Know: $l(p_1, p_2) = \binom{n}{x}\left(\frac{p_1+p_2}{2}\right)^{x}\left(1-\frac{p_1+p_2}{2}\right)^{n-x} \cdot \binom{m_1}{x_1} p_1^{x_1}(1-p_1)^{m_1-x_1}$
$\cdot \binom{m_2}{x_2} p_2^{x_2}(1-p_2)^{m_2-x_2}$

Derivation of

- log likelihood function:

$$ll = \operatorname{Log} l(p_1,p_2) = \operatorname{Log}\binom{n}{x}\left(\frac{p_1+p_2}{2}\right)^{x}\left(1-\frac{p_1+p_2}{2}\right)^{n-x} + \operatorname{Log}\binom{m_1}{x_1}p_1^{x_1}(1-p_1)^{m_1-x_1}$$
$$+ \operatorname{Log}\binom{m_2}{x_2}p_2^{x_2}(1-p_2)^{m_2-x_2}$$
$$= \log\binom{n}{x} + \log\binom{m_1}{x_1} + \log\binom{m_2}{x_2} + x\log\frac{p_1+p_2}{2} + (n-x)\log\left(1-\frac{p_1+p_2}{2}\right)$$
$$+ x_1\log p_1 + (m_1-x_1)\log(1-p_1) + x_2\log p_2 + (m_2-x_2)\log(1-p_2)$$

- gradient vector

$$\frac{\partial ll}{\partial p_1} = 0+0+0+x\cdot\frac{2}{p_1+p_2}\cdot\frac{1}{2} + (n-x)\frac{1}{1-\frac{p_1+p_2}{2}}\cdot\left(-\frac{1}{2}\right) + \frac{x_1}{p_1} + (m_1-x_1)\frac{1}{1-p_1}(-1)$$
$$= \frac{x}{p_1+p_2} + (n-x)\frac{1}{p_1+p_2-2} + \frac{x_1}{p_1} + (m_1-x_1)\frac{1}{p_1-1}$$

$$\frac{\partial ll}{\partial p_2} = \frac{x}{p_1+p_2} + (n-x)\frac{1}{p_1+p_2-2} + \frac{x_2}{p_2} + (m_2-x_2)\frac{1}{p_2-1}$$

so the gradient vector is $\left(\frac{\partial ll}{\partial p_1}, \frac{\partial ll}{\partial p_2}\right)^T$

- Hessian matrix

$$H(ll) = \begin{pmatrix} \frac{\partial^2 ll}{\partial p_1 \partial p_1} & \frac{\partial^2 ll}{\partial p_1 \partial p_2} \\ \frac{\partial^2 ll}{\partial p_2 \partial p_1} & \frac{\partial^2 ll}{\partial p_2 \partial p_2} \end{pmatrix}$$

where $\frac{\partial^2 ll}{\partial p_1 \partial p_1} = -(p_1+p_2)^{-2}x - (n-x)(p_1+p_2-2)^{-2} - x_1 p_1^{-2} - (m_1-x_1)(p_1-1)^{-2}$

$$\frac{\partial^2 ll}{\partial p_1 \partial p_2} = -(p_1+p_2)^{-2}x - (n-x)(p_1+p_2-2)^{-2}$$

$$\frac{\partial^2 ll}{\partial p_2 \partial p_1} = -(p_1+p_2)^{-2}x - (n-x)(p_1+p_2-2)^{-2}$$

$$\frac{\partial^2 ll}{\partial p_2 \partial p_2} = -(p_1+p_2)^{-2}x - (n-x)(p_1+p_2-2)^{-2} - x_2 p_2^{-2} - (m_2-x_2)(p_2-1)^{-2}$$