

1. Field Basic Lemmas

1.1. Overview

Lemmas about division (semi)rings and (semi)fields, focusing on division and fraction operations.

1.2. Division Semiring

1.2.1. Addition and Division

$$\text{add_div: } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

- Division distributes over addition

$$\text{div_add_div_same: } \frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

- Common denominator addition

$$\text{same_add_div, div_add_same: For } b \neq 0$$

- $\frac{b+a}{b} = 1 + \frac{a}{b}$
- $\frac{a+b}{b} = \frac{a}{b} + 1$

1.2.2. Inverse Addition

$$\text{inv_add_inv': For } a, b \neq 0$$

- $a^{\{-1\}} + b^{\{-1\}} = a^{\{-1\}} * (a + b) * b^{\{-1\}}$
- See `inv_add_inv` for commutative version

$$\text{add_div', div_add': Mixed addition with division}$$

- $b + \frac{a}{c} = \frac{b*c+a}{c}$ for $c \neq 0$
- $\frac{a}{c} + b = \frac{a+b*c}{c}$ for $c \neq 0$

1.2.3. Commutative Versions

$$\text{Commute.div_add_div: If } b \text{ commutes with } c \text{ and } d, \text{ and } b, d \neq 0$$

$$\bullet \frac{a}{b} + \frac{c}{d} = \frac{a*d+b*c}{b*d}$$

$$\text{Commute.inv_add_inv: If } a \text{ commutes with } b \text{ and both nonzero}$$

$$\bullet a^{\{-1\}} + b^{\{-1\}} = \frac{a+b}{a*b}$$

1.2.4. Halves

$$\text{For } [\text{NeZero } (2 : K)]:$$

$$\text{add_self_div_two: } \frac{a+a}{2} = a$$

$$\text{add_halves: } \frac{a}{2} + \frac{a}{2} = a$$

1.3. Division Ring

1.3.1. Negative Division

$$\text{div_neg_self: } \frac{a}{-a} = -1 \text{ for } a \neq 0$$

$$\text{neg_div_self: } \frac{-a}{a} = -1 \text{ for } a \neq 0$$

1.3.2. Subtraction and Division

$$\text{div_sub_div_same: } \frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$$

- Division distributes over subtraction

$$\text{same_sub_div: } \frac{b-a}{b} = 1 - \frac{a}{b} \text{ for } b \neq 0$$

$$\text{div_sub_same: } \frac{a-b}{b} = \frac{a}{b} - 1 \text{ for } b \neq 0$$

$$\text{sub_div: } \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$$

1.3.3. Inverse Subtraction

`inv_sub_inv'`: For $a, b \neq 0$

- $a^{\{-1\}} - b^{\{-1\}} = a^{\{-1\}} * (b - a) * b^{\{-1\}}$

`Commute.inv_sub_inv`: If a commutes with b and both nonzero

- $a^{\{-1\}} - b^{\{-1\}} = \frac{b-a}{a*b}$

1.3.4. Half Subtraction

For `[NeZero (2 : K)]`:

`sub_half`: $a - \frac{a}{2} = \frac{a}{2}$

`half_sub`: $\frac{a}{2} - a = -(\frac{a}{2})$

1.3.5. Domain Instance

`DivisionRing.isDomain`: Every division ring is an integral domain

- No zero divisors
- Priority 100 to allow more specific instances

1.4. Semifield

1.4.1. General Division Addition

`div_add_div`: For $b, d \neq 0$

- $\frac{a}{b} + \frac{c}{d} = \frac{a*d+b*c}{b*d}$

- Uses commutativity

`one_div_add_one_div`: For $a, b \neq 0$

- $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{a*b}$

`inv_add_inv`: For $a, b \neq 0$

- $a^{\{-1\}} + b^{\{-1\}} = \frac{a+b}{a*b}$

- Simpler than `inv_add_inv'` due to commutativity

1.5. Implementation Notes

- `@[field_simps]` attribute for simplification in field contexts
- `Commute` versions handle non-commutative cases
- `NeZero` typeclass for characteristic constraints
- Priority settings for instance resolution