ME 360 Lecture 8

From last class, we know that Newton's Method most of the time converge quadratically, but some time converge linearly when there is multiplicity in roots.

If we know the multiplicity of a root in advance, we can modify Newton's method.

Theorem: If f is $(m+1)$-times continuously differentiable on $[a,b]$, which contains a root r of multiplicity $m>1$, then

Modified Newton's Method

$$x_{i+1} = x_i - \frac{m f(x_i)}{f'(x_i)}$$

converges locally and quadratically to r.

Using this theorem, we can apply to the example $f(x)=x^2$

- Traditional Newton's method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^2}{2x_i} = \frac{2x_i^2 - x_i^2}{2x_i} = \frac{1}{2}x_i$$

- Modified Newton's method:

For $f(x)=x^2$, $f'(x)=2x$, $f''(x)=2$, so $m=2$.

$$x_{i+1} = x_i - \frac{m f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{2x_i^2}{2x_i} = 0$$

Failure of Newton's Method

Similar to FPI, Newton's Method does not guarantee convergence

For example, apply Newton's Method to $f(x) = 4x^4 - 6x^2 - 11/4$ with initial guess $x_0 = 1/2$.

Apply Newton's method:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{4x_i^4 - 6x_i^2 - 11/4}{16x_i^3 - 12x_i}$$

For $x_0 = \frac{1}{2}$, $x_1 = \frac{1}{2} - \frac{4\cdot(0.5)^4 - 6\cdot(0.5)^2 - 11/4}{16\cdot(0.5)^3 - 6}$

$$= 0.5 - \frac{0.25 - 1.5 - 2.75}{-4}$$

$$= 0.5 - (-4)/(-4) = -0.5$$

$x_1 = -\frac{1}{2}$, $x_2 = -0.5 - (-4)/(4) = 0.5$

Using Newton's method, we will be oscillating between $1/2$ and $-1/2$.

Other ways Newton's Method can fail

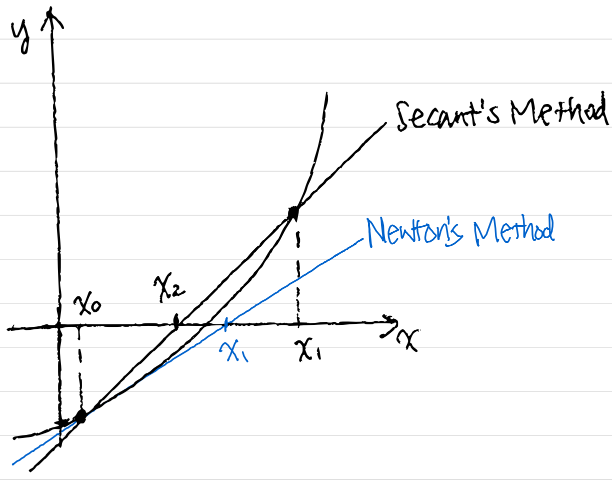
1. $f'(x_i) = 0$
2. iteration diverges to infinity
3. mimics random number generator

Secant Method

Very similar to Newton's method. Instead of the analytical form of the derivative, we use

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Geometric representation of Secant's Method.



- Secant Method:

x_0, x_1 = initial guess

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{for } i = 1, 2, 3, \ldots$$

We need two initial guesses for Secant Method.

Convergence of Secant Method

For function $f(x)$ that converges to r and $f'(r) \neq 0$, the approximate error relationship for Secant Method is

$$e_{i+1} \approx \left| \frac{f''(r)}{2f'(r)} \right| e_i e_{i-1}$$

and this implies that

$$e_{i+1} \approx \left| \frac{f''(r)}{2f'(r)} \right|^{\alpha - 1} e_i^{\alpha} \quad \text{where } \alpha = \frac{1+\sqrt{5}}{2} \approx 1.62$$

Secant's Method converges superlinear, meaning that it lies between linearly and quadratically convergent methods.

Secant's Method has three variants (Sauer P.66-68)

1. Method of False Position
   - a combination of Bisection & Secant
2. Muller's Method
   - instead of 2 initial guess, use 3 to make a parabola.
3. Inverse Quadratic Interpolation
   - Similar to Muller's Method, but build parabola in other direction.

Brent's Method

The Method Python, Matlab uses for root finding.
It's a hybrid method using Inverse Quadratic Interpolation, Secant Method and Bisection Method.
We get fast convergence from IQI and Secant, and guaranteed convergence from Bisection.