Feynman Path Integrals

X.J.J. Xu

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1 How can one formulate the path integral?

One can formulate the path integral with minimal assumptions about the way things work. For instance, one only needs to start with the concept that a certain particle starts at a position in space and then ends up at a new position some time later. From here, a path integral can be built by counting all the possible paths and figuring out the probability for the particle to take a particular path. For example, suppose a young lad named Photon went out to Broadway last night at 11:00 PM and got irreversibly intoxicated at precisely 01:00 AM in Honky Tonk Central. That was when Photon realized, with his last drop of remaining consciousness, that he should go home. As far as we're concerned, Photon's BAC was high enough that he could not tell left from right. As any curious observer would ask: did Photon make it home? Even better, did he make it home at time t'? To attempt to answer the question, let us begin by writing the state of being at home as

$$|\Psi_{Home}\rangle = c_1 |\Psi_1\rangle + c_2 |\Psi_2\rangle + ... + c_n |\Psi_n\rangle$$

$$|\Psi_{Home}\rangle = \sum_i^n c_i |\Psi_i\rangle$$

Where each $|\Psi_i\rangle$ is a possible path that Photon took to get home with probability $|c_i|^2$. The probability amplitude that Photon made it home at 06:00 AM is

$$\langle Home, t' | Honky\ Tonk, 01:00\ AM\rangle$$

(c) Path integral in terms of the action and Lagrangian.

$$\langle x', t' | x_0, t_0\rangle = \int_{x_0}^{x'} \mathscr{D}[x(t)] e^{iS[x(t)]/\hbar} \tag{1}$$

Where

$$S[x(t)] = \int_{t_0}^{t'} dt\ L(x, \dot{x})$$

In continuation of Photon's adventure, we can perhaps interpret the action S as the the 'effort' with which Photon needs to put in to get home. For instance, if he can catch a ride on a horse cart from Broadway, then his journey home will be faster (small

$t' - t_0$). Hence the effort will be less. Moreover, if his journey home is downhill, then the ride will go even smoother (gravitational potential in his favour - small $L(x, \dot{x})$). The opposite would be true if the journey is uphill. However, if Photon is not lucky enough to catch a ride on a horse cart, then he might have to drunk walk his way home, which will take a lot longer (big $t' - t_0$). Therefore the effort will be greater as well. Even worse, if he has to drunk walk uphill, then that'd result in a rather unpleasant journey that requires tremendous effort (big $L(x, \dot{x})$ and big S).
Each path of getting home has a corresponding effort level, or action S, which contributes to the phase factor $e^{iS[x(t)]/\hbar}$. So even though riding a horse cart downhill sounds like the way to go, he may not in phase to catch a horse cart rolling by him. All paths in $\mathscr{D}$ will serve the purpose of bringing Photon home thus a given path is no better than another in that regard. Photon's fate lies within the phase factor.

2 Propagator

Calculate the propagator for a free-particle to transition from position x_0 at time t_0 to position $x^{'}$ at time $t^{'}$.

$$\langle x', t'|x_0, t_0\rangle = \langle x'|\, e^{\frac{-i\hat{H}(t'-t_0)}{\hbar}}\, |x_0\rangle$$

The Hamiltonian for a free particle is $\hat{H} = \frac{\hat{p}_x^2}{2m}$.

$$\langle x', t'|x_0, t_0\rangle = \langle x'|\, e^{\frac{-i\hat{p}_x^2(t'-t_0)}{2m\hbar}}\, |x_0\rangle$$

Switch to momentum basis so that the operator and the ket are in the same basis. x and p are a Fourier transform pair

$$|x\rangle = \int dp\, |p\rangle \langle p|x\rangle \tag{2}$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}} \tag{3}$$

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-ipx}{\hbar}} \tag{4}$$

We're going to treat $\hat{p}_x$ and $\hat{p}$ as the same in the calculations to follow. Use relation (5)

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \left\langle x^{'} \middle| e^{\frac{-i\hat{p}_x^2(t^{'}-t_0)}{2m\hbar}} \int dp\, |p\rangle \langle p|x_0\rangle\right.$$

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \int dp\; e^{\frac{-i\hat{p}_x^2(t^{'}-t_0)}{2m\hbar}} \left\langle x^{'} \middle| p \right\rangle \langle p|x_0\rangle$$

Use relations (6) and (7)

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \int dp\; e^{-i\hat{p}_x^2(t^{'}-t_0)/2m\hbar} \frac{1}{\sqrt{2\pi\hbar}} e^{ipx^{'}/\hbar} \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx_0/\hbar}$$

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \int dp \; e^{-i\hat{p}_x^2(t^{'}-t_0)/2m\hbar} \frac{1}{2\pi\hbar} e^{ip(x^{'}-x_0)/\hbar}$$

Apply appropriate integration boundaries over all x-space

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \; exp\{-\frac{1}{2} i\hat{p}^2(t^{'} - t_0)/m\hbar + ip(x^{'} - x_0)/\hbar\} \tag{5}$$

Let's use a Gaussian integral of the form

$$\int_{-\infty}^{\infty} dp \; e^{-\frac{1}{2}ap^2+bp} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}} \tag{6}$$

Eq (9) is found by completing the square (LHS) in the exponent; changing variables to use the standard Gaussian integral below:

$$\int_{-\infty}^{\infty} dx \; e^{-x^2} = \sqrt{\pi} \tag{7}$$

Apply (9) to (8) by using the following substitutions

$$\begin{cases} a = \frac{i(t^{'}-t_0)}{m\hbar} \\ b = \frac{i(x^{'}-x_0)}{\hbar} \end{cases}$$

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \; e^{-\frac{1}{2}ap^2+bp}$$

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \frac{1}{2\pi\hbar} \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}$$

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \frac{1}{2\pi\hbar} \sqrt{\frac{2\pi}{\frac{i(t^{'}-t_0)}{m\hbar}}} e^{\frac{(\frac{i(x^{'}-x_0)}{\hbar})^2}{2\frac{i(t^{'}-t_0)}{m\hbar}}}$$

$$\left\langle x^{'}, t^{'} \middle| x_0, t_0 \right\rangle = \sqrt{\frac{m}{2\pi\hbar i(t' - t_0)}} e^{\frac{im(x^{'}-x_0)^2}{2\hbar(t^{'}-t_0)}}$$

3 Derive the non-relativistic Schrodinger equation from the path integral formalism

$$\Psi(x', t') = \sqrt{\frac{m}{2\pi i\hbar\Delta t}} \int dx_0 \; exp\{\frac{im(x' - x_0)^2}{2\hbar\Delta t}\}[1 - \frac{i}{\hbar} V(\frac{x' + x_0}{2})\Delta t]\Psi(x_0, t_0)$$

Shift variable $\epsilon = x' - x_0$. Let $A = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}$

$$\Psi(x', t') = A \int d\epsilon \; exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}[1 - \frac{i}{\hbar} V(x' - \frac{\epsilon}{2})\Delta t]\Psi(x' - \epsilon, t_0) \tag{8}$$

Perform two Taylor expansions

$$V(x' - \frac{\epsilon}{2}) = V(x') - \frac{\epsilon}{2}\frac{dV(x')}{dx'} + ... \tag{9}$$

$$\Psi(x' - \epsilon, t_0) = \Psi(x', t_0) - \epsilon\frac{\partial\Psi(x', t_0)}{\partial x'} + \frac{\epsilon^2}{2}\frac{\partial^2\Psi(x', t_0)}{\partial x'^2} + ... \tag{10}$$

Plug (13) and (14) into (12)

$$\Psi(x', t') = A\int d\epsilon\ exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}[1 - \frac{i}{\hbar}V(x')\Delta t][\Psi(x', t_0) - \epsilon\frac{\partial\Psi(x', t_0)}{\partial x'} + \frac{\epsilon^2}{2}\frac{\partial^2\Psi(x', t_0)}{\partial x'^2} + ...]$$

$$\Psi(x', t') = A\int d\epsilon\ exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}[1 - \frac{i}{\hbar}V(x')\Delta t]\Psi(x', t_0) - A\int d\epsilon\ exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}\epsilon\frac{\partial\Psi(x', t_0)}{\partial x'}$$

$$+A\int d\epsilon\ exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}\frac{\epsilon^2}{2}\frac{\partial^2\Psi(x', t_0)}{\partial x'^2}$$

Take a pause and simplify the above expression. The 2nd term can be executed

$$A\int d\epsilon\ exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}\epsilon\frac{\partial\Psi(x', t_0)}{\partial x'}$$

$$= A\frac{\partial\Psi(x', t_0)}{\partial x'}\int_{-\infty}^{\infty} d\epsilon\ exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}\epsilon$$

$$= A\frac{\partial\Psi(x', t_0)}{\partial x'}[exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}\frac{2\hbar\Delta t}{2im}|_{-\infty}^{\infty}]$$

$$= A\frac{\partial\Psi(x', t_0)}{\partial x'}[\infty - \infty] = 0$$

We can execute the 3rd term with a Gaussian integral of the form

$$\int x^2 e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

Simplify the 3rd term and sub $\alpha = -\frac{im}{2\hbar\Delta t}$

$$\frac{A}{2}\frac{\partial^2\Psi(x', t_0)}{\partial x'^2}\int d\epsilon\ \epsilon^2 exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}$$

$$= \frac{A}{2}\frac{\partial^2\Psi(x', t_0)}{\partial x'^2}\int d\epsilon\ \epsilon^2 e^{-\alpha\epsilon^2}$$

$$= \frac{A}{2}\frac{\partial^2\Psi(x', t_0)}{\partial x'^2}\frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

$$= \frac{A}{2}\frac{\partial^2\Psi(x', t_0)}{\partial x'^2}\sqrt{\frac{2\pi\hbar^3\Delta t^3}{im^3}}$$

Return to the wavefunction and sub back in results

$$\Psi(x',t') = A\int d\epsilon\ exp\{\frac{im\epsilon^2}{2\hbar\Delta t}\}[1-\frac{i}{\hbar}V(x')\Delta t]\Psi(x',t_0) + \frac{A}{2}\frac{\partial^2\Psi(x',t_0)}{\partial x'^2}\sqrt{\frac{2\pi\hbar^3\Delta t^3}{im^3}}$$

With another Gaussian integral

$$\int e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$$

Sub in $a = -\frac{im}{2\hbar\Delta t}$ for just first term

$$\Psi(x',t') = A[1-\frac{i}{\hbar}V(x')\Delta t]\Psi(x',t_0)\int d\epsilon\ e^{-a\epsilon^2} + \frac{A}{2}\frac{\partial^2\Psi(x',t_0)}{\partial x'^2}\sqrt{\frac{2\pi\hbar^3\Delta t^3}{im^3}}$$

$$\Psi(x',t') = A[1-\frac{i}{\hbar}V(x')\Delta t]\Psi(x',t_0)\sqrt{\frac{-2\hbar\Delta t\pi}{im}} + \frac{A}{2}\frac{\partial^2\Psi(x',t_0)}{\partial x'^2}\sqrt{\frac{2\pi\hbar^3\Delta t^3}{im^3}}$$

Recall that

$$A = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}$$

$$\Psi(x',t') = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}[1-\frac{i}{\hbar}V(x')\Delta t]\Psi(x',t_0)\sqrt{\frac{-2\hbar\Delta t\pi}{im}} + \frac{\sqrt{\frac{m}{2\pi i\hbar\Delta t}}}{2}\frac{\partial^2\Psi(x',t_0)}{\partial x'^2}\sqrt{\frac{2\pi\hbar^3\Delta t^3}{im^3}}$$

$$\Psi(x',t') = [1-\frac{i}{\hbar}V(x')\Delta t]\Psi(x',t_0) + \frac{\partial^2\Psi(x',t_0)}{\partial x'^2}\frac{i\hbar\Delta t}{2m}$$

$$\Psi(x',t') - \Psi(x',t_0) = -\frac{i}{\hbar}V(x')\Delta t\Psi(x',t_0) + \frac{\partial^2\Psi(x',t_0)}{\partial x'^2}\frac{i\hbar\Delta t}{2m}$$

$$\lim_{\Delta t\to 0}\frac{\Psi(x',t_0+\Delta t)-\Psi(x',t_0)}{\Delta t} = \lim_{\Delta t\to 0}[-\frac{i}{\hbar}V(x')\Psi(x',t_0) + \frac{i\hbar}{2m}\frac{\partial^2\Psi(x',t_0)}{\partial x'^2}]$$

LHS is the definition of a time derivative and the limit has no effect on the RHS

$$\frac{\partial\Psi(x',t')}{\partial t} = -\frac{i}{\hbar}V(x')\Psi(x',t_0) + \frac{i\hbar}{2m}\frac{\partial^2\Psi(x',t_0)}{\partial x'^2}$$

Starting to look very familiar! Rearrange slightly to get the time-dependent non-relativisitic Schrodinger equation

$$i\hbar\frac{\partial\Psi(x',t')}{\partial t} = V(x')\Psi(x',t_0) - \frac{\hbar^2}{2m}\frac{\partial^2\Psi(x',t_0)}{\partial x'^2}$$

$$i\hbar\frac{\partial\Psi(x',t')}{\partial t} = [-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2} + V(x')]\Psi(x',t_0)$$