1) (a) (10 points) Find the general solution of the equation

$$x^2 \frac{dy}{dx} - 4xy = x^6 \cos 2x$$

$y' - \frac{4}{x} y = x^4 \cos 2x$

$\mu(x) = e^{\int -\frac{4}{x} dx} = e^{-4 \ln|x|} = x^{-4}$

$x^{-4} y' - 4x^{-5} y = \cos 2x$

$\frac{d}{dx}(x^{-4} y) = \cos 2x$

$x^{-4} y = \frac{1}{2} \sin 2x + C$

$\boxed{y = \frac{1}{2} x^4 \sin 2x + Cx^4}$

(b) (10 points) Find the explicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{\sqrt{2x+1}}{y^2}, \quad y(4) = 2$$

$y^2 dy = \sqrt{2x+1}\, dx$

$\frac{1}{3} y^3 = \frac{1}{3} (2x+1)^{3/2} + C$

$y^3 = (2x+1)^{3/2} + C$

$y(4) = 2 \Rightarrow \quad 9^{3/2} + C = 8$

$27 + C = 8$

$C = -19$

$y^3 = (2x+1)^{3/2} - 19$

$\boxed{y = \sqrt[3]{(2x+1)^{3/2} - 19}}$

2) (a) (10 points) A 360 g sample of a certain radioactive substance decays to 240 g in 5 months. How long would it take for the sample to decay to 40 g?

$y(t) = 360e^{-kt}$, t in months

$y(5) = 240 \Rightarrow 360e^{-5k} = 240$

$e^{-5k} = \frac{2}{3}$

$k = -\frac{1}{5}\ln\frac{2}{3}$

$y(t) = 360e^{(\frac{1}{5}\ln\frac{2}{3})t} = 40$

$e^{(\frac{1}{5}\ln\frac{2}{3})t} = \frac{1}{9}$

$(\frac{1}{5}\ln\frac{2}{3})t = \ln\frac{1}{9}$

$\boxed{t = \frac{5\ln\frac{1}{9}}{\ln\frac{2}{3}} \text{ months}}$

(b) (10 points) A drink whose temperature is 44°F is placed in a 68°F room. After 8 minutes, the drink is 50°F. How long will it take for the drink to warm to 60°F?

$T(t) = M_0 + (T_0 - M_0)e^{-kt} = 68 + (44-68)e^{-kt}$

$= 68 - 24e^{-kt}$, t in minutes

$T(8) = 50 \Rightarrow 68 - 24e^{-8k} = 50$

$18 = 24e^{-8k}$

$e^{-8k} = \frac{3}{4} \Rightarrow k = -\frac{1}{8}\ln\frac{3}{4}$

$T(t) = 68 - 24e^{(\frac{1}{8}\ln\frac{3}{4})t} = 60$

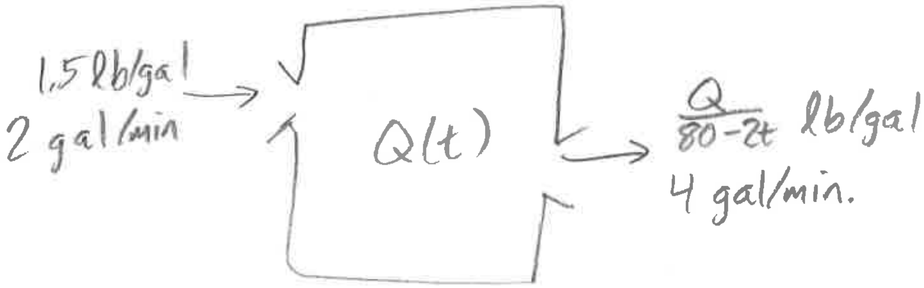
$8 = 24e^{(\frac{1}{8}\ln\frac{3}{4})t}$

$e^{(\frac{1}{8}\ln\frac{3}{4})t} = \frac{1}{3}$

$(\frac{1}{8}\ln\frac{3}{4})t = \ln\frac{1}{3}$

$\boxed{t = \frac{8\ln\frac{1}{3}}{\ln\frac{3}{4}} \text{ minutes}}$

3) (20 points) Initially, a full 80-gallon tank contains a salt solution with a concentration of 0.10 lb/gal. A salt solution with a concentration of 1.5 lb/gal flows into the tank at a rate of 2 gal/min. The mixture is kept well-stirred and flows out of the tank at a rate of 4 gal/min. Find $Q(t)$, the amount of salt in the tank until the tank empties.



$$Q(0) = (0.10\ \text{lb/gal})(80\ \text{gal}) = 8\ \text{lb}$$

$$\frac{dQ}{dt} = (1.5)(2) - \left(\frac{Q}{80-2t}\right)(4)$$

$$\frac{dQ}{dt} = 3 - \frac{2}{40-t}Q \ ; \ Q(0) = 8 ; \ 0 \leq t < 40$$

$$\frac{dQ}{dt} + \frac{2}{40-t}Q = 3$$

$$\mu(t) = e^{\int \frac{2}{40-t}dt} = e^{-2\ln|40-t|} = (40-t)^{-2}$$

$$(40-t)^{-2}\frac{dQ}{dt} + 2(40-t)^{-3}Q = 3(40-t)^{-2}$$

$$\frac{d}{dt}\left[(40-t)^{-2}Q\right] = 3(40-t)^{-2}$$

$$(40-t)^{-2}Q = \int 3(40-t)^{-2}dt$$

$$= 3(40-t)^{-1} + C$$

$$Q(t) = 3(40-t) + C(40-t)^2$$

$$Q(0) = 8 \Rightarrow 120 + 1600C = 8$$

$$C = \frac{-112}{1600} = -\frac{7}{100}$$

$$\boxed{Q(t) = 3(40-t) - \frac{7}{100}(40-t)^2}$$

4) (a) (10 points) Find the explicit solution of the initial value problem

$$\left(2y^3 - \frac{5}{2\sqrt{x}}\right)dx + 6xy^2\,dy = 0, \quad y(4) = -1$$

(The equation is exact. You do not need to verify that first.)

$\frac{\partial F}{\partial x} = 2y^3 - \frac{5}{2\sqrt{x}}$ $\quad\quad$ $\frac{\partial F}{\partial y} = 6xy^2$

$\downarrow$ integrate w.r.t. x $\quad\quad$ $\downarrow$ integrate w.r.t. y

$2xy^3 - 5\sqrt{x}$ $\quad\quad$ $2xy^3$

$F(x,y) = 2xy^3 - 5\sqrt{x}$

$2xy^3 - 5\sqrt{x} = C$

$y(4) = -1 \Rightarrow -8 - 10 = C \Rightarrow C = -18$

$2xy^3 - 5\sqrt{x} = -18$

$2xy^3 = 5\sqrt{x} - 18 \Rightarrow y^3 = \frac{5\sqrt{x}-18}{2x} \Rightarrow \boxed{y = \sqrt[3]{\frac{5\sqrt{x}-18}{2x}}}$

(b) (10 points) A object weighing $mg = 12.8$ lb falls from rest, with coefficient of air resistance $k = 0.2$. Find its velocity when $t = 5$ seconds, and the total distance fallen by the object in the first 5 seconds.

$m = \frac{12.8}{32} = 0.4$

$v(t) = \frac{mg}{k} - \frac{mg}{k}e^{-kt/m} = \frac{12.8}{0.2} - \frac{12.8}{0.2}e^{-0.2t/0.4}$

$= 64 - 64e^{-t/2}$

$\boxed{v(5) = 64 - 64e^{-5/2} \text{ ft/s}}$

$x(t) = \frac{mg}{k}t - \frac{m^2g}{k^2}\left(1 - e^{-kt/m}\right)$

$= 64t - \frac{0.4}{0.2}(64)\left(1 - e^{-t/2}\right)$

$= 64t - 128\left(1 - e^{-t/2}\right)$

$\boxed{x(5) = 320 - 128\left(1 - e^{-5/2}\right) \text{ ft}}$ or $192 + 128e^{-5/2}$ ft

5) **(10 points) Use Euler's method with step size $h = 0.1$ to approximate the solution to the following initial value problem at points $x = 3.1$ and $x = 3.2$.**

$$y' = y^2 + xy, \quad y(3) = 1$$

$X_0 = 3,\ Y_0 = 1$

$X_1 = 3.1,\ Y_1 = Y_0 + hf(x_0, Y_0)$

$= 1 + 0.1[1 + 3]$

$= \boxed{1.4}$

$X_2 = 3.2;\ Y_2 = Y_1 + hf(x_1, Y_1)$

$= 1.4 + 0.1[1.4^2 + 3.1(1.4)]$

$= 1.4 + 0.1[1.96 + 4.34]$

$= 1.4 + 0.1(6.3)$

$= 1.4 + 0.63$

$= \boxed{2.03}$

6) **(a) (10 points) Given that the functions $y_1(x) = e^{-4x}$ and $y_2(x) = xe^{-4x}$ are solutions of the differential equation**

$$y'' + 8y' + 16y = 0$$

(you do not need to verify that), use the Wronskian to show that y_1 and y_2 are linearly independent, and then find the solution of the initial value problem

$$y'' + 8y' + 16y = 0; \quad y(0) = -2, \; y'(0) = \frac{19}{2}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{-4x} & xe^{-4x} \\ -4e^{-4x} & e^{-4x} - 4xe^{-4x} \end{vmatrix} = e^{-8x} - 4xe^{-8x} + 4xe^{-8x}$$

$$= e^{-8x} \neq 0, \text{ so lin. indep.}$$

$$y = c_1 e^{-4x} + c_2 x e^{-4x}$$

$$y' = -4c_1 e^{-4x} + c_2 e^{-4x} - 4c_2 x e^{-4x}$$

$$y(0) = -2 \Rightarrow c_1 = -2$$

$$y'(0) = \frac{19}{2} \Rightarrow -4c_1 + c_2 = \frac{19}{2}$$

$$8 + c_2 = \frac{19}{2}$$

$$c_2 = \frac{3}{2}$$

$$\boxed{y = -2e^{-4x} + \frac{3}{2} x e^{-4x}}$$

(b) (10 points) Given that $y_1(x) = x^4$ is a solution of

$$x^2 y'' + 2xy' - 20y = 0$$

use reduction of order to find a second linearly independent solution for $x > 0$.

$$y_2 = v(x)x^4 \Rightarrow y_2' = v'x^4 + 4vx^3$$

$$y_2'' = v''x^4 + 4v'x^3 + 4v'x^3 + 12vx^2 = v''x^4 + 8v'x^3 + 12vx^2$$

$$x^2(v''x^4 + 8v'x^3 + 12vx^2) + 2x(v'x^4 + 4vx^3) - 20vx^4 = 0$$

$$v''x^6 + 8v'x^5 + \cancel{12vx^4} + 2v'x^5 + \cancel{8vx^4} - \cancel{20vx^4} = 0$$

$$v''x^6 + 10v'x^5 = 0 \Rightarrow v'' = -\frac{10}{x} v'. \text{ Let } w = v'.$$

$$w' = -\frac{10}{x} w$$

$$\frac{1}{w} dw = -\frac{10}{x} dx$$

$$\ln|w| = -10|x| + C$$

$$|w| = e^{-10|x| + C} = e^C x^{-10}$$

$$w = \pm e^C x^{-10} = Cx^{-10}$$

$$v = \int Cx^{-10} dx = Cx^{-9} + D$$

Take $C = 1, D = 0$

$$v(x) = x^{-9}$$

$$y_2(x) = v(x)x^4 = \boxed{x^{-5}}$$