1. Measure Spaces

1.1. Overview

Measure theory provides the rigorous foundation for integration, probability, and analysis. A measure assigns sizes to sets in a way that generalizes length, area, volume, and probability. This module establishes the fundamental theory of measures and measure spaces in Mathlib.

1.2. Core Concepts

1.2.1. Measures

A measure on a measurable space $(\alpha, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ satisfying:
1. Empty set has measure zero: $\mu(\emptyset) = 0$
2. Countable additivity: For pairwise disjoint measurable sets $(E_i)_{\{i \in \mathbb{N}\}}$: μ(⋃{i=1}^∞ E_i) = Σ{i=1}^∞ μ(E_i)

In Mathlib, measures are implemented as outer measures that are countably additive on measurable sets.

1.2.2. Outer Measures

Every measure extends to an outer measure μ^* defined on all subsets:
- $\mu^*(A) = \inf\{\mu(E) : E \supseteq A, E \text{ measurable}\}$
- Provides values for non-measurable sets
- Satisfies countable subadditivity for all sets

1.2.3. Null Sets and Almost Everywhere

Null sets are sets with measure zero:
- N is null if $\mu^*(N) = 0$ (may not be measurable)
- Property holds almost everywhere (a.e.) if it holds except on a null set
- Notation: $\forall^\mu x, P(x)$ means "$P(x)$ holds for μ-almost all x"

1.3. Main Types and Structures

1.3.1. Basic Types

- `Measure α`: Type of measures on α
- `MeasureSpace α`: Class providing a canonical measure called `volume`
- `OuterMeasure α`: Outer measures on α

1.3.2. Key Definitions

Measure from Measurable Sets (`Measure.ofMeasurable`): Define a measure by specifying values only on measurable sets and proving:
- Empty set has measure zero
- Countable additivity for disjoint measurable sets

Measure from Outer Measure (`OuterMeasure.toMeasure`): Convert an outer measure to a measure by verifying all measurable sets are Carathéodory measurable.

1.3.3. Complete Measures

The completion of a measure includes all null-measurable sets:
- A set is null-measurable if it differs from a measurable set by a null set
- Completion is the smallest σ-algebra containing original measurable sets and null sets
- Completed measure assigns 0 to null sets

1.4. Key Theorems

1.4.1. Measure Properties

Monotonicity: If $s \subseteq t$ then $\mu(s) \le \mu(t)$

Countable Subadditivity: For any sequence (s_i): μ(⋃*i s_i) ≤ Σ*i μ(s_i)

Continuity from Below: For increasing (s_n) with $s_n \uparrow s$: μ(s) = lim μ(s_n) space "as" space n ⟶ ∞

Continuity from Above: For decreasing (s_n) with $s_n \downarrow s$ and $\mu(s_1) < \infty$: μ(s) = lim μ(s_n) space "as" space n ⟶ ∞

1.4.2. Operations on Sets

Union-Intersection Formula (`measure_union_add_inter`): For measurable t: μ(s ∪ t) + μ(s ∩ t) = μ(s) + μ(t)

Difference Formula (`measure_diff`): For $s_2 \subseteq s_1$, measurable s_2, $\mu(s_2) < \infty$: μ(s_1 ∖ s_2) = μ(s_1) - μ(s_2)

Symmetric Difference (`measure_symmDiff_eq`): For null-measurable s, t: μ(s Δ t) = μ(s ∖ t) + μ(t ∖ s)

1.4.3. Disjoint Unions

Countable Disjoint Union (`measure_biUnion`): For countable S, pairwise disjoint measurable $(f_i)_{\{i \in S\}}$: μ(⋃*{i in S} f_i) = Σ*{i in S} μ(f_i)

Finite Disjoint Union (`measure_biUnion_finset`): For finite set s, pairwise disjoint measurable $(f_i)_{\{i \in s\}}$: μ(⋃*{i in s} f_i) = Σ*{i in s} μ(f_i)

Uncountable Union Inequality (`tsum_meas_le_meas_iUnion_of_disjoint`): Even for uncountable index sets with pairwise disjoint measurable sets: Σ*i μ(A_i) ≤ μ(*⋃i A_i)

1.4.4. Preimages and Fibers

Singleton Preimage Sum (`tsum_measure_preimage_singleton`): For countable S and measurable fibers: μ(f^{-1}(S)) = Σ "over" y in S : μ(f^{-1}({y}))

Finite Singleton Sum (`sum_measure_singleton`): In spaces with measurable singletons: μ(s) = Σ "over" x in s : μ({x})

1.5. Measure Extension Theorems

1.5.1. π-λ Theorem Applications

Extension from π-System (`ext_of_generateFrom_of_iUnion`): Two measures are equal if they agree on a π-system that:
- Generates the σ-algebra
- Contains an increasing sequence covering the space
- Takes finite values on the sequence

Finite Measure Extension (`ext_of_generate_finite`): Two finite measures are equal if they agree on a π-system generating the σ-algebra.

1.5.2. Uniqueness Criteria

Measurable Sets (`ext`): Two measures are equal iff they agree on all measurable sets.

σ-Finite Case: For σ-finite measures, equality on a generating π-system implies equality everywhere.

1.6. Special Properties

1.6.1. Almost Everywhere Properties

AE Filter (`ae`): The filter of sets whose complements have measure zero:
- $s \in \mathrm{ae}(\mu) \Longleftrightarrow \mu(s^c) = 0$
- Properties hold a.e. if they belong to this filter

AE Disjoint (`AEDisjoint`): Sets are a.e. disjoint if their intersection is null:
- Allows measure additivity even with negligible overlap

1.6.2. Null-Measurable Sets

A set is null-measurable if it equals a measurable set up to a null set:
- Broader than measurable sets
- Closed under measure-preserving operations
- Used in integration theory

1.7. Lattice Structure

Measures form a complete lattice:
- Supremum: $\sup_i \mu_i$ is the smallest measure dominating all μ_i
- Infimum: Defined via outer measures
- Scalar multiplication: $(c \cdot \mu)(s) = c \cdot \mu(s)$ for $c \in [0, \infty]$

1.8. MeasureSpace Class

The `MeasureSpace` class provides:
- A measurable space structure
- A canonical measure called `volume`
- Used for spaces with a natural measure (e.g., Lebesgue measure on $\mathbb{R}^n$)

1.9. Implementation Notes

1.9.1. Design Choices

1. Outer Measure Extension: Measures automatically extend to outer measures
   - Allows applying measures to non-measurable sets
   - Simplifies many proofs

2. Null-Measurable Focus: Many theorems work with null-measurable sets
   - More general than just measurable sets
   - Natural for integration theory

3. Constructor Alternatives:
   - `Measure.ofMeasurable`: Define from measurable sets
   - `OuterMeasure.toMeasure`: Convert from outer measure
   - Direct construction rarely used

1.9.2. Proof Techniques

Measure Equality:
1. Show equality on measurable sets (`ext`)
2. Use π-λ theorem for generating sets
3. For finite measures, use `ext_of_generate_finite`

Null Set Arguments:
- Work modulo null sets using a.e. equivalence
- Extend from measurable to null-measurable

1.10. Applications

1.10.1. Integration Theory

- Measures provide the foundation for the Lebesgue integral
- Null sets determine when functions are equal a.e.
- Completion ensures integrability is preserved under a.e. modifications

1.10.2. Probability Theory

- Probability measures satisfy $\mu(\Omega) = 1$
- Events with probability 0 occur “almost never”
- Independence defined via product measures

1.10.3. Functional Analysis

- Measures on function spaces (e.g., Wiener measure)
- Radon measures on locally compact spaces
- Spectral measures in operator theory

1.10.4. Ergodic Theory

- Invariant measures for dynamical systems
- Birkhoff's ergodic theorem
- Entropy via measure-theoretic techniques

1.11. Historical Context

Measure theory was developed to address limitations of the Riemann integral:

- Lebesgue (1902): Introduced measure and integration
- Carathéodory (1914): Outer measure approach
- Completion: Ensures nice properties for integration

The theory unifies:

- Length, area, volume in geometry
- Probability in stochastics
- Integration in analysis