# 1. Hahn-Banach Extension Theorem

## 1.1. Overview

The Hahn-Banach theorem is one of the cornerstones of functional analysis. It guarantees that continuous linear functionals defined on subspaces can be extended to the entire space while preserving their norm. This result has profound implications for duality theory, separation theorems, and the existence of supporting hyperplanes.

## 1.2. Main Results

### 1.2.1. Real Version

For normed spaces over $\mathbb{R}$: Theorem (`Real.exists_extension_norm_eq`): Let $p$ be a subspace of a normed space $E$ over $\mathbb{R}$, and let $f : p \to \mathbb{R}$ be a continuous linear functional. Then there exists a continuous linear functional $g : E \to \mathbb{R}$ such that:

1. $g|_p = f$ (extension property)
2. $\|g\| = \|f\|$ (norm preservation)

### 1.2.2. Complex Version

For normed spaces over $\mathbb{C}$ (or more generally, $\mathbb{k}$ satisfying `IsRCLikeNormedField`): Theorem (`exists_extension_norm_eq`): The same extension property holds for complex-valued functionals, with the same norm preservation.

The proof strategy:

1. Decompose the complex functional into real and imaginary parts
2. Apply the real Hahn-Banach to the real part
3. Reconstruct the complex extension using the formula: g(x) = g_ℝ(x) - i dot g_ℝ(i x)

## 1.3. Key Corollaries

### 1.3.1. Existence of Dual Vectors

Theorem (`exists_dual_vector`): For any nonzero vector $x \in E$, there exists a continuous linear functional $g$ with:

- $\|g\| = 1$
- $g(x) = \|x\|$

This shows that the dual space "sees" every direction in the primal space.

Variants:

- `exists_dual_vector'`: Works even when $x = 0$ (in nontrivial spaces)
- `exists_dual_vector''`: Ensures $\|g\| \leq 1$ (works in all spaces)

### 1.3.2. Finite-Dimensional Range Extensions

Theorem (`ContinuousLinearMap.exist_extension_of_finiteDimensional_range`): If $f : p \to F$ is a continuous linear map from a submodule to a finite-dimensional space $F$, then $f$ admits an extension to all of $E$.

Note: Unlike the scalar case, this doesn't guarantee norm preservation for vector-valued maps.

### 1.3.3. Closed Complementation

Theorem (`Submodule.ClosedComplemented.of_finiteDimensional`): Every finite-dimensional submodule of a normed space over $\mathbb{R}$ or $\mathbb{C}$ is closed and complemented.

This means we can write $E = p \oplus q$ topologically for finite-dimensional $p$.

## 1.4. Proof Strategy

### 1.4.1. Real Case

The proof uses the extension theorem for sublinear functionals:

1. Define the sublinear functional $p(x) = \|f\| \cdot \|x\|$
2. Verify that $f(x) \leq p(x)$ on the subspace
3. Apply the algebraic Hahn-Banach theorem
4. Show the extension has the same norm

### 1.4.2. Complex Case

The complex case reduces to the real case via:

1. Restriction: View the complex space as a real space
2. Real Extension: Apply real Hahn-Banach to $\mathrm{Re}(f)$
3. Complexification: Recover the complex extension using: g(x) = g_ℝ(x) - i dot g_ℝ(i x)
4. Norm Equality: Verify $\|g\| = \|f\|$ using properties of the real part

## 1.5. Technical Details

### 1.5.1. RCLike Fields

The module works with fields 𝕜 satisfying `RCLike 𝕜`, which includes:

- ℝ (real numbers)
- ℂ (complex numbers)
- Any field with conjugation and norm satisfying appropriate axioms

### 1.5.2. Strong Dual

The notation `StrongDual 𝕜 E` refers to the space of continuous linear functionals $E \to 𝕜$ with the operator norm.

### 1.5.3. Sublinear Functionals

A functional $p : E \to \mathbb{R}$ is sublinear if:

- $p(cx) = c \cdot p(x)$ for $c \geq 0$
- $p(x + y) \leq p(x) + p(y)$

The norm $\| \cdot \|$ is the canonical example.

## 1.6. Applications

### 1.6.1. Separation Theorems

Hahn-Banach leads to separation of convex sets:

- Strict separation of disjoint convex sets
- Supporting hyperplanes for convex sets
- Farkas' lemma and linear programming duality

### 1.6.2. Dual Space Theory

- Reflexivity: Characterization via James' theorem
- Weak Topologies: Weak and weak-star convergence
- Bidual Embedding: Natural embedding $E \to E^{\{**\}}$

### 1.6.3. Optimization

- Lagrange Multipliers: Existence in infinite dimensions
- Subdifferentials: Existence of subgradients
- Convex Duality: Fenchel-Legendre transform

#### 1.6.4. Operator Theory

- Closed Range Theorem: Characterization via annihilators
- Fredholm Theory: Index calculations
- Spectral Theory: Functional calculus

### 1.7. Geometric Interpretation

The Hahn-Banach theorem can be viewed geometrically:

1. Supporting Hyperplanes: Every point on the boundary of a convex set has a supporting hyperplane
2. Separation: Disjoint convex sets can be separated by hyperplanes
3. Dual Characterization: The norm of $x$ equals the supremum of $|f(x)|$ over unit functionals

### 1.8. Historical Note

The theorem was proved independently by:

- Hans Hahn (1927) for the real case
- Stefan Banach (1929/1932) for the general case

It marked a turning point in functional analysis, providing the tools to extend finite-dimensional intuition to infinite dimensions.

### 1.9. Design Choices

The formalization makes several design choices:

1. Unified Treatment: Using `RCLike` to handle $\mathbb{R}$ and $\mathbb{C}$ uniformly
2. Norm Preservation: Explicit equality rather than just bounds
3. Constructive Elements: Providing explicit extensions via `extendTo𝕜`
4. Submodule Focus: Working with submodules rather than general subsets