements are the binary strings where the number of bits is a multiple of three.

(b) In the model M_2 the true statements are the binary strings where there are an even number of 0's.

(c) In model M_3 the true statements are the binary strings where there are at least two 0's or at least two 1's.

11. (6 points)

(a) Show that the set

$$\{(x, y) \mid x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$$

which consists of all possible pairs of integers is recursively enumerable.

(b) Consider a proof rule such as:

Proof Rule: From the statement $\forall x\ P(x)$ one can deduce the countably infinite number of statements $P(0)$, $P(1)$, $P(2)$, $P(3)$, etc.

Somewhat more formally, you can assume that there is a program T that takes as input a statement S, and will output a sequences all the statements one can conclude from S using this proof rule. That is, T will only output statements we can deduce from S by the proof rule, and every statement that we can deduce from S by the proof rule will eventually be output. Show that the following language is still recursively enumerable:

$$L = \{S \mid \text{statement } S \text{ is provable from the finite set } A \text{ of axioms}\}$$

Hint: Now the nodes in the tree of all proofs can have infinitely many children. So breadth first search will no longer work. So you will have to find another way to search this tree so that every node is eventually reached.

12. (8 points)

(a) Show that the set of compressible strings is recursively enumerable. A string is compressible if and only if it is not incompressible.

(b) Show that the set of incompressible strings contains no infinite subset that is recursively enumerable.