# Vectors & Matrices

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- Gilbert Strang, Linear Algebra and its Applications, 5th Edition, Chapter 1.

# Vectors and Matrices

# Matrices

**Matrix:** a rectangular array of numbers or functions. E.g.,

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 0.5 \end{bmatrix}$$

**Elements**: The numbers (or functions) are the *elements* of the matrix.

**Rows**: The horizontal lines of entries.

**Columns**: The vertical lines of entries.

# Matrices

**Square Matrices**: as many rows as columns (3 below).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

**Indices of an element:** The first index is the number of the row and the second is the number of the column.

**Example**: $a_{23}$ is the entry in Row 2 and Column 3.

# Vectors

**Vectors:** Matrices having just a single row or column.

**Row vector**:

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

**Column vector**:

$$\begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

**Elements of a vector:** Only one index is needed, e.g. the 2nd element of the row vector above is denoted by $a_2$.

# Column Vector Notation

A **column vector** is of the form

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \qquad \text{For instance,} \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

When we refer to a vector, our default will be a column vector.

We will denote them with boldface letters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$

# Row Vector Notation

Denote vectors by ***lowercase*** boldface letters $\boldsymbol{a}, \boldsymbol{b}, \ldots$ or by its general component in brackets, $\boldsymbol{a} = [a_j]$.

A **row vector** is of the form

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}. \quad \text{For instance,} \quad \begin{bmatrix} -2 & 5 & 0.8 & 0 & 1 \end{bmatrix}.$$

Row vectors can be seen as **the transpose** of the corresponding column vector: E.g.

$$\mathbf{a}^T = \begin{bmatrix} -2 & 5 & 0.8 & 0 & 1 \end{bmatrix}, \mathbf{a} = \begin{bmatrix} -2 \\ 5 \\ 0.8 \\ 0 \\ 1 \end{bmatrix}$$

# Column Vector Notation

Often column vectors are represented in a row form (in parentheses) to save space.

Thus $\boldsymbol{v} = (1, 1, -1)$ is not a row vector!

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ is also written as } \boldsymbol{v} = (1, \quad 1, \quad -1)$$

# Matrix Notation

Denote matrices by capital boldface letters $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ , or as $\boldsymbol{A} = [a_{jk}]$.

By an $\boldsymbol{m} \times \boldsymbol{n}$ **matrix** we mean a matrix with $m$ rows and $n$ columns

$$\mathbf{A} = \left[a_{jk}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

# Linear Combinations of Vectors

# Linear Combination of Vectors

Vector addition

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \text{ add to: } \boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

Vector multiplication with a scalar

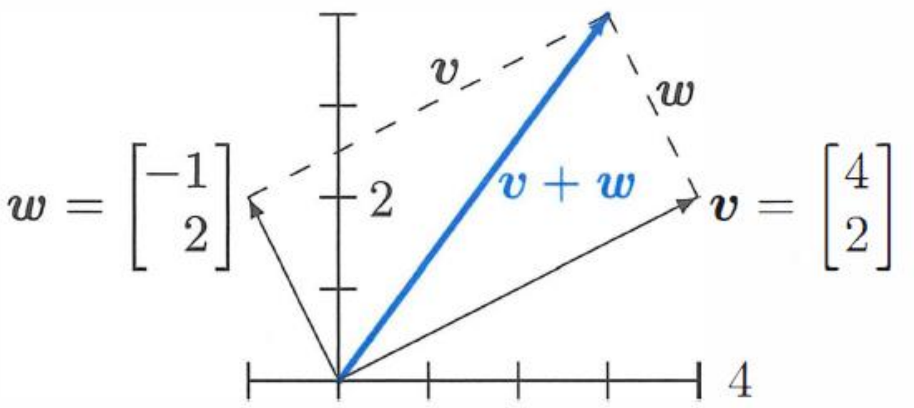
$$2\boldsymbol{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \boldsymbol{v} + \boldsymbol{v} \text{ and } -\boldsymbol{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$$

Linear combination of vectors. For $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$c\boldsymbol{v} + d\boldsymbol{w} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c + 2d \\ c + 3d \end{bmatrix}$$

# Vector Addition

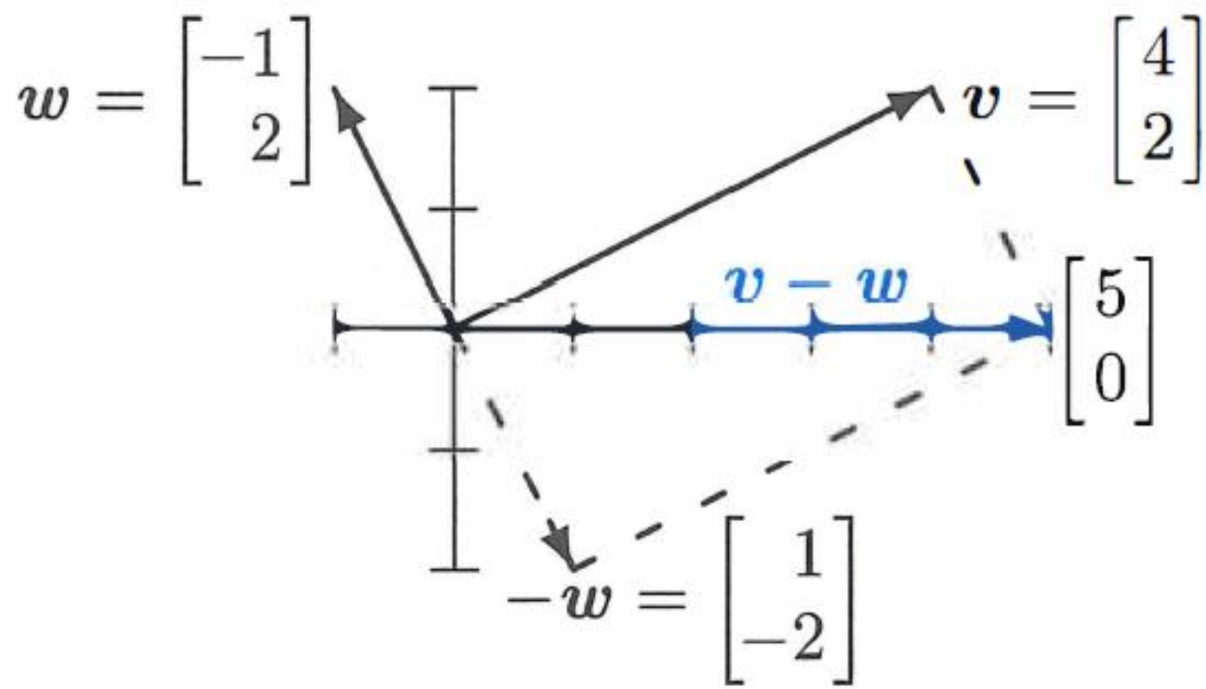
Vector addition




$$\boldsymbol{v} + \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

# Vector Subtraction

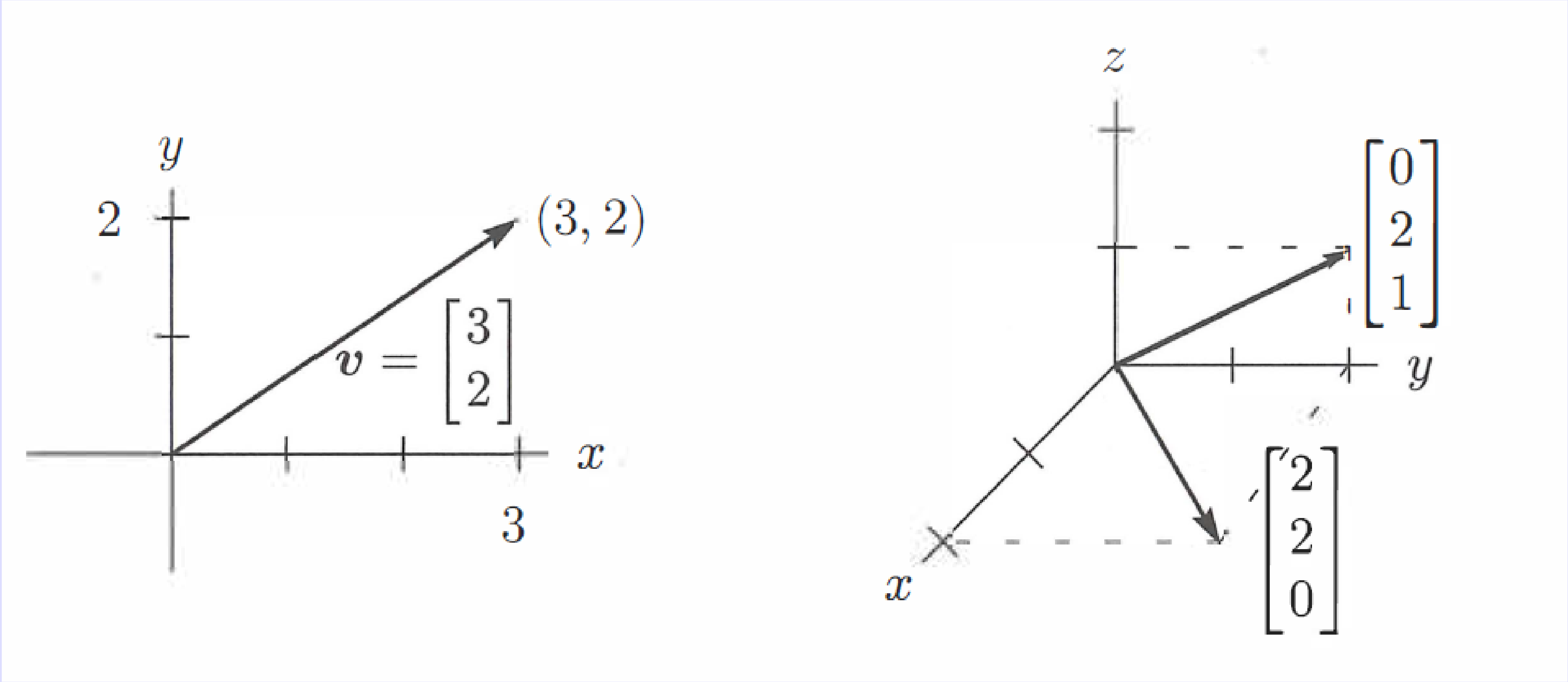
Vector subtraction




$$\boldsymbol{v} - \boldsymbol{w} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

# Vector Representation

There is a perfect match between the ***column vector*** and the ***arrow from the origin*** and the ***point where the arrow ends.***

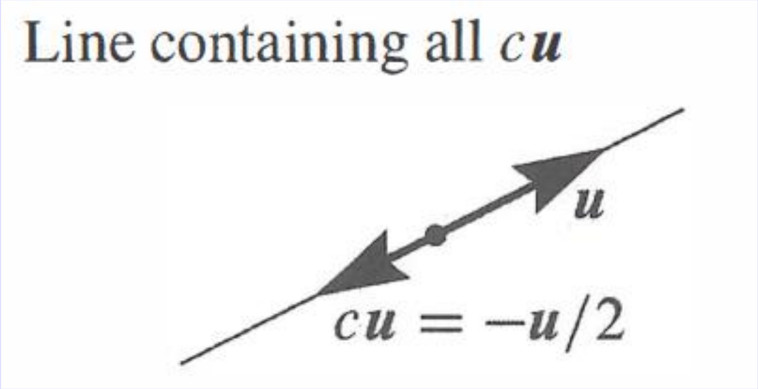


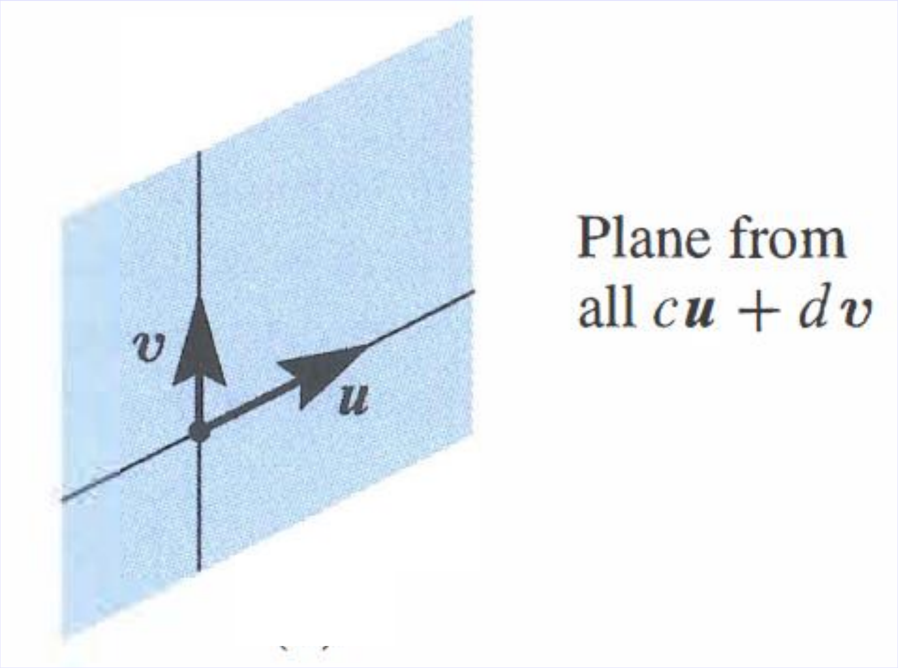
# Linear Combination of Vectors

1. The combinations $c\boldsymbol{u}$ fill a ***line through*** $(0, 0, 0)$.

2. The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ fill a ***plane through*** $(0, 0, 0)$.

3. The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ fill ***the 3D space*** [assume $\boldsymbol{w}$ not in the plane of $\boldsymbol{u}$ and $\boldsymbol{v}$]



# Inner Product of Vectors

# Inner Product of Vectors

**The dot product or inner product of $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$** is the number $\boldsymbol{v} \cdot \boldsymbol{w}$

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 w_1 + v_2 w_2$$

**Perpedicular vectors:**

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 4(-1) + 2(2) = 0$$

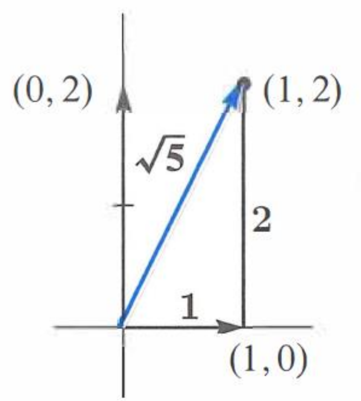
**Dot product** $v \cdot v$ $\|\boldsymbol{v}\|^2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = 14$

**Length squared**

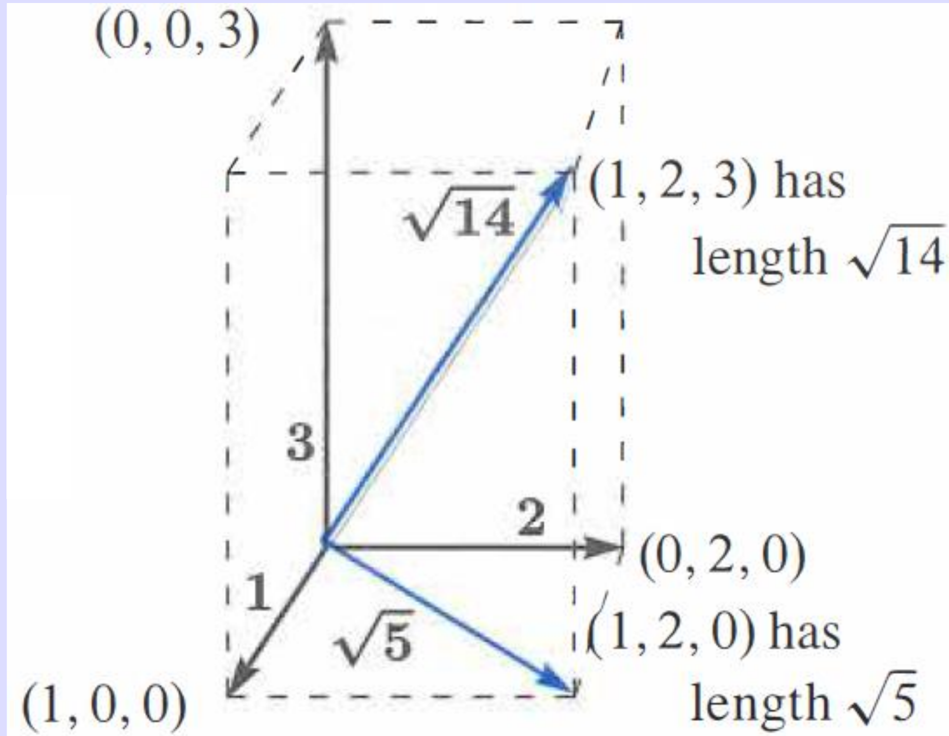
# Length of Vectors

The ***length*** $\|\boldsymbol{v}\|$ of $\boldsymbol{v}$ *is* the square root of $\boldsymbol{v}\cdot\boldsymbol{v}$ *:*

$$length = \|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$



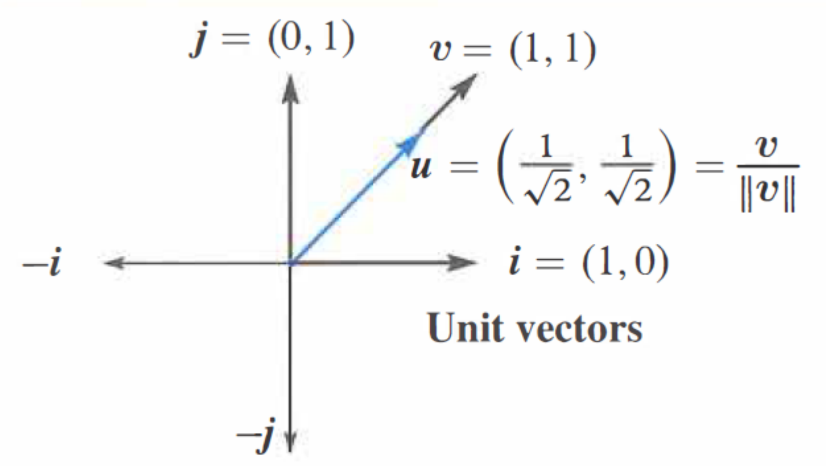

$$\begin{aligned} \boldsymbol{v}\cdot\boldsymbol{v} &= v_1^2 + v_2^2 + v_3^2 \\ 5 &= 1^2 + 2^2 \\ 14 &= 1^2 + 2^2 + 3^2 \end{aligned}$$

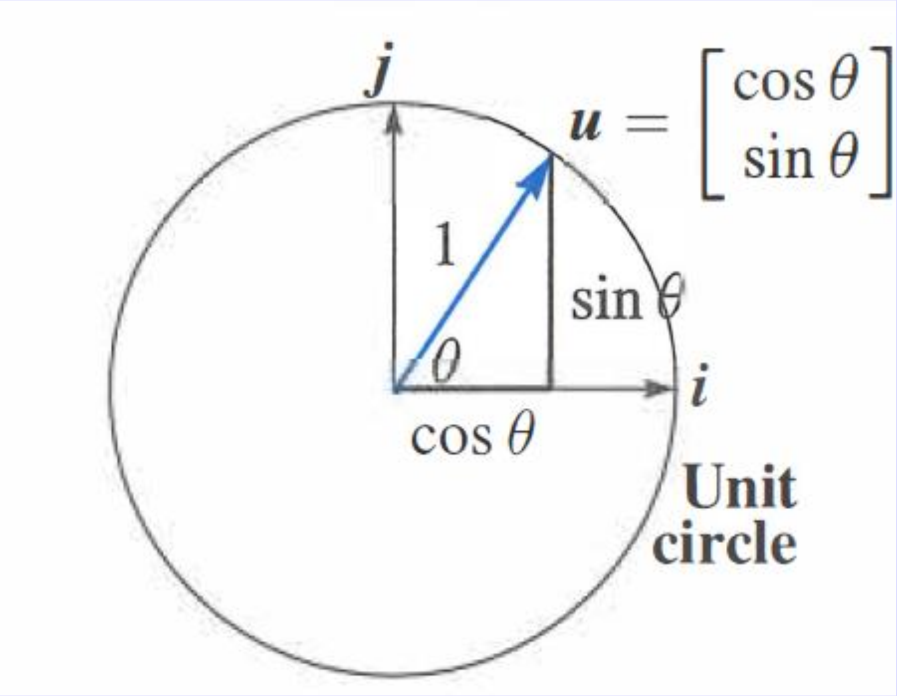



***A unit vector*** $\boldsymbol{u}$ *is a vector with length one, i.e.* $\boldsymbol{u}\cdot\boldsymbol{u} = 1$.

# Unit Vectors

$\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$ is a unit vector in the same direction as $\boldsymbol{v}$.

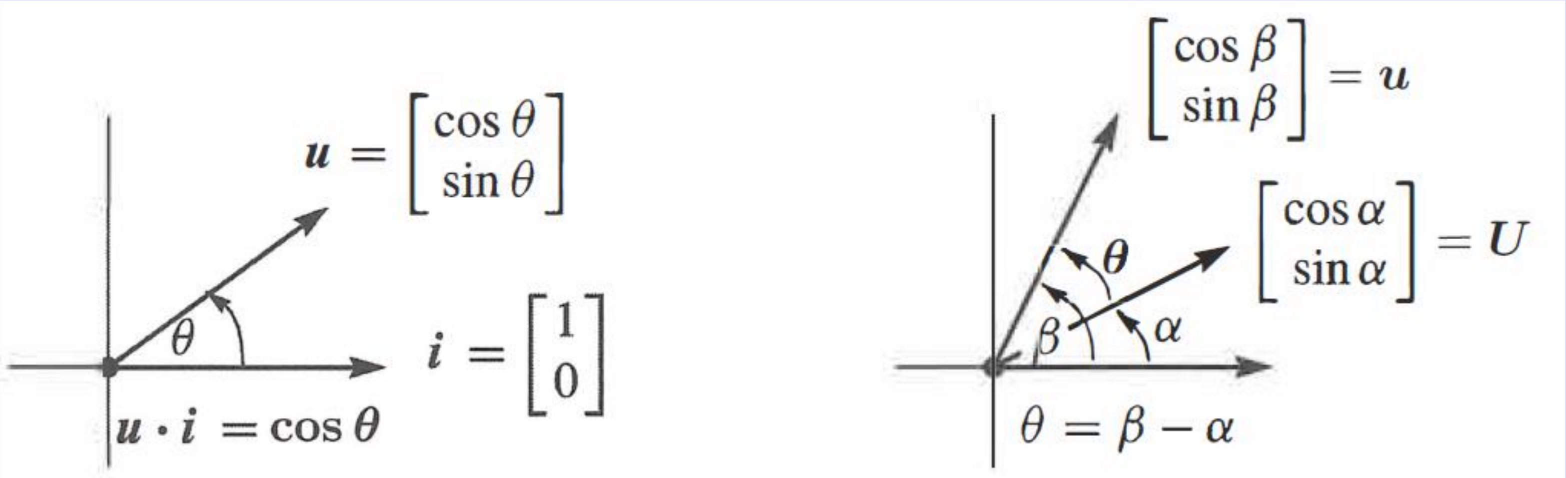


# The Angle Between Two Vectors

Unit vectors $\boldsymbol{u}$ and $\boldsymbol{U}$ at angle $\theta$ have $\boldsymbol{u} \cdot \boldsymbol{U} = \cos\theta$.

Certainly $|\boldsymbol{u} \cdot \boldsymbol{U}| \leq 1$




For non-zero vectors $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\|\|\boldsymbol{w}\|\cos\theta$

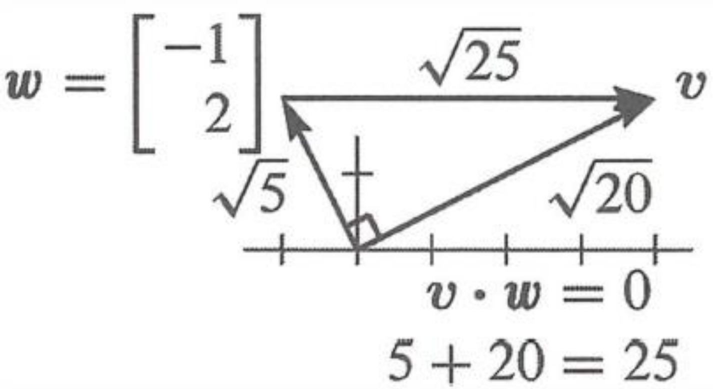
Schwartz Inequality $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\|$

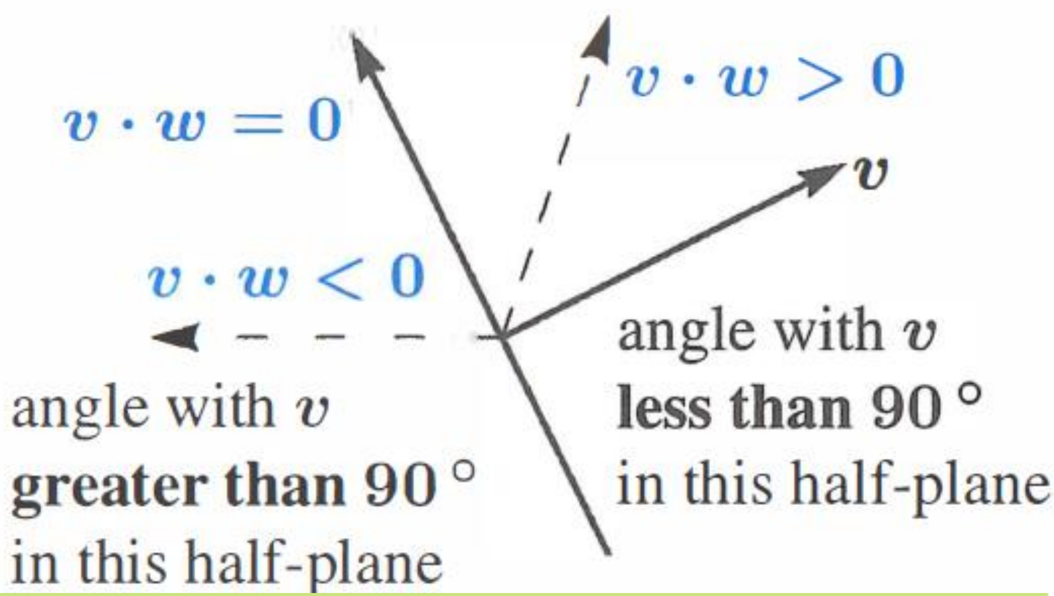
Triangle Inequality $\|\boldsymbol{v}+\boldsymbol{w}\| \leq \|\boldsymbol{v}\|+\|\boldsymbol{w}\|$

# Angle Between Two Vectors

*The dot product is* $\boldsymbol{v}\cdot\boldsymbol{w}=0$ *when* $\boldsymbol{v}$ *is perpendicular to* $\boldsymbol{w}$

*Pythagoras Theorem:* $\|\boldsymbol{v}\|^2+\|\boldsymbol{w}\|^2=\|\boldsymbol{v}-\boldsymbol{w}\|^2$



# Three Different Views of Linear Equations

# Three Different Views of $Ax = b$

The fundamental problem of linear algebra is to solve $n$ linear equations in $n$ unknowns; for example for $n = 2$ consider:

$$2x - y = 0\,, -x + 2y = 3$$

## Matrix Picture

We can write the above 2 Eqs. as a matrix $(2 \times 2)$
Eq. in the form:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Solving $\boldsymbol{Ax} = \boldsymbol{b}$ is fundamental to the first part of our course.

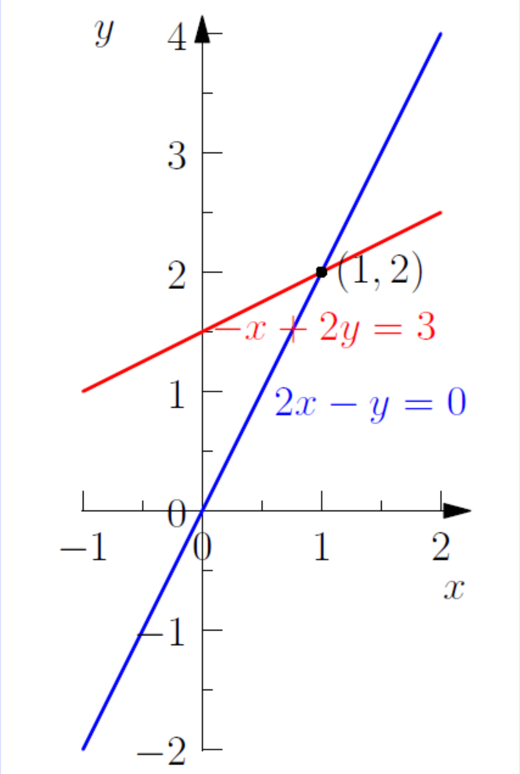
# Three Different Views of $Ax = b$

The fundamental problem of linear algebra is to solve $n$ linear equations in $n$ unknowns; for example for $n = 2$ consider:

$$2x - y = 0\,, -x + 2y = 3$$

## Row Picture

Plot the points that satisfy each equ.
The intersection of the plots (if they do intersect) represents the solution to the system of eqs.




$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\times 1 - 1\times 2 \\ -1\times 1 + 2\times 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

The solution to a 3D system of eqs.
is the intersection of 3 planes (if there is one).

# Three Different Views of $Ax = b$

The fundamental problem of linear algebra is to solve $n$ linear equations in $n$ unknowns; for example for $n = 2$ consider:

$$2x - y = 0, -x + 2y = 3$$

**Column Picture**

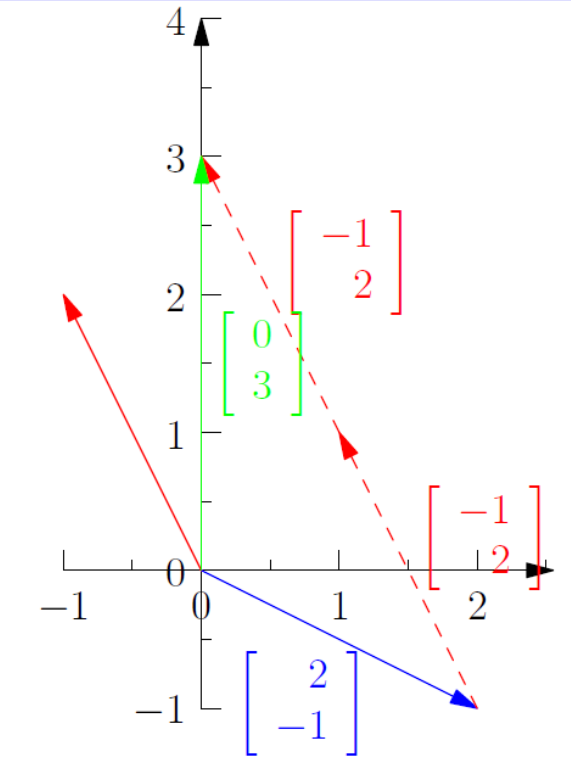
We write the above Eqs as:

$$x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We are looking for **the linear combination**

of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, that is equal to $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



# Three Different Views of $Ax = b$

The 3 views of linear equations also suggest different ways for performing matrix × vector multiplication:

**Linear Combination of Column Vectors**

We can write the above 2 Eqs. as a matrix $(2 \times 2)$ Eq. in the form:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

**Inner Product of the rows of *A* with *x***

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \times 1 - 1 \times 2 \\ -1 \times 1 + 2 \times 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

# Can we solve $Ax = b$ for any $b$?

# Three Different Views of $Ax = b$

In the column and matrix views of $\boldsymbol{Ax} = \boldsymbol{b}$, the r.h.s. of the Eq. is a vector $\boldsymbol{b}$.

Given $\boldsymbol{A}$, can we solve: $\boldsymbol{Ax} = \boldsymbol{b}$ for every possible **b**?

Do the linear combinations of the column vectors fill the *xy*-plane (or space, in 3D)?

If the answer is no, we say that $\boldsymbol{A}$ is a *singular matrix*.

In this singular case its column vectors are *linearly dependent*; all linear combinations of those vectors lie on a point or line (in 2D) or on a point, line or plane (in 3D).

The combinations don't fill the whole space.

# Difference Matrix

Matrices arise from the combination of column vectors.

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\boldsymbol{u}} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\boldsymbol{v}} + x_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{w}} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or $x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boldsymbol{Ax}, \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The matrix $\boldsymbol{A}$ is called **a difference matrix**. It acts on the vector $\boldsymbol{x}$.

The output $\boldsymbol{Ax}$ is a combination $\boldsymbol{b}$ of the columns of $\boldsymbol{A}$.

# Difference Matrix

$\boldsymbol{Ax}$ is also dot products with rows:

$$\boldsymbol{Ax} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1 \quad 0 \quad 0) \cdot (x_1 \quad x_2 \quad x_3) \\ (-1 \quad 1 \quad 0) \cdot (x_1 \quad x_2 \quad x_3) \\ (0 \quad -1 \quad 1) \cdot (x_1 \quad x_2 \quad x_3) \end{bmatrix}$$

The new way is to work with $\boldsymbol{Ax}$ *a column at a time*

$$\boldsymbol{Ax} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

# When Can We Solve $Ax = b$?

Up to now, the numbers $x_1, x_2, x_3$ were known. The right hand side $\boldsymbol{b} = \boldsymbol{Ax}$ was not known.

**Now we think of $\boldsymbol{b}$ as known and we look for $\boldsymbol{x}$.**

*Old question:* Compute the linear combination $x_1\,\boldsymbol{u} + x_2\,\boldsymbol{v} + x_3\,\boldsymbol{w}$ to find $\boldsymbol{b}$.

*New question:* Which combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ produces a particular vector $\boldsymbol{b}$?

**This is the *inverse problem:* Find the input $\boldsymbol{x}$ that gives the desired output $\boldsymbol{b} = \boldsymbol{Ax}$.**

# When Can We Solve $Ax = b$?

I*nverse problem*-find the input $\boldsymbol{x}$ that gives the desired output $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}$.

System of Eqs:
$$\begin{aligned} x_1 \quad\quad\quad\quad &= b_1 \\ -x_1 + x_2 \quad\quad &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned}$$

*Solution* $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$:
$$\begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \end{aligned}$$

If the difference of the $\boldsymbol{x}$'s are the $\boldsymbol{b}$'s, the sum of the $\boldsymbol{b}$'s are now the $\boldsymbol{x}$'s.

*Here the matrix* $\boldsymbol{A}$ *is invertible: From given* $\mathbf{b}$*, we can find* $\boldsymbol{x}$*.*

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

# Note: Difference Matrix and Finite Differences

There is a useful analogy between the differences $\boldsymbol{Ax}$ and $\frac{dx}{dt} = b(t)$, and between $\boldsymbol{A}^{-1}\boldsymbol{b}$ and $x(t) = \int_0^t b(t)dt$. Let $x(t) = t^2$. Then $\frac{dx}{dt} = b(t) = 2t$.

This analogy is not quite right. Consider $t = 1,2,3$, then $dx/dt$ gives the squares (2,4,6). But $\boldsymbol{Ax} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

The right answer to make an analogy to $dx/dt$ would have been $2t - 1$!

*Backwards difference approximation gives the right analogy.*

*Backwards Difference:* $\frac{dx}{dt} \approx \frac{x(t+\Delta t)-x(t)}{\Delta t} = \frac{(t+1)^2-t^2}{1} = 2t - 1$

# When Can We Solve $Ax = b$?

Consider the following combination of vectors (note $\boldsymbol{w}^* =- - \boldsymbol{u} - \boldsymbol{v}$):

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\boldsymbol{u}} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\boldsymbol{v}} + x_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{w}^*} = \begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \equiv \boldsymbol{b}$$

or $\boldsymbol{Cx} = \boldsymbol{b}$ where

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

What is the inverse solution when $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$? $\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

There are infinite solutions $\boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$, $c$ an arbitrary constant

# When Can We Solve $Ax = b$?

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

What is the inverse solution when $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$? $\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

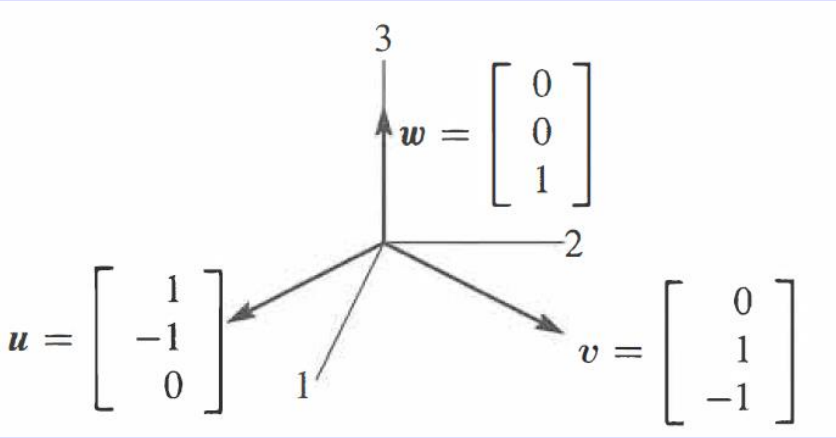
The sum of the entries on the left add to zero but on the right add to 9!

It is clear that now there are NO solutions!

No combination of $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}^*$ will produce the vector $\boldsymbol{b} = (1, 3, 5)$. The combinations don't fill the whole $\mathbb{R}^3$.

# Geometric Interpretation

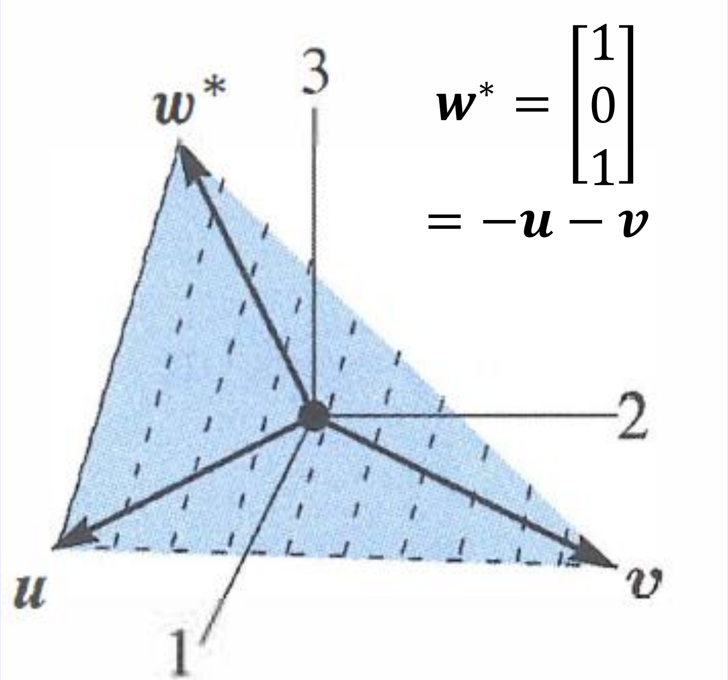
We can see this geometrically.




**Linear combinations $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}$** fill the whole space.

$\boldsymbol{w}$ *is not on the plane of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$.

Independent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.




**All linear combinations $x_1\boldsymbol{u} + x_2\boldsymbol{v} + x_3\boldsymbol{w}^*$ lie on the plane given by $\boldsymbol{b_1} + \boldsymbol{b_2} + \boldsymbol{b_3} = \boldsymbol{0}$.**

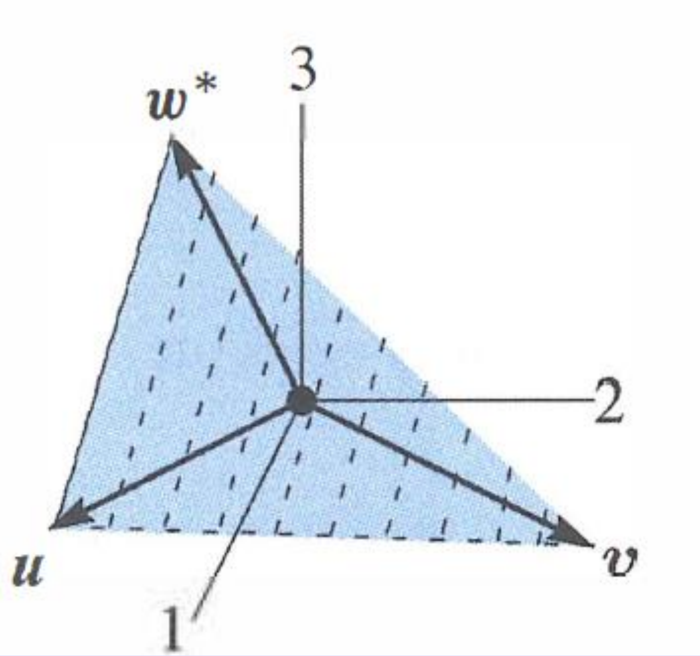
$\boldsymbol{w}^*$ *is on the plane of* $\boldsymbol{u}$ *and* $\boldsymbol{v}$.

Dependent vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ in a plane

# Subspaces

The columns of $\boldsymbol{C}$ lie in the same plane (they are *dependent*).

Many vectors in $\mathbb{R}^3$ do not lie in that plane. Those vectors cannot be written as a linear combination of the columns of $\boldsymbol{C}$ and correspond to values of $\boldsymbol{b}$ for which $\boldsymbol{Cx} = \boldsymbol{b}$ has no solution $\boldsymbol{x}$.




The linear combinations of the columns of $\boldsymbol{C}$ forms a 2D *subspace* of $\mathbb{R}^3$.

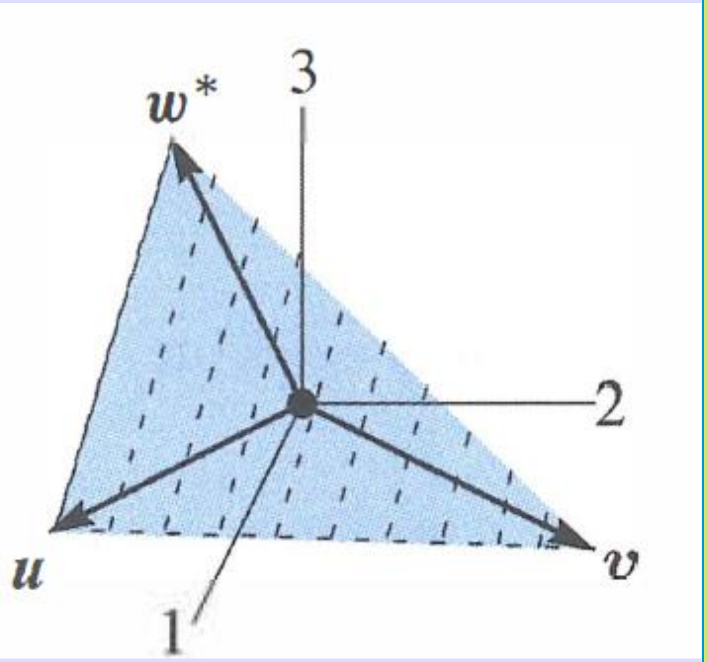
More on subspaces in a forthcoming lecture.

# Subspaces

The plane of combinations of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}^*$ is defined as all vectors $\boldsymbol{Cx}$.

But we know that the vectors **b** for which $\boldsymbol{Cx} = \boldsymbol{b}$ satisfy $b_1 + b_2 + b_3 = 0.$

So the plane of all combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ consists of all vectors whose components sum to $\mathbf{0}$.



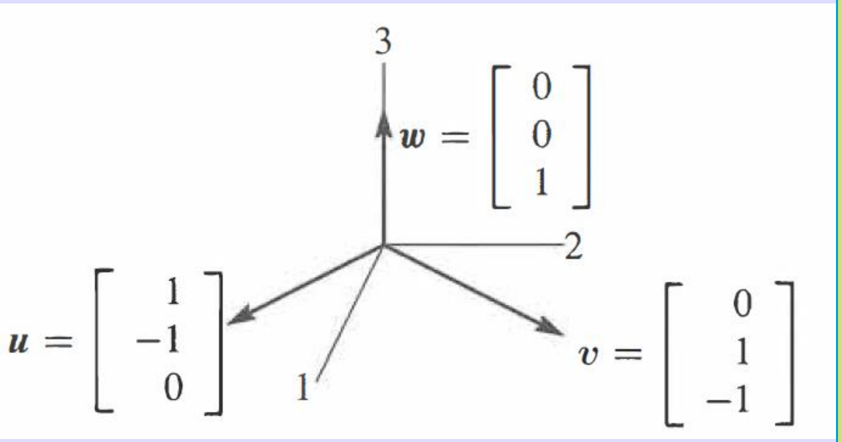
# Basis, Linear Independent Vectors

# Basis

If we take all combinations of:

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we get the entire space $\mathbb{R}^3$; the equation $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$ in $\mathbb{R}^3$.




We say that $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ form a *basis* for $\mathbb{R}^3$.

A *basis* for $\mathbb{R}^n$ is a collection of *n* independent vectors in $\mathbb{R}^n$.

Equivalently: (i) a basis is a collection of *n* vectors whose combinations cover the whole space (ii) a collection of vectors forms a basis whenever a matrix which has those vectors as its columns is invertible.

# Linear Independence of Vectors

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are **independent.** No combination except $0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w} = \mathbf{0}$ gives $\boldsymbol{b} = \mathbf{0}$.

$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}^*$ are **dependent.** Other combinations like $\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w}^*$ give $\boldsymbol{b} = \mathbf{0}$.

Generalize to *n* vectors in n-dimensional space. *Independence* is key point. The vectors go into the columns of an n by n matrix:

Independent columns: $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ has one solution. $\boldsymbol{A}$ is an **invertible matrix.**

Dependent columns: $\boldsymbol{C}\boldsymbol{x} = \mathbf{0}$ has many solutions. $\boldsymbol{C}$ is a **singular matrix.**

# Linear Independence of Vectors

Given any set of $m$ vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(m)}$ (with the same number of components), a **linear combination** of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + \ldots + c_m\mathbf{a}_{(m)}$$

where $c_1, c_2, \ldots, c_m$ are any scalars.

Now consider the equation

$$c_1\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + \ldots + c_m\mathbf{a}_{(m)} = \mathbf{0} \qquad \textbf{(*)}$$

Clearly, *this vector Eq. holds if we choose all $c_j$'s zero, because then it becomes $\mathbf{0} = \mathbf{0}$.*

If this is the only $m$-tuple of scalars for which (*) holds, then our vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(m)}$ are said to form a *linearly independent set* or, more briefly, we call them **linearly independent**.

# Linear Independence of Vectors

$$c_1\mathbf{a}_{(1)} + c_2\,\mathbf{a}_{(2)} + \ldots + c_m\mathbf{a}_{(m)} = \mathbf{0} \quad \mathbf{(*)}$$

Otherwise, if the Eq. above also holds with scalars not all zero, we call these vectors **linearly dependent**.

This means that we can express at least one of the vectors as a linear combination of the other vectors. For instance, if (*) holds with, say, $c_1 \neq 0$, we can solve (*) for $\mathbf{a}_{(1)}$:

$$\mathbf{a}_{(1)} = k_2\,\mathbf{a}_{(2)} + \ldots + k_m\mathbf{a}_{(m)} \text{ where } k_j = -c_j\,/c_1.$$

# Matrices: Addition and Scalar Multiplication

# Matrix Equality

Two matrices $\boldsymbol{A} = [a_{jk}]$ and $\boldsymbol{B} = [b_{jk}]$ are **equal**, $\boldsymbol{A} = \boldsymbol{B}$, if and only if

(1) they have the same size and

(2) the corresponding entries are equal, that is, $a_{ij} = b_{ij}$

Matrices that are not equal are called **different**.

Matrices of different sizes are always different.

# Addition of Matrices

**Definition**

The **sum** of two matrices $\boldsymbol{A} = [a_{jk}]$ and $\boldsymbol{B} = [b_{jk}]$ ***of the same size*** is written $\boldsymbol{A} + \boldsymbol{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Matrices of different sizes cannot be added.

# Scalar Multiplication

**Definition**

The **product** of any $m \times n$ matrix $\boldsymbol{A} = [a_{jk}]$ and any **scalar** $c$ is written $c\boldsymbol{A}$ and is the $m \times n$ matrix $c\boldsymbol{A} = [ca_{jk}]$ obtained by multiplying each entry of $\boldsymbol{A}$ by $c$.

# Rules for Matrix Addition/Scalar Multuplication

For matrices of the same size $m \times n$, namely, the following apply:

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ (written $\mathbf{A} + \mathbf{B} + \mathbf{C}$)

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}.$

Hence matrix addition is *commutative* and *associative* [by (a) and (b)].

Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero.

# Rules for Matrix Addition/Scalar Multuplication

For scalar multiplication we obtain the rules

$$\text{(a)}\ c(\mathbf{A}+\mathbf{B}) = c\mathbf{A}+c\mathbf{B}$$

$$\text{(b)}\ (c+k)\mathbf{A} = c\mathbf{A}+k\mathbf{A}$$

$$\text{(c)}\quad c(k\mathbf{A}) = (ck)\mathbf{A} \qquad \text{(written } ck\mathbf{A}\text{)}$$

$$\text{(d)}\qquad 1\mathbf{A} = \mathbf{A}.$$

# Matrix Multiplication, Transpose of a Matrix, Special Matrices

# Matrix Multiplication

## Definition

The **product** $\boldsymbol{C} = \boldsymbol{AB}$ (in this order) of an $m \times n$ matrix $\boldsymbol{A} = [a_{jk}]$ times an $r \times p$ matrix $\boldsymbol{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\boldsymbol{C} = [c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \cdots + a_{jn} b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{array}$$

# Matrix Multiplication

The condition $r = n$ means that $\boldsymbol{B}$, must have as many rows as the first factor has columns, namely $n$.

$$\begin{array}{ccccc} \boldsymbol{A} & & \boldsymbol{B} & = & \boldsymbol{C} \\ & & & & \\ [m \times n] & & [n \times p] & = & [m \times p] \end{array}$$

# Matrix Multiplication

The entry $c_{jk}$ is obtained by multiplying each entry in the $j$th row of $\boldsymbol{A}$ by the corresponding entry in the $k$th column of $\boldsymbol{B}$ and then adding these $n$ products. E.g.,

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1}$$

$$
m = 4\left\{ \overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}}^{n=3} \right. \overbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}^{p=2} = \left. \overbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \\ c_{41} & c_{42} \end{bmatrix}}^{p=2} \right\} m = 4
$$

# Matrix Multiplication

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & \boxed{14} & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$$c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$$

$$c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$$

The product $\boldsymbol{BA}$ is not defined.

# $AB \neq BA$ in General

For instance,

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

This also shows that $\boldsymbol{AB} = \mathbf{0}$ does ***not*** necessarily imply $\boldsymbol{BA} = \mathbf{0}$ or $\boldsymbol{A} = \mathbf{0}$ or $\boldsymbol{B} = \mathbf{0}$.

# Matrix Multiplication

In matrix products ***the order of factors must always be observed very carefully***. Otherwise matrix multiplication satisfies rules similar to those for numbers, namely.

$$\text{(a)} \quad (k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B}) \qquad \textit{written } k\mathbf{AB} \textit{ or } \mathbf{A}k\mathbf{B}$$

$$\text{(b)} \quad \mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} \qquad \textit{written } \mathbf{ABC}$$

$$\text{(c)} \ (\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$$

$$\text{(d)} \ \mathbf{C}(\mathbf{A}+\mathbf{B}) = \mathbf{CA}+\mathbf{CB}$$

provided $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are such that the expressions on the left are defined; here, $k$ is any scalar.

(b) is called the **associative law**.

(c) and (d) are called the **distributive laws**.

**Parallel processing of products on the computer**

$$\boldsymbol{AB} = \boldsymbol{A}[\boldsymbol{b}_1 \ \ \boldsymbol{b}_2 \ \ \ldots \ \ \boldsymbol{b}_p] = [\boldsymbol{Ab}_1 \ \ \boldsymbol{Ab}_2 \ \ \ldots \ \ \boldsymbol{Ab}_p].$$

Columns of $\boldsymbol{B}$ are then assigned to different processors (individually or several to each processor), which simultaneously compute the columns of the product matrix $\boldsymbol{Ab}_1, \boldsymbol{Ab}_2$, etc.

# Matrix Transpose

We obtain the **transpose of a matrix** by writing its rows as columns (or equivalently its columns as rows).

This also applies to the transpose of vectors. Thus, a row vector becomes a column vector and vice versa.

Can also "reflect" the elements along the main diagonal. Hence $a_{12}$ becomes $a_{21}$, $a_{31}$ becomes $a_{13}$, and so forth.

If $\boldsymbol{A}$ is the given matrix, then we denote its transpose by $\boldsymbol{A}^T$.

# Matrix Transpose

**Definition**

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on. Thus the transpose of $\mathbf{A}$ is $\mathbf{A}^\mathsf{T} = [a_{kj}]$, written out

$$\mathbf{A}^\mathsf{T} = \left[a_{kj}\right] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Transposition converts row vectors to column vectors and conversely.

# Rules for Transposition

(a) $(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$

(b) $(\mathbf{A}+\mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$

(c) $(c\mathbf{A})^{\mathsf{T}} = c\mathbf{A}^{\mathsf{T}}$

(d) $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}.$

Note that in (d) the transposed matrices are ***in reversed order.***

# Symmetric and Skew-Matrices

## Definition

**Symmetric** matrices are square matrices whose transpose equals the matrix itself.

$$\boldsymbol{A}^T = \boldsymbol{A} \quad (\text{thus } a_{kj} = a_{jk})$$ Symmetric Matrix

**Skew-symmetric** matrices are square matrices whose transpose equals ***minus*** the matrix.

$$\boldsymbol{A}^T = -\boldsymbol{A} \quad (\text{thus } a_{kj} = -a_{jk}, \text{ hence } a_{jj} = 0).$$

Skew-Symmetric Matrix

# Symmetric and Skew Matrices

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$$ is symmetric, and

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$ is skew-symmetric.

# Triangular Matrices

**Definition**

**Upper triangular matrices** are square matrices that can have nonzero entries only on and ***above*** the main diagonal, whereas any entry below the diagonal must be zero.

Similarly, **lower triangular matrices** can have nonzero entries only on and ***below*** the main diagonal. Any entry on the main diagonal of a triangular matrix may be zero or not.

# Upper and Lower Triangular Matrices

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}$$

**Upper triangular** **Lower triangular**

# Diagonal Matrices

**Definition**

These are square matrices that can have nonzero entries only on the main diagonal.

Any entry above or below the main diagonal must be zero.

# Scalar and Identity Matrices

If all the diagonal entries of a diagonal matrix **S** are equal, say, $c$, we call $\boldsymbol{S}$ a **scalar matrix.** Multiplication of any square matrix $\boldsymbol{A}$ of the same size by $\boldsymbol{S}$ has the same effect as the multiplication by $c$, that is,

$$\boldsymbol{AS} = \boldsymbol{SA} = c\boldsymbol{A}$$

A scalar matrix, whose entries on the main diagonal are all 1, is called an indenty matrix denoted by $\mathbf{I}_n$ or simply by **I**. For **I**, the above formula becomes

$$\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$$

# Rank of a Matrix

# Rank of a Matrix

## Definition

The **rank** of a matrix $\boldsymbol{A}$ is the maximum number of linearly independent row vectors of $\boldsymbol{A}$.

It is denoted by rank $\boldsymbol{A}$.

# Row Operations, Row-Equivalent

We call a matrix $\boldsymbol{A}_1$ **row-equivalent** to a matrix $\boldsymbol{A}_2$ if $\boldsymbol{A}_1$ can be obtained from $\mathbf{A}_2$ by (finitely many!) elementary row operations.

The maximum number of linearly independent row vectors of a matrix does not change if

- we change the order of rows or
- multiply a row by a nonzero $c$ or
- take a linear combination by adding a multiple of a row to another row.

This shows that rank is **invariant** under elementary row operations.

# Row-Equivalent Matrices

## Theorem

**Row-Equivalent Matrices**

*Row-equivalent matrices have the same rank.*

# Determination of the Rank

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{matrix} \\ \text{Row 2} + 2 \text{ Row 1} \\ \text{Row 3} - 7 \text{ Row 1} \end{matrix}$$

***(continued)***

# Determination of Rank

***(continued)***

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{Row 3} + \frac{1}{2}\,\text{Row 2.}$$

The last matrix is in row-echelon form and has two nonzero rows.

Hence $rank\ \boldsymbol{A} = 2$.

# Linear Independence and Rank

## Theorem

**Linear Independence and Dependence of Vectors**

*Consider p vectors that each have n components. Then these vectors are linearly independent if the matrix formed, with these vectors as row vectors, has rank p.*

*However, these vectors are linearly dependent if that matrix has rank less than p.*

# Rank and Linear Independence of Columns

## Theorem

**Rank in Terms of Column Vectors**

*The rank $r$ of a matrix $\boldsymbol{A}$* also *equals the maximum number of linearly independent* ***column*** *vectors of $\boldsymbol{A}$.*

*Hence $\boldsymbol{A}$ and its transpose $\boldsymbol{A}^T$ have the same rank.*

# Linear Independence

## Theorem

**Linear Dependence of Vectors**

*Consider $p$ vectors each having $n$ components.*

*If $n < p$,* then these vectors *are linearly dependent.*

# Vector Spaces and Subspaces

# Vector Spaces and Subspaces

A *vector space* is a collection of vectors that is closed under linear combinations.

A *subspace* is a vector space inside another vector space; a plane through the origin in $\mathbb{R}^3$ is an example of a subspace.

A subspace could be equal to the space it's contained in; the smallest subspace contains only the zero vector.

The subspaces of $\mathbb{R}^3$ are:

- the origin,
- a line through the origin,
- a plane through the origin,
- all of $\mathbb{R}^3$.

# Vector Space

Consider a nonempty set $V$ of vectors where each vector has the same number of components.

(1) If, for any $\boldsymbol{a}$ and $\boldsymbol{b}$ in $V$, we have that all their linear combinations $\alpha\boldsymbol{a} + \beta\boldsymbol{b}$ ($\alpha, \beta$ any real numbers) are also elements of $V$, and

(2) if, furthermore, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in $V$ satisfy

(a) $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$

(b) $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}=\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})$

(c) $\boldsymbol{a}+\boldsymbol{0}=\boldsymbol{a}$

(d) $\boldsymbol{a}+(-\boldsymbol{a})=\boldsymbol{0}.$

(a) $c(\boldsymbol{a}+\boldsymbol{b})=c\boldsymbol{a}+c\boldsymbol{b}$

(b) $(c+k)\boldsymbol{a}=c\boldsymbol{a}+k\boldsymbol{a}$

(c) $c(k\boldsymbol{a})=(ck)\boldsymbol{a}$

(d) $1\boldsymbol{a}=\boldsymbol{a}$

………then $V$ is a **vector space**!

# Real Vector Space

## Definition

A nonempty set $V$ of elements $\boldsymbol{a}, \boldsymbol{b}$, ... is called a **real vector space** (or *real linear space*), and these elements are called **vectors** if, in $V$, there are defined two algebraic operations (*vector addition* and *scalar multiplication*) as follows.

**I. Vector addition** associates with every pair of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of $V$ a unique vector of $V$, called the *sum* of $\boldsymbol{a}$ and $\boldsymbol{b}$ and denoted by $\boldsymbol{a} + \boldsymbol{b}$, such that the following axioms are satisfied.

# Real Vector Space

## Definition (continued)

**I.1** *Commutativity.* For any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ of $V$,

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}.$$

**I.2** *Associativity.* For any three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ of $V$,

$$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) \quad \text{(written } \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}\text{)}.$$

**I.3** There is a unique vector in $V$, called the *zero vector* and denoted by $\mathbf{0}$, such that for every $\mathbf{a}$ in $V$,

$$\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}.$$

**I.4** For every $\mathbf{a}$ in $V$, there is a unique vector in $V$ that is denoted by $-\boldsymbol{a}$ and is such that

$$\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}.$$

# Real Vector Space

## Definition (continued)

**II. Scalar multiplication.** Scalar multiplication associates with every $\boldsymbol{a}$ in $V$ and every scalar $c \in \mathbb{R}$ a unique vector of $V$, called the *product* of $c$ and $\boldsymbol{a}$ and denoted by $c\boldsymbol{a}$ (or $\boldsymbol{a}c$) such that the following axioms are satisfied.

**II.1** *Distributivity.* For every scalar $c$ and vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $V$,

$$c(\boldsymbol{a} + \boldsymbol{b}) = c\boldsymbol{a} + c\boldsymbol{b}.$$

**II.2** *Distributivity.* For all scalars $c$ and $k$ and every **a** in $V$,

$$(c + k)\boldsymbol{a} = c\boldsymbol{a} + k\boldsymbol{a}.$$

**II.3** *Associativity.* For all scalars $c$ and $k$ and every **a** in $V$,

$$c(k\boldsymbol{a}) = (ck)\boldsymbol{a} \quad \text{(written } ck\boldsymbol{a}\text{)}.$$

**II.4** For every $\boldsymbol{a}$ in $V$,

$$1\boldsymbol{a} = \boldsymbol{a}.$$

# Vector Space and Basis

*The maximum number of linearly independent vectors in $V$* is called the **dimension** of $V$ and is denoted by dim $V$. Here we assume the dimension to be finite.

*A linearly independent set in $V$ consisting of a maximum possible number of vectors in $V$* is called a **basis** for $V$.

*The number of vectors of a basis for $V$* equals dim $V$.

# Span

The set of all linear combinations of given vectors $\mathbf{a}_{(1)}, \ldots,$ $\mathbf{a}_{(p)}$ with the same number of components is called the **span** of these vectors.

Obviously, a span is a vector space. If in addition, the given vectors $\mathbf{a}_{(1)}, \ldots, \mathbf{a}_{(p)}$ are linearly independent, then they form a basis for that vector space.

This then leads to another equivalent definition of basis. A set of vectors is a **basis** for a vector space $V$ if

(1) the vectors in the set are linearly independent, and if

(2) any vector in $V$ can be expressed as a linear combination of the vectors in the set. We then say that the set of vectors **spans** the vector space $V$.

# Vector Space $\mathbb{R}^n$

## Theorem

**Vector Space $\mathbb{R}^n$**

*The vector space $\mathbb{R}^n$ consisting of all vectors with n components (n real numbers) has dimension n.*

# Subspace of a Vector Space

By a **subspace** of a vector space $V$ we mean

"a nonempty subset of $V$ (including $V$ itself) that forms a vector space with respect to the two algebraic operations (addition and scalar multiplication) defined for the vectors of $V$."

# Subspace

**DEFINITION**

A ***subspace*** of a vector space is a set of vectors (including **0**) that satisfies two requirements:

***If v and w are vectors in the subspace and c is any scalar, then***

(1) $\boldsymbol{v} + \boldsymbol{w}$ is in the subspace

(2) $c\boldsymbol{v}$ is in the subspace.

# Subspaces of $\mathbb{R}^3$

Here is a list of all the possible subspaces of $\mathbb{R}^3$.

(L) Any line through $(0,0,0)$

(P) Any plane through $(0,0,0)$

($\mathbb{R}^3$) The whole space

($\mathcal{Z}$) The single vector $(0,0,0)$

# A First Look at Three of the Fundamental Spaces of $A$

# The Column Space of $A$

**DEFINITION** The ***column space*** consists of ***all linear combinations of the columns.***

Consider $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. The combinations $\boldsymbol{Ax}$ fill the column space $C(\boldsymbol{A})$.

***To solve $\boldsymbol{Ax} = \boldsymbol{b}$ is to express b as a combination of the columns.*** The right side $\boldsymbol{b}$ has to be *in the column space* produced by $\boldsymbol{A}$ for a solution to exist.

***The system $\boldsymbol{Ax} = \boldsymbol{b}$ is solvable** if **and only** if **$\boldsymbol{b}$ is in** $C(\boldsymbol{A})$*

When $\boldsymbol{b}$ is in the column space, it is a combination of the columns. The coefficients in that combination give us a solution $\boldsymbol{x}$ to the system $\boldsymbol{Ax} = \boldsymbol{b}$.

For an $m \times n$ matrix $\boldsymbol{A}$, this column space $C(\boldsymbol{A})$ of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$.

# The Null Space of A: $Ax = 0$

We are interested in the subspace containing all solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. The $m$ by $n$ matrix $A$ can be square or rectangular.

*One immediate solution is* $\boldsymbol{x} = \boldsymbol{0}$. For invertible matrices this is the only solution.

For not invertible matrices there are nonzero solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. *Each solution* $\boldsymbol{x}$ *belongs to the nullspace of* $\boldsymbol{A}$. As we will see later on, elimination will find all solutions and identify this very important subspace.

***The nullspace*** $N(\boldsymbol{A})$ ***consists of all solutions to*** $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$

***These vectors*** $\boldsymbol{x}$ ***are in*** $\mathbb{R}^{\boldsymbol{n}}$. The null space $N(\boldsymbol{A})$ of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$.

# The Row Space

**DEFINITION** The *row space* of a matrix is the subspace of $\mathbb{R}^n$, spanned by the rows.

***The row space of A is $C(\boldsymbol{A}^T)$. It is the column space of $\boldsymbol{A}^T$***

The rows of an $m$ by $n$ matrix have $n$ components. They are vectors in $\mathbb{R}^n$-or they would be if they were written as column vectors.

*Transpose the matrix*. Instead of the rows of $\boldsymbol{A}$**,** look at the columns of $\boldsymbol{A}^T$ in the column space**.**

This row space $C(\boldsymbol{A}^T)$ of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$ .

# Row Space and Column Space

## Theorem

**Row Space and Column Space**

*The row space and the column space of a matrix* $\mathbf{A}$ *have the same dimension, equal to rank* $\mathbf{A}$.

# Null Space

For a given matrix **A** *the solution set of the homogeneous system* $\mathbf{A}\boldsymbol{x} = \mathbf{0}$ *is a vector space*, called the **null space** of **A**, and *its dimension* is called the **nullity** of **A**.

We will show in follow up lectures that the following relation holds:

$$\text{rank } \mathbf{A} + \text{nullity } \mathbf{A} = \text{Number of columns of } \mathbf{A}$$

$$r \quad + \quad (n-r) = n$$

# Inner Product Spaces

# (Real) Inner Product Spaces

## Definition

A real vector space $V$ is called a **real inner product space** (or *real pre-Hilbert space*) if it has the following property.

With every pair of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $V$ there is associated a real number, which is denoted by $(\boldsymbol{a}, \boldsymbol{b})$ and is called the **inner product** of $\boldsymbol{a}$ and $\boldsymbol{b}$, such that the following axioms are satisfied.

**I.** For all scalars $q_1$ and $q_2$ and all vectors **a**, **b**, **c** in $V$,

$$(q_1\boldsymbol{a} + q_2\boldsymbol{b}, \boldsymbol{c}) = q_1(\boldsymbol{a}, \boldsymbol{c}) + q_2(\boldsymbol{b}, \boldsymbol{c}) \qquad (\textit{Linearity}).$$

**II.** For all vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $V$,

$$(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{b}, \boldsymbol{a}) \qquad (\textit{Symmetry}).$$

# (Real) Inner Product Spaces

## Real Inner Product Space (continued)

**III.** For every $\boldsymbol{a}$ in $V$,

$$\left.\begin{aligned} &(\boldsymbol{a},\boldsymbol{a}) \ \geq \ 0, \\ &(\boldsymbol{a},\boldsymbol{a}) \ = \ 0 \text{ if and only if } \boldsymbol{a} \ = \ \boldsymbol{0} \end{aligned}\right\} (Positive-definiteness).$$

# Cauchy-Schwarz Inequality

Vectors whose inner product is zero are called **orthogonal**.

The *length* or **norm** of a vector in $V$ is defined by

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a},\mathbf{a})} \qquad (\geq 0).$$

A vector of norm 1 is called a **unit vector**. From these axioms and definition above one can derive

$$\|\mathbf{a},\mathbf{b}\| \leq \|\mathbf{a}\|\|\mathbf{b}\| \qquad (\textit{Cauchy - Schwarz inequality})$$

From this follows

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \qquad (\textit{Triangle inequality}).$$

A simple direct calculation gives

$$\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2) \qquad (\textit{Parallelogram equality}).$$