1. Fréchet Derivatives

1.1. Overview

The Fréchet derivative generalizes the notion of derivative to functions between normed vector spaces. It provides a linear approximation to a function at a point that is "best" in a precise sense. This foundational concept underlies differentiation in infinite-dimensional spaces and is essential for optimization, dynamical systems, and the calculus of variations.

1.2. Core Definitions

1.2.1. Fréchet Differentiability

A function $f : E \to F$ between normed spaces is Fréchet differentiable at $x \in E$ if there exists a continuous linear map $f'(x) : E \to F$ such that:

f(x + h) = f(x) + f'(x)(h) + o(‖h‖)

More precisely: (‖f(x + h) - f(x) - f'(x)(h)‖) / ‖h‖ ⟶ 0 space "as" space h ⟶ 0

The linear map $f'(x)$ is called the Fréchet derivative of f at x.

1.2.2. Differentiability Variants

Within a Set (`HasFDerivWithinAt`):
- f has derivative f' at x within set s
- The limit is taken as $h \to 0$ with $x + h \in s$
- Useful for functions defined on manifolds or with boundaries

At a Point (`HasFDerivAt`):
- f has derivative f' at x in the whole space
- Equivalent to `HasFDerivWithinAt f f' univ x`

Strict Differentiability (`HasStrictFDerivAt`):
- Stronger notion: $f(y) - f(z) - f'(y - z) = o(\|y - z\|)$ as $y, z \to x$
- Uniform approximation in a neighborhood
- Required for the inverse function theorem

1.3. Main Types and Notation

1.3.1. Type Classes

- `HasFDerivWithinAt f f' s x`: Derivative within a set
- `HasFDerivAt f f' x`: Derivative at a point
- `HasStrictFDerivAt f f' x`: Strict derivative
- `DifferentiableWithinAt 𝕜 f s x`: Existence of derivative within s
- `DifferentiableAt 𝕜 f x`: Existence of derivative
- `DifferentiableOn 𝕜 f s`: Differentiable on set s
- `Differentiable 𝕜 f`: Globally differentiable

1.3.2. Derivative Functions

- `fderivWithin 𝕜 f s x`: The derivative within s at x
- `fderiv 𝕜 f x`: The derivative at x
- These return `0` when the derivative doesn't exist

1.4. Key Properties

1.4.1. Uniqueness

Unique Differentiability (`UniqueDiffWithinAt`): A point x satisfies this if the tangent cone at x within s spans a dense subspace. This ensures:

- The derivative within s is unique when it exists
- Important for manifolds and constrained optimization

Uniqueness Theorem: If `HasFDerivWithinAt f f₁' s x` and `HasFDerivWithinAt f f₂' s x` and `UniqueDiffWithinAt 𝕜 s x`, then $f_1' = f_2'$.

1.4.2. Basic Properties

Continuity:
- Differentiability implies continuity
- `HasFDerivAt f f' x → ContinuousAt f x`

Linearity of Derivative:
- The derivative is linear in the function
- $(\alpha f + \beta g)' = \alpha f' + \beta g'$

Locality:
- Differentiability is a local property
- Depends only on behavior in a neighborhood

1.4.3. Convergence Characterization

Rescaling Lemma (`HasFDerivWithinAt.lim`): If f has derivative f' at x, and $c_n \to \infty$ while $c_n \cdot d_n \to v$, then: c_n · (f(x + d_n) - f(x)) ⟶ f'(v)

This characterizes the derivative through rescaled difference quotients.

1.5. Differentiation Rules

1.5.1. Elementary Functions

Constants: The derivative of a constant function is zero

Identity: The derivative of the identity map is the identity linear map

Linear Maps: A continuous linear map is its own derivative

1.5.2. Arithmetic Operations

Addition Rule: (f + g)'(x) = f'(x) + g'(x)

Scalar Multiplication: (c · f)'(x) = c · f'(x)

Subtraction: (f - g)'(x) = f'(x) - g'(x)

1.5.3. Product Rule

For scalar functions: (f · g)'(x) = f'(x) · g(x) + f(x) · g'(x)

For bilinear maps $B : E \times F \to G$: D(B(f, g))(x) = B(f'(x), g(x)) + B(f(x), g'(x))

1.5.4. Chain Rule

General Form: If $g : E \to F$ is differentiable at x and $f : F \to G$ is differentiable at $g(x)$: (f ∘ g)'(x) = f'(g(x)) ∘ g'(x)

Within Sets: The chain rule also holds for derivatives within sets, with appropriate domain conditions.

1.5.5. Inverse Function

If f has an inverse g near x, and $f'(x)$ is invertible: g'(f(x)) = (f'(x))^{-1}

1.6. Special Cases

1.6.1. One-Dimensional Derivatives

For $f : 𝕜 \to E$ where 𝕜 is the scalar field:

- The derivative can be identified with an element of E
- `deriv f x` denotes this scalar derivative
- Related by: `fderiv 𝕜 f x 1 = deriv f x`

1.6.2. Partial Derivatives

For $f : E_1 \times E_2 \to F$:

- Partial derivatives are projections of the total derivative
- `∂₁f(x₁, x₂) = fderiv 𝕜 f (x₁, x₂) ∘ inl`
- `∂₂f(x₁, x₂) = fderiv 𝕜 f (x₁, x₂) ∘ inr`

1.6.3. Directional Derivatives

The derivative in direction v: D_v f(x) = f'(x)(v)

Characterized by: D_v f(x) = lim "as" space t ⟶ 0 : space (f(x + tv) - f(x))/t

1.7. Automation and Tactics

1.7.1. Simplifier Setup

The simplifier can automatically prove differentiability:

```
example (x : ℝ) : Differentiable ℝ (fun x ↦ sin (exp (3 + x²)) - 5 * cos x) := by simp
```

For divisions, provide non-vanishing proofs:

```
example (x : ℝ) (h : 1 + sin x ≠ 0) :
  DifferentiableAt ℝ (fun x ↦ exp x / (1 + sin x)) x := by simp [h]
```

1.7.2. Lemma Organization

Tagged Lemmas:

- Basic operations tagged with `@[simp]`
- Composition handled specially to avoid matching issues
- Function-specific lemmas (e.g., `differentiable_exp`)

Naming Convention:

- `hasFDerivAt_*`: Existence of specific derivative
- `differentiable_*`: Differentiability assertions
- `fderiv_*`: Computing the derivative

1.8. Advanced Topics

1.8.1. Mean Value Theorems

Rolle's Theorem: For differentiable $f : [a, b] \to \mathbb{R}$ with $f(a) = f(b)$, there exists $c \in (a, b)$ with $f'(c) = 0$.

Mean Value Theorem: There exists $c \in (a, b)$ with: f(b) - f(a) = f'(c)(b - a)

Vector-Valued MVT: More subtle for vector-valued functions; involves integration or special estimates.

1.8.2. Taylor's Theorem

For sufficiently smooth f: f(x + h) = Σ "from" space k=0 space "to" space n : (1/k!) D^k f(x)(h^k) + R_n(h)

Where $R_n(h) = o(\|h\|^n)$ is the remainder.

1.8.3. Implicit Function Theorem

If $F : E \times F \to G$ satisfies:

1. $F(x_0, y_0) = 0$
2. $\partial_y F(x_0, y_0)$ is invertible
3. F is continuously differentiable

Then locally there exists $g : E \to F$ with $F(x, g(x)) = 0$ and: g'(x₀) = -(∂y *F(x₀, y₀))^{-1}* ∘ ∂x F(x₀, y₀)

1.9. Applications

1.9.1. Optimization

First-Order Conditions: At a local extremum x^* of differentiable f: f'(x^*) = 0

Second-Order Conditions: For twice differentiable f:
- Minimum: $f''(x^*)$ positive definite
- Maximum: $f''(x^*)$ negative definite

1.9.2. Differential Equations

The Fréchet derivative appears in:
- Linearization of nonlinear ODEs
- Variational equations
- Stability analysis via linearization

1.9.3. Functional Analysis

Calculus of Variations: Finding extrema of functionals uses Fréchet derivatives in function spaces.

Newton's Method in Banach spaces: Next iterate: $x' = x - [f'(x)]^{-1} f(x)$

1.10. Implementation Notes

1.10.1. Design Principles

1. Definitional Choices: Multiple equivalent definitions; chose for computational convenience
2. Bundled Derivatives: Derivatives are continuous linear maps, not just linear maps
3. Filters: Use filter language for flexibility in stating limits

1.10.2. File Organization

The theory is split across multiple files:
- `Defs.lean`: Basic definitions
- `Basic.lean`: Fundamental properties (this file)
- `Linear.lean`: Linear and bounded linear maps
- `Comp.lean`: Chain rule
- `Mul.lean`: Products and scalar multiplication
- Special functions in `SpecialFunctions/`

1.11. Historical Context

The Fréchet derivative was introduced by Maurice Fréchet in 1925, generalizing the derivative to infinite-dimensional spaces. It built upon earlier work by:
- Volterra (1887): Functional derivatives
- Gâteaux (1913): Directional derivatives in function spaces
- Fréchet (1925): The modern formulation

This concept enabled rigorous calculus in:
- Functional analysis
- Partial differential equations
- Infinite-dimensional optimization
- Modern differential geometry