1. Bochner Integral

1.1. Overview

The Bochner integral extends the Lebesgue integral to functions taking values in Banach spaces (complete normed vector spaces). This construction is fundamental for functional analysis, partial differential equations, and probability theory in infinite-dimensional spaces. It provides a robust integration theory that preserves the essential properties of the Lebesgue integral while working in abstract spaces.

1.2. Core Construction

1.2.1. Motivation

The naive approach of componentwise integration fails for infinite-dimensional spaces since:
- There's no canonical basis
- Convergence issues arise with infinite sums
- Coordinate-free definition is needed

The Bochner integral solves these issues through a measure-theoretic construction.

1.2.2. Construction Steps

The Bochner integral is built in stages:

1. Simple Functions: Functions taking finitely many values s = sum*(i=1)^n x_i dot chi*{A_i} where $x_i \in E$ and A_i are measurable sets.
2. Integral of Simple Functions: integral s space d mu = sum "from" i=1 "to" n : mu(A_i) dot x_i
3. L¹ Space: Equivalence classes of integrable functions
   - Functions with finite norm: $\|f\|_1 = \int \|f\| \quad d\mu < \infty$
   - Identified up to equality almost everywhere
4. Extension to L¹: The integral extends by continuity from simple functions to L¹
5. General Functions: For $f : \alpha \to E$: integral f space d mu = cases("L¹ integral of" [f] quad &"if" f in L¹, 0 quad &"otherwise")

1.3. Main Definitions

1.3.1. The Integral

Notation:
- $\int a, f(a) \ d\mu$ or $\int f \ d\mu$: integral with respect to measure μ
- $\int a, f(a)$ or $\int f$: integral with respect to default measure (volume)

Definition: The Bochner integral is defined as:

```
integral f ∂μ = setToFun (dominatedFinMeasAdditive_weightedSMul μ) f
```

1.3.2. Integrability

A function $f : \alpha \to E$ is integrable if:
1. f is strongly measurable
2. $\int \|f\| \quad d\mu < \infty$

Equivalently: $f \in L^1(\mu, E)$

1.3.3. Simple Functions

Notation: $\alpha \underset{s}{\to} E$ denotes simple functions from α to E

Simple functions form a dense subspace of L^1, enabling approximation arguments.

1.4. Fundamental Properties

1.4.1. Linearity

The integral is a linear operator:

Additivity: integral (f + g) space d mu = integral f space d mu + integral g space d mu

Scalar Multiplication: integral (c dot f) space d mu = c dot integral f space d mu

Negation: integral (-f) space d mu = - integral f space d mu

Subtraction: integral (f - g) space d mu = integral f space d mu - integral g space d mu

1.4.2. Almost Everywhere Properties

Congrunce: If $f \overset{\mu}{=} g$ (equal μ-almost everywhere): integral f space d mu = integral g space d mu

Null Sets: If $\mu(A) = 0$: integral_A f space d mu = 0

1.4.3. Norm Inequalities

Triangle Inequality: ‖integral f space d mu‖ ≤ integral ‖f‖ space d mu

This fundamental inequality relates the norm of the integral to the integral of the norm.

Hölder's Inequality (for p, q conjugate exponents): integral |f dot g| space d mu ≤ ‖f‖*p dot ‖g*‖q

1.5. Order Properties

For real-valued functions in ordered Banach spaces:

1.5.1. Monotonicity

Pointwise Order: If $f \leq g$ everywhere: integral f space d mu ≤ integral g space d mu

Almost Everywhere Order: If $f \overset{\mu}{\leq} g$: integral f space d mu ≤ integral g space d mu

1.5.2. Sign Properties

Nonnegativity: If $0 \overset{\mu}{\leq} f$: 0 ≤ integral f space d mu

Nonpositivity: If $f \overset{\mu}{\leq} 0$: integral f space d mu ≤ 0

1.6. Connection to Lebesgue Integral

1.6.1. For Nonnegative Functions

When $f : \alpha \to \mathbb{R}$ with $0 \leq f$: integral f space d mu = integral^- f space d mu

where $\int^-$ denotes the Lebesgue integral for extended nonnegative real functions.

1.6.2. Decomposition Formula

For real-valued f: integral f space d mu = integral^- f^+ space d mu - integral^- f^- space d mu

where:
- $f^+ = \max(f, 0)$ is the positive part
- $f^- = \max(-f, 0)$ is the negative part

1.7. Convergence Theorems

1.7.1. Dominated Convergence Theorem

If:
1. $f_n \to f$ pointwise
2. $|f_n| \leq g$ for integrable g
3. Each f_n is measurable

Then: lim “as” n ⟶ ∞ : integral f_n space d mu = integral f space d mu

1.7.2. Monotone Convergence

For increasing sequence $0 \leq f_n \uparrow f$: lim “as” n ⟶ ∞ : integral f_n space d mu = integral f space d mu

1.7.3. Fatou’s Lemma

For nonnegative functions: integral liminf f_n space d mu ≤ liminf integral f_n space d mu

1.8. Integration with Linear Maps

1.8.1. Continuous Linear Maps

If $T : E \to F$ is continuous linear: T(integral f space d mu) = integral T ∘ f space d mu

This shows integration commutes with continuous linear maps.

1.8.2. Isometries

If $T : E \to F$ is a linear isometry: integral T ∘ f space d mu = T(integral f space d mu) ‖integral f space d mu‖ = ‖integral T ∘ f space d mu‖

1.9. Set Integration

1.9.1. Definition

The integral over a set s: integral_s f space d mu = integral chi_s dot f space d mu

where χ_s is the characteristic function of s.

1.9.2. Properties

Additivity over disjoint sets: If $A \cap B = \emptyset$: integral “over” space A union B : f space d mu = integral_A f space d mu + integral_B f space d mu

Absolute continuity: mu(A) ⟶ 0 implies integral_A f space d mu ⟶ 0

1.10. Proof Techniques

1.10.1. Density Arguments

To prove property P for all integrable functions:

1. Show P holds for simple functions
   - Often straightforward by definition
2. Show the set $\{f \in L^1 : P(f)\}$ is closed
   - Use continuity of operations
3. Apply density
   - Simple functions are dense in L^1
   - Closed set containing dense subset equals the whole space

1.10.2. Induction Principles

Integrable.induction: Allows proving properties by:
- Verifying for indicators of finite measure sets
- Showing closure under operations
- Checking limit preservation

Lp.induction: Similar principle for L^p spaces

1.10.3. Reduction to Nonnegative Case

Many proofs proceed by:
1. Proving for nonnegative functions using monotone convergence

2. Decomposing general functions: $f = f^+ - f^-$
3. Applying linearity

1.11. Applications

1.11.1. Probability Theory

Expectation: For random variable $X : \Omega \to E$: mathbb(E)[X] = integral X space d mathbb(P)

Conditional Expectation: Projection in L^2 onto sub-σ-algebra

1.11.2. PDEs and Functional Analysis

Weak Solutions: Test against smooth functions integral nabla u dot nabla phi = integral f phi

Sobolev Spaces: Functions with integrable weak derivatives

1.11.3. Harmonic Analysis

Fourier Transform: hat(f)(xi) = integral f(x) e^(-2 pi i x dot xi) space d x

Requires Bochner integral for vector-valued functions.

1.12. Implementation Details

1.12.1. Type Classes and Notation

- `Integrable f μ`: typeclass for integrable functions
- `L1 α E μ`: the L^1 space
- `∫ x, f x ∂μ`: integral notation
- `∫ x in s, f x ∂μ`: set integral

1.12.2. Definitional Equality

The integral is definitionally equal to:

```
setToFun (dominatedFinMeasAdditive_weightedSMul μ) f
```

This connects to the abstract extension framework.

1.12.3. Measurability Assumptions

Functions must be:

- Strongly measurable: Limits of simple functions
- Not just weakly measurable (measurable when composed with functionals)

1.13. Historical Context

The Bochner integral was developed by Salomon Bochner in 1933, extending Lebesgue's theory to vector-valued functions. Key developments:

- 1933: Bochner's original construction
- 1938: Pettis integral (weaker notion)
- 1940s: Dunford-Schwartz systematic treatment
- Modern: Essential for infinite-dimensional analysis

Alternative approaches:

- Pettis integral: Uses weak measurability
- Gel'fand-Dunford integral: For weak-star measurable functions
- McShane integral: Gauge-based approach

The Bochner integral remains the standard for its strong properties and natural extension of Lebesgue theory.