Chapter I: The Fundamental Group

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1 Basic Constructions

Lemma (1.15)**.** If a space X is the union of a collection of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

Exercise 1. Obvious

Exercise 2. Obvious

Exercise 3. Obvious

Exercise 4. Obvious

Exercise 5. Obvious

Exercise 6. Obvious

Exercise 7. Obvious

Exercise 8. Consider $S^1 \times S^1 \twoheadrightarrow S^1 \hookrightarrow \mathbb{R}^2$, we have

$$f(x,y) \neq f(-x,-y) \tag{1}$$

.

Exercise 9. Define $S^2 \to \mathbb{R}^2$ as follows:

for each $\boldsymbol{n} \in S^2$, choose an average $t(\boldsymbol{n}) \in \mathbb{R}$ such that plane $p_{\boldsymbol{n}} : \boldsymbol{n} \cdot \boldsymbol{x} = t$ cut A_3 in equal half, by "average" we mean the average of the maximal possible value and the minimal possible value so that $t(\boldsymbol{n}) = -t(-\boldsymbol{n})$, i.e. $p_{\boldsymbol{n}} = p_{-\boldsymbol{n}}$. Let $x(\boldsymbol{n}), y(\boldsymbol{n})$ be the area of A_1, A_2 intersecting with the upper space divided by $p_{\boldsymbol{n}}$ along $\boldsymbol{n}$. We then have $x(\boldsymbol{n}) + x(-\boldsymbol{n}) = 1$ and $y(\boldsymbol{n}) + y(-\boldsymbol{n}) = 1$. Then apply Borsuk-Ulam theorem to $\boldsymbol{n} \mapsto (x(\boldsymbol{n}), y(\boldsymbol{n}))$ we are done.

Exercise 10. Obvious

Exercise 11. Obvious

Exercise 12. Obvious

Exercise 13. Obvious

Exercise 14. Obvious

Exercise 15. Obvious

Exercise 16. Obvious

Exercise 17. Obvious

Exercise 18. Straightforward