A Fully Bayesian Model for RNA-seq Data

Will Landau

Department of Statistics
Iowa State University

June 4, 2013

Contents

1 The Model

Let $y_{g,n}$ be the expression level of gene g ($g = 1, \ldots, G$) in library n ($n = 1, \ldots, N$). Let $\mu(n, \phi_g, \alpha_g, \delta_g)$ be the function given by:

$$\mu(n, \phi_g, \alpha_g, \delta_g) = \begin{cases} \phi_g - \alpha_g & \text{library } n \text{ is in treatment group 1} \\ \phi_g + \delta_g & \text{library } n \text{ is in treatment group 2} \\ \phi_g + \alpha_g & \text{library } n \text{ is in treatment group 3} \end{cases}$$

Then:

$$\begin{aligned}
y_{g,n} &\sim \text{Poisson}(\exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \\
c_n &\sim \text{N}(0, \sigma_c^2) \\
\sigma_c &\sim \text{U}(0, \sigma_{c0}) \\
\varepsilon_{g,n} &\sim \text{N}(0, \sigma_g^2) \\
\sigma_g^2 &\sim \text{Inv-Gamma}\left(\text{shape} = \frac{d}{2},\ \text{rate} = \frac{d \cdot \tau^2}{2}\right) \\
d &\sim \text{U}(0, d_0) \\
\tau^2 &\sim \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\
\phi_g &\sim \text{N}(\theta_\phi, \sigma_\phi^2) \\
\theta_\phi &\sim \text{N}(0, \gamma_\phi^2) \\
\sigma_\phi &\sim \text{U}(0, \sigma_{\phi 0}) \\
\alpha_g &\sim \pi_\alpha^{1 - I(\alpha_g)}[(1 - \pi_\alpha)\text{N}(\theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \\
\theta_\alpha &\sim \text{N}(0, \gamma_\alpha^2) \\
\sigma_\alpha &\sim \text{U}(0, \sigma_{\alpha 0}) \\
\pi_\alpha &\sim \text{Beta}(a_\alpha, b_\alpha) \\
\delta_g &\sim \pi_\delta^{1 - I(\delta_g)}[(1 - \pi_\delta)\text{N}(\theta_\delta, \sigma_\delta^2)]^{I(\delta_g)} \\
\theta_\delta &\sim \text{N}(0, \gamma_\delta^2) \\
\sigma_\delta &\sim \text{U}(0, \sigma_{\delta 0}) \\
\pi_\delta &\sim \text{Beta}(a_\delta, b_\delta)
\end{aligned}$$

where:

- $I(x) = 0$ if $x = 0$ and 1 otherwise.
- Independence is implied unless otherwise specified.
- The parameters to the left of the "$\sim$" are implicitly conditioned on the parameters to the right.

2 Full Conditional Distributions

Let $k(n)$ be the treatment group of library n. Then:

$$p(c_n \mid \cdots) = \left[\prod_{g=1}^{G} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \cdot \text{N}(c_n \mid 0, \sigma_c^2)$$

$$p\left(\sigma_c \mid \cdots\right) = \left[\prod_{n=1}^{N} \text{N}(c_n \mid 0, \sigma_c^2)\right] \cdot \text{U}(\sigma_c \mid 0, \sigma_{c0})$$

$$p(\varepsilon_{g,n} \mid \cdots) = \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)$$

$$p(\sigma_g^2 \mid \cdots) = \left[\prod_{n=1}^{N} \text{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)\right] \cdot \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)$$

$$p(d \mid \cdots) = \left[\prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)\right] \cdot \text{U}(d \mid 0, d_0)$$

$$p(\tau^2 \mid \cdots) = \left[\prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)\right] \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau)$$

$$p(\phi_g \mid \cdots) = \left[\prod_{n=1}^{N} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)$$

$$p(\theta_\phi \mid \cdots) = \left[\prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)\right] \cdot \text{N}(\theta_\phi \mid 0, \gamma_\phi^2)$$

$$p(\sigma_\phi \mid \cdots) = \left[\prod_{g=1}^{G} \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)\right] \cdot \text{U}(\sigma_\phi \mid 0, \sigma_{\phi 0})$$

$$p(\alpha_g \mid \cdots) = \left[\prod_{k(n) \neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \pi_\alpha^{1 - I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}$$

$$p(\theta_\alpha \mid \cdots) = \left[\prod_{g=1}^{G} \pi_\alpha^{1 - I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2)$$

$$p(\sigma_\alpha \mid \cdots) = \left[\prod_{g=1}^{G} \pi_\alpha^{1 - I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \text{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0})$$

$$p(\pi_\alpha \mid \cdots) = \left[\prod_{g=1}^{G} \pi_\alpha^{1 - I(\alpha_g)} [(1 - \pi_\alpha) \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \text{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha)$$

$$p(\delta_g \mid \cdots) = \left[\prod_{k(n)=2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \delta_g, \delta_g)))\right] \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}$$

$$p(\theta_\delta \mid \cdots) = \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2)$$

$$p(\sigma_\delta \mid \cdots) = \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \text{U}(\sigma_\delta \mid 0, \sigma_{\delta 0})$$

$$p(\pi_\delta \mid \cdots) = \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \text{Beta}(\pi_\delta \mid a_\delta, b_\delta)$$

3 Simplifying and Sampling From the Full Conditionals

3.1 $p(c_n \mid \cdots)$: Metropolis

$$p(c_n \mid \cdots) = \left[\prod_{g=1}^{G} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \cdot \text{N}(c_n \mid 0, \sigma_c^2)$$

which is nothing immediately recognizable. I will sample from this distribution in a Metropolis step using the kernel of this distribution (taking $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$):

$$\begin{aligned} &\left[\prod_{g=1}^{G} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n})\right] \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right) \\ &= \exp\left(\sum_{g=1}^{G} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{c_n^2}{2\sigma_c^2}\right) \end{aligned}$$

where the sum inside the exponent can be parallelized on the GPU.

3.2 $p\left(\frac{1}{\sigma_c^2} \mid \cdots\right)$ Truncated Gamma

$$\begin{aligned}
p\left(\sigma_c \mid \cdots\right) &= \left[\prod_{n=1}^{N} \mathrm{N}(c_n \mid 0, \sigma_c^2)\right] \cdot \mathrm{U}(\sigma_c \mid 0, \sigma_{c0}) \\
&\propto \prod_{n=1}^{N} \frac{1}{\sqrt{\sigma_c^2}} \exp\left(-\frac{c_n^2}{2\sigma_c^2}\right) \cdot \mathrm{I}(0 < \sigma_c < \sigma_{c0}) \\
&= (\sigma_c^2)^{-N/2} \exp\left(-\frac{1}{\sigma_c^2}\frac{1}{2}\sum_{n=1}^{N} c_n^2\right) \cdot \mathrm{I}(0 < \sigma_c < \sigma_{c0})
\end{aligned}$$

which, for constants a and $b = \frac{1}{2}\sum_{n=1}^{N} c_n^2$, can be written as

$$p\left(\sigma_c \mid \cdots\right) = a \cdot (\sigma_c^2)^{-N/2} \exp\left(-\frac{1}{\sigma_c^2} b\right) \mathrm{I}(0 < \sigma_c^2 < \sigma_{c0}^2)$$

Transformation: let $z = g(\sigma_c) = \sigma_c^2$ so that $g^{-1}(z) = \sqrt{z}$ and:

$$\begin{aligned}
p\left(\sigma_c^2 = z \mid \cdots\right) &= p(\sigma_c = g^{-1}(z) \mid \cdots) \left|\frac{dg^{-1}(z)}{dz}\right| \\
&= a \cdot z^{-N/2} \exp\left(-\frac{1}{(\sqrt{z})^2} b\right) I\left(0 < z < \sigma_{c0}^2\right) \left|-\frac{1}{2} z^{-1/2}\right| \\
&= \frac{a}{2} z^{-(N/2 - 1/2 + 1)} \exp\left(-\frac{1}{z} b\right) I\left(0 < z < \sigma_{c0}^2\right) \\
&= \text{Inv-Gamma}\left(z \mid \text{shape} = \frac{N-1}{2},\ \text{rate} = b\right) I\left(0 < z < \sigma_{c0}^2\right)
\end{aligned}$$

Recalling that $b = \frac{1}{2}\sum_{n=1}^{N} c_n^2$,

$$p\left(\frac{1}{\sigma_c^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_c^2} \mid \text{shape} = \frac{N-1}{2},\ \text{rate} = \frac{1}{2}\sum_{n=1}^{N} c_n^2\right) I\left(\frac{1}{\sigma_c^2} > \frac{1}{\sigma_{c0}^2}\right)$$

3.3 $p(\varepsilon_{g,n} \mid \cdots)$: Metropolis

$$p(\varepsilon_{g,n} \mid \cdots) = \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))) \cdot \mathrm{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)$$

which is nothing immediately recognizable. I will sample from this distribution in a Metropolis step using the kernel:

$$
\begin{aligned}
&\lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \exp\left(-\frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right) \\
&= \exp\left(y_{g,n} \log \lambda_{g,n} - \lambda_{g,n} - \frac{\varepsilon_{g,n}^2}{2\sigma_g^2}\right)
\end{aligned}
$$

where $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$. The $\varepsilon_{g,n}$'s will be sampled in parallel across genes on the GPU.

3.4 $p\left(\frac{1}{\sigma_g^2} \mid \cdots\right)$ Gamma

$$
\begin{aligned}
p(\sigma_g^2 \mid \cdots) &= \left[\prod_{n=1}^N \mathrm{N}(\varepsilon_{g,n} \mid 0, \sigma_g^2)\right] \cdot \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right) \\
&\propto \left[\prod_{n=1}^N (\sigma_g^2)^{-1/2} \exp\left(-\frac{1}{\sigma_g^2}\frac{\varepsilon_{g,n}^2}{2}\right)\right] \cdot (\sigma_g^2)^{-(d/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{d \cdot \tau^2}{2}\right) \\
&= \left[(\sigma_g^2)^{-N/2} \exp\left(-\frac{1}{\sigma_g^2}\frac{1}{2}\sum_{n=1}^N \varepsilon_{g,n}^2\right)\right] \cdot (\sigma_g^2)^{-(d/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{d \cdot \tau^2}{2}\right) \\
&= (\sigma_g^2)^{-((N+d)/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{1}{2}\left(d \cdot \tau^2 + \sum_{n=1}^N \varepsilon_{g,n}^2\right)\right)
\end{aligned}
$$

The last line is the kernel of an inverse gamma distribution with shape parameter $\frac{N+d}{2}$ and rate parameter $\frac{1}{2}\left(d \cdot \tau^2 + \sum_{n=1}^N \varepsilon_{g,n}^2\right)$. Hence:

$$
p\left(\frac{1}{\sigma_g^2} \mid \cdots\right) = \text{Gamma}\left(\frac{1}{\sigma_g^2} \mid \text{shape} = \frac{N+d}{2},\ \text{rate} = \frac{1}{2}\left(d \cdot \tau^2 + \sum_{n=1}^N \varepsilon_{g,n}^2\right)\right)
$$

The $1/\sigma_g^2$'s will be sampled in parallel on the GPU.

3.5 $p(d \mid \cdots)$: Metropolis

$$
\begin{aligned}
p(d \mid \cdots) &= \left[\prod_{g=1}^G \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)\right] \cdot \mathrm{U}(d \mid 0, d_0) \\
&\propto \prod_{g=1}^G \Gamma\left(d/2\right)^{-1} \left(\frac{d \cdot \tau^2}{2}\right)^{d/2} \left(\sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{1}{\sigma_g^2}\frac{d \cdot \tau^2}{2}\right) I(0 < d < d_0) \\
&\propto \Gamma\left(d/2\right)^{-G} \left(\frac{d \cdot \tau^2}{2}\right)^{Gd/2} \left(\prod_{g=1}^G \sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{d \cdot \tau^2}{2}\sum_{g=1}^G \frac{1}{\sigma_g^2}\right) I(0 < d < d_0)
\end{aligned}
$$

I will sample d with a Metropolis step using the above kernel. Sums and products over g ($g = 1, \dots, G$) will be done in parallel on the GPU.

3.6 $p(\tau^2 \mid \cdots)$: Gamma

$$
\begin{aligned}
p(\tau^2 \mid \cdots) &= \left[\prod_{g=1}^{G} \text{Inv-Gamma}\left(\sigma_g^2 \mid \text{shape} = \frac{d}{2}, \text{rate} = \frac{d \cdot \tau^2}{2}\right)\right] \cdot \text{Gamma}(\tau^2 \mid \text{shape} = a_\tau, \text{rate} = b_\tau) \\
&\propto \left[\Gamma\left(d/2\right)^{-G} \left(\frac{d \cdot \tau^2}{2}\right)^{Gd/2} \left(\prod_{g=1}^{G} \sigma_g^2\right)^{-(d/2+1)} \exp\left(-\frac{d \cdot \tau^2}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)\right] \cdot (\tau^2)^{a_\tau - 1} \exp\left(-b_\tau \tau^2\right) \\
&\propto \left[\left(\tau^2\right)^{Gd/2} \exp\left(-\tau^2 \cdot \frac{d}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)\right] \cdot (\tau^2)^{a_\tau - 1} \exp\left(-b_\tau \tau^2\right) \\
&= (\tau^2)^{Gd/2 + a_\tau - 1} \exp\left(-\tau^2 \left(b_\tau + \frac{d}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)\right)
\end{aligned}
$$

Hence:

$$
p(\tau^2 \mid \cdots) = \text{Gamma}\left(\tau^2 \mid \text{shape} = a_\tau + \frac{Gd}{2}, \ \text{rate} = b_\tau + \frac{d}{2} \sum_{g=1}^{G} \frac{1}{\sigma_g^2}\right)
$$

3.7 $p(\phi_g \mid \cdots)$: Metropolis

$$
\begin{aligned}
p(\phi_g \mid \cdots) &= \left[\prod_{n=1}^{N} \text{Poisson}(y_{g,n} \mid \lambda_{g,n})\right] \cdot \text{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2) \\
&\propto \left[\prod_{n=1}^{N} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n})\right] \cdot \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \\
&\exp\left(\sum_{n=1}^{N} \left[y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}\right] - \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right)
\end{aligned}
$$

where $\lambda_{g,n} = \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g))$. I will sample the ϕ_g's in parallel using Metropolis steps.

3.8 $p(\theta_\phi \mid \cdots)$: Normal

$$
\begin{aligned}
p(\theta_\phi \mid \cdots) &= \left[\prod_{g=1}^{G} \mathrm{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)\right] \cdot \mathrm{N}(\theta_\phi \mid 0, \gamma_\phi^2) \\
&\propto \left[\prod_{g=1}^{G} \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right)\right] \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\sum_{g=1}^{G} \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\sum_{g=1}^G \phi_g^2 - 2\theta_\phi \sum_{g=1}^G \phi_g + G\theta_\phi^2}{2\sigma_\phi^2}\right) \exp\left(-\frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\sum_{g=1}^G \phi_g^2 - 2\theta_\phi \sum_{g=1}^G \phi_g + G\theta_\phi^2}{2\sigma_\phi^2} - \frac{\theta_\phi^2}{2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g^2 - 2\gamma_\phi^2 (\sum_{g=1}^G \phi_g)\theta_\phi + G\gamma_\phi^2\theta_\phi^2}{2\sigma_\phi^2\gamma_\phi^2} - \frac{\sigma_\phi^2\theta_\phi^2}{2\sigma_\phi^2\gamma_\phi^2}\right) \\
&= \exp\left(-\frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g^2 - 2\gamma_\phi^2 (\sum_{g=1}^G \phi_g)\theta_\phi + (G\gamma_\phi^2 + \sigma_\phi^2)\theta_\phi^2}{2\sigma_\phi^2\gamma_\phi^2}\right) \\
&\propto \exp\left(-\frac{(G\gamma_\phi^2 + \sigma_\phi^2)\left(\theta_\phi - \frac{\gamma_\phi^2(\sum_{g=1}^G \phi_g)}{G\gamma_\phi^2 + \sigma_\phi^2}\right)^2}{2\sigma_\phi^2\gamma_\phi^2}\right)
\end{aligned}
$$

Hence:

$$
p(\theta_\phi \mid \cdots) = \mathrm{N}\left(\frac{\gamma_\phi^2 \sum_{g=1}^G \phi_g}{G\gamma_\phi^2 + \sigma_\phi^2},\ \frac{\gamma_\phi^2\sigma_\phi^2}{G\gamma_\phi^2 + \sigma_\phi^2}\right)
$$

3.9 $p\left(\frac{1}{\sigma_\phi^2} \mid \ldots\right)$: Truncated Gamma

$$\begin{aligned}
p(\sigma_\phi \mid \cdots) &= \left[\prod_{g=1}^{G} \mathrm{N}(\phi_g \mid \theta_\phi, \sigma_\phi^2)\right] \cdot \mathrm{U}(\sigma_\phi \mid 0, \sigma_{\phi 0}) \\
&\propto \prod_{g=1}^{G} (\sigma_\phi^2)^{-1/2} \exp\left(-\frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2) \\
&= (\sigma_\phi^2)^{-G/2} \exp\left(-\sum_{g=1}^{G} \frac{(\phi_g - \theta_\phi)^2}{2\sigma_\phi^2}\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2) \\
&= (\sigma_\phi^2)^{-G/2} \exp\left(-\frac{1}{\sigma_\phi^2} \frac{1}{2} \sum_{g=1}^{G} (\phi_g - \theta_\phi)^2\right) \mathrm{I}(0 < \sigma_\phi^2 < \sigma_{\phi 0}^2)
\end{aligned}$$

Transformation: let $z = g(\sigma_\phi) = \sigma_\phi^2$ so that $g^{-1}(z) = \sqrt{z}$. Then for some proportionality constant, a:

$$\begin{aligned}
p(\sigma_\phi^2 = z \mid \cdots) &= p(\sigma_\phi = g^{-1}(z) \mid \cdots) \left|\frac{g^{-1}(z)}{dz}\right| \\
&= p(\sigma_\phi = \sqrt{z} \mid \cdots) \left|\frac{1}{2} z^{-1/2}\right| \\
&= a \cdot (z)^{-G/2} \exp\left(-\frac{1}{z} \frac{1}{2} \sum_{g=1}^{G} (\phi_g - \theta_\phi)^2\right) \mathrm{I}(0 < z < \sigma_{\phi 0}^2) z^{-1/2} \\
&= \frac{a}{2} \cdot z^{-(G/2 - 1/2 + 1)} \exp\left(-\frac{1}{z} \frac{1}{2} \sum_{g=1}^{G} (\phi_g - \theta_\phi)^2\right) \mathrm{I}(0 < z < \sigma_{\phi 0}^2)
\end{aligned}$$

which is a truncated inverse gamma distribution with shape parameter $\frac{G-1}{2}$ and rate parameter $\frac{1}{2}\sum_{g=1}^{G}(\phi_g - \theta_\phi)^2$. Thus:

$$p\left(\frac{1}{\sigma_\phi^2} \mid \cdots\right) = \text{Gamma}\left(\text{shape} = \frac{G-1}{2},\ \text{rate} = \frac{1}{2}\sum_{g=1}^{G}(\phi_g - \theta_\phi)^2\right) \mathrm{I}\left(\frac{1}{\sigma_\phi^2} > \frac{1}{\sigma_{\phi 0}^2}\right)$$

I will sample $1/\sigma_\phi^2$ using the inverse cdf method.

3.10 $p(\alpha_g \mid \cdots)$: Metropolis

$$p(\alpha_g \mid \cdots) = \left[\prod_{k(n) \neq 2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \alpha_g, \delta_g)))\right] \pi_\alpha^{1 - I(\alpha_g)} [(1 - \pi_\alpha) \mathrm{N}(\theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}$$

Draw $u_g \sim U(0,1)$.

1. Case 1: if $u_g < \pi_\alpha$, use a Metropolis step to draw ϕ_g from a distribution with the kernel:

$$\begin{aligned}
&\prod_{k(n)\neq 2} \text{Poisson}(y_{g,n} \mid \lambda_{g,n}) \\
&\propto \prod_{k(n)\neq 2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \\
&= \exp\left(\sum_{k(n)\neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] \right)
\end{aligned}$$

2. Case 2: if $u_g \geq \pi_\alpha$, use a Metropolis step to draw ϕ_g from a distribution with the kernel:

$$\begin{aligned}
&\left[\prod_{k(n)\neq 2} \text{Poisson}(y_{g,n} \mid \lambda_{g,n})) \right] \cdot \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2) \\
&\propto \left[\prod_{k(n)\neq 2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \right] \exp\left(-\frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} \right) \\
&= \exp\left(\sum_{k(n)\neq 2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2} \right)
\end{aligned}$$

3.11 $p(\theta_\alpha \mid \cdots)$: Normal

$$\begin{aligned}
p(\theta_\alpha \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)} [(1-\pi_\alpha)\text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)} \right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2) \\
&\propto \left[\prod_{\alpha_g \neq 0} \text{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)] \right] \cdot \text{N}(\theta_\alpha \mid 0, \gamma_\alpha^2)
\end{aligned}$$

From algebra similar to the derivation of $p(\theta_\phi \mid \cdots)$, we get:

$$p(\theta_\alpha \mid \cdots) = N\left(\frac{\gamma_\alpha^2 \sum_{\alpha_g \neq 0} \alpha_g}{G'\gamma_\alpha^2 + \sigma_\alpha^2}, \ \frac{\gamma_\alpha^2 \sigma_\alpha^2}{G'\gamma_\alpha^2 + \sigma_\alpha^2} \right)$$

where G' is the number of genes for which $\alpha_g \neq 0$.

3.12 $p(\frac{1}{\sigma_\alpha} \mid \cdots)$: Truncated Gamma

$$
\begin{aligned}
p(\sigma_\alpha \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)}[(1-\pi_\alpha)\mathrm{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \mathrm{U}(\sigma_\alpha \mid 0, \sigma_{\alpha 0}) \\
&\propto \prod_{\alpha_g \neq 0} N(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2) \\
&\propto \prod_{\alpha_g \neq 0} (\sigma_\alpha^2)^{-1/2} \exp\left(-\frac{(\alpha_g - \theta_\alpha)^2}{2\sigma_\alpha^2}\right) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2) \\
&= (\sigma_\alpha^2)^{-G'/2} \exp\left(-\frac{1}{\theta_\alpha^2}\frac{1}{2}\sum_{\alpha_g \neq 0}(\alpha_g - \theta_\alpha)^2\right) \cdot I(0 < \sigma_\alpha^2 < \sigma_{\alpha 0}^2)
\end{aligned}
$$

where G' is the number of genes for which $\alpha_g \neq 0$. Transformation: let $z = \sigma_\alpha^2 = g(\sigma_\alpha)$ so that $g^{-1}(z) = \sqrt{z}$. Then:

$$
\begin{aligned}
p(\sigma_\alpha^2 = z \mid \cdots) = p(\sigma_\alpha = \sqrt{z} \mid \cdots) \left|\frac{d}{dz}\sqrt{z}\right|_z \\
&= (z)^{-G'/2} \exp\left(-\frac{1}{z}\frac{1}{2}\sum_{\alpha_g \neq 0}(\alpha_g - \theta_\alpha)^2\right) \cdot I(0 < z < \sigma_{\alpha 0}^2) \left|-\frac{1}{2}z^{-1/2}\right| \\
&= (z)^{-(G'/2 - 1/2 + 1)} \exp\left(-\frac{1}{z}\frac{1}{2}\sum_{\alpha_g \neq 0}(\alpha_g - \theta_\alpha)^2\right) \cdot I(0 < z < \sigma_{\alpha 0}^2)
\end{aligned}
$$

which is the kernel of an Inverse-Gamma distribution. Hence:

$$
p(\frac{1}{\sigma_\alpha^2} \mid \cdots) = \mathrm{Gamma}\left(\frac{1}{\sigma_\alpha^2} \mid \text{shape} = \frac{G'-1}{2},\ \text{rate} = \frac{1}{2}\sum_{\alpha_g \neq 0}(\alpha_g - \theta_\alpha)^2\right)
$$

3.13 $p(\pi_\alpha \mid \cdots)$: Beta

$$
\begin{aligned}
p(\pi_\alpha \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\alpha^{1-I(\alpha_g)}[(1-\pi_\alpha)\mathrm{N}(\alpha_g \mid \theta_\alpha, \sigma_\alpha^2)]^{I(\alpha_g)}\right] \cdot \mathrm{Beta}(\pi_\alpha \mid a_\alpha, b_\alpha) \\
&\propto [\pi_\alpha^{G-G'}(1-\pi_\alpha)^{G'}]\pi_\alpha^{a_\tau - 1}(1-\pi_\alpha)^{b_\tau - 1} \\
&= \pi_\alpha^{G-G'+a_\tau-1}(1-\pi_\alpha)^{G'+b_\tau-1}
\end{aligned}
$$

where G' is the number of genes for which $\alpha_g \neq 0$. Hence:

$$p(\pi_\alpha \mid \cdots) = \text{Beta}(G - G' + \alpha_\tau,\ G' + b_\tau)$$

3.14 $p(\delta_g \mid \cdots)$: Metropolis

$$p(\delta_g \mid \cdots) = \left[\prod_{k(n)=2} \text{Poisson}(y_{g,n} \mid \exp(c_n + \varepsilon_{g,n} + \mu(n, \phi_g, \delta_g, \delta_g)))\right] \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\text{N}(\theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}$$

Draw $u_g \sim U(0,1)$.

1. Case 1: if $u_g < \pi_\delta$, use a Metropolis step to draw ϕ_g from a distribution with the kernel:

$$\begin{aligned} &\prod_{k(n)=2} \text{Poisson}(y_{g,n} \mid \lambda_{g,n}) \\ \propto &\prod_{k(n)=2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n}) \\ = &\exp\left(\sum_{k(n)=2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}]\right) \end{aligned}$$

2. Case 2: if $u_g \geq \pi_\delta$, use a Metropolis step to draw ϕ_g from a distribution with the kernel:

$$\begin{aligned} &\left[\prod_{k(n)=2} \text{Poisson}(y_{g,n} \mid \lambda_{g,n}))\right] \cdot \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2) \\ \propto &\left[\prod_{k(n)=2} \lambda_{g,n}^{y_{g,n}} \exp(-\lambda_{g,n})\right] \exp\left(-\frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2}\right) \\ = &\exp\left(\sum_{k(n)=2} [y_{g,n} \log \lambda_{g,n} - \lambda_{g,n}] - \frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2}\right) \end{aligned}$$

3.15 $p(\theta_\delta \mid \cdots)$: Normal

$$\begin{aligned} p(\theta_\delta \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2) \\ &\propto \left[\prod_{\delta_g \neq 0} \text{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]\right] \cdot \text{N}(\theta_\delta \mid 0, \gamma_\delta^2) \end{aligned}$$

From algebra similar to the derivation of $p(\theta_\phi \mid \cdots)$, we get:

$$p(\theta_\delta \mid \cdots) = N\left(\frac{\gamma_\delta^2 \sum_{\delta_g \neq 0} \delta_g}{G'\gamma_\delta^2 + \sigma_\delta^2},\ \frac{\gamma_\delta^2 \sigma_\delta^2}{G'\gamma_\delta^2 + \sigma_\delta^2}\right)$$

where G' is the number of genes for which $\delta_g \neq 0$.

3.16 $p(\frac{1}{\sigma_\delta} \mid \cdots)$: Truncated Gamma

$$\begin{aligned}
p(\sigma_\delta \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\mathrm{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \mathrm{U}(\sigma_\delta \mid 0, \sigma_{\delta 0}) \\
&\propto \prod_{\delta_g \neq 0} N(\delta_g \mid \theta_\delta, \sigma_\delta^2) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2) \\
&\propto \prod_{\delta_g \neq 0} (\sigma_\delta^2)^{-1/2} \exp\left(-\frac{(\delta_g - \theta_\delta)^2}{2\sigma_\delta^2}\right) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2) \\
&= (\sigma_\delta^2)^{-G'/2} \exp\left(-\frac{1}{\theta_\delta^2}\frac{1}{2}\sum_{\delta_g \neq 0}(\delta_g - \theta_\delta)^2\right) \cdot I(0 < \sigma_\delta^2 < \sigma_{\delta 0}^2)
\end{aligned}$$

where G' is the number of genes for which $\delta_g \neq 0$. Transformation: let $z = \sigma_\delta^2 = g(\sigma_\delta)$ so that $g^{-1}(z) = \sqrt{z}$. Then:

$$\begin{aligned}
p(\sigma_\delta^2 = z \mid \cdots) = p(\sigma_\delta = \sqrt{z} \mid \cdots) \left|\frac{d}{dz}\sqrt{z}\right|_z \\
&= (z)^{-G'/2} \exp\left(-\frac{1}{z}\frac{1}{2}\sum_{\delta_g \neq 0}(\delta_g - \theta_\delta)^2\right) \cdot I(0 < z < \sigma_{\delta 0}^2) \left|-\frac{1}{2}z^{-1/2}\right| \\
&= (z)^{-(G'/2 - 1/2 + 1)} \exp\left(-\frac{1}{z}\frac{1}{2}\sum_{\delta_g \neq 0}(\delta_g - \theta_\delta)^2\right) \cdot I(0 < z < \sigma_{\delta 0}^2)
\end{aligned}$$

which is the kernel of an Inverse-Gamma distribution. Hence:

$$p(\frac{1}{\sigma_\delta^2} \mid \cdots) = \text{Gamma}\left(\frac{1}{\sigma_\delta^2} \mid \text{shape} = \frac{G'-1}{2},\ \text{rate} = \frac{1}{2}\sum_{\delta_g \neq 0}(\delta_g - \theta_\delta)^2\right)$$

3.17 $p(\pi_\delta \mid \cdots)$: Beta

$$\begin{aligned}
p(\pi_\delta \mid \cdots) &= \left[\prod_{g=1}^{G} \pi_\delta^{1-I(\delta_g)}[(1-\pi_\delta)\mathrm{N}(\delta_g \mid \theta_\delta, \sigma_\delta^2)]^{I(\delta_g)}\right] \cdot \mathrm{Beta}(\pi_\delta \mid a_\delta, b_\delta) \\
&\propto [\pi_\delta^{G-G'}(1-\pi_\delta)^{G'}]\pi_\delta^{a_\tau - 1}(1-\pi_\delta)^{b_\tau - 1} \\
&= \pi_\delta^{G-G'+a_\tau-1}(1-\pi_\delta)^{G'+b_\tau-1}
\end{aligned}$$

where G' is the number of genes for which $\delta_g \neq 0$. Hence:

$$p(\pi_\delta \mid \cdots) = \mathrm{Beta}(G - G' + \delta_\tau,\ G' + b_\tau)$$