1) (a) (10 points) Find the explicit solution of the initial value problem

$$x^2\frac{dy}{dx} + 3xy = 3x^3 - 2x^2; \quad y(1) = 1$$

$y' + \frac{3}{x}y = 3x - 2 \Rightarrow \mu(x) = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = x^3$

$x^3y' + 3x^2y = 3x^4 - 2x^3$

$\frac{d}{dx}(x^3y) = 3x^4 - 2x^3$

$x^3y = \int(3x^4 - 2x^3)dx = \frac{3}{5}x^5 - \frac{1}{2}x^4 + C$

$y = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{C}{x^3}$

$y(1) = 1 \Rightarrow \frac{3}{5} - \frac{1}{2} + C = 1 \Rightarrow C = 1 + \frac{1}{2} - \frac{3}{5} = \frac{10+5-6}{10} = \frac{9}{10}$

$\boxed{y = \frac{3}{5}x^2 - \frac{1}{2}x + \frac{9}{10x^3}}$

(b) (10 points) Find the explicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{y-1}{x-3}, \quad y(-1) = 0$$

$\frac{1}{y-1}dy = \frac{1}{x-3}dx$

$\int \frac{1}{y-1}dy = \int \frac{1}{x-3}dx$

$\ln|y-1| = \ln|x-3| + C$

$|y-1| = e^{\ln|x-3|+C} = e^C e^{\ln|x-3|} = e^C|x-3|$

$y - 1 = \pm e^C(x-3) = C(x-3)$

$y = C(x-3) + 1$

$y(-1) = 0 \Rightarrow -4C + 1 = 0 \Rightarrow C = \frac{1}{4}$

$y = \frac{1}{4}(x-3) + 1$

$\boxed{y = \frac{1}{4}x + \frac{1}{4}}$ Note: $y = \frac{1}{4}(x-3) + 1$ is acceptable

2) (a) (10 points) A culture with an initial population of 80 bacteria grows to 300 bacteria in 6 hours. How long will it take for the population to reach 4000 bacteria?

$y(t) = 80e^{kt}$, t in hours

$y(6) = 300 \Rightarrow 80e^{6k} = 300$

$e^{6k} = \frac{300}{80} = \frac{15}{4}$

$6k = \ln\frac{15}{4} \Rightarrow k = \frac{\ln\frac{15}{4}}{6}$

$y(t) = 80e^{\left(\frac{\ln\frac{15}{4}}{6}\right)t} = 4000$

$e^{\left(\frac{\ln\frac{15}{4}}{6}\right)t} = \frac{4000}{80} = 50$

$\left(\frac{\ln\frac{15}{4}}{6}\right)t = \ln 50 \Rightarrow \boxed{t = \frac{6\ln 50}{\ln\frac{15}{4}} \text{ hours}}$

(b) (10 points) A glass of water whose temperature is 45°F is taken outside on an 80°F day. After 5 minutes, the water is 50°F. How long will it take for the water to warm to 65°F?

$T(t) = M_0 + (T_0 - M_0)e^{-kt}$

$= 80 + (45-80)e^{-kt}$

$= 80 - 35e^{-kt}$, t in minutes

$T(5) = 50 \Rightarrow 80 - 35e^{-5k} = 50$

$30 = 35e^{-5k}$

$\frac{6}{7} = e^{-5k} \Rightarrow k = -\frac{\ln\frac{6}{7}}{5}$

$T(t) = 80 - 35e^{\left(\frac{\ln\frac{6}{7}}{5}\right)t} = 65$

$15 = 35e^{\left(\frac{\ln\frac{6}{7}}{5}\right)t}$

$\frac{3}{7} = e^{\left(\frac{\ln\frac{6}{7}}{5}\right)t}$

$\ln\frac{3}{7} = \left(\frac{\ln\frac{6}{7}}{5}\right)t$

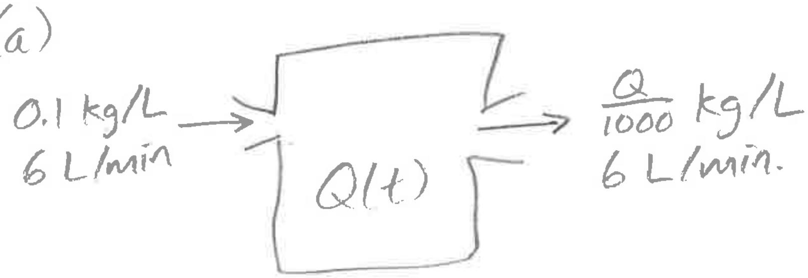
$\boxed{t = \frac{5\ln\frac{3}{7}}{\ln\frac{6}{7}} \text{ minutes}}$

3) **(20 points) Initially, a large tank contains 1000 L of pure water. A salt solution with a concentration of 0.1 kg/L flows into the tank at a rate of 6 L/min. The mixture is kept well-stirred and flows out of the tank at a rate of 6 L/min.**

(a) Find $Q(t)$, the amount of salt in the tank after t minutes.

(b) How long will it take for the amount of salt in the tank to reach 50 kg?

(a)



$Q(0) = 0$ kg (pure water)

$$\frac{dQ}{dt} = (0.1)(6) - \left(\frac{Q}{1000}\right)(6)$$

$$= 0.6 - 0.006Q \; ; \quad Q(0) = 0$$

$$\frac{dQ}{dt} + 0.006Q = 0.6$$

$$\mu(t) = e^{\int 0.006\,dt} = e^{0.006t}$$

$$e^{0.006t}\frac{dQ}{dt} + 0.006e^{0.006t}Q = 0.6e^{0.006t}$$

$$\frac{d}{dt}\left(e^{0.006t}Q\right) = 0.6e^{0.006t}$$

$$e^{0.006t}Q = \int 0.6e^{0.006t}\,dt = \frac{0.6}{0.006}e^{0.006t} + C = 100e^{0.006t} + C$$

$$Q(t) = 100 + Ce^{-0.006t}$$

$$Q(0) = 0 \Rightarrow 100 + C = 0 \Rightarrow C = -100$$

$$\boxed{Q(t) = 100 - 100e^{-0.006t}}$$

(b) $100 - 100e^{-0.006t} = 50$

$$50 = 100e^{-0.006t}$$

$$\frac{1}{2} = e^{-0.006t} \Rightarrow \boxed{t = -\frac{\ln\frac{1}{2}}{0.006} \text{ minutes}} \quad \text{or} \quad \frac{\ln 2}{0.006} \text{ minutes}$$

4) (a) (10 points) Find an implicit solution of the initial value problem

$$\left(\frac{1}{x} + 2xy^2\right) dx + (2x^2y - \cos y)\, dy = 0, \quad y(1) = \pi$$

(The equation is exact. You do not need to verify that first.)

Want $F(x,y)$ such that

$\frac{\partial F}{\partial x} = \frac{1}{x} + 2xy^2$ and $\frac{\partial F}{\partial y} = 2x^2y - \cos y$

$\downarrow$ integrate w.r.t. x $\quad\quad$ $\downarrow$ integrate w.r.t. y

$\ln|x| + x^2y^2$ $\quad\quad$ $x^2y^2 - \sin y$

$F(x,y) = \ln|x| + x^2y^2 - \sin y$

$\ln|x| + x^2y^2 - \sin y = C$

$y(1) = \pi \Rightarrow 0 + \pi^2 - 0 = C \Rightarrow C = \pi^2$

$$\boxed{\ln|x| + x^2y^2 - \sin y = \pi^2}$$

(b) (10 points) A object weighing $mg = 3.2$ lb is dropped from rest from the top of a tall building, with coefficient of air resistance $k = \frac{1}{15}$. How long will it take for the object to reach 50% of its terminal velocity?

$v(t) = \frac{mg}{k} - \frac{mg}{k} e^{-kt/m}$

$mg = 3.2 \Rightarrow m = \frac{3.2}{32} = \frac{1}{10}$

$v(t) = \frac{3.2}{1/15} - \frac{3.2}{1/15} e^{-\frac{1}{15}t/\frac{1}{10}} = 48 - 48e^{-2t/3}$

terminal velocity: $\frac{mg}{k} = 48$ ft/s, so $50\% = 24$ ft/s

$48 - 48e^{-2t/3} = 24$

$24 = 48e^{-2t/3}$

$\frac{1}{2} = e^{-2t/3}$

$\ln\frac{1}{2} = -\frac{2t}{3} \Rightarrow \boxed{t = \frac{-3\ln\frac{1}{2}}{2} \text{ seconds}}$ or $\frac{3\ln 2}{2}$ seconds

5) **(10 points) Use Euler's method with step size $h = 0.1$ to approximate the solution to the following initial value problem at points $x = 0.1$ and $x = 0.2$.**

$$y' = y(2 - y), \quad y(0) = 3$$

$f(x,y) = y(2-y)$; $h = 0.1$; $(x_0, y_0) = (0, 3)$

$$
\begin{aligned}
x_1 = 0.1,\quad y_1 &= y_0 + hf(x_0, y_0) \\
&= 3 + 0.1f(0,3) \\
&= 3 + 0.1[3(-1)] \\
&= 3 + 0.1(-3) \\
&= 3 - 0.3 \\
&= \boxed{2.7}
\end{aligned}
$$

$$
\begin{aligned}
x_2 = 0.2,\quad y_2 &= y_1 + hf(x_1, y_1) \\
&= 2.7 + 0.1f(0.1, 2.7) \\
&= 2.7 + 0.1[2.7(-0.7)] \\
&= 2.7 + 0.1(-1.89) \\
&= 2.7 - 0.189 \\
&= \boxed{2.511}
\end{aligned}
$$

6) (a) (10 points) Given that the functions $y_1(x) = x$ and $y_2(x) = x\ln x$ are solutions of the differential equation

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0, \quad x > 0$$

(you do not need to verify that), use the Wronskian to show that y_1 and y_2 are linearly independent, and then find the solution of the initial value problem

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0; \quad y(1) = \frac{3}{2}, \; y'(1) = -\frac{1}{2}$$

$$W(y_1,y_2) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x(1+\ln x) - x\ln x = x + x\ln x - x\ln x$$

$$= x$$

$\neq 0$, so lin. indep.

$y = C_1 x + C_2 x \ln x$

$y' = C_1 + C_2 + C_2 \ln x$

$y(1) = \frac{3}{2} \Rightarrow C_1 = \frac{3}{2}$

$y'(1) = -\frac{1}{2} \Rightarrow C_1 + C_2 = -\frac{1}{2} \Rightarrow C_2 = -\frac{1}{2} - C_1 = -2$

$\boxed{y = \frac{3}{2}x - 2x\ln x}$

(b) (10 points) Given that $y_1(x) = x^{-2}$ is a solution of

$$x^2y'' + 6xy' + 6y = 0$$

use reduction of order to find a second linearly independent solution for $x > 0$.

$y_2 = v(x)x^{-2}$

$y_2' = v'x^{-2} - 2vx^{-3}$

$y_2'' = v''x^{-2} - 2v'x^{-3} - 2v'x^{-3} + 6vx^{-4} = v''x^{-2} - 4v'x^{-3} + 6vx^{-4}$

$x^2(v''x^{-2} - 4v'x^{-3} + 6vx^{-4}) + 6x(v'x^{-2} - 2vx^{-3}) + 6vx^{-2} = 0$

$v'' - 4v'x^{-1} + \cancel{6vx^{-2}} + 6v'x^{-1} - \cancel{12vx^{-2}} + \cancel{6vx^{-2}} = 0$

$v'' + 2v'x^{-1} = 0$

$v'' = -\frac{2}{x}v'$. Let $w = v'$. Then, $w' = -\frac{2}{x}w$.

$\frac{1}{w}dw = -\frac{2}{x}dx$

$\ln|w| = -2\ln|x| + C$

$|w| = e^{-2\ln|x| + C} = e^C e^{-2\ln|x|} = e^C x^{-2}$

$w = \pm e^C x^{-2} = Cx^{-2}$

$v = \int Cx^{-2}dx = -Cx^{-1} + D$

Take $v(x) = x^{-1}$.

$y_2 = v(x)x^{-2} = \boxed{x^{-3}}$