Top Notes

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1 Fixed point top

Euler angle

$$\begin{aligned}\omega_x =& \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi && (1)\\ \omega_y =& \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi && (2)\\ \omega_z =& \dot{\phi}\cos\theta + \dot{\psi} && (3)\end{aligned}$$

Kinetic energy term

$$\begin{aligned}T =& \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2\\ =& \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2\end{aligned} \tag{4}$$

Potential energy term

$$\begin{aligned}V =& -MRg\\ =& Mgl\cos\theta\end{aligned} \tag{5}$$

Lagrangian

$$\mathcal{L} = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta \tag{6}$$

First integrals of the motion

$$\begin{aligned}p_\psi =& \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = I_3\omega_z = I_1 a && (7)\\ p_\phi =& \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = (I_1\sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta = I_1 b && (8)\\ E =& T + V = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}I_3\omega_z^2 + Mgl\cos\theta && (9)\end{aligned}$$

Quadratures

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \tag{10}$$

$$\dot{\psi} = \frac{I_1 a}{I_3} - \cos\theta \frac{b - a\cos\theta}{\sin^2\theta} \tag{11}$$

$$\tag{12}$$

2 The tippe top revisited (1974)

$$\begin{aligned}
\hat{e}_n =& \cos\phi \hat{e}_x + \sin\phi \hat{e}_y & (13)\\
\hat{e}_{n'} =& -\cos\theta\sin\phi\hat{e}_x + \cos\theta\cos\phi\hat{e}_y + \sin\theta\hat{e}_z & (14)\\
\hat{e}_3 =& \sin\theta\sin\phi\hat{e}_x - \sin\theta\cos\phi\hat{e}_y + \cos\theta\hat{e}_z & (15)
\end{aligned}$$

$$\begin{aligned}
\hat{e}_x =& \cos\phi\hat{e}_n - \cos\theta\sin\phi\hat{e}_{n'} + \sin\theta\sin\phi\hat{e}_3 & (16)\\
\hat{e}_y =& \sin\phi\hat{e}_n + \cos\theta\cos\phi\hat{e}_{n'} - \sin\theta\cos\phi\hat{e}_3 & (17)\\
\hat{e}_z =& \sin\theta\hat{e}_{n'} + \cos\theta\hat{e}_3 & (18)
\end{aligned}$$

$$\vec{r} = -R\sin\theta\hat{e}_{n'} + (a - R\cos\theta)\hat{e}_3 \tag{19}$$

$$\begin{aligned}
\vec{v}_{TCP} =& [(R\dot{\psi} + a\dot{\phi})\sin\theta\cos\phi + \dot{\theta}\sin\phi(a\cos\theta - R)]\hat{e}_x\\
&+ [(R\dot{\psi} + a\dot{\phi})\sin\theta\sin\phi + \dot{\theta}\cos\phi(R - a\cos\theta)]\hat{e}_y
\end{aligned} \tag{20}$$

$$\begin{aligned}
\vec{v}_{TP} =& \vec{u}_{CP} + \vec{v}_{TCP}\\
=& [\dot{x} + (R\dot{\psi} + a\dot{\phi})\sin\theta\cos\phi + \dot{\theta}\sin\phi(a\cos\theta - R)]\hat{e}_x\\
&+ [\dot{y} + (R\dot{\psi} + a\dot{\phi})\sin\theta\sin\phi + \dot{\theta}\cos\phi(R - a\cos\theta)]\hat{e}_y
\end{aligned} \tag{21}$$

$$\vec{F}_f = -\frac{\vec{v}_{TP}}{|\vec{v}_{TP}|}\mu|F_N| \tag{22}$$

$$\vec{F}_N = (mg + m\ddot{z})\hat{e}_z \tag{23}$$

$$\begin{aligned}
z =& R - a\cos\theta & (24)\\
\dot{z} =& a\dot{\theta}\sin\theta & (25)\\
\ddot{z} =& a(\dot{\theta}^2 + \ddot{\theta}\sin\theta) & (26)
\end{aligned}$$

$$\vec{F}_T = \vec{F}_N + \vec{F}_f \tag{27}$$

$$\vec{N} = \vec{r} \times \vec{F}_T \tag{28}$$

$$\begin{aligned}
N_3 =& I_3(\ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta + \ddot{\psi}) & (29)\\
N_n =& I\ddot{\theta} + I_3\dot{\phi}\dot{\psi}\sin\theta + (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta & (30)\\
I_{n'} =& I\ddot{\phi}\sin\theta + (2I - I_3)\dot{\theta}\dot{\phi}\cos\theta - I_3\dot{\theta}\dot{\psi} & (31)
\end{aligned}$$

$$\begin{aligned}
\ddot{\phi} =& \frac{1}{I\sin\theta}[I_{n'} - (2I - I_3)\dot{\theta}\dot{\phi}\cos\theta + I_3\dot{\theta}\dot{\psi}] & (32)\\
\ddot{\theta} =& \frac{1}{I}[N_n - I_3\dot{\phi}\dot{\psi}\sin\theta - (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta] & (33)\\
\ddot{\psi} =& \frac{N_3}{I_3} - \ddot{\phi}\cos\theta + \dot{\phi}\dot{\theta}\sin\theta & (34)
\end{aligned}$$

3 Solving coupled equation

$$\ddot{z} = a(\dot{\theta}^2 + \ddot{\theta}\sin\theta) \tag{35}$$

$$\vec{F}_N = (mg + m\ddot{z})\hat{e}_z \tag{36}$$

$$\begin{aligned} \Rightarrow \vec{F}_N =& [mg + ma(\dot{\theta}^2 + \ddot{\theta}\sin\theta)]\hat{e}_z \\ =& m[g + a(\dot{\theta}^2 + \ddot{\theta}\sin\theta)]\hat{e}_z \end{aligned} \tag{37}$$

$$\vec{F}_f = -\mu m \frac{\vec{v}_{TP}}{|v_{TP}|}[g + a(\dot{\theta}^2 + \ddot{\theta}\sin\theta)] \tag{38}$$

$$\begin{aligned} \vec{F}_T =& \vec{F}_N + \vec{F}_f \\ =& m[g + a(\dot{\theta}^2 + \ddot{\theta}\sin\theta)][\hat{e}_z - \mu\frac{\vec{v}_{TP}}{|v_{TP}|}] \end{aligned} \tag{39}$$

$$\begin{aligned} \vec{N} =& \vec{r} \times \vec{F}_T \\ =& m[g + a(\dot{\theta}^2 + \ddot{\theta}\sin\theta)] \\ & \times [-R\sin\theta\hat{e}_{n'} + (a - R\cos\theta\hat{e}_3] \times [\hat{e}_z - \mu\frac{\vec{v}_{TP}}{|v_{TP}|}] \\ =& m[g + a(\dot{\theta}^2 + \ddot{\theta}\sin\theta)][C_n\hat{e}_n + C_{n'}\hat{e}_{n'} + C_3\hat{e}_3] \end{aligned} \tag{40}$$

Since

$$N_n = I\ddot{\theta} + I_3\dot{\phi}\dot{\psi}\sin\theta + (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta \tag{41}$$

$$\Rightarrow m[g + a(\dot{\theta}^2 + \ddot{\theta}\sin\theta)]C_n = I\ddot{\theta} + I_3\dot{\phi}\dot{\psi}\sin\theta + (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta \tag{42}$$

$$mgC_n + ma\dot{\theta}^2\sin\theta C_n + ma\ddot{\theta}\sin\theta C_n = I\ddot{\theta} + I_3\dot{\phi}\dot{\psi}\sin\theta + (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta \tag{43}$$

$$(ma\sin\theta C_n - I)\ddot{\theta} = I_3\dot{\phi}\dot{\psi}\sin\theta + (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta - mgC_n - ma\dot{\theta}^2\sin\theta C_n \tag{44}$$

$$\Rightarrow \ddot{\theta} = \frac{I_3\dot{\phi}\dot{\psi}\sin\theta + (I_3 - I)\dot{\phi}^2\sin\theta\cos\theta - mgC_n - ma\dot{\theta}^2\sin\theta C_n}{ma\sin\theta C_n - I} \tag{45}$$

4 Solving steps

What we get is

$$\phi, \dot{\phi}, \psi, \dot{\psi}, \theta, \dot{\theta}, x, \dot{x}, y, \dot{y} \tag{46}$$

1. Calculate v_{TP} in x-basis.
2. Calculate $\vec{F}_T$ in x-basis and then turns it into n-basis.
3. Calculate $\vec{N}$ in n-basis, get C_1, C_2, C_3.
4. Using equation 45, calculate $\ddot{\theta}$, and plug it into equation 35, to get $\ddot{z}$.
5. Plug them into equation 40 to calculate $\vec{N}$.
6. Finally, using equation 34 and 32 to get the other two ODEs.
7. The last two trivial ODEs are horizontal translation in x and y direction.