Kan Seminar Talk1: Brown Representability Theorem about Cohomology Theories

Xiyu Zhai

Contents

1 Introduction **1**
1.1 Notions . . . 1
1.2 Motivations . . . 1
1.3 Axioms . . . 2
1.4 Main Results . . . 2

2 Proof **3**
2.1 Lemmas . . . 3
2.2 Prove that Lemma 1 $\wedge$ Lemma 2 $\Rightarrow$ Theorem 2 . . . 3
2.3 Prove that Theorem 2 $\Rightarrow$ Theorem 1 . . . 4
2.4 Prove Lemma 1 $\Rightarrow$ Lemma 2 . . . 5
2.5 Prove Lemma 1 . . . 5

1 Introduction

1.1 Notions

Definition (Reduced Suspension)**.** $\Sigma X := SX/\sim, (x_0, t) \sim (x_0, 1)$.

Definition (Mapping Cylinder)**.** $M_f := [0,1] \times X \sqcup Y/\sim, (x,1) \sim f(x), (x_0, t) \sim y_0$.

Remark. *M_f (strongly) deformation retracts to Y and X is a subspace of M_f.*

Definition (Reduced Mapping Cylinder)**.** You can guess.

1.2 Motivations

First motivation is the following fundamental relationship between singular cohomology and Eilenberg-MacLane spaces:

Theorem (Theorem 4.57 the Homotopy Construction of Cohomology)**.** There are natural bijections $T : \langle X, K(G,n)\rangle \to H^n(X;G)$ for all CW complexes X and all $n > 0$, with G any abelian group. Such a T has the form $T([f]) = f^*(\alpha)$ for a certain distinguished class $\alpha \in H^n(K(G,n);G)$.

Sketch of Proof $K(G,n)$ is very concrete, use them to argue that $T\langle -, K(G,n)\rangle$ must agree with $H^n(K;G)$.

Question: in general, what condition we can put on a sequence of CW complexes K_n such that $\langle X, K_n\rangle$ behaves like cohomology?

The natural isomorphism $h^n(X) \approx h^{n+1}(\Sigma X)$ leads to

$$\langle X, K_n\rangle \approx \langle \Sigma X, K_{n+1}\rangle \approx \langle X, \Omega K_{n+1}\rangle \tag{1}$$

Definition (Ω-spectrum)**.** A sequence of CW complexes K_n such that K_n is weak homotopic to ΩK_{n+1}.

Theorem (Theorem 4.58 in Allen Hatcher)**.** If $\{K_n\}$ is an Ω-spectrum, then the functors $X \mapsto h^n(X) = \langle X, K_n\rangle$, $n \in \mathbb{Z}$, define a reduced cohomology theory on the category of basepointed CW complexes and basepoint-preserving maps.

Example (Bott periodicty-topological version)**.**

$$BU \times \mathbb{Z} \sim \Omega^2 BU. \tag{2}$$

1.3 Axioms

For a contravariant functor h from C (category of CW complexes with basepoint) to abelian groups, we define the following axioms

Axiom 1(homotopy invariance)

Axiom 2 (exact sequence) for $A \hookrightarrow X$, exact sequence

$$h(X/A) \to h(X) \to h(A) \text{ is exact} \tag{3}$$

Axiom 3 (wedge) $h(\vee_\alpha X_\alpha) \simeq \prod_\alpha h(X_\alpha)$ induced by i_α

Remark. *These axioms imply Mayer-Vietoris, just like page 162 in Allen Hatcher's book.*

Suppose $X = A \cup B$, $a \in h(A), b \in h(B)$ *and* $a|_{A\cap B} = b|_{A\cap B}$ *then there exists* $x \in h(X)$ *such that* $a = x|_A, b = x|_B$.

1.4 Main Results

Definition (A Reduced Cohomology Theory on the category of CW complexes with basepoint)**.** A sequence of functors $h^n, n \in \mathbb{Z}$ from $\mathcal{C}$ to abelian groups, together with natural isomorphisms $h^n(X) \approx h^{n+1}(\Sigma X)$ for all X in $\mathcal{C}$ such that axiom (i) (ii) (iii) all hold for each h^n.

Theorem (Theorem 1)**.** Every reduced cohomology theory on the category of basepointed CW complexes and basepoint-preserving maps has the form $h^n(X) = \langle X, K_n \rangle$ for some Ω-spectrum $\{K_n\}$.

Theorem (Theorem 2)**.** If h is a contravariant functor from the category of connected base-pointed CW complexes to the category of pointed sets, satisfying the homotopy axiom (i), the Mayer-Vietoris axiom, and the wedge axiom (iii), then there exists a connected CW complex K and an element $u \in h(K)$ such that the transformation $T_u : \langle X, K \rangle \to h(X), T_u(f) = f^*(u)$, is a bijection for all X.

2 Proof

2.1 Lemmas

Definition (π_*-universal)**.** For a pair (K, u) with K a basepointed connected CW complex and $u \in h(K)$ where h satisfies the axioms of theorem 2, we call it π_*-universal if for any i, $T_u : \pi_i(K) \to h(S^i)$ is an isomorphism.

Lemma 1. Given any pair (Z, z) with Z a connected CW complex and $z \in h(Z)$ there exists a π_*-universal pair (K, u) with Z a subcomplex of K and $u|_Z = z$.

Sketch of Proof Make K by attaching cells to create and to kill.

Lemma 2. Let (K, u) be a π_*-universal pair and let (X, A) be a basepointed CW pair. Then for each $x \in h(X)$ and each map $f : A \to K$ with $f^*(u) = x|_A$ there exists a map $g : X \to K$ extending f with $g^*(u) = x$.

We shall proceed in four steps:

(1) prove that Lemma 1 $\wedge$ Lemma 2 $\Rightarrow$ Theorem 2

(2) prove that Theorem 2 $\Rightarrow$ Theorem 1

(3) prove Lemma 2 $\Rightarrow$ Lemma 1

(4) prove Lemma 1

2.2 Prove that Lemma 1 $\wedge$ Lemma 2 $\Rightarrow$ Theorem 2

There are two things to prove, surjectiveness and injectiveness.

Use lemma 1 to get a π_*-universal pair (K, u).

Apply lemma 2 to (X, pt) to get surjectiveness.

Apply lemma 2 to $(X \times I, X \times \partial I)$ to get injectiveness. Suppose that $T_u(f_0) = T_u(f_1)$, that is $f_0^*(u) = f_1^*(u)$. Combine f_0 and f_1 to form a map from $X \times \partial I \to K$ and taking x to be $p^* f_0^*(u) = p^* f_1^*(u)$ where p is the projection $X \times I \to X$. Here $X \times I$ should be the reduced product, with basepoint $\times I$ collapsed to a point. Then the lemma gives a homotopy from f_0 to f_1.

2.3 Prove that Theorem 2 $\Rightarrow$ Theorem 1

Restrict to connected CW complexes. This is okay because suspension is an isomorphism in any reduced cohomology theory, and the suspension of any CW complex is connected.

Obtain weak homotopy equivalence. Since h is a cohomology theory, we have naturally

$$h^n(X) \approx h^{n+1}(\Sigma X), \tag{4}$$

then naturally

$$\langle X, K_n \rangle \approx \langle \Sigma X, K_{n+1} \rangle \approx \langle X, \Omega K_{n+1} \rangle \tag{5}$$

Apply Yoneda's lemma to the natural bijection $\Phi : \langle X, K_n \rangle \approx \langle X, \Omega K_{n+1} \rangle$ we obtain $\epsilon_n : K_n \to \Omega K_{n+1}$ such that ϵ_n induces Φ. Then it follows easily that ϵ_n is a weak homotopy equivalence (by taking X to be S_n).

(Note: Allen Hatcher doesn't use Yoneda here. He basically reproves Yoneda. Maybe he wants to be clearer.)

Now we get an Ω-spectrum.

Verify $h^n(X) \approx \langle X, K_n \rangle$ is a group isomorphism. Here $\langle X, K_n \rangle$ has the group structure that comes from identifying it with $\langle X, \Omega K_{n+1} \rangle \approx \langle \Sigma X, \Omega K_n \rangle$.

We have

$$\psi : \Sigma X \to \Sigma X \vee \Sigma X \tag{6}$$

by collapsing the middle.

This map induces the group structure on the right obviously.

But this also induces the group structure for h.

Consider p_1, p_2 that collapse one of ΣX in $\Sigma X \vee \Sigma X$ resp.

The key trick is: $p_1 \circ \psi \sim p_2 \circ \psi \sim \mathbb{1}$.

Then just apply Mayer-Vietoris to see that

(here all + means the original group structure)

$\psi^*((a, b)) = \psi^*((a, 0)) + \psi^*((0, b)) = \psi^* p_1^* a + \psi^* p_2^* b = a + b$.

In Allen Hatcher's book, this was already proved in some form in a previous section.

2.4 Prove Lemma 1 $\Rightarrow$ Lemma 2

Let (K, u) be a π_*-universal pair and let (X, A) be a basepointed CW pair. We want to prove for each $x \in h(X)$ and each map $f : A \to K$ with $f^*(u) = x|_A$ there exists a map $g : X \to K$ extending f with $g^*(u) = x$.

Reduce to the case f is the inclusion of a subcomplex. This is okay because we can replace K by the reduced mapping cylinder of f.

Sketch of Proof Put K and X together to a common larger space K' using Lemma 1 with some $u' \in h(K')$ such that $u'|_K = u$ and $u'|_X = x$. Then prove that $X \hookrightarrow K'$ deformation retracts to $X \to K \subset K'$.

Apply Lemma 1 to get (K', u') from $Z = X \cup K$ with the two copies of A identified. By Mayer-Vietoris axiom, there exists $z \in h(Z)$ with $z|_X = x$ and $z|_K = u$. Introduce a π_*-universal pair (K', u') extending from (Z, z) by applying Lemma 1.

Show $X \hookrightarrow K'$ is homotopic to $g : X \to K$ rel A. The inclusion $(K, u) \hookrightarrow (K', u')$ induces an isomorphism on homotopy groups since both u and u' are π_*-universal, so K' deformation retracts onto K by whitehead. Examining the long exact sequence for the homotopy of the triple (A, K, K') it is not hard to see that the conditions for a relative version of whitehead theorem is satisfied, so K' deformation retracts onto K rel A (this part is not mentioned in Hatchers' book, he just directly reaches this), then $X \hookrightarrow K'$ is homotopic to $g : X \to K \subset K'$ rel A.

Then $g^*(u) = x$ holds because

$$g^*(u) = g^*(u'|_K) = u'|_X = (u'|_Z)|_K = z|_K = u. \tag{7}$$

2.5 Prove Lemma 1

We first propose a weaker notion than π_*-universal.

Definition (n-universal)**.** For a pair (K, u) with K a basepointed connected CW complex and $u \in h(K)$ where h satisfies the axioms of theorem 2, we call it n-universal if for any $i < n$, $T_u : \pi_i(K) \to h(S^i)$ is an isomorphism and for $i = n$ is surjective.

Remark. *π_*-universal is n-universal for all n.*

We construct K from Z by an inductive process of attaching cells.

Let $K_1 = Z \vee \bigvee_\alpha S^1_\alpha$ where α ranges over the elements of $h(S^1)$.

Prove (K_1, u_1) is 1-universal. Just by wedge axiom.

For the inductive step, suppose we already have (K_n, u_n) with u_n n-universal, $Z \subset K_n$ and $u_n|Z = z$.

Represent each element α in the kernel of $T_{u_n} : \pi_n(K_n) \to h(S^n)$ by a map $f_\alpha : S^n \to K_n$. Let $f = \vee_\alpha f_\alpha : \vee_\alpha S^n_\alpha \to K_n$. Consider $C_f = M_f / \vee_\alpha S^n_\alpha$. Then set $K_{n+1} = C_f \vee_\beta S^{n+1}_\beta$ where β ranges over $h(S^n + 1)$.

We now choose $u_{n+1} \in h(K_{n+1})$ such that $u_{n+1}|_{K_n} = u_n$. This is possible by

(i) first extend u_n to C_f using Mayer-Vietoris. C_f can be cut in half, with one deforms to K_n and another to a point and the intersection is $\vee_\alpha S^n_\alpha$.

(ii) then extend u_n over K_{n+1} by wedge axiom.

Prove (K_{n+1}, u_{n+1}) is $n+1$-universal. We have two goals:

(i) For $i \leq n$, $T_{u_{n+1}} : \pi_i(K) \to h(S^i)$ is an isomorphism.

(ii) For $i = n+1$ $T_{u_{n+1}} : \pi_i(K) \to h(S^i)$ is surjective.

Note that K_{n+1} is obtained from attaching $n+1$-cells, $i < n$ is obvious because nothing changes.

For $i = n$, leave as an exercise. It's like a very basic manuvre.

For $i = n+1$ also quite obvious.