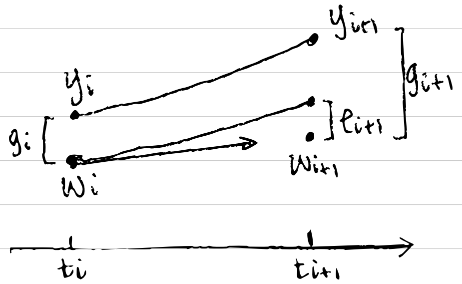
ME 360 Lecture 25

6.2 Analysis of IVP Solvers

6.2.1 Local and global truncation error



$$\begin{cases} y' = f(t,y) \\ y(a) = y_a \\ t \text{ in } [a,b] \end{cases}$$

Global truncation error: $g_i = |w_i - y_i|$

Local truncation error: $e_{i+1} = |w_{i+1} - z(t_{i+1})|$

What is $z(t_{i+1})$? $z(t_{i+1})$ is the correct solution of the "one-step initial value problem"

$$\begin{cases} y' = f(t,y) \\ y(t_i) = w_i \\ t \text{ in } [t_i, t_{i+1}] \end{cases}$$

$$y' = t^3/y^2 + ty^2$$

Euler's Method.

$$w_0 = y_0$$
$$w_{i+1} = w_i + hf(t_i, w_i)$$

Example: Find the local truncation error for Euler's Method.

Assuming y'' is continuous, the exact solution at $t_{i+1} = t_i + h$ is

$$y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(c)$$

based on Taylor's theorem, where $t_i < c < t_{i+1}$.

Since $y(t_i) = w_i$, $y'(t_i) = f(t_i, w_i)$, this can be written as

$$y(t_{i+1}) = w_i + hf(t_i, w_i) + \frac{h^2}{2} y''(c)$$

Meanwhile, we know from Euler's Method

$$w_{i+1} = w_i + hf(t_i, w_i)$$

Subtracting the two equations yields the local truncation error

$$e_{i+1} = |w_{i+1} - y(t_{i+1})| = \frac{h^2}{2} |y''(c)|$$

for some c in the interval.

If M is an upper bound for y'' on $[a, b]$, then the local truncation error satisfies $e_i \leq Mh^2/2$.

Example: Find the global truncation error for Euler's Method.

At the initial condition, $y(a) = y_a$, the global error is

$$g_0 = |w_0 - y_0| = |y_a - y_a| = 0$$

After one step, the global error is the local error,

$$g_1 = e_1 = |w_1 - y_1|$$

After two steps, we can break down g_2 into local truncation error plus the accumulated error.

Define $z(t)$ to be the exact solution of the initial value problem

$$\begin{cases} y' = f(t, y) \\ y(t_1) = w_1 \\ t \text{ in } [t_1, t_2] \end{cases}$$

$z(t_2)$: exact value of the solution starting at initial condition (t_1, w_1)

At step 2: local truncation error $e_2 = |w_2 - z(t_2)|$.

accumulated error is $|z(t_2) - y_2| \leq e^{Lh} g_1$

(note, covered by content in Ch. 6.1.2).

Combining everything together

$$g_2 = |w_2 - y_2| = |w_2 - z(t_2) + z(t_2) - y_2|$$
$$\leq |w_2 - z(t_2)| + |z(t_2) - y_2|$$
$$\leq e_2 + e^{Lh} g_1$$
$$= e_2 + e^{Lh} e_1$$

(L: Lipschitz constant).

For step 3: we can use the same argument

$$g_3 = |w_3 - y_3| \leq e_3 + e^{Lh} g_2 \leq e_3 + e^{Lh} e_2 + e^{2Lh} e_1$$

For step i; we have

$$g_i = |w_i - y_i| \leq e_i + e^{Lh} e_{i-1} + e^{2Lh} e_{i-2} + \cdots + e^{(i-1)Lh} e_1$$

From the previous example, we know that the local truncation error is proportional to h^2, more generally, assume the local truncation error satisfies:

$$e_i \leq C h^{k+1}$$

for some integer k and a constant $C > 0$.

Plug this in. we have

$$g_i \leq C h^{k+1} (1 + e^{Lh} + \cdots + e^{(i-1)Lh})$$
$$= C h^{k+1} \frac{e^{iLh} - 1}{e^{Lh} - 1} \leq C h^{k+1} \frac{e^{L(t_i - a)} - 1}{Lh}$$
$$= \frac{C h^k}{L} (e^{L(t_i - a)} - 1)$$

Global truncation error:

$$g_i = |w_i - y_i| \leq \frac{C h^k}{L} (e^{L(t_i - a)} - 1)$$

If an ODE solver satisfies the above equation as $h \to 0$, we say that the solver has order k.

Since the local truncation error of Euler's Method is of size bounded by $Mh^2/2$, so the order of Euler's Method is 1.

Example: Find an error bound for Euler's method applied to

$$\begin{cases} y' = ty + t^3 \\ y(0) = y_0 \\ t \text{ in } [0,1] \end{cases}$$

The Lipschitz constant on $[0,1]$ is $L=1$, and the analytical solution for the ODE is $y(t) = 3e^{t^2/2} - t^2 - 2$

Recall the local truncation error expression

$$e_i \le \frac{Mh^2}{2}, \quad \text{where } M = \max |y''(c)| \quad a<c<b$$

$$y''(t) = (t^2+2)e^{t^2/2} - 2 \Rightarrow M = \max|y''(c)| = |y''(1)| = 3\sqrt{e} - 2$$

Plug in everything, we have

$$g \le \frac{Mh}{2L}\left(e^{L(t-a)} - 1\right) = \frac{(3\sqrt{e}-2)h}{2}(e-1) \approx 2.53h$$

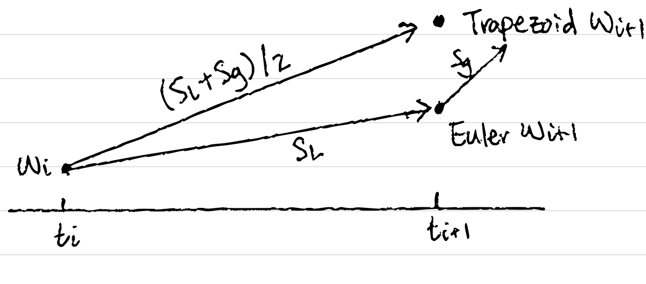
(Note, the book has a different answer, which has a bigger bound).

Many times, Euler's Method converges very slowly and we will need more sophisticated method to reduce compute time for the same accuracy.

Explicit Trapezoid Method:

$$W_0 = y_0$$

$$W_{i+1} = W_i + \frac{h}{2}\left(f(t_i, W_i) + f(t_i + h, W_i + hf(t_i, W_i))\right)$$



Where does the name come from?

If $f(t, y)$ is independent of y, we have

$$W_{i+1} = W_i + \frac{h}{2}\left(f(t_i) + f(t_i + h)\right)$$

It is similar to solve the integral $\int_{t_i}^{t_i+h} f(t)\,dt$ using Trapezoid Rule.

Example: Apply the Explicit Trapezoid Method to the initial value problem

$$\begin{cases} y' = ty + t^3 \\ y(0) = 1 \\ t \text{ in } [0, 1] \end{cases}$$

$$W_0 = y_0 = 1$$

$$W_{i+1} = W_i + \frac{h}{2}\left(f(t_i, W_i) + f(t_i + h), W_i + hf(t_i, W_i))\right)$$

$$= W_i + \frac{h}{2}\left(t_i W_i + t_i^3 + (t_i + h)\left(W_i + hf(t_i, W_i)\right) + (t_i + h)^3\right)$$

$$= W_i + \frac{h}{2}\left(t_i W_i + t_i^3 + (t_i + h)(W_i + ht_i W_i + ht_i^3) + (t_i + h)^3\right)$$

(Note: typo in text book)

We can use the same Taylor expansion method to compute the local and global truncation error.

At the end of the day, the local truncation error is $\sim O(h^3)$, the global truncation error is $\sim O(h^2)$, thus the Explicit Trapezoid method is of order two.