1) (a) (10 points) Solve the initial value problem

$$2y'' - 9y' - 5y = 0; \quad y(0) = \frac{12}{5}, \quad y'(0) = 1$$

$2m^2 - 9m - 5 = 0 \Rightarrow (2m+1)(m-5) = 0$

$m = -\frac{1}{2}, 5$

$y = c_1 e^{-x/2} + c_2 e^{5x}$

$y' = -\frac{1}{2} c_1 e^{-x/2} + 5c_2 e^{5x}$

$y(0) = \frac{12}{5} \Rightarrow c_1 + c_2 = \frac{12}{5}$

$y'(0) = 1 \Rightarrow -\frac{1}{2} c_1 + 5c_2 = 1$

$-5c_1 - 5c_2 = -12$

$-\frac{11}{2} c_1 = -11 \Rightarrow c_1 = 2, \; c_2 = \frac{12}{5} - c_1 = \frac{2}{5}$

$\boxed{y = 2e^{-x/2} + \frac{2}{5} e^{5x}}$

(b) (10 points) Solve the initial value problem

$$y'' - 2y' + 5y = 0; \quad y(0) = 3, \quad y'(0) = 1$$

$m^2 - 2m + 5 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$y = c_1 e^x \cos 2x + c_2 e^x \sin 2x$

$y' = c_1 e^x \cos 2x - 2c_1 e^x \sin 2x + c_2 e^x \sin 2x + 2c_2 e^x \cos 2x$

$y(0) = 3 \Rightarrow c_1 = 3$

$y'(0) = 1 \Rightarrow c_1 + 2c_2 = 1 \Rightarrow c_2 = -1$

$\boxed{y = 3e^x \cos 2x - e^x \sin 2x}$

2) (20 points) Use the method of undetermined coefficients to find the general solution of the equation

$$y'' + 4y' + 4y = 4e^{-2t}; \quad y(0) = 5, \; y'(0) = -4$$

$m^2 + 4m + 4 = 0$

$(m+2)^2 = 0 \Rightarrow m = -2$

$Y_h = C_1 e^{-2t} + C_2 t e^{-2t}$

$Y_p = At^2 e^{-2t}$

$Y_p' = 2Ate^{-2t} - 2At^2 e^{-2t}$

$Y_p'' = 2Ae^{-2t} - 4Ate^{-2t} - 4Ate^{-2t} + 4At^2 e^{-2t}$

$= 2Ae^{-2t} - 8Ate^{-2t} + 4At^2 e^{-2t}$

$Y_p'' + 4Y_p' + 4Y_p = 2Ae^{-2t} - 8Ate^{-2t} + 4At^2 e^{-2t} + 8Ate^{-2t}$
$\quad - 8At^2 e^{-2t} + 4At^2 e^{-2t}$

$= 2Ae^{-2t}$

$2A = 4$

$A = 2$

$Y_p = 2t^2 e^{-2t}$

$\boxed{Y = C_1 e^{-2t} + C_2 t e^{-2t} + 2t^2 e^{-2t}}$

3) (20 points) Use variation of parameters to find the general solution of the equation

$$y'' + 9y = \sec 3x$$

$$m^2 + 9 = 0 \Rightarrow m = \pm 3i$$

$$y_h = c_1 \cos 3x + c_2 \sin 3x$$

$y_p = v_1 \cos 3x + v_2 \sin 3x$, where

$$\begin{bmatrix} v_1' \cos 3x + v_2' \sin 3x = 0 \\ -3v_1' \sin 3x + 3v_2' \cos 3x = \sec 3x \end{bmatrix}$$

$$3v_1' \sin 3x \cos 3x + 3v_2' \sin^2 3x = 0$$

$$-3v_1' \sin 3x \cos 3x + 3v_2' \cos^2 3x = 1$$

$$3v_2' = 1$$

$$v_2' = \frac{1}{3} \Rightarrow v_2 = \frac{1}{3}x$$

$$v_1' = -\frac{\sin 3x}{\cos 3x} v_2' = -\frac{1}{3}\tan 3x$$

$$v_1 = -\frac{1}{9}\ln|\sec 3x|$$

OR

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3$$

$$v_2' = \frac{y_1 f}{W} = \frac{\cos 3x \sec 3x}{3} = \frac{1}{3}$$

$$v_1' = -\frac{y_2 f}{W} = -\frac{\sin 3x \sec 3x}{3} = -\frac{1}{3}\tan 3x$$

$$y_p = -\frac{1}{9}\cos 3x \ln|\sec 3x| + \frac{1}{3}x \sin 3x$$

$$\boxed{y = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{9}\cos 3x \ln|\sec 3x| + \frac{1}{3}x \sin 3x}$$

4) **(15 points) An 8-lb weight is attached to a frictionless spring, that in turn is suspended from the ceiling. The weight stretches the spring $\frac{8}{9}$ ft and comes to rest in its equilibrium position. The weight is then pushed up 4 inches and released with a downward velocity of $2\sqrt{3}$ ft/sec. Find the initial value problem that describes the motion of the weight, and solve it, writing your solution in the form $u(t) = R\cos(\omega_0 t - \delta)$.**

$mg = 8 \Rightarrow m = \frac{1}{4}$

$d = 0$

$\Delta L = \frac{8}{9} \Rightarrow k = \frac{mg}{\Delta L} = \frac{8}{8/9} = 9$

$\frac{1}{4}\ddot{u} + 9u = 0 \; ; \; u(0) = -\frac{1}{3}, \; \dot{u}(0) = 2\sqrt{3}$

$\ddot{u} + 36u = 0$

$r^2 + 36 = 0 \Rightarrow r = \pm 6i$

$u(t) = c_1\cos 6t + c_2 \sin 6t$

$\dot{u}(t) = -6c_1 \sin 6t + 6c_2 \cos 6t$

$u(0) = -\frac{1}{3} \Rightarrow \quad c_1 = -\frac{1}{3}$

$\dot{u}(0) = 2\sqrt{3} \Rightarrow \quad 6c_2 = 2\sqrt{3} \Rightarrow c_2 = \frac{\sqrt{3}}{3}$

$u(t) = -\frac{1}{3}\cos 6t + \frac{\sqrt{3}}{3}\sin 6t = R\cos(6t - \delta)$, where

$R = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{1}{9} + \frac{3}{9}} = \frac{2}{3}$

$$\left.\begin{aligned} \sin\delta &= \frac{c_2}{R} = \frac{\sqrt{3}/3}{2/3} = \frac{\sqrt{3}}{2} \\ \cos\delta &= \frac{c_1}{R} = \frac{-1/3}{2/3} = -\frac{1}{2} \end{aligned}\right\} \; \delta = \frac{2\pi}{3}$$

$$\boxed{u(t) = \frac{2}{3}\cos\left(6t - \frac{2\pi}{3}\right)}$$

5) **(15 points) A 32-lb weight is attached to a spring suspended from the ceiling. The spring constant is $k = 5$, and the damping constant is $d = 4$. The weight is then pushed down 3 inches, and is released with a downward velocity of 9 in./sec.**

(a) Determine the motion of the weight, simplifying your answer into a single term.

(b) Determine the damped amplitude, damped frequency, and damped period of the motion.

(a) $mg = 32 \Rightarrow m = 1$

$\ddot{u} + 4\dot{u} + 5u = 0; \; u(0) = \frac{1}{4}, \; \dot{u}(0) = \frac{3}{4}$

$r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

$u(t) = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t$

$\dot{u}(t) = -2C_1 e^{-2t} \cos t - C_1 e^{-2t} \sin t - 2C_2 e^{-2t} \sin t + C_2 e^{-2t} \cos t$

$u(0) = \frac{1}{4} \Rightarrow C_1 \quad = \frac{1}{4}$

$\dot{u}(0) = \frac{3}{4} \Rightarrow -2C_1 + C_2 = \frac{3}{4} \Rightarrow C_2 = \frac{5}{4}$

$u(t) = \frac{1}{4} e^{-2t} \cos t + \frac{5}{4} e^{-2t} \sin t = R e^{-2t} \cos(t - \delta)$, where

$R = \sqrt{C_1^2 + C_2^2} = \sqrt{\frac{1}{16} + \frac{25}{16}} = \frac{\sqrt{26}}{4}$

$\left.\begin{array}{l} \sin\delta = \frac{C_2}{R} = \frac{5/4}{\sqrt{26}/4} = \frac{5}{\sqrt{26}} \\ \cos\delta = \frac{C_1}{R} = \frac{1/4}{\sqrt{26}/4} = \frac{1}{\sqrt{26}} \end{array}\right\}$ 1st quadrant
$\delta = \tan^{-1} 5$ (or $\sin^{-1}\frac{5}{\sqrt{26}}$ or $\cos^{-1}\frac{1}{\sqrt{26}}$)

$\boxed{u(t) = \frac{\sqrt{26}}{4} e^{-2t} \cos(t - \tan^{-1} 5)}$

(b) Damped amplitude $= \frac{\sqrt{26}}{4} e^{-2t}$

Damped frequency $= \frac{1}{2\pi}$

Damped period $= 2\pi$

6) **(10 points) A mass of 9 slugs is hanging at rest on a frictionless spring whose constant is $k = 16$. Beginning at time $t = 0$, an external force of $F(t) = 4\cos\omega t$ is applied to the system.**

(a) What is the angular frequency of the forcing function that is in resonance with the system?

(b) Find the equation of motion of the mass with resonance.

(a) $\omega = \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{9}} = \boxed{\frac{4}{3}}$

(b) $F(t) = \frac{F_0}{2m\omega_0} t \sin\omega_0 t$

$= \frac{4}{2(9)\left(\frac{4}{3}\right)} t \sin\frac{4}{3}t$

$= \boxed{\frac{1}{6} t \sin\frac{4}{3}t}$

7) **(10 points) Use the definition of Laplace transform to find the Laplace transform of the function**

$$f(t) = \begin{cases} e^{2t}, & 0 \le t < 2 \\ t, & t \ge 2 \end{cases}$$

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-(s-2)t}\,dt + \lim_{b\to\infty} \int_2^b t e^{-st}\,dt$$

$u = t \qquad dv = e^{-st}dt$

$du = dt \qquad v = -\frac{e^{-st}}{s}$

$$= -\frac{e^{-(s-2)t}}{s-2}\Big|_{t=0}^{2} + \lim_{b\to\infty}\left[-\frac{te^{-st}}{s}\Big|_2^b + \int_2^b \frac{e^{-st}}{s}\,dt\right]$$

$$= -\frac{e^{4-2s}}{s-2} + \frac{1}{s-2} + \lim_{b\to\infty}\left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]\Big|_2^b$$

$$= \frac{1-e^{4-2s}}{s-2} + \lim_{b\to\infty}\left[\underbrace{-\frac{be^{-sb}}{s} - \frac{e^{-sb}}{s^2}}_{\to 0 \text{ if } s>0} - \left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right)\right]$$

$$= \boxed{\frac{1-e^{4-2s}}{s-2} + \frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2}}$$