1. Fourier Transform

1.1. Overview

This module establishes the Fourier transform for complex-valued functions on finite-dimensional spaces. The framework is highly general, supporting arbitrary rings with measures and bilinear forms, while also providing specialized notation for the familiar real and complex cases.

1.2. General Framework (VectorFourier)

1.2.1. Setup

The general Fourier transform is defined with:
- $\Bbbk$: A commutative ring
- V, W: Modules over $\Bbbk$ (source and target spaces)
- e: An additive character $\Bbbk \to$ Circle (unitary character)
- μ: A measure on V
- L: A bilinear form $V \times W \to \Bbbk$
- E: A complete normed $\mathbb{C}$-vector space (for values)

1.2.2. Definition

The Fourier integral transforms $f : V \to E$ to a function $W \to E$: "fourierIntegral"(e, μ, L, f)(w) = integral_V e(-L(v, w)) dot f(v) space d μ(v)

This general definition encompasses:
- Classical Fourier transform (when W is dual of V)
- Fourier transform on inner product spaces
- Discrete and continuous variants

1.3. Special Cases

1.3.1. Scalar Case (Namespace Fourier)

When $V = W = \Bbbk$ and L is multiplication: hat(f)(ξ) = integral e(-x ξ) f(x) space d μ(x)

1.3.2. Real Fourier Transform

The most familiar case: $V = W = \mathbb{R}$, with:
- Character: $e(x) = \exp(2\pi i x)$ (denoted $\boldsymbol{e}$)
- Measure: Lebesgue measure
- Transform: $\mathcal{F} f(\xi) = \int \exp(-2\pi i x \xi) f(x)\ dx$

1.3.3. Inner Product Spaces

For $V = W$ an inner product space over $\mathbb{R}$:
- Bilinear form: $L(v, w) = \langle v, w \rangle$
- Notation: $\mathcal{F}$ for transform, $\mathcal{F}^{-1} f(v) = \mathcal{F} f(-v)$ for inverse

1.4. Key Properties

1.4.1. Norm Bounds

The Fourier transform is bounded: ‖"fourierIntegral"(e, μ, L, f)(w)‖ ≤ integral_V ‖f(v)‖ space d μ(v)

This shows the transform maps L^1 functions to bounded functions.

1.4.2. Translation Formula

Right translation becomes phase multiplication: cal(F)(f compose (+v_0))(w) = e(L(v_0, w)) dot cal(F) f(w)

This is the fundamental translation-modulation duality.

1.4.3. Linearity

The Fourier transform is linear:

- $\mathcal{F}(f+g) = \mathcal{F}f + \mathcal{F}g$
- $\mathcal{F}(c \cdot f) = c \cdot \mathcal{F}f$ for $c \in \mathbb{C}$

1.4.4. Continuity

For integrable f, the Fourier transform $\mathcal{F}f$ is continuous.

Requirements:

- e is continuous
- L is continuous
- W has first-countable topology

1.5. Self-Adjointness

1.5.1. Fubini's Theorem Application

The Fourier transform satisfies a self-adjointness property: integral_W M(cal(F) f(ξ), g(ξ)) space d ν(ξ) = integral_V M(f(x), cal(F) g(x)) space d μ(x)

where $M : E \times F \to G$ is a continuous bilinear form.

Special case for inner products: angle.l cal(F) f, g angle.r = angle.l f, cal(F) g angle.r

This shows the Fourier transform is its own adjoint (up to normalization).

1.6. Convergence

1.6.1. Integrability Criterion

The Fourier integral converges if and only if f is integrable: "Integrable"(v ↦ e(-L(v, w)) dot f(v)) ⟺ "Integrable"(f)

This equivalence holds for each fixed $w \in W$.

1.6.2. Dominated Convergence

The continuity proof uses dominated convergence:

- Pointwise limit exists for each w
- Dominated by $\|f\|$, which is integrable
- Character e is continuous

1.7. Applications

1.7.1. Signal Processing

- Time-frequency duality
- Spectral analysis
- Filter design via convolution theorem

1.7.2. PDEs

- Heat equation: Solutions via Fourier transform
- Wave equation: Dispersion relations
- Schrödinger equation: Momentum representation

1.7.3. Harmonic Analysis

- Plancherel theorem (in extended modules)
- Pontryagin duality
- Representation theory of locally compact groups

1.7.4. Probability

- Characteristic functions
- Central limit theorem via Fourier methods
- Lévy processes

1.8. Design Choices

1.8.1. Generality vs Familiarity

The module balances:

- Maximum generality (arbitrary rings and measures)
- Familiar special cases (real/complex with standard choices)
- Clean notation for common uses

1.8.2. Character Convention

Using additive characters $e : \Bbbk \to \mathrm{Circle}$ provides:

- Uniform treatment of discrete/continuous cases
- Natural connection to Pontryagin duality
- Clean formulation of periodicity

1.8.3. Bilinear Form Flexibility

The bilinear form L allows:

- Standard Fourier transform (multiplication)
- Fourier transform on groups (pairing with dual)
- Fourier-Stieltjes transform
- Fractional Fourier transform

1.9. Future Development

The module sets foundations for:

- Fourier inversion theorem
- Plancherel/Parseval theorems
- Convolution theorem
- Uncertainty principles
- Fourier series as special case
- Tempered distributions