STA32 Homework 5 Solution

Book Homework

1. Let X be the time in years until a major flood. Then, $X \sim Exp(\lambda = 1/3)$.
   (a) $P(X < 5) = 1 - e^{-5/3} = 0.8111$.
   (b) $P(X > 2) = e^{-2/3} = 0.5134$
   (c) $P(X < 6|X > 4) = P(X < 2) = 1 - P(X > 2) = 1 - 0.5134 = 0.4866$.
   (d) $\mu_X = 1/\lambda = 3, \sigma = \sqrt{1/\lambda^2} = 3$.
   (e) $X_{50} = \ln(\frac{100-50}{100})/(-1/3) = 2.0794$.

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2. Let X be the time in minutes until a car accident. Then, $X \sim Exp(\lambda = 1/60)$.
   (a) $P(X < 120) = 1 - e^{-120/60} = 0.8647$.
   (b) $P(45 < X < 75) = P(X > 45) - P(X > 75) = e^{-45/60} - e^{-75/60} = 0.1859$.
   (c) $\sigma_X^2 = 1/\lambda^2 = 60^2 = 3600$.
   (d) $P(X < 300|X > 180) = P(X < 120) = 0.8647$.
   (e) Let Y be the number of car accidents in the two hours. Then, $\mu_Y = 2 = \lambda$, so $Y \sim Poission(2)$. $P(Y = 3) = \frac{\lambda^x e^{-\lambda}}{x!} = 0.1804$.

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3. Let X be the wait time at the fast food restaurant, and $X \sim Uniform(1, 15)$.
   (a) $P(X < 6) = \frac{6-1}{15-1} = 0.3571$.
   (b) $P(5 < X < 10) = \frac{10-1}{15-1} - \frac{5-1}{15-1} = 0.3571$.
   (c) $\mu_X = \frac{1+15}{2} = 8, \sigma_X^2 = \frac{(15-1)^2}{12} = 16.3333$.
   (d) $P(X < 10|X > 5) = \frac{P(X<10)}{P(X>5)} = \frac{\frac{10-1}{15-1}}{\frac{15-5}{15-1}} = 0.9$.

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4. Let X be the weight of female cat, and $X \sim Normal(4.1, 0.6)$
   (a) $P(3.7 < X < 4) = P(-0.67 < Z < -0.17) = P(Z < -0.17) - P(-0.67 < Z) = 0.4325 - 0.2514 = 0.1811$.
   (b) $P(X > \mu + 0.5\sigma) = P(Z > 0.5) = 1 - P(Z \leq 0.5) = 1 - 0.6915 = 0.3085$.
   (c) $P(X > 4.5) = P(Z > 0.67) = 1 - P(Z \leq 0.67) = 1 - 0.7486 = 0.2514$
   (d) Let Y be the number of cats who weigh more than 4.5 kg out of 6 cats. Then, $Y \sim Binomial(n = 6, p = 0.2514)$. $P(Y = 1) = {}_6C_1(0.2514)(1 - 0.2514)^5 = 0.354622$.

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5. Let X be the IQ score, $X \sim N(100, 15)$
   (a) $P(X > 120) = P(Z > \frac{120-100}{15}) = P(Z > 1.33) = 1 - P(Z < 1.33) = 1 - 0.9082 = 0.0918$.
   (b) The first quartile for a Z-score, Z_{25} is -0.67. The first quartile for IQ is $X_{25} = Z_{25}\sigma + \mu = -0.67(15) + 100 = 89.95$. The third quartile for a Z-score, Z_{75} is 0.67. The third quartile for IQ is $X_{75} = Z_{75}\sigma + \mu = 0.67(15) + 100 = 110.05$.. The median (the second quartile) of a normal distribution is the mean, so the median is 100.

(c) The lower cutoff for IQ: $89.95 - 1.5(110.05 - 89.95) = 59.8$
The upper cutoff for IQ: $110.05 + 1.5(110.05 - 89.95) = 140.2$
$P(Y > 2) = 1 - P(Y \leq 2) = 1 - P(Y = 0) - P(Y = 1) - P(Y = 2) = 0.322.$

(d) $P(X > 140.2 \text{ or } X < 59.8) = P(Z > 2.68) + P(Z < -2.68) = 2P(Z < -2.68) = 0.0074.$

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6. We are given $X_1 \sim N(200, 50)$ and $X_2 \sim N(230, 10)$. Then,
$Y_1 = (X_1 - X_2) \sim N(-30, \sqrt{50^2 + 10^2} = 50.9902)$
$Y_2 = (X_1 + X_2) \sim N(430, \sqrt{50^2 + 10^2} = 50.9902)$
$\bar{X} = (X_1 + X_2)/2 \sim N(215, \sqrt{50^2 + 10^2}/2 = 25.4951)$

(a) $P(X_1 > X_2) = P(X_1 - X_2 > 0) = P(Y_1 > 0) == P(Z > \frac{0-(-30)}{50.99}) = P(Z > 0.59) = 1 - P(Z \leq 0.59) = 1 - 0.7224 = 0.2776.$

(b) $P(X_1 + X_2 > 400) = P(Y_2 > 400) = P(Z > \frac{400-(430)}{50.99}) = P(Z > -0.59) = 1 - P(Z \leq -0.59) = 1 - 0.2776 = 0.7224.$

(c) $P(220 < \bar{X} < 250) = P(\frac{220-215}{25.495} < X < \frac{250-215}{25.495}) = P(Z < 1.37) - P(Z < 0.2) = 0.9147 - 0.5793 = 0.3354.$

(d) The 30^{th} percentile for Z distribution, Z_{30}, is -0.52. Thus, the 30^{th} percentile for the sample mean is: $\bar{X}_{30} = (-0.52)25.495 + 215 = 201.74$

(e) The 60^{th} percentile for Z distribution, Z_{60}, is 0.25. Thus, the 60^{th} percentile for the total price is: $X_{60} = (0.25)50.99 + 430 = 442.75$

(f) We want the 90^{th} percentile for the total, and the 90^{th} percentile for Z distribution, Z_{90}, is 1.28. Thus, the 90^{th} percentile for the total is: $X_{2,90} = (1.28)50.99 + 495.27.$