1. Order Theory Basics

1.1. Overview

Order theory studies binary relations that capture notions of comparison, precedence, and hierarchy. From the natural ordering of numbers to the inclusion of sets, ordered structures appear throughout mathematics. This module establishes fundamental concepts including partial orders, lattices, and order-preserving maps.

1.2. Preorders and Partial Orders

1.2.1. Preorder

A **preorder** (or quasi-order) on a set P is a relation $\leq$ that is:
- **Reflexive**: $a \leq a$ for all $a \in P$
- **Transitive**: $a \leq b$ and $b \leq c$ implies $a \leq c$

Examples:
- Divisibility on integers
- Preference relations
- Reachability in directed graphs

1.2.2. Partial Order

A **partial order** (or poset) is a preorder that is also:
- **Antisymmetric**: $a \leq b$ and $b \leq a$ implies $a = b$

Notation: $(P, \leq)$ or just P when the order is clear.

1.2.3. Strict Order

The **strict order** $<$ is defined by: a < b arrow.l.r.double a ≤ b ∧ a ≠ b

Properties:
- Irreflexive: $a \nless a$
- Asymmetric: $a < b \Rightarrow b \nless a$
- Transitive: $a < b < c \Rightarrow a < c$

1.3. Examples of Partial Orders

1.3.1. Numerical Orders

Natural numbers: $(\mathbb{N}, \leq)$ with usual ordering

Integers: $(\mathbb{Z}, \leq)$ extends natural ordering

Rationals and Reals: $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are dense linear orders

Complex numbers: No natural total order, but partial orders exist

1.3.2. Set-Theoretic Orders

Power set: $(\mathcal{P}(X), \subseteq)$ ordered by inclusion

Partitions: Refinement order on partitions

Subgroups: Subgroups of a group under inclusion

1.3.3. Divisibility

On positive integers: $a \mid b$ if a divides b
- Not antisymmetric on $\mathbb{Z}$ (consider ± 1)
- Forms a partial order on $\mathbb{N}^+$

1.4. Special Elements

1.4.1. Bounds

For subset $S \subset P$:

Upper bound: $u \in P$ where $s \leq u$ for all $s \in S$

Lower bound: $\ell \in P$ where $\ell \leq s$ for all $s \in S$

Supremum (least upper bound): $\sup S$ is the least upper bound

Infimum (greatest lower bound): $\inf S$ is the greatest lower bound

1.4.2. Extremal Elements

Maximum: $\max S \in S$ where $s \leq \max S$ for all $s \in S$

Minimum: $\min S \in S$ where $\min S \leq s$ for all $s \in S$

Maximal element: $m \in S$ where no $s \in S$ has $m < s$

Minimal element: $m \in S$ where no $s \in S$ has $s < m$

Note: Maximum implies maximal, but not conversely in general posets.

1.5. Chains and Antichains

1.5.1. Chain

A **chain** is a totally ordered subset: C subset P "where" ∀a, b in C, a ≤ b ∨ b ≤ a

The **height** of P is the supremum of chain lengths.

1.5.2. Antichain

An **antichain** is a set of mutually incomparable elements: A subset P "where" ∀a, b in A, a ≠ b arrow.r.double a ≰ b ∧ b ≰ a

The **width** of P is the supremum of antichain sizes.

1.5.3. Dilworth's Theorem

For finite posets: "width"(P) = "minimum number of chains covering" P

1.6. Total Orders

1.6.1. Definition

A **total order** (or linear order) satisfies: ∀a, b in P, a ≤ b ∨ b ≤ a

Every pair is comparable.

1.6.2. Properties

In a total order:
- Every finite subset has a maximum and minimum
- Maximal = maximum, minimal = minimum
- Every subset is a chain

1.6.3. Well-Orders

A **well-order** is a total order where every non-empty subset has a minimum.

Examples:
- $(\mathbb{N}, \leq)$ is well-ordered
- $(\mathbb{Z}, \leq)$ is not well-ordered
- Every finite total order is well-ordered

1.7. Lattices

1.7.1. Definition

A **lattice** is a poset where every pair $\{a, b\}$ has:

- A supremum (join): $a \vee b$
- An infimum (meet): $a \wedge b$

1.7.2. Properties

Lattice operations satisfy:

- **Commutativity**: $a \vee b = b \vee a$, $a \wedge b = b \wedge a$
- **Associativity**: $(a \vee b) \vee c = a \vee (b \vee c)$
- **Absorption**: $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$
- **Idempotence**: $a \vee a = a$, $a \wedge a = a$

1.7.3. Examples

Power set lattice: $(\mathcal{P}(X), \subseteq)$ with $\cup$ and $\cap$

Divisibility lattice: Positive integers with gcd and lcm

Subspace lattice: Subspaces of a vector space

Partition lattice: Partitions under refinement

1.8. Complete Lattices

1.8.1. Definition

A **complete lattice** is a poset where every subset has a supremum and infimum.

Equivalently: Every subset has a supremum (infima follow by duality).

1.8.2. Examples

Power set: $(\mathcal{P}(X), \subseteq)$ is complete

Extended reals: $[-\infty, +\infty]$ with usual order

Closed sets: Closed subsets of a topological space

Complete Boolean algebras: Used in forcing

1.8.3. Fixed Point Theorems

Knaster-Tarski: Every monotone function on a complete lattice has a fixed point.

The set of fixed points forms a complete lattice.

1.9. Distributive Lattices

1.9.1. Definition

A lattice is **distributive** if: a ∧ (b ∨ c) = (a ∧ b) ∨ (a ∧ c)

Equivalently: a ∨ (b ∧ c) = (a ∨ b) ∧ (a ∨ c)

1.9.2. Characterization

A lattice is distributive if and only if it contains no sublattices isomorphic to:

- M_3: Three-element diamond
- N_5: Five-element pentagon

1.9.3. Examples

Boolean algebras: All Boolean algebras are distributive

Linear orders: Every totally ordered set forms a distributive lattice

Young's lattice: Partitions under dominance (not distributive)

1.10. Boolean Algebras

1.10.1. Definition

A **Boolean algebra** is a distributive lattice with:
- Top element: 1
- Bottom element: 0
- Complement: For each a, exists $\neg a$ where:
  - $a \wedge \neg a = 0$
  - $a \vee \neg a = 1$

1.10.2. Properties

Boolean algebras satisfy:
- **De Morgan's laws**: $\neg(a \wedge b) = \neg a \vee \neg b$
- **Double negation**: $\neg\neg a = a$
- **Distributivity**: Both join and meet distribute

1.10.3. Stone's Theorem

Every Boolean algebra is isomorphic to a field of sets.

Every finite Boolean algebra has 2^n elements for some n.

1.11. Order-Preserving Maps

1.11.1. Monotone Functions

A function $f : P \to Q$ is **monotone** (order-preserving) if: a ≤ b arrow.r.double f(a) ≤ f(b)

Antitone (order-reversing) if: a ≤ b arrow.r.double f(b) ≤ f(a)

1.11.2. Order Isomorphism

An **order isomorphism** is a bijection $f : P \to Q$ where: a ≤ b arrow.l.r.double f(a) ≤ f(b)

Order-isomorphic posets have identical order structure.

1.11.3. Galois Connections

A **Galois connection** between posets P and Q consists of:
- $f : P \to Q$ and $g : Q \to P$
- $f(p) \leq q \Leftrightarrow p \leq g(q)$

Properties:
- f and g are monotone
- $p \leq g(f(p))$ and $f(g(q)) \leq q$
- $f \circ g \circ f = f$ and $g \circ f \circ g = g$

1.12. Closure Operators

1.12.1. Definition

A **closure operator** on poset P is a function $c : P \to P$ that is:
- **Extensive**: $x \leq c(x)$
- **Monotone**: $x \leq y \Rightarrow c(x) \leq c(y)$
- **Idempotent**: $c(c(x)) = c(x)$

1.12.2. Examples

Topological closure: Closure in topological spaces

Span: Linear span in vector spaces

Transitive closure: In relations

Convex hull: In real vector spaces

1.12.3. Closed Elements

An element x is **closed** if $c(x) = x$.

The closed elements form a complete lattice.

1.13. Zorn's Lemma

1.13.1. Statement

If every chain in a poset has an upper bound, then the poset has a maximal element.

1.13.2. Applications

Basis existence: Every vector space has a basis

Maximal ideals: Every proper ideal extends to a maximal ideal

Well-ordering: Every set can be well-ordered

Hahn-Banach: Linear functionals extend to the whole space

1.14. Order Dimension

1.14.1. Definition

The **order dimension** of P is the minimum number of linear orders whose intersection gives P.

1.14.2. Properties

- Linear orders have dimension 1
- Standard example: $(\mathbb{N}^n, \leq)$ has dimension n
- Computing dimension is NP-complete

1.15. Applications

1.15.1. Computer Science

Sorting algorithms: Based on comparing elements

Lattice-based cryptography: Security from lattice problems

Program analysis: Abstract interpretation uses lattices

Databases: Query optimization with partial orders

1.15.2. Algebra

Ideal lattices: Ideals form complete lattices

Subgroup lattices: Reveal group structure

Congruence lattices: Universal algebra

1.15.3. Topology

Specialization order: Points in topological spaces

Alexandrov topology: One-to-one with preorders

Continuous lattices: Domain theory

1.16. Special Topics

1.16.1. Interval Orders

Orders representable by intervals on the real line.

Forbidden: $2 + 2$ as induced subposet.

1.16.2. Bruhat Order

On symmetric group: Related to reduced decompositions.

On Weyl groups: Geometry of flag varieties.

1.16.3. Tamari Lattice

On binary trees or parenthesizations.

Related to associahedra and Catalan numbers.

1.17. Implementation Notes

1.17.1. Mathlib Structure

```
class Preorder (α : Type*) extends LE α, LT α where
  le_refl : ∀ a : α, a ≤ a
  le_trans : ∀ a b c : α, a ≤ b → b ≤ c → a ≤ c
  lt_iff_le_not_le : ∀ a b : α, a < b ↔ a ≤ b ∧ ¬b ≤ a

class PartialOrder (α : Type*) extends Preorder α where
  le_antisymm : ∀ a b : α, a ≤ b → b ≤ a → a = b
```

1.17.2. Lattice Operations

```
class Lattice (α : Type*) extends PartialOrder α, Sup α, Inf α where
  sup_le : ∀ a b c : α, a ≤ c → b ≤ c → a ⊔ b ≤ c
  le_sup_left : ∀ a b : α, a ≤ a ⊔ b
  le_sup_right : ∀ a b : α, b ≤ a ⊔ b
  -- dual for inf
```

1.18. Historical Context

Order theory developed through:

- **Dedekind (1897)**: Lattices from ideals
- **Birkhoff (1940)**: "Lattice Theory" systematization
- **Stone (1936)**: Stone duality for Boolean algebras
- **Tarski (1955)**: Fixed point theorems
- **Dilworth (1950)**: Chain decomposition theorem
- **Scott (1970s)**: Domain theory for computation

Modern applications:
- Formal concept analysis
- Quantum logic
- Rough set theory
- Preference modeling