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ON THE DISTANCE BETWEEN TWO ELLIPSOIDS*

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Abstract. We study the fundamental geometric problem of finding the distance between two ellipsoids. An algorithm is proposed for computing the distance and locating the two closest points. The algorithm is based on a local approximation of the two ellipsoids by balls. It is simple, geometric in nature, and has excellent convergence properties.

Key words. distance, ellipsoid, convex, optimization, algorithm

AMS subject classifications. 49M37, 65K05, 90C25, 90C30

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1. Introduction.

In this paper we are concerned with the geometric problem of finding the distance between two ellipsoids. The problem is a fundamental optimization problem which can be expressed in the following form:

$$\begin{array}{rl} \min & \|x-y\| \\ \text{subject to} & x \in E_1, \\ & y \in E_2, \end{array} \tag{1.1}$$

where $E_1 := \{x : q_1(x) \le 0\}$ and $E_2 : \{y : q_2(y) \le 0\}$ are two given ellipsoids determined by the two quadratic functions

$$q_1(x) := \frac{1}{2}x^T A_1 x + b_1^T x + \alpha_1 \quad \text{and} \quad q_2(y) := \frac{1}{2}y^T A_2 y + b_2^T y + \alpha_2,$$

with positive definite symmetric matrices A_1 and A_2, vectors b_1 and b_2, and scalars α_1 and α_2. The norm $\|\cdot\|$ is the Euclidean norm. Though simple, the problem is nonlinear in nature and cannot be effectively solved by any linear technique such as a linear programming or quadratic programming method. On the other hand, it is highly structured, and any general nonlinear programming method that fails to exploit its structure may also not be very efficient. As an attempt to overcome such difficulties, we present in this paper a special algorithm for solving problem (1.1). The algorithm, which is an extension of our algorithm for finding the projection of a point on an ellipsoid [4], is extremely simple, easy to implement, and has excellent convergence properties.

In section 2 we describe our algorithm. In section 3, to facilitate our analysis of the algorithm, we characterize the optimal solution in terms of angles between some vectors. The convergence analysis of the algorithm is given in section 4, and some comments about the implementation and computational experiment are given in section 5. A proof for a key lemma is given in the appendix.

2. The algorithm.

To describe the algorithm, we first introduce some notation. As usual, the angle between two nonzero vectors x and y is defined to be

$$\theta(x,y) := \arccos\left(\frac{x^T y}{\|x\|\|y\|}\right), \qquad 0 \le \theta(x,y) \le \pi,$$

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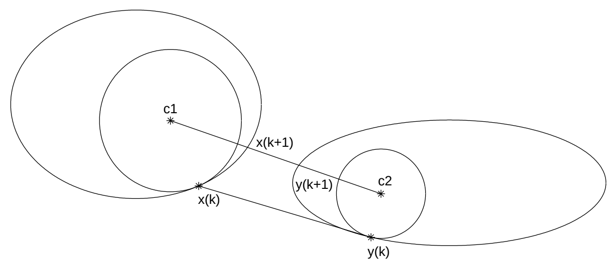

FIG. 1.

and a ball with center y and radius β is defined by

$$B(y;\beta) := \{x : \|y - x\| \leq \beta\}.$$

Also, we use $d(E_1, E_2)$ to denote the distance between E_1 and E_2.

In the algorithm we will generate two sequences of points $\{x_k\}$ and $\{y_k\}$ on the boundaries of the two ellipsoids $\Omega(E_1)$ and $\Omega(E_2)$, respectively, and it will be shown that $\lim_{k\to\infty} \|x_k - y_k\| = d(E_1, E_2)$.

The algorithm is an iterative process. To avoid some cumbersome superscripts or subscripts, we use the undaunted symbols such as x and y to denote our current computed vectors, and the barred symbols $\bar{x}$ and $\bar{y}$ to denote the new vectors. More specifically, we use x, y for x^k, y^k and use $\bar{x}$, $\bar{y}$ for x^{k+1}, y^{k+1}, etc.

The algorithm can now be described geometrically as follows. At the kth iteration, having two points $x \in \Omega(E_1)$ and $y \in \Omega(E_2)$, we construct a ball $B(c_1; r_1)$ completely inside the ellipsoid E_1 and tangent to E_1 at x, and a ball $B(c_2; r_2)$ completely inside the ellipsoid E_2 and tangent to E_2 at y (see Figure 1). Then we check whether the line segment $[c_1, c_2]$ between the two centers is entirely contained in $E_1 \cup E_2$. If it is, then the two ellipsoids have a nonempty intersection and the distance $d(E_1, E_2) = 0$; otherwise, we continue and compute the new point $\bar{x}$ as the intersection of the line segment $[c_1, c_2]$ with the boundary $\Omega(E_1)$, and also $\bar{y}$ as the intersection of $[c_1, c_2]$ with the boundary $\Omega(E_2)$.

Two issues need to be addressed to make the algorithm viable. First, can the two balls $B(c_1, r_1)$ and $B(c_2, r_2)$ be easily constructed? Second, how can we check $[c_1, c_2] \subset E_1 \cup E_2$ and compute new estimates $\bar{x}$ and $\bar{y}$? The first issue can be resolved by the following lemma. (A slightly different but equivalent version of the lemma is given in [4]. However, because the proof is short and because we want the paper to be self-contained, we include its proof here.)

LEMMA 2.1. *Let* $E := \{w : \frac{1}{2}w^T Aw + b^T w + \alpha \leq 0\}$ *be a nonempty ellipsoid determined by a positive definite symmetric matrix* A*, a vector* b*, and a scalar* α*. Let* z *be a point on the boundary* $\Omega(E)$ *of* E*; then for any* $0 < \gamma \leq \frac{1}{\rho(A)}$

$$B(z - \gamma(Az + b); \gamma\|Az + b\|) \subset E,$$

where $\rho(A)$ *is the spectral radius of* A*.*

Proof. Let $q(x) := \frac{1}{2}x^T Ax + b^T x + \alpha$, and let y be any vector on the boundary of the ball $B(z - \gamma(Az + b); \gamma\|Az + b\|)$. It suffices to show $q(y) \leq 0$. We have

$$\|y - z + \gamma(Az + b)\|^2 = \gamma^2\|Az + b\|^2.$$

By expanding both sides of the above equality, we get

$$\|y - z\|^2 + 2\gamma(y - z)^T(Az + b) = 0.$$

Equivalently, we have

$$\nabla q(z)^T(y - z) = -\frac{1}{2\gamma}\|y - z\|^2.$$

Now, we can show $q(y) \leq 0$ when $0 < \gamma \leq \frac{1}{\rho(A)}$ by

$$\begin{aligned} q(y) &= q(z) + \nabla q(z)^T(y - z) + \frac{1}{2}(y - z)^T A(y - z) \\ &\leq q(z) - \frac{1}{2\gamma}\|y - z\|^2 + \frac{\rho(A)}{2}\|y - z\|^2 \\ &= \frac{1}{2}\left(\rho(A) - \frac{1}{\gamma}\right)\|y - z\|^2 \\ &\leq 0. \qquad \square \end{aligned}$$

From the above lemma we can construct the two balls in the ellipsoids by choosing the centers c_1 and c_2 as $x - \gamma_1(A_1x + b_1)$ and $y - \gamma_2(A_2y + b_2)$ and choosing the radii r_1 and r_2 as $\gamma_1\|A_1x + b_1\|$ and $\gamma_2\|A_2y + b_2\|$, respectively, with γ_1 and γ_2 satisfying

$$0 < \tau \leq \gamma_1 \leq \frac{1}{\rho(A_1)} \quad \text{and} \quad 0 < \tau \leq \gamma_2 \leq \frac{1}{\rho(A_2)}. \tag{2.1}$$

Here τ is a prescribed small fixed number. Such a lower bound is needed because, for the method to work properly, it is also required that the radii of the balls be bounded away from zero. Obviously, we can choose a matrix norm $\|\cdot\|$ and set γ_1 and γ_2 to be

$$\gamma_1 = \frac{1}{\|A_1\|}, \qquad \gamma_2 = \frac{1}{\|A_2\|}.$$

For such a choice, the γ's are independent of the iteration, and the lower bound τ can be

$$\tau = \min\left\{\frac{1}{\|A_1\|}, \frac{1}{\|A_2\|}\right\}.$$

Of course, the 1-norm and the ∞-norm are particularly useful here because of their ease of computation.

As for the second issue of checking the condition $[c_1, c_2] \subset E_1 \cup E_2$ and the computation of new estimates $\bar{x}$ and $\bar{y}$, we compute two stepsizes t_1 and t_2 by

$$\begin{aligned} t_1 &= \max\{t \in [0, 1] : (1 - t)c_1 + tc_2 \in E_1\}, \\ t_2 &= \min\{t \in [0, 1] : (1 - t)c_1 + tc_2 \in E_2\}. \end{aligned} \tag{2.2}$$

Because $[c_1, c_2] \cap E_1$ and $[c_1, c_2] \cap E_2$ are both nonempty closed sets, t_1 and t_2 are well defined and can be easily computed by solving two one-dimensional quadratic equations.

If $t_2 \leq t_1$, then E_1 and E_2 have a nonempty intersection. In this case $d(E_1, E_2) = 0$ and we are done. If $t_2 > t_1$, then we let $\bar{x} = c_1 + t_1(c_2 - c_1)$ and $\bar{y} = c_1 + t_2(c_2 - c_1)$.

In this case, we have $\bar{x} \in \Omega(E_1)$, $\bar{y} \in \Omega(E_2)$, and the open line segment $(\bar{x}, \bar{y})$ has no intersection with the union $E_1 \cup E_2$.

Now we summarize the above discussion and give a precise description of the algorithm below.

Algorithm 1.

Initiation. Start from an interior point c_1 in E_1 and an interior point c_2 in E_2. The natural choices for these two points are the centers $-A_1^{-1}b_1$ and $-A_2^{-1}b_2$ of the two ellipsoids, respectively.

General steps. At the kth iteration, having an interior point c_1 of E_1 and an interior point c_2 of E_2, we proceed as follows:

1. We solve two one-dimensional quadratic equations to get the stepsizes t_1 and t_2 as given in (2.2).
2. If $t_2 \le t_1$, we are done, and we set $d(E_1, E_2) = 0$. In this case, any point $c_1 + t(c_2 - c_1)$ with $t_2 \le t \le t_1$ is in $E_1 \cap E_2$. Otherwise, we compute new points $\bar{x}$ and $\bar{y}$ by

$$\bar{x} = c_1 + t_1(c_2 - c_1), \qquad \bar{y} = c_1 + t_2(c_2 - c_1).$$

3. We compute θ_1 and θ_2 by

$$\theta_1 = \theta(\bar{y} - \bar{x}, A_1\bar{x} + b_1), \qquad \theta_2 = \theta(\bar{x} - \bar{y}, A_2\bar{y} + b_2).$$

If $\theta_1 = \theta_2 = 0$, then terminate.

4. We compute the new centers $\bar{c}_1$ and $\bar{c}_2$ by

$$\bar{c}_1 = \bar{x} - \gamma_1(A_1\bar{x} + b_1), \qquad \bar{c}_2 = \bar{y} - \gamma_2(A_2\bar{y} + b_2),$$

with γ_1 and γ_2 satisfying (2.1).

We note here that the algorithm will terminate in a finite number of iterations when any of the two situations occurs: (1) $t_2 \le t_1$ or (2) $t_2 > t_1$ but $\theta_1 = \theta_2 = 0$. When $t_2 \le t_1$, as mentioned before, an intersection point of E_1 and E_2 is found and we stop. When $t_2 > t_1$ and $\theta_1 = \theta_2 = 0$, the new points $\bar{x}$ and $\bar{y}$ are distinct, and it will be shown in Corollary 3.6 that, in this case, the two sets E_1 and E_2 are disjoint and the pair $(\bar{x}, \bar{y})$ is the unique optimal solution of problem (1.1). We will also justify the usage of the angle values θ_1 and θ_2 for determining convergence in Theorem 3.5. Of course, in practice we check $\theta_1 \le \epsilon$ and $\theta_2 \le \epsilon$ instead of $\theta_1 = \theta_2 = 0$.

3. Optimality conditions via angles. To study the convergence properties of the algorithm, we need to characterize the optimal solution of problem (1.1). Recall that a pair (x^*, y^*) is a Karush–Kuhn–Tucker point of problem (1.1) if there exist Lagrange multipliers λ and μ such that

$$\begin{cases} x^* - y^* + \lambda(A_1x^* + b_1) & = & 0, \\ y^* - x^* + \mu(A_2y^* + b_2) & = & 0, \\ \lambda(\frac{1}{2}x^{*T}A_1x^* + b_1^Tx^* + \alpha_1) & = & 0, \\ \mu(\frac{1}{2}y^{*T}A_2y^* + b_2^Ty^* + \alpha_2) & = & 0, \\ \frac{1}{2}x^{*T}A_1x^* + b_1^Tx^* + \alpha_1 & \le & 0, \\ \frac{1}{2}y^{*T}A_2y^* + b_2^Ty^* + \alpha_2 & \le & 0, \\ \lambda \ge 0 \quad \text{and} \quad \mu \ge 0. \end{cases}$$

We assume that the matrices A_1 and A_2 are positive definite and that both ellipsoids E_1 and E_2 are nonempty. Under these assumptions, the problem is a well-defined separable convex optimization problem. Therefore, a pair (x^*, y^*) is an optimal solution of problem (1.1) if and only if (x^*, y^*) is also a Karush–Kuhn–Tucker point [1, 5]. When $E_1 \cap E_2 \neq \emptyset$, then for any z in $E_1 \cap E_2$ the pair (z, z) is a Karush–Kuhn–Tucker point with both multipliers $\lambda = 0$ and $\mu = 0$. In this section, we are more concerned with the case in which $E_1 \cap E_2 = \emptyset$. In this case, the optimal value $d(E_1, E_2) > 0$, and the optimal solution (x^*, y^*) is unique.

For our convergence analysis of the algorithm, we need to consider an equivalent optimality condition in terms of angles. Since $E_1 \cap E_2 = \emptyset$, we must have $x^* \neq y^*$. Hence it follows from the first two Karush–Kuhn–Tucker equations that λ and μ must be strictly positive. This implies that $x^* \in \Omega(E_1)$ and $y^* \in \Omega(E_2)$. Moreover, it again follows from the first two equations that the three vectors $y^* - x^*$, $A_1 x^* + b_1$, and $-(A_2 y^* + b_2)$ are all in the same direction. Therefore, under the assumption that $E_1 \cap E_2 = \emptyset$, the optimal solution (x^*, y^*) also satisfies the following angle conditions:

$$\begin{cases} x^* \in \Omega(E_1) \quad \text{and} \quad y^* \in \Omega(E_2), \\ \theta(y^* - x^*, A_1 x^* + b_1) = 0, \\ \theta(x^* - y^*, A_2 y^* + b_2) = 0. \end{cases} \tag{3.1}$$

We note here that the above angle condition is also sufficient for optimality. Actually, we need to establish the stronger and more useful result that, if two points $x \in \Omega(E_1)$ and $y \in \Omega(E_2)$ have small angles $\theta(y - x, A_1 x + b_1)$ and $\theta(x - y, A_2 y + b_2)$, the pair (x, y) is close to the optimal solution (x^*, y^*). To show this we need two simple lemmas.

LEMMA 3.1. *If u and v are two nonzero vectors in R^n with $\|u\| = \|v\|$ and $\theta = \theta(u, v)$, then*

$$\|u - v\| = 2\|u\| \sin \frac{1}{2}\theta.$$

Proof. The result follows immediately from the following equations:

$$\begin{aligned} \|u - v\|^2 &= \|u\|^2 - 2u^T v + \|v\|^2 \\ &= \|u\|^2 - 2\|u\|\|v\| \cos\theta + \|v\|^2 \\ &= (\|u\| - \|v\|)^2 + 2\|u\|\|v\|(1 - \cos\theta) \\ &= 4\|u\|^2 \sin^2 \frac{\theta}{2}. \qquad \square \end{aligned}$$

LEMMA 3.2. *Let z, p, and h be vectors in R^n with $p \neq 0$, $h \neq 0$, and let $\theta = \theta(p, h)$ and ϕ be a scalar. If $z^T p \leq \phi$, then*

$$z^T h \leq 2\|z\|\|h\| \sin \frac{\theta}{2} + \frac{\|h\|}{\|p\|}\phi.$$

Proof. It follows from the previous lemma that

$$\begin{aligned} z^T h &= z^T \left(h - \frac{\|h\|}{\|p\|} p \right) + \frac{\|h\|}{\|p\|} z^T p \\ &\leq \|z\| \left\| h - \frac{\|h\|}{\|p\|} p \right\| + \frac{\|h\|}{\|p\|}\phi \\ &= 2\|z\|\|h\| \sin \frac{\theta}{2} + \frac{\|h\|}{\|p\|}\phi. \qquad \square \end{aligned}$$

To facilitate our presentation of our next key theorem, we introduce some notation. Let κ_1 and κ_2 be the smallest eigenvalues of A_1 and A_2, respectively. Then, for any $z \in R^n$,

$$\kappa_1||z||^2 \leq z^T A_1 z \quad \text{and} \quad \kappa_2||z||^2 \leq z^T A_2 z.$$

We also define η_1 and η_2 as the following bounds:

$$\eta_1 := \max_{z\in\Omega(E_1)} \|A_1 z + b_1\| \quad \text{and} \quad \eta_2 := \max_{z\in\Omega(E_2)} \|A_2 z + b_2\|.$$

Notice that η_i is zero if and only if the corresponding ellipsoid is a singleton. We will assume that both ellipsoids contain interior points and thus that both η's are positive. Now we give the key theorem below.

THEOREM 3.3. *Let both E_1 and E_2 have interior points. If $x \in \Omega(E_1)$, $y \in \Omega(E_2)$, and $x \neq y$, then for any $u \in E_1$ and $v \in E_2$*

$$\|y-x\| + \frac{\kappa_1}{2\eta_1}\|u-x\|^2 + \frac{\kappa_2}{2\eta_2}\|v-y\|^2 \leq \|u-v\| + 2\|u-x\|\sin\frac{\theta_1}{2} + 2\|v-y\|\sin\frac{\theta_2}{2},$$

where $\theta_1 = \theta(y-x, A_1x+b_1)$ and $\theta_2 = \theta(x-y, A_2y+b_2)$.

Proof. As before, we let $q_1(w) := \frac{1}{2}w^T A_1 w + b_1^T w + \alpha_1$. Because $x \in \Omega(E_1)$ and $u \in E_1$, we have

$$\begin{aligned} 0 &\geq q_1(u) - q_1(x) \\ &= (u-x)^T \nabla q_1(x) + \frac{1}{2}(u-x)^T A_1(u-x) \\ &\geq (u-x)^T(A_1x+b_1) + \frac{\kappa_1}{2}\|u-x\|^2. \end{aligned}$$

Therefore, we have

$$(u-x)^T(A_1x+b_1) \leq -\frac{\kappa_1}{2}\|u-x\|^2.$$

We apply Lemma 3.2 to the above inequality with $\phi = -\frac{\kappa_1}{2}\|u-x\|^2$, $z = (u-x)$, $p = A_1x+b_1$, and $h = y-x$ to get

$$(u-x)^T(y-x) \leq 2\|u-x\|\|y-x\|\sin\frac{\theta_1}{2} - \frac{\kappa_1\|y-x\|}{2\eta_1}\|u-x\|^2.$$

Similarly, for the case of E_2, we interchange the roles of x and y and replace u by v to get

$$(v-y)^T(x-y) \leq 2\|v-y\|\|x-y\|\sin\frac{\theta_2}{2} - \frac{\kappa_2\|y-x\|}{2\eta_2}\|v-y\|^2.$$

Adding the above two inequalities and rearranging terms, we have

$$\begin{aligned} &\|y-x\|^2 + \frac{\kappa_1\|y-x\|}{2\eta_1}\|u-x\|^2 + \frac{\kappa_2\|y-x\|}{2\eta_2}\|v-y\|^2 \\ &\quad\leq (v-u)^T(y-x) + 2\|u-x\|\|y-x\|\sin\frac{\theta_1}{2} + 2\|v-y\|\|x-y\|\sin\frac{\theta_2}{2} \\ &\quad\leq \|u-v\|\|y-x\| + 2\|u-x\|\|y-x\|\sin\frac{\theta_1}{2} + 2\|v-y\|\|x-y\|\sin\frac{\theta_2}{2}. \end{aligned}$$

The desired result follows immediately when we divide the above inequality by $\|y - x\|$. ∎

Some useful results follow from the above theorem. First, we show that the angle condition (3.1) is a necessary and sufficient condition for optimality.

THEOREM 3.4. *Let E_1 and E_2 have nonempty interiors and $E_1 \cap E_2 = \emptyset$. The pair (x^*, y^*) satisfies the angle condition* (3.1) *if and only if (x^*, y^*) is the optimal solution of problem* (1.1).

Proof. The necessary part has already been given in the paragraph prior to the angle condition (3.1). Now we show the sufficient part. We apply the above theorem to (x^*, y^*) and use the assumption that $\theta_1 = \theta_2 = 0$ to get the result that for any $u \in E_1$ and $v \in E_2$, $\|x^* - y^*\| \le \|u - v\|$. Therefore the pair (x^*, y^*) is optimal. ∎

For our convergence analysis the following result is useful.

THEOREM 3.5. *Let E_1 and E_2 have nonempty interiors and $E_1 \cap E_2 = \emptyset$, and let (x^*, y^*) be the optimal solution of problem* (1.1). *Then, for any $x \in \Omega(E_1)$ and $y \in \Omega(E_2)$,*

$$\frac{\kappa_1}{\eta_1}\|x - x^*\|^2 + \frac{\kappa_2}{\eta_2}\|y - y^*\|^2 \le 4\sigma \left(\sin\frac{\theta_1}{2} + \sin\frac{\theta_2}{2} \right),$$

where $\theta_1 = \theta(y - x, A_1 x + b_1)$, $\theta_2 = \theta(x - y, A_2 y + b_2)$, and

$$\sigma := \max\{\|u - v\| : u \in E_1 \quad \text{and} \quad v \in E_2\}. \tag{3.2}$$

Proof. This follows from Theorem 3.3 by letting $u = x^*$ and $v = y^*$ and using the fact that $\|x - y\| \ge \|x^* - y^*\|$. ∎

The following corollary gives a justification for using $\theta_1 = \theta_2 = 0$ as the stopping criterion of the algorithm.

COROLLARY 3.6. *Let E_1 and E_2 have nonempty interiors. If $x \in \Omega(E_1)$ and $y \in \Omega(E_2)$ such that $x \ne y$ and $\theta(y - x, A_1 x + b_1) = \theta(x - y, A_2 y + b_2) = 0$, then $E_1 \cap E_2 = \emptyset$ and the pair (x, y) is the unique optimal solution of problem* (1.1).

Proof. Suppose that $E_1 \cap E_2 \ne \emptyset$; then let $z \in E_1 \cap E_2$. We apply Theorem 3.3 to the pair (x, y) with $u = v = z$. It follows from the inequality of Theorem 3.3 that at least one of θ_1 and θ_2 is nonzero, which contradicts our assumption that $\theta_1 = \theta_2 = 0$. Therefore, we have $E_1 \cap E_2 = \emptyset$, and problem (1.1) has a unique optimal solution. Then the optimality of the pair (x, y) follows immediately from Theorem 3.5. ∎

4. Convergence analysis. The distance between any two balls $B(c_1, r_1)$ and $B(c_2, r_2)$ is easy to find. Indeed, if we assume that $r_1 > 0$, $r_2 > 0$ and the two balls are disjoint, then the closest two points on the boundaries $\Omega(B(c_1, r_1))$ and $\Omega(B(c_2, r_2))$ are given respectively by

$$\hat{x} = c_1 + \frac{r_1}{\|c_1 - c_2\|}(c_2 - c_1) \quad \text{and} \quad \hat{y} = c_2 + \frac{r_2}{\|c_1 - c_2\|}(c_1 - c_2). \tag{4.1}$$

Our algorithm can be viewed as if the two ellipsoids were being iteratively approximated locally by balls. Therefore, for our convergence analysis, we first analyze how the distance between any two points on the boundaries of the two balls, say $x \in \Omega(B(c_1, r_1))$ and $y \in \Omega(B(c_2, r_2))$, is related to the shortest distance $\|\hat{x} - \hat{y}\|$ in terms of the angles $\theta(y - x, x - c_1)$ and $\theta(x - y, y - c_2)$. This result is contained in the following fundamental lemma, which is not only useful for our analysis but also geometrically interesting in its own right. But the proof is long and provides little insight into the algorithm; hence we include the proof in the appendix.

LEMMA 4.1. *Let $B(c_1, r_1)$ and $B(c_2, r_2)$ be two disjoint balls with $r_1 > 0$ and $r_2 > 0$, and let $\hat{x}$ and $\hat{y}$ be defined as in* (4.1). *For any two points $x \in \Omega(B(c_1, r_1))$ and $y \in \Omega(B(c_2, r_2))$,*

$$\|x - y\| - \|\hat{x} - \hat{y}\| \geq \frac{4}{\eta}\left(r_1 d \sin^2 \frac{\theta_1}{2} + r_2 d \sin^2 \frac{\theta_2}{2} + r_1 r_2 \sin^2 \frac{(\theta_1 - \theta_2)}{2}\right),$$

where $d = \|x - y\|$, $\theta_1 = \theta(y - x, x - c_1)$, $\theta_1 = \theta(x - y, y - c_2)$, and $\eta = r_1 + r_2 + d + \|c_1 - c_2\|$.

We now give our convergence theorem. It is noted here that if $E_1 \cap E_2 \neq \emptyset$, then any point in the intersection is considered a solution.

THEOREM 4.2. *If E_1 and E_2 have nonempty interiors, then either any sequence generated by the algorithm terminates at, or any accumulation point of the sequence is a solution of, problem* (1.1). *Furthermore, if $E_1 \cap E_2 = \emptyset$, then the sequence converges to the unique solution of problem* (1.1).

Proof. Suppose the algorithm terminates in a finite number of iterations. In this case, as mentioned in the paragraph following the description of the algorithm, either $t_1 \geq t_1$ or $t_1 < t_2$, but $\theta_1 = \theta_2 = 0$. If $t_1 \geq t_2$, then all the points $c_1 + t(c_2 - c_1)$ with $t \in [t_2, t_1]$ are in $E_1 \cap E_2$. If $t_1 < t_2$ but $\theta_1 = \theta_2 = 0$, then it follows from Corollary 3.6 that $E_1 \cap E_2 = \emptyset$ and the new pair $(\bar{x}, \bar{y})$ is the unique optimal solution of problem (1.1).

We now consider the case in which the generated sequence is infinite. In this case, we have $t_1 < t_2$ in each iteration, and the two balls $B(c_1, r_1)$ and $B(c_2, r_2)$ are mutually disjoint and entirely contained in the ellipsoids E_1 and E_2, respectively. Therefore, we have

$$\|c_1 - c_2\| \geq r_1 + r_2 + \|\bar{x} - \bar{y}\|.$$

On the other hand, by the triangle inequality, we also have

$$\begin{aligned} \|c_1 - c_2\| &= \|c_1 - x + y - y + x - c_2\| \\ &\leq \|c_1 - x\| + \|x - y\| + \|y - c_2\| \\ &\leq r_1 + r_2 + \|x - y\|. \end{aligned}$$

Then, from the two inequalities above, we have the monotonicity property:

$$\|x - y\| \geq \|\bar{x} - \bar{y}\|.$$

Therefore, the sequence of distances $\{\|x^k - y^k\|\}$ is monotone and hence converges, say to d^*. Consider the two cases $d^* = 0$ and $d^* \neq 0$.

For the case $d^* = 0$, let (x^*, y^*) be an accumulation point of $\{(x^k, y^k)\}$. Then there is a subsequence $\{(x^{k_m}, y^{k_m})\}$ converging to (x^*, y^*). Clearly $x^* = y^*$ because

$$\lim_{k_m \to \infty} \|x^{k_m} - y^{k_m}\| = d^* = 0.$$

The fact $x^* \in E_1 \cap E_2$ follows from $\{(x^{k_m})\} \subset E_1$, $\{(y^{k_m})\} \subset E_2$, and that E_1 and E_2 are two closed sets.

We now consider the case $d^* \neq 0$. Because the boundaries $\Omega(E_1)$ and $\Omega(E_2)$ are compact and because the stepsizes γ's are bounded away from zero, the radii r_1 and r_2 remain bounded below from zero throughout the computation. More specifically, there exists a positive number δ such that $r_1 \geq \delta$ and $r_2 \geq \delta$ in each iteration. Let

$\sigma = \max\{\|u - v\| : u \in E_1 \text{ and } v \in E_2\}$ be defined as in (3.2). Clearly, σ is an upper bound for all the generated quantities r_1, r_2, $\|c_1 - c_2\|$, and $\|x - y\|$. In our algorithm, we have $x \in \Omega(E_1)$ and $y \in \Omega(E_2)$. Then it follows from the previous lemma that

$$\begin{aligned}\|x - y\| - \|\bar{x} - \bar{y}\| &\geq \|x - y\| - \|\hat{x} - \hat{y}\| \\ &\geq \frac{1}{\sigma}\left(\delta d^* \sin^2\frac{\theta_1}{2} + \delta d^* \sin^2\frac{\theta_2}{2} + \delta^2 \sin^2\frac{(\theta_1 - \theta_2)}{2}\right).\end{aligned}$$

From the convergence of $\{\|x^k - y^k\|\}$, we have that

$$\lim_{k\to\infty}(\|x^k - y^k\| - \|x^{k+1} - y^{k+1}\|) = 0.$$

This result, combined with the above inequality, implies that

$$\lim_{k\to\infty}\theta(y^k - x^k, A_1x^k + b_1) = 0 \quad \text{and} \quad \lim_{k\to\infty}\theta(x^k - y^k, A_2y^k + b_2) = 0.$$

Then it follows immediately from Theorem 3.5 that $\{(x^k, y^k)\}$ converges to the unique optimal solution of problem (1.1). □

5. Discussion. The algorithm proposed here is simple and easy to implement. The major work in each iteration is merely the solution of two one-dimensional quadratic equations and the computation of two angles. We implemented it in MATLAB and did some preliminary testing. We found the algorithm to be very reliable and to work very well generally. But the convergence may become a little slow when the ellipsoids are small, thin, and far apart. This is mainly because the balls generated inside the ellipsoids become too tiny to produce substantial improvement in each iteration. Though the requirement that the balls be completely inside the ellipsoids is sufficient for convergence, it is not necessary. It is desirable to design an acceleration technique that can avoid such restriction and allow more adequate improvements.

The algorithm is well suited for large sparse matrices, because the computation involves only the matrices themselves and does not need factorization. Of course, for this situation, we have to choose vectors other than the centers of the ellipsoids as the initial points.

According to our convergence theorem, the algorithm will work when both ellipsoids have interiors. Therefore, we can check this condition by evaluating the two quadratic functions $q_1(x)$ and $q_2(x)$ at the centers $A_1^{-1}b_1$ and $A_2^{-1}b_2$, respectively. We apply the algorithm only when both values are strictly negative. If one is negative and the other is zero, the problem is reduced to the projection of a point to an ellipsoid. For this case, an algorithm such as the one proposed in [2, 3, 4, 6, 7, 8, 9] should apply. Of course, when both values are nonnegative, this is just a trivial case and no further computation is needed.

Appendix. The proof of Lemma 4.1. Here we give a proof for our main lemma. For doing this, we define two parallel hyperplanes which pass through the centers c_1 and c_2 of the two balls $B(c_1; r_1)$ and $B(c_2; r_2)$, respectively, and have a common normal vector $w = (x - y)/\|x - y\|$:

$$H_1 = \{z : w^T z = w^T c_1\} \quad \text{and} \quad H_2 = \{z : w^T z = w^T c_2\}.$$

The orthogonal projectors onto these two hyperplanes H_1 and H_2 are

$$\begin{aligned}P_1(z) &= (w^T c_1)w + (I - ww^T)z, \\ P_2(z) &= (w^T c_2)w + (I - ww^T)z.\end{aligned}$$

We now give some useful lemmas.

LEMMA A.1. *Let $B(c_1,r_1)$ and $B(c_2,r_2)$ be disjoint and $x \in \Omega(B(c_1,r_1))$, $y \in \Omega(B(c_2,r_2))$, and $d = \|x-y\|$. Let $\theta_1 = \theta(y-x, x-c_1)$ and $\theta_2 = \theta(x-y, y-c_2)$. Then*

(1) $\|P_1(x) - c_1\| = r_1 \sin\theta_1$ *and* $\|P_2(y) - c_2\| = r_2 \sin\theta_2$*;*

(2) $P_1(x) - P_2(y) = ww^T(c_1 - c_2) = (r_1\cos\theta_1 + r_2\cos\theta_2 + d)w$.

Proof. We use the fact that the two vectors $ww^T(x-c_1)$ and $(I-ww^T)(x-c_1)$ are orthogonal to get

$$\begin{aligned}
r_1^2 &= \|x-c_1\|^2 \\
&= \|ww^T(x-c_1) + (I-ww^T)(x-c_1)\|^2 \\
&= \|ww^T(x-c_1)\|^2 + \|(I-ww^T)(x-c_1)\|^2 \\
&= (\|x-c_1\|\cos\theta_1)^2 + \|ww^Tc_1 + (I-ww^T)x - c_1\|^2 \\
&= (r_1\cos\theta_1)^2 + \|P_1(x) - c_1\|^2.
\end{aligned}$$

Therefore, it follows that $\|P_1(x) - c_1\| = (r_1^2 - r_1^2\cos^2\theta_1)^{\frac{1}{2}} = r_1\sin\theta_1$.

The result $\|P_2(y) - c_2\| = r_2\sin\theta_2$ can be proven similarly. To prove (2), we notice that $(I-ww^T)(x-y) = 0$, and we have

$$\begin{aligned}
P_1(x) - P_2(y) &= (w^Tc_1)w + (I-ww^T)x - (w^Tc_2)w - (I-ww^T)y \\
&= ww^T(c_1-c_2) + (I-ww^T)(x-y) \\
&= ww^T(c_1-c_2).
\end{aligned}$$

Then the above equality, in turn, implies that

$$\begin{aligned}
P_1(x) - P_2(y) &= ww^T((c_1-x) + (x-y) + (y-c_2)) \\
&= w^T(c_1-x)w + w^T(x-y)w + w^T(y-c_2)w \\
&= (r_1\cos\theta_1 + \|x-y\| + r_2\cos\theta_2)w. \qquad \square
\end{aligned}$$

LEMMA A.2. *Let the assumptions of the previous lemma hold; then*

$$\|c_1-c_2\|^2 \le (r_1\cos\theta_1 + r_2\cos\theta_2 + d)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2.$$

Proof. A direct calculation shows that the two vectors $P_1(x) - P_2(y)$ and $c_1 - P_1(x) + P_2(y) - c_2$ are orthogonal. Therefore, by the previous lemma and the triangle inequality,

$$\begin{aligned}
\|c_1-c_2\|^2 &= \|P_1(x) - P_2(y) + c_1 - P_1(x) + P_2(y) - c_2\|^2 \\
&= \|P_1(x) - P_2(y)\|^2 + \|c_1 - P_1(x) + P_2(y) - c_2\|^2 \\
&\le (r_1\cos\theta_1 + r_2\cos\theta_2 + d)^2 + (\|c_1 - P_1(x)\| + \|P_2(y) - c_2\|)^2 \\
&= (r_1\cos\theta_1 + r_2\cos\theta_2 + d)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2. \qquad \square
\end{aligned}$$

We now give a proof of our main lemma.

Proof of Lemma 4.1. We use the fact that $\|c_1-c_2\| = r_1 + r_2 + \|\hat{x}-\hat{y}\|$ to get

$$\begin{aligned}
\|x-y\| - \|\hat{x}-\hat{y}\| &= (d + r_1 + r_2) - (r_1 + r_2 + \|\hat{x}-\hat{y}\|) \\
&= (d + r_1 + r_2) - \|c_1-c_2\| \\
&= \frac{1}{\eta}((d + r_1 + r_2)^2 - \|c_1-c_2\|^2).
\end{aligned}$$

Using Lemma A.2 and by expansion and simplification, we get

$$\begin{aligned}
&(d + r_1 + r_2)^2 - \|c_1 - c_2\|^2 \\
&\qquad \geq (d + r_1 + r_2)^2 - (r_1 \cos\theta_1 + r_2 \cos\theta_2 + d)^2 - (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2 \\
&\qquad = 4\left(r_1 d \sin^2 \frac{\theta_1}{2} + r_2 d \sin^2 \frac{\theta_2}{2} + r_1 r_2 \sin^2 \frac{(\theta_1 - \theta_2)}{2} \right).
\end{aligned}$$

Incorporating the above inequality into the previous equality, we then get the desired result. □

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