1. Normed Spaces and Modules

1.1. Overview

This module defines normed spaces (vector spaces with compatible norms) and normed algebras over normed fields. The key structure is `NormedSpace`, which combines a module structure with a norm satisfying the fundamental scaling property $\|c \cdot x\| = \|c\| \cdot \|x\|$.

1.2. Main Definition

1.2.1. NormedSpace

A normed space over a normed field 𝕜 is a vector space E with a norm satisfying: ‖a dot b‖ ≤ ‖a‖ dot ‖b‖

In fact, equality always holds (proved as `norm_smul`), but we only require the inequality in the definition for flexibility.

Key insight: This works for "semi-normed spaces" too, since we only require `SeminormedAddCommGroup` rather than `NormedAddCommGroup`.

1.3. Core Properties

1.3.1. Scalar Multiplication and Norms

The fundamental property of normed spaces:
- $\|c \cdot x\| = \|c\| \cdot \|x\|$ (exact equality, though only $\leq$ is required)
- $\|n \cdot x\| = |n| \cdot \|x\|$ for integer scalars $n \in \mathbb{Z}$
- $\|n \cdot x\| = n \cdot \|x\|$ for natural scalars $n \in \mathbb{N}$

1.3.2. Continuity Properties

Scalar multiplication interacts continuously with limits:
- If $f_n \to 0$ and $\|g_n\|$ is bounded, then $f_n \cdot g_n \to 0$
- If $\|f_n\|$ is bounded and $g_n \to 0$, then $f_n \cdot g_n \to 0$

These properties are essential for analysis in normed spaces.

1.4. Standard Instances

1.4.1. Field as Normed Space

Every normed field 𝕜 is naturally a normed space over itself: ‖a dot b‖ = ‖a b‖ = ‖a‖ dot ‖b‖

1.4.2. Product Spaces

The product $E \times F$ of normed spaces is a normed space with the sup norm: ‖(x, y)‖ = max(‖x‖, ‖y‖) ‖c dot (x, y)‖ = (‖c dot x‖, ‖c dot y‖) = ‖c‖ dot max(‖x‖, ‖y‖)

1.4.3. Pi Types

For finitely many normed spaces E_i, the product $\prod_i E_i$ is a normed space: ‖f‖ = sup_i ‖f_i‖ This generalizes the finite product case.

1.4.4. Subspaces

Every submodule of a normed space inherits the normed space structure:
- The norm is simply restricted from the ambient space
- The scaling property is inherited automatically

1.4.5. Induced Structures

Given a linear map $f : E \to G$ where G is a normed space, we can induce a normed space structure on E via: ‖x‖*E = ‖f(x)‖*G

1.5. Special Structures

1.5.1. Opposite Spaces

The multiplicative opposite E^{op} of a normed space is itself a normed space with the same norm.

1.5.2. Separation Quotient

The separation quotient (identifying points at distance 0) preserves the normed space structure.

1.5.3. ULift

The universe lift of a normed space retains its normed space structure.

1.6. Nontrivially Normed Spaces

When 𝕜 is a nontrivially normed field and E is nontrivial:

1.6.1. Unboundedness

For any $c \in \mathbb{R}$, there exists $x \in E$ with $\|x\| > c$.

Proof idea: Take any nonzero x_0, then scale by large $r \in$ 𝕜: ‖r dot x_0‖ = ‖r‖ dot ‖x_0‖ Since 𝕜 is unbounded, we can make this arbitrarily large.

1.6.2. Noncompactness

Consequences of unboundedness:
- The universal set is not bounded
- The cobounded filter is non-trivial (NeBot)
- The space is not compact

1.6.3. Infinitude

Any nontrivially normed field must be infinite (cannot be finite).

1.7. Discrete Subgroups

1.7.1. Integer Multiples

For a normed space over $\mathbb{Q}$, the additive subgroup $\mathbb{Z} \cdot e$ (integer multiples of $e \neq 0$) has discrete topology.

This is because $\|k \cdot e\| = |k| \cdot \|e\|$, so distinct integer multiples are separated by at least $\|e\|$.

1.8. Type Class Hierarchy

The instance hierarchy:

```
NormedSpace 𝕜 E
    ↓
NormSMulClass 𝕜 E
    ↓
IsBoundedSMul 𝕜 E
```

Special instances with priorities:
- Priority 100: `NormedSpace.toNormSMulClass`
- Priority 75: `SubmoduleClass.toNormedSpace`

1.9. Applications

Normed spaces are fundamental for:
- **Functional Analysis**: Banach spaces, Hilbert spaces
- **Differential Calculus**: Derivatives in infinite dimensions
- **Operator Theory**: Bounded linear operators
- **Approximation Theory**: Best approximation problems
- **PDEs**: Solution spaces for differential equations

1.10. Design Notes

The definition requires only the inequality $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ rather than equality, making it easier to verify instances. The equality is then proved as a theorem. This design choice simplifies the construction of normed spaces while maintaining full strength in applications.