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# A Fully Bayesian Model for Gene Expression Heterosis in RNA-seq Data

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## 1 The Model

Let  $y_{g,n}$  be the expression level of gene  $g$  ( $g = 1, \dots, G$ ) in library  $n$  ( $n = 1, \dots, N$ ). Let  $\mu(n, \phi_g, \alpha_g, \delta_g)$  be the function given by:

$$\mu(n, \phi_g, \alpha_g, \delta_g) = \begin{cases} \phi_g - \alpha_g & \text{library } n \text{ is in treatment group 1} \\ \phi_g + \delta_g & \text{library } n \text{ is in treatment group 2} \\ \phi_g + \alpha_g & \text{library } n \text{ is in treatment group 3} \end{cases}$$

Then:

$$\begin{aligned} y_{g,n} &\stackrel{\text{ind}}{\sim} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \\ c_n &\stackrel{\text{ind}}{\sim} \text{N}(c_n \mid 0, \sigma_c^2) \\ \sigma_c &\sim \text{U}(\sigma_c \mid 0, \sigma_{c0}) \\ \varepsilon_{g,n} &\stackrel{\text{ind}}{\sim} \text{N}(\varepsilon_{g,n} \mid 0, \eta_g^2) \\ \eta_g^2 &\stackrel{\text{ind}}{\sim} \text{Inv-Gamma}\left(\eta_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right) \\ d &\sim \text{U}(d \mid 0, d_0) \\ \tau^2 &\sim \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\ \phi_g &\stackrel{\text{ind}}{\sim} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \\ \theta_\phi &\sim \text{N}(\theta_\phi \mid 0, \gamma_\phi^2) \\ \sigma_\phi &\sim \text{U}(\sigma_\phi \mid 0, \sigma_{\phi 0}) \\ \alpha_g &\stackrel{\text{ind}}{\sim} \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \\ \theta_\alpha &\sim \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2) \\ \sigma_\alpha &\sim \text{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0}) \\ \pi_\alpha &\sim \text{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha) \\ \delta_g &\stackrel{\text{ind}}{\sim} \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \\ \theta_\delta &\sim \text{N}(\theta_\delta \mid 0, \gamma_\delta^2) \\ \sigma_\delta &\sim \text{U}(\sigma_\delta \mid 0, \sigma_{\delta 0}) \\ \pi_\delta &\sim \text{Beta}(\pi_\delta \mid a_\delta, b_\delta) \end{aligned}$$

where:

- $I(x) = 0$  if  $x = 0$  and 1 otherwise.
- Conditional independence is implied unless otherwise specified.
- The parameters to the left of the “ $\sim$ ” are implicitly conditioned on the parameters to the right.

## 2 Full Conditional Distributions

Define:

- $k(n)$  = treatment group of library  $n$ .
- $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$
- $G_\alpha$  = number of genes for which  $\alpha_g \neq 0$
- $G_\delta$  = number of genes for which  $\delta_g \neq 0$
- $I(x) = 0$  if  $x = 0$  and 1 otherwise.

Then:

$$\begin{aligned}
p(c_n \mid \dots) &\propto \exp \left( c_n G \bar{y}_{\cdot, n} - \exp(c_n) \sum_{g=1}^G \exp(\varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{c_n^2}{2\sigma_c^2} \right) \\
p \left( \frac{1}{\sigma_c^2} \mid \dots \right) &= \text{Gamma} \left( \frac{1}{\sigma_c^2} \mid \text{shape} = \frac{N-1}{2}, \text{rate} = \frac{1}{2} \sum_{n=1}^N c_n^2 \right) I \left( \frac{1}{\sigma_c^2} > \frac{1}{\sigma_{c0}^2} \right) \\
p(\varepsilon_{g,n} \mid \dots) &\propto \exp \left( y_{g,n} \varepsilon_{g,n} - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{\varepsilon_{g,n}^2}{2\eta_g^2} \right) \\
p \left( \frac{1}{\eta_g^2} \mid \dots \right) &= \text{Gamma} \left( \frac{1}{\eta_g^2} \mid \text{shape} = \frac{N+d}{2}, \text{rate} = \frac{1}{2} \left( d \cdot \tau^2 + \sum_{n=1}^N \varepsilon_{g,n}^2 \right) \right) \\
p(d \mid \dots) &\propto \Gamma(d/2)^{-G} \left( \frac{d \cdot \tau^2}{2} \right)^{Gd/2} \left( \prod_{g=1}^G \eta_g^2 \right)^{-(d/2+1)} \exp \left( -\frac{d \cdot \tau^2}{2} \sum_{g=1}^G \frac{1}{\eta_g^2} \right) I(0 < d < d_0) \\
p(\tau^2 \mid \dots) &= \text{Gamma} \left( \tau^2 \mid \text{shape} = a_\tau + \frac{Gd}{2}, \text{rate} = b_\tau + \frac{d}{2} \sum_{g=1}^G \frac{1}{\eta_g^2} \right) \\
p(\phi_g \mid \dots) &\propto \exp \left( \sum_{n=1}^N [y_{g,n} \mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2} \right) \\
p(\theta_\phi \mid \dots) &= \text{N} \left( \theta_\phi \mid \frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g}{G\gamma_\phi^2 + \sigma_\phi^2}, \frac{\gamma_\phi^2 \sigma_\phi^2}{G\gamma_\phi^2 + \sigma_\phi^2} \right) \\
p \left( \frac{1}{\sigma_\phi^2} \mid \dots \right) &= \text{Gamma} \left( \frac{1}{\sigma_\phi^2} \mid \text{shape} = \frac{G-1}{2}, \text{rate} = \frac{1}{2} \sum_{g=1}^G (\phi_g - \theta_\phi)^2 \right) I \left( \frac{1}{\sigma_\phi^2} > \frac{1}{\sigma_{\phi 0}^2} \right)
\end{aligned}$$

$$\begin{aligned}
p(\alpha_g \mid \cdots) &\propto \exp \left( \sum_{k(n) \neq 2} [y_{g,n} \cdot \mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] \right. \\
&\quad \left. - I(\alpha_g) \left( \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + \log(1 - \pi_\alpha) \right) + (1 - I(\alpha_g)) \log \pi_\alpha \right) \\
p(\theta_\alpha \mid \cdots) &= N \left( \theta_\alpha \mid \frac{\gamma_\alpha^2 \sum_{\alpha_g \neq 0} \alpha_g}{G_\alpha \gamma_\alpha^2 + \sigma_\alpha^2}, \frac{\gamma_\alpha^2 \sigma_\alpha^2}{G_\alpha \gamma_\alpha^2 + \sigma_\alpha^2} \right) \\
p \left( \frac{1}{\sigma_\alpha^2} \mid \cdots \right) &= \text{Gamma} \left( \frac{1}{\sigma_\alpha^2} \mid \text{shape} = \frac{G_\alpha - 1}{2}, \text{rate} = \frac{1}{2} \sum_{\alpha_g \neq 0} (\alpha_g - \theta_\alpha)^2 \right) \mathbf{I} \left( \frac{1}{\sigma_\alpha^2} > \frac{1}{\sigma_{\alpha 0}^2} \right) \\
p(\pi_\alpha \mid \cdots) &= \text{Beta}(\pi_\alpha \mid G - G_\alpha + \alpha_\tau, G_\alpha + b_\tau) \\
p(\delta_g \mid \cdots) &\propto \exp \left( \sum_{k(n) \neq 2} [y_{g,n} \cdot \mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] \right. \\
&\quad \left. - I(\delta_g) \left( \frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2} + \log(1 - \pi_\delta) \right) + (1 - I(\delta_g)) \log \pi_\delta \right) \\
p(\theta_\delta \mid \cdots) &= N \left( \theta_\delta \mid \frac{\gamma_\delta^2 \sum_{\delta_g \neq 0} \delta_g}{G_\delta \gamma_\delta^2 + \sigma_\delta^2}, \frac{\gamma_\delta^2 \sigma_\delta^2}{G_\delta \gamma_\delta^2 + \sigma_\delta^2} \right) \\
p \left( \frac{1}{\sigma_\delta^2} \mid \cdots \right) &= \text{Gamma} \left( \frac{1}{\sigma_\delta^2} \mid \text{shape} = \frac{G_\delta - 1}{2}, \text{rate} = \frac{1}{2} \sum_{\delta_g \neq 0} (\delta_g - \theta_\delta)^2 \right) \mathbf{I} \left( \frac{1}{\sigma_\delta^2} > \frac{1}{\sigma_{\delta 0}^2} \right) \\
p(\pi_\delta \mid \cdots) &= \text{Beta}(\pi_\delta \mid G - G_\delta + \delta_\tau, G_\delta + b_\tau) \\
p(\phi_g, \alpha_g, \delta_g \mid \cdots) &\propto \exp \left( \sum_{n=1}^N [y_{g,n} \mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2} \right. \\
&\quad \left. - I(\alpha_g) \left( \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + \log(1 - \pi_\alpha) \right) + (1 - I(\alpha_g)) \log \pi_\alpha \right. \\
&\quad \left. - I(\delta_g) \left( \frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2} + \log(1 - \pi_\delta) \right) + (1 - I(\delta_g)) \log \pi_\delta \right)
\end{aligned}$$

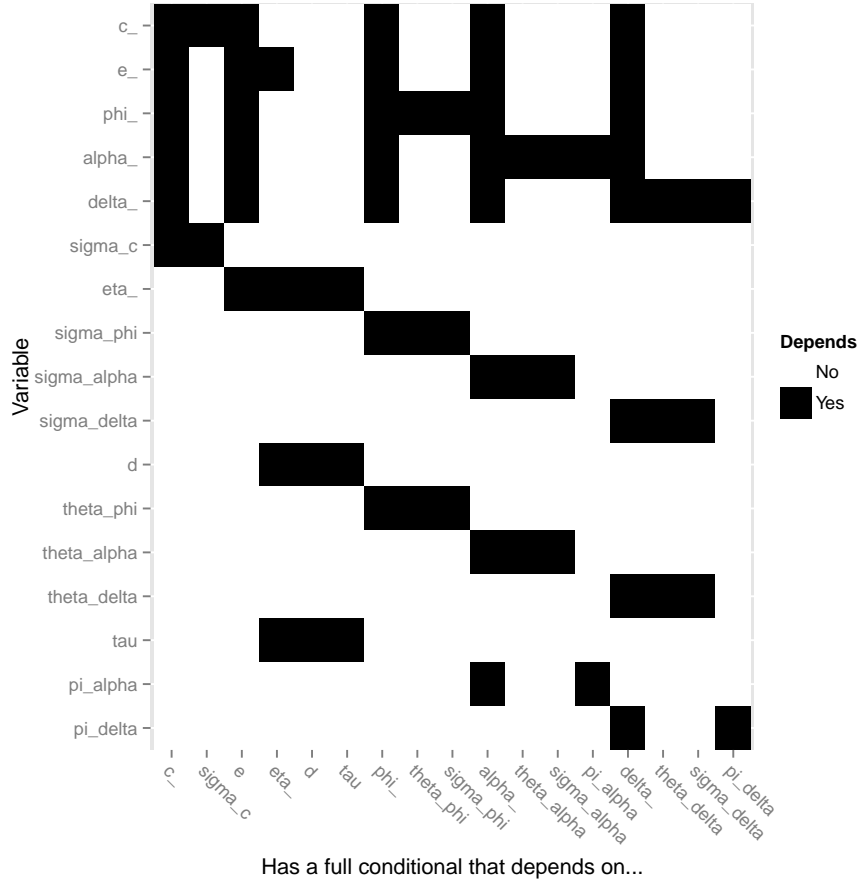
### 3 The Gibbs Sampler

For certain parameters, the full conditional distribution is independent of other key parameters. For example, the full conditional distribution of  $c_1$  does not contain  $c_2$ . Hence,  $c_1$  and  $c_2$  can be sampled in parallel in a single Gibbs step. Obvious sets of parameters that can be jointly sampled are:

- $c_1, \dots, c_N$
- $\varepsilon_{1,1}, \varepsilon_{1,2}, \dots, \varepsilon_{1,N}, \varepsilon_{2,N}, \dots, \varepsilon_{G,N}$
- $\eta_1^2, \dots, \eta_G^2$

- $\phi_1, \dots, \phi_G$
- $\alpha_1, \dots, \alpha_G$
- $\delta_1, \dots, \delta_G$

The following raster plot gives us a more complete idea of which parameters can be jointly sampled:



Hence, each of the following sets of parameters can be jointly sampled:

1.  $c_1, \dots, c_N$
2.  $\tau, \pi_\alpha, \pi_\delta$
3.  $d, \theta_\phi, \theta_\alpha, \theta_\delta$
4.  $\sigma_c, \sigma_\phi, \sigma_\alpha, \sigma_\delta, \eta_1^2, \dots, \eta_G^2$
5.  $\varepsilon_{1,1}, \varepsilon_{1,2}, \dots, \varepsilon_{1,N}, \varepsilon_{2,N}, \dots, \varepsilon_{G,N}$

6.  $\phi_1, \dots, \phi_G$

7.  $\alpha_1, \dots, \alpha_G$

8.  $\delta_1, \dots, \delta_G$

In order, these are the 8 steps of the Gibbs sampler. Alternatively, one could sample each triplet  $(\phi_g, \alpha_g, \delta_g)$  jointly in a single Metropolis step using  $p(\phi_g, \alpha_g, \delta_g \dots)$ .

## A Derivations of the Full Conditionals

Recall:

- $k(n)$  = treatment group of library  $n$ .
- $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$
- $G_\alpha$  = number of genes for which  $\alpha_g \neq 0$
- $G_\delta$  = number of genes for which  $\delta_g \neq 0$
- $I(x) = 0$  if  $x = 0$  and 1 otherwise.

Then from the model in Section 1, we get:

$$p(c_n \mid \dots) \propto \left[ \prod_{g=1}^G \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \right] \cdot \text{N}(c_n \mid 0, \sigma_c^2)$$

$$p(\sigma_c \mid \dots) = \left[ \prod_{n=1}^N \text{N}(c_n \mid 0, \sigma_c^2) \right] \cdot \text{U}(\sigma_c \mid 0, \sigma_{c0})$$

$$p(\varepsilon_{g,n} \mid \dots) \propto \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\varepsilon_{g,n} \mid 0, \eta_g^2)$$

$$p(\eta_g^2 \mid \dots) \propto \left[ \prod_{n=1}^N \text{N}(\varepsilon_{g,n} \mid 0, \eta_g^2) \right] \cdot \text{Inv-Gamma} \left( \eta_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2} \right)$$

$$p(d \mid \dots) \propto \left[ \prod_{g=1}^G \text{Inv-Gamma} \left( \eta_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2} \right) \right] \cdot \text{U}(d \mid 0, d_0)$$

$$p(\tau^2 \mid \dots) \propto \left[ \prod_{g=1}^G \text{Inv-Gamma} \left( \eta_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2} \right) \right] \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau)$$

$$p(\phi_g \mid \dots) \propto \left[ \prod_{n=1}^N \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \right] \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)$$

$$p(\theta_\phi \mid \dots) \propto \left[ \prod_{g=1}^G \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \right] \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2)$$

$$p(\sigma_\phi \mid \dots) \propto \left[ \prod_{g=1}^G \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \right] \cdot \text{U}(\sigma_\phi \mid 0, \sigma_{\phi 0})$$



$$\begin{aligned}
p(\alpha_g \mid \dots) &\propto \left[ \prod_{k(n) \neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \right] \\
&\quad \times \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \\
p(\theta_\alpha \mid \dots) &\propto \left[ \prod_{g=1}^G \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2) \\
p(\sigma_\alpha \mid \dots) &\propto \left[ \prod_{g=1}^G \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \right] \cdot \text{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0}) \\
p(\pi_\alpha \mid \dots) &\propto \left[ \prod_{g=1}^G \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \right] \cdot \text{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha) \\
p(\delta_g \mid \dots) &\propto \left[ \prod_{k(n)=2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \delta_g, \delta_g))) \right] \\
&\quad \times \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \\
p(\theta_\delta \mid \dots) &\propto \left[ \prod_{g=1}^G \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2) \\
p(\sigma_\delta \mid \dots) &\propto \left[ \prod_{g=1}^G \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \right] \cdot \text{U}(\sigma_\delta \mid 0, \sigma_{\delta 0}) \\
p(\pi_\delta \mid \dots) &\propto \left[ \prod_{g=1}^G \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \right] \cdot \text{Beta}(\pi_\delta \mid a_\delta, b_\delta) \\
p(\phi_g, \alpha_g, \delta_g \mid \dots) &\propto \left[ \prod_{n=1}^N \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \right] \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \\
&\quad \times \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \times \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}
\end{aligned}$$

### A.1 Transformations of Standard Deviations

Let  $\sigma$  be a standard deviation parameter and let  $p(\sigma \mid \dots)$  be its full conditional distribution. Then, by a transformation of variables,

$$\begin{aligned}
p(\sigma^2 \mid \dots) &= p(\sqrt{\sigma^2} \mid \dots) \cdot \left| \frac{d}{d\sigma^2} \sqrt{\sigma^2} \right| \\
&= p(\sigma \mid \dots) \frac{1}{2} (\sigma^2)^{-1/2}
\end{aligned}$$

I use this transformation several times in the next sections.

## A.2 $p(c_n \mid \dots)$ : Metropolis

$$\begin{aligned}
p(c_n \mid \dots) &\propto \left[ \prod_{g=1}^G \text{Poisson}(y_{g,n} \mid \lambda_{g,n}) \right] \cdot \text{N}(c_n \mid 0, \sigma_c^2) \\
&\propto \left[ \prod_{g=1}^G \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \right] \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right) \\
&= \exp\left(\sum_{g=1}^G [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{c_n^2}{2\sigma_c^2}\right) \\
&= \exp\left(\sum_{g=1}^G [y_{g,n}(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{c_n^2}{2\sigma_c^2}\right) \\
&= \exp\left(c_n G \bar{y}_{\cdot,n} + \sum_{g=1}^G [y_{g,n}(\varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \sum_{g=1}^G \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{c_n^2}{2\sigma_c^2}\right) \\
&\propto \exp\left(c_n G \bar{y}_{\cdot,n} - \exp(c_n) \sum_{g=1}^G \exp(\varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{c_n^2}{2\sigma_c^2}\right)
\end{aligned}$$

## A.3 $p\left(\frac{1}{\sigma_c^2} \mid \dots\right)$ Truncated Gamma

$$\begin{aligned}
p(\sigma_c^2 \mid \dots) &= p(\sigma_c \mid \dots) \frac{1}{2} (\sigma_c^2)^{-1/2} \quad (\text{transformation in Section A.1}) \\
&\propto \left[ \prod_{n=1}^N \text{N}(c_n \mid 0, \sigma_c^2) \right] \cdot \text{U}(\sigma_c \mid 0, \sigma_{c0}) \frac{1}{2} (\sigma_c^2)^{-1/2} \\
&\propto \prod_{n=1}^N \left[ \frac{1}{\sqrt{\sigma_c^2}} \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right) \right] \cdot \text{I}(0 < \sigma_c < \sigma_{c0}) (\sigma_c^2)^{-1/2} \\
&= (\sigma_c^2)^{-N/2} \exp\left(-\frac{1}{\sigma_c^2} \frac{1}{2} \sum_{n=1}^N c_n^2\right) \cdot \text{I}(0 < \sigma_c < \sigma_{c0}) (\sigma_c^2)^{-1/2} \\
&= (\sigma_c^2)^{-(N/2-1/2+1)} \exp\left(-\frac{1}{\sigma_c^2} \frac{1}{2} \sum_{n=1}^N c_n^2\right) \cdot \text{I}(0 < \sigma_c < \sigma_{c0})
\end{aligned}$$

which is the kernel of a truncated inverse gamma distribution. Hence:

$$p\left(\frac{1}{\sigma_c^2} \mid \dots\right) = \text{Gamma}\left(\frac{1}{\sigma_c^2} \mid \text{shape} = \frac{N-1}{2}, \text{rate} = \frac{1}{2} \sum_{n=1}^N c_n^2\right) I\left(\frac{1}{\sigma_c^2} > \frac{1}{\sigma_{c0}^2}\right)$$

#### A.4 $p(\varepsilon_{g,n} \mid \dots)$ Metropolis

$$\begin{aligned} p(\varepsilon_{g,n} \mid \dots) &= \text{Poisson}(y_{g,n} \mid \lambda_{g,n}) \cdot \text{N}(\varepsilon_{g,n} \mid 0, \eta_g^2) \\ &\propto \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \exp\left(-\frac{\varepsilon_{g,n}^2}{2\eta_g^2}\right) \\ &= \exp\left(y_{g,n} \log \lambda_{g,n} - \lambda_{g,n} - \frac{\varepsilon_{g,n}^2}{2\eta_g^2}\right) \\ &= \exp\left(y_{g,n}(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{\varepsilon_{g,n}^2}{2\eta_g^2}\right) \\ &= \exp\left(y_{g,n}\varepsilon_{g,n} - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{\varepsilon_{g,n}^2}{2\eta_g^2}\right) \end{aligned}$$

#### A.5 $p\left(\frac{1}{\eta_g^2} \mid \dots\right)$ Gamma

$$\begin{aligned} p(\eta_g^2 \mid \dots) &= \left[ \prod_{n=1}^N \text{N}(\varepsilon_{g,n} \mid 0, \eta_g^2) \right] \cdot \text{Inv-Gamma}\left(\eta_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right) \\ &\propto \left[ \prod_{n=1}^N (\eta_g^2)^{-1/2} \exp\left(-\frac{1}{\eta_g^2} \frac{\varepsilon_{g,n}^2}{2}\right) \right] \cdot (\eta_g^2)^{-(d/2+1)} \exp\left(-\frac{1}{\eta_g^2} \frac{d \cdot \tau^2}{2}\right) \\ &= \left[ (\eta_g^2)^{-N/2} \exp\left(-\frac{1}{\eta_g^2} \frac{1}{2} \sum_{n=1}^N \varepsilon_{g,n}^2\right) \right] \cdot (\eta_g^2)^{-(d/2+1)} \exp\left(-\frac{1}{\eta_g^2} \frac{d \cdot \tau^2}{2}\right) \\ &= (\eta_g^2)^{-((N+d)/2+1)} \exp\left(-\frac{1}{\eta_g^2} \frac{1}{2} \left(d \cdot \tau^2 + \sum_{n=1}^N \varepsilon_{g,n}^2\right)\right) \end{aligned}$$

which is the kernel of an inverse gamma distribution. Hence:

$$p\left(\frac{1}{\eta_g^2} \mid \dots\right) = \text{Gamma}\left(\frac{1}{\eta_g^2} \mid \text{shape} = \frac{N+d}{2}, \text{rate} = \frac{1}{2} \left(d \cdot \tau^2 + \sum_{n=1}^N \varepsilon_{g,n}^2\right)\right)$$

### A.6 $p(d \mid \dots)$ : Metropolis

$$\begin{aligned}
p(d \mid \dots) &= \left[ \prod_{g=1}^G \text{Inv-Gamma} \left( \eta_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2} \right) \right] \cdot \text{U}(d \mid 0, d_0) \\
&\propto \prod_{g=1}^G \left[ \Gamma(d/2)^{-1} \left( \frac{d \cdot \tau^2}{2} \right)^{d/2} (\eta_g^2)^{-(d/2+1)} \exp \left( -\frac{1}{\eta_g^2} \frac{d \cdot \tau^2}{2} \right) \right] I(0 < d < d_0) \\
&\propto \Gamma(d/2)^{-G} \left( \frac{d \cdot \tau^2}{2} \right)^{Gd/2} \left( \prod_{g=1}^G \eta_g^2 \right)^{-(d/2+1)} \exp \left( -\frac{d \cdot \tau^2}{2} \sum_{g=1}^G \frac{1}{\eta_g^2} \right) I(0 < d < d_0)
\end{aligned}$$

### A.7 $p(\tau^2 \mid \dots)$ : Gamma

$$\begin{aligned}
p(\tau^2 \mid \dots) &= \left[ \prod_{g=1}^G \text{Inv-Gamma} \left( \eta_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2} \right) \right] \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\
&\propto \left[ \Gamma(d/2)^{-G} \left( \frac{d \cdot \tau^2}{2} \right)^{Gd/2} \left( \prod_{g=1}^G \eta_g^2 \right)^{-(d/2+1)} \exp \left( -\frac{d \cdot \tau^2}{2} \sum_{g=1}^G \frac{1}{\eta_g^2} \right) \right] \cdot (\tau^2)^{a_\tau-1} \exp(-b_\tau \tau^2) \\
&\propto \left[ (\tau^2)^{Gd/2} \exp \left( -\tau^2 \cdot \frac{d}{2} \sum_{g=1}^G \frac{1}{\eta_g^2} \right) \right] \cdot (\tau^2)^{a_\tau-1} \exp(-b_\tau \tau^2) \\
&= (\tau^2)^{Gd/2+a_\tau-1} \exp \left( -\tau^2 \left( b_\tau + \frac{d}{2} \sum_{g=1}^G \frac{1}{\eta_g^2} \right) \right)
\end{aligned}$$

Hence:

$$p(\tau^2 \mid \dots) = \text{Gamma} \left( \tau^2 \mid \text{shape} = a_\tau + \frac{Gd}{2}, \text{rate} = b_\tau + \frac{d}{2} \sum_{g=1}^G \frac{1}{\eta_g^2} \right)$$

### A.8 $p(\phi_g \mid \dots)$ : Metropolis

$$\begin{aligned}
p(\phi_g \mid \dots) &= \left[ \prod_{n=1}^N \text{Poisson}(y_{g,n} \mid \lambda_{g,n}) \right] \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \\
&\propto \left[ \prod_{n=1}^N \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \right] \cdot \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \\
&= \exp\left(\sum_{n=1}^N [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \\
&= \exp\left(\sum_{n=1}^N [y_{g,n}(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \\
&\propto \exp\left(\sum_{n=1}^N [y_{g,n}\mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right)
\end{aligned}$$

### A.9 $p(\theta_\phi \mid \dots)$ : Normal

$$\begin{aligned}
p(\theta_\phi \mid \dots) &= \left[ \prod_{g=1}^G \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \right] \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2) \\
&\propto \left[ \prod_{g=1}^G \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \right] \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\sum_{g=1}^G \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\sum_{g=1}^G \phi_g^2 - 2\theta_\phi \sum_{g=1}^G \phi_g + G\theta_\phi^2}{2\sigma_\phi^2}\right) \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\sum_{g=1}^G \phi_g^2 - 2\theta_\phi \sum_{g=1}^G \phi_g + G\theta_\phi^2}{2\sigma_\phi^2} - \frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g^2 - 2\gamma_\phi^2 (\sum_{g=1}^G \phi_g) \theta_\phi + G\gamma_\phi^2 \theta_\phi^2}{2\sigma_\phi^2 \gamma_\phi^2} - \frac{\sigma_\phi^2 \theta_\phi^2}{2\sigma_\phi^2 \gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g^2 - 2\gamma_\phi^2 (\sum_{g=1}^G \phi_g) \theta_\phi + (G\gamma_\phi^2 + \sigma_\phi^2) \theta_\phi^2}{2\sigma_\phi^2 \gamma_\phi^2}\right) \\
&\propto \exp\left(-\frac{(G\gamma_\phi^2 + \sigma_\phi^2) \left(\theta_\phi - \frac{\gamma_\phi^2 (\sum_{g=1}^G \phi_g)}{G\gamma_\phi^2 + \sigma_\phi^2}\right)^2}{2\sigma_\phi^2 \gamma_\phi^2}\right)
\end{aligned}$$

Hence:

$$p(\theta_\phi \mid \dots) = N \left( \theta_\phi \mid \frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g}{G\gamma_\phi^2 + \sigma_\phi^2}, \frac{\gamma_\phi^2 \sigma_\phi^2}{G\gamma_\phi^2 + \sigma_\phi^2} \right)$$

#### A.10 $p\left(\frac{1}{\sigma_\phi^2} \mid \dots\right)$ : Truncated Gamma

$$\begin{aligned} p(\sigma_\phi^2 \mid \dots) &= p(\sigma_\phi \mid \dots) \frac{1}{2} (\sigma_\phi^2)^{-1/2} \quad (\text{transformation in Section A.1}) \\ &\propto \left[ \prod_{g=1}^G N(\phi_g \mid \theta_\phi, \sigma_\phi^2) \right] \cdot U(\sigma_\phi \mid 0, \sigma_{\phi 0}) (\sigma_\phi^2)^{-1/2} \\ &\propto \left[ \prod_{g=1}^G (\sigma_\phi^2)^{-1/2} \exp \left( -\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2} \right) \right] I(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2) (\sigma_\phi^2)^{-1/2} \\ &= (\sigma_\phi^2)^{-G/2} \exp \left( -\sum_{g=1}^G \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2} \right) I(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2) (\sigma_\phi^2)^{-1/2} \\ &= (\sigma_\phi^2)^{-(G/2-1/2+1)} \exp \left( -\frac{1}{\sigma_\phi^2} \frac{1}{2} \sum_{g=1}^G (\phi_g - \theta_\phi)^2 \right) I(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2) \end{aligned}$$

which is the kernel of a truncated inverse gamma distribution. Hence:

$$p \left( \frac{1}{\sigma_\phi^2} \mid \dots \right) = \text{Gamma} \left( \text{shape} = \frac{G-1}{2}, \text{rate} = \frac{1}{2} \sum_{g=1}^G (\phi_g - \theta_\phi)^2 \right) I \left( \frac{1}{\sigma_\phi^2} > \frac{1}{\sigma_{\phi 0}^2} \right)$$

### A.11 $p(\alpha_g \mid \dots)$ : Metropolis

$$\begin{aligned}
p(\alpha_g \mid \dots) &= \left[ \prod_{k(n) \neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \right] \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \\
&\propto \left[ \prod_{k(n) \neq 2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \right] \exp \left( -\frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} \right)^{I(\alpha_g)} \pi_\alpha^{1-I(\alpha_g)} (1 - \pi_\alpha)^{I(\alpha_g)} \\
&= \exp \left( \sum_{k(n) \neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - I(\alpha_g) \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + (1 - I(\alpha_g)) \log \pi_\alpha + I(\alpha_g) \log(1 - \pi_\alpha) \right) \\
&= \exp \left( \sum_{k(n) \neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - I(\alpha_g) \left( \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + \log(1 - \pi_\alpha) \right) + (1 - I(\alpha_g)) \log \pi_\alpha \right) \\
&= \exp \left( \sum_{k(n) \neq 2} [y_{g,n} (c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] \right. \\
&\quad \left. - I(\alpha_g) \left( \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + \log(1 - \pi_\alpha) \right) + (1 - I(\alpha_g)) \log \pi_\alpha \right) \\
&\propto \exp \left( \sum_{k(n) \neq 2} [y_{g,n} \cdot \mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] \right. \\
&\quad \left. - I(\alpha_g) \left( \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + \log(1 - \pi_\alpha) \right) + (1 - I(\alpha_g)) \log \pi_\alpha \right)
\end{aligned}$$

### A.12 $p(\theta_\alpha \mid \dots)$ : Normal

$$\begin{aligned}
p(\theta_\alpha \mid \dots) &= \left[ \prod_{g=1}^G \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2) \\
&\propto \left[ \prod_{\alpha_g \neq 0} \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2) \right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2)
\end{aligned}$$

From algebra similar to the derivation of  $p(\theta_\phi \mid \dots)$ , we get:

$$p(\theta_\alpha \mid \dots) = N \left( \theta_\alpha \mid \frac{\gamma_\alpha^2 \sum_{\alpha_g \neq 0} \alpha_g}{G_\alpha \gamma_\alpha^2 + \sigma_\alpha^2}, \frac{\gamma_\alpha^2 \sigma_\alpha^2}{G_\alpha \gamma_\alpha^2 + \sigma_\alpha^2} \right)$$

### A.13 $p\left(\frac{1}{\sigma_\alpha^2} \mid \dots\right)$ : Truncated Gamma

$$\begin{aligned}
p(\sigma_\alpha^2 \mid \dots) &= p(\sigma_\alpha \mid \dots) \frac{1}{2} (\sigma_\alpha^2)^{-1/2} \quad (\text{transformation in Section A.1}) \\
&\propto \left[ \prod_{g=1}^G \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) N(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \right] \cdot U(\sigma_\alpha \mid 0, \sigma_{\alpha 0}) (\sigma_\alpha^2)^{-1/2} \\
&\propto \prod_{\alpha_g \neq 0} N(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2) (\sigma_\alpha^2)^{-1/2} \\
&\propto \prod_{\alpha_g \neq 0} (\sigma_\alpha^2)^{-1/2} \exp\left(-\frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2) (\sigma_\alpha^2)^{-1/2} \\
&= (\sigma_\alpha^2)^{-G_\alpha/2} \exp\left(-\frac{1}{\theta_\alpha^2} \frac{1}{2} \sum_{\alpha_g \neq 0} (\alpha_g - \theta_\alpha)^2\right) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2) (\sigma_\alpha^2)^{-1/2} \\
&= (\sigma_\alpha^2)^{-(G_\alpha/2 - 1/2 + 1)} \exp\left(-\frac{1}{\theta_\alpha^2} \frac{1}{2} \sum_{\alpha_g \neq 0} (\alpha_g - \theta_\alpha)^2\right) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2)
\end{aligned}$$

which is the kernel of a truncated inverse gamma distribution. Hence:

$$p\left(\frac{1}{\sigma_\alpha^2} \mid \dots\right) = \text{Gamma}\left(\frac{1}{\sigma_\alpha^2} \mid \text{shape} = \frac{G_\alpha - 1}{2}, \text{rate} = \frac{1}{2} \sum_{\alpha_g \neq 0} (\alpha_g - \theta_\alpha)^2\right) I\left(\frac{1}{\sigma_\alpha^2} > \frac{1}{\sigma_{\alpha 0}^2}\right)$$

### A.14 $p(\pi_\alpha \mid \dots)$ : Beta

$$\begin{aligned}
p(\pi_\alpha \mid \dots) &= \left[ \prod_{g=1}^G \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) N(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \right] \cdot \text{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha) \\
&\propto [\pi_\alpha^{G-G_\alpha} (1 - \pi_\alpha)^{G_\alpha}] \pi_\alpha^{a_\tau-1} (1 - \pi_\alpha)^{b_\tau-1} \\
&= \pi_\alpha^{G-G_\alpha+a_\tau-1} (1 - \pi_\alpha)^{G_\alpha+b_\tau-1}
\end{aligned}$$

Hence:

$$p(\pi_\alpha \mid \dots) = \text{Beta}(G - G_\alpha + a_\tau, G_\alpha + b_\tau)$$



### A.15 $p(\delta_g \mid \dots)$ : Metropolis

$$\begin{aligned}
p(\delta_g \mid \dots) &= \left[ \prod_{k(n) \neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \right] \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) \text{N}(\theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \\
&\propto \left[ \prod_{k(n) \neq 2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \right] \exp \left( -\frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2} \right)^{I(\delta_g)} \pi_\delta^{1-I(\delta_g)} (1 - \pi_\delta)^{I(\delta_g)} \\
&= \exp \left( \sum_{k(n) \neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - I(\delta_g) \frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2} + (1 - I(\delta_g)) \log \pi_\delta + I(\delta_g) \log(1 - \pi_\delta) \right) \\
&= \exp \left( \sum_{k(n) \neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - I(\delta_g) \left( \frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2} + \log(1 - \pi_\delta) \right) + (1 - I(\delta_g)) \log \pi_\delta \right) \\
&= \exp \left( \sum_{k(n) \neq 2} [y_{g,n} (c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] \right. \\
&\quad \left. - I(\delta_g) \left( \frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2} + \log(1 - \pi_\delta) \right) + (1 - I(\delta_g)) \log \pi_\delta \right) \\
&\propto \exp \left( \sum_{k(n) \neq 2} [y_{g,n} \cdot \mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] \right. \\
&\quad \left. - I(\delta_g) \left( \frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2} + \log(1 - \pi_\delta) \right) + (1 - I(\delta_g)) \log \pi_\delta \right)
\end{aligned}$$

### A.16 $p(\theta_\delta \mid \dots)$ : Normal

$$\begin{aligned}
p(\theta_\delta \mid \dots) &= \left[ \prod_{g=1}^G \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2) \\
&\propto \left[ \prod_{\delta_g \neq 0} \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2) \right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2)
\end{aligned}$$

From algebra similar to the derivation of  $p(\theta_\phi \mid \dots)$ , we get:

$$p(\theta_\delta \mid \dots) = N \left( \frac{\gamma_\delta^2 \sum_{\delta_g \neq 0} \delta_g}{G_\delta \gamma_\delta^2 + \sigma_\delta^2}, \frac{\gamma_\delta^2 \sigma_\delta^2}{G_\delta \gamma_\delta^2 + \sigma_\delta^2} \right)$$

where  $G_\delta$  is the number of genes for which  $\delta_g \neq 0$ .

### A.17 $p\left(\frac{1}{\sigma_\delta^2} \mid \dots\right)$ : Truncated Gamma

$$\begin{aligned}
p(\sigma_\delta^2 \mid \dots) &= p(\sigma_\delta \mid \dots) \frac{1}{2} (\sigma_\delta^2)^{-1/2} \quad (\text{transformation in Section A.1}) \\
&\propto \left[ \prod_{g=1}^G \pi_\delta^{1-I(\delta_g)} [(1-\pi_\delta)N(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \right] \cdot U(\sigma_\delta \mid 0, \sigma_{\delta 0}^2) (\sigma_\delta^2)^{-1/2} \\
&\propto \prod_{\delta_g \neq 0} N(\delta_g \mid \theta_\delta, \sigma_\delta^2) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2) (\sigma_\delta^2)^{-1/2} \\
&\propto \prod_{\delta_g \neq 0} (\sigma_\delta^2)^{-1/2} \exp\left(-\frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2}\right) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2) (\sigma_\delta^2)^{-1/2} \\
&= (\sigma_\delta^2)^{-G_\delta/2} \exp\left(-\frac{1}{\theta_\delta^2} \frac{1}{2} \sum_{\delta_g \neq 0} (\delta_g - \theta_\delta)^2\right) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2) (\sigma_\delta^2)^{-1/2} \\
&= (\sigma_\delta^2)^{-(G_\delta/2-1/2+1)} \exp\left(-\frac{1}{\theta_\delta^2} \frac{1}{2} \sum_{\delta_g \neq 0} (\delta_g - \theta_\delta)^2\right) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2)
\end{aligned}$$

which is the kernel of a truncated inverse gamma distribution. Hence:

$$p\left(\frac{1}{\sigma_\delta^2} \mid \dots\right) = \text{Gamma}\left(\frac{1}{\sigma_\delta^2} \mid \text{shape} = \frac{G_\delta - 1}{2}, \text{rate} = \frac{1}{2} \sum_{\delta_g \neq 0} (\delta_g - \theta_\delta)^2\right) I\left(\frac{1}{\sigma_\delta^2} > \frac{1}{\sigma_{\delta 0}^2}\right)$$

### A.18 $p(\pi_\delta \mid \dots)$ : Beta

$$\begin{aligned}
p(\pi_\delta \mid \dots) &= \left[ \prod_{g=1}^G \pi_\delta^{1-I(\delta_g)} [(1-\pi_\delta)N(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \right] \cdot \text{Beta}(\pi_\delta \mid a_\delta, b_\delta) \\
&\propto [\pi_\delta^{G-G_\delta} (1-\pi_\delta)^{G_\delta}] \pi_\delta^{a_\tau-1} (1-\pi_\delta)^{b_\tau-1} \\
&= \pi_\delta^{G-G_\delta+a_\tau-1} (1-\pi_\delta)^{G_\delta+b_\tau-1}
\end{aligned}$$

where  $G_\delta$  is the number of genes for which  $\delta_g \neq 0$ . Hence:

$$p(\pi_\delta \mid \dots) = \text{Beta}(G - G_\delta + a_\tau, G_\delta + b_\tau)$$

### A.19 $p(\phi_g, \alpha_g, \delta_g \mid \dots)$ : Metropolis

$$\begin{aligned}
p(\phi_g, \alpha_g, \delta_g \mid \dots) &\propto \left[ \prod_{n=1}^N \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \right] \cdot N(\phi_g \mid \theta_\phi, \sigma_\phi^2) \\
&\quad \times \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) N(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \times \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) N(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \\
&\propto \exp \left( \sum_{n=1}^N [y_{g,n} \mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2} \right) \\
&\quad \times \pi_\alpha^{1-I(\alpha_g)} [(1 - \pi_\alpha) N(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \times \pi_\delta^{1-I(\delta_g)} [(1 - \pi_\delta) N(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \\
&\propto \exp \left( \sum_{n=1}^N [y_{g,n} \mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2} \right. \\
&\quad \left. - I(\alpha_g) \left( \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + \log(1 - \pi_\alpha) \right) + (1 - I(\alpha_g)) \log \pi_\alpha \right. \\
&\quad \left. - I(\delta_g) \left( \frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2} + \log(1 - \pi_\delta) \right) + (1 - I(\delta_g)) \log \pi_\delta \right)
\end{aligned}$$

## B Derivations of Metropolis proposals for point mass mixtures

### B.1 $\alpha_g$

I choose a proposal for  $\alpha_g$  with the form,

$$q(\alpha_g \mid \theta'_\alpha, \sigma'_\alpha, \pi'_\alpha) = I(\alpha_g = 0) \pi'_\alpha + I(\alpha_g \neq 0) (1 - \pi'_\alpha) N(\alpha_g \mid \theta'_\alpha, (\sigma'_\alpha)^2),$$

which resembles the prior for  $\alpha_g$  except that the parameters are updated to reflect the data,  $\underline{y} = (y_{1,1}, \dots, y_{G,N})$  (except for  $\pi'_\alpha$ , for which we simply use  $\pi_\alpha$ ). To find  $\theta'_\alpha$  and  $\sigma'_\alpha$ , we pretend that  $\alpha_g$  has a  $N(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)$  conditional likelihood,  $\theta_\alpha$  has a  $N(\theta_\alpha \mid 0, \gamma_\alpha^2)$  prior, and  $\sigma_\alpha$  is fixed. From the rule on pages 46 and 47 of Gelman's book, the conditional posterior distribution of  $\theta_\alpha$  is

$$N \left( \theta_\alpha \mid \frac{\sigma_\alpha^{-2} \alpha_g}{\gamma_\alpha^{-2} + \sigma_\alpha^{-2}}, (\gamma_\alpha^{-2} + \sigma_\alpha^{-2})^{-1} \right)$$

Hence, we let

$$\begin{aligned}
\theta'_\alpha &= \frac{\sigma_\alpha^{-2} \alpha_g}{\gamma_\alpha^{-2} + \sigma_\alpha^{-2}} \\
(\sigma_\alpha^2)' &= \text{Var}(\alpha_g) \\
&= \text{Var}(E(\alpha_g \mid \theta_\alpha)) + E(\text{Var}(\alpha_g \mid \theta_\alpha)) \\
&= \underbrace{\text{Var}(\theta_\alpha)}_{\text{Use prior variance.}} + E(\sigma_\alpha^2) \\
&= \gamma_\alpha^2 + \sigma_\alpha^2
\end{aligned}$$

## B.2 $\delta_g$

The proposal for  $\delta_g$  is analogous to that of  $\alpha_g$ :

$$q(\delta_g \mid \theta'_\delta, \sigma'_\delta, \pi'_\delta) = I(\delta_g = 0) \pi'_\delta + I(\delta_g \neq 0) (1 - \pi'_\delta) N(\delta_g \mid \theta'_\delta, (\sigma'_\delta)^2),$$

where:

$$\begin{aligned}
\theta'_\delta &= \frac{\sigma_\delta^{-2} \delta_g}{\gamma_\delta^{-2} + \sigma_\delta^{-2}} \\
(\sigma'_\delta)^2 &= \gamma_\delta^2 + \sigma_\delta^2 \\
\pi'_\delta &= \pi_\delta
\end{aligned}$$

## Bibliography