PRYM REPRESENTATIONS AND TWISTED COHOMOLOGY OF THE MAPPING CLASS GROUP WITH LEVEL STRUCTURES

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Abstract. We compute the twisted cohomology of the mapping class group with level structures, coefficients being r-tensors of the Prym representations for any positive integer r. It is not stable when the genus is large, unlike the rational cohomology of the mapping class group with level structures. As a corollary to our computations, we prove that higher Prym representations of any finite abelian cover of a non-closed finite-type surface is locally rigid.

1. Introduction

Let $\Sigma_{g,p}^b$ be a genus-g surface with p punctures and b boundary components. We denote the mapping class group $\Sigma_{g,p}^b$ by

$$\mathrm{Mod}_{g,p}^b := \mathrm{Mod}(\Sigma_{g,p}^b) = \mathrm{Diffeo}^+(\Sigma_{g,p}^b)/\mathrm{Diffeo}_0(\Sigma_{g,p}^b).$$

We omit p or b when it is 0. Given an integer $l \geq 2$, the level-l mapping class group of $\Sigma_{g,p}^b$ is the subgroup of $\mathrm{Mod}_{g,p}^b$ which acts trivially on $H_1(\Sigma_{g,p}^b;\mathbb{Z}/l)$:

$$\mathrm{Mod}_{g,p}^b(l) := \mathrm{Ker}(\mathrm{Mod}_{g,p}^b \to \mathrm{Aut}(H_1(\Sigma_{g,p}^b;\mathbb{Z}/l)).$$

Harer proved ([10]) that $\mathrm{Mod}_{g,p}^b$ satisfies homological stability, which means $H_k(\mathrm{Mod}_{g,p}^b;\mathbb{Z})$ is independent of the genus g when $g \gg k$. People wondered if the finite-index subgroup $\mathrm{Mod}_{g,p}^b(l)$ of $\mathrm{Mod}_{g,p}^b$ has the same stable homology as $\mathrm{Mod}_{g,p}^b$. The answer for integral homology is false: see Perron [19], Sato [26], and Putman [21] for exotic torsion elements. The answer for rational homology is true, proved by Putman ([23, Theorem A]):

$$H_k(\mathrm{Mod}_{g,p}^b(l);\mathbb{Q}) \cong H_k(\mathrm{Mod}_{g,p}^b;\mathbb{Q}) \text{ if } g \geq 2k^2+7k+2.$$

For twisted homology groups of $\mathrm{Mod}_{g,p}^b$ and $\mathrm{Mod}_{g,p}^b(l)$, we need to specify their representations first. One common standard representation of $\mathrm{Mod}_{g,p}^b$ and $\mathrm{Mod}_{g,p}^b(l)$ is $H_1(\Sigma_{g,p}^b;\mathbb{Q})$, where $\mathrm{Mod}_{g,p}^b(l)$ acts on $H_1(\Sigma_{g,p}^b;\mathbb{Q})$ via the inclusion $\mathrm{Mod}_{g,p}^b(l) \hookrightarrow \mathrm{Mod}_{g,p}^b$. Given an integer $r \geq 1$, we can consider the r-tensor power $H_1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes r}$. Putman ([23, Theorem B]) proved the twisted homology groups of $\mathrm{Mod}_{g,p}^b$ and $\mathrm{Mod}_{g,p}^b(l)$ are isomorphic when the genus g is large enough:

$$H_k(\mathrm{Mod}_{g,p}^b(l);H_1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes r}) \cong H_k(\mathrm{Mod}_{g,p}^b;H_1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes r}),$$

if $g \geq 2(k+r)^2+7k+6r+2$.

Another interesting representation of $\mathrm{Mod}_{g,p}^b(l)$ is the Prym representation. Assume $p+b \geq 1$, and let $\mathcal{D} = H_1(\Sigma_g;\mathbb{Z}/l)$. We consider the regular cover $\Sigma_{g,p}^b[\mathcal{D}] \to \Sigma_{g,p}^b$ corresponding to the group homomorphism

$$\pi_1(\Sigma_{g,p}^b) \to H_1(\Sigma_{g,p}^b;\mathbb{Z}/l) \to H_1(\Sigma_g;\mathbb{Z}/l) = \mathcal{D}.$$

Here $H_1(\Sigma_{g,p}^b;\mathbb{Z}/l) \to H_1(\Sigma_g;\mathbb{Z}/l)$ is induced by the map $\Sigma_{g,p}^b \to \Sigma_g$ obtained by gluing disks to all boundary components and filling in all punctures of $\Sigma_{g,p}^b$. Since $\mathrm{Mod}_{g,p}^b(l)$ acts trivially on $H_1(\Sigma^b_{g,p};\mathbb{Z}/l)$, we know $\mathrm{Mod}^b_{g,p}(l)$ also acts trivially on $H_1(\Sigma_g;\mathbb{Z}/l)$. Hence any diffeomorphism of $\Sigma^b_{g,p}$ lifts to the cover $\Sigma^b_{g,p}[\mathcal{D}]$, fixing all punctures and boundary components pointwise. Thus $\mathrm{Mod}^b_{g,p}(l)$ acts on $\mathfrak{H}^b_{g,p}(l;\mathbb{Q}) := H_1(\Sigma^b_{g,p}[\mathcal{D}];\mathbb{Q})$, which is called the **Prym representation**. Given a positive integer r, the cover $\Sigma^b_{g,p}[\mathcal{D}] \to \Sigma^b_{g,p}$ induces a map $\mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes r} \to H_1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes r}$, so it induces a map of twisted homology:

$$H_k(\mathrm{Mod}^b_{g,p}(l);\mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes r}) \to H_k(\mathrm{Mod}^b_{g,p};H_1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes r}).$$

For $r=1$, the above map has been proved to be a isomorphism when $g \geq 2(k+1)^2+7k+8$ by Putman ([23, Theorem C]). Putman also conjectured that it is not an isomorphism for $r \geq 2$. In our paper, we will compute both sides in a range when $g \gg k$, and see directly that they are not isomorphic when $r \geq 2$. Instead of dealing with homology, we will deal with cohomology. The above theorems of Putman are still true for cohomology, since our coefficients are always $\mathbb{Q}$-vector spaces, and there is a duality between group homology and group cohomology (will state this formally later in section 2). We will use the same notation $\mathfrak{H}^b_{g,p}(l;\mathbb{Q})$ for the cohomology group $H^1(\Sigma^b_{g,p}[\mathcal{D}];\mathbb{Q})$, and we also call it a Prym representation.

We will state our computational results in the following subsections. We will begin with easy-to-state results for small r, and state the result for general r after putting some effort into introducing some new notation.

1.1. **1-Tensor Power as Coefficients.** Our computation starts with the case $r=1$, where we list the two cohomology groups below. In the polynomial expressions of the two cohomology groups, the generators $\kappa_i \in H^{2i}(\Sigma^b_{g,p};\mathbb{Q})$ $(i \geq 1)$ are the Miller-Morita-Mumford classes ([15]), and the generators $c_1,\cdots,c_p \in H^2(\Sigma^b_{g,p};\mathbb{Q})$ are the first Chern classes corresponding to those p punctures of $\Sigma_{g,p}$. The generator u_1 of degree two in the two cohomology groups has different meaning in them, which we will illustrate in the two theorems. The subscript (k) of the polynomial ring indicates the degree-k part.

First we have $H^*(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Q}))$ as follows.

Theorem 1.1. *For integers g,p,b such that $p+b \geq 1$, we have the following isomorphism*

$$H^{k-1}(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Q})) \cong (\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\otimes u_1\,\mathbb{Q}[u_1])_{(k)}\,,$$

in degree $k \leq \frac{2}{3}(g-1)$, where u_1 of degree 2 corresponds to the first Chern class of a marked point in $\Sigma^b_{g,p}$.

Remark: Looijenga ([14]) computed this for closed surfaces Σ_g and got

$$H^{*-1}(\mathrm{Mod}_g;H^1(\Sigma_g;\mathbb{Q})) \cong \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_1^2[u_1],\ \text{in degree}\ \leq \frac{2}{3}(g-1),$$

which turns out to be different from our non-closed case. Kawazumi ([13]) was able to compute this over $\mathbb{Z}$ when $b \geq 1$:

$$H^*(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Z})) \cong \left(\bigoplus_{i\geq 1} H^*(\mathrm{Mod}^1_g;\mathbb{Z})m_i\right)\otimes_{H^*(\mathrm{Mod}^1_g;\mathbb{Z})} H^*(\mathrm{Mod}^b_{g,p};\mathbb{Z}),$$

in degree $\leq \frac{2}{3}(g-1)-1$, where $m_i \in H^{2i-1}(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Z}))$ are the twisted Miller-Morita-Mumford classes. Kawazumi's result implies Theorem 1.1 above for $b \geq 1$ after tensoring with $\mathbb{Q}$, and identifying m_i with u_1^i. In section 3, we will start with computing $H^*(\mathrm{Mod}_{g,1};H^1(\Sigma_{g,1};\mathbb{Q}))$ using Looijenga's method, and rely on Kawazumi's work to get the result for general surfaces $\Sigma^b_{g,p}$ with $p+b \geq 1$.

Next, we consider level-l structures, and the cohomology group $H^*(\mathrm{Mod}^b_{g,p}(l);\mathfrak{H}^b_{g,p}(l;\mathbb{Q}))$ is as follows.

Theorem 1.2. *For integers g,p,b,l such that $p+b\geq 1$ and $l\geq 2$, we have the following isomorphism*

$$H^{k-1}(\mathrm{Mod}^b_{g,p}(l);\mathfrak{H}^b_{g,p}(l;\mathbb{Q}))\cong(\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\otimes u_1\,\mathbb{Q}[u_1])_{(k)}\,,$$

if $g\geq 2k^2+7k+2$, where u_1 of degree 2 corresponds to the first Chern class of a marked point in the regular $\mathcal{D}$-cover $\Sigma^b_{g,p}[\mathcal{D}]$.

We observe that both twisted cohomology groups are independent of the genus g and the number of boundary components b when $g\gg k$, while they do depend on the number of punctures p. Combining these two results, noticing $\max(\frac{3}{2}k+1,2k^2+7k+2)=2k^2+7k+2$, we have the isomorphism

$$H^k(\mathrm{Mod}^b_{g,p}(l);\mathfrak{H}^b_{g,p}(l;\mathbb{Q}))\cong H^k(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Q})),$$

when $g\geq 2(k+1)^2+7(k+1)+2=2(k+1)^2+7k+9$, differing by 1 with Putman's range $2(k+1)^2+7k+8$.

1.2. **2-Tensor Power as Coefficients.** We have the following results for $r=2$. The cohomology classes $c_1,\cdots,c_p$ and κ_i are the same as before.

For $H^*(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes 2})$, the coefficients are $H^1(\Sigma^b_{g,p};\mathbb{Q})\otimes H^1(\Sigma^b_{g,p};\mathbb{Q})$, so we need to consider two ordered marked points in $\Sigma^b_{g,p}$. There are some new generators related to the positions of two marked points. Below u_i $(i=1,2)$ is the first Chern class of the i-th marked points in $\Sigma^b_{g,p}$. The class $a_{\{1,2\}}$ of degree 2 is the Poincaré dual of the subvariety of the moduli space of two ordered marked points in Riemann surfaces homeomorphic to $\Sigma^b_{g,p}$, whose two marked points coincide. The symbol $\mathbb{Q}[u_{\{1,2\}}]a_{\{1,2\}}$ is the $\mathbb{Q}[u_1,u_2]$-module generated by $a_{\{1,2\}}$ subject to the relation $u_1a_{\{1,2\}}=u_2a_{\{1,2\}}$.

Theorem 1.3. *For integers g,p,b such that $p+b\geq 1$, we have*

$$H^{k-2}(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes 2})\cong\left(\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\otimes\begin{matrix}u_1u_2\,\mathbb{Q}[u_1,u_2]\\ \oplus\,\mathbb{Q}[u_{\{1,2\}}]a_{\{1,2\}}\end{matrix}\right)_{(k)}$$

in degree $k\leq\frac{2}{3}(g-1)$.

Remark: Looijenga ([14]) computed this for closed surfaces Σ_g and got

$$H^{*-2}(\mathrm{Mod}_g;H^1(\Sigma_g;\mathbb{Q})^{\otimes 2})\cong\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\left(u_1^2u_2^2[u_1,u_2]\oplus\mathbb{Q}[u_{\{1,2\}}]a_{\{1,2\}}\right),$$

in degree $\leq\frac{2}{3}(g-1)$, which is also different from our non-closed case. Kawazumi ([13]) was able to compute this over $\mathbb{Z}$ when $b\geq 1$:

$$\begin{aligned}H^*(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Z})^{\otimes 2})\cong&\left(\bigoplus_{i\geq 0}H^*(\mathrm{Mod}^1_g;\mathbb{Z})m_{\{1,2\},i}\right)\otimes_{H^*(\mathrm{Mod}^1_g;\mathbb{Z})}H^*(\mathrm{Mod}^b_{g,p};\mathbb{Z})\\&\oplus\left(\bigoplus_{i_1,i_2\geq 1}H^*(\mathrm{Mod}^1_g;\mathbb{Z})m_{(\{1\},i_1),(\{2\},i_2)}\right)\otimes_{H^*(\mathrm{Mod}^1_g;\mathbb{Z})}H^*(\mathrm{Mod}^b_{g,p};\mathbb{Z}),\end{aligned}$$

in degree $\leq\frac{2}{3}(g-1)-2$, where

$$m_{\{1,2\},i}\in H^{2i}(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Z})^{\otimes 2}),\ m_{(\{1\},i_1),(\{2\},i_2)}\in H^{2i_1+2i_2-2}(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Z})^{\otimes 2})$$

are the twisted Miller-Morita-Mumford classes. Kawazumi's result implies Theorem 1.3 above for $b\geq 1$ after tensoring with $\mathbb{Q}$, and identifying $m_{\{1,2\},i}$ with $u^i_{\{1,2\}}a_{\{1,2\}}$, and identifying $m_{(\{1\},i_1),(\{2\},i_2)}$ with $u_1^{i_1}u_2^{i_2}$. In section 3, we will first compute $H^*(\mathrm{Mod}_{g,1};H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes 2})$

using Looijenga's method, and rely on Kawazumi's work to get the result for general surfaces $\Sigma_{g,p}^b$ with $p+b\geq 1$.

For $H^*(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 2})$, instead of considering two ordered marked points in $\Sigma_{g,p}^b$, we consider two ordered marked points in the regular $\mathcal{D}$-cover $\Sigma_{g,p}^b[\mathcal{D}]$. The new generator u_i $(i=1,2)$ is the first Chern classes of the i-th marked points in the regular $\mathcal{D}$-cover $\Sigma_{g,p}^b[\mathcal{D}]$. The new generator $a_{\{1,2\},d}$, of degree 2, is the Poincaré dual of the subvariety of the moduli space of two ordered marked points in the regular $\mathcal{D}$ cover of Riemann surfaces homeomorphic to $\Sigma_{g,p}^b$, whose two marked points in $(\Sigma_{g,p}^b)[\mathcal{D}]$ lie in the same $\mathcal{D}$-orbit and the second marked point differs from the first one by the action $d\in\mathcal{D}$. The symbol $\mathbb{Q}[u_{\{1,2\},d}]a_{\{1,2\},d\}}$ is the $\mathbb{Q}[u_1,u_2]$-module generated by $u_{\{1,2\},d}$ subject to the relation $u_1a_{\{1,2\},d}=u_2a_{\{1,2\},d}$.

Theorem 1.4. *For integers g,p,b,l such that $p+b\geq 1$ and $l\geq 2$, we have*

$$H^{k-2}(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 2})\cong\left(\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\otimes\left(\begin{array}{c}u_1u_2\,\mathbb{Q}[u_1,u_2]\\ \oplus\bigoplus_{d\in\mathcal{D}}\mathbb{Q}[u_{\{1,2\},d}]a_{\{1,2\},d}\end{array}\right)\right)_{(k)},$$

if $g\geq 2k^2+7k+2$.

Thus we see that

$$H^k(\mathrm{Mod}_{g,p}^b;H^1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes 2})\ncong H^k(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 2})$$

when $g\gg k$. In particular, $H^k(\mathrm{Mod}_{g,p}^b;H^1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes 2})$ is independent of g when the genus g is large enough, but $H^k(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 2})$ does depend on the genus g so it is not homologically stable! This is because for $H^k(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 2})$, we have a nontrivial summand for each $d\in\mathcal{D}=(\mathbb{Z}/l)^{2g}$.

We will have more insight on the combinatorial information when we see the 3-tensor power case.

1.3. **3-Tensor Power as Coefficients.** The two cohomology groups are listed below. The cohomology classes $c_1,\cdots,c_p$ and κ_i are the same as before.

First, for $H^*(\mathrm{Mod}_{g,p}^b;H^1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes 3})$, we need to consider three ordered marked points in $\Sigma_{g,p}^b$. The generator u_i $(i=1,2,3)$ is the first Chern classes of the i-th marked point. The generators $a_{\{1,2\}}$ of degree two is the Poincaré dual of the subvariety of of the moduli space of three ordered marked points in Riemann surfaces homeomorphic to $\Sigma_{g,p}^b$, whose first marked point coincide with the second one, and $\mathbb{Q}[u_{\{1,2\}},u_3]a_{\{1,2\}}$ is the $\mathbb{Q}[u_1,u_2,u_3]$-module generated by $a_{\{1,2\}}$ subject to the relation $u_1a_{\{1,2\}}=u_2a_{\{1,2\}}$; and it is similar for other such types. The generator $a_{\{1,2,3\}}$, of degree 4, is the Poincaré dual of the subvariety whose three marked points coincide, and $\mathbb{Q}[u_{\{1,2,3\}}]a_{\{1,2,3\}}$ indicates the $\mathbb{Q}[u_1,u_2,u_3]$-module generated by $a_{\{1,2,3\}}$ subject to the relations $u_1a_{\{1,2,3\}}=u_2a_{\{1,2,3\}}=u_3a_{\{1,2,3\}}$.

Theorem 1.5. *For integers g,p,b such that $p+b\geq 1$, we have*

$$H^{k-2}(\mathrm{Mod}_{g,p}^b;H^1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes 3})\cong\left(\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\otimes\left(\begin{array}{c}u_1u_2u_3\,\mathbb{Q}[u_1,u_2,u_3]\\ \oplus u_3\,\mathbb{Q}[u_{\{1,2\}},u_3]a_{\{1,2\}}\\ \oplus u_2\,\mathbb{Q}[u_{\{1,3\}},u_2]a_{\{1,3\}}\\ \oplus u_1\,\mathbb{Q}[u_{\{2,3\}},u_1]a_{\{2,3\}}\\ \oplus\,\mathbb{Q}[u_{\{1,2,3\}}]a_{\{1,2,3\}}\end{array}\right)\right)_{(k)},$$

in degree $k\leq\frac{2}{3}(g-1)$.

Remark: Looijenga ([14]) computed this for closed surfaces Σ_g and got

$$H^{*-3}(\mathrm{Mod}_g; H^1(\Sigma_g;\mathbb{Q})^{\otimes 3}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\cdots] \otimes \begin{pmatrix} u_1^2u_2^2u_3^2[u_1,u_2,u_3] \\ \oplus u_3^2\,\mathbb{Q}[u_{\{1,2\}},u_3]a_{\{1,2\}} \\ \oplus u_2^2\,\mathbb{Q}[u_{\{1,3\}},u_2]a_{\{1,3\}} \\ \oplus u_1^2\,\mathbb{Q}[u_{\{2,3\}},u_1]a_{\{2,3\}} \\ \oplus\,\mathbb{Q}[u_{\{1,2,3\}}]a_{\{1,2,3\}} \end{pmatrix},$$

in degree $\le \frac{2}{3}(g-1)$, which is different from our non-closed case. Kawazumi ([13]) was able to compute this over $\mathbb{Z}$ when $b \ge 1$:

$$\begin{aligned} H^*(\mathrm{Mod}_{g,p}^b;H^1(\Sigma_{g,p}^b;\mathbb{Z})^{\otimes 3}) \cong & \left(\bigoplus_{i\ge 0} H^*(\mathrm{Mod}_g^1;\mathbb{Z})m_{\{1,2,3\},i}\right) \otimes_{H^*(\mathrm{Mod}_g^1;\mathbb{Z})} H^*(\mathrm{Mod}_{g,p}^b;\mathbb{Z}) \\ & \oplus \left(\bigoplus_{i_1\ge 0,i_2\ge 1} H^*(\mathrm{Mod}_g^1;\mathbb{Z})m_{(\{1,2\},i_1),(\{3\},i_2)}\right) \otimes_{H^*(\mathrm{Mod}_g^1;\mathbb{Z})} H^*(\mathrm{Mod}_{g,p}^b;\mathbb{Z}) \\ & \oplus \left(\bigoplus_{i_1\ge 0,i_2\ge 1} H^*(\mathrm{Mod}_g^1;\mathbb{Z})m_{(\{1,3\},i_1),(\{2\},i_2)}\right) \otimes_{H^*(\mathrm{Mod}_g^1;\mathbb{Z})} H^*(\mathrm{Mod}_{g,p}^b;\mathbb{Z}) \\ & \oplus \left(\bigoplus_{i_1\ge 0,i_2\ge 1} H^*(\mathrm{Mod}_g^1;\mathbb{Z})m_{(\{2,3\},i_1),(\{1\},i_2)}\right) \otimes_{H^*(\mathrm{Mod}_g^1;\mathbb{Z})} H^*(\mathrm{Mod}_{g,p}^b;\mathbb{Z}) \\ & \oplus \left(\bigoplus_{i_1,i_2,i_3\ge 1} H^*(\mathrm{Mod}_g^1;\mathbb{Z})m_{(\{1\},i_1),(\{2\},i_2),(\{3\},i_3)}\right) \otimes_{H^*(\mathrm{Mod}_g^1;\mathbb{Z})} H^*(\mathrm{Mod}_{g,p}^b;\mathbb{Z}), \end{aligned}$$

in degree $\le \frac{2}{3}(g-1)-3$. Here

$$m_{\{1,2,3\},i} \in H^{2i+1}(\mathrm{Mod}_{g,p}^b;H^1(\Sigma_{g,p}^b;\mathbb{Z})^{\otimes 3}),$$
$$m_{(\{1,2\},i_1),(\{3\},i_2)}, m_{(\{1,3\},i_1),(\{2\},i_2)}, m_{(\{2,3\},i_1),(\{1\},i_2)} \in H^{2i_1+2i_2-1}(\mathrm{Mod}_{g,p}^b;H^1(\Sigma_{g,p}^b;\mathbb{Z})^{\otimes 3}),$$
$$m_{(\{1\},i_1),(\{2\},i_2),(\{3\},i_3)} \in H^{2i_1+2i_2+2i_3-3}(\mathrm{Mod}_{g,p}^b;H^1(\Sigma_{g,p}^b;\mathbb{Z})^{\otimes 3})$$

are the twisted Miller-Morita-Mumford classes. Kawazumi's result implies Theorem 1.5 for $b \ge 1$ after tensoring with $\mathbb{Q}$, and identifying $m_{\{1,2,3\},i}$ with $u_{\{1,2,3\}}^i a_{\{1,2,3\}}$, and identifying $m_{(\{1\},i_1),(\{2\},i_2),(\{3\},i_3)}$ with $u_1^{i_1}u_2^{i_2}u_3^{i_3}$, and identifying $m_{(\{1,2\},i_1),(\{3\},i_2)}$ with $u_3^{i_2}u_{\{1,2\}}^{i_1}a_{\{1,2\}}$, etc. In section 3, we will first compute $H^*(\mathrm{Mod}_{g,1};H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes 3})$ using Looijenga's method, and rely on Kawazumi's work to get the result for general surfaces $\Sigma_{g,p}^b$ with $p+b\ge 1$.

Next, for $H^*(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 3})$, we need to consider all possible positions of three ordered marked points in the regular $\mathcal{D}$-cover $\Sigma_{g,p}^b[\mathcal{D}]$. Not only do we need to track whether some marked points are in the same $\mathcal{D}$-orbit, but we also need to record the action of the deck group $\mathcal{D}$ for points in the same orbit. The new generators u_i $(i=1,2,3)$, of degree 2, is the first Chern class of the i-th marked point in $\Sigma_{g,p}^b[\mathcal{D}]$. The new generator $a_{\{1,2\},d}$, of degree 2, is the Poincaré dual of the subvariety of the moduli space of three ordered marked points in the regular $\mathcal{D}$ cover of Riemann surfaces homeomorphic to $\Sigma_{g,p}^b$, whose second marked point is in the same $\mathcal{D}$-orbit as the first marked point, differing by the deck action $d\in\mathcal{D}$. The symbol $\mathbb{Q}[u_{\{1,2\},d}]a_{\{1,2\},d}$ is the $\mathbb{Q}[u_1,u_2,u_3]$-module generated by $a_{\{1,2\},d}$ subject to the relation $u_1a_{\{1,2\},d} = u_2a_{\{1,2\},d}$. It is similar for all other such types. The new generator $a_{\{1,2,3\},(d,d')}$, of degree 4, is the Poincaré dual of the subvariety whose three marked points in $\Sigma_{g,p}^b[\mathcal{D}]$ are in the same $\mathcal{D}$-orbit, and the second point is $d\cdot$ the first point, and the third point is $d'\cdot$ the first point. The symbol $\mathbb{Q}[u_{\{1,2,3\},(d,d')}]a_{\{1,2,3\},(d,d')}$ is the $\mathbb{Q}[u_1,u_2,u_3]$-module generated by $a_{\{1,2,3\},(d,d')}$ subject to the relations $u_1 a_{\{1,2,3\},(d,d')} = u_2 a_{\{1,2,3\},(d,d')} = u_3 a_{\{1,2,3\},(d,d')}$.

Theorem 1.6. *For integers g,p,b,l such that $p+b\geq 1$ and $l\geq 2$, we have*

$$\begin{aligned} &H^{k-2}(\mathrm{Mod}^b_{g,p}(l);\mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes 3}) \\ \cong &\left(\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\otimes\left(\begin{array}{c} u_1u_2u_3\,\mathbb{Q}[u_1,u_2,u_3] \\ \oplus\left(\bigoplus_{d\in\mathcal{D}} u_3\,\mathbb{Q}[u_{\{1,2\},d}]a_{\{1,2\},d}\right) \\ \oplus\left(\bigoplus_{d\in\mathcal{D}} u_2\,\mathbb{Q}[u_{\{1,3\},d}]a_{\{1,3\},d}\right) \\ \oplus\left(\bigoplus_{d\in\mathcal{D}} u_1\,\mathbb{Q}[u_{\{2,3\},d}]a_{\{2,3\},d}\right) \\ \oplus\left(\bigoplus_{d,d'\in\mathcal{D}} \mathbb{Q}[u_{\{1,2,3\},(d,d')}]a_{\{1,2,3\},(d,d')}\right)\end{array}\right)\right)_{(k)}, \end{aligned}$$

if $g\geq 2k^2+7k+2$.

1.4. **r-Tensor Power as Coefficients.** Observing the above cases for small r, we should expect for general r:

- $H^k(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes r})$ is described in terms of all possible positions of r ordered marked points in $\Sigma^b_{g,p}$.
- $H^k(\mathrm{Mod}^b_{g,p}(l);\mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes r})$ is described in terms of all possible positions of r ordered marked points in the regular $\mathcal{D}$ cover $\Sigma^b_{g,p}[\mathcal{D}]$, simultaneously keeping tracking of the data of $\mathcal{D}$ actions if some points lie in the same $\mathcal{D}$-orbit.

In the following Theorem A about $H^k(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes r})$, the notation $P|[r]$ means $P=\{S_1,S_2,\cdots,S_\nu\}$ is a partition of the set $[r]=\{1,2,\cdots,r\}$, i.e. $[r]$ is a disjoint union of $S_1,S_2,\cdots,S_\nu$. The cohomology class u_i $(1\leq i\leq r)$ of degree 2 is the first Chern class of the i-th marked point in $\Sigma^b_{g,p}$. For $I\subset[r]$ with $|I|\geq 2$, the class a_I, of degree $2|I|-2$, is the Poincaré dual of the subvariety of the moduli space of r ordered marked points in Riemann surfaces homeomorphic to $\Sigma^b_{g,p}$, whose marked points indexed by I coincide.

Theorem A. *For $p+b\geq 1$ and $r\geq 1$, we have*

$$H^{*-r}(\mathrm{Mod}^b_{g,p};H^1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes r})\cong\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}\Big(\prod_{\{i\}\in P}u_i\Big)\,\mathbb{Q}[u_I:I\in P]a_P\right)$$

in degree $*\leq\frac{2}{3}(g-1)$, where $a_P=\prod_{I\in P,|I|\geq 2}a_I$ (with the convention that $a_P=1$ if P is the partition into singletons), and $\mathbb{Q}[u_I:I\in P]a_P$ is the $\mathbb{Q}[u_i:1\leq i\leq r]$-module generated by a_P subject to the relations

$$u_i a_P = u_j a_P,\ \text{ if } i,j\in I,\ \text{ and } I\in P.$$

Remark: Looijenga ([14]) computed this for closed surfaces Σ_g and got

$$H^{*-2}(\mathrm{Mod}_g;H^1(\Sigma_g;\mathbb{Q})^{\otimes r})\cong\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}\Big(\prod_{\{i\}\in P}u_i^2\Big)\mathbb{Q}[u_I:I\in P]a_P\right),$$

in degree $\leq \frac{2}{3}(g-1)$, which is different from our non-closed case. Kawazumi ([13]) was able to compute this over $\mathbb{Z}$ when $b \geq 1$:

$$H^*(\mathrm{Mod}_{g,p}^b; H^1(\Sigma_{g,p}^b;\mathbb{Z})^{\otimes r}) \cong \left(\bigoplus_{\hat{P} \in \mathcal{P}_r} H^*(\mathrm{Mod}_g^1;\mathbb{Z}) m_{\hat{P}} \right) \otimes_{H^*(\mathrm{Mod}_g^1;\mathbb{Z})} H^*(\mathrm{Mod}_{g,p}^b;\mathbb{Z}),$$

in degree $\leq \frac{2}{3}(g-1) - r$, where $m_{\hat{P}}$ are the twisted Miller-Morita-Mumford classes for weighted partitions $\hat{P}$. We will recall the definitions of weighted partitions, and state the degrees of the twisted Miller-Morita-Mumford classes in section 3. Kawazumi's result implies Theorem A for $b \geq 1$ after tensoring with $\mathbb{Q}$, and identifying Kawazumi's notation with ours in the right way. In section 3, we will first compute $H^*(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})$ using Looijenga's method, and rely on Kawazumi's work to get the result for general surfaces $\Sigma_{g,p}^b$ with $p + b \geq 1$.

Expressing $H^k(\mathrm{Mod}_{g,p}^b(l); \mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes r})$ nicely is more complicated since we need to include the action of $\mathcal{D}$ in addition to the partition. Therefore we introduce the following notion of $\mathcal{D}$-weighted partitions:

Definition 1.7. *Fix a group $\mathcal{D}$. A set $\widetilde{P} = \{(S_1, \vec{d_1}), (S_2, \vec{d_2}), \cdots, (S_\nu, \vec{d_\nu})\}$ is called a $\mathcal{D}$-weighted partition of the index set $[r] = \{1, 2, \cdots, r\}$, if*

1. *The set $\{S_1, S_2, \cdots, S_\nu\}$ is a partition of the set $\{1, 2, \cdots, r\}$.*
2. *For each $1 \leq a \leq \nu$, the element $\vec{d_a}$ is a tuple $(d_a^{(1)}, d_a^{(2)}, \cdots, d_a^{(|S_a|-1)})$, with $d_a^{(i)} \in \mathcal{D}$. By convention, $\vec{d_a}$ is empty if $|S_a| = 1$.*

We denote by $\mathcal{P}_r^{\mathcal{D}}$ the set of all $\mathcal{D}$-weighted partitions of the index set $\{1, 2, \cdots, r\}$. For $I = (S_a, \vec{d_a})$. We define $|I|$ to be $|S_a|$.

Using this, we can describe $H^k(\mathrm{Mod}_{g,p}^b(l); \mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes r})$ as below. Consider r ordered marked points in the regular $\mathcal{D}$-cover $\Sigma_{g,p}^b[\mathcal{D}]$. The cohomology class u_i $(1 \leq i \leq r)$ of degree 2 is the first Chern class of the i-th marked point in $\Sigma_{g,p}^b[\mathcal{D}]$. Consider a $\mathcal{D}$-weighted partition $\widetilde{P}$ of $[r]$ and any $I = (S_a, \vec{d_a}) \in \widetilde{P}$. Denote by a_I the Poincaré dual of the subvariety of the moduli space of r ordered marked points in the regular $\mathcal{D}$ cover of Riemann surfaces homeomorphic to $\Sigma_{g,p}^b$, whose marked points indexed by S_a lie in the same $\mathcal{D}$ orbit and these points differ by the deck action $1, d_a^{(1)}, d_a^{(2)}, \cdots, d_a^{(|S_a|-1)}$. Then a_I has degree $2|S_a| - 2 = 2|I| - 2$.

Theorem B. *For $p + b \geq 1$ and $l \geq 2$, we have*

$$H^{k-r}(\mathrm{Mod}_{g,p}^b(l); \mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes r}) \cong \left(\begin{array}{c} \mathbb{Q}[c_1, \cdots, c_p, \kappa_1, \kappa_2, \cdots] \otimes \\ \left(\bigoplus_{\widetilde{P} \in \mathcal{P}_r^{\mathcal{D}}} \left(\prod_{\{i\} \in \widetilde{P}} u_i \right) \mathbb{Q}[u_I : I \in \widetilde{P}] a_{\widetilde{P}} \right) \end{array} \right)_{(k)}$$

if $g \geq 2k^2 + 7k + 2$. Here $a_{\widetilde{P}} = \prod_{I \in \widetilde{P}, |I| \geq 2} a_I$ (with the convention that $a_{\widetilde{P}} = 1$ if $\widetilde{P}$ is composed of singletons), and $\mathbb{Q}[u_I : I \in \widetilde{P}] a_{\widetilde{P}}$ is the $\mathbb{Q}[u_i : 1 \leq i \leq r]$-module generated by $a_{\widetilde{P}}$ subject to the relations

$$u_i a_{\widetilde{P}} = u_j a_{\widetilde{P}}, \text{ if } i, j \in S_a, I = (S_a, \vec{d_a}) \in \widetilde{P}.$$

The ideas for our computations are inspired by Looijenga's paper [14]. We shall start with a fibration obtained by a map between moduli spaces of curves (with level structures) with marked points. We then try to compactify the fibration and apply the Leray spectral sequence. Deligne's ([5]) degeneration theorem tells us our spectral sequence degenerates at

E_2. After carefully studying the E_2-page, we will be able to compute our twisted cohomology groups. Putman's work [23] also helps us avoid repeated calculations when $p+b$ is large.

1.5. **Local Rigidity of Prym Representations.** As a corollary to our calculation of $H^k(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes r})$, we will prove in section 5 that the Prym representation for $\Sigma_{g,p}^b[\mathcal{D}] \to \Sigma_{g,p}^b$ is locally rigid for $g \geq 41$, and moreover the Prym representations for all finite abelian covers of $\Sigma_{g,p}^b$ are locally rigid.

Given g,p,b such that $p+b \geq 1$, let $S_K \to \Sigma_{g,p}^b$ be an arbitrary finite abelian cover corresponding to a finite-index normal subgroup $K < \pi_1(\Sigma_{g,p}^b)$. Let $\widehat{S_K}$ be the closed surface obtained by gluing disks to all boundary components and filling in all punctures of S_K, and let $V_K = H_1(\widehat{S_K};\mathbb{Q})$. Denote the finite abelian group $\pi_1(\Sigma_{g,p}^b)/K$ by A, and denote by $\mathrm{Mod}_{g,p}^b(A)$ the subgroup of $\mathrm{Mod}_{g,p}^b$ which takes K to K and acts trivially on A. Thus $\mathrm{Mod}_{g,p}^b(A)$ acts on V_K, so we obtain a map $\Phi : \mathrm{Mod}_{g,p}^b(A) \to \mathrm{Aut}_{\mathbb{R}}(V_K \otimes \mathbb{R})$. The image of Φ is contained in the Lie group $G_K = Sp(2h;\mathbb{R})^A$, where h is the genus of $\widehat{S_K}$. We call the map

$$\Phi : \mathrm{Mod}_{g,p}^b(A) \to G_K$$

a **Prym representation** of K.

A homomorphism $\Phi : \Gamma \to G$ from a finitely generated group Γ to a Lie group G is called **locally rigid** if Φ is an isolated point in $Hom(\Gamma, G)/G$, i.e. any $\Phi' \in Hom(\Gamma, G)$ sufficiently close to Φ is conjugate to Φ.

We have the following theorem:

Theorem C. *For all finite abelian covers $S_K \to \Sigma_{g,p}^b$ with deck group A, the Prym representation $\Phi : \mathrm{Mod}_{g,p}^b(A) \to G_K$ is locally rigid when $g \geq 41$.*

There is a very useful criterion by Weil ([28]) to see when a homomorphism $\Phi : \Gamma \to G$ is locally rigid. He showed this holds when $H^1(\Gamma;\mathfrak{g}) = 0$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G. We will first make use of Weil's criterion as well as $H^1(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 2}) = 0$ to show the Prym representation for our regular $\mathcal{D}$-cover of $\Sigma_{g,p}^b$ is locally rigid. Then we will generalize the result to all finite abelian covers as shown in the above Theorem C.

1.6. **Outline:** In section 2, we will introduce some preliminaries about the stable cohomology of mapping class groups, level-l mapping class groups, Deligne's degeneration theorem, and some basic facts about group cohomology and mixed Hodge theory. In section 3, we will compute $H^k(\mathrm{Mod}_{g,p}^b; H^1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes r})$. In section 4, we will compute $H^k(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes r})$, the twisted cohomology of level-l mapping class groups with coefficients the r-tensor power of the Prym representations. In section 5, we will use the result we proved in section 4, to prove that for all finite abelian covers of $\Sigma_{g,p}^b$, the higher Prym representations are locally rigid.

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2. Preliminaries

2.1. **Stable Cohomology.** Harer Stability ([10]) states that the cohomology $H^k(\mathrm{Mod}_{g,p}^b;\mathbb{Z})$ is independent of g and b when $g \gg k$. The stable cohomology of the mapping class group does depend on the number of punctures p, and each puncture corresponds to a degree-2 generator in the following way (see Looienga's [14, Proposition 2.2]):

Theorem 2.1 (Looijenga [14]). *Let $c_i \in H^2(\mathrm{Mod}^b_{g,p};\mathbb{Z})$ be the first Chern class corresponding to the i-th puncture of $\Sigma^b_{g,p}$, for $1 \leq i \leq p$. The ring homomorphism*

$$H^*(\mathrm{Mod}_g;\mathbb{Z})[c_1, c_2, \cdots, c_p] \to H^*(\mathrm{Mod}(\Sigma^b_{g,p};\mathbb{Z})$$

is an isomorphism in degree $\leq N(g)$.

Remark: Here the number $N(g)$ is the maximal degree N such that the two homomorphisms $H^N(\mathrm{Mod}^b_{g+1,p};\mathbb{Z}) \to H^N(\mathrm{Mod}^{b+1}_{g,p};\mathbb{Z})$ and $H^N(\mathrm{Mod}^b_{g,p};\mathbb{Z}) \to H^N(\mathrm{Mod}^{b+1}_{g,p};\mathbb{Z})$ are isomorphisms. Roughly, it is the bound for Harer stability. The range of $N(g)$ first given by Harer ([10]) is $N(g;\mathbb{Z}) \geq \frac{1}{3}g$, which was later improved by Ivanov ([12]), Boldsen ([2]) and Randal-Williams ([25]) to $N(g) \geq \frac{2}{3}(g-1)$.

The stable integral cohomology of the mapping class group is complicated, but the stable rational cohomology has a beautiful form. The Mumford conjecture ([18]) says that the stable rational cohomology of the mapping class group is isomorphic to a polynomial ring in a certain range. The generators for the polynomial ring are $\kappa_i \in H^{2i}(\mathrm{Mod}_g;\mathbb{Q})$, called Miller-Morita-Mumford classes They are characteristic classes of surface bundles. Constructions can be found in [16], [17]. The Mumford conjecture was first proved by Madsen and Weiss ([15]), stated below as the Madsen-Weiss Theorem. See also [11], [7], and [27] for alternate proofs and expositions.

Theorem 2.2 (Madsen-Weiss [15]). *We have*

$$H^*(\mathrm{Mod}_g;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \cdots]$$

for $(*)$ up to degree $\frac{2}{3}(g-1)$.

In particular, if we combine two theorems above, for $\mathbb{Q}$ coefficients, we have:

$$H^*(\mathrm{Mod}^b_{g,p};\mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, \cdots, c_p, \kappa_1, \kappa_2, \cdots]$$

for $(*)$ up to degree $N(g)$.

2.2. **Level-l Mapping Class Groups.** Recall the level-l mapping class group is $\mathrm{Mod}^b_{g,p}(l) = \mathrm{Ker}(\mathrm{Mod}^b_{g,p} \to \mathrm{Aut}(H_1(\Sigma^b_{g,p};\mathbb{Z}/l))$. It has many similar properties to $\mathrm{Mod}^b_{g,p}$. For example:

Proposition 2.3 ([23, Proposition 2.10]). *Fix some $g, p, b \geq 0$ such that $\pi_1(\Sigma^{b+1}_{g,p})$ is non-abelian, and let ∂ be a boundary component of $\Sigma^{b+1}_{g,p}$. Let $l \geq 2$. Then there is a central extension*

$$1 \to \mathbb{Z} \to \mathrm{Mod}^{b+1}_{g,p}(l) \to \mathrm{Mod}^b_{g,p+1}(l) \to 1,$$

where the central $\mathbb{Z}$ is generated by the Dehn twist T_∂.

We also have the level l-version of the Birman exact sequence:

Theorem 2.4 (Mod-l Birman exact sequence [23]). *Fix $g, p, b \geq 0, l \geq 2$ such that $\pi_1(\Sigma^b_{g,p})$ is non-abelian. Let x_0 be a puncture of $\Sigma^b_{g,p+1}$. There is a short exact sequence obtained by forgetting x_0:*

$$1 \to PP_{x_0}(\Sigma^b_{g,p};l) \to \mathrm{Mod}^b_{g,p+1}(l) \to \mathrm{Mod}^b_{g,p}(l) \to 1,$$

where the level-l point pushing group $PP_{x_0}(\Sigma^b_{g,p};l)$ is as follows:

- *If $p = b = 0$, then $PP_{x_0}(\Sigma^b_{g,p};l) = \pi_1(\Sigma^b_{g,p}, x_0)$.*
- *If $p+b \geq 1$, then $PP_{x_0}(\Sigma^b_{g,p};l) = \mathrm{Ker}(\pi_1(\Sigma^b_{g,p}, x_0) \to H_1(\Sigma^b_{g,p};\mathbb{Z}/l) \to H_1(\Sigma_g;\mathbb{Z}/l))$.*

2.3. **Deligne's degeneration theorem.** Deligne's degeneration theorem can be found in [5], and I will use the version in Griffiths and Schimid's survey [8]:

Theorem 2.5 (Deligne's degeneration theorem [5])**.** *Let E be a Kähler manifold, X a complex manifold, and $f : E \to X$ a smooth, proper holomorphic mapping, which means f is a differential fiber bundle whose fibers $X_b, b \in B$ are compact Kähler manifolds. The corresponding Leray spectral sequence*

$$E_2^{p,q} = H^p(B, R^q_{f_*}(\mathbb{Q})) \Rightarrow H^{p+q}(E;\mathbb{Q}),$$

degenerates at page 2, i.e. $E_2 = E_\infty$. Here

$$R^q_{f_*}(\mathbb{Q}) \text{ comes from the presheaf } U \mapsto H^*(f^{-1}(U);\mathbb{Q}).$$

Remark: Smooth quasi-projective varieties are Kähler manifolds. In later applications, we will apply Deligne's degeneration theorem to quasi-projective orbifolds. We can do that since Deligne's degeneration theorem is still true for orbifolds, roughly because we are working with $\mathbb{Q}$-coefficients and the action of a finite group can be passed down through the Leray spectral sequence.

2.4. **Useful Facts about Group Cohomology.** First, we will transform the theorems in the introduction from homology to cohomology. There is a useful duality between group homology and group cohomology when the coefficients are $\mathbb{Q}$-vector spaces:

Proposition 2.6 ([22, Theorem 2.2])**.** *Let G be a group and let M be a G-vector space over $\mathbb{Q}$. Define $M' = Hom(M, \mathbb{Q})$. Then for every $k \geq 0$, there is a natural isomorphism $H^k(G; M') \cong Hom(H_k(G; M), \mathbb{Q})$.*

Consider a group G and a subgroup H, along with a $\mathbb{Z}[G]$-module M. There is a natural map $\mathrm{Res}^G_H : H^k(G; M) \to H^k(H; M)$ in group cohomology obtained from the inclusion $\mathbb{Z}[H] \hookrightarrow \mathbb{Z}[G]$. We call it the restriction map. There is a "wrong-way" map, called the transfer map, $\mathrm{cor}^G_H : H^k(H; M) \to H^k(G; M)$ which satisfies:

Proposition 2.7 ([3, Proposition 9.5])**.** *If H is a finite index subgroup of G with index $[G : H]$, then the composition of transfer maps and restriction maps is the multiplication map by $[G : H]$, i.e. $cor^G_H \cdot \mathrm{Res}^G_H = [G : H]\mathrm{id}$.*

Remark: In particular, supposing H is a finite-index subgroup of G, if M is a $\mathbb{Q}$(or $\mathbb{R}$)-vector space, we see that cor^G_H is surjective and Res^G_H is injective.

The following Gysin Sequence ([9]) can be deduced from the Hochschild-Serre spectral sequence ([3]) of a short exact sequence of groups:

Proposition 2.8 (Gysin Sequence [9])**.** *Consider a central extension:*

$$1 \to \mathbb{Z} \to G \to K \to 1,$$

and a $\mathbb{Z}[K]$ module M (thus M is also a $\mathbb{Z}[G]$ module through the map $G \to K$). We have the following long exact sequence:

$$\cdots \to H^{k-2}(G; M) \to H^k(K; M) \to H^k(G; M) \to H^{k-1}(H; M) \to H^{k+1}(H; M) \to \cdots,$$

where $H^{k-2}(G; M) \to H^k(K; M)$ is the differential on the E_2-page of the Hochschild-Serre spectral sequence.

Remark: The geometric version of the Gysin sequence is that, for an oriented sphere bundle $S^d \hookrightarrow E \to M$, we have the following long exact sequence

$$\cdots \to H^{k-d-1}(M) \to H^k(M) \to H^k(E) \to H^{k-d}(M) \to H^{k+1}(M) \to \cdots,$$

where the map $H^{k-d-1}(M) \to H^k(M)$ is the wedge product with the Euler class, and the map $H^k(E) \to H^{k-d}(M)$ is fiberwise integration.

In our later computations, we will apply the following Thom-Gysin Sequence ([1]) multiple times, which is derived from the Thom Isomorphism Theorem and the long exact sequence of relative cohomology:

Proposition 2.9 (Thom-Gysin Sequence [1]). *Let X be a complex variety, and let Y be an open subvariety of X of (real) codimension d. Letting R be a commutative ring, we then have the following long exact sequence:*

$$\cdots \to H^{k-d}(Y;R) \to H^k(X;R) \to H^k(X \setminus Y;R) \to H^{k-d+1}(Y;R) \to H^{k+1}(X;R) \to \cdots .$$

2.5. **Mixed Hodge Theory.** Mixed Hodge theory is used in the proof of Deligne's degeneration theorem, and is also a powerful tool for determining terms in spectral sequences. We will introduce some basic properties according to the survey [8] by Griffiths and Schimid. First, we start with definitions of pure Hodge structures.

Definition 2.10 ([8, Definition 1.1, 1.2]). *Let $H_{\mathbb{R}}$ be a finite dimensional real vector space, and $H_{\mathbb{Z}}$ be a lattice in $H_{\mathbb{R}}$. Let $H = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification.*

(1) *A* ***Hodge structure of weight*** *m on H consists of a direct sum decomposition*

$$H = \bigoplus_{p+q=m} H^{p,q}, \text{ with } H^{q,p} = \bar{H}^{p,q},$$

where $\bar{H}^{p,q}$ denotes the complex conjugate of $H^{p,q}$.

(2) *A* ***morphism of Hodge structures of type*** *(r,r) is a linear map (defined over $\mathbb{Q}$ relative to the lattices $H_{\mathbb{Z}}$, $H'_{\mathbb{Z}}$)*

$$\varphi : H \to H', \text{ with } \varphi(H^{p,q}) \subset (H')^{p+r,q+r}.$$

(3) *A Hodge structure H of weight m is* ***polarized*** *by a non-degenerate integer bilinear form Q on $H_{\mathbb{Z}}$ if the extended bilinear form Q on H satisfies the following conditions*

$$Q(v,w) = (-1)^m Q(w,v), \forall v,w \in H,$$
$$Q(H^{p,q}, H^{p',q'}) = 0, \text{ unless } p = q', q = p',$$
$$\sqrt{-1}^{p-q} Q(v,\bar{v}) > 0, \text{ for } v \in H^{p,q}, v \neq 0.$$

Remark: Let H be a Hodge structure of weight m and H' be a Hodge structure of weight m'. The tensor product $H \otimes H'$ inherits a Hodge structure of weight $m + m'$:

$$H \otimes H' = \sum_{p+q=m+m'} H^{p,m-p} \otimes (H')^{q,m'-q}.$$

Moreover, if H is polarized by Q and H' is polarized by Q', then $H \otimes H'$ is polarized by the induced bilinear form $Q \otimes Q'$.

The above definitions (except the last one) are also in one-to-one correspondence with the following:

Proposition 2.11 ([8, p35]). *Let $H, H_{\mathbb{R}}, H_{\mathbb{Z}}$ be the same as above.*

(1) *There is a Hodge stucture of weight m on H if and only if H has a* ***Hodge filtration***

$$H \supset \cdots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \cdots \supset 0,$$

$$\text{with } F^p \oplus \bar{F}^{m-p+1} \xrightarrow{\cong} H, \text{ for all } p.$$

(2) *A map* $\varphi : H \to H'$ *is a morphism of Hodge structures of type* (r,r) *if and only if* φ *preserves the Hodge filtration with a shift by* r*, i.e.*

$$\varphi(F^p) \subset (F')^{p+r},\ \textit{for all } p.$$

In particular, a morphism of Hodge structures of type (r,r) *preserves the Hodge filtration strictly:*

$$\varphi(F^p) = (F')^{p+r} \cap Im(\varphi),\ \textit{for all } p.$$

A mixed Hodge structure is a generalization of a Hodge structure.

Definition 2.12 ([8, Definition 1.11])**.** *Let* $H_{\mathbb{Z}}$ *be a finitely generated free abelian group.*

(1) *A mixed Hodge structure is a triple* $(H_{\mathbb{Z}}, W_\bullet, F^\bullet)$ *such that*

(a) *The* ***weight filtration*** $W_\bullet$ *is*

$$0 \subset \cdots \subset W_{m-1} \subset W_m \subset W_{m+1} \subset \cdots \subset H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} = H_{\mathbb{Q}}.$$

(b) *The* ***Hodge filtration*** $F^\bullet$ *is*

$$H = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \supset \cdots \supset F^{p-1} \supset F^p \supset F^{p+1} \supset \cdots \supset 0.$$

(c) *For each* $m \in \mathbb{Z}$*, on the graded piece* $Gr_m(W_\bullet) = W_m/W_{m-1}$*, the induced filtration by* $F^\bullet$ *defines a Hodge structure of weight* m*.*

(2) *A morphism of mixed Hodge structures of type* (r,r) *consists of a linear map*

$$\varphi : H_{\mathbb{Q}} \to (H')_{\mathbb{Q}} \textit{ with } \varphi(W_m) \subset (W')_{m+2r},\ \textit{and } \varphi(F^p) \subset (F')^{p+r}.$$

The morphisms of mixed Hodge structures are also strict in the following sense.

Lemma 2.13 ([8, Lemma 1.13])**.** *A morphism of type* (r,r) *between mixed Hodge structures is strict with respect to both the weight and Hodge filtrations. More precisely,*

$$\varphi(W_m) = (W')_{m+2r} \cap Im(\varphi),\ \varphi(F^p) = (F')^{p+r} \cap Im(\varphi).$$

Remark: Let $(H_{\mathbb{Z}}, W_\bullet, F^\bullet)$ and $(\tilde{H}_{\mathbb{Z}}, \tilde{W}_\bullet, \tilde{F}^\bullet)$ be two mixed Hodge structures. Their tensor product $H \otimes \tilde{H}$ inherits a mixed Hodge structure with the weight filtration

$$0 \subset \cdots \subset \sum_{a+b \le m-1} W_a \otimes \tilde{W}_b \subset \sum_{a+b \le m} W_a \otimes \tilde{W}_b \subset \sum_{a+b \le m+1} W_a \otimes \tilde{W}_b \subset \cdots \subset H_{\mathbb{Q}} \otimes \tilde{H}_{\mathbb{Q}},$$

and the Hodge filtration

$$H \otimes \tilde{H} \supset \cdots \supset \sum_{a+b \ge p-1} F^a \otimes \tilde{F}^b \supset \sum_{a+b \ge p} F^a \otimes \tilde{F}^b \supset \sum_{a+b \ge p+1} F^a \otimes \tilde{F}^b \supset \cdots \supset 0.$$

We are interested in the cohomology of complex varieties, which has a canonical polarizable mixed Hodge structure by the following theorem of Deligne (see section 4.2 of [4]):

Theorem 2.14 (Deligne [4])**.** *Let* X *be a complex algebraic variety. Then* $H^*(X;\mathbb{Q})$ *carries a canonical polarizable mixed Hodge structure.*

Here a polarizable mixed Hodge structure means all graded pieces $\mathrm{Gr}_m(W_\bullet)$ are polarizable Hodge structures. We can decompose a polarized Hodge structure into a direct sum of simple objects by the following theorem:

Theorem 2.15 ([20, Corollary 2.12])**.** *The category of polarizable Hodge structures of weight* m *is semi-simple.*

The semi-simplicity will help us to identify components of a direct sum decomposition of $H^*(X;\mathbb{Q})$ without additional issues.

3. Stable Cohomology of $\mathrm{Mod}(\Sigma_{g,p}^b)$ with Coefficients in $H^1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes r}$

In this section, we will compute $H^*(\Sigma_{g,p}^b; H^1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes r})$ and prove Theorem A.

For a closed surface Σ_g and an integer $r \geq 1$, the cohomology of Mod_g with coefficients $H^1(\Sigma_g;\mathbb{Q})^{\otimes r}$ is completely computed by Looijenga ([14]). One important step in Looigenga's paper is making use of the stable cohomology of the following moduli space of r marked points in closed genus-g Riemann surfaces:

Definition 3.1. *Let $[r] = \{1, 2, \cdots, r\}$. Write $P|[r]$ if P is a partition of $[r]$, i.e. $P = \{I_1, I_2, \cdots, I_a\}$ where I_i are disjoint nonempty subsets of $[r]$, and $I_1 \cup I_2 \cup \cdots \cup I_a = [r]$. We denote by $\mathcal{C}_g^r$ the moduli space of pairs (C, x) where C is a compact Riemann surface of genus g and $x : [r] \to C$ is a map.*

Using the notation of partitions, Looijenga ([14]) first introduced a graded algebra $A_r^\bullet$. For $1 \leq i \leq r$, let $u_i \in H^2(\mathcal{C}_g;\mathbb{Q})$ be the first Chern class of $\theta_i = f_i^*(\theta)$, where $f_i : \mathcal{C}_g^r \to \mathcal{C}_g^1$ is the map $f(C, x) = (C, x(i))$, and θ is the relative tangent sheaf of $\pi : \mathcal{C}_g^1 \to \mathcal{M}_g$. For each subset I of $[r]$ with $|I| \geq 2$, let $a_I \in H^{2|I|-2}(\mathcal{C}_g^r;\mathbb{Q})$ be the Poincaré dual of the subvariety of $\mathcal{C}_g^r$ whose $x : [r] \to C$ takes elements in I to the same point. Due to Lemma 2.4 of [14], these cohomology classes u_i, a_I satisfy the following relations

$$u_i a_I = u_j a_I \text{ if } i, j \in I,$$

$$a_I a_J = u_i^{|I\cap J|-1} a_{I\cup J} \text{ if } i \in I \cap J \neq \emptyset.$$

Let $A_r^\bullet$ be the $\mathbb{Q}[u_i : 1 \leq i \leq r]$-algebra generated by all a_I and 1 subject to the above relations. The second relation tells us $A_r^\bullet$ is the $\mathbb{Q}[u_i : 1 \leq i \leq r]$-module generated by the elements

$$a_P = \prod_{I\in P, |I|\geq 2} a_I$$

as P ranges over all partitions of $[r]$ (with the convention that $a_P = 1$ if P is the partition into singletons). Here for each $I \subset [r]$ such that $|I| \geq 2$, the element a_I is equal to the element a_P, where P is the partition of $[r]$ into I and singletons. Next, for each $I \subset [r]$ with $|I| \geq 2$, let u_I be a formal symbol. Then the first relation tells us the $\mathbb{Q}[u_i : 1 \leq i \leq r]$-module generated by a_P is isomorphic to $\mathbb{Q}[u_I : I \in P]a_P$ taking $u_i a_P$ to $u_I a_P$, where $I \in P$ contains i. Thus we have the isomorphism

$$A_r^\bullet \cong \bigoplus_{P|[r]} \mathbb{Q}[u_I : I \in P]a_P.$$

Assuming that u_i is of degree 2 and a_I is of degree $2|I| - 2$, the grading of $A_r^\bullet$ is

$$A_r^\bullet = \bigoplus_{m=0}^{\infty} A_r^{2m},$$

where A_r^{2m} is the degree $2m$ part. Note that $A_r^\bullet$ has a trivial mixed Hodge structure where A_r^{2m} has Hodge type (m, m). The stable cohomology of mapping class groups $H^*(\mathrm{Mod}_\infty;\mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \cdots]$ also has a canonical mixed Hodge structure (see e.g. [14]). Therefore the tensor product $H^*(\mathcal{M}_\infty;\mathbb{Q}) \otimes A_r^\bullet$ has a mixed Hodge structure. Looijenga proved the following theorem.

Theorem 3.2 (Looijenga [14, Theorem 2.3])**.** *There is an algebra homomorphism, that is also a morphism of mixed Hodge structures*

$$\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}\mathbb{Q}[u_I : I\in P]a_P\right)\to H^*(\mathcal{C}_g^r;\mathbb{Q})$$

which is an isomorphism in degree $\leq N(g)$*.*

To get the cohomology of Mod_g with coefficients $H^1(\Sigma_g;\mathbb{Q})^{\otimes r}$, Looigenga's idea is to apply Deligne's Theorem 2.5 to the Leray spectral sequence of the projection $\mathcal{C}_g^r\to\mathcal{M}_g$ and his result is as follows:

Theorem 3.3 (Looijenga [14, Corollary 3.3])**.** *There is a natural graded map*

$$\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}(\prod_{\{i\}\in P}u_i^2)\mathbb{Q}[u_I : I\in P]a_P\right)\to H^{*-r}(\mathrm{Mod}_g;(H^1(\Sigma_g;\mathbb{Q})^{\otimes r}),$$

which is an isomorphism in degree $\leq N(g)$*, where* $a_P=\prod_{I\in P}a_I$*.*

Thus we know from the above theorem that the cohomology of Mod_g with coefficients $H^1(\Sigma_g;\mathbb{Q})^{\otimes r}$ is stable when the genus g is large enough. We will imitate Looijenga's method to compute the cohomology for surfaces with punctures, which turns out not to be stable.

We first focus on the case of one puncture and no boundary components, and we start our computation with $r=1$:

Proposition 3.4. *We have an isomorphism*

$$H^{*-1}(\mathrm{Mod}_{g,1};H^1(\Sigma_{g,1};\mathbb{Q}))\cong\mathbb{Q}[u_1,\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_2]$$

in degree $\leq N(g)$*.*

Proof. Recall the moduli space $\mathcal{C}_g^r$ of r points in the surface Σ_g is

$$\mathcal{C}_g^r=\{(C,x:[r]\to C)|C\in\mathcal{M}_g\}.$$

When those points in the surface are distinct, we have the usual moduli space with punctures:

$$\mathcal{M}_{g,r}=\{(C,x:[r]\to C)|C\in\mathcal{M}_g,\ x\text{ is injective}\}.$$

Then we have a fibration as follows:

$$\begin{tikzcd} \Sigma_g \arrow[r] & \mathcal{C}_g^2 \arrow[d] & (C,x:[2]\to C) \arrow[d, mapsto] \\ & \mathcal{M}_{g,1} & (C,x(1)) \end{tikzcd}\qquad .\tag{1}$$

We can apply the Leray spectral sequence with $\mathbb{Q}$-coefficients to get:

$$E_2^{p,q}=H^p(\mathcal{M}_{g,1};H^q(\Sigma_g;\mathbb{Q}))\Rightarrow H^{p+q}(\mathcal{C}_g^2;\mathbb{Q}).$$

Since the fibration above is a fibration of complex varieties whose fiber Σ_g is compact, Deligne's Theorem 2.5 applies, which means the above spectral sequence degenerates at page 2. Since we are working over $\mathbb{Q}$, there are no extension issues, so:

$$H^k(\mathcal{C}_g^2;\mathbb{Q})\cong\bigoplus_{p+q=k}H^p(\mathcal{M}_{g,1};H^q(\Sigma_g;\mathbb{Q})).$$

Since $\mathcal{M}_{g,1}$ has the same rational cohomology as $\mathrm{Mod}_{g,1}$, we can rewrite this as

$$H^k(\mathcal{C}_g^2;\mathbb{Q}) \cong \bigoplus_{p+q=k} H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g;\mathbb{Q})).$$

That is

$$H^k(\mathrm{Mod}_{g,1}; H^0(\Sigma_g;\mathbb{Q})) \oplus H^{k-1}(\mathrm{Mod}_{g,1}; H^1(\Sigma_g;\mathbb{Q})) \oplus H^{k-2}(\mathrm{Mod}_{g,1}; H^2(\Sigma_g;\mathbb{Q})). \tag{2}$$

Note that the Leray filtration respects the mixed Hodge structure of $H^*(\mathcal{C}_g^2;\mathbb{Q})$, so the E_2 page terms $E_2^{p,q} = H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g;\mathbb{Q}))$ inherit mixed Hodge structures.

By Looijenga's Theorem 3.3, there is an isomorphism of mixed Hodge structures

$$H^*(\mathcal{C}_g^2;\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\cdots] \otimes \left(\bigoplus_{P|[2]} \mathbb{Q}[u_I : I \in P] a_P \right) \tag{3}$$

in degree $\leq N(g)$, where $a_P = \prod_{I\in P} u_I^{|I|-1}$. We will get $H^{k-1}(\mathrm{Mod}_{g,1}; H^1(\Sigma_g;\mathbb{Q}))$ by identifying terms in (2) with (3).

For partitions $P|[2]$, there are only two possibilities: $P = \{\{1\},\{2\}\}$, or $P = \{\{1,2\}\}$, corresponding to the two marked points being distinct or identical. Thus we can also write 3 as

$$H^*(\mathcal{C}_g^2;\mathbb{Q}) \cong \begin{array}{c} (\mathbb{Q}[\kappa_1,\kappa_2,\cdots] \otimes \mathbb{Q}[u_1,u_2]) \\ \oplus(\mathbb{Q}[\kappa_1,\kappa_2,\cdots] \otimes \mathbb{Q}[u_{\{1,2\}}]a_{\{1,2\}}) \end{array}$$

in degree $\leq N(g)$. The geometric reason for this form is that we can think of $\mathcal{C}_g^2$ as the disjoint union of $\mathcal{M}_{g,2} = \{(C, x : [2] \to C) | C \in \mathcal{M}_g, x(1) \neq x(2)\}$ and $\mathcal{M}_{g,1} = \{(C, x : [2] \to C) | C \in \mathcal{M}_g, x(1) = x(2)\}$. Then we can apply Thom-Gysin sequence (Proposition 2.9) to obtain the rational cohomology of $\mathcal{C}_g^2$.

The $H^k(\mathrm{Mod}_{g,1}; H^0(\Sigma_g;\mathbb{Q}))$ component of $H^k(\mathcal{C}_g^2;\mathbb{Q})$ from the decomposition (2) is

$$E_\infty^{k,0} = E_2^{k,0} = H^k(\mathcal{M}_{g,1}; H^0(\Sigma_g;\mathbb{Q})) = \mathrm{Image}(H^k(\mathrm{Mod}_{g,1};\mathbb{Q}) \to H^k(\mathcal{C}_g^2;\mathbb{Q})),$$

where the map $H^k(\mathrm{Mod}_{g,1};\mathbb{Q}) \to H^k(\mathcal{C}_{g,2};\mathbb{Q})$ is induced by the projection $\mathcal{C}_g^2 \to \mathcal{M}_{g,1}$ in the fibration (1). Thus we have

$$H^*(\mathrm{Mod}_{g,1};\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\cdots] \otimes \mathbb{Q}[u_1] \subset \mathbb{Q}[\kappa_1,\kappa_2,\cdots] \otimes \mathbb{Q}[u_1,u_2],$$

in degree $* \leq N(g)$, where u_1 is the first Chern class of the pullback (through $\mathcal{C}_g^2 \to \mathcal{M}_{g,1}$) of the relative tangent sheaf of $\mathcal{M}_{g,1} \to \mathcal{M}_g$ as in Theorem 3.2.

The $H^{k-2}(\mathrm{Mod}_{g,1}; H^2(\Sigma_g;\mathbb{Q}))$ component of $H^k(\mathcal{C}_g^2;\mathbb{Q})$ from the decomposition (2) is

$$E_\infty^{k-2,2} = E_2^{k-2,2} = H^{k-2}(\mathcal{M}_{g,1}; H^2(\Sigma_g;\mathbb{Q})).$$

The map

$$H^k(\mathcal{C}_g^2;\mathbb{Q}) \to E_\infty^{k-2,2} \to E_2^{k-2,2} = H^{k-2}(\mathcal{M}_{g,1}; H^2(\Sigma_g;\mathbb{Q}))$$

is the integration along fibers for the fibration (1). Let $a_{\{1,2\}}$ be the Poincaré dual of $\mathcal{M}_{g,1}$ in $\mathcal{C}_g^2$ as in Theorem 3.2. Then for any $\omega \in H^{k-2}(\mathcal{M}_{g,1}; H^2(\Sigma_g;\mathbb{Q}))$, letting $\tilde{\omega}$ be its preimage in $H^k(\mathcal{C}_g^2;\mathbb{Q})$, we have

$$\int_{\mathcal{C}_g^2} \tilde{\omega} \wedge a_{\{1,2\}} = \int_{\mathcal{M}_{g,1}} \omega.$$

Here $\mathcal{M}_{g,1}$ embeds into $\mathcal{C}_g^2$ via the trivial section of the fibration (1):

$$\mathcal{M}_{g,1} \to \mathcal{C}_g^2, (C, x(1)) \mapsto (C, x : [2] \to C, x(2) = x(1)),$$

i.e. $\mathcal{M}_{g,1}$ is the subvariety of $\mathcal{C}_g^2$ whose two marked points are identical. Thus we have

$$H^{*-2}(\mathrm{Mod}_{g,1};\mathbb{Q}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\cdots] \otimes \mathbb{Q}[u_{\{1,2\}}]a_{\{1,2\}},$$

in degree $* - 2 \leq N(g)$.

Since all the above maps are morphisms of (polarized) mixed Hodge structures, and all above objects are semi-simple by Theorem 2.15, we can obtain the rest of the components in (2) by excluding the other two components from $H^k(\mathcal{C}_g^2)$, which is

$$H^{*-1}(\mathrm{Mod}_{g,1}; H^1(\Sigma_g; \mathbb{Q})) \cong \mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes u_2\, \mathbb{Q}[u_1, u_2],$$

in degree $* \leq N(g)$.

Since $H^1(\Sigma_g; \mathbb{Q}) \cong H^1(\Sigma_{g,1}; \mathbb{Q})$, we can rewrite the above as

$$H^{*-1}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1}; \mathbb{Q})) \cong \mathbb{Q}[u_1, \kappa_1, \kappa_2, \cdots] \otimes u_2\, \mathbb{Q}[u_2]$$

in degree $\leq N(g)$. □

Applying the above method to a more general fibration, through more computations, we will get the stable cohomology of $M_{g,1}$ with coefficients $H^1(\Sigma_{g,1}; \mathbb{Q})^{\otimes r}$, for general $r \geq 1$ (recall the notations $\mathbb{Q}[u_I : I \in P]a_P$ in the beginning of this section):

Theorem 3.5. *For $r \geq 1$, we have*

$$H^{*-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1}; \mathbb{Q})^{\otimes r}) \cong \mathbb{Q}[c_1, \kappa_1, \kappa_2, \cdots] \otimes \left(\bigoplus_{P|[r]} \Big(\prod_{\{i\} \in P} u_i \Big)\, \mathbb{Q}[u_I : I \in P]a_P \right)$$

in degree $\leq N(g)$.

Proof. We have a fibration as follows:

$$(4) \qquad \begin{tikzcd} \Sigma_g^{\times r} \arrow[r] & \mathcal{C}_g^{r+1} \arrow[d] & (C, x : [r+1] \to C) \arrow[d, mapsto] \\ & \mathcal{M}_{g,1} & (C, x(1)) \end{tikzcd} .$$

We can apply the Leray spectral sequence with coefficients $\mathbb{Q}$ to get

$$E_2^{p,q} = H^p(\mathcal{M}_{g,1}; H^q(\Sigma_g^{\times r}; \mathbb{Q})) \Rightarrow H^{p+q}(\mathcal{C}_g^{r+1}).$$

Since the fibration above is a fibration of complex varieties whose fiber $\Sigma_g^{\times r}$ is compact, Deligne's Theorem 2.5 applies, which means the above spectral sequence degenerates at page 2. Just like before, this implies that

$$H^k(\mathcal{C}_g^{r+1}) \cong \bigoplus_{p+q=k} H^p(\mathcal{M}_{g,1}; H^q(\Sigma_g^{\times r}; \mathbb{Q})).$$

Since $\mathcal{M}_{g,1}$ has the same rational cohomology as $\mathrm{Mod}_{g,1}$, we can rewrite this as

$$H^k(\mathcal{C}_g^{r+1}) \cong \bigoplus_{p+q=k} H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g^{\times r}; \mathbb{Q})).$$

The Leray filtration preserves the mixed Hodge structure of $H^*(\mathcal{C}_g^{r+1}; \mathbb{Q})$, therefore the E_2-page terms $E_2^{p,q} = H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g^{\times r}; \mathbb{Q}))$ inherit mixed Hodge structures.

Looijenga's Theorem 3.3 gives us the following isomorphism of mixed Hodge structures:

$$H^*(\mathcal{C}_g^{r+1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes \left(\bigoplus_{P|[r+1]} \mathbb{Q}[u_I : I \in P]a_P \right)$$

in degree $\leq N(g)$.

We then proceed the proof by induction on r.

The base case $r = 1$ is Proposition 3.4.

For $r \geq 2$, we suppose it is true for any $s \leq r-1$ that:

$$H^{*-s}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes s}) \cong \mathbb{Q}[c_1,\kappa_1,\kappa_2,\cdots] \otimes \left(\bigoplus_{P|[s]} (\prod_{\{i\}\in P} u_i)\, \mathbb{Q}[u_I : I \in P] a_P\right)$$

in degree $\leq N(g)$.

We have $H^q(\Sigma_g^{\times r};\mathbb{Q}) \cong H^q(\Sigma_{g,1}^{\times r};\mathbb{Q})$ as $\mathrm{Mod}_{g,1}$ modules, and by the Künneth formula,

$$H^q(\Sigma_{g,1}^{\times r};\mathbb{Q}) \cong \bigoplus_{i_1+i_2+\cdots+i_r=q} H^{i_1}(\Sigma_{g,1};\mathbb{Q}) \otimes H^{i_2}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{i_r}(\Sigma_{g,1};\mathbb{Q})$$

Thus we have

$$\begin{aligned} & H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g^{\times r};\mathbb{Q})) \\ \cong & \bigoplus_{i_1+i_2+\cdots+i_r=q} H^p(\mathrm{Mod}_{g,1}; H^{i_1}(\Sigma_{g,1};\mathbb{Q}) \otimes H^{i_2}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{i_r}(\Sigma_{g,1};\mathbb{Q})) \end{aligned}$$

Observe that $H^{k-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})$ is the component of $H^{k-r}(\mathrm{Mod}_{g,1}; H^r(\Sigma_g^{\times r};\mathbb{Q}))$ with $i_1 = i_2 = \cdots = i_r = 1$. Let's think about what the remaining components of $H^{k-r}(\mathrm{Mod}_{g,1}; H^r(\Sigma_g^{\times r};\mathbb{Q}))$ are:

(1) When some $i_j = 0$, the component

$$H^p(\mathrm{Mod}_{g,1}; H^{i_1}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{i_j}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{i_r}(\Sigma_{g,1};\mathbb{Q}))$$

must be a component of $H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g^{\times r};\mathbb{Q}))$ whose $q \leq 2r-2$. Notice the fibration (4) factors through the map $\psi_i : \mathcal{C}_g^{r+1} \to \mathcal{C}_g^r$ by forgetting the i-th ($2 \leq i \leq r+1$) marked point, so we can get the following map:

$$\begin{tikzcd} \Sigma_g^{\times r} \arrow[r] \arrow[d] & \mathcal{C}_g^{r+1} \arrow[r] \arrow[d, "\psi_i"] & \mathcal{M}_{g,1} \arrow[d, "id"] \\ \Sigma_g^{\times (r-1)} \arrow[r] & \mathcal{C}_g^r \arrow[r] & \mathcal{M}_{g,1} \end{tikzcd}.$$

The map $\psi_i^* : H^*(\mathcal{C}_g^r;\mathbb{Q}) \to H^*(\mathcal{C}_g^{r+1};\mathbb{Q})$ induces maps between items in the two Leray spectral sequences. That is

$$H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g^{\times (r-1)};\mathbb{Q})) \to H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g^{\times r};\mathbb{Q})).$$

When $q \leq 2r-2$, the image of the above map is clear by the Künneth formula and the induction on r.

(2) When some $i_j = 2$, the cup product

$$\begin{array}{c} H^p(\mathrm{Mod}_{g,1}; H^{i_1}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{\widehat{i_j}}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{i_r}(\Sigma_{g,1};\mathbb{Q})) \\ \otimes H^0(\mathrm{Mod}_{g,1}; H^2(\Sigma_{g,1};\mathbb{Q})) \\ \downarrow \\ H^p(\mathrm{Mod}_{g,1}; H^{i_1}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{i_j}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{i_r}(\Sigma_{g,1};\mathbb{Q})) \end{array}$$

turns out to be an isomorphism by direct computations. The term

$$H^p(\mathrm{Mod}_{g,1}; H^{i_1}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{\widehat{i_j}}(\Sigma_{g,1};\mathbb{Q}) \otimes \cdots \otimes H^{i_r}(\Sigma_{g,1};\mathbb{Q}))$$

is a component of $H^p(\mathrm{Mod}_{g,1}; H^*(\Sigma_g^{\times(r-1)};\mathbb{Q}))$, so it is known by induction. Besides, in the proof of Proposition 3.4, we know that $H^0(\mathrm{Mod}_{g,1}; H^2(\Sigma_{g,1};\mathbb{Q}))$ is $\mathbb{Q}$, generated by $u_{\{1,j\}}$. If we make use of the following relations before Theorem 3.2:

$$u_i a_I = u_j a_I \text{ if } i,j \in I,$$

$$a_I a_J = u_i^{|I\cap J|-1} a_{I\cup J} \text{ if } i \in I\cap J,$$

we can express the cohomology in a desired way (e.g. write $a_{\{1,2\}}a_{\{1,3\}} = a_{\{1,2,3\}}$).

The maps in (1) and (2) are morphisms of mixed Hodge structures. All the Hodge structures involved are polarizable, hence semi-simple by Theorem 2.15. Therefore, after carefully writing terms of the above two types in terms of partitions P, we can exclude them to get $H^{k-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})$ in Table 1 as follows. (For the polynomials in the table, we mean the degree k parts of them. For the $j_1, j_2\cdots$ indices in the table, they should be distinct and between 2 and $r+1$. The order listed is by increasing q. As a shorthand, we denote $K=\mathbb{Q}[\kappa_1,\kappa_2,\cdots]$. We always have degree $k\le N(g)$.)

Table 1: Rational cohomology of $\mathcal{C}_g^{r+1}$ written in two ways

$P\mid[r+1]$	$H^k(\mathcal{C}_g^{r+1};\mathbb{Q})$	$\bigoplus_{p+q=k} H^p(\mathrm{Mod}_{g,1}; H^q(\Sigma_g^{\times r};\mathbb{Q}))$
$\{1\},\cdots,\{r+1\}$	$K\otimes\mathbb{Q}[u_1,u_2,\cdots,u_{r+1}]$	$K\otimes\mathbb{Q}[u_1]$ $K\otimes u_{j_1}\,\mathbb{Q}[u_1,u_{j_1}]$ $K\otimes u_{j_1}\cdot u_{j_2}\,\mathbb{Q}[u_1,u_{j_1},u_{j_2}]$ $\vdots$ $K\otimes u_{j_1}\cdots u_{j_{r-1}}\,\mathbb{Q}[u_1,u_{j_1},\cdots,u_{j_{r-1}}]$ $?\subset H^{k-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})$
$\{1\}$, $I_2=\{2,\cdots,r+1\}$	$K\otimes\mathbb{Q}[u_1,u_{I_2}]a_{I_2}$	$?\subset H^{k-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})$
$\{1\}$, $P\setminus\{1\}$ not as above	$K\otimes\mathbb{Q}[u_I: I\in P]a_P$	$K\otimes\mathbb{Q}[u_1,u_I: I\in P,\lvert I\rvert\ge 2]a_P$ $K\otimes u_{j_1}\,\mathbb{Q}[u_1,u_{j_1},u_I:\{j_1\}\in P, I\in P,\lvert I\rvert\ge 2]a_P$ (if $\sum_{\lvert I\rvert\ge 2}\lvert I\rvert<r-1$) $\vdots$ $K\otimes u_{j_1}\cdots u_{j_m}\,\mathbb{Q}[u_1,u_{j_1},\cdots,u_{j_m},u_I:$ $:\{j_1\},\cdots\{j_m\}, I\in P,\lvert I\rvert\ge 2]a_P$ $(m+\sum_{I\in P,\lvert I\rvert\ge 2}\lvert I\rvert=r-1)$ $?\subset H^{k-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})$
$1\in I_1,\lvert I_1\rvert\ge 2$	$K\otimes\mathbb{Q}[u_I: I\in P]a_P$	(*Take the degree $k-2(\lvert I_1\rvert-1)$ part of polynomials.) $K\otimes\mathbb{Q}[u_I: I\in P,\lvert I\rvert\ge 2]\prod_{I\in P,I\ne I_1}a_I)$ $K\otimes u_{j_1}\,\mathbb{Q}[u_{j_1},u_I:\{j_1\}\in P, I\in P,\lvert I\rvert\ge 2]\prod_{I\in P,I\ne I_1}a_I$ $K\otimes u_{j_1}\cdots u_{j_n}\,\mathbb{Q}[u_{j_1},\cdots,u_{j_n},u_I:$ $:\{j_1\},\cdots,\{j_n\}, I\in P,\lvert I\rvert\ge 2]\prod_{I\in P,I\ne I_1}a_I$ $(n+\sum_{I\in P,\lvert I\rvert\ge 2}\lvert I\rvert=r+1)$ (*The above is equivalent to the degree k part of the polynomial multipled by $u_{I_1}^{\lvert I_1\rvert-1}$, thus equivalent to polynomials whose last term is a_P.)

Thus from the table, we can get:

$$H^{*-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})\cong\mathbb{Q}[c_1,\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}\Big(\prod_{\{i\}\in P}u_i\Big)\mathbb{Q}[u_I: I\in P]a_P\right)$$

in degree $\leq N(g)$. $\square$

Kawazumi has a different computation with $\mathbb{Z}$-coefficients ([13]), whose result turns out to be same as ours in $\mathbb{Q}$-coefficients. He presented the results in a slightly different language using the following weighted partitions:

Definition 3.6 (Kawazumi[13])**.** *A set* $\hat{P} = \{(S_1, i_1), (S_2, i_2), \cdots, (S_\nu, i_\nu)\}$ *is a weighted partition of the index set* $\{1, 2, \cdots, r\}$ *if*

1. *The set* $\{S_1, S_2, \cdots, S_\nu\}$ *is a partition of the set* $\{1, 2, \cdots, r\}$.
2. $i_1, i_2, \cdots, i_\nu \geq 0$ *are non-negative integers.*
3. *Each* $(S_a, i_a), 1 \leq a \leq \nu$, *satisfies:* $i_a + |S_a| \geq 2$.

We denote by $\mathcal{P}_r$ *the set of all weighted partitons of* $\{1, 2, \cdots, r\}$.

Kawazumi first computed for surfaces with one boundary component, and then used induction to generalize to $\Sigma_{g,p}^b$ with $p + b \geq 1$, in the following two theorems:

Theorem 3.7 (Kawazumi [13])**.** *We have*

$$H^*(\mathrm{Mod}_g^1; H^1(\Sigma_g^1; \mathbb{Z})^{\otimes r}) \cong \bigoplus_{\hat{P} \in \mathcal{P}_r} H^*(\mathrm{Mod}_g^1; \mathbb{Z}) m_{\hat{P}}$$

in degree $\leq N(g) - n$. *Here* $m_{\hat{P}}$ *is the twisted Miller-Morita-Mumford class whose degree is* $2(\sum_{a=1}^{\nu} i_a) + r - 2\nu$.

Theorem 3.8 (Kawazumi [13])**.** *For* $b \geq 1, p \geq 0$, *we have*

$$H^*(\mathrm{Mod}_{g,p}^b; H^1(\Sigma_{g,p}^b; \mathbb{Z})^{\otimes r}) \cong H^*(\mathrm{Mod}_g^1; H^1(\Sigma_g^1; \mathbb{Z})^{\otimes r}) \otimes_{H^*(\mathrm{Mod}_g^1; \mathbb{Z})} H^*(\mathrm{Mod}_{g,p}^b; \mathbb{Z})$$

in degree $\leq N(g) - n$.

Tensoring Theorem 3.8 with $\mathbb{Q}$, the result is equivalent to Theorem A in the special case where $b \geq 1$. Recall the general case of Theorem A says that for $p + b \geq 1$, we have

$$H^{*-r}(\mathrm{Mod}_{g,p}^b; H^1(\Sigma_{g,p}^b; \mathbb{Q})^{\otimes r}) \cong \mathbb{Q}[c_1, \cdots, c_p, \kappa_1, \kappa_2, \cdots] \otimes \left(\bigoplus_{P|[r]} \Big(\prod_{\{i\} \in P} u_i \Big) \mathbb{Q}[u_I : I \in P] a_P \right)$$

in degree $* \leq N(g)$.

The equivalence for the special case can be seen as follows. We can identify Kawazumi's weight i_a with the exponent of our u_{S_a}. His condition $i_a + |S_a| \geq 2$ is equivalent to u_i having exponent greater than 1 if $\{i\} \in P$. Given a weighted partition

$$\hat{P} = \{(S_1, i_1), (S_2, i_2), \cdots, (S_\nu, i_\nu)\}$$

of $\{1, 2 \cdots, r\}$, let $P = \{S_1, S_2, \cdots, S_\nu\}$. The degree of $\prod_{S_a \in P} u_{S_a}^{i_a} \cdot a_P$, where $a_P = \prod_{S_a \in P} a_{S_a}$, is $2(\sum_{a=1}^{\nu} i_a) + 2\sum_{a=1}^{\nu}(|S_a| - 1) = 2(\sum_{a=1}^{\nu} i_a) + 2r - 2\nu$, which is exactly the degree of $m_{\hat{P}}$ minus r (notice Kawazumi computed H^* but our result is H^{*-r}).

Although we believe our method of computing $H^*(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1}; \mathbb{Q})^{\otimes r})$ can be used to compute $H^*(\mathrm{Mod}_{g,p}^b; H^1(\Sigma_{g,p}^b; \mathbb{Q})^{\otimes r})$ for all $p + b \geq 1$, doing that would require lots of computations. We turn to give a quick proof of Theorem A using Theorem 3.8 by Kawazumi ([13]). The difference between Theorem A and Theorem 3.8 with $\mathbb{Q}$-coefficients is that Theorem A includes the case where $b = 0, p \geq 1$.

Proof of Theorem A. Recall we have computed in Theorem 3.5 that

$$H^{*-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r}) \cong \mathbb{Q}[c_1,\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}(\prod_{\{i\}\in P}u_i)\,\mathbb{Q}[u_I : I\in P]a_P\right)$$

in degree $\le N(g)$, where $a_P = \prod_{I\in P,|I|\ge 2} a_I$.

The Gysin sequence (Proposition 2.8) of $1\to\mathbb{Z}\to\mathrm{Mod}_g^1\to\mathrm{Mod}_{g,1}\to 1$ with coefficients $H^1(\Sigma_g^1;\mathbb{Q})^{\otimes r}\cong H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r}$ is

$$\begin{aligned}\cdots\to H^{*-r-2}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})\to H^{*-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})\to\\ \to H^{*-r}(\mathrm{Mod}_g^1; H^1(\Sigma_g^1;Q)^{\otimes r})\to H^{*-r-1}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})\cdots,\end{aligned}$$

where the map

$$H^{*-r-2}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})\to H^{*-r}(\mathrm{Mod}_{g,1}; H^1(\Sigma_{g,1};\mathbb{Q})^{\otimes r})$$

is the multiplication by the first Chern class $c_1\in H^2(\mathrm{Mod}_{g,1};\mathbb{Q})$. Thus we have

$$H^{*-r}(\mathrm{Mod}_g^1; H^1(\Sigma_g^1;\mathbb{Q})^{\otimes r}) \cong \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}(\prod_{\{i\}\in P}u_i)\,\mathbb{Q}[u_I : I\in P]a_P\right)$$

in degree $\le N(g)$, where $a_P = \prod_{I\in P,|I|\ge 2} a_I$.

Now Theorem 3.8 by Kawazumi ([13]) with $\mathbb{Q}$-coefficients tells us, for $b\ge 1, p\ge 0$,

$$\begin{aligned}&H^{*-r}(\mathrm{Mod}_{g,p}^b; H^1(\Sigma_{g,p}^b;\mathbb{Q})^{\otimes r}) \cong H^*(\mathrm{Mod}_g^1; H^1(\Sigma_g^1;\mathbb{Q})^{\otimes r})\otimes_{H^*(\mathrm{Mod}_g^1;\mathbb{Q})} H^*(\mathrm{Mod}_{g,p}^b;\mathbb{Q})\\ &\cong \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}(\prod_{\{i\}\in P}u_i)\,\mathbb{Q}[u_I : I\in P]a_P\right)\bigotimes_{\mathbb{Q}[\kappa_1,\kappa_2,\cdots]}\mathbb{Q}[c_1,\cdots,c_p,\kappa_1\kappa_2,,\cdots]\\ &\cong \mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{P|[r]}(\prod_{\{i\}\in P}u_i)\,\mathbb{Q}[u_I : I\in P]a_P\right)\end{aligned}$$

in degree $*\le N(g)$, where $a_P = \prod_{I\in P,|I|\ge 2} a_I$.

The remaining case we need to prove is when $b=0, p\ge 1$. We just need to apply the Gysin sequence (Proposition 2.8) of $1\to\mathbb{Z}\to\mathrm{Mod}_{g,p-1}^1\to\mathrm{Mod}_{g,p}\to 1$ with coefficients $H^1(\Sigma_{g,p-1}^1;\mathbb{Q})^{\otimes r}\cong H^1(\Sigma_{g,p};\mathbb{Q})^{\otimes r}$:

$$\begin{aligned}\cdots\to H^{*-r-2}(\mathrm{Mod}_{g,p}; H^1(\Sigma_{g,p};\mathbb{Q})^{\otimes r})\to H^{*-r}(\mathrm{Mod}_{g,p}; H^1(\Sigma_{g,p};\mathbb{Q})^{\otimes r})\to\\ H^{*-r}(\mathrm{Mod}_{g,p-1}^1; H^1(\Sigma_{g,p-1}^1;\mathbb{Q})^{\otimes r})\to H^{*-r-1}(\mathrm{Mod}_{g,p}; H^1(\Sigma_{g,p};\mathbb{Q})^{\otimes r})\to\cdots,\end{aligned}$$

where the map

$$H^{*-r-2}(\mathrm{Mod}_{g,p}; H^1(\Sigma_{g,p};\mathbb{Q})^{\otimes r})\to H^{*-r}(\mathrm{Mod}_{g,p}; H^1(\Sigma_{g,p};\mathbb{Q})^{\otimes r})$$

is the multiplication by $c_p \in H^2(\mathrm{Mod}_{g,p};\mathbb{Q})$, the first Chern class for the p-th puncture of $\Sigma_{g,p}$ obtained by capping a punctured disk to $\Sigma^1_{g,p-1}$. Thus we have

$$\begin{aligned} &H^{*-r}(\mathrm{Mod}_{g,p}; H^1(\Sigma_{g,p};\mathbb{Q})^{\otimes r}) \cong H^{*-r}(\mathrm{Mod}^1_{g,p-1}; H^1(\Sigma^1_{g,p-1};\mathbb{Q})^{\otimes r}) \otimes \mathbb{Q}[c_p] \\ &\cong \mathbb{Q}[c_1,\cdots,c_{p-1},\kappa_1,\kappa_2,\cdots] \otimes \left(\bigoplus_{P|[r]} (\prod_{\{i\}\in P} u_i)\, \mathbb{Q}[u_I : I \in P] a_P \right) \otimes \mathbb{Q}[c_p] \\ &\cong \mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots] \otimes \left(\bigoplus_{P|[r]} (\prod_{\{i\}\in P} u_i)\, \mathbb{Q}[u_I : I \in P] a_P \right) \end{aligned}$$

in degree $* \le N(g)$, where $a_P = \prod_{I\in P, |I|\ge 2} a_I$. $\square$

4. Twisted Cohomology of Level-l Mapping Class Groups

In this section, we will compute the twisted cohomology of the level-l mapping class group with coefficients in the r-tensor power of Prym representations. Recall that

$$\mathcal{D} = H^1(\Sigma^1_g;\mathbb{Z}/l) = H^1(\Sigma_g;\mathbb{Z}/l) = H_1(\Sigma^1_g;\mathbb{Z}/l) = H_1(\Sigma_g;\mathbb{Z}/l) = (\mathbb{Z}/l)^{2g}.$$

For $p+b\ge 1$, let $\Sigma^b_{g,p}[\mathcal{D}] \to \Sigma^b_{g,p}$ be the regular cover with deck group $\mathcal{D}$, arising from the group homomorphism

$$\pi_1(\Sigma^b_{g,p}) \to H_1(\Sigma^b_{g,p};\mathbb{Z}/l) \to H_1(\Sigma_g;\mathbb{Z}/l),$$

where the second map is induced by gluing disks to all boundary components and filling in all punctures.

Let the level-l subgroup of the mapping class group be

$$\mathrm{Mod}^b_{g,p}(l) = \mathrm{Ker}(\mathrm{Mod}(\Sigma^b_{g,p}) \to \mathrm{Aut}(H_1(\Sigma^b_{g,p};\mathbb{Z}/l)).$$

Since $\mathrm{Mod}^b_{g,p}(l)$ acts trivially on $H_1(\Sigma^b_{g,p};\mathbb{Z}/l)$, it also acts trivially on $H_1(\Sigma_g;\mathbb{Z}/l) = \mathcal{D}$. Therefore the action of $\mathrm{Mod}^b_{g,p}(l)$ lifts to the cover $\Sigma^b_{g,p}[\mathcal{D}]$. This gives us an action of $\mathrm{Mod}^b_{g,p}(l)$ on $\mathfrak{H}^b_{g,p}(l;\mathbb{Q}) = H^1(\Sigma^b_{g,p}[\mathcal{D}];\mathbb{Q})$, which is called the Prym representation.

For $r\ge 1$, the regular cover $\Sigma^b_{g,p}[\mathcal{D}] \to \Sigma^b_{g,p}$ induces a map

$$H^1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes r} \to H^1(\Sigma^b_{g,p}[\mathcal{D}];\mathbb{Q})^{\otimes r} = \mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes r},$$

which is equivariant with respect to the group inclusion $\mathrm{Mod}^b_{g,p}(l) \to \mathrm{Mod}^b_{g,p}$. This induces a homomorphism on group cohomology with twisted coefficients:

$$H^k(\mathrm{Mod}^b_{g,p}; H^1(\Sigma^b_{g,p};\mathbb{Q})^{\otimes r}) \to H^k(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes r}).$$

This map was proved to be an isomorphism by Putman([23]) for $r = 0,1$, and Putman also conjectured that this map is not an isomorphism when $r\ge 2$. Although he dealt with homology, we can translate into cohomology using the duality in Proposition 2.6 and get:

Theorem 4.1 (Putman [23])**.** *Let $g,p,b\ge 0$ and $l\ge 2$. Then the map $H^k(\mathrm{Mod}^b_{g,p};\mathbb{Q}) \to H^k(\mathrm{Mod}^b_{g,p}(l);\mathbb{Q})$, induced by the inclusion $\mathrm{Mod}^b_{g,p}(l) \hookrightarrow \mathrm{Mod}^b_{g,p}$ is an isomorphism if $g \ge 2k^2+7k+2$.*

Theorem 4.2 (Putman [23])**.** *Let $g, p, b \geq 0$ and $l \geq 2$ be such that $p + b \geq 1$. Then the map*

$$H^k(\mathrm{Mod}_{g,p}^b; H^1(\Sigma_{g,p}^b; \mathbb{Q})) \to H^k(\mathrm{Mod}_{g,p}^b(l); \mathfrak{H}_{g,p}^b(l; \mathbb{Q}))$$

is an isomorphism if $g \geq 2(k+1)^2 + 7k + 8$.

For $r \geq 2$, knowing $H^k(\mathrm{Mod}_{g,p}^b; H^1(\Sigma_{g,p}^b; \mathbb{Q})^{\otimes r})$ from last section, we will compute $H^k(\mathrm{Mod}_{g,p}^b(l); \mathfrak{H}_{g,p}^b(l; \mathbb{Q})^{\otimes r})$ directly. We will start with computing $H^*(\mathrm{Mod}_{g,1}; \mathfrak{H}_{g,1}(\mathbb{Q})^{\otimes 2})$, then calculate $H^*(\mathrm{Mod}_{g,1}; \mathfrak{H}_{g,1}(\mathbb{Q})^{\otimes r})$ by induction on r, and at last generalize the result to $H^k(\mathrm{Mod}_{g,p}^b(l); \mathfrak{H}_{g,p}^b(l; \mathbb{Q})^{\otimes r})$.

Fix g, p, b with $p + b \geq 1$. Recall $\Sigma_{g,p}^b[\mathcal{D}]$ is the regular $\mathcal{D}$ cover of $\Sigma_{g,p}^b$, and if we glue disks to all the boundary components of $\Sigma_{g,p}^b[\mathcal{D}]$ and fill in all its punctures, we get a closed surface, denoted by $\Sigma_g[\mathcal{D}]$. Note that $\Sigma_g[\mathcal{D}]$ is the regular $\mathcal{D}$-cover of Σ_g. Moreover, $\mathrm{Mod}_{g,p}^b(l)$ also acts on $H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})$ via the inclusion $H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q}) \to H^1(\Sigma_{g,p}^b[\mathcal{D}]; \mathbb{Q})$, since $\mathrm{Mod}_{g,p}^b$ preserves the boundary components and punctures of $\Sigma_{g,p}^b$.

The idea to compute the twisted cohomology of the level-l mapping class group with coefficients in the Prym representation, is to get a fibration of level-l moduli spaces and compute in the way we did in the last section for the case with no level structures.

Recall that the moduli space $\mathcal{M}_{g,p}$ can be viewed as $\mathrm{Teich}(\Sigma_{g,p})/\mathrm{Mod}_{g,p}$, where

$$\mathrm{Teich}(\Sigma_{g,p}) = \{\text{complex structures on } \Sigma_{g,p}\}/\text{homotopy}$$

is the Teichmüller space of $\Sigma_{g,p}$. When the Euler characteristic $\chi(\Sigma_{g,p}) = 2 - 2g - p < 0$, it is known that $\mathrm{Teich}(\Sigma_{g,p})$ is homeomorphic to $\mathbb{R}^{6-6g+2p}$, and $\mathrm{Mod}_{g,p}$ acts properly discontinuously on $\mathrm{Teich}(\Sigma_{g,p})$ (see e.g. [6]). From this we know $\mathcal{M}_{g,p}$ has the same rational cohomology as $\mathrm{Mod}_{g,p}$. When we consider the finite-index subgroup $\mathrm{Mod}_{g,p}(l)$ of $\mathrm{Mod}_{g,p}$, we also have the moduli space with the level-l structure $\mathcal{M}_{g,p}(l) := \mathrm{Teich}(\Sigma_{g,p})/\mathrm{Mod}_{g,p}(l)$, which is a finite cover of $\mathcal{M}_{g,p}$. For the same reason, $\mathcal{M}_{g,p}(l)$ has the same rational cohomology as $\mathrm{Mod}_{g,p}(l)$.

In last section, we implicitly use fibration

$$\begin{array}{ccc} \Sigma_{g,p} \longrightarrow \mathcal{M}_{g,p+1} & & (C, x : [p+1] \to C) \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,p} & & (C, x|[p]) \end{array},$$

which coincide with the Birman exact sequence

$$1 \to \pi_1(\Sigma_{g,p}) \to \mathrm{Mod}_{g,p+1} \to \mathrm{Mod}_{g,1} \to 1.$$

We also have the mod-l Birman exact sequence for $b = 0, p \geq 1$ (Theorem 2.4):

$$1 \to \pi_1(\Sigma_{g,p}^b[\mathcal{D}]) \to \mathrm{Mod}_{g,p+1}(l) \to \mathrm{Mod}_{g,p}(l) \to 1.$$

Here we use that $\pi_1(\Sigma_{g,p}^b[\mathcal{D}], x_0)$ is exactly the kernel of $\pi_1(\Sigma_{g,p}) \to H_1(\Sigma_g; \mathbb{Z}/l) = \mathcal{D}$. This short exact sequence tells us that the fundamental group of the fiber of the fibration $\mathrm{Teich}(\Sigma_{g,p+1})/\mathrm{Mod}_{g,p+1}(l) \to \mathrm{Teich}(\Sigma_{g,p})/\mathrm{Mod}_{g,p}(l)$ is isomorphic to $\pi_1(\Sigma_{g,p}^b[\mathcal{D}])$. Moreover, the fiber has dimension 2 by counting dimensions of $\mathrm{Teich}(\Sigma_{g,p+1})$ and $\mathrm{Teich}(\Sigma_{g,p})$, so the fiber is actually homeomorphic to $\Sigma_{g,p}^b[\mathcal{D}]$. Thus we get a fibration

$$\begin{array}{c} \Sigma_{g,p}[\mathcal{D}] \longrightarrow \mathcal{M}_{g,p+1}(l) \\ \downarrow \\ \mathcal{M}_{g,p}(l) \end{array}.$$

As the number of punctures increases from 1 to p, we have many such fibrations, and when we try to combine these fibrations to get a large fibration, the new fiber turns out to be an **orbit configuration space**

$$Conf_p^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]) = \{(x_1, x_2, \cdots, x_p) | x_i \in \Sigma_{g,1}[\mathcal{D}]; \text{if } i \neq j, \forall d \in \mathcal{D}, x_i \neq d \cdot x_j\},$$

which is the space of p ordered points in $\Sigma_{g,1}[\mathcal{D}]$ in different $\mathcal{D}$-orbits. See the following proposition.

Proposition 4.3. *For $p \geq 1$, we have the following fibration:*

$$\begin{tikzcd} Conf_p^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]) \arrow[r] & \mathcal{M}_{g,p+1}(l) \arrow[d] \\ & \mathcal{M}_{g,1}(l) \end{tikzcd}.$$

Proof. We will prove this by induction on p.

For $p = 1$, we have $Conf_1^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]) = \Sigma_{g,1}[\mathcal{D}]$, so the desired fibration has been obtained before this proposition.

For $p \geq 2$, assume that we have the fibration

$$\begin{tikzcd} Conf_{p-1}^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]) \arrow[r] & \mathcal{M}_{g,p}(l) \arrow[d] \\ & \mathcal{M}_{g,1}(l) \end{tikzcd}.$$

We also have the fibration

$$\begin{tikzcd} \Sigma_{g,p}[\mathcal{D}] \arrow[r] & \mathcal{M}_{g,p+1}(l) \arrow[d] \\ & \mathcal{M}_{g,p}(l) \end{tikzcd}.$$

Then the fiber of the composition map $\mathcal{M}_{g,p+1}(l) \to \mathcal{M}_{g,p}(l) \to \mathcal{M}_{g,1}(l)$ is

$$\{(x_1, x_2, \cdots, x_{p-1}) \in Conf_{p-1}^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]), x_p \in \Sigma_{g,p}[\mathcal{D}]\}$$

Here $x_1, x_2, \cdots, x_{p-1} \in \Sigma_{g,1}[\mathcal{D}]$ lie in different $\mathcal{D}$-orbits and $x_p \in \Sigma_{g,p}[\mathcal{D}]$ means x_p lies in the regular $\mathcal{D}$-cover of $\Sigma_{g,1} \setminus \{\overline{x_1}, \cdots, \overline{x_{p-1}}\}$, where $\overline{x_1}, \cdots, \overline{x_{p-1}}$ are the images of $x_1, \cdots, x_{p-1}$ under the map $\Sigma_{g,1}[\mathcal{D}] \to \Sigma_{g,1}$. This is equivalent to say that $x_1, \cdots, x_{p-1}, x_p$ lie in different $\mathcal{D}$-orbits in $\Sigma_{g,1}[\mathcal{D}]$. Thus the fiber of $\mathcal{M}_{g,p+1}(l) \to \mathcal{M}_{g,1}(l)$ is the orbit configuration space $Conf_p^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])$. □

Remark: Observing this, we can think of $\mathcal{M}_{g,p}(l)$ as the following moduli space:

$$\mathcal{M}_{g,p}(l) = \{(C, x_1, \cdots, x_p) | C \in \mathcal{M}_g, x_1, \cdots, x_p \in C[\mathcal{D}] \text{ in different } \mathcal{D} \text{ orbits}\},$$

where the notation $C[\mathcal{D}]$ refers to the regular $\mathcal{D}$ cover of C. In this way, the vertical map in the above fibration is forgetting all but one marked point. If we allow points to be in the same $\mathcal{D}$-orbit, we get a larger moduli space containing $\mathcal{M}_{g,p}(l)$:

$$\mathcal{C}_{g,p}(l) = \{C, x_1, \cdots, x_p) | C \in \mathcal{M}_g, x_1, \cdots, x_p \in C[\mathcal{D}]\}.$$

To decompose $\mathcal{C}_{g,p}(l)$ as a union of $\mathcal{M}_{g,*}(l)$, we need to record when marked points are in different $\mathcal{D}$-orbits and how the points differ by the $\mathcal{D}$ action, being in the same $\mathcal{D}$-orbit. Thus we introduce the following notation of $\mathcal{D}$-weighted partitions to express our results. The idea of this notation combines Looijenga's notation of partitions and Kawazumi's notation of weighted partitions.

Definition 4.4. *Given a group $\mathcal{D}$, a set $\widetilde{P} = \{(S_1, \vec{d_1}), (S_2, \vec{d_2}), \cdots, (S_\nu, \vec{d_\nu})\}$ is called a $\mathcal{D}$-weighted partition of the index set $\{1, 2, \cdots, r\}$, if*

1. *The set $\{S_1, S_2, \cdots, S_\nu\}$ is a partition of the set $\{1, 2, \cdots, r\}$.*
2. *For each $1 \le a \le \nu$, $\vec{d_a} = (d_a^{(1)}, d_a^{(2)}, \cdots, d_a^{(|S_a|-1)})$, where $d_a^{(i)} \in \mathcal{D}$. By convention, $\vec{d_a}$ is empty if $|S_a| = 1$.*

We denote by $\mathcal{P}_r^{\mathcal{D}}$ the set of all $\mathcal{D}$-weighted partitions of the index set $\{1, 2, \cdots, r\}$. The cardinality of $(S_a, \vec{d_a})$ is assumed to be the cardinality of the set S_a.

The idea for this notation is that the marked points indexed by S_a lie in the same $\mathcal{D}$-orbit and they differ from the first point by the action $1, d_a^{(1)}, d_a^{(2)}, \cdots, d_a^{(|S_a|-1)}$. Using this notation of $\mathcal{D}$-weighted partitions, we start by writing

$$\mathcal{C}_g^2(l) = \mathcal{M}_{g,2}(l) \coprod (\coprod_{d \in \mathcal{D}} \mathcal{M}_{g,1}(l)) = \coprod_{\widetilde{P} \in \mathcal{P}_2^{\mathcal{D}}} \mathcal{M}_{g,|\widetilde{P}|}(l),$$

where $\mathcal{M}_{g,2}(l)$ is the subvariety whose marked points x_1 and x_2 lie in different $\mathcal{D}$-orbits, and for each $d \in \mathcal{D}$, the subvariety $\mathcal{M}_{g,1}(l)$ is embedded in $\mathcal{C}_g^2(l)$ such that $x_2 = d \cdot x_1$. More generally, we have

Lemma 4.5. *As a set,*

$$\mathcal{C}_g^{r+1}(l) = \coprod_{\widetilde{P} \in \mathcal{P}_{r+1}^{\mathcal{D}}} \mathcal{M}_{g,|\widetilde{P}|}(l),$$

where $|\widetilde{P}| = \nu$ for $\widetilde{P} = \{(S_1, \vec{d_1}), \cdots, (S_\nu, \vec{d_\nu})\} \in \mathcal{P}_{r+1}^{\mathcal{D}}$.

The proof is clear since all possible positions of marked points are included. This decomposition allows us to apply the Thom-Gysin sequence (Proposition 2.9) multiple times to get the rational cohomology of $\mathcal{C}_g^{r+1}(l)$ in terms of $H^*(\mathcal{M}_{g,|\widetilde{P}|}(l); \mathbb{Q})$, and we know $H^*(\mathcal{M}_{g,|\widetilde{P}|}(l); \mathbb{Q})$ by Theorem 4.1. To write the cohomology as a direct sum of polynomials, we need the following cohomology classes.

1. For $1 \le i \le r+1$, let u_i denote the pullback of $u_i \in H^2(\mathcal{M}_{g,1}; \mathbb{Q})$ through the composition map
$$f_i : \mathcal{C}_g^{r+1}(l) \to \mathcal{M}_{g,1}(l) \to \mathcal{M}_{g,1},$$
where the first map forgets all but the i-th marked point.
2. For a $\mathcal{D}$-weighted partition $\widetilde{P}$ and $I = (S_a, \vec{d_a}) \in \widetilde{P}$ with $|I| = |S_a| \ge 2$, let $a_I \in H^{2|I|-2}(\mathcal{C}_g^{r+1}(l); \mathbb{Q})$ be the Poincaré dual of the subvariety $\mathcal{C}_g^{r+1}(l)[I]$ of $\mathcal{C}_g^{r+1}(l)$ whose marked points indexed by S_a lie in the same $\mathcal{D}$-orbit and differ by actions $1, d_a^{(1)}, \cdots, d_a^{(|S_a|-1)}$.

These classes u_i, a_I satisfy the following relations due to Lemma 2.4 of [14], just as in Theorem 3.2:

$$u_i a_I = u_j a_I \text{ if } i, j \in I,$$
$$a_I a_J = u_i^{|I \cap J|-1} a_{I \cup J} \text{ if } i \in I \cap J \ne \emptyset.$$

In the context of $\mathcal{D}$-weighted partitions, we need to tell when $i \in I$, and what $I \cap J$ and $I \cup J$ are, by checking what subvariety the intersection $\mathcal{C}_g^{r+1}(l)[I] \cap \mathcal{C}_g^{r+1}(l)[J]$ is. Let's look at some examples.

1. If $I = (\{1, 2\}, d)$, then $1, 2 \in I$. This is because the $f_i^*(\theta)$ and $f_j^*(\theta)$ (where θ is the relative tangent sheaf of $\mathcal{M}_{g,1} \to \mathcal{M}_g$) have isomorphic restrictions to $\mathcal{C}_g^{r+1}(l)[I]$. Generally, we see $i \in I = (S_a, \vec{d_a})$ if $i \in S_a$.

(2) If $I = (\{1,2\}, d)$ and $J = (\{1,3\}, d')$, then $I \cup J = (\{1,2,3\}, (d, d'))$, because

$$\mathcal{C}_g^{r+1}(l)[I] \cap \mathcal{C}_g^{r+1}(l)[J] = \mathcal{C}_g^{r+1}(l)[I \cup J].$$

The intersection is $I \cap J = (\{1\})$. The second relation above applies:

$$a_{(\{1,2\},d)} \cdot a_{(\{1,3\},d')} = 1 \cdot a_{(\{1,2,3\},(d,d'))}.$$

(3) If $I = (\{1,2\}, d)$ and $J = (\{1,2\}, d')$ with $d \neq d'$, then we can see

$$\mathcal{C}_g^{r+1}(l)[I] \cap \mathcal{C}_g^{r+1}(l)[J] = \emptyset,$$

since you can not simultaneously require that the second marked point is $d \cdot x_1$ and $d' \cdot x_1$. In this case we should have $I \cap J = \emptyset$ and we do not have the second relation above for a_I and a_J.

We will later use these relations when computing terms in spectral sequences. First, we define a graded algebra $A^\bullet_{r+1}(l)$ to be the $\mathbb{Q}[u_i : 1 \leq i \leq r+1]$-algebra generated by all a_I and 1, subject to the two relations:

$$u_i a_I = u_j a_I \text{ if } i, j \in I,$$

$$a_I a_J = u_i^{|I \cap J|-1} a_{I \cup J} \text{ if } i \in I \cap J \neq \emptyset.$$

The second relation tells us $A^\bullet_{r+1}(l)$ is the $\mathbb{Q}[u_i : 1 \leq i \leq r+1]$-module generated by the elements

$$a_{\widetilde{P}} = \prod_{I \in \widetilde{P}, |I| \geq 2} a_I$$

for as $\widetilde{P}$ ranges over all $\mathcal{D}$-weighted partitions of the index set $[r]$, with the convention that $a_{\widetilde{P}} = 1$ if $\widetilde{P}$ consists of only singletons. Here the element a_I is equal to the element a_P where P is the union of I and singletons. For each $I = (S_a, \vec{d_a})$ with $|I| = |S_a| \geq 2$, let u_I be a formal symbol. The first relation tells us the $\mathbb{Q}[u_i : 1 \leq i \leq r+1]$-module generated by $a_{\widetilde{P}}$ is isomorphic to $\mathbb{Q}[u_I : I \in \widetilde{P}] a_P$ taking $u_i a_{\widetilde{P}}$ to $u_I a_{\widetilde{P}}$, where $I = (S_a, \vec{d_a}) \in \widetilde{P}$ and S_a contains i. Thus we have the isomorphism

$$A^\bullet_{r+1}(l) \cong \bigoplus_{\widetilde{P} \in \mathcal{P}^{\mathcal{D}}_{r+1}} \mathbb{Q}[u_I : I \in \widetilde{P}] a_{\widetilde{P}}.$$

Assuming u_i is of degree 2 and a_I is of degree $2|I| - 2$, the grading of $A^\bullet_{r+1}(l)$ is

$$A^\bullet_{r+1}(l) = \bigoplus_{m=0}^{\infty} A^{2m}_{r+1}(l)$$

where $A^{2m}_{r+1}(l)$ is the degree $2m$ part. Then $A^\bullet_{r+1}(l)$ has a trivial mixed Hodge structure where $A^{2m}_{r+1}(l)$ has Hodge type (m, m). The rational cohomolgy of $\mathcal{C}_g^{r+1}(l)$ also carries a canonical polarizable mixed Hodge structure by Theorem 2.14. Now we state the rational cohomolgy of $\mathcal{C}_g^{r+1}(l)$ in the following theorem.

Theorem 4.6. *For $r \geq 1$, if $g \geq 2k^2+7k+2$, we have the following isomorphism of mixed Hodge structures:*

$$\begin{aligned} H^k(\mathcal{C}_g^{r+1}(l);\mathbb{Q}) &\cong \bigoplus_{\widetilde{P}\in\mathcal{P}_{r+1}^{\mathcal{D}}} H^{k-2(r+1-|\widetilde{P}|)}(\mathcal{M}_{g,|\widetilde{P}|}(l);\mathbb{Q}) \\ &\cong \bigoplus_{\widetilde{P}\in\mathcal{P}_{r+1}^{\mathcal{D}}} H^{k-2(r+1-|\widetilde{P}|)}(\mathrm{Mod}_{g,|\widetilde{P}|}(l);\mathbb{Q}) \\ &\cong \bigoplus_{\widetilde{P}\in\mathcal{P}_{r+1}^{\mathcal{D}}} H^{k-2(r+1-|\widetilde{P}|)}(\mathrm{Mod}_{g,|\widetilde{P}|};\mathbb{Q}) \\ &\cong \left(\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes \bigoplus_{\widetilde{P}\in\mathcal{P}_{r+1}^{\mathcal{D}}} \mathbb{Q}[u_I : I\in\widetilde{P}]a_{\widetilde{P}}\right)_{(k)}. \end{aligned}$$

Proof. Let's first see the case $r=1$. Recall we can write

$$\mathcal{C}_g^2(l) = \mathcal{M}_{g,2}(l)\coprod(\coprod_{d\in\mathcal{D}}\mathcal{M}_{g,1}(l)) = \coprod_{\widetilde{P}\in\mathcal{P}_2^{\mathcal{D}}}\mathcal{M}_{g,|\widetilde{P}|}(l).$$

The Thom-Gysin sequence for $\coprod_{\mathcal{D}}\mathcal{M}_{g,1}(l)\subset\mathcal{C}_g^2(l)$ gives us a long exact sequence:

$$\cdot\to H^{k-2}(\coprod_{\mathcal{D}}\mathcal{M}_{g,1}(l);\mathbb{Q})\to H^k(\mathcal{C}_g^2(l);\mathbb{Q})\to H^k(\mathcal{M}_{g,2}(l);\mathbb{Q})\to H^{k-1}(\coprod_{\mathcal{D}}\mathcal{M}_{g,1}(l);\mathbb{Q})\to\cdot.$$

Here $H^k(\mathcal{C}_g^2(l);\mathbb{Q})\to H^k(\mathcal{M}_{g,2}(l);\mathbb{Q})$ is surjective when $g\geq 2k^2+7k+2$. This is because $H^k(\mathcal{M}_{g,2}(l);\mathbb{Q})\to H^{k-1}(\coprod_{\mathcal{D}}\mathcal{M}_{g,1}(l);\mathbb{Q})$ is the 0-map since $H^k(\mathcal{M}_{g,p}(l);\mathbb{Q})\cong H^k(\mathrm{Mod}_{g,p}(l);\mathbb{Q})$ is 0 when k is odd by Theorem 4.1. Thus we have:

$$\begin{aligned} H^k(\mathcal{C}_g^2(l);\mathbb{Q}) &\cong H^k(\mathcal{M}_{g,2}(l);\mathbb{Q})\oplus H^{k-2}(\coprod_{\mathcal{D}}\mathcal{M}_{g,1}(l);\mathbb{Q}) \\ &\cong H^k(\mathcal{M}_{g,2}(l);\mathbb{Q})\oplus\left(\bigoplus_{\mathcal{D}}H^{k-2}(\mathcal{M}_{g,1}(l);\mathbb{Q})\right). \end{aligned}$$

Since $\mathcal{M}_{g,p}(l)$ has the same rational cohomology as $\mathrm{Mod}_{g,p}(l)$, we have

$$H^k(\mathcal{M}_{g,2}(l);\mathbb{Q})\oplus\left(\bigoplus_{\mathcal{D}}H^{k-2}(\mathcal{M}_{g,1}(l);\mathbb{Q})\right)\cong H^k(\mathrm{Mod}_{g,2}(l);\mathbb{Q})\oplus\left(\bigoplus_{\mathcal{D}}H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathbb{Q})\right).$$

Moreover, by Theorem 4.1, if $g\geq 2k^2+7k+2$, we have

$$H^k(\mathrm{Mod}_{g,2}(l);\mathbb{Q})\oplus\left(\bigoplus_{\mathcal{D}}H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathbb{Q})\right)\cong H^k(\mathrm{Mod}_{g,2};\mathbb{Q})\oplus\left(\bigoplus_{\mathcal{D}}H^{k-2}(\mathrm{Mod}_{g,1};\mathbb{Q})\right).$$

Thus we have, when $g\geq 2k^2+7k+2$,

$$H^k(\mathcal{C}_g^2(l);\mathbb{Q})\cong\left(\begin{array}{c}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_1,u_2] \\ \oplus\left(\bigoplus_{d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},d)}]a_{(\{1,2\},d)}\right)\end{array}\right)_{(k)},$$

since the Gysin map $H^{k-2}(\coprod_{\mathcal{D}}\mathcal{M}_{g,1}(l);\mathbb{Q})\to H^k(\mathcal{C}_g^2(l);\mathbb{Q})$ is multiplication by the Poincaré dual $a_{(\{1,2\},d)}$ in each component indexed by $d\in\mathcal{D}$. The maps in the Thom-Gysin sequence above are morphisms of mixed Hodge structures, therefore the isomorphism is also an isomorphism of mixed Hodge structures. We prove for general r by applying the Thom-Gysin sequence multiple times, and the reasons for those sequences to split are similar to the case $r = 1$, so we omit it. □

Now if we use the simplest fibration in Proposition 4.3 for $p = 1$, we can repeat the method of Proposition 3.4 to compute $H^k(\mathrm{Mod}_{g,1}(l); \mathfrak{H}_{g,1}(l; \mathbb{Q}))$. The only difference is that we need to first compactify the fiber $\Sigma_{g,1}[\mathcal{D}]$ so that Deligne's Theorem 2.5 applies. The compactified fiber will be $\Sigma_g[\mathcal{D}]$, which is the regular $\mathcal{D}$-cover of Σ_g. We first compute the cohomology of $\mathrm{Mod}_{g,1}(l)$ with coefficients $H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})$:

Proposition 4.7. *If $g \geq 2k^2 + 7k + 2$, we have*

$$H^{k-1}(\mathcal{M}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})) \cong \begin{pmatrix} \mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes u_2\, \mathbb{Q}[u_1, u_2] \\ \oplus \left(\bigoplus_{1 \neq d \in \mathcal{D}} \mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes \mathbb{Q}[u_{(\{1,2\},d)}] a_{(\{1,2\},d)} \right) \end{pmatrix}_{(k)}.$$

Proof. The fibration

$$\begin{array}{ccc} \Sigma_{g,1}[\mathcal{D}] \longrightarrow \mathcal{M}_{g,2}(l) & & (C, x_1, x_2) \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,1}(l) & & (C, x_1) \end{array}$$

can be compactified by filling in all $|\mathcal{D}|$ punctures of $\Sigma_{g,1}[\mathcal{D}]$, to get:

$$\begin{array}{ccc} \Sigma_{g}[\mathcal{D}] \longrightarrow \mathcal{C}_g^2(l) & & (C, x_1, x_2) \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,1}(l) & & (C, x_1) \end{array}.$$

In the first fibration, x_1 and x_2 are in different $\mathcal{D}$-orbits, but in the second fibration, we allow x_1 and x_2 to be in the same $\mathcal{D}$-orbit. Remember $\mathcal{C}_g^2(l) = \mathcal{M}_{g,2}(l) \coprod \left(\coprod_{\mathcal{D}} \mathcal{M}_{g,1}(l) \right)$ as a set. This coincides with the fact that in each fiber, the complement $\Sigma_g[\mathcal{D}] \setminus \Sigma_{g,1}[\mathcal{D}]$ is $|\mathcal{D}|$ discrete points.

Consider the associated Leray spectral sequence with $\mathbb{Q}$-coefficients:

$$E_2^{p,q} = H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}]; \mathbb{Q})) \Rightarrow H^{p+q}(\mathcal{C}_{g,2}(l); \mathbb{Q}).$$

Since the fiber $\Sigma_g[\mathcal{D}]$ is a projective variety, Deligne's Theorem 2.5 applies, so this spectral sequence degenerates at page 2. Therefore we have:

$$H^k(\mathcal{C}_g^2(l); \mathbb{Q}) \cong \bigoplus_{p+q=k} H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}]; \mathbb{Q})).$$

Since the Leray filtration respects the mixed Hodge structure of $H^*(\mathcal{C}_g^2(l); \mathbb{Q})$, the E_2 page terms $E_2^{p,q} = H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}]; \mathbb{Q}))$ inherit mixed Hodge structures.

One the one hand, by Theorem 4.6, when $g \geq 2k^2 + 7k + 2$, we have the following isomorphism of mixed Hodge structures

$$H^k(\mathcal{C}_g^2(l); \mathbb{Q}) \cong \begin{pmatrix} \mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes \mathbb{Q}[u_1, u_2]) \\ \oplus \left(\bigoplus_{d \in \mathcal{D}} \mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes \mathbb{Q}[u_{(\{1,2\},d)}] a_{(\{1,2\},d)} \right) \end{pmatrix}_{(k)}.$$

One the other hand, we have

$$\begin{aligned}&\bigoplus_{p+q=k} H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}];\mathbb{Q}))\\ =&H^k(\mathcal{M}_{g,1}(l); H^0(\Sigma_g[\mathcal{D}];\mathbb{Q})) \oplus H^{k-1}(\mathcal{M}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})) \oplus H^{k-2}(\mathcal{M}_{g,1}(l); H^2(\Sigma_g[\mathcal{D}];\mathbb{Q}))\end{aligned}.$$

In a way almost the same as the proof of Theorem 3.5, we know

$$\begin{aligned}H^k(\mathcal{M}_{g,1}(l); H^0(\Sigma_g[\mathcal{D}];\mathbb{Q})) =&\mathrm{Image}(H^k(\mathrm{Mod}_{g,1}(l);\mathbb{Q}) \to H^k(\mathcal{C}_g^2(l);\mathbb{Q}))\\ \cong&(\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_1])_{(k)};\end{aligned}$$

for any $\omega \in H^{k-2}(\mathcal{M}_{g,1}(l); H^2(\Sigma_g[\mathcal{D}];\mathbb{Q}))$, letting $\tilde{\omega} \in H^k(\mathcal{C}_g^2(l);\mathbb{Q})$ be its preimage via

$$H^k(\mathcal{C}_g^2(l);\mathbb{Q}) \to E_\infty^{k-2,2} \to E_2^{k-2,2} = H^{k-2}(\mathcal{M}_{g,1}(l); H^2(\Sigma_g[\mathcal{D}];\mathbb{Q})),$$

noticing the Poincaré dual of $\mathcal{M}_{g,1}(l)$ in $\mathcal{C}_g^2(l)$ is $a_{(\{1,2\},1\in\mathcal{D})}$, we have

$$\int_{\mathcal{C}_g^2(l)} \tilde{\omega}\wedge a_{(\{1,2\},1\in\mathcal{D})} = \int_{\mathcal{M}_{g,1}(l)} \omega.$$

Thus we have

$$H^{k-2}(\mathcal{M}_{g,1}(l); H^2(\Sigma_g[\mathcal{D}];\mathbb{Q})) \cong \left(\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},1\in\mathcal{D})}]a_{(\{1,2\},1\in\mathcal{D})}\right)_{(k)},$$

if $g \geq 2k^2+7k+2$. Since all Hodge structures here are polarized, so semi-simple, we can eliminating above two terms to get

$$H^{k-1}(\mathcal{M}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})) \cong \left(\begin{matrix}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_1,u_2]\\ \oplus\left(\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},d)}]a_{(\{1,2\},d)}\right)\end{matrix}\right)_{(k)},$$

if $g \geq 2k^2+7k+2$. □

Next we make use of this result to get the twisted cohomology with the coefficients $\mathfrak{H}_{g,1}(l;\mathbb{Q}) = H^1(\Sigma_{g,1}[\mathcal{D}];\mathbb{Q})$.

Proposition 4.8. *When $g \geq 2k^2+7k+2$, we have*

$$H^{k-1}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})) \cong \mathbb{Q}[c_1,\kappa_1,\kappa_2,\cdots]\otimes u_1\,\mathbb{Q}[u_1].$$

Proof. The map $\Sigma_{g,1}[\mathcal{D}] \to \Sigma_g[\mathcal{D}]$ induces a short exact sequence:

$$0 \to H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}) \to H^1(\Sigma_{g,1}[\mathcal{D}];\mathbb{Q}) \to \mathbb{Q}^{|\mathcal{D}|-1} \to 0.$$

This is a short exact sequence of $\mathrm{Mod}_{g,1}(l)$-modules and $\mathbb{Q}^{|\mathcal{D}|-1}$ is a trivial $\mathrm{Mod}_{g,1}(l)$-module, since $\mathrm{Mod}_{g,1}(l)$ preserves the punctures of $\Sigma_{g,1}[\mathcal{D}]$. Therefore it induces a long exact sequence in group cohomology:

$$\begin{aligned}\cdots \to H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathbb{Q}^{|\mathcal{D}|-1}) \to H^{k-1}(\mathrm{Mod}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})) \to\\ \to H^{k-1}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})) \to H^{k-1}(\mathrm{Mod}_{g,1}(l);\mathbb{Q}^{|\mathcal{D}|-1}) \to \cdots\end{aligned}.$$

Let $g \geq 2k^2+7k+2$. Denote by ϕ_{k-2} the map

$$H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathbb{Q}^{|\mathcal{D}|-1}) \to H^{k-1}(\mathrm{Mod}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})).$$

Then the above long exact sequence gives us a short exact squence

$$1 \to \mathrm{Coker}(\phi_{k-2}) \to H^{k-1}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})) \to \mathrm{Ker}(\phi_{k-1}) \to 1.$$

Observe that $H^k(\mathcal{M}_{g,1}(l);\mathbb{Q}^{|\mathcal{D}|-1})$ is zero when k is odd by Putman's Theorem 4.2, and $H^k(\mathrm{Mod}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))$ is zero when k is even, as we computed in Theorem 4.7. Thus the map ϕ_{k-2} is the 0-map when k is odd. When k is even, since $\mathbb{Q}^{|\mathcal{D}|-1}$ is generated by the loops around the 2rd to $|\mathcal{D}|$-th punctures of $\Sigma_{g,1}[\mathcal{D}]$, assuming that the first puncture is the marked point x_1 in $\mathcal{M}_{g,1}(l)$, we can rewrite the map ϕ_{k-2} as:

$$\begin{array}{c}\left(\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},d)}]\right)_{(k-2)}\\ \Big\downarrow{\scriptstyle\cdot a_{(\{1,2\},d)}}\\ (\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_1,u_2])_{(k)}\\ \oplus\left(\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},d)}]a_{(\{1,2\},d)}\right)_{(k)}\end{array},$$

where the subscripts indicate the degrees. Thus after replacing u_1 with c_1 and replacing u_2 with u_1, we have, when $g\geq 2k^2+7k+2$,

$$H^{k-1}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})\cong\mathbb{Q}[c_1,\kappa_1,\kappa_2,\cdots]\otimes u_1\,\mathbb{Q}[u_1].\qquad\square$$

Remark: This result together with the null-level case Proposition 3.4 verifies Putman's theorem 4.2.

Next, let's see the case with the coefficients being second tensor powers.

Proposition 4.9. *We have*

$$H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes 2})\cong\left(\mathbb{Q}[c_1,\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{\widetilde{P}\in\mathcal{P}_2^{\mathcal{D}}}(\prod_{\{i\}\in\widetilde{P}}u_i)\,\mathbb{Q}[u_I:I\in\widetilde{P}]a_{\widetilde{P}}\right)\right)_{(k)}$$

if $g\geq 2k^2+7k+2$, *where* $a_{\widetilde{P}}=\prod_{I\in\widetilde{P},|I|\geq 2}a_I$.

Proof. Taking $p=3$ in the Theorem 4.3, we get a fibration:

$$\begin{array}{ccccc}Conf_2^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])&\longrightarrow&\mathcal{M}_{g,3}(l)&&(C,x_1,x_2,x_3)\\&&\downarrow&&\downarrow\\&&\mathcal{M}_{g,1}(l)&&(C,x_1)\end{array}$$

We can compactify it to get a larger fibration with compact fiber:

$$\begin{array}{ccccc}\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}]&\longrightarrow&\mathcal{C}_{g,3}(l)&&(C,x_1,x_2,x_3)\ .\\&&\downarrow&&\downarrow\\&&\mathcal{M}_{g,1}(l)&&(C,x_1)\end{array}\tag{5}$$

The reason can be seen as follows.

First, the fiber $Conf_2^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])=\{(x_2,x_3)|x_2,x_3\in\Sigma_{g,1}[\mathcal{D}],\forall d\in\mathcal{D},x_2\neq d\cdot x_3\}$ is included in $\Sigma_{g,1}[\mathcal{D}]\times\Sigma_{g,1}[\mathcal{D}]$, and the complement is:

$$\begin{aligned}&\Sigma_{g,1}[\mathcal{D}]\times\Sigma_{g,1}[\mathcal{D}]\setminus Conf_2^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])\\=&\{(x_2,x_3)|x_2,x_3\in\Sigma_{g,1}[\mathcal{D}],\exists d\in\mathcal{D},x_2=d\cdot x_3\}\\=&\coprod_{\mathcal{D}}\Sigma_{g,1}[\mathcal{D}]\end{aligned},$$

which induces a new fibration with fiber $\Sigma_{g,1}[\mathcal{D}] \times \Sigma_{g,1}[\mathcal{D}]$

$$\begin{tikzcd} \Sigma_{g,1}[\mathcal{D}] \times \Sigma_{g,1}[\mathcal{D}] \arrow[r] & \mathcal{M}_{g,3}(l) \coprod \left(\coprod_{\mathcal{D}} \mathcal{M}_{g,2}(l) \right) \arrow[d] \\ & \mathcal{M}_{g,1}(l) \end{tikzcd}.$$

Next, the new fiber has the compactification $\Sigma_g[\mathcal{D}] \times \Sigma_g[\mathcal{D}]$, with complement:

$$\begin{aligned} &\Sigma_g[\mathcal{D}] \times \Sigma_g[\mathcal{D}] \setminus (\Sigma_{g,1}[\mathcal{D}] \times \Sigma_{g,1}[\mathcal{D}]) \\ =&(\mathcal{D} \times \Sigma_{g,1}[\mathcal{D}]) \coprod (\Sigma_{g,1}[\mathcal{D}] \times \mathcal{D}) \coprod (\mathcal{D} \times \mathcal{D}) \end{aligned}.$$

Thus we can derive a new fibration over $\mathcal{M}_{g,1}(l)$ whose fiber is $\Sigma_g[\mathcal{D}] \times \Sigma_g[\mathcal{D}]$:

$$\begin{tikzcd} \Sigma_g[\mathcal{D}] \times \Sigma_g[\mathcal{D}] \arrow[r] & \mathcal{M}_{g,3}(l) \coprod \left(\coprod_{3|\mathcal{D}|} \mathcal{M}_{g,2}(l) \right) \coprod \left(\coprod_{\mathcal{D}\times\mathcal{D}} \mathcal{M}_{g,1}(l) \right) \arrow[d] \\ & \mathcal{M}_{g,1}(l) \end{tikzcd}.$$

This fibration is exactly the same as (5), since

$$\mathcal{C}_g^3(l) = \mathcal{M}_{g,3}(l) \coprod \left(\coprod_{3|\mathcal{D}|} \mathcal{M}_{g,2}(l) \right) \coprod \left(\coprod_{\mathcal{D}\times\mathcal{D}} \mathcal{M}_{g,1}(l) \right).$$

Here the 3 copies of $\coprod_{\mathcal{D}} \mathcal{M}_{g,2}(l)$ have different meaning in terms of three marked points in $\Sigma_g[\mathcal{D}]$. Denote by x_1 the marked point for the base $\mathcal{M}_{g,1}(l)$. Recall one copy of $\coprod_{\mathcal{D}} \mathcal{M}_{g,2}(l)$ comes up when we add $\Sigma_{g,1}[\mathcal{D}] \times \Sigma_{g,1}[\mathcal{D}] \setminus Conf_2^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])$ to the fiber. Therefore, for each $d \in \mathcal{D}$, the corresponding three marked points are

$$\{(x_1, x_2, x_3) \in (\Sigma_g[\mathcal{D}])^{\times 3} | x_3 = d \cdot x_2; \forall d' \in \mathcal{D}, x_2 \neq d' \cdot x_1\}.$$

The other two copies of $\coprod_{\mathcal{D}} \mathcal{M}_{g,2}(l)$ are from adding to the fiber $\Sigma_g[\mathcal{D}] \times \Sigma_g[\mathcal{D}] \setminus (\Sigma_{g,1}[\mathcal{D}] \times \Sigma_{g,1}[\mathcal{D}])$, whose three marked points in $\Sigma_g[\mathcal{D}]$ are:

$$\text{for } d \in \mathcal{D}, \{(x_1, x_2, x_3) \in (\Sigma_g[\mathcal{D}])^{\times 3} | x_2 = d \cdot x_1; \forall d' \in \mathcal{D}, x_3 \neq d' \cdot x_1\};$$

$$\text{for } d \in \mathcal{D}, \{(x_1, x_2, x_3) \in (\Sigma_g[\mathcal{D}])^{\times 3} | x_3 = d \cdot x_1; \forall d' \in \mathcal{D}, x_2 \neq d' \cdot x_1\}.$$

The $\mathcal{D} \times \mathcal{D}$ copies of $\mathcal{M}_{g,1}(l)$ also comes from adding to the fiber $\Sigma_g[\mathcal{D}] \times \Sigma_g[\mathcal{D}] \setminus (\Sigma_{g,1}[\mathcal{D}] \times \Sigma_{g,1}[\mathcal{D}])$. Thus the corresponding three marked points for each $d \in \mathcal{D}$ and $d' \in \mathcal{D}$ are

$$\{(x_1, x_2, x_3) \in (\Sigma_g[\mathcal{D}])^{\times 3} | x_2 = d \cdot x_1, x_3 = d' \cdot x_1\}.$$

Now, we can apply Deligne's degeneration theorem (Theorem 2.5) to the fibration (5), and get the Leray spectral sequence:

$$E_2^{p,q} = H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}] \times \Sigma_g[\mathcal{D}]; \mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{C}_g^3(l); \mathbb{Q}),$$

which degenerates at page 2, so:

$$H^k(\mathcal{C}_g^3(l); \mathbb{Q}) \cong \bigoplus_{p+q=k} H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}] \times \Sigma_g[\mathcal{D}]; \mathbb{Q})).$$

Since the Leray filtration respects the mixed Hodge structure of $H^*(\mathcal{C}_g^3(l);\mathbb{Q})$, the E_2 page terms inherit mixed Hodge structures.

On the one hand, by Theorem 4.6, we have the following isomorphism of mixed Hodge structures, when $g \geq 2k^2+7k+2$,

$$\begin{aligned} H^k(\mathcal{C}_g^3(l);\mathbb{Q}) \cong & (\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_1,u_2,u_3])_{(k)} \\ & \oplus \left(\bigoplus_{d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_1,u_{(\{2,3\},d)}]a_{(\{2,3\},d)}\right)_{(k)} \\ & \oplus \left(\bigoplus_{d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},d)},u_3]a_{(\{1,2\},d)}\right)_{(k)} \\ & \oplus \left(\bigoplus_{d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,3\},d)},u_2]a_{(\{1,3\},d)}\right)_{(k)} \\ & \oplus \left(\bigoplus_{d,d'\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2,3\},(d,d'))}]a_{(\{1,2,3\},(d,d'))}\right)_{(k)} \end{aligned}.$$

On the other hand, using the Künneth formula, the E_2-page terms $H^p(\mathcal{M}_{g,1}(l);H^q(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q}))$ with $p+q=k$, have the following possible cases (for $k\geq 4$):

(1) When $q=0$, when $g\geq 2k^2+7k+2$, we have

$$\begin{aligned} H^k(\mathcal{M}_{g,1}(l);H^0(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q})) &= H^k(\mathcal{M}_{g,1}(l);\mathbb{Q}) \\ &\cong H^k(\mathrm{Mod}_{g,1}(l);\mathbb{Q}) \\ &\cong (\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_1])_{(k)} \end{aligned}$$

(2) When $q=1$, when $g\geq 2k^2+7k+2$ by Proposition 4.7, we have

$$\begin{aligned} H^{k-1}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q})) =& H^{k-1}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))^{\oplus 2} \\ \cong & \begin{pmatrix} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_1,u_2] \\ \oplus\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},d)}]a_{(\{1,2\},d)} \\ \oplus\,\mathbb{Q}[\kappa_1,\kappa_3,\cdots]\otimes u_2\,\mathbb{Q}[u_1,u_3] \\ \oplus\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,3\},d)}]a_{(\{1,3\},d)} \end{pmatrix}_{(k)} \end{aligned}.$$

(3) When $q=2$, when $g\geq 2k^2+7k+2$ by the proof of Proposition 4.7, we have

$$\begin{aligned} H^{k-2}(\mathcal{M}_{g,1}(l);H^2(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q})) =& H^{k-2}(\mathcal{M}_{g,1};H^2(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\oplus 2} \\ & \oplus H^{k-2}(\mathcal{M}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2}) \\ \cong & \begin{pmatrix} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},1)}]a_{(\{1,2\},1)} \\ \oplus\,\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,3\},1)}]a_{(\{1,3\},1)} \end{pmatrix}_{(k)} \\ & \oplus H^{k-2}(\mathcal{M}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2}) \end{aligned}.$$

(4) When $q=3$, recall the following relations in Theorem 4.6,

$$u_i a_I = u_j a_I \text{ if } i,j\in I,$$

$$a_I a_J = u_i^{|I\cap J|-1} a_{I\cup J} \text{ if } i\in I\cap J\neq\emptyset.$$

When $g \geq 2k^2+7k+2$, by the proof of Proposition 4.7 and the above relations, we have

$$\begin{aligned}
H^{k-3}(\mathcal{M}_{g,1}(l);H^3(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q})) &= H^{k-3}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\otimes H^2(\Sigma_g[\mathcal{D};\mathbb{Q}))^{\oplus 2}\\
&\cong \Big(H^{k-3}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))\otimes H^0(\mathcal{M}_{g,1}(l);H^2(\Sigma_g[\mathcal{D}];\mathbb{Q}))\Big)^{\oplus 2}\\
&\cong \begin{pmatrix} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_1,u_2]\cdot a_{(\{1,3\},1)}\\ \oplus\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},d)}]a_{(\{1,2\},d)}\cdot a_{(\{1,3\},1)}\\ \oplus\,\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_3\,\mathbb{Q}[u_1,u_3]\cdot a_{(\{1,2\},1)}\\ \oplus\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,3\},d)}]a_{(\{1,3\},d)}\cdot a_{(\{1,2\},1)}\end{pmatrix}_{(k)}\\
&\cong \begin{pmatrix} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_{(\{1,3\},1)},u_2]a_{(\{1,3\},1)}\\ \oplus\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2,3\},(d,1))}]a_{(\{1,2,3\},(d,1))}\\ \oplus\,\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_3\,\mathbb{Q}[u_{(\{1,2\},1)},u_3]a_{(\{1,2\},1)}\\ \oplus\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2,3\},(1,d))}]a_{(\{1,2,3\},(1,d))}\end{pmatrix}_{(k)}.
\end{aligned}$$

(5) When $q=3$, when $g\geq 2k^2+7k+2$, by the proof of 4.7 and the relations we use in the above step (4), we have

$$\begin{aligned}
H^{k-4}(\mathcal{M}_{g,1}(l);H^4(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q})) &= H^{k-4}(\mathcal{M}_{g,1}(l);H^2(\Sigma_g[\mathcal{D}];\mathbb{Q})\otimes H^2(\Sigma_g[\mathcal{D}];\mathbb{Q}))\\
&\cong H^{k-4}(\mathcal{M}_{g,1}(l);H^2(\Sigma_g[\mathcal{D}];\mathbb{Q}))\otimes H^0(\mathcal{M}_{g,1}(l);H^2(\Sigma_g[\mathcal{D}];\mathbb{Q}))\\
&\cong \big(\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2\},1)}]a_{(\{1,2\},1)}\cdot a_{(\{1,3\},1)}\big)_{(k)}\\
&\cong \big(\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2,3\},(1,1))}]a_{(\{1,2,3\},(1,1))}\big)_{(k)}.
\end{aligned}$$

All the above isomorphisms are isomorphisms of mixed Hodge structures, and they are all polarized. For all above terms, only $H^{k-2}(\mathrm{Mod}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2})$ is unknown to us. Therefore by the semi-simplicity of polarized mixed Hodge structures, we can carefully exclude other components to find it. We arrange $H^p(\mathcal{M}_{g,1}(l);H^q(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q}))$ with $p+q=k$ in the Table 2 below, together with the rational cohomology of $\mathcal{C}_g^3(l)$, in terms of $\mathcal{D}$-partitions of the index $\{1,2,3\}$, when $g\geq 2k^2+7k+2$. (For the polynomials in the table, we actually mean the degree k parts of them. As a shorthand, we denote $K=\mathbb{Q}[\kappa_1,\kappa_2,\cdots]$.)

From the table, we can get, if $g\geq 2k^2+7k+2$,

$$\begin{aligned}
H^{k-2}(\mathrm{Mod}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2}) &\cong (\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2u_3\,\mathbb{Q}[u_1,u_2,u_3])_{(k)}\\
&\oplus\Big(\bigoplus_{d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_1,u_{(\{2,3\},d)}]a_{(\{2,3\},d)}\Big)_{(k)}\\
&\oplus\Big(\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_3\,\mathbb{Q}[u_{(\{1,2\},d)},u_3]a_{(\{1,2\},d)}\Big)_{(k)}\\
&\oplus\Big(\bigoplus_{1\neq d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_{(\{1,3\},d)},u_2]a_{(\{1,3\},d)}\Big)_{(k)}\\
&\oplus\Big(\bigoplus_{1\neq d,1\neq d'\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_{(\{1,2,3\},(d,d'))}]a_{(\{1,2,3\},(d,d'))}\Big)_{(k)}.
\end{aligned}$$

TABLE 2. Rational cohomology of $\mathcal{C}_g^3(l)$ written in two ways

$\widetilde{P} \in \mathcal{P}_3^{\mathcal{D}}$	$H^k(\mathcal{C}_g^3(l);\mathbb{Q})$	$\bigoplus_{p+q=k} H^p(\mathcal{M}_{g,1}(l);H^q(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q}))$
$\{1\},\{2\},\{3\}$	$K\otimes\mathbb{Q}[u_1,u_2,u_3]$	$K\otimes\mathbb{Q}[u_1]$ $K\otimes u_2\,\mathbb{Q}[u_1,u_2]$ $K\otimes u_3\,\mathbb{Q}[u_1,u_3]$ $?\subset H^{k-2}(\mathrm{Mod}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2})$
$\{1\},(\{2,3\},d)$	$K\otimes\mathbb{Q}[u_1,u_{(\{2,3\},d)}]a_{(\{2,3\},d)}$	$?\subset H^{k-2}(\mathrm{Mod}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2})$
$(\{1,2\},d),\{3\}$	$K\otimes\mathbb{Q}[u_{(\{1,2\},d)},u_3]a_{(\{1,2\},d)}$	$\bigoplus_{d\neq 1} K\otimes\mathbb{Q}[u_{(\{1,2\},d)}]a_{(\{1,2\},d)}$ $K\otimes\mathbb{Q}[u_{(\{1,2\},d=1)}]a_{(\{1,2\},d=1)}$ $K\otimes u_3\,\mathbb{Q}[u_{(\{1,2\},d=1)},u_3]a_{(\{1,2\},d=1)}$ $?\subset H^{k-2}(\mathrm{Mod}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2})$
$(\{1,3\},d),\{2\}$	$K\otimes\mathbb{Q}[u_{(\{1,3\},d)},u_2]a_{(\{1,3\},d)}$	$\bigoplus_{d\neq 1} K\otimes\mathbb{Q}[u_{(\{1,3\},d)}]a_{(\{1,3\},d)}$ $K\otimes\mathbb{Q}[u_{(\{1,3\},d=1)}]a_{(\{1,3\},d=1)}$ $K\otimes u_2\,\mathbb{Q}[u_{(\{1,3\},d=1)},u_2]a_{(\{1,3\},d=1)}$ $?\subset H^{k-2}(\mathrm{Mod}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2})$
$(\{1,2,3\},(d,d'))$	$K\otimes\mathbb{Q}[u_{(\{1,2,3\},(d,d'))}]a_{(\{1,2,3\},(d,d'))}$	$\bigoplus_{d\neq 1} K\otimes\mathbb{Q}[u_{(\{1,2,3\},(d,1))}]a_{(\{1,2,3\},(d,1))}$ $\bigoplus_{d'\neq 1} K\otimes\mathbb{Q}[u_{(\{1,2,3\},(1,d'))}]a_{(\{1,2,3\},(1,d'))}$ $K\otimes\mathbb{Q}[u_{(\{1,2,3\},(d=1,d'=1))}]a_{(\{1,2,3\},(d=1,d'=1))}$ $?\subset H^{k-2}(\mathrm{Mod}_{g,1};H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2})$

Our next goal is to calculate $H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes 2})$ by making use of the following short exact sequence in two ways:

$$0\to H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\to\mathfrak{H}_{g,1}(l;\mathbb{Q})\to\mathbb{Q}^{|\mathcal{D}|-1}\to 0.$$

Specifically, we can tensor the above short exact sequence with $H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})$, and $\mathfrak{H}_{g,1}(l;\mathbb{Q})$ respectively (one from right and the other from left), and get two exact sequences:

$$0\to H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2}\to\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\to\mathbb{Q}^{|\mathcal{D}|-1}\otimes H^1(\Sigma_g[\mathcal{D}];Q)\to 0,$$

$$0\to\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\to\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes 2}\to\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes\mathbb{Q}^{|\mathcal{D}|-1}\to 0.$$

As short exact sequences of $\mathrm{Mod}_{g,1}(l)$-modules, they induce two long exact seuqences of twisted cohomology of $\mathrm{Mod}_{g,1}(l)$:

$$\begin{array}{l}\cdots\to H^{k-3}(\mathrm{Mod}_{g,1}(l);\mathbb{Q}^{|\mathcal{D}|-1}\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))\to H^{k-2}(\mathrm{Mod}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2})\to\\ \to H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))\to H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathbb{Q}^{|\mathcal{D}|-1}\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))\to\cdots\end{array},$$

and

$$\begin{array}{l}\cdots\to H^{k-3}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes\mathbb{Q}^{|\mathcal{D}|-1})\to H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))\to\\ \to H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes 2})\to H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes\mathbb{Q}^{|\mathcal{D}|-1})\to\cdots\end{array}.$$

For the first long exact sequence, the map

$$H^{k-3}(\mathrm{Mod}_{g,1}(l);\mathbb{Q}^{|\mathcal{D}|-1}\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))\to H^{k-2}(\mathrm{Mod}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2})$$

has image (when $g \geq 2k^2+7k+2$)

$$\left(\bigoplus_{1\neq d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_3\,\mathbb{Q}[u_1,u_3]a_{(\{1,2\},d)}\right)_{(k)}$$

$$\oplus\left(\bigoplus_{1\neq d,1\neq d'\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes \mathbb{Q}[u_{(\{1,3\},d')}]a_{(\{1,3\},d')}\cdot a_{(\{1,2\},d)}\right)_{(k)},$$

which is isomorphic to (by the relations in Theorem 4.6)

$$\left(\bigoplus_{1\neq d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_3\,\mathbb{Q}[u_{(\{1,2\},d)},u_3]a_{(\{1,2\},d)}\right)_{(k)}$$

$$\oplus\left(\bigoplus_{1\neq d,1\neq d'\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes \mathbb{Q}[u_{(\{1,2,3\},(d,d'))}]a_{(\{1,2,3\},(d,d'))}\right)_{(k)}.$$

Thus we get that $H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))$ is isomorphic to (when $g\geq 2k^2+7k+2$)

$$(\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2u_3\,\mathbb{Q}[u_1,u_2,u_3])_{(k)}$$

$$\oplus\left(\bigoplus_{d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes \mathbb{Q}[u_1,u_{(\{2,3\},d)}]a_{(\{2,3\},d)}\right)_{(k)}$$

$$\oplus\left(\bigoplus_{1\neq d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_{(\{1,3\},d)},u_2]a_{(\{1,3\},d)}\right)_{(k)}.$$

For the second long exact sequence, the map

$$H^{k-3}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes\mathbb{Q}^{|\mathcal{D}|-1})\to H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}))$$

has image

$$\left(\bigoplus_{1\neq d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_1,u_2]a_{(\{1,3\},d)}\right)_{(k)}$$

$$\cong\left(\bigoplus_{1\neq d\in\mathcal{D}} \mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2\,\mathbb{Q}[u_2,u_{(\{1,3\},d)}]a_{(\{1,3\},d)}\right)_{(k)}.$$

Therefore we can finally get:

$$H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes 2})\cong\begin{pmatrix}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes u_2u_3\,\mathbb{Q}[u_1,u_2,u_3]\\ \oplus\left(\bigoplus_{d\in\mathcal{D}}\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\mathbb{Q}[u_1,u_{(\{2,3\},d)}]a_{(\{2,3\},d)}\right)\end{pmatrix}_{(k)}.$$

We modify the notation by changing u_1 to c_1 (the first Chern class for the base $\mathrm{Mod}_{g,1}(l)$), and changing indexes $\{2,3\}$ to $\{1,2\}$. In this way, we only have weighted partitions of the index set $\{1,2\}$ in the above cohomology, then we can write the cohomology as:

$$H^{k-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes 2})\cong\left(\mathbb{Q}[c_1,\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{\widetilde{P}\in\mathcal{P}_2^{\mathcal{D}}}\Big(\prod_{\{i\}\in\widetilde{P}}u_i\Big)\,\mathbb{Q}[u_I:I\in\widetilde{P}]a_{\widetilde{P}}\right)\right)_{(k)}$$

if $g \geq 2k^2 + 7k + 2$, where $a_{\widetilde{P}} = \prod_{I \in \widetilde{P}, |I| \geq 2} a_I$. $\square$

The ideas for computations are almost illustrated in the tensor-2 case. Now we state the tensor-r case:

Theorem 4.10. *We have*

$$H^{k-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r}) \cong \left(\mathbb{Q}[c_1, \kappa_1, \kappa_2, \cdots] \otimes \left(\bigoplus_{\widetilde{P} \in \mathcal{P}_r^{\mathcal{D}}} \Big(\prod_{\{i\} \in \widetilde{P}} u_i \Big) \mathbb{Q}[u_I : I \in \widetilde{P}] a_{\widetilde{P}} \right) \right)_{(k)}$$

if $g \geq 2k^2 + 7k + 2$, *and* $a_{\widetilde{P}} = \prod_{I \in \widetilde{P}, |I| \geq 2} a_I$.

The idea is to prove by induction, while in the process the coefficients we will first see are the r-tensor power of $H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})$. In order to derive the cohomology with the coefficients $\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r}$, as in the last step of the proof of the tensor-2 case, we will need to know all cohomology groups of $\mathrm{Mod}_{g,1}(l)$ with the coefficients being multiple tensor powers of both $H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})$ and $\mathfrak{H}_{g,1}(l;\mathbb{Q})$. To describe such coefficients, we introduce the following notation. First recall $\mathcal{D} = H^1(\Sigma_{g,1}; \mathbb{Z}/l)$, and $\Sigma_{g,1}[\mathcal{D}]$ is the regular $\mathcal{D}$-cover of $\Sigma_{g,1}$. Then $\Sigma_g[\mathcal{D}]$ is the closed surface obtained by filling in all punctures of $\Sigma_{g,1}[\mathcal{D}]$.

Definition 4.11. *Define notations* $f(1) = H^1(\Sigma_{g,1}[\mathcal{D}]; \mathbb{Q})$ *and* $f(0) = H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})$. *Given* $r \geq 1$, *let* $J = (J_1, J_2, \cdots, J_r)$ *be an array with* $J_i \in \{0, 1\}$ *for all* i. *We define*

$$\mathfrak{H}^r(J) := f(J_1) \otimes f(J_2) \otimes \cdots \otimes f(J_r),$$

which is an r*-tensor product of* $H^1(\Sigma_{g,1}[\mathcal{D}]; \mathbb{Q})$ *and* $H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})$, *ordered by* J.

To better describe the cohomology of $\mathrm{Mod}_{g,1}(l)$ with coefficients $\mathfrak{H}^r(J)$, we define when a $\mathcal{D}$-weighted partition is compatible with J:

Definition 4.12. *Given* $\mathcal{D}$, r, J, $\mathfrak{H}^r(J)$ *as above. Recall the Definition 1.7 about* $\mathcal{D}$*-weighted partitions. We say a* $\mathcal{D}$*-weighted partition* $\widetilde{P} = \{(S_1, \vec{d_1}), (S_2, \vec{d_2}), \cdots, (S_\nu, \vec{d_\nu})\}$, *indexed by* $\{1, 2, \cdots, r+1\}$, *is compatible with* J *if:*

1. *By convention, we assume* $1 \in S_1$.
2. *For* $\vec{d_1} = (d_1^{(1)}, d_1^{(2)}, \cdots, d_1^{(|S_1|-1)})$, *for all* $1 \leq i \leq |S_1| - 1$, $d_1^{(i)}$ *is not the unit* 1 *in* $\mathcal{D}$.
3. S_1 *does not contain* $2 \leq a \leq r+1$ *such that* $J_{a-1} = 1$.

We denote by $\mathcal{P}_{r+1}^{\mathcal{D}}(J)$ *the set of all* $\mathcal{D}$*-weighted partitions compatible with* J, *indexed by* $\{1, 2, \cdots, r+1\}$.

Now we have $H^{*-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}^r(J))$ as follows:

Theorem 4.13. *We have*

$$H^{*-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}^r(J)) \cong \mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes \left(\bigoplus_{\widetilde{P} \in \mathcal{P}_{r+1}^{\mathcal{D}}(J)} \Big(\prod_{\{i\} \in \widetilde{P}, i \neq 1} u_i \Big) \mathbb{Q}[u_I : I \in \widetilde{P}] a_{\widetilde{P}} \right)$$

if $g \geq 2k^2 + 7k + 2$, *and* $a_{\widetilde{P}} = \prod_{I \in \widetilde{P}, |I| \geq 2} a_I$.

Proof. We prove it by induction on r:

For $r = 1$, $\mathfrak{H}^1(J)$ is either $\mathfrak{H}_{g,1}(l;\mathbb{Q})$ if $J = (1)$, or $H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})$ if $J = (0)$. Our theorem is true in these two cases, by Proposition 4.7 and the Proposition 4.8.

Now for $r \geq 2$, we assume by induction that our theorem is true for cases $\leq (r-1)$.

As what we did in the tensor-2 case, we start with a fibration by letting $p = r+1$ in Theorem 4.3:

$$\begin{tikzcd} Conf_r^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]) \arrow[r] & \mathcal{M}_{g,r+1}(l) \arrow[d] \\ & \mathcal{M}_{g,1}(l) \end{tikzcd} .$$

Then we can compactify the fiber $Conf_r^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]) \subset (\Sigma_g[\mathcal{D}])^{\times r}$, and get a fibration with projective fiber:

$$\begin{tikzcd} \Sigma_g[\mathcal{D}]^{\times r} \arrow[r] & \mathcal{C}_g^{r+1}(l) \arrow[d] \\ & \mathcal{M}_{g,1}(l) \end{tikzcd} , \tag{6}$$

where $\mathcal{C}_g^{r+1}(l)$ is the moduli space of $C \in \mathcal{M}_g$ and $(r+1)$ marked points in $\Sigma_g[\mathcal{D}]$.

Since the fiber $\Sigma_g[\mathcal{D}]^{\times r}$ is projective, we can apply Deligne's degeneration theorem (Theorem 2.5) to the fibration (6) and get the Leray spectral sequence which degenerates at page 2:

$$E_2^{p,q} = H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}]^{\times r}; \mathbb{Q})) \Rightarrow H^{p+q}(\mathcal{C}_g^{r+1}(l); \mathbb{Q}).$$

Thus we have:

$$H^k(\mathcal{C}_g^{r+1}(l); \mathbb{Q}) \cong \bigoplus_{p+q=k} H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}]^{\times r}; \mathbb{Q})).$$

Since the Leray filtration respects the mixed Hodge structure of $H^*(\mathcal{C}_g^{r+1}(l); \mathbb{Q})$, the E_2 page terms inherit mixed Hodge structures.

On the one hand, by Theorem 4.6, we have the cohomology of $H^k(\mathcal{C}_g^{r+1}(l); \mathbb{Q})$ when $g \geq 2k^2 + 7k + 2$ by an isomorphism of mixed Hodge structures

$$H^k(\mathcal{C}_g^{r+1}(l); \mathbb{Q}) \cong \left(\bigoplus_{\widetilde{P} \in \mathcal{P}_{r+1}^{\mathcal{D}}} \mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes \mathbb{Q}[u_I : I \in \widetilde{P}] a_{\widetilde{P}} \right)_{(k)},$$

where $a_{\widetilde{P}} = \prod_{I \in \widetilde{P}, |I| \geq 2} a_I$ has degree $\sum_{I \in \widetilde{P}} (2|I| - 2) = 2(r + 1 - |\widetilde{P}|)$.

The weighted partitions have meanings in terms of the $(r+1)$ marked points in $\mathcal{C}_g^{r+1}(l)$. Like in the tensor-2 case, they are closely related to how we build the fibration (6). By convention, let 1 indexes the marked point in the base $\mathcal{M}_{g,1}(l)$, and index the remaining r points in the fiber by $\{2, 3, \cdots, r+1\}$. Thus the initial fiber $Conf_r^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])$ corresponds to the $\mathcal{D}$-weighted partition $\{\{1\}, \{2\}, \cdots, \{r+1\}\}$, since none of these $(r+1)$ points are in the same $\mathcal{D}$-orbit.

We compactify the fiber $Conf_r^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])$ through two steps:

$$Conf_r^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]) \subset (\Sigma_{g,1}[\mathcal{D}])^{\times r}; \ (\Sigma_{g,1}[\mathcal{D}])^{\times r} \subset \Sigma_g[\mathcal{D}]^{\times r}.$$

The complement in the first step is

$$(\Sigma_{g,1}[\mathcal{D}])^{\times r} \setminus Conf_r^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}]) = \{(x_2, x_3, \cdots, x_{r+1}) | x_i \in \Sigma_{g,1}[\mathcal{D}]; \exists d \in \mathcal{D}, j \neq k, x_j = d \cdot x_k\},$$

which corresponds to $\mathcal{D}$-partitions $\widetilde{P} = \{\{1\}, (S_2, \vec{d_2}), \cdots, (S_\nu, \vec{d_\nu})\} \in \mathcal{P}_{r+1}^{\mathcal{D}}$, where there exists some S_a which has at least two elements. Points indexed by S_i are in the same $\mathcal{D}$-orbit, and these points differ from the first point indexed by S_i respectively by $1, d_a^{(1)}, d_a^{(2)}, \cdots, d_a^{(|S_a|-1)}$. Each $\widetilde{P}$ corresponds to a fibration $\mathcal{M}_{g,\nu}(l) \to \mathcal{M}_{g,1}(l)$ with fiber $Conf_{\nu-1}^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])$.

The complement in the second step is

$$\Sigma_g[\mathcal{D}]^{\times r} \setminus (\Sigma_{g,1}[\mathcal{D}])^{\times r} = \{(x_2, x_3, \cdots, x_{r+1} | x_i \in \Sigma_g[\mathcal{D}]; \exists d \in \mathcal{D}, j \geq 2, x_j = d \cdot x_1\},$$

which corresponds to $\mathcal{D}$-partitions $\widetilde{P} = \{(S_1, \vec{d_1}), (S_2, \vec{d_2}), \cdots, (S_\nu, \vec{d_\nu})\} \in \mathcal{P}_{r+1}^{\mathcal{D}}$, where $1 \in S_1$ and $|S_1| \geq 2$. Each such $\widetilde{P}$ corresponds to a fibration $\mathcal{M}_{g,\nu}(l) \to \mathcal{M}_{g,1}(l)$ with fiber $Conf_{\nu-1}^{\mathcal{D}}(\Sigma_{g,1}[\mathcal{D}])$, and the union of all the above fibrations gives us the fibration (6). Note that all $\mathcal{D}$-weighted partitions arise as above.

For the terms $H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}]^{\times r}; \mathbb{Q}))$ on the E_2 page of the Leray spectral sequence, after we expand $H^q(\Sigma_g[\mathcal{D}]^{\times r}; \mathbb{Q})$ by Künneth theorem, it turns out that only

$$H^{k-r}(\mathcal{M}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})^{\otimes r})$$

is unknown to us. This is because by our induction, we know all

$$H^*(\mathcal{M}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})^{\otimes m}), m \leq r-1,$$

which correspond to $\mathcal{D}$-weighted partitions of $(m+1)$ points including x_1 in the base $\mathcal{M}_{g,1}(l)$, compatible with the array $J = (0, 0, \cdots, 0)$, since

$$H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})^{\otimes m} = \mathfrak{H}^m((0, \cdots, 0)).$$

More precisely, these partitions are $\widetilde{P} = \{(S_1, \vec{d_1}), \cdots, (S_\nu, \vec{d_\nu})\} \in \mathcal{P}_{m+1}^{\mathcal{D}}$ such that $1 \in S_1$ and for $\vec{d_1} = (d_1^{(1)}, d_1^{(2)}, \cdots, d_1^{(|S_1|-1)})$, $d_1^{(i)}$ is not 1 for all $1 \leq i \leq |S_1| - 1$. The $\mathcal{D}$-weighted partitions provide us a nice way to arrange those cohomology groups we already know.

In the process of calculating these cohomology groups, terms in the 0-th-$(2r-2)$-th rows in the E_2 page of the above type can be deduced through the forgetting maps for all $2 \leq i \leq r+1$:

$$\psi_i : \mathcal{C}_g^{r+1}(l) \to \mathcal{C}_g^r(l),\ (C, x_1, \cdots, x_{r+1}) \mapsto (C, x_1, \cdots, \hat{x_i}, \cdots x_{r+1}).$$

The fibration (6) factors through these forgetting maps:

$$\begin{tikzcd}
\Sigma_g[\mathcal{D}]^{\times r} \arrow[r] \arrow[d] & \mathcal{C}_g^{r+1}(l) \arrow[r] \arrow[d, "\psi_i"] & \mathcal{M}_{g,1}(l) \arrow[d, "id"] \\
\Sigma_g[\mathcal{D}]^{\times(r-1)} \arrow[r] & \mathcal{C}_g^r(l) \arrow[r] & \mathcal{M}_{g,1}(l)
\end{tikzcd},$$

which induces maps between the E_2 terms of two Leray spectral sequences

$$H^{k-m}(\mathcal{M}_{g,1}(l); H^m(\Sigma_g[\mathcal{D}]^{\times(r-1)}; \mathbb{Q})) \to H^{k-m}(\mathcal{M}_{g,1}(l); H^m(\Sigma_g[\mathcal{D}]^{\times r}; \mathbb{Q})), 0 \leq m \leq 2r-2.$$

The images of these maps, along with our induction, can help us identify all terms in the 0-th-$(2r-2)$-th rows except

$$H^{k-r}(\mathcal{M}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})^{\otimes r}).$$

For the top two rows of the E_2 page, we need to use the cup product

$$\begin{aligned}
&H^*(\mathcal{M}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})^{\otimes m}) \otimes H^0(\mathcal{M}_{g,1}(l); H^2(\Sigma_g[\mathcal{D}]; \mathbb{Q})) \\
\to &H^*(\mathcal{M}_{g,1}(l); H^1(\Sigma_g[\mathcal{D}]; \mathbb{Q})^{\otimes m} \otimes H^2(\Sigma_g[\mathcal{D}]; \mathbb{Q})).
\end{aligned}$$

We also need the relations in Theorem 4.6 to help us simplify our results, just like we did in the tensor-2 case.

See the Table 3 above for the calculation results. (For the polynomials in the table, we actually mean the degree k parts of them. As a shorthand, we denote $K = \mathbb{Q}[\kappa_1, \kappa_2, \cdots]$. Let $g \geq 2k^2 + 7k + 2$.)

TABLE 3. Rational cohomology of $\mathcal{C}_g^{r+1}(l)$ written in two Ways

$\widetilde{P} \in \mathcal{P}_{r+1}^{\mathcal{D}}$	$H^k(\mathcal{C}_g^3(l);\mathbb{Q})$	$\bigoplus_{p+q=k} H^p(\mathcal{M}_{g,1}(l); H^q(\Sigma_g[\mathcal{D}]\times\Sigma_g[\mathcal{D}];\mathbb{Q}))$
$\{1\},\{2\},\cdots,\{r+1\}$	$K\otimes\mathbb{Q}[u_1,u_2,\cdots,u_{r+1}]$	$K\otimes\mathbb{Q}[u_1]$ $K\otimes u_a\,\mathbb{Q}[u_1,u_a],\forall a\geq 2$ $K\otimes u_b u_c\,\mathbb{Q}[u_1,u_b,u_c], 2\leq b<c$ $\vdots$ $u_{a_1}u_{a_2}\cdots u_{a_{r-1}}[u_1,u_{a_1},u_{a_2},\cdots,u_{a_{r-1}}]$, $2\leq a_1<a_2<\cdots<a_{r-1}$ $?\subset H^{k-r}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes r})$
$\{1\}, I_2=(S_2,\vec{d_2}),\cdots$ $\cdots, I_\nu=(S_\nu,\vec{d_\nu})$ $\lvert S_j\rvert\geq 2,\forall j\geq 2$	$K\otimes\mathbb{Q}[u_1,u_{I_j}:j\geq 2]a_{\widetilde{P}}$	$?\subset H^{k-r}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes r})$
$\{1\}, S_2=\{s_2\},\cdots$ $\cdots, S_{m-1}=\{s_{m-1}\}$, $I_m=(S_m,\vec{d_m}),\cdots$ $\cdots, I_\nu=(S_\nu,\vec{d_\nu})$ $m>2,\forall j\geq m,\lvert S_j\rvert\geq 2$	$K\otimes\mathbb{Q}[u_1,u_{s_2},\cdots$ $\cdots,u_{s_m},u_{I_j}:j\geq m]a_{\widetilde{P}}$	$K\otimes\mathbb{Q}[u_1,u_{I_j}:j\geq m]a_{\widetilde{P}}$ $K\otimes u_a\,\mathbb{Q}[u_1,u_{s_a},u_{I_j}:j\geq m]a_{\widetilde{P}}$, $2\leq a\leq m-1, m>3$ $\vdots$ $u_{s_{a_1}}\cdots u_{s_{a_{m-3}}}\,\mathbb{Q}[u_1,u_{s_{a_1}},\cdots$ $\cdots,u_{s_{a_{m-3}}},u_{I_j}:j\geq m]a_{\widetilde{P}}$ $(2\leq a_1<\cdots<a_{m-3}\leq m-1.)$ $?\subset H^{k-r}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes r})$
$(S_1,\vec{d_1}),\cdots,(S_\nu,\vec{d_\nu})$ $1\in S_1,\lvert S_1\rvert\geq 2$	$K\otimes\mathbb{Q}[u_{I_j}:1\leq j\leq\nu]a_{\widetilde{P}}$	If for $\vec{d_1}=(d_1^{(1)},d_1^{(2)},\cdots,d_1^{(\lvert S_1\rvert-1)})$, $\exists i$ such that $d_1^{(i)}=1$: all can be realized outside $H^{k-r}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes r})$; if $\forall i,\ d_1^{(i)}\neq 1$: all except $(\prod_{\{i\}\in\widetilde{P},i\neq 1} u_i)\,\mathbb{Q}[u_I:I\in\widetilde{P}]a_{\widetilde{P}}$ can be realized outside $H^{k-r}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes r})$.

Since all isomorphisms in the table are isomorphims of (polarized) mixed Hodge structures, by semi-simplicity, we have, when $g\geq 2k^2+7k+2$,

$$H^{k-r}(\mathcal{M}_{g,1}(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes r})\cong\left(K\otimes\left(\bigoplus_{\widetilde{P}\in\mathcal{P}_{r+1}^{\mathcal{D}}(J)}\Big(\prod_{\{i\}\in\widetilde{P},i\neq 1}u_i\Big)\,\mathbb{Q}[u_I:I\in\widetilde{P}]a_{\widetilde{P}}\right)\right)_{(k)},$$

where $J=(0,0,\cdots,0)$, and $K=\mathbb{Q}[\kappa_1,\kappa_2,\cdots]$.

Next, we start another induction on $\sum_{i=1}^{r}J_i$, with $J=(J_1,J_2,\cdots,J_r)$. The computation we just did shows that our theorem is true for $\sum_{i=1}^{r}J_i=0$.

Now, we fix $m>0$, and $J=(J_1,J_2,\cdots,J_r)$ satisfying $\sum_{i=1}^{r}J_i=m$. For J, since $\sum_{i=1}^{r}J_i>0$, there must be some J_t which is 1. We replace the t-th term in J by 0 to get a new array $\widetilde{J}$ whose sum of coordinates is $m-1$.

Recall the short exact sequence of $\mathcal{M}_{g,1}(l)$-modules:

$$0\to H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\to\mathfrak{H}_{g,1}(l;\mathbb{Q})\to\mathbb{Q}^{\lvert\mathcal{D}\rvert-1}\to 0.$$

Observe that

$$\mathfrak{H}^r(J) = f(J_1) \otimes \cdots \otimes f(J_t = 1) \otimes \cdots \otimes f(J_r),$$
$$\mathfrak{H}^r(\widetilde{J}) = f(J_1) \otimes \cdots \otimes f(\widetilde{J}_t = 0) \otimes \cdots \otimes f(J_r).$$

Here $f(0) = H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})$ and $f(1) = \mathfrak{H}_{g,1}(l;\mathbb{Q})$. We can derive a new short exact sequence by tensoring the above short exact sequence on the left with $f(J_1) \otimes \cdots \otimes f(J_{t-1})$ and on the right with $f(J_{t+1}) \otimes \cdots \otimes f(J_r)$ which turns out to be

$$0 \to \mathfrak{H}^r(\widetilde{J}) \to \mathfrak{H}^r(J) \to f(J_1) \otimes \cdots \otimes f(J_{t-1}) \otimes \mathbb{Q}^{|\mathcal{D}|-1} \otimes f(J_{t+1}) \otimes \cdots \otimes f(J_r) \to 0.$$

This short exact sequence of $\mathrm{Mod}_{g,1}(l)$-modules induces a long exact sequence in group cohomology:

$$\begin{aligned} \cdots \to H^{k-r-1}&(\mathrm{Mod}_{g,1}(l); f(J_1) \otimes \cdots \otimes f(J_{t-1}) \otimes \mathbb{Q}^{|\mathcal{D}|-1} \otimes f(J_{t+1}) \otimes \cdots \otimes f(J_r)) \to \\ &\to H^{k-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}^r(\widetilde{J})) \to H^{k-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}^r(J)) \to \\ &\to H^{k-r}(\mathrm{Mod}_{g,1}(l); f(J_1) \otimes \cdots \otimes f(J_{t-1}) \otimes \mathbb{Q}^{|\mathcal{D}|-1} \otimes f(J_{t+1}) \otimes \cdots \otimes f(J_r)) \to \cdots \end{aligned}.$$

To find $H^{k-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}^r(J))$, we only need to figure out the map

$$\begin{aligned} &H^{k-r-1}(\mathrm{Mod}_{g,1}(l); f(J_1) \otimes \cdots \otimes f(J_{t-1}) \otimes \mathbb{Q}^{|\mathcal{D}|-1} \otimes f(J_{t+1}) \otimes \cdots \otimes f(J_r)) \\ \to &H^{k-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}^r(\widetilde{J})). \end{aligned}$$

On the one hand,

$$\begin{aligned} &H^{k-r-1}(\mathrm{Mod}_{g,1}(l); f(J_1) \otimes \cdots \otimes f(J_{t-1}) \otimes \mathbb{Q}^{|\mathcal{D}|-1} \otimes f(J_{t+1}) \otimes \cdots \otimes f(J_r)) \\ \cong &\bigoplus_{|\mathcal{D}|-1} H^{k-r-1}(\mathrm{Mod}_{g,1}(l); f(J_1) \otimes \cdots \otimes f(J_{t-1}) \otimes f(J_{t+1}) \otimes \cdots \otimes f(J_r)) \end{aligned}.$$

The coefficients are $(r-1)$ tensor powers of $H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})$ and $\mathfrak{H}_{g,1}(l;\mathbb{Q})$, so by induction on r we know this cohomology. The $|\mathcal{D}|-1$ components range over all $1 \neq d \in \mathcal{D}$ for the marked point x_{t+1}, i.e. we have $x_{t+1} = d \cdot x_1$ in these cases. Therefore

$$\bigoplus_{|\mathcal{D}|-1} H^{k-r-1}(\mathrm{Mod}_{g,1}(l); f(J_1) \otimes \cdots \otimes f(J_{t-1}) \otimes f(J_{t+1}) \otimes \cdots \otimes f(J_r))$$

contains all summands whose $\mathcal{D}$-weighted partitions of $\{1, 2, \cdots, r+1\}$ are of the form: $\widetilde{P} = \{(S_1, \vec{d_1}), \cdots, (S_\nu, \vec{d_\nu})\}$, $\{1, t+1\} \subset S_1$ and $d_1^{(t)} = d$ for some $1 \neq d \in \mathcal{D}$ s.t.

(1) For $\vec{d_1} = (d_1^{(1)}, d_1^{(2)}, \cdots, d_1^{(|S_1|-1)})$, for all $1 \leq i \leq |S_1| - 1$, $d_1^{(i)}$ is not the unit 1 in $\mathcal{D}$.
(2) S_1 does not contain $2 \leq a \leq r+1$ such that $J_{a-1} = 1$.

For each such $\tilde{P} = \{I_1 = (S_1, \vec{d_1}), \cdots, I_\nu\}$, there is a summand which is the degree $(k-2)$ part of

$$\mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes (\prod_{\{i\} \in \widetilde{P}, i \neq 1} u_i)\, \mathbb{Q}[u_I : I \in \widetilde{P}] a_{I_1 \setminus \{t+1\}} \prod_{m \geq 2} a_{I_m}.$$

The generator $a_{I_1 \setminus \{t+1\}} \prod_{m \geq 2} a_{I_m}$ is of this form because we should delete the marked point $x_{t+1} = d \cdot x_1$ from the index set when we consider this cohomology with $(r-1)$-tensors.

On the other hand, by induction (since $\sum_{i=1}^{r+1} \tilde{J}_i = m - 1 < m$), we know the cohomology with the coefficients $\mathfrak{H}^r(\widetilde{J})$ as follows when $g \geq 2k^2 + 7k + 2$:

$$H^{k-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}^r(\widetilde{J})) \cong \left(\mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes \left(\bigoplus_{\widetilde{P} \in \mathcal{P}^{\mathcal{D}}_{r+1}(\widetilde{J})} (\prod_{\{i\} \in \widetilde{P}, i \neq 1} u_i)\, \mathbb{Q}[u_I : I \in \widetilde{P}] a_{\widetilde{P}} \right) \right)_{(k)}.$$

The map

$$\begin{aligned}&H^{k-r-1}(\mathrm{Mod}_{g,1}(l); f(J_1)\otimes\cdots\otimes f(J_{t-1})\otimes\mathbb{Q}^{|\mathcal{D}|-1}\otimes f(J_{t+1})\otimes\cdots\otimes f(J_r))\\ \to &H^{k-r}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}^r(\widetilde{J}))\end{aligned}$$

restricted to the component (for each $1\neq d\in\mathcal{D}$)

$$H^{k-r-1}(\mathrm{Mod}_{g,1}(l); f(J_1)\otimes\cdots\otimes f(J_{t-1})\otimes f(J_{t+1})\otimes\cdots\otimes f(J_r))$$

is the multiplication by $H^0(\mathcal{M}_{g,1}(l);\mathbb{Q})$, which is $\mathbb{Q}$ generated by $a_{(\{1,t+1\},d)}$. Using the relation (Theorem 4.6) for $I_1=(S_1,\vec{d_1})=(\{1,t+1\},d)\cup(I_1\setminus\{t+1\})$

$$a_{(\{1,t+1\},d)}\cdot a_{I_1\setminus\{t+1\}}=a_{I_1},$$

we know the image of the whole map is

$$\left(\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{1\neq d\in\mathcal{D}}\bigoplus_{\widetilde{P}\in\mathcal{P}^{\mathcal{D}}_{r+1}(\widetilde{J}),\{1,t+1\}\in S_1,d_1^{(t)}=d}(\prod_{\{i\}\in\widetilde{P},i\neq 1}u_i)\,\mathbb{Q}[u_I:I\in\widetilde{P}]a_{\widetilde{P}}\right)\right)_{(k)}.$$

Thus $H^{k-r}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}^r(J))$ is isomorphic to the cokernal of this map, whose summands range over all $\mathcal{D}$-weighted partitions of $\{1,2,\cdots,r+1\}$ compatible with $\widetilde{J}$ and whose S_1 does not contain $t+1$. These are, by the definition of compatibility (Definition 4.12), exactly all $\mathcal{D}$-weighted partitions of $\{1,2,\cdots,r+1\}$ compatible with J, whose $J_t=1$. □

Remark: In the case $J=(1,1,\cdots,1)$, Theorem 4.13 implies Theorem 4.10 above, so the case with 1 puncture is clear.

Our next goal is to generalize the result for any non-closed compact surfaces $\Sigma^b_{g,p}$. The intuition is that the result should be independent of the number of boundary components b, and that adding a puncture will make the result differ by tensoring with the corresponding first Chern class. We will later prove this general statement.

We start by computing the cohomology for surfaces with one boundary component as follows:

Corollary 4.14. *Given $l\geq 2$, we have the following isomorphism*

$$H^{k-r}(\mathrm{Mod}^1_g(l);\mathfrak{H}^1_g(l;\mathbb{Q})^{\otimes r})\cong\left(\mathbb{Q}[\kappa_1,\kappa_2,\cdots]\otimes\left(\bigoplus_{\widetilde{P}\in\mathcal{P}^{\mathcal{D}}_r}(\prod_{\{i\}\in\widetilde{P}}u_i)\,\mathbb{Q}[u_I:I\in\widetilde{P}]a_{\widetilde{P}}\right)\right)_{(k)},$$

if $g\geq 2k^2+7k+2$. Here $a_{\widetilde{P}}=\prod_{I\in\widetilde{P},|I|\geq 2}a_I$.

Proof. We have the following short exact sequence by gluing a punctured disk to the boundary of Σ^1_g by Proposition 2.3:

$$1\to\mathbb{Z}\to\mathrm{Mod}^1_g(l)\to\mathrm{Mod}_{g,1}(l)\to 1.$$

The corresponding Gysin sequence (Proposition 2.8) with coefficients $\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r}\cong\mathfrak{H}^1_g(l;\mathbb{Q})^{\otimes r}$ is:

$$\begin{aligned}\cdots\to &H^{k-r-2}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r})\to H^{k-r}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r})\to\\ \to &H^{k-r}(\mathrm{Mod}^1_g(l);\mathfrak{H}^1_g(l;\mathbb{Q})^{\otimes r})\to\\ \to &H^{k-r-1}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r})\to H^{k-r+1}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r})\to\cdots.\end{aligned}$$

Here the map

$$\phi_{k-r-1}:H^{k-r-1}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r})\to H^{k-r+1}(\mathrm{Mod}_{g,1}(l);\mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r})$$

is the multiplication by the first Chern class c_1 corresponding to the marked point of the base $\mathrm{Mod}_{g,1}(l)$, and in particular it is injective. From the short exact sequence

$$1 \to \mathrm{Coker}(\phi_{k-r-2}) \to H^{k-r}(\mathrm{Mod}_g^1(l); \mathfrak{H}_g^1(l;\mathbb{Q})^{\otimes r}) \to \mathrm{Ker}(\phi_{k-r-1}) \to 1,$$

knowing $H^{k-r}(\mathrm{Mod}_{g,1}(l); \mathfrak{H}_{g,1}(l;\mathbb{Q})^{\otimes r})$ in Theorem 4.10, we get

$$H^{k-r}(\mathrm{Mod}_g^1(l); \mathfrak{H}_g^1(l;\mathbb{Q})^{\otimes r}) \cong \left(\mathbb{Q}[\kappa_1, \kappa_2, \cdots] \otimes \left(\bigoplus_{\widetilde{P}\in\mathcal{P}_r^{\mathcal{D}}} (\prod_{\{i\}\in\widetilde{P}} u_i)\, \mathbb{Q}[u_I : I \in \widetilde{P}] a_{\widetilde{P}}\right)\right)_{(k)},$$

if $g \geq 2k^2 + 7k + 2$. ☐

To further prove that this cohomology is independent of the number of boundary components, the ideas are inspired by Putman's paper [23]. First, we can decompose $\mathfrak{H}_{g,p}^b(l;\mathbb{C})$, the Prym representation with $\mathbb{C}$-coefficients, into a direct sum of isotypic components. This is because $\mathcal{D} \cong (\mathbb{Z}/l)^{2g}$ is a finite group acting on $\mathfrak{H}_{g,p}^b(l;\mathbb{C})$ via deck transformations . The irreducible representations of $\mathcal{D}$ are characterized by characters. A character $\chi : \mathcal{D} \to \mathbb{C}\backslash\{0\}$ gives a irreducible representation $\mathbb{C}_\chi$ with the action:

$$d \cdot \vec{v} = \chi(d)\vec{v}, d \in \mathcal{D}, \vec{v} \in \mathbb{C}_\chi.$$

We denote by $\widehat{\mathcal{D}}$ the abelian group of all characters of $\mathcal{D}$, and let $\mathfrak{H}_{g,p}^b(\chi)$ to be the $\mathbb{C}_\chi$-isotypic component of $\mathfrak{H}_{g,p}^b(l;\mathbb{C})$. Thus we have a direct sum decomposition:

$$\mathfrak{H}_{g,p}^b(l;\mathbb{C}) = \bigoplus_{\chi\in\widehat{\mathcal{D}}} \mathfrak{H}_{g,p}^b(\chi).$$

Since the action of $\mathrm{Mod}_{g,p}^b(l)$ on $\mathfrak{H}_{g,p}^b(l;\mathbb{C})$ commutes with the action of $\mathcal{D}$ by definitions, we know $\mathrm{Mod}_{g,p}^b(l)$ preserves the $\mathcal{D}$-isotypic components, so the above decomposition is also a decomposition of $\mathrm{Mod}_{g,p}^b(l)$-modules. Taking the r-tensor power of $\mathfrak{H}_{g,p}^b(l;\mathbb{C})$,we get:

$$\mathfrak{H}_{g,p}^b(l;\mathbb{C})^{\otimes r} = \bigoplus_{\chi_1,\cdots,\chi_r\in\widehat{\mathcal{D}}} \mathfrak{H}_{g,p}^b(\chi_1) \otimes \cdots \otimes \mathfrak{H}_{g,p}^b(\chi_r).$$

A subgroup $H < H^1(\Sigma_{g,p}^b; \mathbb{Z}/l)$ is called a symplectic subgroup if the algebraic intersection pairing on $H < H^1(\Sigma_{g,p}^b; \mathbb{Z}/l)$ restricts to a non-degenerate pairing on H. Thus we have the isomorphism $H \cong (\mathbb{Z}/l)^{2h}$, where h is the genus of H. A symplectic subgroup is called **compatible** with r characters $\chi_1, \cdots, \chi_r \in \widehat{\mathcal{D}}$ if all χ_i $(1 \leq i \leq r)$ factor through the map

$$\mathcal{D} \to H^1(\Sigma_{g,p}^b; \mathbb{Z}/l) \to H \oplus H^\perp \to H.$$

Then we have the following lemma:

Lemma 4.15. *For $g > r$, given $\chi_1, \cdots, \chi_r \in \widehat{\mathcal{D}}$, there exists a symplectic subgroup H of genus r compatible with $\chi_1, \cdots, \chi_r$.*

Proof. Given $\chi_1, \cdots, \chi_r \in \widehat{\mathcal{D}}$, we define a group homomorphism

$$\mu_r : H^1(\Sigma_g; \mathbb{Z}) \to H^1(\Sigma_g; \mathbb{Z}/l) = \mathcal{D} \to (\mathbb{C} \setminus \{0\})^r, x \mapsto \bar{x} \to (\chi_1(\bar{x}), \cdots, \chi_r(\bar{x})).$$

Notice that elements in $\mathcal{D} \cong (\mathbb{Z}/l)^{2g}$ have order divisible by l, then the images of all χ_i lie in the cyclic group of order l of all l-th roots of unity. Thus we can rewrite μ_r as

$$\mu_r : H^1(\Sigma_g; \mathbb{Z}) \to (\mathbb{Z}/l)^r.$$

By Lemma 3.5 in Putman's paper [24], there exists a symplectic subspace V of $H^1(\Sigma_g; \mathbb{Z})$ of genus $(g-r)$ such that $\mu_r|_V = 0$. Then we can just take H to be the orthgonal complement of the image of V under the map $H^1(\Sigma_g; \mathbb{Z}) \to \mathcal{D}$. ☐

Given a symplectic subrgoup H of $H^1(\Sigma^b_{g,p};\mathbb{Z}/l)$, we can define $\mathrm{Mod}^b_{g,p}(H)$ to be the subgroup of $\mathrm{Mod}^b_{g,p}$ which fixes H pointwise. Denote by $\Sigma^b_{g,p}[H]$ the regular H cover of $\Sigma^b_{g,p}$ induced by

$$\pi_1(\Sigma^b_{g,p}) \to H_1(\Sigma^b_{g,p};\mathbb{Z}/l) \to H^1(\Sigma^b_{g,p};\mathbb{Z}/l) \to H.$$

Here $H_1(\Sigma^b_{g,p};\mathbb{Z}/l) \to H^1(\Sigma^b_{g,p};\mathbb{Z}/l)$ is Poincare duality isomorphism. Therefore $\mathrm{Mod}^b_{g,p}(H)$ acts on $H^1(\Sigma^b_{g,p}[H];\mathbb{C})$. The following lemma by Putman decomposes $H^1(\Sigma^b_{g,p}[H];\mathbb{C})$:

Lemma 4.16 (Putman [23, Lemma 6.5])**.** *Fix $g, p, b \geq 0$ and $l \geq 2$ with $p+b \geq 1$. Let $\widehat{H}$ to be the set of all characters of $\mathcal{D}$ compatible with H. Then we have the following isomorphism of $\mathrm{Mod}^b_{g,p}(H)$ modules:*

$$\mathfrak{H}^b_{g,p}(H;\mathbb{C}) := H^1(\Sigma^b_{g,p}[H];\mathbb{C}) \cong \bigoplus_{\chi\in\widehat{H}} \mathfrak{H}^b_{g,p}(\chi).$$

From this lemma, we know $\mathrm{Mod}^b_{g,p}(H)$ acts on $\mathfrak{H}^b_{g,p}(\underline{\chi}) := \mathfrak{H}^b_{g,p}(\chi_1)\otimes\cdots\otimes\mathfrak{H}^b_{g,p}(\chi_r)$ if H is compatible with characters $\chi_1,\cdots,\chi_r$. By definition, $\mathrm{Mod}^b_{g,p}(l) < \mathrm{Mod}^b_{g,p}(H)$ since $H < H^1(\Sigma^b_{g,p};\mathbb{Z}/l)$. The following theorem of Putman ([23]) helps us to restrict the cohomology of $\mathrm{Mod}^b_{g,p}(l)$ with coefficient $\mathfrak{H}^b_{g,p}(\underline{\chi}) = \mathfrak{H}^b_{g,p}(\chi_1)\otimes\cdots\otimes\mathfrak{H}^b_{g,p}(\chi_r)$ to the cohomology of $\mathrm{Mod}^b_{g,p}(H)$:

Theorem 4.17 (Putman [23])**.** *Let $g, p, b \geq 0$ and $l \geq 2$ be such that $p+b \geq 1$. Let $\chi_1,\cdots,\chi_r \in \mathcal{D}$ be r characters and let H be a symplectic subgroup of $H^1(\Sigma^b_g;\mathbb{Z}/l)$ compatible with $\chi_1,\cdots,\chi_r$. Assume that $g \geq 2(k+r)^2+7k+6r+2$. Then the map*

$$H^k(\mathrm{Mod}^b_{g,p}(H);\mathfrak{H}^b_{g,p}(\underline{\chi})) \to H^k(\mathrm{Mod}^b_{g,p}(l);\mathfrak{H}^b_{g,p}(\underline{\chi}))$$

induced by the inclusion $\mathrm{Mod}^b_{g,p}(l) \hookrightarrow \mathrm{Mod}^b_{g,p}(H)$ is an isomorphism.

The following theorem of Putman serves as a important ingredient in our later proof. It implies the twisted cohomology groups are independent of the number of boundary components, when the number of punctures is 0.

Theorem 4.18 (Putman [23])**.** *Let $\iota : \Sigma^b_g \to \Sigma^{b'}_{g'}$ be an orientation-preserving embedding between surfaces with nonempty boundary. For some $l \geq 2$, let H be a genus-h symplectic subgroup of $H^1(\Sigma^b_g;\mathbb{Z}/l)$. Fix some $k, r \geq 0$, and assume that $g \geq (2h+2)(k+r)+(4h+2)$. Then the induced map*

$$H^k(\mathrm{Mod}^{b'}_{g'}(H);\mathfrak{H}^{b'}_{g'}(H;\mathbb{C})^{\otimes r}) \to H^k(\mathrm{Mod}^b_g(H);\mathfrak{H}^b_g(H;\mathbb{C})^{\otimes r})$$

is an isomorphism.

Now we recall and prove our Theorem B: for $p+b \geq 1$ we have

$$H^{k-r}(\mathrm{Mod}^b_{g,p}(l);\mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes r}) \cong \left(\begin{array}{c}\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots]\\ \otimes\left(\bigoplus_{\widetilde{P}\in\mathcal{P}^{\mathcal{D}}_r}\left(\prod_{\{i\}\in\widetilde{P}} u_i\right)\mathbb{Q}[u_I : I\in\widetilde{P}]a_{\widetilde{P}}\right)\end{array}\right)_{(k)}$$

if $g \geq 2k^2+7k+2$. Here $a_{\widetilde{P}} = \prod_{I\in\widetilde{P},|I|\geq 2} a_I$.

Proof of Theorem B. We will prove it by induction on p. We will show, for each fixed p, the cohomology is independent of b when $p+b \geq 1$.

We start with the case $p = 0$, and it suffices to prove with $\mathbb{C}$-coefficients.

When $b = 1$, our theorem is true by Corollary 4.14.

When $b \geq 2$, observe that there is an orientation-preserving embedding $\Sigma_g^1 \hookrightarrow \Sigma_g^b$ by gluing a surface homeomorphic to Σ_0^{b+1} to the boundary of Σ_g^1. Then by Theorem 4.18, we have the following isomorphism for any symplectic subgroup H of genus h:

$$H^{k-r}(\mathrm{Mod}_g^b(H); \mathfrak{H}_g^b(H;\mathbb{C})^{\otimes r}) \cong H^{k-r}(\mathrm{Mod}_g^1(H); \mathfrak{H}_g^1(H;\mathbb{C})^{\otimes r}),$$

if $g \geq (2h+2)k + (4h+2)$.

Lemma 4.16 gives us a direct sum decomposition of the coefficients in terms of characters on H:

$$\begin{aligned}\mathfrak{H}_g^1(H;\mathbb{C})^{\otimes r} &\cong (\bigoplus_{\chi\in\widehat{H}} \mathfrak{H}_g^1(\chi))^{\otimes r} = \bigoplus_{\chi_1,\cdots,\chi_r\in\widehat{H}} \mathfrak{H}_g^1(\chi_1)\otimes\cdots\otimes\mathfrak{H}_g^1(\chi_r);\\ \mathfrak{H}_g^b(H;\mathbb{C})^{\otimes r} &\cong (\bigoplus_{\chi\in\widehat{H}} \mathfrak{H}_g^b(\chi))^{\otimes r} = \bigoplus_{\chi_1,\cdots,\chi_r\in\widehat{H}} \mathfrak{H}_g^b(\chi_1)\otimes\cdots\otimes\mathfrak{H}_g^b(\chi_r)\end{aligned}.$$

Since the action of Mod_g^b (resp. Mod_g^1) commutes with the action of H, the Künneth formula gives us an isomorphism for each direct sum component, that is :

$$H^{k-r}(\mathrm{Mod}_g^b(H); \mathfrak{H}_g^b(\chi_1)\otimes\cdots\otimes\mathfrak{H}_g^b(\chi_r)) \cong H^{k-r}(\mathrm{Mod}_g^1(H); \mathfrak{H}_g^1(\chi_1)\otimes\cdots\otimes\mathfrak{H}_g^1(\chi_r)),$$

if $g \geq (2h+2)k + (4h+2)$.

By Lemma 4.15, for each pair $\underline{\chi} = (\chi_1, \cdots, \chi_r)$, where $\chi_1, \cdots, \chi_r \in \widehat{\mathcal{D}}$, there exists a genus-$r$ symplectic subgroup, which we denote by $H_{\underline{\chi}}$, compatible with $\chi_1, \cdots, \chi_r$. Theorem 4.17 gives us an isomorphism of the cohomology of $\mathrm{Mod}_g^b(H)$ with the cohomology of $\mathrm{Mod}_g^b(l)$, with coefficients $\mathfrak{H}_g^b(\underline{\chi}) = \mathfrak{H}_g^b(\chi_1)\otimes\cdots\otimes\mathfrak{H}_g^b(\chi_r)$:

$$H^{k-r}(\mathrm{Mod}_g^b(H_{\underline{\chi}}); \mathfrak{H}_g^b(\underline{\chi})) \cong H^{k-r}(\mathrm{Mod}_g^b(l); \mathfrak{H}_g^b(\underline{\chi})),$$

if $g \geq 2k^2 + 7k - r + 2$. This is also true for $b = 1$.

Summarizing all the facts above, we have:

$$\begin{aligned}H^{k-r}(\mathrm{Mod}_g^b(l); \mathfrak{H}_g^b(l;\mathbb{C})^{\otimes r}) &\cong H^{k-r}(\mathrm{Mod}_g^b(l); (\bigoplus_{\chi\in\widehat{\mathcal{D}}} \mathfrak{H}_g^b(\chi))^{\otimes r})\\ &\cong H^{k-r}(\mathrm{Mod}_g^b(l); \bigoplus_{\chi_1,\cdots,\chi_r\in\widehat{\mathcal{D}}} \mathfrak{H}_g^b(\chi_1)\otimes\cdots\otimes\mathfrak{H}_g^b(\chi_r))\\ &\cong \bigoplus_{\underline{\chi}\in(\widehat{\mathcal{D}})^{\times r}} H^{k-r}(\mathrm{Mod}_g^b(l); \mathfrak{H}_g^b(\underline{\chi}))\\ &\cong \bigoplus_{\underline{\chi}\in(\widehat{\mathcal{D}})^{\times r}} H^{k-r}(\mathrm{Mod}_g^b(H_{\underline{\chi}}); \mathfrak{H}_g^b(\underline{\chi}))\\ &\cong \bigoplus_{\underline{\chi}\in(\widehat{\mathcal{D}})^{\times r}} H^{k-r}(\mathrm{Mod}_g^1(H_{\underline{\chi}}); \mathfrak{H}_g^1(\underline{\chi}))\\ &\cong \bigoplus_{\underline{\chi}\in(\widehat{\mathcal{D}})^{\times r}} H^{k-r}(\mathrm{Mod}_g^1(l); \mathfrak{H}_g^1(\underline{\chi}))\\ &\cong H^{k-r}(\mathrm{Mod}_g^1(l); \bigoplus_{\chi_1,\cdots,\chi_r\in\widehat{\mathcal{D}}} \mathfrak{H}_g^1(\chi_1)\otimes\cdots\otimes\mathfrak{H}_g^1(\chi_r))\\ &\cong H^{k-r}(\mathrm{Mod}_g^1(l); (\bigoplus_{\chi\in\widehat{\mathcal{D}}} \mathfrak{H}_g^1(\chi))^{\otimes r})\\ &\cong H^{k-r}(\mathrm{Mod}_g^1(l); \mathfrak{H}_g^1(l;\mathbb{C})^{\otimes r})\end{aligned}.$$

The 4-th and 6-th isomorphisms, due to Theorem 4.17, happen when $g \geq 2k^2+7k-r+2$. The 5-th isomorphism, due to Lemma 4.16 and what we discussed, is true when $g \geq (2r+2)k+(4r+2)$. The other isomorphisms are always true. Since $k \geq r$, we have $(2r+2)k+(4r+2) \leq 2k^2+7k-r+2$. Corollary 4.14 shows that the theorem is true for Σ_g^1 when $g \geq 2k^2+7k+2$. Thus the above isomorphism tells us the theorem is also true for Σ_g^b ($b \geq 2$), when $g \geq 2k^2+7k+2$ since $2k^2+7k+2 \geq 2k^2+7k-r+2$.

Next, we discuss the case when $p \geq 1$. By induction the theorem is true for $\Sigma_{g,p-1}^{b+1}$, with $b \geq 0$. Recall from Proposition 2.3 that we have a short exact sequence induced by gluing a punctured disk:

$$1 \to \mathbb{Z} \to \mathrm{Mod}_{g,p-1}^{b+1}(l) \to \mathrm{Mod}_{g,p}^{b}(l) \to 1.$$

This induces a Gysin sequence (Proposition 2.8) with coefficients $\mathfrak{H}_{g,p-1}^{b+1}(l;\mathbb{Q})^{\otimes r} \cong \mathfrak{H}_{g,p}^{b}(l;\mathbb{Q})^{\otimes r}$:

$$\begin{aligned}\cdots \to & H^{k-r}(\mathrm{Mod}_{g,p}^{b}(l);\mathfrak{H}_{g,p}^{b}(l;\mathbb{Q})^{\otimes r}) \to H^{k-r}(\mathrm{Mod}_{g,p-1}^{b+1}(l);\mathfrak{H}_{g,p-1}^{b+1}(l;\mathbb{Q})^{\otimes r}) \to \\ &\to H^{k-r-1}(\mathrm{Mod}_{g,p}^{b}(l);\mathfrak{H}_{g,p}^{b}(l;\mathbb{Q})^{\otimes r}) \to H^{k-r+1}(\mathrm{Mod}_{g,p}^{b}(l);\mathfrak{H}_{g,p}^{b}(l;\mathbb{Q})^{\otimes r}) \to \cdots .\end{aligned}$$

Here the map

$$\phi_{k-r-1} : H^{k-r-1}(\mathrm{Mod}_{g,p}^{b}(l);\mathfrak{H}_{g,p}^{b}(l;\mathbb{Q})^{\otimes r}) \to H^{k-r+1}(\mathrm{Mod}_{g,p}^{b}(l);\mathfrak{H}_{g,p}^{b}(l;\mathbb{Q})^{\otimes r})$$

is the multiplication by the first Chern class c_p, and it is injective. From the short exact sequence

$$1 \to \mathrm{Coker}(\phi_{k-r-2}) \to H^{k-r}(\mathrm{Mod}_{g,p-1}^{b+1}(l);\mathfrak{H}_{g,p-1}^{b+1}(l;\mathbb{Q})^{\otimes r}) \to \mathrm{Ker}(\phi_{k-r-1}) \to 1,$$

we have

$$H^{*}(\mathrm{Mod}_{g,p}^{b}(l);\mathfrak{H}_{g,p}^{b}(l;\mathbb{Q})^{\otimes r}) \cong H^{*}(\mathrm{Mod}_{g,p-1}^{b+1}(l);\mathfrak{H}_{g,p-1}^{b+1}(l;\mathbb{Q})^{\otimes r})[c_p].$$

Knowing $H^{*}(\mathrm{Mod}_{g,p-1}^{b+1}(l);\mathfrak{H}_{g,p-1}^{b+1}(l;\mathbb{Q})^{\otimes r})$ by induction, we get:

$$H^{k-r}(\mathrm{Mod}_{g,p}^{b}(l);\mathfrak{H}_{g,p}^{b}(l;\mathbb{Q})^{\otimes r}) \cong \left(\begin{array}{c}\mathbb{Q}[c_1,\cdots,c_p,\kappa_1,\kappa_2,\cdots] \\ \otimes\left(\bigoplus_{\widetilde{P}\in\mathcal{P}_r^{\mathcal{D}}}(\prod_{\{i\}\in\widetilde{P}} u_i)\,\mathbb{Q}[u_I : I\in\widetilde{P}]a_{\widetilde{P}}\right)\end{array}\right)_{(k)}$$

if $g \geq 2k^2+7k+2$. Here $a_{\widetilde{P}} = \prod_{I\in\widetilde{P},|I|\geq 2} a_I$. Note that this cohomology is independent of b (the number of boundary components) but dependent on p (the number of punctures) since we have generators coming from first Chern classes of all punctures. $\square$

5. Applications

Consider a surface $\Sigma_{g,p}^b$ with $p+b \geq 1$, and a finite-index normal subgroup $K < \pi_1(\Sigma_{g,p}^b)$. Then K corresponds to a finite cover $S_K \to \Sigma_{g,p}^b$. Although $\mathrm{Mod}_{g,p}^b$ may not be liftable to S_K, we can find a finite-index subgroup $\Gamma < \mathrm{Mod}_{g,p}^b$ which acts on S_K (e.g. we can take Γ to be the subgroup which acts trivially on $\pi_1(\Sigma_{g,p}^b)/K$ and preserves the base-point of S_K). The group Γ acts on $H_1(S_K;\mathbb{Q})$. We then take B to be the subspace of $H_1(S_K;\mathbb{Q})$ spanned by the homology classes of the boundary components of S_K and loops around the punctures of S_K, and define:

$$V_K := H_1(S_K;\mathbb{Q})/B.$$

The group Γ also acts on V_K. We call the resulting representation $\Gamma \to \mathrm{Aut}(V_K \otimes \mathbb{R})$ a Prym representation of $\Gamma < \mathrm{Mod}_{g,p}^b$.

We will first consider the special covers $\Sigma_{g,p}^b[\mathcal{D}] \to \Sigma_{g,p}^b$, and then generalize our result to all finite abelian covers.

First, we consider the finite-index subgroup K to be $\mathrm{Ker}(\pi_1(\Sigma_{g,p}^b) \to H_1(\Sigma_g;\mathbb{Z}/l) \cong \mathcal{D}$. Then the corresponding cover is the regular $\mathcal{D}$-cover $\Sigma_{g,p}^b[\mathcal{D}] \to \Sigma_{g,p}^b$. We can take Γ to be $\mathrm{Mod}_{g,p}^b(l)$. Then $V_K = H_1(\Sigma_{g,p}^b[\mathcal{D}];\mathbb{Q})/B = H_1(\Sigma_g[\mathcal{D}];\mathbb{Q})$, where $\Sigma_g[\mathcal{D}]$ is the regular $\mathcal{D}$-cover of Σ_g. Then we have the following Prym representations with coefficients in $\mathbb{R}$:

$$\Phi : \mathrm{Mod}_{g,p}^b(l) \to \mathrm{Aut}_{\mathbb{R}}(V_K \otimes \mathbb{R}) = \mathrm{Aut}_{\mathbb{R}}(H_1(\Sigma_g[\mathcal{D}];\mathbb{R})).$$

The action of $\mathrm{Mod}_{g,p}^b(l)$ on $H_1(\Sigma_g[\mathcal{D}];\mathbb{R})$ preserves the algebraic intersection form on $\Sigma_g[\mathcal{D}]$, and commutes with the action of the deck group $\mathcal{D}$. Therefore, we have the inclusion $\mathrm{Im}(\Phi) \hookrightarrow (Sp(2h;\mathbb{R}))^{\mathcal{D}}$, where h is the genus of $\Sigma_g[\mathcal{D}]$. The notation $Sp(2h;\mathbb{R}))^{\mathcal{D}}$ means the centralizer of $\mathcal{D}$ in $Sp(2h;\mathbb{R})$. Here $\mathcal{D}$ is the image of $\mathcal{D}$ in $\mathrm{Aut}(H^1(\Sigma_g[\mathcal{D}];\mathbb{R})$ via deck transformations. We define the Lie group $G := Sp(2h;\mathbb{R}))^{\mathcal{D}}$, and get a representation which we still call Φ:

$$\Phi : \mathrm{Mod}_{g,p}^b(l) \to G.$$

We will show that this representation is locally rigid. The following theorem of Weil ([28]) shows that it is enough to show that $H^1(\mathrm{Mod}_{g,p}^b(l);\mathfrak{g}) = 0$, where $\mathfrak{g}$ is the Lie algebra of G. Here $\mathfrak{g}$ is a $\mathrm{Mod}_{g,p}^b(l)$-module via the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.

Theorem 5.1 (Weil [28])**.** *Consider a homomorphism $\Phi : \Gamma \to G$, where Γ is a finitely generated group, and G is a Lie group. Then ϕ is locally rigid if $H^1(\Gamma;\mathfrak{g}) = 0$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G.*

For our Lie group $G = Sp(2h;\mathbb{R}))^{\mathcal{D}}$, we first determine its Lie algebra:

Lemma 5.2. *For the Lie group $G = Sp(2h;\mathbb{R}))^{\mathcal{D}}$, its Lie algebra is $\mathfrak{g} = (\mathfrak{sp}(2h;\mathbb{R})^{\mathcal{D}}$.*

Proof. To simplify our notation, denote by $\mathcal{D}$ the image of $\mathcal{D} \to \mathrm{Aut}(H_1(\Sigma_g[\mathcal{D}];\mathbb{R}))$. We can describe our Lie group as:

$$G = \{A \in GL(2h;\mathbb{R}) | A^T J A = J, AD = DA, \forall D \in \mathcal{D}\},$$

where

$$J = \begin{pmatrix} 0 & I_h \\ -I_h & 0 \end{pmatrix}.$$

The Lie algebra of the matrix Lie group G is:

$$\mathfrak{g} = \{X \in Mat(2h;\mathbb{R}) | e^{tX} \in G, \forall t \in \mathbb{R}\}.$$

The Lie algebra of $Sp(2h;\mathbb{R})$ is $\mathfrak{sp}(2h;\mathbb{R})$, which satisfies $X^T J + JX = 0$. It remains to check what the condition $AD = DA, \forall D \in \mathcal{D}$ descends to. Let $X \in \mathfrak{g}$. We should have $e^{tX} D = De^{tX}$, and after plugging in $e^{tX} = \sum_{j=o}^{\infty} \frac{(tX)^j}{j!}$ and ignoring all order ≥ 2 polynomials of t, we get: $tXD = DtX$ for all $t \in \mathbb{R}$. Thus $XD = DX$ for all $D \in \mathcal{D}$, so we have $\mathfrak{g} = (\mathfrak{sp}(2h;\mathbb{R})^{\mathcal{D}}$. □

Now observe that our Lie algebra $\mathfrak{g}$ is a $\mathrm{Mod}_{g,p}^b(l)$-submodule of $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2} \cong (\mathbb{R}^{2h})^{\otimes 2}$ in the following way:

- $\mathfrak{g} = (\mathfrak{sp}(2h;\mathbb{R})^{\mathcal{D}} \subset \mathrm{Mat}(2h;\mathbb{R}) \cong (\mathbb{R}^{2h})^* \otimes \mathbb{R}^{2h} \cong (\mathbb{R}^{2h})^{\otimes 2}$, where the isomorphism of the dual space $(\mathbb{R}^{2h})^* \cong \mathbb{R}^{2h}$ is induced by the nondegenerate algebraic intersection form $i : H_1(\Sigma_g[\mathcal{D}];\mathbb{R}) \times H_1(\Sigma_g[\mathcal{D}];\mathbb{R}) \to \mathbb{R}$.
- The action of $\mathrm{Mod}_{g,p}^b(l)$ on $\mathfrak{g}$ is compatible with the action of $\mathrm{Mod}_{g,p}^b(l)$ on $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}$. Consider $f \in \mathrm{Mod}_{g,p}^b(l)$, and denote by F the corresponding matrix for f acting on $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})$. Taking $X \in \mathfrak{g}$, we see $f \cdot X = FAF^{-1}$. We will translate this to an

identity in $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}$. First we take a symplectic basis $\{\alpha_1,\beta_1,\cdots,\alpha_h,\beta_h\}$ of $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})$. We can write

$$X=\sum_{j=1}^{h}(\alpha_j)^*\otimes X\alpha_j+\sum_{j=1}^{h}(\beta_j)^*\otimes X\beta_j\in(\mathbb{R}^{2h})^*\otimes\mathbb{R}^{2h}.$$

Then

$$f\cdot X=\sum_{j=1}^{h}(\alpha_j)^*\otimes FXF^{-1}\alpha_j+\sum_{j=1}^{h}(\beta_j)^*\otimes FXF^{-1}\beta_j\in(\mathbb{R}^{2h})^*\otimes\mathbb{R}^{2h}.$$

The isomorphism $(\mathbb{R}^{2h})^*\cong\mathbb{R}^{2h}$ lets us write, in $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}$,

$$X=\sum_{j=1}^{h}\beta_j\otimes X\alpha_j+\sum_{j=1}^{h}(-\alpha_j)\otimes X\beta_j,$$

and

$$\begin{aligned}f\cdot X&=\sum_{j=1}^{h}\beta_j\otimes FXF^{-1}\alpha_j+\sum_{j=1}^{h}(-\alpha_j)\otimes FXF^{-1}\beta_j\\&=\sum_{j=1}^{h}F(F^{-1}\beta_j)\otimes FX(F^{-1}\alpha_j)+\sum_{j=1}^{h}F(-F^{-1}\alpha_j)\otimes FX(F^{-1}\beta_j)\end{aligned}$$

which is exactly the action of $f\in\mathrm{Mod}_{g,p}^b(l)$ on $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})$ since $\{f^{-1}(\alpha_1),f^{-1}(\beta_1),\cdots,f^{-1}(\alpha_h),f^{-1}(\beta_h)\}$ is still a symplectic basis.

We will deduce that the first homology group of $\mathrm{Mod}_{g,p}^b(l)$ with coefficients $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}$ is 0 as a corollary of our computations in last section:

Corollary 5.3. *For integers $g\geq 0, p+b\geq 1, l\geq 2,$ we have*

$$H^1(\mathrm{Mod}_{g,p}^b(l);H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2})=0,$$

if $g\geq 41$.

Proof. Recall that we have computed $H^{k-r}(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes r})$ in Theorem B. Letting $r=2$ and $k=3$, we see this polynomial algebra only has even-degree terms, so we have:

$$H^1(\mathrm{Mod}_{g,p}^b(l);\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 2})=0,$$

when $g\geq 2k^2+7k+2=41$.

Since $\mathfrak{H}_{g,p}^b(l)=H^1(\Sigma_{g,p}^b[\mathcal{D}];\mathbb{Q})$, and $\Sigma_g[\mathcal{D}]$ is obtained by filling in all punctures and gluing disks to all boundary components of $\Sigma_{g,p}^b[\mathcal{D}]$, we have the following short exact sequence induced by the map $\Sigma_{g,p}^b[\mathcal{D}]\to\Sigma_g[\mathcal{D}]$:

$$0\to H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\to\mathfrak{H}_{g,p}^b(l;\mathbb{Q})\to\mathbb{Q}^{(p+b)\cdot|\mathcal{D}|-1}\to 0.$$

Here $\mathfrak{H}_{g,p}^b(l;\mathbb{Q})=H^1(\Sigma_{g,p}^b[\mathcal{D}];\mathbb{Q})$. We can tensor the above short exact sequence with $H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})$ (on the right) and $\mathfrak{H}_{g,p}^b(l;\mathbb{Q})$ (on the left) respectively and get the following two short exact sequences:

$$0\to H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2}\to\mathfrak{H}_{g,p}^b(l;\mathbb{Q})\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\to\mathbb{Q}^{(p+b)\cdot|\mathcal{D}|-1}\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\to 0;$$
$$0\to\mathfrak{H}_{g,p}^b(l;\mathbb{Q})\otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})\to\mathfrak{H}_{g,p}^b(l;\mathbb{Q})^{\otimes 2}\to\mathbb{Q}^{(p+b)\cdot|\mathcal{D}|-1}\otimes\mathfrak{H}_{g,p}^b(l;\mathbb{Q})\to 0.$$

These terms are all $\mathrm{Mod}_{g,p}^b(l)$-modules and we know the cohomology of $\mathrm{Mod}_{g,p}^b(l)$ with these coefficients:

- $H^1(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes 2}) = 0$,when $g \geq 41$, by Theorem B.
- $H^2(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q})) = 0$, when $g \geq 41$, by Theorem B.
- $H^2(\mathrm{Mod}^b_{g,p}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})) = 0$, when $g \geq 41$. This is because the short exact sequence

$$0 \to H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}) \to \mathfrak{H}^b_{g,p}(l;\mathbb{Q}) \to \mathbb{Q}^{(p+b)\cdot|\mathcal{D}|-1} \to 0$$

induces a long exact sequence

$$\to H^3(\mathrm{Mod}^b_{g,p}(l); \mathbb{Q}^{(p+b)\cdot|\mathcal{D}|-1}) \to H^2(\mathrm{Mod}^b_{g,p}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})) \to H^2(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q})) \to .$$

Here $H^3(\mathrm{Mod}^b_{g,p}(l); \mathbb{Q}^{(p+b)\cdot|\mathcal{D}|-1})$ is isomorphic to $H^3(\mathrm{Mod}^b_{g,p}(l); \mathbb{Q})^{(p+b)\cdot|\mathcal{D}|-1}$ which is 0; and as above $H^2(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q})) = 0$, when $g \geq 41$.

Thus we can make use of these in the two long exact sequences induced by the the above two short exact sequences, and get:

(1) $H^1(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q}) \otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q}) = 0$, when $g \geq 41$. This is because

$$\begin{aligned} &H^2(\mathrm{Mod}^b_{g,p}(l); \mathbb{Q}^{(p+b)\cdot|\mathcal{D}|-1} \otimes \mathfrak{H}^b_{g,p}(l;\mathbb{Q})) \to H^1(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q}) \otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})) \to \\ &\to H^1(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q})^{\otimes 2}) \end{aligned}$$

is exact and both ends are 0.

(2) $H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2}) = 0$, when $g \geq 41$. This is because

$$\begin{aligned} &H^2(\mathrm{Mod}^b_{g,p}(l); \mathbb{Q}^{(p+b)\cdot|\mathcal{D}|-1} \otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})) \to H^1(\mathrm{Mod}^b_{g,p}(l); H_1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}) \to \\ &\to H^1(\mathrm{Mod}^b_{g,p}(l); \mathfrak{H}^b_{g,p}(l;\mathbb{Q}) \otimes H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})) \end{aligned}$$

is exact and both ends are 0.

Tensoring $H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{Q})^{\otimes 2}) = 0$ with $\mathbb{R}$, we conclude that for $g \geq 41$:

$$H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}) = 0. \qquad \square$$

Knowing $H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}) = 0$ and the fact that $\mathfrak{g}$ is a submodule of $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}$, we can deduce:

Theorem 5.4. *For integers p, b, l such that $p+b \geq 1, l \geq 2$, we have $H^1(\mathrm{Mod}^b_{g,p}(l); \mathfrak{g}) = 0$, if $g \geq 41$. Therefore the Prym representation $\Phi : \mathrm{Mod}^b_{g,p}(l) \to Sp(2h;\mathbb{R})^{\mathcal{D}}$ is locally rigid when $g \geq 41$.*

Proof. By Weil's Theorem 5.1, it is enough to prove $H^1(\mathrm{Mod}^b_{g,p}(l); \mathfrak{g}) = 0$ when $g \geq 41$. Knowing $H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}) = 0$ from Corollary 5.3, it suffices to prove that $\mathfrak{g}$ is a direct summond of $H^1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}$, as $\mathrm{Mod}^b_{g,p}(l)$-modules. From Lemma 5.2, we have $\mathfrak{g} = \mathfrak{sp}(2h;\mathbb{R})^{\mathcal{D}} \subset \mathfrak{sp}(2h;\mathbb{R})$. Note that $\mathfrak{sp}(2h;\mathbb{R})$ is a direct summand of $H_1(\Sigma_g[\mathcal{D}];\mathbb{R})^{\otimes 2}$, being the space of symmetric two-tensors. Thus it suffices to prove $\mathfrak{sp}(2h;\mathbb{R})^{\mathcal{D}}$ is a direct summand of $\mathfrak{sp}(2h;\mathbb{R})$ as $\mathrm{Mod}^b_{g,p}(l)$-modules. By Maschke's Theorem, the $\mathcal{D}$-representation $\mathfrak{sp}(2h;\mathbb{R})$ decomposes into a direct sum of isotypic components, one of which is the trivial sub-representation $\mathfrak{sp}(2h;\mathbb{R})^{\mathcal{D}}$. Since the actions of $\mathrm{Mod}^b_{g,p}(l)$ and $\mathcal{D}$ commute, the group $\mathrm{Mod}^b_{g,p}(l)$ preserves the isotypic decomposition of $\mathfrak{sp}(2h;\mathbb{R})$. $\square$

We now generalize this local-rigidity result to all finite abelian covers of $\Sigma^b_{g,p}$ with $p+b \geq 1$, which is our Theorem C.

Proof of Theorem C. Recall that for a finite abelian cover $S_K \to \Sigma^b_{g,p}$ induced by $K < \pi_1(\Sigma^b_{g,p})$, we have a Prym representation

$$\Phi : \Gamma \to \mathrm{Aut}_{\mathbb{R}}(V_K \otimes \mathbb{R}),$$

where $\Gamma < \mathrm{Mod}^b_{g,p}$ is a finite-index subgroup which acts on S_K. Denote by A the finite abelian group $\pi_1(\Sigma^b_{g,p})/K$. We then take Γ to be $\mathrm{Mod}^b_{g,p}(A) < \mathrm{Mod}^b_{g,p}$, which preserves K and acts trivially on A. We also restrict the image of the above Prym representation to the Lie group $G_K = Sp(2h; \mathbb{R})^A$, where h is the genus of $\widehat{S_K}$. Then we have a more precise Prym representation

$$\Phi : \mathrm{Mod}^b_{g,p}(A) \to G_K,$$

which will be proved to be locally rigid when $g \geq 41$. By Weil's Theorem 5.1, it suffices to prove $H^1(\mathrm{Mod}^b_{g,p}(A); \mathfrak{g}_K) = 0$, where $\mathfrak{g}_K$ is the Lie algebra of G_K.

The cover $S_K \to \Sigma^b_{g,p}$ corresponds to the homomorphism $\pi_1(\Sigma^b_{g,p}) \to \pi_1(\Sigma^b_{g,p})/K = A$. Since A is abelian, this map factors through $\pi_1(\Sigma^b_{g,p}) \to H_1(\Sigma^b_{g,p}; \mathbb{Z})$. Letting $l = |A|$, this map furthermore factors through $\pi_1(\Sigma^b_{g,p}) \to H_1(\Sigma^b_{g,p}; \mathbb{Z}/l)$. Therefore $\mathrm{Mod}^b_{g,p}(l)$ is a finite-index subgroup of $\mathrm{Mod}^b_{g,p}(A)$. Let $\Sigma_g[\mathcal{D}]$ be the regular $H_1(\Sigma_g; \mathbb{Z}/l)$-cover of Σ_g. Then $\Sigma_g[\mathcal{D}]$ is a finite cover of $\widehat{S_K}$, since $\widehat{S_K} \to \Sigma_g$ is a cover (by filling in all punctures and gluing disks to all boundary components of $S_K \to \Sigma^b_{g,p}$) with deck group $\hat{A} :=$ Image of A in $H_1(\Sigma^b_{g,p}; \mathbb{Z}/l) \to H_1(\Sigma_g; \mathbb{Z}/l)$. From Corollary 5.3, we have:

$$H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\Sigma_g[\mathcal{D}]; \mathbb{R})^{\otimes 2}) = 0,$$

if $g \geq 41$. Observe that $H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2}$ is a direct summand of $H^1(\Sigma_g[\mathcal{D}]; \mathbb{R})^{\otimes 2}$, as $\mathrm{Mod}^b_{g,p}(l)$-modules, by Maschke's Theorem and the fact that the actions of $\mathrm{Mod}^b_{g,p}(l)$ and $\hat{A}$ commute. Thus we get, when $g \geq 41$:

$$H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2}) = 0.$$

The transfer map (see Proposition 2.7) of $\mathrm{Mod}^b_{g,p}(l) < \mathrm{Mod}^b_{g,p}(A)$ shows that the composition map

$$H^1(\mathrm{Mod}^b_{g,p}(A); H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2}) \to H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2}) \to H^1(\mathrm{Mod}^b_{g,p}(A); H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2})$$

is the multiplication by the index $[\mathrm{Mod}^b_{g,p}(A) : \mathrm{Mod}^b_{g,p}(l)]$. Since the coefficients $H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2}$ is an $\mathbb{R}$-vector space, the transfer map

$$H^1(\mathrm{Mod}^b_{g,p}(l); H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2}) \to H^1(\mathrm{Mod}^b_{g,p}(A); H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2})$$

is surjective. Thus we have, for $g \geq 41$,

$$H^1(\mathrm{Mod}^b_{g,p}(A); H^1(\widehat{S_K}; \mathbb{R})^{\otimes 2}) = 0.$$

Then by an argument identical to the last step of the proof of Theorem 5.4, we have $H^1(\mathrm{Mod}^b_{g,p}(A); \mathfrak{g}_K) = 0$ when $g \geq 41$. □

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