1) (a) (10 points) Solve the initial value problem

$$y'' - y' - 6y = 0; \quad y(0) = \frac{21}{4}, \quad y'(0) = \frac{39}{2}$$

$m^2 - m - 6 = 0 \Rightarrow (m-3)(m+2) = 0 \Rightarrow m = 3, -2$

$y = c_1 e^{3x} + c_2 e^{-2x}$

$y' = 3c_1 e^{3x} - 2c_2 e^{-2x}$

$y(0) = \frac{21}{4} \Rightarrow \quad c_1 + c_2 = \frac{21}{4}$

$y'(0) = \frac{39}{2} \Rightarrow \quad 3c_1 - 2c_2 = \frac{39}{2}$

$2c_1 + 2c_2 = \frac{21}{2}$

$5c_1 = 30 \Rightarrow c_1 = 6, \; c_2 = \frac{21}{4} - c_1 = \frac{-3}{4}$

$$\boxed{y = 6e^{3x} - \frac{3}{4}e^{-2x}}$$

(b) (10 points) Solve the initial value problem

$$y'' + 2y' + 3y = 0; \quad y(0) = -3, \quad y'(0) = 5$$

$m^2 + 2m + 3 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm 2\sqrt{2}\,i}{2}$

$= -1 \pm \sqrt{2}\,i$

$y = c_1 e^{-x} \cos\sqrt{2}x + c_2 e^{-x} \sin\sqrt{2}x$

$y' = -c_1 e^{-x} \cos\sqrt{2}x - \sqrt{2}c_1 e^{-x} \sin\sqrt{2}x - c_2 e^{-x} \sin\sqrt{2}x + \sqrt{2}c_2 e^{-x} \cos\sqrt{2}x$

$y(0) = -3 \Rightarrow \quad c_1 = -3$

$y'(0) = 5 \Rightarrow -c_1 + \sqrt{2}c_2 = 5 \Rightarrow \sqrt{2}c_2 = 2 \Rightarrow c_2 = \sqrt{2}$

$$\boxed{y = -3e^{-x} \cos\sqrt{2}x + \sqrt{2}e^{-x} \sin\sqrt{2}x}$$

2) (20 points) Use the method of undetermined coefficients to find the general solution of the equation

$$y'' - 4y = x\cos 2x$$

$$m^2 - 4 = 0 \Rightarrow m = \pm 2 \Rightarrow y_h = c_1 e^{2x} + c_2 e^{-2x}$$

$$y_p = (Ax+B)\cos 2x + (Cx+D)\sin 2x$$

$$y_p' = (-2Ax-2B)\sin 2x + A\cos 2x + (2Cx+2D)\cos 2x + C\sin 2x$$
$$= (2Cx + A + 2D)\cos 2x + (-2Ax - 2B + C)\sin 2x$$

$$y_p'' = (-4Cx - 2A - 4D)\sin 2x + 2C\cos 2x + (-4Ax - 4B + 2C)\cos 2x - 2A\sin 2x$$
$$= (-4Ax - 4B + 4C)\cos 2x + (-4Cx - 4A - 4D)\sin 2x$$

$y'' - 4y = x\cos 2x$ becomes

$$(-4Ax - 4B + 4C - 4Ax - 4B)\cos 2x + (-4Cx - 4A - 4D - 4Cx - 4D)\sin 2x = x\cos 2x$$

$$(-8Ax - 8B + 4C)\cos 2x + (-8Cx - 4A - 8D)\sin 2x = x\cos 2x$$

$$\left[\begin{array}{llll} -8A & & & = 1 \\ & -8B + 4C & & = 0 \\ & & -8C & = 0 \\ -4A & & -8D & = 0 \end{array}\right] \begin{array}{l} \Rightarrow A = -\frac{1}{8} \\ \Rightarrow B = \frac{1}{2}C = 0 \\ \Rightarrow C = 0 \\ \Rightarrow D = -\frac{1}{2}A = \frac{1}{16} \end{array}$$

$$y_p = -\frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x$$

general sol'n.:

$$\boxed{y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x}$$

3) (20 points) Use variation of parameters to find the general solution of the equation

$$y'' - 2y' + y = \frac{e^x}{x}, \quad x > 0$$

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$

$$y_h = c_1 e^x + c_2 x e^x$$

$y_p = v_1 e^x + v_2 x e^x$, where:

$$\left[\begin{aligned} v_1' e^x + v_2' x e^x &= 0 \\ v_1' e^x + v_2'(e^x + xe^x) &= \frac{e^x}{x} \end{aligned}\right]$$

$$-v_2' e^x = -\frac{e^x}{x}$$

$$v_2' = \frac{1}{x}$$

$$v_1' = -\frac{xe^x}{e^x} v_2' = -x v_2' = -1$$

OR: $W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}$

$= e^x(e^x + xe^x) - xe^x(e^x)$

$= e^{2x} + xe^{2x} - xe^{2x}$

$= e^{2x}$

$v_2' = \frac{y_1 f}{W} = \frac{e^x\left(\frac{e^x}{x}\right)}{e^{2x}} = \frac{1}{x}$

$v_1' = -\frac{y_2 f}{W} = \frac{-xe^x\left(\frac{e^x}{x}\right)}{e^{2x}} = -1$

$v_2 = \int \frac{1}{x}\, dx = \ln|x|$

$v_1 = \int -1\, dx = -x$

$y_p = -xe^x + xe^x \ln|x|$

$y = c_1 e^x + c_2 x e^x - xe^x + xe^x \ln|x|$

combine

$$\boxed{y_1 = c_1 e^x + c_2 x e^x + xe^x \ln|x|}$$

absolute value not needed, since $x > 0$

4) **(15 points)** **A 16-lb weight is attached to a frictionless spring, that in turn is suspended from the ceiling. The weight stretches the spring $\frac{8}{9}$ ft and comes to rest in its equilibrium position. The weight is then pushed up 1 foot and released with an upward velocity of 6 ft/s. Find the initial value problem that describes the motion of the weight, and solve it, writing your solution in the form $u(t) = R\cos(\omega_0 t - \delta)$.**

$mg = 16 \Rightarrow m = \frac{16}{32} = \frac{1}{2}$

$d = 0$

$\Delta L = \frac{8}{9} \Rightarrow k = \frac{mg}{\Delta L} = \frac{16}{8/9} = 16\left(\frac{9}{8}\right) = 18$

$\frac{1}{2}\ddot{u} + 18u = 0\ ;\quad u(0) = -1,\ \dot{u}(0) = -6$

$\ddot{u} + 36u = 0$

$r^2 + 36 = 0 \Rightarrow r = \pm 6i$

$u(t) = c_1\cos 6t + c_2 \sin 6t$

$\dot{u}(t) = -6c_1 \sin 6t + 6c_2 \cos 6t$

$u(0) = -1 \Rightarrow c_1 = -1$

$\dot{u}(0) = -6 \Rightarrow 6c_2 = -6 \Rightarrow c_2 = -1$

$u(t) = -\cos 6t - \sin 6t$

$= R\cos(6t - \delta)$, where

$R = \sqrt{c_1^2 + c_2^2} = \sqrt{1+1} = \sqrt{2}$

$\left.\begin{array}{l} \sin\delta = \frac{c_2}{R} = \frac{-1}{\sqrt{2}} \\ \cos\delta = \frac{c_1}{R} = \frac{-1}{\sqrt{2}} \end{array}\right\}\ \delta = \frac{5\pi}{4}$

$\boxed{u(t) = \sqrt{2}\cos\left(6t - \frac{5\pi}{4}\right)}$

5) **(15 points) An 8-lb weight is attached to a spring suspended from the ceiling. The spring constant is $k = \frac{9}{4}$, and the damping constant is $d = 1$. The weight is then pushed down $\underbrace{\text{3 inches}}_{=\frac{1}{4}\text{ft.}}$, and is released with a downward velocity of $\frac{\sqrt{15}-2}{4}$ ft/s.**

(a) Determine the motion of the weight, simplifying your answer into a single term.

(b) Determine the damped amplitude, damped frequency, and damped period of the motion.

(a) $mg = 8 \Rightarrow m = \frac{8}{32} = \frac{1}{4}$; $d = 1$; $k = \frac{9}{4}$

$\frac{1}{4}\ddot{u} + \dot{u} + \frac{9}{4}u = 0$; $u(0) = \frac{1}{4}$, $\dot{u}(0) = \frac{\sqrt{15}-2}{4}$

$\ddot{u} + 4\dot{u} + 9u = 0$

$r^2 + 4r + 9 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{16-36}}{2} = \frac{-4 \pm 2\sqrt{5}\,i}{2} = -2 \pm \sqrt{5}\,i$

$u(t) = c_1 e^{-2t}\cos\sqrt{5}t + c_2 e^{-2t}\sin\sqrt{5}t$

$\dot{u}(t) = -2c_1 e^{-2t}\cos\sqrt{5}t - \sqrt{5}c_1 e^{-2t}\sin\sqrt{5}t - 2c_2 e^{-2t}\sin\sqrt{5}t + \sqrt{5}c_2 e^{-2t}\cos\sqrt{5}t$

$u(0) = \frac{1}{4} \Rightarrow c_1 = \frac{1}{4}$

$\dot{u}(0) = \frac{\sqrt{15}-2}{4} \Rightarrow -2c_1 + \sqrt{5}c_2 = \frac{\sqrt{15}-2}{4} \Rightarrow \sqrt{5}c_2 = \frac{\sqrt{15}-2}{4} + \frac{2}{4} = \frac{\sqrt{15}}{4}$

$c_2 = \frac{\sqrt{3}}{4}$

$u(t) = \frac{1}{4}e^{-2t}\cos\sqrt{5}t + \frac{\sqrt{3}}{4}e^{-2t}\sin\sqrt{5}t$

$= Re^{-2t}\cos(\sqrt{5}t - \delta)$, where $R = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$\left.\begin{array}{l} \sin\delta = \frac{c_2}{R} = \frac{\sqrt{3}/4}{1/2} = \frac{\sqrt{3}}{2} \\ \cos\delta = \frac{c_1}{R} = \frac{1/4}{1/2} = \frac{1}{2} \end{array}\right\} \delta = \frac{\pi}{3}$

$$\boxed{u(t) = \frac{1}{2}e^{-2t}\cos\left(\sqrt{5}t - \frac{\pi}{3}\right)}$$

(b)
Damped amplitude: $\frac{1}{2}e^{-2t}$
Damped frequency: $\frac{\sqrt{5}}{2\pi}$
Damped period: $\frac{2\pi}{\sqrt{5}}$ or $\frac{2\pi\sqrt{5}}{5}$

6) **(10 points) A mass of 0.4 slugs is hanging at rest on a frictionless spring whose constant is $k = 10$. Beginning at time $t = 0$, an external force of $F(t) = 12\cos\omega t$ is applied to the system.**

(a) What is the angular frequency of the forcing function that is in resonance with the system?

(b) Find the equation of motion of the mass with resonance.

(a) $\omega = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{0.4}} = \sqrt{\frac{10}{2/5}} = \sqrt{\frac{50}{2}} = \sqrt{25} = \boxed{5}$

(b) $u(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$

$= \frac{12}{2(0.4)(5)} t \sin 5t$

$= \frac{12}{4} t \sin 5t$

$= \boxed{3t \sin 5t}$

7) (10 points) Use the definition of Laplace transform to find the Laplace transform of the function

$$f(t) = \begin{cases} 4t, & 0 \le t < 3 \\ e^{3t}, & t \ge 3 \end{cases}$$

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^3 4te^{-st}\,dt + \int_3^\infty e^{-st}e^{3t}\,dt$$

$$= \int_0^3 4te^{-st}\,dt + \lim_{b\to\infty} \int_3^b e^{-(s-3)t}\,dt$$

$u = 4t \qquad dv = e^{-st}dt$

$du = 4dt \qquad v = -\frac{e^{-st}}{s}$

$$= -\frac{4te^{-st}}{s}\Big|_{t=0}^{3} + \int_0^3 \frac{4e^{-st}}{s}\,dt + \lim_{b\to\infty} -\frac{e^{-(s-3)t}}{s-3}\Big|_{t=3}^{b}$$

$$= \left(-\frac{4te^{-st}}{s} - \frac{4e^{-st}}{s^2}\right)\Big|_{t=0}^{3} + \lim_{b\to\infty}\left[\underbrace{-\frac{e^{-(s-3)b}}{s-3}}_{\to 0 \text{ if } s>3} + \frac{e^{-3(s-3)}}{s-3}\right]$$

$$= \frac{-12e^{-3s}}{s} - \frac{4e^{-3s}}{s^2} - \left(0 - \frac{4}{s^2}\right) + 0 + \frac{e^{-3s+9}}{s-3}$$

$$= \boxed{-\frac{12e^{-3s}}{s} + \frac{4-4e^{-3s}}{s^2} + \frac{e^{-3s+9}}{s-3}}$$