ME 360 Lecture 7

Recap of last lecture

- For Fixed Point Iteration (FPI) method, the convergence rate is $S = |g'(r)|$. If $S < 1$, then $g(x)$ locally converges.

Some Exercises for FPI

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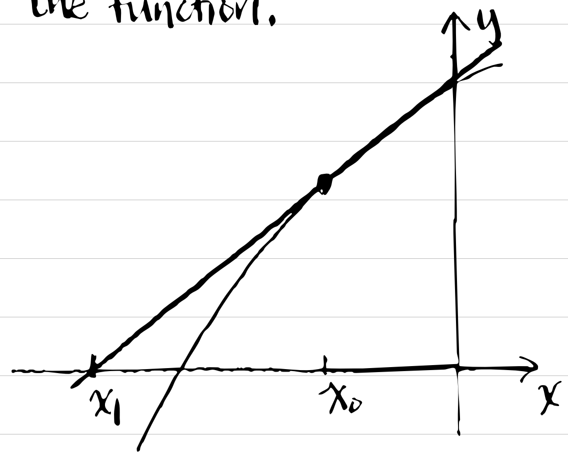
Newton - Raphson Method

Usually converges much faster than methods like Bisection or FPI. We will be using information from the derivative of the function.



One step of Newton's method.

1. start with x_0
2. find the tangent line @ $f(x_0)$
3. find the intersection between the tangent line & x-axis, which is x_1

- Algebraic Representation of Newton's Method.

We are trying to solve the value of x_1.

We know that $(x_1, 0)$ and $(x_0, f(x_0))$ are both points on the tangent line, and the tangent line's slope is $f'(x_0)$.

Put everything together, we have

$$\frac{f(x_0) - 0}{x_0 - x_1} = f'(x_0) \Rightarrow (x_0 - x_1) f'(x_0) = f(x_0)$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Iterative formula for Newton's Method

$$x_0 = \text{initial guess}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{for } i = 0, 1, 2, \ldots$$

Example: Find the Newton's Method formula for $x^3 + x - 1 = 0$.

Here: $f(x) = x^3 + x - 1$, $f'(x) = 3x^2 + 1$

$$x_{i+1} = x_i - \frac{x_i^3 + x_i - 1}{3x_i^2 + 1}$$

$$= \frac{3x_i^3 + x_i - x_i^3 - x_i + 1}{3x_i^2 + 1} = \boxed{\frac{2x_i^3 + 1}{3x_i^2 + 1}}$$

Exercise in class:

Quadratic Convergence of Newton's Method

Newton's Method converges faster than Bisection and FPI.

- **Definition of Quadratically convergent**

  Let e_i denote the error after step i of an iterative method. The iteration is quadratically convergent if $M = \lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} < \infty$

- **Theorem on Quadratic Convergence of Newton's Method**

  Let f be twice continuously differentiable and $f(r)=0$. If $f'(r)\neq 0$, then Newton's method is locally and quadratically convergent to r. The error e_i at step i satisfies

$$\lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} = M, \quad \text{where} \quad M = \frac{f''(r)}{2f'(r)}.$$

(Note: if you are curious of the proof, it's on page 56 of Sauer)

Value M is less critical regarding the convergence speed of Newton's Method.

Linear Convergence of Newton's Method

Newton's method does not always converge quadratically. Some time, we only get linear convergence.

For example, $f(x) = x^2$.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2}{2x_i} = \frac{x_i}{2}$$

Here, convergence rate of Newton's Method is $S = \frac{1}{2}$

The more general case. $f(x) = x^m$

$$x_{i+1} = x_i - \frac{x_i^m}{m x_i^{m-1}} = \frac{m-1}{m} x_i$$

We know that root is at $r=0$, so $e_i = |x_i - r| = x_i$

$$e_{i+1} = Se_i \Rightarrow S = \frac{e_{i+1}}{e_i} = \frac{m-1}{m}$$

Theorem:

Assume that the $(m+1)$-times continuously differentiable function f on $[a,b]$ has a multiplicity m root at r. Then Newton's Method is locally convergent to r, and the error e_i at step i satisfies $\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = S$, where $S = \frac{m-1}{m}$.

Let's apply this theorem to an example:

$$f(x) = \sin x + x^2 \cos x - x^2 - x$$

Find the multiplicity (m) of root $r=0$, and then determine the convergence rate S.

1. Finding m.

$$f(0) = 0 + 0 - 0 - 0 = 0$$

$$f'(x) = \cos x + 2x\cos x - x^2 \sin x - 2x - 1$$

$$f'(0) = 1 + 0 - 0 - 0 - 1 = 0$$

$f''(x) = -\sin x + 2\cos x - 2x\sin x - 2x\sin x - x^2\cos x - 2$

$f''(0) = 0 + 2 - 0 - 0 - 0 - 2 = 0$

$f'''(x) = -\cos x - 2\sin x - 4\sin x - 4x\cos x - 2x\cos x + x^2\sin x$

$f'''(0) = -1 - 0 - 0 - 0 - 0 + 0 = -1$

Since the third derivative of $f(x)$ is not zero, $r=0$ is a triple root.

$\boxed{m=3}$

2. Convergence rate S.

By theorem, $S = \frac{m-1}{m} = \boxed{\frac{2}{3}}$