1. Linear Algebra Basics

1.1. Overview

Linear algebra studies vector spaces and linear transformations, providing the mathematical framework for solving systems of equations, understanding geometric transformations, and modeling countless phenomena in science and engineering. This module establishes fundamental concepts including linear independence, bases, dimension, and linear maps.

1.2. Vector Spaces

1.2.1. Definition

A **vector space** over a field $\mathbb{F}$ is a set V with:
- Addition: $V \times V \to V$
- Scalar multiplication: $\mathbb{F} \times V \to V$

Satisfying eight axioms including commutativity, associativity, distributivity, and the existence of additive identity and inverses.

1.2.2. Examples

Standard spaces:
- $\mathbb{F}^n$: n-tuples with component-wise operations
- $\mathbb{F}[X]$: Polynomials over $\mathbb{F}$
- $C([a,b])$: Continuous functions on $[a,b]$
- $M_{m \times n}(\mathbb{F})$: Matrices

Function spaces:
- $L^2(\Omega)$: Square-integrable functions
- $C^k(M)$: k-times differentiable functions
- ℓ^p: p-summable sequences

1.3. Linear Independence

1.3.1. Definition

Vectors $v_1, …, v_n$ are **linearly independent** if: a_1 v_1 + … + a_n v_n = 0 arrow.r.double a_1 = … = a_n = 0

A set S is linearly independent if every finite subset is.

1.3.2. Properties

Extension: If S is linearly independent and $v \notin \text{span}(S)$, then $S \cup \{v\}$ is linearly independent.

Extraction: Every spanning set contains a basis.

Exchange: If B is a basis and $v = \Sigma a_i b_i$ with $a_j \neq 0$, then $\left(B \setminus \left\{b_j\right\}\right) \cup \{v\}$ is a basis.

1.4. Basis

1.4.1. Definition

A **basis** is a linearly independent spanning set.

Equivalently, B is a basis if every vector has a unique representation: v = sum_B a_b b with only finitely many $a_b \neq 0$.

1.4.2. Standard Bases

Euclidean basis for $\mathbb{F}^n$: e_1 = (1,0,…,0), space e_2 = (0,1,…,0), space …, space e_n = (0,0,…,1)

Monomial basis for $\mathbb{F}[X]_n$: 1, X, X^2, …, X^n

Matrix basis for $M_{\{m \times n\}}(\mathbb{F})$: $E_{\{ij\}}$: matrix with 1 in position (i, j), 0 elsewhere

1.4.3. Existence

Finite-dimensional: Every finite-dimensional vector space has a basis.

Infinite-dimensional: Every vector space has a basis (requires Axiom of Choice via Zorn's Lemma).

1.5. Dimension

1.5.1. Definition

The **dimension** of V is the cardinality of any basis.

Well-defined by the Steinitz exchange theorem.

1.5.2. Properties

Invariance: All bases of V have the same cardinality.

Subspace dimension: If $W \subset V$, then $\dim(W) \leq \dim(V)$.

Sum formula: dim(U + W) = dim(U) + dim(W) - dim(U ∩ W)

Product formula: dim(V × W) = dim(V) + dim(W)

1.6. Linear Maps

1.6.1. Definition

A function $T : V \to W$ is **linear** if:
1. $T(v_1 + v_2) = T(v_1) + T(v_2)$ (additivity)
2. $T(cv) = cT(v)$ (homogeneity)

The set of linear maps is denoted $L(V, W)$ or $\mathrm{Hom}(V, W)$.

1.6.2. Kernel and Image

Kernel: ker(T) = {v in V : T(v) = 0}

Image: im(T) = {T(v) : v in V}

Both are subspaces of their respective spaces.

1.6.3. Rank-Nullity Theorem

For $T : V \to W$ with V finite-dimensional: dim(V) = dim(ker(T)) + dim(im(T))

where:
- $\mathrm{nullity}(T) = \dim(\ker(T))$
- $\mathrm{rank}(T) = \dim(\mathrm{im}(T))$

1.7. Matrix Representation

1.7.1. Definition

Given bases $B = \{b_1, ..., b_n\}$ of V and $C = \{c_1, ..., c_m\}$ of W, the matrix of $T : V \to W$ is: [T]*B^C = (a*{i j}) where $T(b_j) = \sum_{\{i=1\}}^{m} a_{\{ij\}} c_i$.

1.7.2. Change of Basis

If P is the change of basis matrix from B to B': [T]*(B')^(C') = Q^(-1) [T]*B^C P where Q changes from C to C'.

1.7.3. Similar Matrices

Matrices A and B are **similar** if: B = P^(-1) A P for some invertible P.

Similar matrices represent the same linear operator in different bases.

1.8. Dual Space

1.8.1. Definition

The **dual space** V^* consists of linear functionals $V \to \mathbb{F}$.

For finite-dimensional V: dim(V^star) = dim(V)

1.8.2. Dual Basis

Given basis $b_1, ..., b_n$ of V, the **dual basis** $b^1, ..., b^n$ of V^* satisfies: b^i(b_j) = delta_i_j

1.8.3. Double Dual

The natural map $V \to V^{**}$ given by: v ↦ (φ ↦ φ(v)) is an isomorphism for finite-dimensional spaces.

1.9. Direct Sums

1.9.1. Internal Direct Sum

$V = U \oplus W$ if:
- $V = U + W$
- $U \cap W = \{0\}$

Every $v \in V$ uniquely decomposes as $v = u + w$.

1.9.2. External Direct Sum

The direct sum $V \oplus W$ consists of pairs (v, w) with component-wise operations.

1.9.3. Projection Operators

If $V = U \oplus W$, the projection $P_U : V \to U$ satisfies:
- $P_U^2 = P_U$ (idempotent)
- $\ker(P_U) = W$
- $\mathrm{im}(P_U) = U$

1.10. Eigenvalues and Eigenvectors

1.10.1. Definition

A scalar λ is an **eigenvalue** of $T : V \to V$ if there exists $v \neq 0$ with: T(v) = λ v

Such v is an **eigenvector**.

1.10.2. Characteristic Polynomial

For a matrix A: p_A(λ) = det(λ I - A)

Eigenvalues are roots of the characteristic polynomial.

1.10.3. Diagonalization

T is **diagonalizable** if there exists a basis of eigenvectors.

Equivalently, the matrix of T in some basis is diagonal.

1.11. Inner Product Spaces

1.11.1. Definition

An **inner product** on a real vector space is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that is:
- Symmetric: $\langle v, w \rangle = \langle w, v \rangle$
- Linear in first argument
- Positive definite: $\langle v, v \rangle > 0$ for $v \neq 0$

For complex spaces, use sesquilinearity instead of symmetry.

1.11.2. Orthogonality

Vectors v, w are **orthogonal** if $\langle v, w \rangle = 0$.

An **orthonormal basis** satisfies: ⟨e_i, e_j⟩ = delta_i_j

1.11.3. Gram-Schmidt Process

Converts any basis to an orthonormal basis:
1. $u_1 = v_1 / \|v_1\|$
2. Orthogonalize and normalize subsequent vectors

1.12. Applications

1.12.1. Systems of Linear Equations

The system $Ax = b$ has:
- Unique solution if $\text{rank}(A) = n$
- Infinitely many solutions if $\text{rank}(A) < n$ and consistent
- No solution if inconsistent

1.12.2. Least Squares

For overdetermined systems, minimize: ||A x - b||^2

Solution: $x = \left(A^T A\right)^{-1} A^T b$ (when $A^T A$ invertible)

1.12.3. Principal Component Analysis

Find orthogonal directions of maximum variance:
1. Center data
2. Compute covariance matrix
3. Find eigenvectors (principal components)

1.12.4. Quantum Mechanics

States are unit vectors in Hilbert space:
- Observables are Hermitian operators
- Measurements yield eigenvalues
- Evolution via unitary operators

1.13. Computational Aspects

1.13.1. Matrix Factorizations

LU decomposition: $A = LU$ for solving systems

QR decomposition: $A = QR$ for least squares

SVD: $A = U \Sigma V^T$ for dimensionality reduction

Eigendecomposition: $A = PDP^{-1}$ for diagonalizable matrices

1.13.2. Complexity

- Matrix multiplication: $O\left(n^3\right)$ classical, $O\left(n^{2.373}\right)$ theoretical
- Gaussian elimination: $O\left(n^3\right)$
- Eigenvalues: $O\left(n^3\right)$ iterative methods
- Determinant: $O\left(n^3\right)$ via LU decomposition

1.14. Advanced Topics

1.14.1. Tensor Products

The tensor product $V \otimes W$ satisfies:
- Bilinearity: $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$

- Dimension: $\dim(V \otimes W) = \dim(V) \times \dim(W)$

1.14.2. Exterior Algebra

The exterior product ∧ creates antisymmetric tensors: v ∧ w = -w ∧ v

Used in differential forms and determinant theory.

1.14.3. Representation Theory

Groups act on vector spaces via linear operators: ρ: G ⟶ GL(V)

Decompose into irreducible representations.

1.15. Implementation Notes

1.15.1. Type Classes in Mathlib

```
class VectorSpace (K : Type*) (V : Type*) [Field K] extends Module K V

instance : VectorSpace ℝ (EuclideanSpace ℝ (Fin n))
```

1.15.2. Finite Dimensions

```
class FiniteDimensional (K V : Type*) [Field K] [VectorSpace K V] : Prop where
  exists_basis_finset : ∃ s : Finset V, Basis s K V
```

1.15.3. Linear Maps

```
structure LinearMap (K : Type*) (V W : Type*) [Field K] [VectorSpace K V] [VectorSpace K
W] where
  toFun : V → W
  map_add : ∀ x y, toFun (x + y) = toFun x + toFun y
  map_smul : ∀ (c : K) x, toFun (c • x) = c • toFun x
```

1.16. Historical Context

Linear algebra developed through:

- **Leibniz (1693)**: Determinants for solving systems
- **Gauss (1810s)**: Gaussian elimination
- **Cayley (1858)**: Matrix algebra
- **Grassmann (1844)**: Exterior algebra
- **Jordan (1870)**: Jordan normal form
- **Hilbert (1900s)**: Infinite-dimensional spaces
- **von Neumann (1930s)**: Operator theory

Modern applications span:
- Machine learning (neural networks, PCA)
- Computer graphics (transformations, rendering)
- Engineering (control theory, signal processing)
- Physics (quantum mechanics, relativity)