A Fully Bayesian Model for RNA-seq Data

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Contents

1 The Model

Let $y_{g,n}$ be the expression level of gene g ($g = 1, \ldots, G$) in library n ($n = 1, \ldots, N$). Let $\mu(n, \phi_g, \alpha_g, \delta_g)$ be the function given by:

$$\mu(n, \phi_g, \alpha_g, \delta_g) = \begin{cases} \phi_g - \alpha_g & \text{library } n \text{ is in treatment group 1} \\ \phi_g + \delta_g & \text{library } n \text{ is in treatment group 2} \\ \phi_g + \alpha_g & \text{library } n \text{ is in treatment group 3} \end{cases}$$

Then:

$$\begin{aligned}
y_{g,n} &\sim \text{Poisson}(\exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \\
&\quad c_n \sim \text{N}(0, \sigma_c^2) \\
&\quad\quad \sigma_c \sim \text{U}(0, \sigma_{c0}) \\
&\quad \varepsilon_{g,n} \sim \text{N}(0, \sigma_g^2) \\
&\quad\quad \sigma_g^2 \sim \text{Inv-Gamma}\left(\text{shape} = \frac{d \cdot \tau^2}{2},\ \text{rate} = \frac{d}{2}\right) \\
&\quad\quad\quad d \sim \text{U}(0, d_0) \\
&\quad\quad\quad \tau^2 \sim \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\
&\quad \phi_g \sim \text{N}(\theta_\phi, \sigma_\phi^2) \\
&\quad\quad \theta_\phi \sim \text{N}(0, \gamma_\phi^2) \\
&\quad\quad \sigma_\phi \sim \text{U}(0, \sigma_{\phi 0}) \\
&\quad \alpha_g \sim \pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\theta_\alpha, \sigma_\alpha^2) \\
&\quad\quad \theta_\alpha \sim \text{N}(0, \gamma_\alpha^2) \\
&\quad\quad \sigma_\alpha \sim \text{U}(0, \sigma_{\alpha 0}) \\
&\quad\quad \pi_\alpha \sim \text{Beta}(a_\alpha, b_\alpha) \\
&\quad \delta_g \sim \pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\theta_\delta, \sigma_\delta^2) \\
&\quad\quad \theta_\delta \sim \text{N}(0, \gamma_\delta^2) \\
&\quad\quad \sigma_\delta \sim \text{U}(0, \sigma_{\delta 0}) \\
&\quad\quad \pi_\delta \sim \text{Beta}(a_\delta, b_\delta)
\end{aligned}$$

where:

- Independence is implied unless otherwise specified.
- The parameters to the left of the "$\sim$" are implicitly conditioned on the parameters to the right.

2 Full Conditional Distributions

Let $k(n)$ be the treatment group of library n. Then:

$$
\begin{aligned}
p(c_n \mid \cdots) &= \prod_{g=1}^{G} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(c_n \mid 0, \sigma_c^2) \\
p\left(\sigma_c \mid \cdots\right) &= \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(c_n \mid 0, \sigma_c^2) \cdot \text{U}(\sigma_c \mid 0, \sigma_{c0}) \\
p(\varepsilon_{g,n} \mid \cdots) &= \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2) \\
p(\sigma_g^2 \mid \cdots) &= \prod_{n=1}^{N} \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2) \cdot \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \\
p(d \mid \cdots) &= \prod_{n=1}^{N} \prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \cdot \text{U}(d \mid 0, d_0) \\
p(\tau^2 \mid \cdots) &= \prod_{n=1}^{N} \prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\
p(\phi_g \mid \cdots) &= \prod_{n=1}^{N} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \\
p(\theta_\phi \mid \cdots) &= \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2) \\
p(\sigma_\phi \mid \cdots) &= \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \cdot \text{U}(\sigma_\phi \mid 0, \sigma_{\phi 0}) \\
p(\alpha_g \mid \cdots) &= \prod_{k(n) \neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \\
&\qquad \times (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \\
p(\theta_\alpha \mid \cdots) &= \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2) \\
p(\sigma_\alpha \mid \cdots) &= \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0})) \\
p(\pi_\alpha \mid \cdots) &= \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha)
\end{aligned}
$$

$$
\begin{aligned}
p(\delta_g \mid \cdots) &= \prod_{k(n)=2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \delta_g, \delta_g))) \\
&\quad \times (\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)) \\
p(\theta_\delta \mid \cdots) &= \prod_{k(n)=2} \prod_{g=1}^{G} (\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)) \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2) \\
p(\sigma_\delta \mid \cdots) &= \prod_{k(n)=2} \prod_{g=1}^{G} (\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)) \cdot \text{U}(\sigma_\delta \mid 0, \sigma_{\delta 0})) \\
p(\pi_\delta \mid \cdots) &= \prod_{k(n)=2} \prod_{g=1}^{G} (\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)) \cdot \text{Beta}(\pi_\delta \mid a_\delta, b_\delta)
\end{aligned}
$$

3 Simplifying and Sampling From the Full Conditionals

3.1 $p(c_n \mid \cdots)$: Metropolis

$$
p(c_n \mid \cdots) = \prod_{g=1}^{G} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(c_n \mid 0, \sigma_c^2)
$$

which is nothing immediately recognizable. I will sample from this distribution in a Metropolis step using the kernel of this distribution (taking $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$):

$$
\begin{aligned}
&\prod_{g=1}^{G} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right) \\
=&\prod_{g=1}^{G} \exp\left(y_{g,n} \log \lambda_{g,n} - \lambda_{g,n} - \frac{c_n^2}{2\sigma_c^2}\right) \\
=&\exp\left(\sum_{g=1}^{G} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{Gc_n^2}{2\sigma_c^2}\right)
\end{aligned}
$$

where the sum inside the exponent can be parallelized on the GPU.

3.2 $p\left(\frac{1}{\sigma_c^2} \mid \cdots\right)$: inverse cdf

$$\begin{aligned}
p\left(\sigma_c \mid \cdots\right) &= \prod_{n=1}^{N}\prod_{g=1}^{G} \mathrm{N}(c_n \mid 0, \sigma_c^2) \cdot \mathrm{U}(\sigma_c \mid 0, \sigma_{c0}) \\
&\propto \prod_{n=1}^{N}\prod_{g=1}^{G} \frac{1}{\sqrt{\sigma_c^2}} \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right) \cdot \mathrm{I}(0 < \sigma_c < \sigma_{c0}) \\
&= (\sigma_c^2)^{-NG/2} \exp\left(-\frac{1}{\sigma_c^2}\frac{G}{2}\sum_{n=1}^{N} c_n^2\right) \cdot \mathrm{I}(0 < \sigma_c < \sigma_{c0})
\end{aligned}$$

which, for constants a and $b = \frac{G}{2}\sum_{n=1}^{N} c_n^2$, can be written as

$$a \cdot (\sigma_c^2)^{-NG/2} \exp\left(-\frac{1}{\sigma_c^2} b\right) \mathrm{I}(0 < \sigma_c^2 < \sigma_{c0}^2)$$

Transformation: let $z = g(\sigma_c) = \sigma_c^2$ so that $g^{-1}(z) = \sqrt{z}$ and:

$$\begin{aligned}
p\left(\sigma_c^2 = z \mid \cdots\right) &= p(\sigma_c = g^{-1}(z) \mid \cdots) \left|\frac{dg^{-1}(z)}{dz}\right| \\
&= a \cdot z^{-NG/2} \exp\left(-\frac{1}{(\sqrt{z})^2} b\right) I\left(0 < z < \sigma_{c0}^2\right) \left|\frac{1}{2} z^{-1/2}\right| \\
&= \frac{a}{2} z^{-(NG/2-1/2+1)} \exp\left(-\frac{1}{z} b\right) I\left(0 < z < \sigma_{c0}^2\right) \\
&= \text{Inv-Gamma}\left(z \mid \text{shape} = \frac{NG-1}{2},\ \text{rate} = \frac{1}{b}\right) I\left(0 < z < \sigma_{c0}^2\right)
\end{aligned}$$

Recalling that $\frac{1}{b} = \frac{2}{G}\frac{1}{\sum_{n=1}^{N} c_n^2}$,

$$p\left(\frac{1}{\sigma_c^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_c^2} \mid \text{shape} = \frac{NG-1}{2},\ \text{rate} = \frac{2}{G}\frac{1}{\sum_{n=1}^{N} c_n^2}\right) I\left(\frac{1}{\sigma_c^2} < \frac{1}{\sigma_{c0}^2}\right)$$

3.3 $p(\varepsilon_{g,n} \mid \cdots)$: Metropolis

$$p(\varepsilon_{g,n} \mid \cdots) = \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \mathrm{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)$$

which is nothing immediately recognizable. I will sample from this distribution in a Metropolis step using the kernel:

$$\begin{aligned}
&\lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \exp\left(-\frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right) \\
&= \exp\left(y_{g,n} \log \lambda_{g,n} - \lambda_{g,n} - \frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right)
\end{aligned}$$

where $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$. The $\varepsilon_{g,n}$'s will be sampled in parallel across genes on the GPU.

3.4 $p\left(\frac{1}{\sigma_g^2} \mid \cdots\right)$ inverse cdf

$$\begin{aligned}
p\left(\sigma_g^2 \mid \cdots\right) &= \prod_{n=1}^{N} \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2) \cdot \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \\
&\propto \prod_{n=1}^{N} (\sigma_g^2)^{-1/2} \exp\left(-\frac{1}{\sigma_g^2} \frac{\varepsilon_{g,n}^2}{2}\right) \cdot \left(\sigma_g^2\right)^{-(d\tau^2/2+1)} \exp\left(-\frac{1}{\sigma_g^2} \frac{2}{d}\right) \\
&= \prod_{n=1}^{N} (\sigma_g^2)^{-(d\tau^2+3)/2} \exp\left(-\frac{1}{\sigma_g^2}\left(\frac{\varepsilon_{g,n}^2}{2} + \frac{2}{d}\right)\right) \\
&= (\sigma_g^2)^{-N(d\tau^2+3)/2} \exp\left(-\frac{1}{\sigma_g^2}\left(\frac{2N}{d} + \frac{1}{2}\sum_{n=1}^{N} \varepsilon_{g,n}^2\right)\right)
\end{aligned}$$

The last line is the kernel of an inverse gamma distribution with shape parameter $N\frac{d\tau^2+3}{2} - 1$ and rate parameter $\left(\frac{2N}{d} + \frac{1}{2}\sum_{n=1}^{N} \varepsilon_{g,n}^2\right)^{-1}$. Hence:

$$p\left(\frac{1}{\sigma_g^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_g^2} \mid \text{shape} = N\frac{d\tau^2+3}{2} - 1, \ \text{rate} = \left(\frac{2N}{d} + \frac{1}{2}\sum_{n=1}^{N} \varepsilon_{g,n}^2\right)^{-1}\right)$$

The $1/\sigma_g^2$'s will be sampled in parallel on the GPU.

3.5 $p(d \mid \cdots)$: Metropolis

$$
\begin{aligned}
p(d \mid \cdots) &= \prod_{n=1}^{N} \prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \cdot \text{U}(d \mid 0, d_0) \\
&= \prod_{n=1}^{N} \prod_{g=1}^{G} \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-1} \left(\frac{d}{2}\right)^{-d\tau^2/2} (\sigma_g^2)^{-(d\tau^2/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{2}{d}\right) I(0 < d < d_0) \\
&= \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-GN} \left(\frac{d}{2}\right)^{-GNd\tau^2/2} \prod_{n=1}^{N} \prod_{g=1}^{G} (\sigma_g^2)^{-(d\tau^2/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{2}{d}\right) I(0 < d < d_0) \\
&= \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-GN} \left(\frac{d}{2}\right)^{-GNd\tau^2/2} \prod_{g=1}^{G} (\sigma_g^2)^{-N(d\tau^2/2+1)} \exp\left(-\frac{N}{\sigma_g^2}\frac{2}{d}\right) I(0 < d < d_0) \\
&= \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-GN} \left(\frac{d}{2}\right)^{-GNd\tau^2/2} \left(\prod_{g=1}^{G} \sigma_g^2\right)^{-N(d\tau^2/2+1)} \exp\left(-\frac{2N}{d}\sum_{g=1}^{G}\frac{1}{\sigma_g^2}\right) I(0 < d < d_0)
\end{aligned}
$$

I will sample d with a Metropolis step using the above density. Sums and products over g ($g = 1, \ldots, G$) will be done in parallel on the GPU.

3.6 $p(\tau^2 \mid \cdots)$: Metropolis

$$
\begin{aligned}
p(\tau^2 \mid \cdots) &= \prod_{n=1}^{N} \prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\
p(\tau^2 \mid \cdots) &\propto \prod_{n=1}^{N} \prod_{g=1}^{G} \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-1} \left(\frac{d}{2}\right)^{-d\tau^2/2} (\sigma_g^2)^{-(d\tau^2/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{2}{d}\right) \\
&\quad \times (\tau^2)^{a_\tau - 1} \exp\left(-\tau^2 b_\tau\right) \\
&= (\tau^2)^{a_\tau - 1} \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-GN} \left(\frac{d}{2}\right)^{-GNd\tau^2/2} \left(\prod_{g=1}^{G} \sigma_g^2\right)^{-N(d\tau^2/2+1)} \exp\left(-GN\tau^2 b_\tau - \frac{2N}{d}\sum_{g=1}^{G}\frac{1}{\sigma_g^2}\right)
\end{aligned}
$$

I will sample τ^2 with a Metropolis step using the above density. Sums and products over g ($g = 1, \ldots, G$) will be done in parallel on the GPU.

3.7 $p(\phi_g \mid \cdots)$: Metropolis

$$\begin{aligned}
p(\phi_g \mid \cdots) &= \prod_{n=1}^{N} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \\
&\propto \prod_{n=1}^{N} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \\
&= \exp\left(\sum_{n=1}^{N} \left[y_{g,n} \log \lambda_{g,n} - \lambda_{g,n} - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right]\right) \\
&= \exp\left(-N\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2} + \sum_{n=1}^{N} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}]\right)
\end{aligned}$$

where $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$. I will sample the ϕ_g's in parallel using Metropolis steps.

3.8 $p(\theta_\phi \mid \cdots)$: Box-Mueller

$$\begin{aligned}
p(\theta_\phi \mid \cdots) &= \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2) \\
&\propto \prod_{n=1}^{N} \prod_{g=1}^{G} \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \prod_{n=1}^{N} \prod_{g=1}^{G} \exp\left(-\frac{\phi_g^2 - 2\phi_g\theta_\phi + \theta_\phi^2}{2\sigma_\phi^2} - \frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \prod_{n=1}^{N} \prod_{g=1}^{G} \exp\left(-\frac{2\gamma_\phi^2\phi_g^2 - 4\gamma_\phi^2\phi_g\theta_\phi + 2\gamma_\phi^2\theta_\phi^2 + 2\sigma_\phi^2\theta_\phi^2}{4\sigma_\phi^2\gamma_\phi^2}\right) \\
&\propto \prod_{n=1}^{N} \prod_{g=1}^{G} \exp\left(-\frac{(\gamma_\phi^2 + \sigma_\phi^2)\theta_\phi^2 - 2\gamma_\phi^2\phi_g\theta_\phi}{2\sigma_\phi^2\gamma_\phi^2}\right) \\
&= \exp\left(-\sum_{n=1}^{N} \sum_{g=1}^{G} \frac{(\gamma_\phi^2 + \sigma_\phi^2)\theta_\phi^2 - 2\gamma_\phi^2\phi_g\theta_\phi}{2\sigma_\phi^2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{NG(\gamma_\phi^2 + \sigma_\phi^2)\theta_\phi^2 - 2N\gamma_\phi^2\left(\sum_{g=1}^{G}\phi_g\right)\theta_\phi}{2\sigma_\phi^2\gamma_\phi^2}\right) \\
&\propto \exp\left(-\frac{NG(\gamma_\phi^2 + \sigma_\phi^2)\left[\theta_\phi - \frac{\gamma_\phi^2\sum_{g=1}^{G}\phi_g}{G(\gamma_\phi^2 + \sigma_\phi^2)}\right]^2}{2\sigma_\phi^2\gamma_\phi^2}\right)
\end{aligned}$$

which is the kernel of a normal distribution:

$$p(\theta_\phi \mid \cdots) = \mathrm{N}\left(\frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g}{G(\gamma_\phi^2 + \sigma_\phi^2)}, \frac{\sigma_\phi^2 \gamma_\phi^2}{NG(\sigma_\phi^2 + \gamma_\phi^2)}\right)$$

3.9 $p\left(\frac{1}{\sigma_\phi^2} \mid \ldots\right)$: inverse cdf

$$\begin{aligned}
p(\sigma_\phi \mid \cdots) &= \prod_{n=1}^N \prod_{g=1}^G \mathrm{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \cdot \mathrm{U}(\sigma_\phi \mid 0, \sigma_{\phi 0}) \\
&\propto \prod_{n=1}^N \prod_{g=1}^G (\sigma_\phi^2)^{-1/2} \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2) \\
&= (\sigma_\phi^2)^{-NG/2} \exp\left(-\sum_{n=1}^N \sum_{g=1}^G \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2) \\
&= (\sigma_\phi^2)^{-NG/2} \exp\left(-\frac{1}{\sigma_\phi^2} \frac{N}{2} \sum_{g=1}^G (\phi_g - \theta_\phi)^2\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2)
\end{aligned}$$

Transformation: let $z = g(\sigma_\phi) = \sigma_\phi^2$ so that $g^{-1}(z) = \sqrt{z}$. Then for some proportionality constant, a:

$$\begin{aligned}
p(\sigma_\phi^2 = z \mid \cdots) &= p(\sigma_\phi = g^{-1}(z) \mid \cdots) \left|\frac{g^{-1}(z)}{dz}\right| \\
&= p(\sigma_\phi = \sqrt{z} \mid \cdots) \left|\frac{1}{2} z^{-1/2}\right| \\
&= a \cdot (z)^{-NG/2} \exp\left(-\frac{1}{z} \frac{N}{2} \sum_{g=1}^G (\phi_g - \theta_\phi)^2\right) \mathrm{I}(0 < z < \sigma_{\phi 0}^2) z^{-1/2} \\
&= \frac{a}{2} \cdot z^{-(NG/2 - 1/2 + 1)} \exp\left(-\frac{1}{z} \frac{N}{2} \sum_{g=1}^G (\phi_g - \theta_\phi)^2\right) \mathrm{I}(0 < z < \sigma_{\phi 0}^2)
\end{aligned}$$

which is a truncated inverse gamma distribution with shape parameter $\frac{NG}{2} - \frac{1}{2}$ and rate parameter $\left(\frac{N}{2} \sum_{g=1}^G (\phi_g - \theta_\phi)^2\right)^{-1}$. Thus:

$$p\left(\frac{1}{\sigma_\phi^2} \mid \cdots\right) = \mathrm{Gamma}\left(\text{shape} = \frac{NG}{2} - \frac{1}{2}, \text{ rate} = \left(\frac{N}{2} \sum_{g=1}^G (\phi_g - \theta_\phi)^2\right)^{-1}\right) \mathrm{I}\left(\frac{1}{\sigma_\phi^2} > \frac{1}{\sigma_{\phi 0}^2}\right)$$

I will sample $1/\sigma_\phi^2$ using the inverse cdf method.

3.10 $p(\alpha_g \mid \cdots)$: Metropolis

$$p(\alpha_g \mid \cdots) = \prod_{k(n)\neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))$$
$$\times (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2))$$

Draw u_g $U(0, 1)$.

1. Case 1: if $u_g < \pi_\alpha$, use a Metropolis step to draw ϕ_g from a distribution with the kernel:

$$\begin{aligned} &\prod_{k(n)\neq 2} \text{Poisson}(y_{g,n} \mid \lambda_{g,n}) \\ \propto &\prod_{k(n)\neq 2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \\ = &\exp\left(\sum_{k(n)\neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}]\right) \end{aligned}$$

2. Case 2: if $u_g \geq \pi_\alpha$, use a Metropolis step to draw ϕ_g from a distribution with the kernel:

$$\text{Poisson}(y_{g,n} \mid \lambda_{g,n})) \cdot \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)$$

which, from similar work on the ϕ_g's, simplifies to:

$$\exp\left(-N_{13}\frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + \sum_{k(n)\neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}]\right)$$

where N_{13} is the number of libraries not in treatment group 2.

3.11 $p(\theta_\alpha \mid \cdots)$: Metropolis

$$p(\theta_\alpha \mid \cdots) = \prod_{k(n)\neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2)$$

Since some of the α_g's are 0 and some are nonzero, this expression cannot simplify any further. I will use a Metropolis step here.

3.12 $p(\sigma_\alpha \mid \cdots)$: Metropolis

$$p(\sigma_\alpha \mid \cdots) = \prod_{k(n)\neq 2} \prod_{g=1}^{G} (\pi_\alpha \mathrm{I}(\alpha_g = 0) + (1-\pi_\alpha)\mathrm{I}(\alpha_g \neq 0)\mathrm{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \mathrm{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0}))$$

Since some of the α_g's are 0 and some are nonzero, this expression cannot simplify any further. I will use a Metropolis step here.

3.13 $p(\pi_\alpha \mid \cdots)$: Metropolis

$$p(\pi_\alpha \mid \cdots) = \prod_{k(n)\neq 2} \prod_{g=1}^{G} (\pi_\alpha \mathrm{I}(\alpha_g = 0) + (1-\pi_\alpha)\mathrm{I}(\alpha_g \neq 0)\mathrm{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \mathrm{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha)$$

Since some of the α_g's are 0 and some are nonzero, this expression cannot simplify any further. I will use a Metropolis step here.

3.14 $p(\delta_g \mid \cdots)$: Metropolis

$$\begin{aligned} p(\delta_g \mid \cdots) = \prod_{k(n)\neq 2} & \mathrm{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \\ & \times (\pi_\delta \mathrm{I}(\delta_g = 0) + (1-\pi_\delta)\mathrm{I}(\delta_g \neq 0)\mathrm{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)) \end{aligned}$$

Draw u_g $U(0,1)$.

1. Case 1: if $u_g < \pi_\delta$, use a Metropolis step to draw ϕ_g from a distribution with the kernel:

$$\begin{aligned} & \prod_{k(n)\neq 2} \mathrm{Poisson}(y_{g,n} \mid \lambda_{g,n}) \\ \propto & \prod_{k(n)\neq 2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \\ = & \exp\left(\sum_{k(n)\neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] \right) \end{aligned}$$

2. Case 2: if $u_g \geq \pi_\delta$, use a Metropolis step to draw ϕ_g from a distribution with the kernel:

$$\mathrm{Poisson}(y_{g,n} \mid \lambda_{g,n})) \cdot \mathrm{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)$$

which, from similar work on the ϕ_g's, simplifies to:

$$\exp\left(-N_{13}\frac{(\delta_g-\theta_\delta)^2}{2\sigma_\delta^2}+\sum_{k(n)\neq 2}\left[y_{g,n}\log\lambda_{g,n}-\lambda_{g,n}\right]\right)$$

where N_{13} is the number of libraries not in treatment group 2.

3.15 $p(\theta_\delta \mid \cdots)$: Metropolis

$$p(\theta_\delta \mid \cdots) = \prod_{k(n)\neq 2}\prod_{g=1}^{G}(\pi_\delta \mathrm{I}(\delta_g=0)+(1-\pi_\delta)\mathrm{I}(\delta_g\neq 0)\mathrm{N}(\delta_g \mid \theta_\delta,\sigma_\delta^2))\cdot \mathrm{N}(\theta_\delta \mid 0,\gamma_\delta^2)$$

Since some of the δ_g's are 0 and some are nonzero, this expression cannot simplify any further. I will use a Metropolis step here.

3.16 $p(\sigma_\delta \mid \cdots)$: Metropolis

$$p(\sigma_\delta \mid \cdots) = \prod_{k(n)\neq 2}\prod_{g=1}^{G}(\pi_\delta \mathrm{I}(\delta_g=0)+(1-\pi_\delta)\mathrm{I}(\delta_g\neq 0)\mathrm{N}(\delta_g \mid \theta_\delta,\sigma_\delta^2))\cdot \mathrm{U}(\sigma_\delta \mid 0,\sigma_{\delta 0}))$$

Since some of the δ_g's are 0 and some are nonzero, this expression cannot simplify any further. I will use a Metropolis step here.

3.17 $p(\pi_\delta \mid \cdots)$: Metropolis

$$p(\pi_\delta \mid \cdots) = \prod_{k(n)\neq 2}\prod_{g=1}^{G}(\pi_\delta \mathrm{I}(\delta_g=0)+(1-\pi_\delta)\mathrm{I}(\delta_g\neq 0)\mathrm{N}(\delta_g \mid \theta_\delta,\sigma_\delta^2))\cdot \mathrm{Beta}(\pi_\delta \mid a_\delta,b_\delta)$$

Since some of the δ_g's are 0 and some are nonzero, this expression cannot simplify any further. I will use a Metropolis step here.