1. Inner Product Spaces - Basic Properties

1.1. Overview

This module establishes fundamental properties of inner product spaces over real or complex fields. Inner product spaces generalize the notion of angle and length from Euclidean spaces to arbitrary vector spaces, providing the foundation for Hilbert space theory and functional analysis.

1.2. Core Properties

1.2.1. Conjugate Symmetry

The inner product satisfies conjugate symmetry: angle.l y, x angle.r^† = angle.l x, y angle.r

For real inner products, this reduces to commutativity: angle.l y, x angle.r_ℝ = angle.l x, y angle.r_ℝ

1.2.2. Self-Inner Product

For any vector x:
- The imaginary part vanishes: $\mathrm{Im}\ \langle x, x\rangle = 0$
- The value is always real and non-negative

1.3. Linearity Properties

1.3.1. Additivity

The inner product is additive in both arguments:
- Left additivity: $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
- Right additivity: $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$

1.3.2. Scalar Multiplication

For scalar multiplication by $r \in \Bbbk$:
- Left: $\langle r \cdot x, y\rangle = r^{\dagger} \cdot \langle x, y\rangle$
- Right: $\langle x, r \cdot y\rangle = r \cdot \langle x, y\rangle$

For real scalars $r \in \mathbb{R}$:
- Both sides: $\langle r \cdot x, y\rangle = \langle x, r \cdot y\rangle = r \cdot \langle x, y\rangle$

1.3.3. General Algebra Actions

For a commutative semiring 𝕝 acting on the space:
- With star structure: $\langle r \cdot x, y\rangle = r^{\dagger} \cdot \langle x, y\rangle$
- Trivial star (e.g., $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$): $\langle r \cdot x, y\rangle = r \cdot \langle x, y\rangle$

1.4. Symmetry Properties

1.4.1. Real and Imaginary Parts

- Real part symmetry: $\mathrm{Re}\ \langle x, y\rangle = \mathrm{Re}\ \langle y, x\rangle$
- Imaginary part antisymmetry: $\mathrm{Im}\ \langle x, y\rangle = -\mathrm{Im}\ \langle y, x\rangle$

1.4.2. Zero Inner Product

The condition $\langle x, y\rangle = 0$ is symmetric: angle.l x, y angle.r = 0 ⟺ angle.l y, x angle.r = 0

1.5. Sesquilinear and Bilinear Forms

1.5.1. Sesquilinear Form

The inner product defines a sesquilinear form: “sesqFormOfInner” : E →𝕜 *E* →⋆[𝕜] 𝕜

This captures the conjugate-linearity in the first argument and linearity in the second.

1.5.2. Bilinear Form (Real Case)

For real inner product spaces, we get a bilinear form: “bilinFormOfRealInner” : “BilinForm” space ℝ space F

Note the argument order is preserved (unlike the sesquilinear form).

1.6. Summation Formulas

1.6.1. Finite Sums

Inner products distribute over finite sums:

- Left sum: $\langle \sum_{i \in s} f_i, x \rangle = \sum_{i \in s} \langle f_i, x \rangle$
- Right sum: $\langle x, \sum_{i \in s} f_i \rangle = \sum_{i \in s} \langle x, f_i \rangle$

1.6.2. Finsupp Sums

For finitely supported functions $l : \iota \underset{0}{\to} \Bbbk$: angle.l sum_i l_i dot v_i, x angle.r = sum_i “conj”(l_i) dot angle.l v_i, x angle.r

1.7. Main Theorems (Referenced)

1.7.1. Cauchy-Schwarz Inequality

The fundamental inequality (proved later in the file): |angle.l x, y angle.r|^2 ≤ angle.l x, x angle.r dot angle.l y, y angle.r

Equality holds if and only if x and y are linearly dependent.

1.7.2. Polarization Identity

Expresses inner product in terms of norms: angle.l x, y angle.r = frac(‖x + y‖^2 - ‖x - y‖^2 + i(‖x + i y‖^2 - ‖x - i y‖^2), 4)

For real spaces: angle.l x, y angle.r_ℝ = frac(‖x + y‖^2 - ‖x - y‖^2, 4)

1.8. Applications

These properties are fundamental for:

- **Hilbert space theory**: Complete inner product spaces
- **Orthogonality**: Perpendicular vectors and projections
- **Spectral theory**: Self-adjoint operators and eigenvalues
- **Quantum mechanics**: State spaces and observables
- **Signal processing**: Fourier analysis and filtering

1.9. Design Notes

The module uses `RCLike` to handle both real and complex cases uniformly. The notation $\langle x, y \rangle$ represents the inner product, with subscript ℝ for explicitly real inner products. The conjugate operation is denoted by † (dagger).

The sesquilinear form approach provides a clean categorical perspective on inner products, facilitating the development of abstract theory while maintaining computational convenience.