ME 360 Lecture 5

Ch.1 Solving Equations (Root Finding)

We will learn the following methods to find roots

1. The Bisection Method
2. Fixed Point Iteration
3. Newton-Raphson Method : uses derivative
4. Secant Method — variation
5. Brent's Method : combines the best part of the above methods.

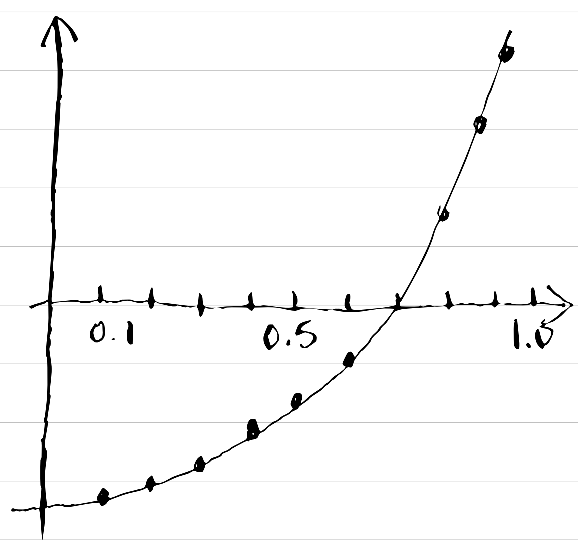
1. Bisection Method

- **Definition of root:**

  The function $f(x)$ has a root at $x=r$ if $f(r)=0$.

- **Theorem:**

  Let f be a continuous function on $[a,b]$, satisfying $f(a)f(b)<0$. Then f has a root between a and b, that is, there exists a number r satisfying $a<r<b$ and $f(r)=0$.



The bisection method is built on this theorem. We want to keep checking the sign of $f(a)f(b)$ and narrow down the interval of $[a,b]$.

to build an algorithm for Bisection Method, we need

- starting interval $[a,b]$ where $f(a)f(b)<0$
- a tolerance criteria for stopping the algorithm.

Pseudo code for Bisection Method

```
Given initial interval [a,b] such that f(a)f(b)<0.
while (b-a)/2 > tol          (tol: tolerance & stopping criteria)
    c = (a+b)/2
    if f(c)=0, stop, end     (note: in Python, we don't use word
    if f(a)f(c) < 0                 "end" to indicate ending of
        b = c                       a loop, we use indentation
    else                            instead).
        a = c
    end
end
root ≈ (a+b)/2
```

Example: let's find the root of $f(x) = x^2 - 2$ in $[0, 2]$ using bisection method

- We know that $x^2 - 2 = 0$ has two roots, $-\sqrt{2}$ & $\sqrt{2}$, let's try to go through the numerical method to estimate $\sqrt{2}$.
- for the interest of time, we will set $tol = 0.1$

a	b	(b-a)/2	c	f(a)	f(b)	f(c)
0	2	1	1	-2	2	-1
1	2	0.5	1.5	-1	2	0.25
1	1.5	0.25	1.25	-1	0.25	-0.4375
1.25	1.5	0.125	1.375	-0.4375	0.25	-0.109375
1.375	1.5	0.0625 < 0.1				

now $(b-a)/2 = 0.0625 < tol = 0.1$, so our estimated root is

$$\boxed{root \approx (a+b)/2 = 1.4375}$$

we know that $\sqrt{2} \approx 1.414$, so our result is within the 0.1 tolerance.

Exercise in class:

When we evaluate a numerical algorithm, we want to learn a few things:

1. accuracy
2. convergence speed (how fast to solution).

We will discuss these two factors when we compare bisection and fixed point method.

2. Fixed Point Iteration

- **Definition of fixed point**

  The real number r is a fixed point of the function g if $g(r) = r$.

Using Fixed Point Iteration to solve for $g(x) = x$ is the equivalent of finding root of $g(x) - x = 0$.

Fixed point iteration algorithm:

- pick an initial guess x_0.
- then $x_{i+1} = g(x_i)$ for $i = 0, 1, 2, 3, \ldots$

To write the algorithm step by step, we have

$$x_1 = g(x_0)$$
$$x_2 = g(x_1)$$
$$x_3 = g(x_2)$$
$$\vdots$$

Note: the sequence x_i may or may not converge as i (the number of steps) goes to infinity, but, if g is continuous and x_i converges to a number r, then r is a fixed point.

$$g(r) = g(\lim_{i \to \infty} x_i) = \lim_{i \to \infty} g(x_i) = \lim_{i \to \infty} x_{i+1} = r$$

(you can prove the above relationship using calculus).

How do we use fixed point iteration to do root finding?
Can we turn an equation of $f(x)=0$ to $g(x)=x$?
Let's use our bisection method example

$$f(x) = x^2 - 2$$

We can re-write the equation as

$$x^2 + x - 2 = x \Rightarrow g(x) = x^2 + x - 2$$

Of course, there is an easier way to solve it, we can directly have $x = \pm\sqrt{2}$. here, $g(x) = \pm\sqrt{2}$.

But, for the sake of practicing the method, let's use the scenario of $g(x) = x^2 + x - 2$. with $x_0 = 1.0$

i	x_i
0	1.0
1	0.0
2	-2
3	0.0
4	-2

now we are alternating between 0 & 2.0.
fixed point iteration failed.

There could be many different scenario happening with fixed point iteration. Let's look at the example from the text book (page 34).

We have a function: $f(x) = x^3 + x - 1 = 0$.

To convert it to the form of $g(x) = x$, we have a few options:

1. $x = 1 - x^3 \Rightarrow g(x) = 1 - x^3$
2. $x^3 = 1 - x$, $x = (1-x)^{1/3} \Rightarrow g(x) = (1 - x)$
3. (a) add $2x^3$ to both side of the equation

$$3x^3 + x - 1 = 2x^3$$
$$(3x^2 + 1)x = 2x^3 + 1$$
$$x = \frac{1+2x^3}{1+3x^2} \Rightarrow g(x) = \frac{1+2x^3}{1+3x^2}$$

For option 1: $g(x) = 1 - x^3$, $x_0 = 0.5$

i	x_i
0	0.5
1	0.875
2	0.3301
3.	0.9640
4.	0.1041
5	0.9989
6	0.0034
7	0.9999
8	0.0000
9	1.0
10	0.0
11	1.0
12	0.0

$\}$ alternating between 0 & 1
method failed.

For option 2: $g(x) = (1-x)^{1/3}$, $x_0 = 0.5$

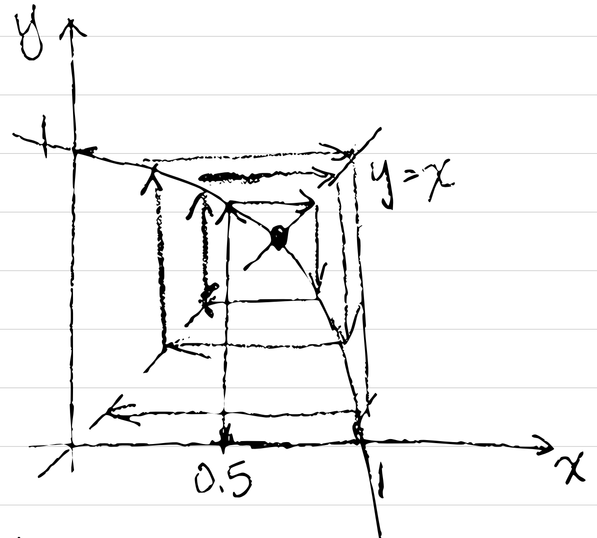
- have good convergence of $r = 0.6823$ after 25 iteration

For option 3: $g(x) = (1+2x^3)/(1+3x^2)$, $x_0 = 0.5$.

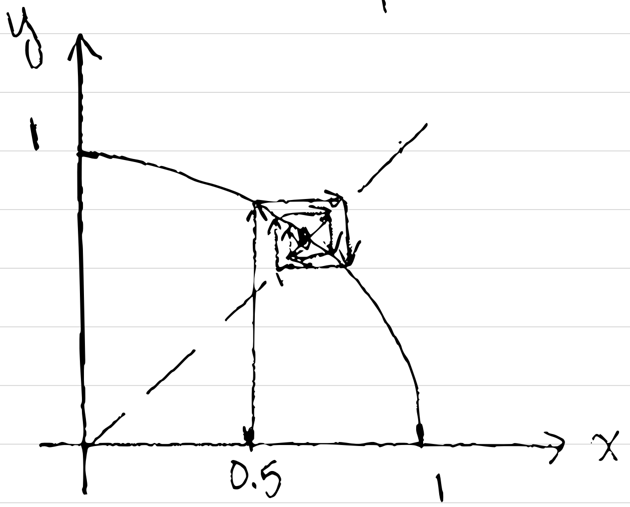
i	x_i
0	0.5
1	0.7143
2	0.6832
3	0.6823
4	0.6823
$\vdots$	

You get the first four digits correct after only 4 iterations!

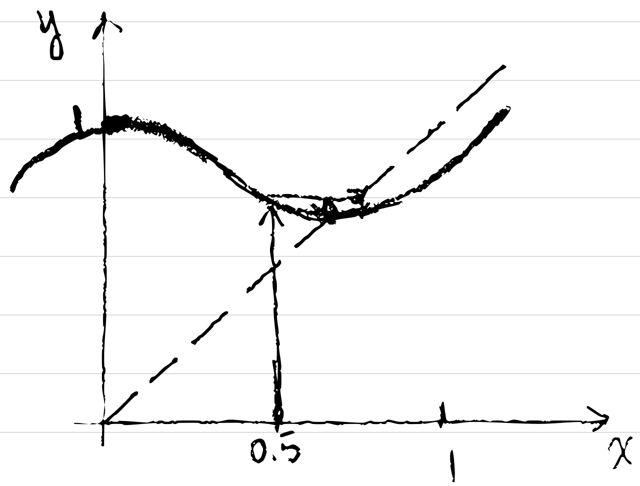
Why these three options behave so differently?
Let's look at the shape of these three different $g(x)$ functions.



$g(x) = 1 - x^3$



$g(x) = (1-x)^{1/3}$



$$g(x) = \frac{1+2x^3}{1+3x^2}$$