STA 32 Homework 3 Solutions

Book Homework

1. (a) FALSE. $A \cap B^C$ is the same as saying "In A but not in B". $S-(A \cap B)$ means "everything but not in the intersection of A and B". Thus, $P(A \cap B^C) \neq P(S) - P(A \cap B)$
   (b) TRUE. When A and B are independent, $P(A \cap B) = P(A)P(B)$. Then, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = P(A) + P(B)(1 - P(A))$
   (c) TRUE. For example, $P(X > 3) = P(X \geq 4)$ because X takes on integer values.
   (d) TRUE. Since $F(a) = P(X \leq a) = 1 - P(X > a)$

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2. (a) $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.4 + 0.3 + 0.15 = 0.85$.
   (b) $P(X > 1) = P(X = 2) + P(X = 3) + P(X = 4) = 0.15 + 0.1 + 0.05 = 0.3$.
   (c) $\mu_X = 0(0.4) + 1(0.3) + 2(0.15) + 3(0.1) + 4(0.05) = 1.1$
   (d) $\sigma_X^2 = (0 - 1.1)^2(0.4) + (1 - 1.1)^2(0.3) + (2 - 1.1)^2(0.15) + (3 - 1.1)^2(0.1) + (4 - 1.1)^2(0.05) = 1.39$

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3. (a) $\mu_X = 1(0.4) + 2(0.2) + 3(0.2) + 4(0.1) + 5(0.1) = 2.3$.
   (b) $\sigma_X^2 = (1 - 2.3)2(0.4) + (2 - 2.3)2(0.2) + (3 - 2.3)2(0.2) + (4 - 2.3)2(0.1) + (5 - 2.3)2(0.1) = 1.81$. Alternatively, $\sigma_X^2 = 12(0.4) + 22(0.2) + 32(0.2) + 42(0.1) + 52(0.1) - 2.32 = 1.81$. Moreover, $\sigma_X = \sqrt{1.81} = 1.345$.
   (c) $Y = 10X$ The pmf of y:

y	10	20	30	40	50
$p(y)$	0.4	0.2	0.2	0.1	0.1

   (d) $\mu_Y = 10(0.4) + 20(0.2) + 30(0.2) + 40(0.1) + 50(0.1) = 23$.

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4. (a) The appropriate pdf is:

x	-2	-1	0	1	2
$p(x)$	0.1	0.05	0.50	0.20	0.15

   (b) This is the expected value, or μ_X. $\mu_X = xp(x) = (-2)0.1 + (-1)0.05 + (0)0.5 + (1)0.2 + (2)0.15 = 0.25$
   (c) $P(X = 1|X > 0) = \frac{P(X=1)}{P(X>0)} = \frac{0.2}{0.35} = 0.571$
   (d) $P(X < 0) = P(X = -1) + P(X = -2) = 0.15$

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5. (a) $P(X < 0.02) = \int_0^{0.02} 625x dx = \frac{625x^2}{2}\Big|_0^{0.02} = 0.125$
   (b) $\mu = \int_0^{0.04} 625x^2 dx + \int_{0.04}^{0.08} (50x - 625x^2) dx = \frac{625x^3}{3}\Big|_0^{0.04} + \left(25x^2 - \frac{625x^3}{3}\right)\Big|_{0.04}^{0.08} = 0.04$
   (c) $\sigma^2 = \int_0^{0.04} 625x^3 dx + \int_{0.04}^{0.08} (50x^2 - 625x^3) dx - \mu^2 = \frac{625x^4}{4}\Big|_0^{0.04} + \left(\frac{25x^3}{3} - \frac{625x^4}{4}\right)\Big|_{0.04}^{0.08} - 0.04^2 = 0.0002667$. Thus, $\sigma = 0.0163$.

(d) $F(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 625x^2/2 & \text{for } 0 < x \leq 0.04 \\ 50x - 625x^2/2 - 1 & \text{for } 0.04 < x \leq 0.08 \\ 1 & \text{for } 0.08 < x \end{cases}$

(e) The median x_m solves $F(x_m) = 0.5$. $F(0.04) = 625(0.04)^2/2 = 0.5$. So $x_m = 0.04$.

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6. (a) $\mu_X = \int_4^6 \left(\frac{3}{4}x - \frac{3x(x-5)^2}{4}\right)dx = \left(-9x^2 + \frac{5x^3}{2} - \frac{3x^4}{16}\right)\Big|_4^6 = 5$

(b) $\sigma_X^2 = \int_4^6 \left(\frac{3(x-5)^2}{4} - \frac{3(x-5)^4}{4}\right)dx = \left(\frac{(x-5)^3}{4} - \frac{3(x-5)^5}{20}\right)\Big|_4^6 = 0.2$

(c) $\mu_Y = 0.0394(\mu_X) = 0.0394(5) = 0.197$, $\sigma_Y^2 = (0.0394)^2\sigma_X^2 = (0.0394)^2(0.2) = 0.00031$.

(d) Let X_1, X_2 and X_3 be the three thicknesses, in millimeters. Then $S = X_1 + X_2 + X_3$ is the total thickness. $\mu_S = \mu_{X_1} + \mu_{X_2} + \mu_{X_3} = 3(5) = 15$. $\sigma_S^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 = 3(0.2) = 0.6$.

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7. (a) $\int f(x)dx = \int_{-1}^1 c(x^2+2)dx = c(x^3/3 + 2x)|_{-1}^1 = c(1/3+2) - c(-1/3-2) = c(1/3+6/3+1/3+6/3) = 14/3c$. Thus, for $f(x)$ to integrate to 1, $c = 3/14$.

(b) $F(x) = P(X \leq x) = \int_{-1}^x f(t)dt = \int_{-1}^x c(t^2+2)dt = c(t^2+2)dt = c(t^3/3+2x)|_{-1}^x = c(x^3/3+2x) - c(-1/3-2)$. Recall $c = 3/14$, and simplify to $F(x) = \frac{x^3+6x+7}{14}$, where $-1 \leq x \leq 1$, $F(x) = 0$ for $x < -1$, and $F(x) = 1$ for $x > 1$.

(c) $P(0 < X < 0.5) = F(0.50) - F(0) = \frac{2^3+6*2+7}{14} - \frac{0^3+6*0+7}{14} = 0.2232143$.

(d) $\mu_X = \int_{-1}^1 xf(x)dx = \int_{-1}^1 \frac{3}{14}(x^3+2x)dx = \frac{3}{14}(x^4/4+x^2)|_{-1}^1 = 0$
$\sigma_X^2 = \int_{-1}^1 x^2 f(x)dx - \mu_x^2 = \int_{-1}^1 \frac{3}{14}(x^4+2x^2)dx - 0^2 = \frac{3}{14}(x^5/5+2x^3/3)|_{-1}^1 = 0.3714286$. Thus, $\sigma_X = \sqrt{0.3714286} = 0.6094494$.

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8. (a) $\mu_{Z_1} = 3\mu_X = (3)(9.5) = 28.5$ and $\sigma_{Z_1} = 3\sigma_X = 3(0.4) = 1.2$

(b) $\mu_{Z_2} = \mu_Y - \mu_X = 6.8 - 9.5 = -2.7$, and $\sigma_{Z_2}^2 = \sqrt{\sigma_Y^2 + \sigma_X^2} = \sqrt{0.1^2 + 0.4^2} = 0.412$

(c) $\mu_{Z_3} = \mu_X + 4\mu_Y = 9.5 + 4(6.8) = 36.7$, and $\sigma_{Z_3}^2 = \sqrt{\sigma_X^2 + 16\sigma_Y^2} = \sqrt{0.4^2 + 16(0.1)^2} = 0.566$.

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9. (a) The median is the value, X_{50} such that $F(X_{50}) = 0.50$. So $1 - \exp(-2X_{50}) = 0.50$, $\exp(-2X_{50}) = 0.50$, $-2X_{50} = \ln(0.50)$. Thus, $X_{50} = -\ln(0.50)/2 = 0.3466$.

(b) The 10^{th} percentile is the value, X_{10} such that $F(X_{10}) = 0.10$. So $1 - \exp(-2X_{10}) = 0.10$, $\exp(-2X_{10}) = 0.90$, $-2X_{10} = \ln(0.90)$. Thus, $X_{10} = -\ln(0.90)/2 = 0.0527$.

(c) $P(1 < X < 3) = F(3) - F(1) = (1 - \exp(-2 \cdot 3)) - (1 - \exp(-2 \cdot 1)) = 0.9975 - 0.8647 = 0.1328$.

(d) $P(X < 3) = 1 - F(3) = 1 - (1 - \exp(-2 \cdot 3)) = 0.00248$.