ME 491 Lecture 7.

1.5. Principal Component Analysis (PCA)

PCA is a method to represent a series of correlated data using uncorrelated (orthogonal) coordinates (principal components).

- A geometric representation can be found in Ch. 1.3.

Similar to SVD, we are interested in ordering PCs from the most important to least important.

* Derivation

For a matrix of data $X \in \mathbb{C}^{n \times m}$, we compute the average row $\bar{x}$

$$\bar{x}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}$$

mean matrix is:

$$\bar{X} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \bar{x}$$

Let

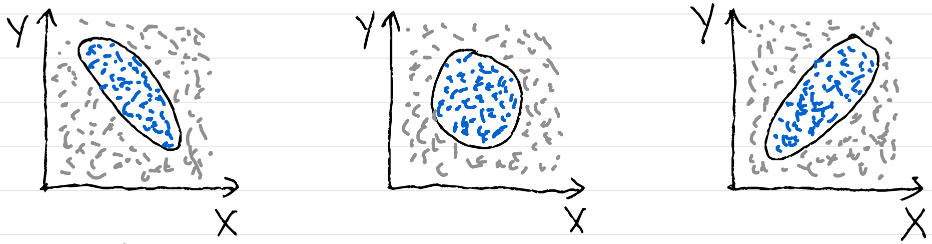
$$B = X - \bar{X}$$

we can define the covariance matrix of B.

$$C = \frac{1}{n-1} B^* B$$

(• covariance matrix : a square matrix giving the covariance between each pair of elements of a given random vector.

• covariance : measure of the joint variability of two random variables)



$\mathrm{cov}(X,Y)<0$ $\quad$ $\mathrm{cov}(X,Y)\approx 0$ $\quad$ $\mathrm{cov}(X,Y)>0$

Note that it's $(n-1)$ instead of n, which is called the Bessel correction.

C: symmetric, positive semi-definite.

$$b^* C b \geq 0 \text{ for all } b \in \mathbb{C}^n \neq 0.$$

We can then define the following change of coordinates.

$$CV = VD \Rightarrow C = VDV^* \Rightarrow D = V^* C V$$

V: eigenvectors of C, also are the principal components

D: diagonal matrix, variance of data along each column of V.

V: also the right singular vectors of B.

Let $B = U\Sigma V^*$, we have

$$C = \frac{1}{n-1} B^* B = \frac{1}{n-1} V\Sigma \overset{I}{\cancel{U^* U}} \Sigma V^* = \frac{1}{n-1} V \Sigma^2 V^*$$

$$D = V^* C V = \frac{1}{n-1} \overset{I}{\cancel{V^* V}} \Sigma^2 \overset{I}{\cancel{V^* V}} = \frac{1}{n-1} \Sigma^2$$

The diagonal values λ_k of D is related to the singular values.

$$\lambda_k = \frac{\sigma_k^2}{n-1}$$