MATH 6280 - CLASS 15

Contents

These notes are based on

- *Algebraic Topology from a Homotopical Viewpoint*, M. Aguilar, S. Gitler, C. Prieto
- *A Concise Course in Algebraic Topology*, J. Peter May
- *More Concise Algebraic Topology*, J. Peter May and Kate Ponto
- *Algebraic Topology*, A. Hatcher

1. Long exact sequence on homotopy groups for a pair

Recall that

$$\begin{aligned}
\pi_n(X, A) &= \pi_n(X, A, *) \\
&= \pi_{n-1} P_i \\
&= [(I^n, \partial I^n, J^n), (X, A, *)] \\
&= [(I^n, \partial I^n), (X, A)] \\
&= [(D^n, S^{n-1}), (X, A)].
\end{aligned}$$

It follows immediately from the long exact sequence for homotopy groups of a fiber sequence that there is a long exact sequence:

$$\ldots \to \pi_2(A) \to \pi_2(X) \to \pi_2(X, A) \to \pi_1(A) \to \pi_1(X) \to \pi_1(X, A) \to \pi_0(A) \to \pi_0(X)$$

The maps $\pi_n(A) \to \pi_n(X)$ and $\pi_n(X) \to \pi_n(X, A)$ are the obvious restrictions

$$\pi_n(X, A) \to \pi_{n-1}(A)$$

sends

$$(D^n, S^{n-1}, *) \to (X, A, *)$$

to the restriction $(S^{n-1}, *) \to (A, *)$.

Remark 1.1. For $* \in B \subseteq A \subseteq X$ we get a long exact sequence for a triple:

$$\ldots \to \pi_3(X,A) \to \pi_2(A,B) \to \pi_2(X,B) \to \pi_2(X,A) \to \pi_1(A,B) \to \pi_1(X,B) \to \pi_1(X,A)$$

Theorem 1.2. *Let B be path-connected. If $p : E \to B$ is a Serre fibration, with fiber $F = p^{-1}(*)$, then*

$$\pi_n(E,F) \to \pi_n B$$

is an isomorphism for $n \geq 1$, where the map takes $(D^n, S^{n-1}) \xrightarrow{f} (E,F)$ to $(D^n, S^{n-1}) \xrightarrow{p \circ f} (B, *)$. In particular, for a Serre fibration, there is a long exact sequence on homotopy groups:

$$\ldots \to \pi_2 B \to \pi_1 F \to \pi_1 E \to \pi_1 B \to \pi_0 F \to \pi_0 E \to *$$

Proof. Let $(I^n, \partial I^n) \xrightarrow{f} (B, *)$ be a homotopy class. Then $I^{n-1} \times \{0\} \subset \partial I^n$, so we can form the commutative diagram

$$\begin{tikzcd} I^{n-1} \times \{0\} \arrow[r, "*"] \arrow[d] & E \arrow[d, "p"] \\ I^n \arrow[r, "f"'] \arrow[ur, dotted, "\widetilde{f}"] & B \end{tikzcd}$$

Since p is a Serre fibration, there is a lift $\widetilde{f} : I^n \to E$. Such that $p\widetilde{f} = f$. Since $f(\partial I^n) = *$, we must have

$$\widetilde{f}(\partial I^n) \subset p^{-1}(*) = F,$$

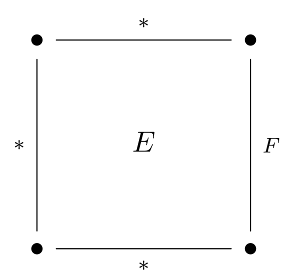
so

$$\widetilde{f} : (I^n, \partial I^n) \to (E, F)$$

and $\pi_n(E,F) \to \pi_n B$ is surjective.

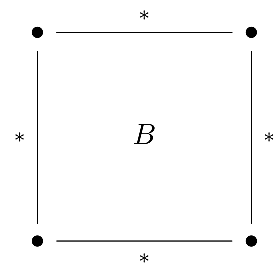
Now, suppose that $(I^n, \partial I^n, J^n) \xrightarrow{\widetilde{f}} (E, F, *)$ is such that $p\widetilde{f} \simeq *$.

$\widetilde{f}$:



and

$p\widetilde{f}$:



We can find a commutative diagram

$$\begin{tikzcd}
I^n \times \{0\} \arrow[r, "\cong"] \arrow[d] & J^{n+1} \cong I^n \times \{0\} \cup I^n \times \{1\} \cup J^n \times I \arrow[d] \\
I^n \times I \arrow[r, "\cong"] & I^n \times I
\end{tikzcd}$$

where the horizontal arrows are homeomorphisms. So, $E \to B$ has the HLP with respect to the inclusion $J^{n+1} \to I^n \times I$.

Choose a null-homotopy $H : (I^n, \partial I^n) \times I \to (B, *)$ from $p\widetilde{f}$ to $*$. We let

$$h : J^{n+1} \to E$$

be $\widetilde{f}$ on $I^n \times \{0\}$ and $*$ on $I^n \times \{1\} \cup J^n \times I$. We have a diagram with a lift

$$\begin{tikzcd}
J^{n+1} \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\
I^n \times I \arrow[r, "H"'] \arrow[ur, "\widetilde{H}"] & B
\end{tikzcd}$$

One now only needs to check that $\widetilde{H}$ is a null-homotopy from $\widetilde{f}$ to the constant map.

So, we get a long exact sequence on homotopy groups coming from the long exact sequence for the pair (E, F):

$$\ldots \to \pi_2 B \to \pi_1 F \to \pi_1 E \to \pi_1 B \to \pi_0 F \to \pi_0 E$$

We just need to check surjectivity at the end. Suppose that e is a point in E. Choose a path γ from $p(e)$ to $*$ in B. Choose a lift $\widetilde{\gamma}$ which starts at e. Then $p(\widetilde{\gamma}(1)) = *$ so $\widetilde{\gamma}(1)$ is in the same path component as e and also in the fiber F. Hence, $\pi_0 F \to \pi_0 E$ is surjective. □

2. Relationship between fibrations and cofibrations

There is a the following close connection between fibrations a cofibration.

Proposition 2.1.
- *A map $A \to X$ is a cofibration with closed image if and only if it has the left lifting property with respect to all Hurewicz fibrations which are also homotopy equivalences. That is, given any fibration $p : E \xrightarrow{\simeq} B$*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

there is map $X \to E$ making the diagram commute.
- *A map $E \to B$ is a Hurewicz fibration if and only if it has the right lifting property with respect to all cofibrations with closed image which are also homotopy equivalences. That is, given any cofibrations with closed image $i : A \xrightarrow{\simeq} X$*

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there is map $X \to E$ making the diagram commute.

3. Homotopy pushouts

Definition 3.1. The homotopy pushout of

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & Y \\ X & \end{tikzcd}$$

is the double mapping cylinder, which is the pushout of

$$\begin{tikzcd} A \sqcup A \arrow[r, "f \sqcup g"] \arrow[d, "i_0 \sqcup i_1"'] & Y \sqcup X \arrow[d] \\ A \times I \arrow[r] & M_{f,g} \end{tikzcd}$$

Proposition 3.2. *If f is a cofibration, then $M_{f,g} \to X \cup_A Y$ is a homotopy equivalence.*

Proof. Let $R : X \times I \to X \times \{0\} \cup A \times I$ be the retract which exists since $A \to X$ is a cofibration. Let $q : M_{f,g} \to X \cup_A Y$ be the quotient and $p : X \cup_A Y \to M(f,g)$ be the composite

$$X \cup_A Y \xrightarrow{i_1} X \times I \cup_{A \times \{1\}} Y \xrightarrow{R \cup_{g(A)} \mathrm{id}_Y} M_{f,g}.$$

Let $r_s : M_{f,g} \to M_{f,g}$ be defined by

$$r_s(z) = \begin{cases} y & z \in Y \\ R(x, (1-s)t + s) & z = (x,t) \in X \times \{0\} \cup A \times I. \end{cases}$$

Then, $r_0(z) = \mathrm{id}_{M_{f,g}}$ and $r_1(z) = p \circ q$. Conversely, let $H : X \times I \times J \to X \times I$ be the homotopy between $\mathrm{id}_{X \times I}$ to R. Then let $h_t : X \cup_A Y \times I \to X \cup_A Y$ be

$$h_s(z) = \begin{cases} y & z \in Y \\ \pi_X(H_s(x,1)) & z \in X. \end{cases}$$

Then one can verify that h is a homotopy between the identity and $q \circ p$. $\square$

Exercise 3.3. Define coequalizers as pushouts and colimits of diagrams $* \to * \to \ldots$ as coequalizers. Use this to define homotopy coequalizers and homotopy colimits of diagrams $* \to * \to \ldots$.

Definition 3.4. The homotopy pullback of

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is the double mapping path space, which is the pull-back of

$$\begin{tikzcd} E_{f,g} \arrow[d] & X \times Y \arrow[d, "f \times g"] \\ Z^I \arrow[r, "(ev_0, ev_1)"] & Z \times Z \end{tikzcd}$$

That is,

$$E_{f,g} = \{(x, \alpha, y) \mid \alpha(0) = x, \alpha(1) = y\} \subset X \times Z^I \times Y$$

Exercise 3.5. Prove that if f is a fibration, then the natural map $X \times_Z Y \to E_{f,g}$ which sends (x,z,y) to (x, α_z, y), for α_z the constant path at z, is a homotopy equivalence.

Exercise 3.6. Define equalizers as pull-backs and limits of diagrams $\ldots \to * \to *$ as equalizers. Use this to define homotopy equalizers and homotopy limits of diagrams $\ldots \to * \to *$.