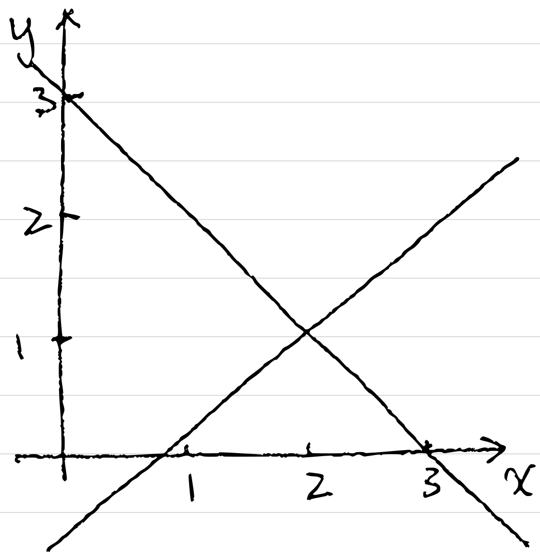
# ME 360 Lecture 12

## Ch. 2. Systems of Equations

In the last chapter, we looked at single equation solutions. In this chapter, we will look at multiple equations. We will start with looking at multiple linear equations.

### 2.1 Gaussian Elimination




Consider the system of equations

$$\begin{cases} x + y = 3 \\ 3x - 4y = 2 \end{cases}$$

For this equation, we can solve it by hand

$$y = 3 - x$$
$$3x - 4(3 - x) = 2$$
$$3x - 12 + 4x = 2$$
$$7x = 14 \Rightarrow \boxed{x = 2,\ y = 3 - 2 = 1}$$

What happens when you have N equations and N unknowns?

#### 2.1.1. Naive Gaussian Elimination

Why Naive?

Because it has the simplest form of Gaussian Elimination, and it does not guarantee completion or accurate solution.

Three useful operations for equation solving

(1). Swap one equation for another

(2). Add or subtract a multiple of one equation from another

(3). Multiply an equation by a nonzero constant.

Our solution from above used operation (1). let's try to use operation (2).

Subtracting $3\cdot[x+y=3]$ from second equation $[3x-4y=2]$, we get

$$(3x-4y)-3\cdot(x+y)=2-3\cdot 3$$
$$-7y=-7 \Rightarrow y=1$$
$$x+y=3 \Rightarrow x+(1)=3 \Rightarrow x=2.$$
$$\therefore (x,y)=(2,1).$$

We can write the same elimination is tableau form

$$\begin{array}{cc} x & y \end{array} \qquad\qquad\qquad\qquad\qquad \begin{array}{cc} x & y \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -4 & 2 \end{array}\right] \Rightarrow \text{subtract } 3\times \text{row } 1 \text{ from row } 2 \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -7 & -7 \end{array}\right]$$

Advantage of tableau form: variables are hidden during elimination

When the square array on the left of the tableau is "triangular", we can backsolve for the solution, starting at the bottom.

**Example:** apply Gaussian elimination in tableau form for the system of three equations with three unknowns.

$$\begin{cases} x+2y-z=3 \\ 2x+y-2z=3 \\ -3x+y+z=-6 \end{cases}$$

Step 1. write in tableau form

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right]$$

our goal is to have something like the following

$$\left[\begin{array}{ccc|c} & & & \\ 0 & & & \\ 0 & 0 & & \end{array}\right]$$

Step 2: Eliminate column 1

(a) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right] \rightarrow$ Subtract $2 \times$ row 1 from row 2 $\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ -3 & 1 & 1 & -6 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ -3 & 1 & 1 & -6 \end{array}\right] \rightarrow$ Subtract $-3 \times$ row 1 from row 3 $\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right]$

Note: operation (a) & (b) are independent, and can be done at the same time

Step 3: Eliminate column 2

$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right] \rightarrow$ subtract $-7/3 \times$ row 2 from row 3 $\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -2 & -4 \end{array}\right]$

Step 4: return to equation form

$$\begin{cases} x + 2y - z = 3 \\ \quad -3y = -3 \\ \quad -2z = -4 \end{cases}$$

Step 5: Solve variables in the order $z, y, x$.

(This step is also called back substitution, or back-solving)

$-2z = -4 \Rightarrow z = 2$

$-3y = -3 \Rightarrow y = 1$

$x + 2y - z = 3 \Rightarrow x + 2 \times (1) - (2) = 3 \Rightarrow x = 3$.

$\therefore (x, y, z) = (3, 1, 2)$

In class exercise

Solution:

## General form of Gaussian Elimination

Imagine we have n equations with n unknowns.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \cdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

We can write it in tableau format

$$\left[\begin{array}{cccc|c} a_{11}, & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21}, & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1}, & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

To solve for $x_1, x_2, \ldots x_n$, we need to convert the tableau into the triangular format. and to do that, we need to eliminate columns.

A rough sketch of pseudo code

```
for j = 1:n-1
    eliminate column j
end
```

Now let's write out what do we mean by "eliminate column $j$".

- We are trying to use row operation to put a zero in each location below the main diagonal, which has locations $a_{j+1,j}, a_{j+2,j}, \dots, a_{nj}$. As an example, to carry out elimination on column 1, we need to put zeros in $a_{21}, a_{31}, \dots, a_{n1}$

A slightly more detailed pseudo code.

```
for j = 1 : n-1
    for i = j : n
        eliminate entry a(i,j).
    end
end.
```

How do we eliminate entry $a(i,j)$?

Let's look at how to eliminate $a_{21}$ entry.

To make $a_{21}$ zero, we need to subtract $a_{21}/a_{11}$ times row 1 from row 2, assuming $a_{11} \neq 0$, we will have

$$\left[\begin{array}{cccc|c} a_{11}, & a_{12}, & \dots & a_{1n} & b_1 \\ a_{21}, & a_{22} & \dots & a_{2n} & b_2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} a_{11}, & a_{12}, & \dots & a_{1n} & b_1 \\ 0, & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & \dots & a_{2n} - \frac{a_{21}}{a_{11}} a_{1n} & b_2 - \frac{a_{21}}{a_{11}} b_1 \end{array}\right]$$

To make it more general, the row operation used to eliminate entry $a_{i1}$ of the first column is

$$\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & a_{i2} - \frac{a_{i1}}{a_{11}} a_{12} & \cdots & a_{in} - \frac{a_{i1}}{a_{11}} a_{1n} & b_i - \frac{a_{i1}}{a_{11}} b_1 \end{array}$$

**Pivot**: The numbers that are eventually divisors in Gaussian elimination are called **pivots**.

Now, let's find out the general form of eliminate entry $a_{i,j}$.

$$\begin{array}{cccccc|c} 0 & 0 & a_{jj} & a_{j,j+1} & \cdots & a_{jn} & b_j \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & a_{i,j+1} - \frac{a_{ij}}{a_{jj}} a_{j,j+1} & \cdots & a_{in} - \frac{a_{ij}}{a_{jj}} a_{jn} & b_i - \frac{a_{ij}}{a_{jj}} b_j \end{array}$$

Put everything together in pseudo code

```
for j = 1:n-1
  if abs(a(j,j)) < eps
    error('zero pivot encountered')
  end
  for i = j+1:n
    mult = a(i,j)/a(j,j)
    for k = j+1:n
      a(i,k) = a(i,k) - mult * a(j,k)
    end
    b(i) = b(i) - mult * b(j)
  end
end
```

Note: Ch. 2.1.2 (P. 77-80) talks in depth about how many steps of Gaussian elimination one needs for $n$ equations with $n$ unknowns. We will not go through those motions in class, but if you are interested, read the relevant pages in the textbook.

**Naive Gaussian Elimination Complexity.**

For $n$ equations with $n$ unknowns, Naive Gaussian Elimination is an $O(n^3)$ process. This means that to solve $n$ equations with $n$ unknowns, Naive Gaussian Elimination will take $n^3$ operations. in the leading order.

**Example**: if Gaussian elimination took 3s for a system of 500 equations with 500 unknowns on a computer, how long would Gaussian elimination take for a system of 1000 equations with 1000 unknowns on the same computer?

**Solution**: Since Gaussian elimination scales as $n^3$, we have

$$\frac{3s}{x} = \left(\frac{500}{1000}\right)^3 \Rightarrow x = 3s \times 8 = \boxed{24s}$$