# 1. Topology Basics

## 1.1. Overview

Topology provides the mathematical framework for studying continuity, convergence, and geometric properties that are preserved under continuous deformations. This module establishes the fundamental concepts of topological spaces, open and closed sets, and their basic properties. Topology serves as the foundation for analysis, differential geometry, and algebraic topology.

## 1.2. Core Definitions

### 1.2.1. Topological Space

A topological space is a pair $(X, \tau)$ where:
- $X$ is a set (the underlying set)
- $\tau$ is a collection of subsets of $X$ (the topology)

The topology $\tau$ must satisfy:
1. Empty and Total: $\emptyset, X \in \tau$
2. Arbitrary Unions: If $\{U_i\}_{\{i \in I\}} \subset \tau$, then $\cup_i U_i \in \tau$
3. Finite Intersections: If $U_1, \ldots, U_n \in \tau$, then $\cap_{\{i=1\}}^{n} U_i \in \tau$

Elements of $\tau$ are called open sets.

### 1.2.2. Open Sets

Definition: A set $U \subset X$ is open if $U \in \tau$.

In Mathlib:

```
IsOpen s : Prop
```

Key properties:
- The empty set and the whole space are open
- Arbitrary unions of open sets are open
- Finite intersections of open sets are open

### 1.2.3. Closed Sets

Definition: A set $F \subset X$ is closed if its complement $F^c$ is open.

In Mathlib:

```
IsClosed s ↔ IsOpen sᶜ
```

Key properties:
- The empty set and the whole space are closed
- Arbitrary intersections of closed sets are closed
- Finite unions of closed sets are closed

### 1.2.4. Locally Closed Sets

A set $S$ is locally closed if it can be written as the intersection of an open set and a closed set: S = U ∩ F where $U$ is open and $F$ is closed.

Properties:
- Every open set is locally closed
- Every closed set is locally closed
- Locally closed sets form a Boolean algebra

## 1.3. Construction Methods

### 1.3.1. From Closed Sets

Theorem (`TopologicalSpace.ofClosed`): A topology can be constructed by specifying the closed sets and verifying:
1. $\emptyset$ is closed
2. Arbitrary intersections of closed sets are closed
3. Finite unions of closed sets are closed

This dual approach is sometimes more convenient than specifying open sets.

### 1.3.2. Extensionality

Theorem (`TopologicalSpace.ext`): Two topologies on $X$ are equal if and only if they have the same open sets: τ_1 = τ_2 ⟺ ∀ S subset X, S "is open in" τ_1 ⟺ S "is open in" τ_2

This provides the standard way to prove equality of topologies.

## 1.4. Operations on Open Sets

### 1.4.1. Unions

Arbitrary Union (`isOpen_iUnion`): If $(U_i)_{\{i \in I\}}$ are open, then $\cup_i U_i$ is open.

Indexed Union (`isOpen_biUnion`): If $f : \alpha \to \text{Set } X$ and each $f(i)$ is open for $i \in S$, then: union "over" i in S : f(i) "is open"

Binary Union (`IsOpen.union`): If $U$ and $V$ are open, then $U \cup V$ is open.

### 1.4.2. Intersections

Finite Intersection (`IsOpen.inter`): If $U$ and $V$ are open, then $U \cap V$ is open.

Finite Indexed Intersection (`isOpen_biInter_finset`): For a finite set $S$ and open sets $f(i)$: sect "over" i in S : f(i) "is open"

Infinite Intersection (generally not open): The intersection of infinitely many open sets need not be open. Example: $\cap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ in $\mathbb{R}$.

## 1.5. Operations on Closed Sets

### 1.5.1. Intersections

Arbitrary Intersection (`isClosed_iInter`): If $(F_i)_{\{i \in I\}}$ are closed, then $\cap_i F_i$ is closed.

Indexed Intersection (`isClosed_biInter`): If each $f(i)$ is closed for $i \in S$, then: sect "over" i in S : f(i) "is closed"

### 1.5.2. Unions

Finite Union (`IsClosed.union`): If $F$ and $G$ are closed, then $F \cup G$ is closed.

Finite Indexed Union (by De Morgan): Finite unions of closed sets are closed.

## 1.6. Special Cases

### 1.6.1. Constant Sets

Theorem (`isOpen_const`, `isClosed_const`): For a proposition $p$:
- The set $\{x : p\}$ is either $\emptyset$ or $X$
- Both are open and closed (clopen)

### 1.6.2. Empty and Universe

Properties:

- `isOpen_empty`: $\emptyset$ is open
- `isClosed_empty`: $\emptyset$ is closed
- `isOpen_univ`: $X$ is open
- `isClosed_univ`: $X$ is closed

These sets are always clopen (both open and closed).

## 1.7. Cover Properties

### 1.7.1. Open Cover Criterion

Theorem (`isOpen_iff_of_cover`): Let $(U_i)$ be an open cover of $X$ (i.e., $\cup_i U_i = X$). Then $S$ is open if and only if $S \cap U_i$ is open for each $i$.

This localizes the property of being open to a cover.

## 1.8. Filter-Based Approach

### 1.8.1. Implementation Note

Mathlib heavily uses filters for topology, more than traditional approaches:
- Neighborhoods are described via neighborhood filters
- Convergence is expressed through filter convergence
- This provides a unified framework for limits

The filter approach:
- Generalizes sequences and nets
- Handles uncountable index sets naturally
- Unifies pointwise and uniform convergence

## 1.9. Key Theorems

### 1.9.1. Complement Relationships

Open-Closed Duality (`isOpen_compl_iff`): S "is open" ⟺ S^c "is closed"

This fundamental duality allows transferring results between open and closed sets.

### 1.9.2. De Morgan's Laws

For topology: (union_i U_i)^c = sect_i U_i^c (sect_i U_i)^c = union_i U_i^c

These connect unions/intersections with complements.

### 1.9.3. Finite vs Infinite

Key Distinction:
- Arbitrary unions of open sets are open
- Only finite intersections of open sets are guaranteed open
- Dually for closed sets (arbitrary intersections, finite unions)

## 1.10. Applications

### 1.10.1. Continuity

A function $f : X \to Y$ is continuous if: f^{-1}(U) "is open in" X "for all open" U subset Y

Open sets are the fundamental tool for defining continuity.

### 1.10.2. Convergence

A sequence $(x_n)$ converges to $x$ if: For every open set $U$ containing $x$, there exists $N$ such that $x_n \in U$ for all $n \geq N$.

### 1.10.3. Separation Axioms

Topological spaces are classified by separation properties:
- $T_0$: Points are topologically distinguishable
- $T_1$: Points are closed
- $T_2$ (Hausdorff): Points have disjoint neighborhoods
- Regular, Normal: Stronger separation conditions

### 1.10.4. Compactness

A space is compact if every open cover has a finite subcover. This generalizes finite sets and closed bounded sets in $\mathbb{R}^n$.

## 1.11. Examples

### 1.11.1. Discrete Topology

$\tau = \mathcal{P}(X)$ (all subsets are open)
- Finest topology on $X$
- Every function from $X$ is continuous

### 1.11.2. Indiscrete Topology

$\tau = \{\emptyset, X\}$ (only trivial sets are open)
- Coarsest topology on $X$
- Only constant functions to it are continuous

### 1.11.3. Metric Topology

For a metric space $(X, d)$: $U$ is open if for every $x \in U$, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$.

### 1.11.4. Zariski Topology

On algebraic varieties: Closed sets are zero sets of polynomials.

## 1.12. Design Principles

### 1.12.1. Bundled Structures

Mathlib uses:
- `TopologicalSpace X`: bundled topology on `X`
- `IsOpen`, `IsClosed`: predicates for sets
- This allows multiple topologies on the same type

### 1.12.2. Constructors

Multiple ways to define topologies:
1. Specify open sets directly
2. Specify closed sets (`ofClosed`)
3. Generate from a basis
4. Generate from a subbasis
5. Induce from functions

### 1.12.3. Extensibility

The design allows:
- Product topologies
- Subspace topologies
- Quotient topologies
- Weak topologies
- Uniform spaces as special cases

## 1.13. Historical Context

Topology emerged from several sources:

- Euler (1736): Seven Bridges of Königsberg - topology of graphs
- Riemann (1851): Riemann surfaces - analysis on manifolds
- Cantor (1870s): Point-set topology from set theory
- Poincaré (1895): Algebraic topology foundations
- Hausdorff (1914): Systematic treatment of topological spaces
- Bourbaki (1940s): Modern axiomatic approach with filters

The subject unifies:

- Geometric intuition (rubber sheet geometry)
- Analytic rigor (limits and continuity)
- Algebraic methods (homology and homotopy)