ME 360 Lecture 14

2.2.2 Back substitution with the LU factorization

Now we have our matrices LU, how do they help us find the solution x?

Once L and U are known, the problem $Ax=b$ can be written as $LUx=b$. Define a new "auxiliary" vector $c=Ux$. Then back substitution is a two-step procedure:

(a) Solve $Lc=b$ for c

(b). Solve $Ux=c$ for x.

Both steps are straightforward since L and U are triangular matrices.

Example: $$LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = A$$

The right hand $b=[3,2]$.

Step (a): solve $Lc=b$

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 3 \\ 3c_1 + c_2 = 2 \end{matrix} \Rightarrow \begin{matrix} c_1 = 3 \\ c_2 = -7 \end{matrix}$$

Step (b): Solve $Ux=c$

$$\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_2 = 3 \\ -7x_2 = -7 \end{matrix} \Rightarrow \boxed{\begin{matrix} x_1 = 2 \\ x_2 = 1 \end{matrix}}$$

Example:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = A$$

$b=[3,3,-6]$

step (a): solve c for $Lc=b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7/3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 3 \\ 2c_1 + c_2 = 3 \\ -3c_1 - 7/3\, c_2 + c_3 = -6 \end{matrix} \Rightarrow \begin{matrix} c_1 = 3 \\ c_2 = -3 \\ c_3 = -4 \end{matrix}$$

Step (b): Solve x for $Ux = C$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ -3x_2 &= -3 \\ -2x_3 &= -4 \end{aligned} \Rightarrow \boxed{\begin{aligned} x_1 &= 3 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}}$$

In class exercise

2.2.3 Complexity of the LU factorization

It seems that the steps to do LU factorization is the same as naive Gaussian Elimination. Why do we still want to do it?

- Gaussian Elimination

  To solve $Ax=b$, not only we modify A, we also modify b at the same time, so if we want to solve $Ax=b_2$, we need to start fresh.

- LU Factorization

  We first solve $A=LU$. In this step, we don't need to know what b is. So if we want to solve Ax for different b's, we only need to do back substitution.

In reality, we often need to solve a series of equations.

$$Ax = b_1$$
$$Ax = b_2$$
$$\vdots$$
$$Ax = b_n$$

and these can be accelerated by using LU factorization.

Of course, no method works on all problems. Now let's see how LU factorization fail.

Example: show that $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ does not have an LU factorization.

LU factorization must have the form:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} b & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} b & c \\ ab & ac+d \end{bmatrix}$$

to make the equality true, we need $b=0$ & $ab=1$, which contradicts each other.

We will introduce new method later on that overcome the above problem.

2.3. Sources of Error

(a) ill conditioning

(2). swamping

2.3.1 Error magnification and condition number

Similar to root finding, there can be great differences between forward and backward error in systems of equations as well.

- **Infinity/maximum norm definition**

The infinity norm, or maximum norm, of the vector $x=(x_1,\dots,x_n)$ is $\|x\|_\infty = \max|x_i|$, $i=1,\dots,n$, that is, the maximum of the absolute values of the components of x.

- residual, forward / backward error definition

Let x_a be an approximate solution of the linear system $Ax=b$. The residual is the vector $r = b - Ax_a$. The backward error is the norm of the residual $\|b - Ax_a\|_\infty$, and the forward error is $\|x - x_a\|_\infty$

Example: Find the backward and forward errors for the approximate solution $x_a = [1, 1]$ of the system

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(a). backward error

$$\|b - Ax_a\|_\infty = \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\|_\infty = \boxed{3}$$

(b) forward error

* We need to know the correct solution first.

For these two equations, the correct solution $x = [2, 1]$

$$\|x - x_a\|_\infty = \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_\infty = \boxed{1}$$

In this example, forward and backward errors are on the same order of magnitude. In other cases, these two errors can be vastly different.

Example: find the forward and backward errors for the approximate solution $[-1, 3.0001]$ of the system

$$\begin{cases} x_1 + x_2 = 2 \\ 1.0001x_1 + x_2 = 2.0001 \end{cases}$$

Step 0: find the exact solution $[x_1, x_2]$. Using Gaussian elimination

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1.0001 & 1 & 2.0001 \end{array}\right] \Rightarrow \text{subtract } 1.0001 \times \text{row 1 from row 2} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -0.0001 & -0.0001 \end{array}\right]$$

Back substituting to get the solution

$$\begin{cases} x_1 + x_2 = 2 \\ -0.0001x_2 = -0.0001 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases}$$

Step 1: backward error

$$b - Ax_a = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3.0001 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2.0001 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.0001 \\ 0.0001 \end{bmatrix}$$

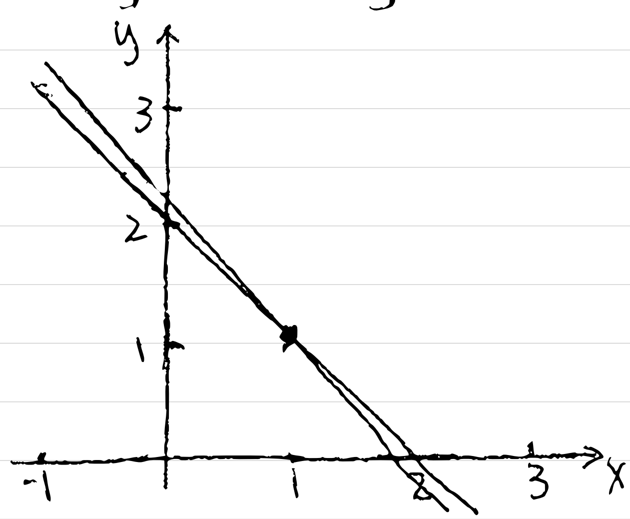
$$\|b - Ax_a\|_\infty = 0.0001$$

Step 2: forward error

$$x - x_a = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3.0001 \end{bmatrix} = \begin{bmatrix} 2 \\ -2.0001 \end{bmatrix}$$

$$\|x - x_a\|_\infty = 2.0001$$

Why such a big difference between forward & backward error?



the plot roughly represent the system of equations mentioned above. We can see that point $(-1, 3.0001)$ nearly misses lying on both lines.