1. Continuous Functions on Subsets

1.1. Overview

This module develops the theory of continuity relative to subsets, providing tools for studying functions that are continuous only on parts of their domain. This is essential for real analysis, manifolds with boundary, and partial differential equations where functions naturally have restricted domains of continuity.

1.2. Core Concepts

1.2.1. Neighborhoods Within Sets

Definition (`nhdsWithin`): The neighborhood filter within a set s at a point x is: cal(N)[s] x = cal(N) x ∩ cal(P)(s)

where:
- $\mathcal{N} x$ is the neighborhood filter at x
- $\mathcal{P}(s)$ is the principal filter of s

Notation:
- `𝓝[s] x`: neighborhoods of x within s
- `𝓝 x`: unrestricted neighborhoods of x
- `𝓟 s`: principal filter of set s

1.2.2. Continuous at a Point Within a Set

Definition (`ContinuousWithinAt`): A function $f : \alpha \to \beta$ is continuous at x within s if: f(cal(N)[s] x) ≤ cal(N) f(x)

Equivalently: For every neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U \cap s) \subset V$.

1.2.3. Continuous on a Set

Definition (`ContinuousOn`): A function $f : \alpha \to \beta$ is continuous on s if it is continuous at every point of s within s: “ContinuousOn” f s ⟺ ∀ x in s, “ContinuousWithinAt” f s x

1.3. Fundamental Properties

1.3.1. Eventually and Frequently

Eventually Within (`eventually_nhdsWithin_iff`): ∀^f x in cal(N)[s] a, P(x) ⟺ ∀^f x in cal(N) a, x in s ⟶ P(x)

A property holds eventually within s if it holds eventually in the full neighborhood for points in s.

Frequently Within (`frequently_nhdsWithin_iff`): ∃^f x in cal(N)[s] a, P(x) ⟺ ∃^f x in cal(N) a, P(x) ∧ x in s

1.3.2. Basis Characterization

Neighborhood Basis Within (`nhdsWithin_hasBasis`): If $(\mathcal{N} a)$ has basis (B_i), then $\mathcal{N}[s] a$ has basis $(B_i \cap s)$.

Open Basis (`nhdsWithin_basis_open`): cal(N)[s] a “has basis” {U : a in U, U “open”} “with sets” {U ∩ s}

1.3.3. Membership Criteria

Set Membership (`mem_nhdsWithin`): t in cal(N)[s] a ⟺ ∃ U “open”, a in U ∧ U ∩ s subset t

Alternative Form (`mem_nhdsWithin_iff_exists_mem_nhds_inter`): t in cal(N)[s] a ⟺ ∃ U in cal(N) a, U ∩ s subset t

1.4. Key Theorems

1.4.1. Filter Relationships

Binding Property (`nhds_bind_nhdsWithin`): (cal(N) a).“bind”(λ x. cal(N)[s] x) = cal(N)[s] a

This shows that the within-neighborhood operation is idempotent under binding.

Eventually Eventually (`eventually_eventually_nhdsWithin`): ∀^f y in cal(N)[s] a, ∀^f x in cal(N)[s] y, P(x) ⟺ ∀^f x in cal(N)[s] a, P(x)

1.4.2. Order Properties

Within Comparison (`nhdsWithin_le_iff`): cal(N)[s] x ≤ cal(N)[t] x ⟺ t in cal(N)[s] x

Smaller filters correspond to larger sets.

Equality Condition (`nhdsWithin_eq_iff_eventuallyEq`): cal(N)[s] x = cal(N)[t] x ⟺ s =^f[cal(N) x] t

Within-neighborhoods are equal if the sets are eventually equal.

1.4.3. Special Cases

Universal Set (`nhdsWithin_univ`): cal(N)[“univ”] a = cal(N) a

The within-neighborhood for the whole space is the full neighborhood.

Self Membership (`self_mem_nhdsWithin`): s in cal(N)[s] a

A set always belongs to its own within-neighborhood filter.

Pure Filter (`pure_le_nhdsWithin`): If $a \in s$, then: “pure” a ≤ cal(N)[s] a

1.5. Operations with Within-Neighborhoods

1.5.1. Set Operations

Intersection (`inter_mem_nhdsWithin`): If $t \in \mathcal{N}a$, then $s \cap t \in \mathcal{N}[s]a$.

Difference (`diff_mem_nhdsWithin_diff`): If $s \in \mathcal{N}[t]x$, then $s \setminus t' \in \mathcal{N}[t \setminus t']x$.

Complement (`diff_mem_nhdsWithin_compl`): If $s \in \mathcal{N}x$, then $s \setminus t \in \mathcal{N}[t^c]x$.

1.5.2. Relationship to Full Neighborhoods

Lifting (`nhds_of_nhdsWithin_of_nhds`): If $s \in \mathcal{N}a$ and $t \in \mathcal{N}[s]a$, then $t \in \mathcal{N}a$.

Restriction (`mem_nhdsWithin_of_mem_nhds`): If $s \in \mathcal{N}a$, then $s \in \mathcal{N}[t]a$ for any t.

1.6. Continuity Properties

1.6.1. Characterizations

Within Continuity: f is continuous at x within s if:
- For every $\varepsilon > 0$, exists $\delta > 0$ such that $y \in s$, $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$
- The preimage of every neighborhood of $f(x)$ is a within-neighborhood of x
- $\lim_{y \to x, y \in s} f(y) = f(x)$

On Continuity: f is continuous on s if:
- The restriction f to s is continuous (with subspace topology)
- For every open V in β, the set $f^{-1}(V) \cap s$ is relatively open in s

1.6.2. Composition

Chain Rule for Within: If f is continuous at x within s and g is continuous at $f(x)$ within $f(s)$: g ∘ f “is continuous at” x “within” s

Restriction Principle: If $f : \alpha \to \beta$ is continuous and $s \subset \alpha$: f “restricted to” s: s ⟶ β “is continuous”

1.7. Closure and Interior

1.7.1. Closure Characterization

Membership in Closure (`mem_closure_ne_iff_frequently_within`): x in “closure”(s ∖ {x}) ⟺ ∃^f y in cal(N)[≠] x, y in s

A point is in the closure of a set minus itself if there are frequently other points of the set nearby.

1.7.2. Relative Interior

The relative interior of s consists of points with within-neighborhoods: “relint”(s) = {x in s : ∃ U in cal(N) x, U ∩ s subset s}

For convex sets, this gives the interior relative to the affine hull.

1.8. Subspace Topology Connection

1.8.1. Coinduced Topology

Preimage Property (`preimage_nhdsWithin_coinduced'`): For the subspace topology on $t \subset \alpha$: If s is a neighborhood of $X(a)$ in the coinduced topology, then $X^{-1}(s) \in \mathcal{N}[t]a$.

This connects within-neighborhoods to the subspace topology.

1.8.2. Subtype Correspondence

For $s \subset \alpha$ with subspace topology:
- `ContinuousOn f s` in α corresponds to `Continuous f` from the subtype s
- Within-neighborhoods in α correspond to full neighborhoods in the subtype

1.9. Applications

1.9.1. Piecewise Functions

Functions defined piecewise can be shown continuous using within-continuity:

```
f(x) = cases(
  g(x) "if" x in A,
  h(x) "if" x in B
)
```

If g is continuous on A and h is continuous on B, and they agree on $A \cap B$, then f is continuous on $A \cup B$.

1.9.2. Boundary Behavior

For domains with boundary:
- Study continuity up to and including the boundary
- Handle functions with different behavior at boundary points
- Prove extension theorems for continuous functions

1.9.3. Partial Derivatives

In multivariable calculus:
- Directional derivatives use within-continuity along lines
- Partial derivatives use within-continuity along coordinate axes
- Differentiability requires stronger conditions than partial continuity

1.9.4. Manifolds with Boundary

For manifolds with boundary:
- Charts are continuous on their domains (including boundary)
- Transition maps are continuous on overlaps
- Within-continuity handles boundary charts properly

1.10. Design Principles

1.10.1. Filter-Based Approach

Mathlib uses filters systematically:

- Within-neighborhoods are intersections of filters
- Continuity is expressed via filter inequalities
- This unifies pointwise and uniform concepts

1.10.2. Notation Conventions

Standard notation:

- `𝓝[s] x`: within-neighborhood filter
- `𝓝[≠] x`: punctured neighborhood (deleted neighborhood)
- `𝓝[Icc a b] x`: neighborhoods within closed interval

1.10.3. Relation to Metrics

In metric spaces: cal(N)[s] x = {t : ∃ ε > 0, B(x, ε) ∩ s subset t}

The within-neighborhood consists of sets containing small balls intersected with s.

1.11. Historical Context

The concept of relative continuity developed from:

- Riemann (1854): Piecewise continuous functions in integration
- Peano (1890): Functions continuous on closed sets
- Kuratowski (1922): Relative topology and continuity
- Bourbaki (1940s): Systematic treatment via filters

Modern applications include:

- Partial differential equations with boundary conditions
- Numerical analysis on irregular domains
- Geometric measure theory on submanifolds
- Stochastic processes with barriers