ME 360 Lecture 16

2.4.2. Permutation Matrices

In order to apply partial pivoting to LU factorization, we first need to know the fundamental properties of permutation matrices.

- **Definition of permutation matrix**

A permutation matrix is an $n \times n$ matrix consisting of all zeros, except for a single 1 in every row and column

Equivalently, a permutation matrix P is created by applying arbitrary row exchanges to the $n \times n$ identity matrix (or arbitrary column exchanges). For example, there are two 2×2 permutation matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \& \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There are six 3×3 permutation matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- **Fundamental Theorem of Permutation Matrices**

Let P be the $n \times n$ permutation matrix formed by a particular set of row exchanges applied to the identity matrix. Then, for any $n \times n$ matrix A, PA is the matrix obtained by applying exactly the same set of row exchanges to A.

For example, the permutation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is formed by exchanging rows 2 & 3 of the identity matrix.
Multiplying an arbitrary matrix on the left with P has the effect of exchanging rows 2 & 3.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

2.4.3 PA=LU factorization

- The PA=LU factorization is the established workhorse for solving systems of linear equations.
- We need to keep track of previous multipliers when a row exchange is made.

Example: Find the PA=LU factorization of the matrix.

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$$

Step 1: exchange row 1 & 2 according to partial pivoting.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix} \Rightarrow \text{exchange row 1 \& 2} \Rightarrow \begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix}$$

Note: We will use the permutation matrix P to keep track of the cumulative permutations.
We will also store the multiplier in the location where we eliminate the column entry, to better keep track row permutation.

Step 2: Column 1 elimination

$$\begin{bmatrix} 4 & 4 & -4 \\ 2 & 1 & 5 \\ 1 & 3 & 1 \end{bmatrix} \Rightarrow \begin{matrix}\text{Subtract } 1/2 \times \text{row } 1 \\ \text{from row } 2\end{matrix} \Rightarrow \begin{bmatrix} 4 & 4 & -4 \\ 1/2 & -1 & 7 \\ 1 & 3 & 1 \end{bmatrix} \Rightarrow \begin{matrix}\text{subtract } 1/4 \times \text{row } 1 \\ \text{from row } 3\end{matrix} \Rightarrow \begin{bmatrix} 4 & 4 & -4 \\ 1/2 & -1 & 7 \\ 1/4 & 2 & 2 \end{bmatrix}$$

Step 3: exchange row 2 & 3 according to partial pivoting.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & -4 \\ 1/2 & -1 & 7 \\ 1/4 & 2 & 2 \end{bmatrix} \Rightarrow \text{exchange row } 2 \,\&\, 3 \Rightarrow \begin{bmatrix} 4 & 4 & -4 \\ 1/4 & 2 & 2 \\ 1/2 & -1 & 7 \end{bmatrix}$$

Step 4: Column 2 elimination

$$\begin{bmatrix} 4 & 4 & -4 \\ 1/4 & 2 & 2 \\ 1/2 & -1 & 7 \end{bmatrix} \Rightarrow \begin{matrix}\text{subtract } -1/2 \times \text{row } 2 \\ \text{from row } 3\end{matrix} \Rightarrow \begin{bmatrix} 4 & 4 & -4 \\ 1/4 & 2 & 2 \\ 1/2 & -1/2 & 8 \end{bmatrix}$$

Step 5: get $PA = LU$ factorization

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}}_{U}$$

- **Two-step back substitution for $PA = LU$**

The procedure here is very similar to the version for $A = LU$. Multiply through the equatio $Ax = b$ by P on the left of both sides, and proceed as before.

$$\begin{cases} PAx = Pb \\ LUx = Pb \end{cases} \Rightarrow \text{solve} \begin{cases} 1.\ Lc = Pb \text{ for } x \\ 2.\ Ux = c \text{ for } x \end{cases}$$

Using the matrix notation, all elimination and permutation are automatically stored in the matrix equations.

Example: Use two-step substitution to solve for the system of equations in previous example, where $b = [5, 0, 6]$

$$\underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix}}$$

Step 1: solve $Lc = Pb$ for c

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{cases} c_1 = 0 \\ 1/4\, c_1 + c_2 = 6 \\ 1/2\, c_1 - 1/2\, c_2 + c_3 = 5 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = 6 \\ c_3 = 8 \end{cases}$$

Step 2: Solve $Ux = c$ for x

$$\begin{bmatrix} 4 & 4 & -4 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 8 \end{bmatrix} \Rightarrow \begin{cases} 4x_1 + 4x_2 - 4x_3 = 0 \\ 2x_2 + 2x_3 = 6 \\ 8x_3 = 8. \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -1 \\ x_2 = 2 \\ x_3 = 1 \end{cases} \Rightarrow \boxed{x = [-1, 2, 1]}$$

Example: Solve the system $\begin{cases} 2x_1 + 3x_2 = 4 \\ 3x_1 + 2x_2 = 1 \end{cases}$ using $PA = LU$

Step 0: Write it in the form of $Ax = b$

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Step 1: PA = LU factorization.

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \Rightarrow \text{exchange row 1 \& 2} \Rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{array}{c} \text{Subtract } 2/3 \times \text{row1} \\ \text{from row 2} \end{array} \Rightarrow \begin{bmatrix} 3 & 2 \\ 2/3 & 5/3 \end{bmatrix}$$

The factorization is

$$\underset{P}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{A}{\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 3 & 2 \\ 0 & 5/3 \end{bmatrix}}$$

Step 2. back substitution.

(a). $Lc = Pb$.

$$\begin{bmatrix} 1 & 0 \\ 2/3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Rightarrow \begin{cases} c_1 = 1 \\ 2/3 c_1 + c_2 = 4 \end{cases}$$

$$\Rightarrow c = \begin{bmatrix} 1 \\ 10/3 \end{bmatrix}$$

(b) $Ux = c$

$$\begin{bmatrix} 3 & 2 \\ 0 & 5/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10/3 \end{bmatrix} \Rightarrow \begin{cases} 3x_1 + 2x_2 = 1 \\ 5/3 x_2 = 10/3 \end{cases} \Rightarrow \begin{cases} x_1 = -1 \\ x_2 = 2 \end{cases}$$

$$\therefore \boxed{x = [-1, 2]}$$

Exercise in class: