1 Gamma and Inverse-Gamma

$$\text{Gamma}(x \mid \text{shape} = \alpha, \text{rate} = \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

Let $y = x^{-1} = g(x)$ so that $g^{-1}(y) = y^{-1}$ and $\frac{dg^{-1}}{dy} \mid_y = -y^{-2}$. Then, we apply the transformation:

$$\begin{aligned}
p(x^{-1} = y \mid \alpha, \beta) &= p(x = g^{-1}(y) \mid \alpha, \beta) \left| \frac{dg^{-1}}{dy} \right|_y \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} (y^{-1})^{\alpha-1} e^{-\beta(y^{-1})} | - y^{-2} | \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} y^{-(\alpha+1)} e^{-\beta/y}
\end{aligned}$$

Hence, the pdf of the Inverse-Gamma distribution is:

$$\text{Inv-Gamma}(x \mid \text{shape} = \alpha, \text{rate} = \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x}$$