A Fully Bayesian Model for RNA-seq Data

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May 30, 2013

Contents

1 The Model

Let $y_{g,n}$ be the expression level of gene g $(g = 1, \ldots, G)$ in library n $(n = 1, \ldots, N)$. Let $\mu(n, \phi_g, \alpha_g, \delta_g)$ be the function given by:

$$\mu(n, \phi_g, \alpha_g, \delta_g) = \begin{cases} \phi_g - \alpha_g & \text{library } n \text{ is in treatment group 1} \\ \phi_g + \delta_g & \text{library } n \text{ is in treatment group 2} \\ \phi_g + \alpha_g & \text{library } n \text{ is in treatment group 3} \end{cases}$$

Then:

$$y_{g,n} \sim \text{Poisson}(\exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))$$
$$c_n \sim \text{N}(0, \sigma_c^2)$$
$$\sigma_c \sim \text{U}(0, \sigma_{c0})$$
$$\varepsilon_{g,n} \sim \text{N}(0, \sigma_g^2)$$
$$\sigma_g^2 \sim \text{Inv-Gamma}\left(\text{shape} = \frac{d \cdot \tau^2}{2}, \text{ rate} = \frac{d}{2}\right)$$
$$d \sim \text{U}(0, d_0)$$
$$\tau^2 \sim \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau)$$
$$\phi_g \sim \text{N}(\theta_\phi, \sigma_\phi^2)$$
$$\theta_\phi \sim \text{N}(0, \gamma_\phi^2)$$
$$\sigma_\phi \sim \text{U}(0, \sigma_{\phi 0})$$
$$\alpha_g \sim \pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\theta_\alpha, \sigma_\alpha^2)$$
$$\theta_\alpha \sim \text{N}(0, \gamma_\alpha^2)$$
$$\sigma_\alpha \sim \text{U}(0, \sigma_{\alpha 0})$$
$$\pi_\alpha \sim \text{Beta}(a_\alpha, b_\alpha)$$
$$\delta_g \sim \pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\theta_\delta, \sigma_\delta^2)$$
$$\theta_\delta \sim \text{N}(0, \gamma_\delta^2)$$
$$\sigma_\delta \sim \text{U}(0, \sigma_{\delta 0})$$
$$\pi_\delta \sim \text{Beta}(a_\delta, b_\delta)$$

where:

- Independence is implied unless otherwise specified.

- The parameters to the left of the "$\sim$" are implicitly conditioned on the parameters to the right.

2   Full Conditional Distributions

Let $k(n)$ be the treatment group of library n. Then:

$$p(c_n \mid \cdots) = \prod_{g=1}^{G} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(c_n \mid 0, \sigma_c^2)$$

$$p(\sigma_c \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(c_n \mid 0, \sigma_c^2) \cdot \text{U}(\sigma_c \mid 0, \sigma_{c0})$$

$$p(\varepsilon_{g,n} \mid \cdots) = \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)$$

$$p(\sigma_g^2 \mid \cdots) = \prod_{n=1}^{N} \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2) \cdot \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right)$$

$$p(d \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \cdot \text{U}(d \mid 0, d_0)$$

$$p(\tau^2 \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau)$$

$$p(\phi_g \mid \cdots) = \prod_{n=1}^{N} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)$$

$$p(\theta_\phi \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2)$$

$$p(\sigma_\phi \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \cdot \text{U}(\sigma_\phi \mid 0, \sigma_{\phi 0})$$

$$p(\alpha_g \mid \cdots) = \prod_{k(n) \neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))$$
$$\times (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2))$$

$$p(\theta_\alpha \mid \cdots) = \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2)$$

$$p(\sigma_\alpha \mid \cdots) = \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0}))$$

$$p(\pi_\alpha \mid \cdots) = \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha)\text{I}(\alpha_g \neq 0)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha)$$

$$p(\delta_g \mid \cdots) = \prod_{k(n)=2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \delta_g, \delta_g)))$$

$$\times \left(\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)\right)$$

$$p(\theta_\delta \mid \cdots) = \prod_{k(n)=2} \prod_{g=1}^{G} \left(\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)\right) \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2)$$

$$p(\sigma_\delta \mid \cdots) = \prod_{k(n)=2} \prod_{g=1}^{G} \left(\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)\right) \cdot \text{U}(\sigma_\delta \mid 0, \sigma_{\delta 0})$$

$$p(\pi_\delta \mid \cdots) = \prod_{k(n)=2} \prod_{g=1}^{G} \left(\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta)\text{I}(\delta_g \neq 0)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)\right) \cdot \text{Beta}(\pi_\delta \mid a_\delta, b_\delta)$$

3 Simplifying the Full Conditionals

3.1 $p(c_n \mid \cdots)$

$$p(c_n \mid \cdots) = \prod_{g=1}^{G} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(c_n \mid 0, \sigma_c^2)$$

which is nothing immediately recognizable. I will sample from this distribution in a metropolis step using the kernel of this distribution (taking $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$):

$$\prod_{g=1}^{G} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right)$$

$$= \prod_{g=1}^{G} \exp\left(y_{g,n} \log \lambda_{g,n} - \lambda_{g,n} - \frac{c_n^2}{2\sigma_c^2}\right)$$

$$= \exp\left(\sum_{g=1}^{G} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{Gc_n^2}{2\sigma_c^2}\right)$$

where the sum inside the exponent can be parallelized on the GPU.

3.2 $p\left(\frac{1}{\sigma_c^2}\mid\cdots\right)$

$$p\left(\sigma_c\mid\cdots\right) = \prod_{n=1}^{N}\prod_{g=1}^{G} \mathrm{N}(c_n\mid 0,\sigma_c^2)\cdot \mathrm{U}(\sigma_c\mid 0,\sigma_{c0})$$

$$\propto \prod_{n=1}^{N}\prod_{g=1}^{G}\frac{1}{\sqrt{\sigma_c^2}}\exp\left(-\frac{c_n^2}{2\sigma_c^2}\right)\cdot \mathrm{I}(0 < \sigma_c < \sigma_{c0})$$

$$= (\sigma_c^2)^{-NG/2}\exp\left(-\frac{1}{\sigma_c^2}\frac{G}{2}\sum_{n=1}^{N}c_n^2\right)\cdot \mathrm{I}(0 < \sigma_c < \sigma_{c0})$$

which, for constants a and $b = \frac{G}{2}\sum_{n=1}^{N}c_n^2$, can be written as

$$a\cdot (\sigma_c^2)^{-NG/2}\exp\left(-\frac{1}{\sigma_c^2}b\right)\mathrm{I}(0 < \sigma_c^2 < \sigma_{c0}^2)$$

Transformation: let $z = g(\sigma_c) = \sigma_c^2$ so that $g^{-1}(z) = \sqrt{z}$ and:

$$p\left(\sigma_c^2 = z\mid\cdots\right) = p(\sigma_c = g^{-1}(z)\mid\cdots)\left|\frac{dg^{-1}(z)}{dz}\right|$$

$$= a\cdot z^{-NG/2}\exp\left(-\frac{1}{(\sqrt{z})^2}b\right)I\left(0 < z < \sigma_{c0}^2\right)\left|\frac{1}{2}z^{-1/2}\right|$$

$$= \frac{a}{2}z^{-(NG/2-1/2+1)}\exp\left(-\frac{1}{z}b\right)I\left(0 < z < \sigma_{c0}^2\right)$$

$$= \mathrm{Inv\text{-}Gamma}\left(z\mid \mathrm{shape} = \frac{NG-1}{2},\ \mathrm{rate} = \frac{1}{b}\right)I\left(0 < z < \sigma_{c0}^2\right)$$

Recalling that $\frac{1}{b} = \frac{2}{G}\frac{1}{\sum_{n=1}^{N}c_n^2}$,

$$p\left(\frac{1}{\sigma_c^2}\mid\cdots\right) = \mathrm{Gamma}\left(\frac{1}{\sigma_c^2}\mid \mathrm{shape} = \frac{NG-1}{2},\ \mathrm{rate} = \frac{2}{G}\frac{1}{\sum_{n=1}^{N}c_n^2}\right)I\left(\frac{1}{\sigma_c^2} < \frac{1}{\sigma_{c0}^2}\right)$$

3.3 $p(\varepsilon_{g,n}\mid\cdots)$

$$p(\varepsilon_{g,n}\mid\cdots) = \mathrm{Poisson}(y_{g,n}\mid \exp(c_n + \varepsilon_{g,n} + \mu(n,\phi_g,\alpha_g,\delta_g)))\cdot \mathrm{N}(\varepsilon_{g,n}\mid 0,\sigma_g^2)$$

which is nothing immediately recognizable. I will sample from this distribution in a metropolis step using the kernel:

$$\lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \exp\left(-\frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right)$$

$$= \exp\left(y_{g,n} \log \lambda_{g,n} - \lambda_{g,n} - \frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right)$$

where $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$. The $\varepsilon_{g,n}$'s will be sampled in parallel across genes on the GPU.

3.4 $\quad p\left(\frac{1}{\sigma_g^2} \mid \cdots\right)$

$$p\left(\sigma_g^2 \mid \cdots\right) = \prod_{n=1}^{N} \mathrm{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2) \cdot \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right)$$

$$\propto \prod_{n=1}^{N} (\sigma_g^2)^{-1/2} \exp\left(-\frac{1}{\sigma_g^2}\frac{\varepsilon_{g,n}^2}{2}\right) \cdot (\sigma_g^2)^{-(d\tau^2/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{2}{d}\right)$$

$$= \prod_{n=1}^{N} (\sigma_g^2)^{-(d\tau^2+3)/2} \exp\left(-\frac{1}{\sigma_g^2}\left(\frac{\varepsilon_{g,n}^2}{2} + \frac{2}{d}\right)\right)$$

$$= (\sigma_g^2)^{-N(d\tau^2+3)/2} \exp\left(-\frac{1}{\sigma_g^2}\left(\frac{2N}{d} + \frac{1}{2}\sum_{n=1}^{N} \varepsilon_{g,n}^2\right)\right)$$

The last line is the kernel of an inverse gamma distribution with shape parameter $N\frac{d\tau^2+3}{2} - 1$ and rate parameter $\left(\frac{2N}{d} + \frac{1}{2}\sum_{n=1}^{N} \varepsilon_{g,n}^2\right)^{-1}$. Hence:

$$p\left(\frac{1}{\sigma_g^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_g^2} \mid \text{shape} = N\frac{d\tau^2+3}{2} - 1, \text{ rate} = \left(\frac{2N}{d} + \frac{1}{2}\sum_{n=1}^{N} \varepsilon_{g,n}^2\right)^{-1}\right)$$

The $1/\sigma_g^2$'s will be sampled in parallel on the GPU.

3.5 $p(d \mid \cdots)$

$$p(d \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \cdot \text{U}(d \mid 0, d_0)$$

$$= \prod_{n=1}^{N} \prod_{g=1}^{G} \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-1} \left(\frac{d}{2}\right)^{-d\tau^2/2} (\sigma_g^2)^{-(d\tau^2/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{2}{d}\right) I(0 < d < d_0)$$

$$= \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-GN} \left(\frac{d}{2}\right)^{-GNd\tau^2/2} \prod_{n=1}^{N} \prod_{g=1}^{G} (\sigma_g^2)^{-(d\tau^2/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{2}{d}\right) I(0 < d < d_0)$$

$$= \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-GN} \left(\frac{d}{2}\right)^{-GNd\tau^2/2} \prod_{g=1}^{G} (\sigma_g^2)^{-N(d\tau^2/2+1)} \exp\left(-\frac{N}{\sigma_g^2}\frac{2}{d}\right) I(0 < d < d_0)$$

$$= \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-GN} \left(\frac{d}{2}\right)^{-GNd\tau^2/2} \left(\prod_{g=1}^{G} \sigma_g^2\right)^{-N(d\tau^2/2+1)} \exp\left(-\frac{2N}{d}\sum_{g=1}^{G}\frac{1}{\sigma_g^2}\right) I(0 < d < d_0)$$

I will sample d with a metropolis step using the above density. Sums and products over g $(g = 1, \ldots, G)$ will be done in parallel on the GPU.

3.6 $p(\tau^2 \mid \cdots)$

$$p(\tau^2 \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d \cdot \tau^2}{2}, \text{rate} = \frac{d}{2}\right) \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau)$$

$$p(\tau^2 \mid \cdots) \propto \prod_{n=1}^{N} \prod_{g=1}^{G} \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-1} \left(\frac{d}{2}\right)^{-d\tau^2/2} (\sigma_g^2)^{-(d\tau^2/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{2}{d}\right)$$

$$\times (\tau^2)^{a_\tau - 1} \exp\left(-\tau^2 b_\tau\right)$$

$$= (\tau^2)^{a_\tau - 1} \left[\Gamma\left(\frac{d\tau^2}{2}\right)\right]^{-GN} \left(\frac{d}{2}\right)^{-GNd\tau^2/2} \left(\prod_{g=1}^{G} \sigma_g^2\right)^{-N(d\tau^2/2+1)} \exp\left(-GN\tau^2 b_\tau - \frac{2N}{d}\sum_{g=1}^{G}\frac{1}{\sigma_g^2}\right)$$

I will sample τ^2 with a metropolis step using the above density. Sums and products over g $(g = 1, \ldots, G)$ will be done in parallel on the GPU.

$$p(\phi_g \mid \cdots) = \prod_{n=1}^{N} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)$$

$$p(\theta_\phi \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2)$$

$$p(\sigma_\phi \mid \cdots) = \prod_{n=1}^{N} \prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \cdot \text{U}(\sigma_\phi \mid 0, \sigma_{\phi 0})$$

$$p(\alpha_g \mid \cdots) = \prod_{k(n) \neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))$$
$$\times \, (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha) \text{I}(\alpha_g \neq 0) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2))$$

$$p(\theta_\alpha \mid \cdots) = \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha) \text{I}(\alpha_g \neq 0) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2)$$

$$p(\sigma_\alpha \mid \cdots) = \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha) \text{I}(\alpha_g \neq 0) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0}))$$

$$p(\pi_\alpha \mid \cdots) = \prod_{k(n) \neq 2} \prod_{g=1}^{G} (\pi_\alpha \text{I}(\alpha_g = 0) + (1 - \pi_\alpha) \text{I}(\alpha_g \neq 0) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)) \cdot \text{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha)$$

$$p(\delta_g \mid \cdots) = \prod_{k(n) = 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \delta_g, \delta_g)))$$
$$\times \, (\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta) \text{I}(\delta_g \neq 0) \text{N}(\delta_g \mid \theta_g, \sigma_\delta^2))$$

$$p(\theta_\delta \mid \cdots) = \prod_{k(n) = 2} \prod_{g=1}^{G} (\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta) \text{I}(\delta_g \neq 0) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)) \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2)$$

$$p(\sigma_\delta \mid \cdots) = \prod_{k(n) = 2} \prod_{g=1}^{G} (\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta) \text{I}(\delta_g \neq 0) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)) \cdot \text{U}(\sigma_\delta \mid 0, \sigma_{\delta 0}))$$

$$p(\pi_\delta \mid \cdots) = \prod_{k(n) = 2} \prod_{g=1}^{G} (\pi_\delta \text{I}(\delta_g = 0) + (1 - \pi_\delta) \text{I}(\delta_g \neq 0) \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)) \cdot \text{Beta}(\pi_\delta \mid a_\delta, b_\delta)$$