ME 360 Lecture 21

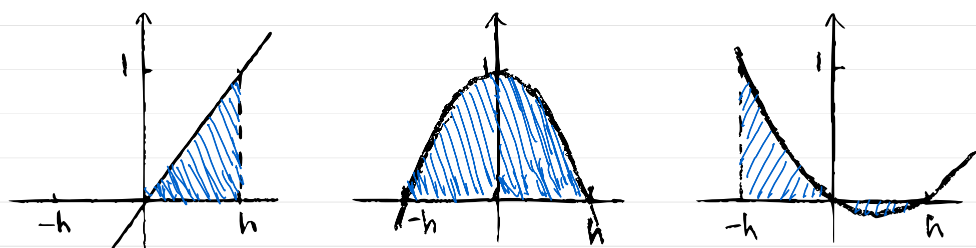
5.2. Newton-Cotes Formulas for Numerical Integration

- Trapezoid Rule - Degree 1
- Simpson's Rule - Degree 2
- Composite Newton-Cotes Formulas.

These series of method is based on approximating the function $f(x)$ as a polynomial. Normally, the higher the polynomial degree, the more accurate of the approximation.

In practice, if the interval is small, degree 1 & 2 polynomial approximation is good enough.

Three helper integrals:



(a) (b) (c)

(a). The shaded region is the line passing through (0,0) and (h,1). The function for the line can be written as x/h.

$$\int_0^h \frac{x}{h}\,dx = \frac{1}{2}\frac{x^2}{h}\Big|_0^h = \frac{1}{2}\frac{h^2}{h} - 0 = \frac{h}{2}$$

(b). The shaded region is the parabola $P(x)$ passing through $(-h,0)$, $(0,1)$, $(h,0)$. $P(x)=(h^2-x^2)/h^2$

$$\int_{-h}^{h} P(x)\,dx = \int_{-h}^{h} \frac{(h^2-x^2)}{h^2}\,dx = \frac{(h^2x-\frac{1}{3}x^3)}{h^2}\Big|_{-h}^{h}$$

$$= \frac{h^3-\frac{1}{3}h^3}{h^2} - \frac{-h^3+\frac{1}{3}h^3}{h^2} = \frac{2}{3}h + \frac{2}{3}h = \frac{4}{3}h$$

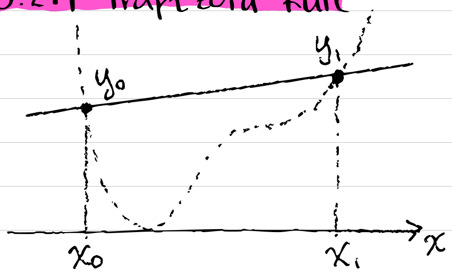
(c). The shaded region is the parabola $P(x)$ passing through $(-h, 1)$, $(0, 0)$, and $(h, 0)$. $P(x) = (x^2 - hx)/2h^2$

$$\int_{-h}^{h} P(x)dx = \int_{-h}^{h} \frac{(x^2-hx)}{2h^2} dx = \frac{(x^3/3 - hx^2/2)}{2h^2} \Big|_{-h}^{h}$$

$$= \frac{(h^3/3 - h^3/2)}{2h^2} - \frac{(-h^3/3 - h^3/2)}{2h^2} = \frac{2h^3/3}{2h^2} = \frac{h}{3}$$

(Note; if you want to know more about how we found the functional forms of the parabolas, check out Ch. 3).

5.2.1 Trapezoid Rule



For function $f(x)$ that passes through (x_0, y_0), (x_1, y_1), we can write the function as a series of polynomial expansions. (Lagrange formulation).

$$f(x) = \underbrace{y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}}_{P(x)} + \underbrace{\frac{(x-x_0)(x-x_1)}{2!} f''(c_x)}_{E(x)}$$

$$= P(x) + E(x)$$

(c_x depends continuously on x).

Integrating over the interval $[x_0, x_1]$, we have

$$\int_{x_0}^{x_1} f(x) = \int_{x_0}^{x_1} P(x) + \int_{x_0}^{x_1} E(x).$$

1. Evaluate $\int_{x_0}^{x_1} P(x)$

$$\int_{x_0}^{x_1} P(x) = \int_{x_0}^{x_1} y_0 \frac{x - x_1}{x_0 - x_1} dx + \int_{x_0}^{x_1} y_1 \frac{x - x_0}{x_1 - x_0} dx$$

Let $x_1 - x_0 = h$, we have

$$\int_{x_0}^{x_1} P(x) = y_0 \int_{x_0}^{x_1} \frac{x_1 - x}{h} dx + y_1 \int_{x_0}^{x_1} \frac{x - x_0}{h} dx$$

$$= y_0 \left(\frac{x_1 x - x^2/2}{h} \right) \Big|_{x_0}^{x_1} + y_1 \left(\frac{x^2/2 - x_0 x}{h} \right) \Big|_{x_0}^{x_1}$$

$$= y_0\left(\frac{x_1^2 - x_1^2/2}{h} - \frac{x_1x_0 - x_0^2/2}{h}\right) + y_1\left(\frac{x_1^2/2 - x_0x_1}{h} - \frac{x_0^2/2 - x_0^2}{h}\right)$$

$$= y_0\left(\frac{x_1^2/2 - x_1x_0 + x_0^2/2}{h}\right) + y_1\left(\frac{x_1^2/2 - x_0x_1 + x_0^2/2}{h}\right)$$

$$= y_0\left(\frac{\frac{1}{2}(x_1 - x_0)^2}{h}\right) + y_1\left(\frac{\frac{1}{2}(x_1 - x_0)^2}{h}\right)$$

$$= y_0\left(\frac{h^2/2}{h}\right) + y_1\left(\frac{h^2/2}{h}\right) = \frac{h}{2}(y_0 + y_1)$$

$h(y_0+y_1)/2$ also computes the area of a trapezoid, hence the name.

2. Evaluate $\int_{x_0}^{x_1} E(x)$

Theorem Detour:

<u>Mean Value Theorem for Integrals (see Sauer P. 23)</u>

Let f be a continuous function on the interval $[a,b]$, and let g be an integrable function that does not change sign on $[a,b]$. Then there exists a number c between a and b such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

$$\int_{x_0}^{x_1} E(x)\,dx = \frac{1}{2!}\int_{x_0}^{x_1} \underbrace{(x-x_0)(x-x_1)}_{g(x),\text{ no sign change}}\underbrace{f''(c(x))}_{f(x)}\,dx$$

$$= \frac{f''(c)}{2}\int_{x_0}^{x_1}(x-x_0)(x-x_1)\,dx$$

$$= \frac{f''(c)}{2}\int_{x_0}^{x_1} x^2 - (x_0+x_1)x + x_0x_1\,dx$$

$$= \frac{f''(c)}{2}\left(\frac{1}{3}x^3 - \frac{x_0+x_1}{2}x^2 + x_0x_1x\right)\Bigg|_{x_0}^{x_1}$$

$$= \frac{f''(c)}{2}\left(\frac{1}{3}x_1^3 - \frac{x_0+x_1}{2}x_1^2 + x_0x_1^2\right) - \frac{f''(c)}{2}\left(\frac{1}{3}x_0^3 - \frac{x_0+x_1}{2}x_0^2 + x_0^2x_1\right)$$

$$= \frac{f''(c)}{2}\left(\frac{1}{3}(x_1^3 - x_0^3) - \frac{x_0+x_1}{2}(x_1^2 - x_0^2) + x_0x_1(x_1-x_0)\right)$$

$$= \frac{f''(c)(x_1-x_0)}{2}\left(\frac{x_1^2+x_1x_0+x_0^2}{3} - \frac{(x_0+x_1)^2}{2} + x_0x_1\right)$$

$$= \frac{f''(c)\cdot h}{2}\left(\frac{2x_1^2+2x_1x_0+2x_0^2-3x_0^2-6x_0x_1-3x_1^2+6x_0x_1}{6}\right)$$

$$= \frac{f''(c)\cdot h}{2}\left(\frac{2x_1x_0 - x_1^2 - x_0^2}{6}\right)$$

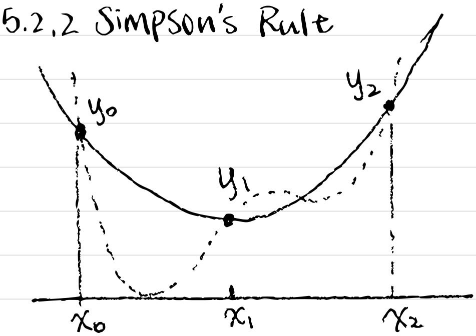
$$= \frac{f''(c)\cdot h}{12}\left(-(x_1-x_2)^2\right) = -\frac{h^3}{12}f''(c)$$

Combine everything together, we have the Trapezoid Rule:

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(y_0+y_1) - \frac{h^3}{12}f''(c)$$

where $h = x_1 - x_0$, $\quad x_0 < c < x_1$

5.2.2 Simpson's Rule



Instead of drawing a linear line between y_0 & y_1, we now get three points to generate a parabola.
(Keep expanding more terms using Lagrange formulation)

$$f(x) = \overbrace{y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_3\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}}^{P(x)}$$

$$+ \underbrace{\frac{(x-x_0)(x-x_1)(x-x_2)}{3!}f'''(c(x))}_{E(x)}$$

Integrating over $[x_0, x_2]$, we have.

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_2} P(x)\,dx + \int_{x_0}^{x_2} E(x)\,dx$$

1. $P(x)$.

$$\int_{x_0}^{x_2} P(x)\,dx = y_0 \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx + y_1 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,dx$$
$$+ y_2 \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,dx$$
$$= y_0 \frac{h}{3} + y_1 \frac{4h}{3} + y_2 \frac{h}{3}$$

$h = x_2 - x_1 = x_1 - x_0$.

2. $E(x)$

$$\int_{x_0}^{x_2} E(x)\,dx = -\frac{h^5}{90} f^{(iv)}(c) \qquad x_0 < c < x_2$$

Simpson's Rule

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} f^{(iv)}(c).$$

Example:

Apply Trapezoid and Simpon's Rule to approximate

$$\int_1^2 \ln x\,dx.$$

and find an upper bound for the error.

(0) Solve the integral using integration by parts as a benchmark.

$$\int_1^2 \ln x\,dx = x\ln x\Big|_1^2 - \int_1^2 dx = 2\ln 2 - 1\ln 1 - 1 \approx 0.386294$$

(1) Trapezoid Rule

$\int_1^2 \ln x\,dx \approx \frac{h}{2}(y_0 + y_1) = \frac{1}{2}(\ln 1 + \ln 2) = \frac{\ln 2}{2} \approx 0.3466$

Error term: $-h^3 f''(c)/12$

$f''(x) = (\ln x)'' = (\frac{1}{x})' = (-\frac{1}{x^2})$, $\quad 1 \le c \le 2$

$-\frac{h^3}{12} f''(1) = \frac{1}{12} \cdot 1 = \frac{1}{12}$; $\quad -\frac{h^3}{12} f''(2) = \frac{1}{12} \cdot \frac{1}{4} = \frac{1}{48}$.

$\therefore$ the largest magnitude of error is $1/12 \approx 0.0834$

$\int_1^2 \ln x\,dx = 0.3466 \pm 0.0834$

This result agrees with the previous solution.

(2). Simpson's Rule

$\int_1^2 \ln x \approx \frac{h}{3}(y_0 + 4y_1 + y_2) = \frac{0.5}{3}(\ln 1 + 4\ln 1.5 + \ln 2)$

≈ 0.3858

Error term: $-h^5 f^{(iv)}(c)/90$

$f^{(iv)}(c) = (-\frac{1}{x^2})'' = (\frac{2}{x^3})' = (-\frac{6}{x^4})$

$-\frac{h^5}{90} f^{(iv)}(c) = \frac{1}{15} \frac{h^5}{c^4} \Rightarrow$ has largest value when $c = 1$

$\frac{1}{15} \frac{(0.5)^5}{(1)^4} \approx 0.0021$

$\int_1^2 \ln x\,dx = 0.3858 \pm 0.0021$

Additional reading: degree of precision. see Sauer P.268-269.