ME 360 Lecture 13

2.2 The LU Factorization

Before we start, let's review some matrix math.

Recall an $m \times n$ matrix A can be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

- **Matrix Addition**

The sum $A+B$ of two $m \times n$ matrices A and B is calculated entrywise: $(A+B)_{ij} = A_{ij} + B_{ij}$ where $1 \le i \le m$ & $1 \le j \le n$

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$$

- **Scalar Multiplication**

The product cA of a number c and a matrix A is computed by multiplying every entry of A by c: $(cA)_{i,j} = c \cdot A_{ij}$.

$$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2\cdot 1 & 2\cdot 8 & 2\cdot(-3) \\ 2\cdot 4 & 2\cdot(-2) & 2\cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

- **Transposition**

The transpose of an $m \times n$ matrix is the $n \times m$ matrix A^T formed by turning rows into columns and vice versa.

$$(A^T)_{ij} = A_{ji}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}$$

- **Matrix Multiplication**

$$[AB]_{ij} = a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \cdots a_{i,n} b_{n,j} = \sum_{r=1}^{n} a_{i,r} b_{r,j}$$

where A: $m \times n$ matrix, B: $n \times m$ matrix.
$1 \le i \le m$, $1 \le j \le n$

For example:

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1000 \\ 1 & 100 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 2\times0+3\times1+4\times0 & 2\times1000+3\times100+4\times10 \\ 0\times1+1\times0+0\times0 & 1\times1000+0\times100+0\times10 \end{bmatrix} = \begin{bmatrix} 3 & 2340 \\ 0 & 1000 \end{bmatrix}$$

• Matrix Math Properties

1. $A+B = B+A$
2. $(cA)^T = c(A)^T$
3. $(A+B)^T = A^T + B^T$
4. $(A^T)^T = A$
5. $(AB)C = A(BC)$
6. $(A+B)C = AC + BC$
7. $C(A+B) = CA + CB$
8. $AB \neq BA$

Now let's go back to Ch. 2.

2.2.1 Matrix form of Gaussian Elimination

Recall the equations we had last time

$$\begin{cases} x + y = 3 \\ 3x - 4y = 2 \end{cases}$$

here, we will rename our unknowns to x_1, x_2.

We can rewrite the equations in matrix format $Ax = b$.

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Here $A = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}$ (coefficient matrix)

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

We want to solve for x such that $Ax = b$.

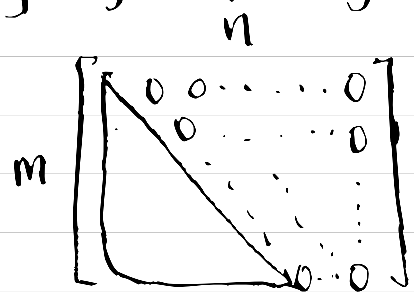
The advantage of writing systems of equations in matrix form is that we can use matrix operations. The LU factorization is a matrix representation of Gaussian Elimination.

L: lower triangular matrix

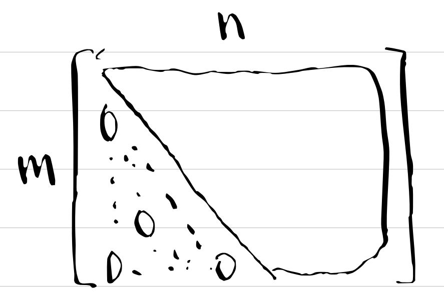
U: upper triangular matrix

Definition.

An $m \times n$ matrix L is **lower triangular** if its entries satisfy $l_{ij} = 0$ for $i < j$. An $m \times n$ matrix U is **upper triangular** if its entries satisfy $u_{ij} = 0$ for $i > j$.



Lower triangular



Upper triangular

Example: find the LU factorization for matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}$

We use the same elimination steps from before.

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \Rightarrow \begin{matrix} \text{subtract } 3 \times \text{row } 1 \\ \text{from row } 2 \end{matrix} = \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix} = U$$

We now need to store the multiplier 3 somewhere as well. Define a lower triangular matrix that has 1 on the diagonal, and multiplier 3 in the (2,1) position.

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1\times1 + 0\times0 & 1\times1 + 0\times(-7) \\ 3\times1 + 1\times0 & 1\times3 + 1\times(-7) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = A$$

Before we discuss why this works, let's look at a 3x3 example.

Example: find the LU factorization of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix}$$

Step 1: Use Gaussian Elimination to find U.

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \rightarrow \begin{matrix}\text{Subtract } 2 \times \text{row } 1 \\ \text{from row } 2\end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ -3 & 1 & 1 \end{bmatrix} \rightarrow \begin{matrix}\text{subtract } -3 \times \text{row } 1 \\ \text{from row } 3\end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 7 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 7 & -2 \end{bmatrix} \rightarrow \begin{matrix}\text{subtract } -7/3 \times \text{row } 2 \\ \text{from row } 3\end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U$$

Step 2: Fill in L

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7/3 & 1 \end{bmatrix}$$

(a). put ones on the diagonal
(b). fill lower left based on elimination in step 1.

How do we decide where to put in the multiplier?
i.e. $-7/3$ is used to eliminate element entry $(3,2)$ of A, so we put it in entry $(3,2)$ of L.

Step 3: Check your L & U.

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -7/3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = A$$

The reason that this procedure gives the LU factorization follows from three facts about lower triangular matrices.

Fact 1: Let $L_{ij}(-c)$ denote the lower triangular matrix whose only nonzero entries are 1's on the main diagonal and $-c$ in the (i,j) position. Then $A \to L_{ij}(-c)A$ represents the row operation "subtracting c times row j from row i".
For example, multiplication by $L_{21}(-c)$ yields

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} - ca_{11} & a_{22} - ca_{12} & a_{23} - ca_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Fact 2: $L_{ij}(-c)^{-1} = L_{ij}(c)$.
(Note. A^{-1} denotes the inverse of matrix A, where $AA^{-1} = I$, where I is the identity matrix)
For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using fact 1 & 2, we can understand the 2x2 matrix example.
The elimination step can be represented by

$$L_{21}(-3)A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 7 \end{bmatrix}.$$

we can multiply both sides on the left by $L_{21}(-3)^{-1}$ to get

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}$$

To handle $n \times n$ matrices for $n > 2$, we need one more fact.

Fact 3: The following matrix product equation holds.

$$\begin{bmatrix} 1 & & \\ c_1 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ c_2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & c_3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ c_1 & 1 & \\ c_2 & c_3 & 1 \end{bmatrix}$$

This fact allows us to collect the inverse L_{ij}'s into one matrix, which becomes the L of the LU factorization. For the 3×3 matrix example, we have

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & 7/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ 3 & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$$A = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ -3 & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & -7/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -3 & -7/3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = LU$$

In class exercise: