1. Module Theory Basics

1.1. Overview

Module theory generalizes vector spaces by allowing scalars from a ring rather than a field. Modules are fundamental to homological algebra, representation theory, and algebraic topology. This module establishes the basic theory of modules over rings, linear maps, and submodules.

1.2. Core Definitions

1.2.1. Module

A **left module** over a ring R is an abelian group $(M, +)$ with a scalar multiplication $R \times M \to M$ satisfying:

1. $(r + s) \cdot m = r \cdot m + s \cdot m$ (distributive over ring addition)
2. $r \cdot (m + n) = r \cdot m + r \cdot n$ (distributive over module addition)
3. $(r \times s) \cdot m = r \cdot (s \cdot m)$ (associativity)
4. $1 \cdot m = m$ (unital)

Right modules have scalar multiplication $M \times R \to M$ with $(m \cdot r) \cdot s = m \cdot (r \times s)$.

1.2.2. Bimodule

An (R, S)-**bimodule** is simultaneously:

- A left R-module
- A right S-module
- With compatibility: $(r \cdot m) \cdot s = r \cdot (m \cdot s)$

1.2.3. Vector Space

A **vector space** is a module over a field.

Key difference: In vector spaces, every non-zero scalar is invertible.

1.3. Examples

1.3.1. Natural Examples

Abelian groups: Modules over $\mathbb{Z}$

- Scalar multiplication: $n \cdot g = g + g + \ldots + g$ (n times)

Ideals: Left ideals are left R-modules

Ring as module: R is a module over itself

Polynomial modules: $R[X]$ is a module over R

1.3.2. Linear Algebra Examples

Column vectors: R^n is a module over R

Matrices: $M_{m \times n}(R)$ is an $(M_m(R), M_n(R))$-bimodule

Dual module: $\mathrm{Hom}_R(M, R)$ with $(r \cdot f)(m) = r \times f(m)$

1.3.3. Geometric Examples

Sections of bundles: Smooth sections form modules over smooth functions

Differential forms: $\Omega^k(M)$ is a module over $C^\infty(M)$

1.4. Submodules

1.4.1. Definition

A **submodule** $N \subset M$ is a subgroup closed under scalar multiplication:

- $0 \in N$
- $m, n \in N \Rightarrow m + n \in N$
- $r \in R, m \in N \Rightarrow r \cdot m \in N$

1.4.2. Operations

Sum: $N_1 + N_2 = \{n_1 + n_2 : n_1 \in N_1, n_2 \in N_2\}$

Intersection: $N_1 \cap N_2$ is a submodule

Product: $R \cdot N = \{r \cdot n : r \in R, n \in N\}$

Span: $\langle S \rangle =$ smallest submodule containing S

1.4.3. Lattice Structure

Submodules form a complete lattice:
- Join: $N_1 \vee N_2 = N_1 + N_2$
- Meet: $N_1 \wedge N_2 = N_1 \cap N_2$
- Modular law: $N_1 \subset N_3 \Rightarrow N_1 + (N_2 \cap N_3) = (N_1 + N_2) \cap N_3$

1.5. Linear Maps

1.5.1. Definition

A function $f : M \to N$ between R-modules is **linear** if:
1. $f(m_1 + m_2) = f(m_1) + f(m_2)$ (additive)
2. $f(r \cdot m) = r \cdot f(m)$ (scalar-linear)

Notation: $\mathrm{Hom}_R(M, N)$ or $L(M, N)$

1.5.2. Properties

Kernel: $\ker(f) = \{m \in M : f(m) = 0\}$ is a submodule of M

Image: $\mathrm{im}(f) = \{f(m) : m \in M\}$ is a submodule of N

Composition: Linear maps compose associatively

Module structure: $\mathrm{Hom}_R(M, N)$ is an R-module with: (f + g)(m) = f(m) + g(m) (r · f)(m) = r · f(m)

1.5.3. Isomorphisms

A linear map $f : M \to N$ is an **isomorphism** if it's bijective.

Notation: $M \cong N$

Properties:
- Isomorphisms preserve all module structure
- Isomorphism is an equivalence relation
- $M \cong N \Rightarrow \dim(M) = \dim(N)$ (for vector spaces)

1.6. Quotient Modules

1.6.1. Construction

For a submodule $N \subset M$, the **quotient module** M/N has:
- Elements: cosets $m + N = \{m + n : n \in N\}$
- Addition: $(m_1 + N) + (m_2 + N) = (m_1 + m_2) + N$
- Scalar multiplication: $r \cdot (m + N) = (r \cdot m) + N$

1.6.2. Universal Property

The projection $\pi : M \to M/N$ satisfies: For any linear map $f : M \to P$ with $N \subset \ker(f)$, there exists unique $|(f) : M/N \to P$ with $f = |(f) \circ \pi$.

1.6.3. Isomorphism Theorems

First: If $f : M \to N$ is linear: M "/" ker(f) ≅ im(f)

Second: If $N \subset P \subset M$: (M "/" N) "/" (P "/" N) ≅ M "/" P

Third: If $N, P \subset M$: (N + P) "/" P ≅ N "/" (N ∩ P)

1.7. Free Modules

1.7.1. Definition

A module M is **free** if it has a basis: M ≅ plus.circle_I R

The **rank** is $|I|$ (when well-defined).

1.7.2. Properties

Over a field (vector spaces):
- Every module is free
- Any two bases have the same cardinality
- Submodules of free modules are free

Over general rings:
- Not all modules are free
- Submodules of free modules need not be free
- Rank may not be well-defined (IBN property)

1.7.3. Construction

The free module on a set S: F(S) = plus.circle_S R

Elements: formal linear combinations with finite support.

1.8. Direct Sums and Products

1.8.1. Direct Sum

The **direct sum** $\oplus_I M_i$ consists of: {(m_i) : m_i = 0 "for all but finitely many" i}

Operations are component-wise.

1.8.2. Direct Product

The **direct product** $\prod_I M_i$ consists of: {(m_i) : m_i in M_i}

For finite index sets: direct sum = direct product.

1.8.3. Internal Direct Sum

$M = N_1 \oplus N_2$ internally if:
- $M = N_1 + N_2$
- $N_1 \cap N_2 = \{0\}$

Equivalently: Every $m \in M$ uniquely writes as $m = n_1 + n_2$.

1.9. Tensor Products

1.9.1. Definition

The **tensor product** $M \otimes_R N$ is the module satisfying:
- Bilinear map: $M \times N \to M \otimes_R N$
- Universal property for bilinear maps

1.9.2. Properties

Distributivity: (m_1 + m_2) ⊗ n = m_1 ⊗ n + m_2 ⊗ n m ⊗ (n_1 + n_2) = m ⊗ n_1 + m ⊗ n_2

Scalar compatibility: (r · m) ⊗ n = m ⊗ (r · n) = r · (m ⊗ n)

Associativity: (M ⊗*R N)* ⊗R P ≅ M ⊗*R (N* ⊗R P)

1.9.3. Applications

Extension of scalars: If $R \to S$ is a ring homomorphism, then $M \otimes_R S$ becomes an S-module.

Flat modules: M is flat if $M \otimes_R (-)$ is exact

1.10. Exact Sequences

1.10.1. Definition

A sequence … $\to M_{i-1} \to M_i \to M_{i+1} \to$ … is **exact** at M_i if the image of the incoming map equals the kernel of the outgoing map.

1.10.2. Short Exact Sequences

0 ⟶ M’ ⟶^i M ⟶^p M’‘ ⟶ 0

This means:
- i is injective
- p is surjective
- $\mathrm{im}(i) = \ker(p)$
- $M/M' \cong M''$

1.10.3. Split Exact Sequences

A short exact sequence **splits** if one of:
- There exists $s : M'' \to M$ with $p \circ s = \mathrm{id}$
- There exists $r : M \to M'$ with $r \circ i = \mathrm{id}$
- $M \cong M' \oplus M''$

1.11. Special Types of Modules

1.11.1. Simple Modules

A module is **simple** if its only submodules are 0 and M.

Examples:
- $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}$-module
- One-dimensional vector spaces
- Irreducible representations

1.11.2. Semisimple Modules

A module is **semisimple** if it’s a direct sum of simple modules.

Theorem (Wedderburn-Artin): Modules over semisimple rings are semisimple.

1.11.3. Projective Modules

A module P is **projective** if for every surjection $M \to\to N$ and map $P \to N$, there exists a lift $P \to M$.

Properties:
- Free modules are projective
- Direct summands of free modules are projective
- Projective modules are flat

1.11.4. Injective Modules

A module I is **injective** if for every injection $N \hookrightarrow M$ and map $N \to I$, there exists an extension $M \to I$.

Examples:
- $\mathbb{Q}/\mathbb{Z}$ as $\mathbb{Z}$-module

- Divisible abelian groups

1.11.5. Noetherian and Artinian

A module is:

- **Noetherian** if every ascending chain of submodules stabilizes
- **Artinian** if every descending chain of submodules stabilizes

Over Noetherian rings, finitely generated modules are Noetherian.

1.12. Applications

1.12.1. Representation Theory

Group representations are modules over group rings:

- G-module = $\mathbb{Z}[G]$-module
- Character theory studies traces of representations

1.12.2. Homological Algebra

Modules enable:

- Chain complexes
- Derived functors (Ext, Tor)
- Cohomology theories

1.12.3. Algebraic Geometry

Coherent sheaves are modules over structure sheaves:

- Vector bundles correspond to locally free modules
- Ideal sheaves give subscheme structure

1.12.4. Number Theory

Ideal class groups are modules:

- Class field theory uses Galois modules
- Iwasawa theory studies $\mathbb{Z}_p[[T]]$-modules

1.13. Implementation Notes

1.13.1. Type Classes

Mathlib hierarchy:

```
class Module (R : Type*) (M : Type*) [Semiring R] [AddCommMonoid M] extends
  DistribMulAction R M where
  add_smul : ∀ (r s : R) (m : M), (r + s) • m = r • m + s • m
  zero_smul : ∀ m : M, (0 : R) • m = 0
```

1.13.2. Scalar Multiplication

Notation: • for scalar multiplication (smul)

```
notation:73 r " • " m => SMul.smul r m
```

1.13.3. Bundled Morphisms

Linear maps bundle the function with linearity:

```
structure LinearMap (R : Type*) (M N : Type*) [Semiring R] [AddCommMonoid M]
  [AddCommMonoid N] [Module R M] [Module R N] where
  toFun : M → N
  map_add : ∀ x y, toFun (x + y) = toFun x + toFun y
  map_smul : ∀ (r : R) x, toFun (r • x) = r • toFun x
```

1.14. Historical Context

Module theory developed through:

- **Dedekind (1890s)**: Modules over Dedekind domains
- **Noether (1920s)**: Abstract module theory
- **Artin & Wedderburn (1920s)**: Semisimple modules
- **Cartan & Eilenberg (1956)**: Homological algebra
- **Serre (1955)**: Coherent sheaves as modules
- **Quillen (1976)**: Algebraic K-theory of modules

Modern developments:

- Derived categories of modules
- Cyclic homology
- Noncommutative geometry (modules over C-star algebras)
- Categorification (module categories)