ME 360 Lecture 22

5.2.3 Composite Newton - Cotes formulas

When computing integrals, we can divide the interval into several subintervals, and summing over the results. This strategy is called composite numerical integration.

• Composite Trapezoid Rule

To approximate $\int_a^b f(x)dx$

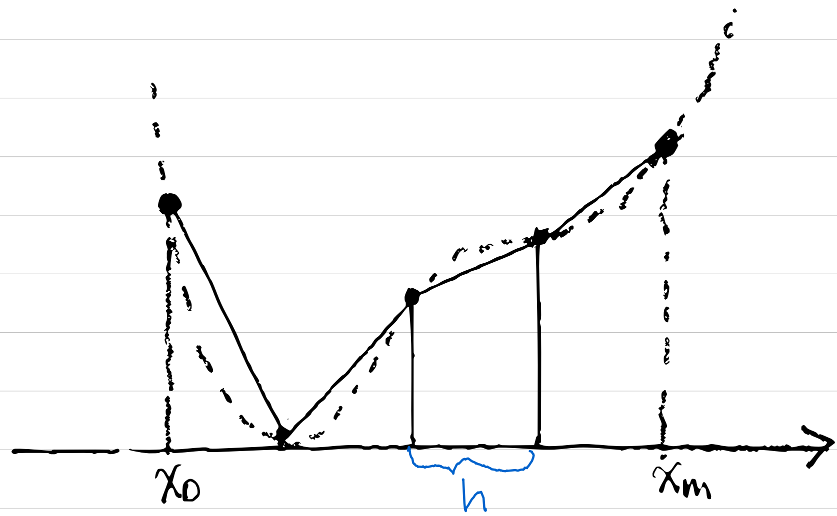
consider an evenly spaced grid

$$a = x_0 < x_1 < x_2 < \cdots < x_{m-2} < x_{m-1} < x_m = b$$

along the horizontal axis, where $h = x_{i+1} - x_i$

Apply Trapezoid's Rule on every subinterval, we have

$$\int_{x_i}^{x_{i+1}} f(x)dx = \frac{h}{2}\left(f(x_i) + f(x_{i+1})\right) - \frac{h^3}{12} f''(c_i)$$



Adding up all subintervals, we have

$$\int_a^b f(x)dx = \frac{h}{2}\Big((f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{m-2}) + f(x_{m-1}))$$
$$+ (f(x_{m-1}) + f(x_m)\Big) - \frac{h^3}{12}\left(f''(c_0) + f''(c_1) + \cdots + f''(c_{m-1})\right)$$
$$= \frac{h}{2}\left[f(a) + f(b) + 2\sum_{i=1}^{m-1} f(x_i)\right] - \sum_{i=0}^{m-1} \frac{h^3}{12} f''(c_i)$$

(Note: $f(x_0) = f(a)$, $f(x_m) = f(b)$ based on our interval definition)

Using the Generalized Intermediate Value Theorem, we can rewrite the error term

$$\frac{h^3}{12}\sum_{i=0}^{m-1} f''(c_i) = \frac{h^3}{12} m f''(c), \quad a<c<b$$

since $mh=(b-a)$, we can rewrite the error term to be

$$\frac{h^2(b-a)}{12} f''(c)$$

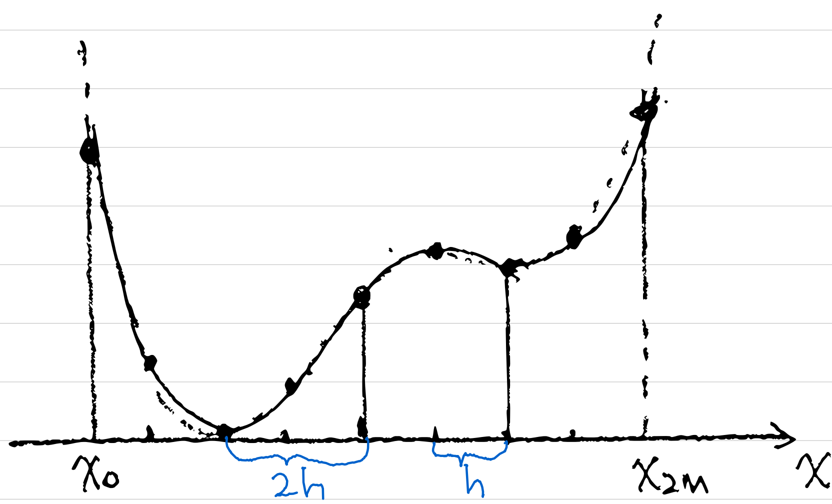
To summarize, if f'' is continuous on $[a,b]$, we have the

Composite Trapezoid Rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(y_0 + y_m + 2\sum_{i=1}^{m-1} y_i\right) - \frac{(b-a)h^2}{12} f''(c)$$

$$h=(b-a)/m, \quad a<c<b$$

- Composite Simpson's Rule



Following the same strategy, consider an evenly spaced grid.

$$a = x_0 < x_1 < x_2 \ldots < x_{2m-2} < x_{2m-1} < x_{2m} = b$$

along the horizontal axis, where $h = x_{i+1} - x_i$ for each i. On each length $2h$ panel $[x_{2i}, x_{2i+2}]$, for $i=0,\ldots,m-1$, a Simpson's method is carried out.

For every subinterval of length $2h$, we have the approximation

$$\int_{x_{2i}}^{x_{2i+2}} f(x)\,dx = \frac{h}{3}\left[f(x_{2i}) + 4f(x_{2i+1}) + f(x_{2i+2})\right] - \frac{h^5}{90} f^{(iv)}(c_i)$$

Adding up over all subintervals, we have

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[\big(f(x_0)+4f(x_1)+f(x_2)\big) + \big(f(x_2)+4f(x_3)+f(x_4)\big) + \cdots$$

$$+ \big(f(x_{2m-2}) + 4f(x_{2m-1}) + f(x_{2m})\big)\Big] - \frac{h^5}{90}\left[f^{(iv)}(c_0)+\cdots+f^{(iv)}(c_{m-1})\right]$$

$$= \frac{h}{3}\left[f(a) + f(b) + 4\sum_{i=1}^{m} f(x_{2i-1}) + 2\sum_{i=1}^{m-1} f(x_{2i})\right] - \sum_{i=0}^{m-1} \frac{h^5}{90} f^{(iv)}(c_i)$$

The error term can be written as

$$\frac{h^5}{90}\sum_{i=0}^{m-1} f^{(iv)}(c_i) = \frac{h^5}{90} m\, f^{(iv)}(c) \qquad a<c<b$$

Since here $m \cdot 2h = (b-a)$, the error term is $(b-a)h^4 f^{(iv)}(c)/180$.

To summarize, if $f^{(iv)}$ is continuous on $[a, b]$, we have

Composite Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[y_0 + y_{2m} + 4\sum_{i=1}^{m} y_{2i-1} + 2\sum_{i=1}^{m-1} y_{2i}\right] - \frac{(b-a)h^4}{180} f^{(iv)}(c)$$

$$h = \frac{(b-a)}{2m}, \qquad a<c<b$$

Example: Carry out four-panel approximations of

$$\int_1^2 \ln x\,dx$$

using composite Trapezoid Rule & composite Simpson's rule.

- Composite Trapezoid Rule.

$a=1$, $b=2$, $m=4$, $h=(b-a)/m = 1/4$.

$$\int_1^2 \ln x\,dx \approx \frac{1/4}{2}\left[y_0 + y_4 + 2\sum_{i=1}^{3} y_i\right]$$

$$= \frac{1}{8}\left[\ln 1 + \ln 2 + 2\left(\ln 5/4 + \ln 3/2 + \ln 7/4\right)\right]$$

$$\approx 0.3837$$

The error at most is

$$\frac{(b-a)h^2}{12}|f''(c)| = \frac{1/16}{12}\frac{1}{c^2} \leq \frac{1}{(16)(12)(1)^2} = \frac{1}{192} \approx 0.0052$$

which is smaller than the single panel Trapezoid Rule.

- Composite Simpson's Rule

  $a=1,\ b=2,\ m=4,\quad h=(b-a)/2m = 1/8$

$$\int_1^2 \ln x\,dx \approx \frac{1/8}{3}\left[y_0 + y_8 + 4\sum_{i=1}^{4} y_{2i-1} + 2\sum_{i=1}^{3} y_{2i}\right]$$

$$= \frac{1}{24}\left[\ln 1 + \ln 2 + 4(y_1+y_3+y_5+y_7) + 2(y_2+y_4+y_6)\right]$$

$$= \frac{1}{24}\left[\ln 1 + \ln 2 + 4(\ln 9/8 + \ln 11/8 + \ln 13/8 + \ln 15/8) + 2(\ln 10/8 + \ln 12/8 + \ln 14/8)\right]$$

$$\approx 0.386292$$

This result agrees within five decimal places with the correct value.

The error at most is

$$\frac{(b-a)h^4}{180}|f^{(iv)}(c)| = \frac{(1/8)^4}{180}\frac{6}{c^4} \leq \frac{6}{180 \times 8^4} \approx 0.000008$$

Using the error estimation method, we can compute how many panel (m) we need for a given accuracy.

Example: Find the number of panels m for the composite Simpson's Rule to approximate

$$\int_0^{\pi} \sin^2 x\,dx$$

within six correct decimal places.

- Within six correct decimal places means that the error needs to be smaller than 0.5×10^{-6}

We need to solve

$$\frac{(\pi-0)h^4}{180}\left|f^{(iv)}(c)\right| < 0.5\times10^{-6}$$

$$f^{(iv)}(x) = (\sin^2 x)^{(iv)} = (2\sin x\cos x)''' = (2\cos^2 x - 2\sin^2 x)''$$

$$= (-8\sin x\cos x)' = 8(\sin^2 x - \cos^2 x) = -8\cos 2x$$

in the range of $[0, \pi]$, $\max\left|f^{(iv)}(c)\right| = 8$

solve $\frac{\pi h^4 \times 8}{180} < 0.5\times10^{-6} \Rightarrow h < 0.0435$

$$M = \frac{(b-a)}{2h} = \frac{\pi}{2\times0.0435} \approx 36.11$$

Finding the nearest integer bigger than 36.11, we get $\boxed{M=37}$

In class exercise: