A Fully Bayesian Model for Gene Expression Heterosis in RNA-seq Data

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Contents

1 The Model

Let $y_{g,n}$ be the expression level of gene g ($g = 1, \ldots, G$) in library n ($n = 1, \ldots, N$). Let $\mu(n, \phi_g, \alpha_g, \delta_g)$ be the function given by:

$$\mu(n, \phi_g, \alpha_g, \delta_g) = \begin{cases} \phi_g - \alpha_g & \text{library } n \text{ is in treatment group 1} \\ \phi_g + \delta_g & \text{library } n \text{ is in treatment group 2} \\ \phi_g + \alpha_g & \text{library } n \text{ is in treatment group 3} \end{cases}$$

Then:

$$\begin{aligned}
y_{g,n} &\stackrel{\text{ind}}{\sim} \text{Poisson}(\exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \\
c_n &\stackrel{\text{ind}}{\sim} \text{N}(0, \sigma_c^2) \\
\sigma_c &\stackrel{\text{ind}}{\sim} \text{U}(0, \sigma_{c0}) \\
\varepsilon_{g,n} &\stackrel{\text{ind}}{\sim} \text{N}(0, \sigma_g^2) \\
\sigma_g^2 &\stackrel{\text{ind}}{\sim} \text{Inv-Gamma}\left(\text{shape} = \frac{d}{2},\ \text{rate} = \frac{d \cdot \tau^2}{2}\right) \\
d &\stackrel{\text{ind}}{\sim} \text{U}(0, d_0) \\
\tau^2 &\stackrel{\text{ind}}{\sim} \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\
\phi_g &\stackrel{\text{ind}}{\sim} \text{N}(\theta_\phi, \sigma_\phi^2) \\
\theta_\phi &\stackrel{\text{ind}}{\sim} \text{N}(0, \gamma_\phi^2) \\
\sigma_\phi &\stackrel{\text{ind}}{\sim} \text{U}(0, \sigma_{\phi 0}) \\
\alpha_g &\stackrel{\text{ind}}{\sim} \pi_\alpha^{1 - I(\alpha_g)}[(1 - \pi_\alpha)\text{N}(\theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \\
\theta_\alpha &\stackrel{\text{ind}}{\sim} \text{N}(0, \gamma_\alpha^2) \\
\sigma_\alpha &\stackrel{\text{ind}}{\sim} \text{U}(0, \sigma_{\alpha 0}) \\
\pi_\alpha &\stackrel{\text{ind}}{\sim} \text{Beta}(a_\alpha, b_\alpha) \\
\delta_g &\stackrel{\text{ind}}{\sim} \pi_\delta^{1 - I(\delta_g)}[(1 - \pi_\delta)\text{N}(\theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \\
\theta_\delta &\stackrel{\text{ind}}{\sim} \text{N}(0, \gamma_\delta^2) \\
\sigma_\delta &\stackrel{\text{ind}}{\sim} \text{U}(0, \sigma_{\delta 0}) \\
\pi_\delta &\stackrel{\text{ind}}{\sim} \text{Beta}(a_\delta, b_\delta)
\end{aligned}$$

where:

- $I(x) = 0$ if $x = 0$ and 1 otherwise.

- Conditional independence is implied unless otherwise specified.
- The parameters to the left of the "$\sim$" are implicitly conditioned on the parameters to the right.

2 Full Conditional Distributions

Define:

- $k(n)$ = treatment group of library n.
- $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$)
- G_α = number of genes for which $\alpha_g \neq 0$
- G_δ = number of genes for which $\delta_g \neq 0$
- $I(x) = 0$ if $x = 0$ and 1 otherwise.

Then:

$$p(c_n \mid \cdots) \propto \exp\left(c_n G \overline{y}_{.n} - \exp(c_n) \sum_{g=1}^{G} \exp(\varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{c_n^2}{2\sigma_c^2}\right)$$

$$p\left(\frac{1}{\sigma_c^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_c^2} \mid \text{shape} = \frac{N-1}{2},\ \text{rate} = \frac{1}{2}\sum_{n=1}^{N} c_n^2\right) I\left(\frac{1}{\sigma_c^2} > \frac{1}{\sigma_{c0}^2}\right)$$

$$p(\varepsilon_{g,n} \mid \cdots) \propto \exp\left(y_{g,n}\varepsilon_{g,n} - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right)$$

$$p\left(\frac{1}{\sigma_g^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_g^2} \mid \text{shape} = \frac{N+d}{2},\ \text{rate} = \frac{1}{2}\left(d \cdot \tau^2 + \sum_{n=1}^{N} \varepsilon_{g,n}^2\right)\right)$$

$$p(d \mid \cdots) \propto \Gamma\left(d/2\right)^{-G} \left(\frac{d \cdot \tau^2}{2}\right)^{Gd/2} \left(\prod_{g=1}^{G} \sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{d \cdot \tau^2}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right) I(0 < d < d_0)$$

$$p(\tau^2 \mid \cdots) = \text{Gamma}\left(\tau^2 \mid \text{shape} = a_\tau + \frac{Gd}{2},\ \text{rate} = b_\tau + \frac{d}{2}\sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)$$

$$p(\phi_g \mid \cdots) \propto \exp\left(\sum_{n=1}^{N} \left[y_{g,n}\mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))\right] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right)$$

$$p(\theta_\phi \mid \cdots) = \text{N}\left(\theta_\phi \mid \frac{\gamma_\phi^2 \sum_{g=1}^{G} \phi_g}{G\gamma_\phi^2 + \sigma_\phi^2},\ \frac{\gamma_\phi^2 \sigma_\phi^2}{G\gamma_\phi^2 + \sigma_\phi^2}\right)$$

$$p\left(\frac{1}{\sigma_\phi^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_\phi^2} \mid \text{shape} = \frac{G-1}{2},\ \text{rate} = \frac{1}{2}\sum_{g=1}^{G} (\phi_g - \theta_\phi)^2\right) \text{I}\left(\frac{1}{\sigma_\phi^2} > \frac{1}{\sigma_{\phi 0}^2}\right)$$

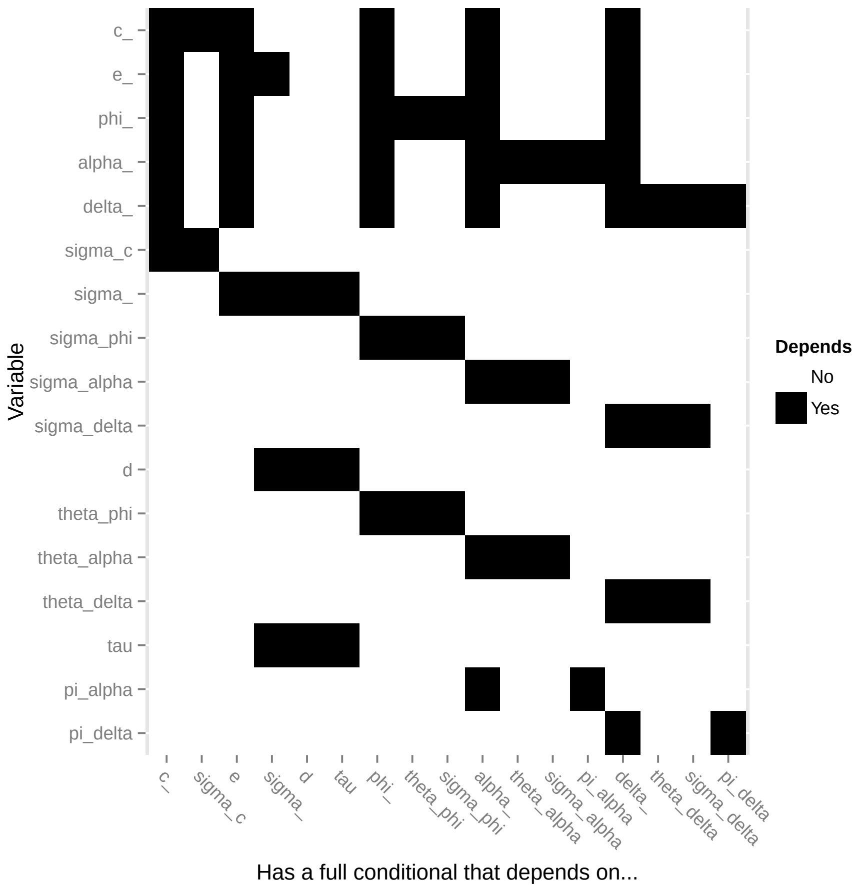
$$p(\alpha_g \mid \cdots) \propto \exp\left(\sum_{k(n)\neq 2} [y_{g,n}\mu(n,\phi_g,\alpha_g,\delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n,\phi_g,\alpha_g,\delta_g))]\right.$$
$$\left. + I(\alpha_g)\left(\log(1-\alpha_g) - \frac{(\alpha_g-\theta_\alpha)^2}{2\sigma_\alpha^2}\right) + (1 - I(\alpha_g))\log \pi_\alpha\right)$$
$$p(\theta_\alpha \mid \cdots) = N\left(\theta_\alpha \mid \frac{\gamma_\alpha^2 \sum_{\alpha_g \neq 0} \alpha_g}{G_\alpha \gamma_\alpha^2 + \sigma_\alpha^2},\ \frac{\gamma_\alpha^2 \sigma_\alpha^2}{G_\alpha \gamma_\alpha^2 + \sigma_\alpha^2}\right)$$
$$p\left(\frac{1}{\sigma_\alpha^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_\alpha^2} \mid \text{shape} = \frac{G_\alpha - 1}{2},\ \text{rate} = \frac{1}{2}\sum_{\alpha_g \neq 0}(\alpha_g - \theta_\alpha)^2\right)$$
$$p(\pi_\alpha \mid \cdots) = \text{Beta}(\pi_\alpha \mid G - G_\alpha + \alpha_\tau,\ G_\alpha + b_\tau)$$
$$p(\delta_g \mid \cdots) \propto \exp\left(\sum_{k(n)= 2} [y_{g,n}\mu(n,\phi_g,\alpha_g,\delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n,\phi_g,\alpha_g,\delta_g))]\right.$$
$$\left. + I(\delta_g)\left(\log(1-\delta_g) - \frac{(\delta_g-\theta_\delta)^2}{2\sigma_\delta^2}\right) + (1 - I(\delta_g))\log \pi_\delta\right)$$
$$p(\theta_\delta \mid \cdots) = N\left(\theta_\delta \mid \frac{\gamma_\delta^2 \sum_{\delta_g \neq 0} \delta_g}{G_\delta \gamma_\delta^2 + \sigma_\delta^2},\ \frac{\gamma_\delta^2 \sigma_\delta^2}{G_\delta \gamma_\delta^2 + \sigma_\delta^2}\right)$$
$$p\left(\frac{1}{\sigma_\delta^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_\delta^2} \mid \text{shape} = \frac{G_\delta - 1}{2},\ \text{rate} = \frac{1}{2}\sum_{\delta_g \neq 0}(\delta_g - \theta_\delta)^2\right)$$
$$p(\pi_\delta \mid \cdots) = \text{Beta}(\pi_\delta \mid G - G_\delta + \delta_\tau,\ G_\delta + b_\tau)$$

3 The Gibbs Sampler

For certain parameters, the full conditional distribution is independent of other key parameters. For example, the full conditional distribution of c_1 does not contain c_2. Hence, c_1 and c_2 can be sampled in parallel in a single Gibbs step. Obvious sets of parameters that can be jointly sampled are:

- $c_1, \ldots, c_N$
- $\varepsilon_{1,1}, \varepsilon_{1,2}, \ldots, \varepsilon_{1,N}, \varepsilon_{2,N}, \ldots, \varepsilon_{G,N}$
- $\sigma_1^2, \ldots, \sigma_G^2$
- $\phi_1, \ldots, \phi_G$
- $\alpha_1, \ldots, \alpha_G$
- $\delta_1, \ldots, \delta_G$

The following raster plot gives us a more complete idea of which parameters can be jointly sampled:



Hence, each of the following sets of parameters can be jointly sampled:

1. $c_1, \ \ldots, \ c_N$
2. $\tau, \ \pi_\alpha, \ \pi_\delta$
3. $d, \ \theta_\phi, \ \theta_\alpha, \ \theta_\delta$
4. $\sigma_c, \ \sigma_\phi, \ \sigma_\alpha, \ \sigma_\delta, \ \sigma_1^2, \ \ldots, \ \sigma_G^2$
5. $\varepsilon_{1,1}, \ \varepsilon_{1,2}, \ \ldots, \ \varepsilon_{1,N}, \ \varepsilon_{2,N}, \ \ldots, \ \varepsilon_{G,N}$
6. $\phi_1, \ \ldots, \ \phi_G$
7. $\alpha_1, \ \ldots, \ \alpha_G$
8. $\delta_1, \ \ldots, \ \delta_G$

In order, these are the 8 steps of the Gibbs sampler.

A Derivations of the Full Conditionals

Recall:

- $k(n)$ = treatment group of library n.
- $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$)
- G_α = number of genes for which $\alpha_g \neq 0$
- G_δ = number of genes for which $\delta_g \neq 0$
- $I(x) = 0$ if $x = 0$ and 1 otherwise.

Then from the model in Section 1, we get:

$$p(c_n \mid \cdots) \propto \left[\prod_{g=1}^{G} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \cdot \text{N}(c_n \mid 0, \sigma_c^2)$$

$$p\left(\sigma_c \mid \cdots\right) = \left[\prod_{n=1}^{N} \text{N}(c_n \mid 0, \sigma_c^2)\right] \cdot \text{U}(\sigma_c \mid 0, \sigma_{c0})$$

$$p(\varepsilon_{g,n} \mid \cdots) \propto \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)$$

$$p(\sigma_g^2 \mid \cdots) \propto \left[\prod_{n=1}^{N} \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)\right] \cdot \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)$$

$$p(d \mid \cdots) \propto \left[\prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)\right] \cdot \text{U}(d \mid 0, d_0)$$

$$p(\tau^2 \mid \cdots) \propto \left[\prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)\right] \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau)$$

$$p(\phi_g \mid \cdots) \propto \left[\prod_{n=1}^{N} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)$$

$$p(\theta_\phi \mid \cdots) \propto \left[\prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)\right] \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2)$$

$$p(\sigma_\phi \mid \cdots) \propto \left[\prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)\right] \cdot \text{U}(\sigma_\phi \mid 0, \sigma_{\phi 0})$$

$$
\begin{aligned}
p(\alpha_g \mid \cdots) &\propto \left[\prod_{k(n)\neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \pi_\alpha^{1-I(\alpha_g)} [(1-\pi_\alpha)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \\
p(\theta_\alpha \mid \cdots) &\propto \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)} [(1-\pi_\alpha)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2) \\
p(\sigma_\alpha \mid \cdots) &\propto \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)} [(1-\pi_\alpha)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \text{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0}) \\
p(\pi_\alpha \mid \cdots) &\propto \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)} [(1-\pi_\alpha)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \text{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha) \\
p(\delta_g \mid \cdots) &\propto \left[\prod_{k(n)= 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \delta_g, \delta_g)))\right] \pi_\delta^{1-I(\delta_g)} [(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \\
p(\theta_\delta \mid \cdots) &\propto \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)} [(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2) \\
p(\sigma_\delta \mid \cdots) &\propto \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)} [(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \text{U}(\sigma_\delta \mid 0, \sigma_{\delta 0}) \\
p(\pi_\delta \mid \cdots) &\propto \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)} [(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \text{Beta}(\pi_\delta \mid a_\delta, b_\delta)
\end{aligned}
$$

A.1 Transformations of Standard Deviations

Let σ be a standard deviation parameter and let $p(\sigma \mid \cdots)$ be its full conditional distribution. Then, by a transformation of variables,

$$
\begin{aligned}
p(\sigma^2 \mid \cdots) &= p(\sqrt{\sigma^2} \mid \cdots) \cdot \left| \frac{d}{d\sigma^2} \sqrt{\sigma^2} \right| \\
&= p(\sigma \mid \cdots) \frac{1}{2} (\sigma^2)^{-1/2}
\end{aligned}
$$

I use this transformation several times in the next sections.

A.2 $p(c_n \mid \cdots)$: Metropolis

$$
\begin{aligned}
p(c_n \mid \cdots) &\propto \left[\prod_{g=1}^{G} \text{Poisson}(y_{g,n} \mid \lambda_{g,n})\right] \cdot \text{N}(c_n \mid 0, \sigma_c^2) \\
&\propto \left[\prod_{g=1}^{G} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n})\right] \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right) \\
&= \exp\left(\sum_{g=1}^{G} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{c_n^2}{2\sigma_c^2}\right) \\
&= \exp\left(\sum_{g=1}^{G} [y_{g,n}(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{c_n^2}{2\sigma_c^2}\right) \\
&= \exp\left(c_n G \overline{y}_{.n} + \sum_{g=1}^{G} [y_{g,n}(\varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \sum_{g=1}^{G} \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{c_n^2}{2\sigma_c^2}\right) \\
&= \exp\left(c_n G \overline{y}_{.n} - \exp(c_n) \sum_{g=1}^{G} \exp(\varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{c_n^2}{2\sigma_c^2}\right)
\end{aligned}
$$

A.3 $p\left(\frac{1}{\sigma_c^2} \mid \cdots\right)$ Truncated Gamma

$$
\begin{aligned}
p(\sigma_c^2 \mid \cdots) &= p(\sigma_c \mid \cdots) \frac{1}{2} (\sigma_c^2)^{-1/2} \qquad \text{(transformation in Section A.1)} \\
&\propto \left[\prod_{n=1}^{N} \text{N}(c_n \mid 0, \sigma_c^2)\right] \cdot \text{U}(\sigma_c \mid 0, \sigma_{c0}) \frac{1}{2} (\sigma_c^2)^{-1/2} \\
&\propto \prod_{n=1}^{N} \frac{1}{\sqrt{\sigma_c^2}} \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right) \cdot \text{I}(0 < \sigma_c < \sigma_{c0}) (\sigma_c^2)^{-1/2} \\
&= (\sigma_c^2)^{-N/2} \exp\left(-\frac{1}{\sigma_c^2} \frac{1}{2} \sum_{n=1}^{N} c_n^2\right) \cdot \text{I}(0 < \sigma_c < \sigma_{c0}) (\sigma_c^2)^{-1/2} \\
&= (\sigma_c^2)^{-(N/2 - 1/2 + 1)} \exp\left(-\frac{1}{\sigma_c^2} \frac{1}{2} \sum_{n=1}^{N} c_n^2\right) \cdot \text{I}(0 < \sigma_c < \sigma_{c0})
\end{aligned}
$$

which is the kernel of a truncated inverse gamma distribution. Hence:

$$p\left(\frac{1}{\sigma_c^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_c^2} \mid \text{shape} = \frac{N-1}{2},\ \text{rate} = \frac{1}{2}\sum_{n=1}^{N} c_n^2\right) I\left(\frac{1}{\sigma_c^2} > \frac{1}{\sigma_{c0}^2}\right)$$

A.4 $p(\varepsilon_{g,n} \mid \cdots)$ Metropolis

$$\begin{aligned}
p(\varepsilon_{g,n} \mid \cdots) &= \text{Poisson}(y_{g,n} \mid \lambda_{g,n}) \cdot \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2) \\
&\propto \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \exp\left(-\frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right) \\
&= \exp\left(y_{g,n} \log \lambda_{g,n} - \lambda_{g,n} - \frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right) \\
&= \exp\left(y_{g,n}(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right) \\
&= \exp\left(y_{g,n}\varepsilon_{g,n} - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right)
\end{aligned}$$

A.5 $p\left(\frac{1}{\sigma_g^2} \mid \cdots\right)$ Gamma

$$\begin{aligned}
p(\sigma_g^2 \mid \cdots) &= \left[\prod_{n=1}^{N} \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)\right] \cdot \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right) \\
&\propto \left[\prod_{n=1}^{N} (\sigma_g^2)^{-1/2} \exp\left(-\frac{1}{\sigma_g^2}\frac{\varepsilon_{g,n}^2}{2}\right)\right] \cdot \left(\sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{d \cdot \tau^2}{2}\right) \\
&= \left[(\sigma_g^2)^{-N/2} \exp\left(-\frac{1}{\sigma_g^2}\frac{1}{2}\sum_{n=1}^{N} \varepsilon_{g,n}^2\right)\right] \cdot \left(\sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{d \cdot \tau^2}{2}\right) \\
&= (\sigma_g^2)^{-((N+d)/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{1}{2}\left(d \cdot \tau^2 + \sum_{n=1}^{N} \varepsilon_{g,n}^2\right)\right)
\end{aligned}$$

which is the kernel of an inverse gamma distribution. Hence:

$$p\left(\frac{1}{\sigma_g^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_g^2} \mid \text{shape} = \frac{N+d}{2},\ \text{rate} = \frac{1}{2}\left(d \cdot \tau^2 + \sum_{n=1}^{N} \varepsilon_{g,n}^2\right)\right)$$

A.6 $p(d \mid \cdots)$: Metropolis

$$
\begin{aligned}
p(d \mid \cdots) &= \left[\prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)\right] \cdot \text{U}(d \mid 0, d_0) \\
&\propto \prod_{g=1}^{G} \Gamma\left(d/2\right)^{-1} \left(\frac{d \cdot \tau^2}{2}\right)^{d/2} \left(\sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{1}{\sigma_g^2} \frac{d \cdot \tau^2}{2}\right) I(0 < d < d_0) \\
&\propto \Gamma\left(d/2\right)^{-G} \left(\frac{d \cdot \tau^2}{2}\right)^{Gd/2} \left(\prod_{g=1}^{G} \sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{d \cdot \tau^2}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right) I(0 < d < d_0)
\end{aligned}
$$

A.7 $p(\tau^2 \mid \cdots)$: Gamma

$$
\begin{aligned}
p(\tau^2 \mid \cdots) &= \left[\prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)\right] \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\
&\propto \left[\Gamma\left(d/2\right)^{-G} \left(\frac{d \cdot \tau^2}{2}\right)^{Gd/2} \left(\prod_{g=1}^{G} \sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{d \cdot \tau^2}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)\right] \cdot (\tau^2)^{a_\tau - 1} \exp\left(-b_\tau \tau^2\right) \\
&\propto \left[\left(\tau^2\right)^{Gd/2} \exp\left(-\tau^2 \cdot \frac{d}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)\right] \cdot (\tau^2)^{a_\tau - 1} \exp\left(-b_\tau \tau^2\right) \\
&= (\tau^2)^{Gd/2 + a_\tau - 1} \exp\left(-\tau^2 \left(b_\tau + \frac{d}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)\right)
\end{aligned}
$$

Hence:

$$
p(\tau^2 \mid \cdots) = \text{Gamma}\left(\tau^2 \mid \text{shape} = a_\tau + \frac{Gd}{2},\ \text{rate} = b_\tau + \frac{d}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)
$$

A.8 $p(\phi_g \mid \cdots)$: Metropolis

$$
\begin{aligned}
p(\phi_g \mid \cdots) &= \left[\prod_{n=1}^{N} \text{Poisson}(y_{g,n} \mid \lambda_{g,n})\right] \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \\
&\propto \left[\prod_{n=1}^{N} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n})\right] \cdot \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \\
&= \exp\left(\sum_{n=1}^{N} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \\
&= \exp\left(\sum_{n=1}^{N} [y_{g,n}(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \\
&\propto \exp\left(\sum_{n=1}^{N} [y_{g,n}\mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right)
\end{aligned}
$$

A.9 $p(\theta_\phi \mid \cdots)$: Normal

$$
\begin{aligned}
p(\theta_\phi \mid \cdots) &= \left[\prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)\right] \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2) \\
&\propto \left[\prod_{g=1}^{G} \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right)\right] \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\sum_{g=1}^{G} \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\sum_{g=1}^{G} \phi_g^2 - 2\theta_\phi \sum_{g=1}^{G} \phi_g + G\theta_\phi^2}{2\sigma_\phi^2}\right) \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\sum_{g=1}^{G} \phi_g^2 - 2\theta_\phi \sum_{g=1}^{G} \phi_g + G\theta_\phi^2}{2\sigma_\phi^2} - \frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\gamma_\phi^2 \sum_{g=1}^{G} \phi_g^2 - 2\gamma_\phi^2 (\sum_{g=1}^{G} \phi_g)\theta_\phi + G\gamma_\phi^2\theta_\phi^2}{2\sigma_\phi^2\gamma_\phi^2} - \frac{\sigma_\phi^2\theta_\phi^2}{2\sigma_\phi^2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\gamma_\phi^2 \sum_{g=1}^{G} \phi_g^2 - 2\gamma_\phi^2 (\sum_{g=1}^{G} \phi_g)\theta_\phi + (G\gamma_\phi^2 + \sigma_\phi^2)\theta_\phi^2}{2\sigma_\phi^2\gamma_\phi^2}\right) \\
&\propto \exp\left(-\frac{(G\gamma_\phi^2 + \sigma_\phi^2)\left(\theta_\phi - \frac{\gamma_\phi^2(\sum_{g=1}^{G} \phi_g)}{G\gamma_\phi^2 + \sigma_\phi^2}\right)^2}{2\sigma_\phi^2\gamma_\phi^2}\right)
\end{aligned}
$$

Hence:

$$p(\theta_\phi \mid \cdots) = \mathrm{N}\left(\frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g}{G\gamma_\phi^2 + \sigma_\phi^2},\ \frac{\gamma_\phi^2 \sigma_\phi^2}{G\gamma_\phi^2 + \sigma_\phi^2}\right)$$

A.10 $p\left(\frac{1}{\sigma_\phi^2} \mid \ldots\right)$: Truncated Gamma

$$\begin{aligned}
p(\sigma_\phi^2 \mid \cdots) &= p(\sigma_\phi \mid \cdots)\frac{1}{2}(\sigma_\phi^2)^{-1/2} \qquad \text{(transformation in Section A.1)} \\
&\propto \left[\prod_{g=1}^G \mathrm{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)\right] \cdot \mathrm{U}(\sigma_\phi \mid 0, \sigma_{\phi 0})(\sigma_\phi^2)^{-1/2} \\
&\propto \prod_{g=1}^G (\sigma_\phi^2)^{-1/2} \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2)(\sigma_\phi^2)^{-1/2} \\
&= (\sigma_\phi^2)^{-G/2} \exp\left(-\sum_{g=1}^G \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2)(\sigma_\phi^2)^{-1/2} \\
&= (\sigma_\phi^2)^{-(G/2-1/2+1)} \exp\left(-\frac{1}{\sigma_\phi^2}\frac{1}{2}\sum_{g=1}^G (\phi_g - \theta_\phi)^2\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2)
\end{aligned}$$

which is a truncated inverse gamma distribution. Hence:

$$p\left(\frac{1}{\sigma_\phi^2} \mid \cdots\right) = \mathrm{Gamma}\left(\text{shape} = \frac{G-1}{2},\ \text{rate} = \frac{1}{2}\sum_{g=1}^G (\phi_g - \theta_\phi)^2\right) \mathrm{I}\left(\frac{1}{\sigma_\phi^2} > \frac{1}{\sigma_{\phi 0}^2}\right)$$

A.11 $p(\alpha_g \mid \cdots)$: Metropolis

$$\begin{aligned}
p(\alpha_g \mid \cdots) &= \left[\prod_{k(n)\neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \pi_\alpha^{1-I(\alpha_g)}[(1-\pi_\alpha)\text{N}(\theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \\
&\propto \left[\prod_{k(n)\neq 2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n})\right] \exp\left(-\frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right)^{I(\alpha_g)} \pi_\alpha^{1-I(\alpha_g)}(1-\pi_\alpha)^{I(\alpha_g)} \\
&= \exp\left(\sum_{k(n)\neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - I(\alpha_g)\frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + (1 - I(\alpha_g)) \log \pi_\alpha + I(\alpha_g) \log(1-\pi_\alpha)\right) \\
&= \exp\left(\sum_{k(n)\neq 2} [y_{g,n}(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))]\right. \\
&\qquad \left. + I(\alpha_g)\left(\log(1-\alpha_g) - \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right) + (1 - I(\alpha_g)) \log \pi_\alpha\right) \\
&\propto \exp\left(\sum_{k(n)\neq 2} [y_{g,n}\mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))]\right. \\
&\qquad \left. + I(\alpha_g)\left(\log(1-\alpha_g) - \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right) + (1 - I(\alpha_g)) \log \pi_\alpha\right)
\end{aligned}$$

A.12 $p\left(\theta_\alpha \mid \cdots\right)$: Normal

$$\begin{aligned}
p(\theta_\alpha \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)}[(1-\pi_\alpha)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2) \\
&\propto \left[\prod_{\alpha_g \neq 0} \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]\right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2)
\end{aligned}$$

From algebra similar to the derivation of $p(\theta_\phi \mid \cdots)$, we get:

$$p(\theta_\alpha \mid \cdots) = N\left(\frac{\gamma_\alpha^2 \sum_{\alpha_g \neq 0} \alpha_g}{G_\alpha \gamma_\alpha^2 + \sigma_\alpha^2}, \ \frac{\gamma_\alpha^2 \sigma_\alpha^2}{G_\alpha \gamma_\alpha^2 + \sigma_\alpha^2}\right)$$

A.13 $p\left(\frac{1}{\sigma_\alpha^2} \mid \cdots\right)$: Truncated Gamma

$$
\begin{aligned}
p(\sigma_\alpha^2 \mid \cdots) &= p(\sigma_\alpha \mid \cdots)\frac{1}{2}(\sigma_\alpha^2)^{-1/2} \qquad \text{(transformation in Section A.1)} \\
&\propto \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)}[(1-\pi_\alpha)\mathrm{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \mathrm{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0})(\sigma_\alpha^2)^{-1/2} \\
&\propto \prod_{\alpha_g \neq 0} N(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2)(\sigma_\alpha^2)^{-1/2} \\
&\propto \prod_{\alpha_g \neq 0} (\sigma_\alpha^2)^{-1/2} \exp\left(-\frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2)(\sigma_\alpha^2)^{-1/2} \\
&= (\sigma_\alpha^2)^{-G_\alpha/2} \exp\left(-\frac{1}{\theta_\alpha^2}\frac{1}{2}\sum_{\alpha_g \neq 0}(\alpha_g - \theta_\alpha)^2\right) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2)(\sigma_\alpha^2)^{-1/2} \\
&= (\sigma_\alpha^2)^{-(G_\alpha/2 - 1/2 + 1)} \exp\left(-\frac{1}{\theta_\alpha^2}\frac{1}{2}\sum_{\alpha_g \neq 0}(\alpha_g - \theta_\alpha)^2\right) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2)
\end{aligned}
$$

which is the kernel of a truncated inverse gamma distribution. Hence:

$$
p(\frac{1}{\sigma_\alpha^2} \mid \cdots) = \mathrm{Gamma}\left(\frac{1}{\sigma_\alpha^2} \mid \text{shape} = \frac{G_\alpha - 1}{2},\ \text{rate} = \frac{1}{2}\sum_{\alpha_g \neq 0}(\alpha_g - \theta_\alpha)^2\right)
$$

A.14 $p(\pi_\alpha \mid \cdots)$: Beta

$$
\begin{aligned}
p(\pi_\alpha \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)}[(1-\pi_\alpha)\mathrm{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \mathrm{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha) \\
&\propto [\pi_\alpha^{G-G_\alpha}(1-\pi_\alpha)^{G_\alpha}]\pi_\alpha^{a_\tau - 1}(1-\pi_\alpha)^{b_\tau - 1} \\
&= \pi_\alpha^{G-G_\alpha+a_\tau-1}(1-\pi_\alpha)^{G_\alpha+b_\tau-1}
\end{aligned}
$$

Hence:

$$
p(\pi_\alpha \mid \cdots) = \mathrm{Beta}(G - G_\alpha + \alpha_\tau,\ G_\alpha + b_\tau)
$$

A.15 $p(\delta_g \mid \cdots)$: Metropolis

$$
\begin{aligned}
p(\delta_g \mid \cdots) &= \left[\prod_{k(n)=2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \pi_\alpha^{1-I(\delta_g)}[(1-\pi_\alpha)\text{N}(\theta_\alpha, \sigma_\alpha^2)]^{I(\delta_g)} \\
&\propto \left[\prod_{k(n)=2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n})\right] \exp\left(-\frac{(\delta_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right)^{I(\delta_g)} \pi_\alpha^{1-I(\delta_g)}(1-\pi_\alpha)^{I(\delta_g)} \\
&= \exp\left(\sum_{k(n)=2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - I(\delta_g)\frac{(\delta_g - \theta_\alpha)^2}{2\sigma_\alpha^2} + (1 - I(\delta_g)) \log \pi_\alpha + I(\delta_g) \log(1-\pi_\alpha)\right) \\
&= \exp\left(\sum_{k(n)=2} [y_{g,n}(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))]\right. \\
&\qquad \left. + I(\delta_g)\left(\log(1-\delta_g) - \frac{(\delta_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right)\right) \\
&\propto \exp\left(\sum_{k(n)=2} [y_{g,n}\mu(n, \phi_g, \alpha_g, \delta_g) - \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))]\right. \\
&\qquad \left. + I(\delta_g)\left(\log(1-\delta_g) - \frac{(\delta_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right)\right)
\end{aligned}
$$

A.16 $p(\theta_\delta \mid \cdots)$: Normal

$$
\begin{aligned}
p(\theta_\delta \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2) \\
&\propto \left[\prod_{\delta_g \neq 0} \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]\right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2)
\end{aligned}
$$

From algebra similar to the derivation of $p(\theta_\phi \mid \cdots)$, we get:

$$
p(\theta_\delta \mid \cdots) = N\left(\frac{\gamma_\delta^2 \sum_{\delta_g \neq 0} \delta_g}{G_\delta \gamma_\delta^2 + \sigma_\delta^2},\ \frac{\gamma_\delta^2 \sigma_\delta^2}{G_\delta \gamma_\delta^2 + \sigma_\delta^2}\right)
$$

where G_δ is the number of genes for which $\delta_g \neq 0$.

A.17 $p\left(\frac{1}{\sigma_\delta^2} \mid \cdots\right)$: Truncated Gamma

$$
\begin{aligned}
p(\sigma_\delta^2 \mid \cdots) &= p(\sigma_\delta \mid \cdots)\frac{1}{2}(\sigma_\delta^2)^{-1/2} \qquad \text{(transformation in Section A.1)} \\
&\propto \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\mathrm{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \mathrm{U}(\sigma_\delta \mid 0, \sigma_{\delta 0})(\sigma_\delta^2)^{-1/2} \\
&\propto \prod_{\delta_g \neq 0} N(\delta_g \mid \theta_\delta, \sigma_\delta^2) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2)(\sigma_\delta^2)^{-1/2} \\
&\propto \prod_{\delta_g \neq 0} (\sigma_\delta^2)^{-1/2} \exp\left(-\frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2}\right) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2)(\sigma_\delta^2)^{-1/2} \\
&= (\sigma_\delta^2)^{-G_\delta/2} \exp\left(-\frac{1}{\theta_\delta^2}\frac{1}{2}\sum_{\delta_g \neq 0}(\delta_g - \theta_\delta)^2\right) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2)(\sigma_\delta^2)^{-1/2} \\
&= (\sigma_\delta^2)^{-(G_\delta/2 - 1/2 + 1)} \exp\left(-\frac{1}{\theta_\delta^2}\frac{1}{2}\sum_{\delta_g \neq 0}(\delta_g - \theta_\delta)^2\right) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2)
\end{aligned}
$$

which is the kernel of a truncated inverse gamma distribution. Hence:

$$
p(\frac{1}{\sigma_\delta^2} \mid \cdots) = \mathrm{Gamma}\left(\frac{1}{\sigma_\delta^2} \mid \text{shape} = \frac{G_\delta - 1}{2},\ \text{rate} = \frac{1}{2}\sum_{\delta_g \neq 0}(\delta_g - \theta_\delta)^2\right)
$$

A.18 $p(\pi_\delta \mid \cdots)$: Beta

$$
\begin{aligned}
p(\pi_\delta \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\mathrm{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \mathrm{Beta}(\pi_\delta \mid a_\delta, b_\delta) \\
&\propto [\pi_\delta^{G-G_\delta}(1-\pi_\delta)^{G_\delta}]\pi_\delta^{a_\tau - 1}(1-\pi_\delta)^{b_\tau - 1} \\
&= \pi_\delta^{G - G_\delta + a_\tau - 1}(1-\pi_\delta)^{G_\delta + b_\tau - 1}
\end{aligned}
$$

where G_δ is the number of genes for which $\delta_g \neq 0$. Hence:

$$
p(\pi_\delta \mid \cdots) = \mathrm{Beta}(G - G_\delta + \delta_\tau,\ G_\delta + b_\tau)
$$