1. Locally Convex Spaces - Basic Concepts

1.1. Overview

This module introduces the fundamental concepts for locally convex topological vector spaces (LCTVS): absorbent and balanced sets. These are the building blocks for understanding neighborhoods of the origin in topological vector spaces, generalizing the notion of balls in normed spaces.

1.2. Core Definitions

1.2.1. Balanced Sets

A set $A \subseteq E$ is **balanced** with respect to 𝕜 if: a dot A subset.eq A text(“ whenever “) ‖a‖ ≤ 1

Intuitively, a balanced set “surrounds the origin uniformly” - it contains all scalings by coefficients of norm at most 1.

Equivalent characterizations:
- Balanced $\Bbbk$ $A \iff \forall a \in \Bbbk, \|a\| \leq 1 \implies a \cdot A \subseteq A$
- Balanced $\Bbbk$ $A \iff \mathrm{closedBall}(0, 1) \cdot A \subseteq A$
- For elements: $x \in A \implies a \cdot x \in A$ whenever $\|a\| \leq 1$

1.2.2. Absorbing Sets

A set A **absorbs** a set B if B is contained in all sufficiently large scalings of A: “Absorbs” space 𝕜 space A space B ⟹ ∃ r > 0, ∀ c in 𝕜, ‖c‖ ≥ r ⟹ B subset.eq c dot A

This captures the idea that A eventually “swallows” B under scaling.

1.2.3. Absorbent Sets

A set A is **absorbent** if it absorbs every singleton (equivalently, every point): “Absorbent” space 𝕜 space A ⟺ ∀ x in E, “Absorbs” space 𝕜 space A space {x}

Absorbent sets are the vector space analogue of neighborhoods of the origin.

1.3. Properties of Balanced Sets

1.3.1. Closure Under Operations

Balanced sets are preserved by:
- Union: If A and B are balanced, so is $A \cup B$
- Intersection: If A and B are balanced, so is $A \cap B$
- Arbitrary unions: $\cup_i A_i$ is balanced if all A_i are balanced
- Arbitrary intersections: $\cap_i A_i$ is balanced if all A_i are balanced

1.3.2. Module Operations

For balanced sets in modules:
- Negation: A balanced $\iff -A$ balanced
- Addition: A, B balanced $\implies A + B$ balanced
- Subtraction: A, B balanced $\implies A - B$ balanced
- Scalar multiplication: If A is balanced and scalars commute, then $a \cdot A$ is balanced

1.3.3. Special Cases

- The empty set $\emptyset$ is balanced
- The universal set is balanced
- The zero set $\{0\}$ is balanced

1.3.4. Symmetry Property

In normed spaces with $\|-1\| = 1$:
- Balanced sets are symmetric: $x \in A \iff -x \in A$

- Therefore: $-A = A$ for balanced sets

1.4. Properties of Absorbing Sets

1.4.1. Neighborhood Characterization

For normed division rings: “Absorbs” space 𝕜 space A space B ⟺ ∀^F c in cal(N)[≠] 0, c dot B subset.eq A

where $\mathcal{N}[\neq]0$ is the punctured neighborhood of 0.

If $0 \in A$, this simplifies to: “Absorbs” space 𝕜 space A space B ⟺ ∀^F c in cal(N) space 0, c dot B subset.eq A

1.4.2. Monotonicity

For balanced sets, scalar multiplication is monotone: “Balanced” space A ∧ ‖a‖ ≤ ‖b‖ ⟹ a dot A subset.eq b dot A

This shows that balanced sets expand monotonically with the norm of the scalar.

1.5. Preimages and Morphisms

1.5.1. Preimage Property

If $f : E \to [𝕜]F$ is a 𝕜-linear map and $S \subseteq F$ is balanced, then: f^(-1)(S) text(“ is balanced in “) E

This is crucial for showing that balanced neighborhoods pull back under continuous linear maps.

1.6. Relationship to Convexity

While this module focuses on balanced and absorbent sets, these concepts are intimately connected to convexity:

- In real vector spaces, balanced convex sets are exactly the symmetric convex sets
- The intersection of all balanced convex neighborhoods of 0 gives the finest locally convex topology
- Absorbent balanced convex sets are exactly the neighborhoods of 0 in locally convex spaces

1.7. Applications

These concepts are fundamental for:

1.7.1. Locally Convex Topology

- Defining the finest locally convex topology
- Characterizing continuous seminorms
- Hahn-Banach theorem in locally convex spaces

1.7.2. Functional Analysis

- Polar sets and bipolar theorem
- Weak and weak-star topologies
- Barrel and bornological spaces

1.7.3. Optimization

- Constraint qualifications in infinite dimensions
- Subdifferential calculus
- Variational analysis

1.8. Design Notes

The module uses filter-based definitions for absorption, allowing seamless integration with Mathlib’s topology library. The use of `NormSMulClass` provides flexibility in the types of scalar multiplication considered.

The deprecated aliases for `nhdsWithin` reflect a recent refactoring to use the cleaner `𝓝[≠] 0` notation for punctured neighborhoods.