Systems & Control Letters 62 (2013) 225–233

Contents lists available at SciVerse ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

# Unit dual quaternion-based feedback linearization tracking problem for attitude and position dynamics[☆]

Xiangke Wang [a,b], Changbin Yu [b,c,*]

[a] College of Mechatronics and Automation, National University of Defense Technology, 410073, Changsha, China
[b] Research School of Engineering, The Australian National University, Canberra ACT 2601, Australia
[c] Shandong Provincial Key Laboratory of Computer Network, Shandong Computer Science Center, Jinan 250014, China

ARTICLE INFO

*Article history:*
Received 29 July 2010
Received in revised form
18 November 2012
Accepted 18 November 2012
Available online 16 January 2013

*Keywords:*
Attitude and position tracking problem
Unit dual quaternion
Dynamics
Regulation problem

ABSTRACT

This paper provides a unified solution for the attitude and position tracking problem of a rigid body in 3-dimensional space, using the concept of the unit dual quaternion. The error dynamics described by a unit dual quaternion are deduced after the dual-quaternion-based dynamics of a single rigid body are given. Then by utilizing the feedback linearization principle, a unit dual quaternion based tracker is proposed based on the error dynamics, which is proven to render the equilibrium point of the closed loop system asymptotically stable, and includes the attitude and position regulation problems as particular cases. Furthermore, to solve the two equilibria problem, a switching parameter is introduced to improve the tracker, which causes the system to converge to the 'nearer' equilibrium with a 'shorter' path. Both the trackers uniquely deal with the rotational and translational dynamics simultaneously with non-singularity and maintain the interconnection between rotation and translation. Finally, the proposed control schemes with applications to the regulation problem and the tracking problem are simulated to illustrate the theoretical results.



## 1. Introduction

Attitude and position control is a critical issue for a rigid body in 3-dimensional (3D) space. It has been studied in many applications, such as autonomous underwater vehicles (AUV's) [1,2], and spacecraft [3,4]. In most existing literature, the dynamic equations of motion are separated into the rotational and the translational motions; consequently, the control problem is divided into the attitude control problem and the position control problem, respectively. Position is specified by a three-dimensional vector, whereas various representations of attitude have been discussed, such as Euler angles, Rodrigues parameters, unit equivalent axis/angle, and the unit quaternion. The unit quaternion may be the most popular tool in attitude control, as it uses the least possible number of parameters (four) to represent attitude globally [5,6]. However, the unit quaternion just focuses on attitude control and translational motion control cannot be addressed. The second method for attitude and position control problem is based on utilizing the geometric structure of SE(3) [7]. All the homogenous transformation matrices construct SE(3). However, the homogenous transformation matrix is a $4 \times 4$ matrix, which leads to the controller being more complex than the controllers obtainable using the methods of our paper. In this paper, we use the unit dual quaternion as the basis to provide a unified solution of the attitude and position control problem in 3D space. Note that a second but different unified solution is provided in [7], where the construction of controllers is achieved with homogenous transformation matrices.

The unit dual quaternion, as a natural extension of the unit quaternion [8] and consisting of the unit quaternion and the translational vector in Plücker coordinates, is better to represent an arbitrary transformation comprising rotation and translation with its non-singularity and compactness using only 8 numbers. It has been revealed by existing works that among the many mathematical approaches such as homogeneous transformation matrices [9], quaternion/vector pairs [10], Lie algebra [7], the unit dual quaternion offers the most compact and most efficient way to express the screw motion, that is both rotational and translational transformation in 3D space [10–12]. The unit dual quaternion has been found useful in many applications, such as computer-aided geometric

[☆] Part of this paper appeared in its preliminary form in the Proceedings of the 2010 IEEE Multi-Conference on Systems and Control, Yokohama, p. 2380–2384, 2010.

* Corresponding author at: Shandong Provincial Key Laboratory of Computer Network, Shandong Computer Science Center, Jinan 250014, China. Tel.: +61 2 61258670.

*E-mail addresses:* xkwang@nudt.edu.cn (X. Wang), brad.yu@anu.edu.au (C. Yu).

0167-6911/$ – see front matter 
doi:10.1016/j.sysconle.2012.11.019

design [13], image-based localization [14], hand-eye calibration [15] and navigation [16]. However, the unit dual quaternion based control problem has not been extensively addressed due to some technical challenges, although dual quaternions offer a potentially significant advantage (for example, compactness, non-singularity, computational efficiency). The primary use of the dual quaternion in control is reported in [17], in which, in a manner parallel (but not equivalent) to [7], the generalized proportional controller on kinematics is derived based on the unit dual quaternion Lie group and its logarithmic mapping. To the best of our knowledge, none of the existing works apply the dual quaternion to the dynamic control of a rigid body. This study provides arguably the first attempt to solve the dynamic control problem with a unit dual quaternion. We first deduce the dual-quaternion-based dynamics of a single rigid body, which can be treated as the basis for the solution of many rigid body dynamic problems, from traditional rotational dynamics and translational dynamics. Then we further deduce the error dynamics for the attitude and position tracking problem. Based on the error dynamics, we propose a tracker by utilizing the feedback linearization principle, which is proven to render the equilibrium point of the closed loop system asymptotically stable. To solve the two equilibria problem, we introduce a switching parameter to improve the proposed tracker; this causes the system to converge to the 'nearer' equilibrium with a 'shorter' path. Compared with conventional decoupled methods, the proposed trackers uniquely deal with the rotational and translational dynamic control simultaneously without decoupling, and maintain the interconnection between rotation and translation. Compared with the SE(3) method in [7], the proposed controllers provide the notional concision and avoid singularities without limitations of parameters. Both proposed trackers include the attitude and position regulation problem as a particular case. Further, the effectiveness of the proposed trackers and the particular regulators are easily validated in simulations.

The remainder of the paper is organized as follows. The mathematical preliminaries of dual quaternions are introduced in Section 2, and then the dynamics of a single rigid body and the error dynamics are both deduced with unit dual quaternion descriptors in Section 3. A new tracker and its stability analysis are established in Section 4. The introduction of a switching parameter to solve two equilibria problem is also discussed in this section. Section 5 demonstrates the simulation results when the trackers are applied to the regulation problem and then the tracking problem. The last section states conclusions.

*Notions*

We employ 0 simply to denote the scalar zero. Three dimensional vector $(0, 0, 0)^T$, dual number $0 + \epsilon 0$ and dual vector $(0, 0, 0)^T + \epsilon(0, 0, 0)^T$ are denoted by $\mathbf{0}$, $\hat{0}$ and $\hat{\mathbf{0}}$, respectively. We denote unit quaternion $[1, 0, 0, 0]$ and unit dual quaternion $[1, 0, 0, 0] + \epsilon[0, 0, 0, 0]$ by $I$ and $\hat{I}$, respectively. If not otherwise stated, a (dual) vector is denoted by the boldface, and its corresponding (dual) vector quaternion is denoted by the normal type, for example, $v = [0, \boldsymbol{v}]$ or $\hat{v} = [\hat{0}, \hat{\boldsymbol{v}}]$.

## 2. Mathematical preliminaries

In this section, a brief introduction to the dual quaternion is given. As two foundations of the dual quaternion, the concepts of the quaternion and the dual number are defined first. More details on these concepts can be found in the literature [14,16,17].

The *quaternion* is an extension of a complex number to $\mathbb{R}^4$. Formally, a quaternion $q$ can be defined as

$$q = a + bi + cj + dk,$$

where $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. A convenient shorthand notation is $q = [s, \boldsymbol{v}]$, where $s$ is a scalar (called *scalar part*), and $\boldsymbol{v}$ is a three-dimensional vector (called *vector part*). Obviously, a three-dimensional vector can be treated as a quaternion with vanishing scalar part, called *vector quaternion*. If $a^2 + b^2 + c^2 + d^2 = 1$, then the quaternion $q$ is called a *unit quaternion*.

A *dual number* is defined as

$$\hat{a} = a + \epsilon b \quad \text{with } \epsilon^2 = 0, \text{ but } \epsilon \neq 0,$$

where $a$ and $b$ are real numbers, called the *real part* and the *dual part*, respectively, and $\epsilon$ is nilpotent, such as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

*Dual vectors* are a generalization of dual numbers whose real and dual parts are both three-dimensional vectors.

To compare two dual vectors, a partial order is defined as follows.

**Definition 1.** Let $\hat{\boldsymbol{v}}_1 = \boldsymbol{v}_{r1} + \epsilon \boldsymbol{v}_{d1}$ and $\hat{\boldsymbol{v}}_2 = \boldsymbol{v}_{r2} + \epsilon \boldsymbol{v}_{d2}$ be dual vectors. If $\boldsymbol{v}_{r1} - \boldsymbol{v}_{r2} \geq (>) \mathbf{0}$ and $\boldsymbol{v}_{d1} - \boldsymbol{v}_{d2} \geq (>) \mathbf{0}$, then $\hat{\boldsymbol{v}}_1 \geq (>) \hat{\boldsymbol{v}}_2$.

A *dual quaternion* is a quaternion with components replaced by dual number components, i.e. $\hat{q} = [\hat{s}, \hat{\boldsymbol{v}}]$, where $\hat{s}$ is a dual number and $\hat{\boldsymbol{v}}$ is a dual vector. Obviously, a dual vector can be treated equivalently as a dual quaternion with vanishing scalar part, and such a dual quaternion is termed a *dual vector quaternion*. The *dot-product* between two dual vector quaternions is defined as follows.

**Definition 2.** For two dual vector quaternions $\hat{v} = [\hat{0}, \hat{\boldsymbol{v}}]$ with $\hat{\boldsymbol{v}} = \boldsymbol{v}_r + \epsilon \boldsymbol{v}_d$ and $\hat{k} = [\hat{0}, \hat{\boldsymbol{k}}]$ with $\hat{\boldsymbol{k}} = \boldsymbol{k}_r + \epsilon \boldsymbol{k}_d = (k_{r1}, k_{r2}, k_{r3})^T + \epsilon(k_{d1}, k_{d2}, k_{d3})^T$, the *dot-product* is defined as $\hat{k} \cdot \hat{v} = [0, K_r \boldsymbol{v}_r] + \epsilon[0, K_d \boldsymbol{v}_d]$, where $K_r$ and $K_d$ are $3 \times 3$ diagonal matrices with diagonal entries $k_{r1}, k_{r2}, k_{r3}$ and $k_{d1}, k_{d2}, k_{d3}$, respectively, namely $K_r = \mathrm{diag}(k_{r1}, k_{r2}, k_{r3})$ and $K_d = \mathrm{diag}(k_{d1}, k_{d2}, k_{d3})$. For convenience and simplicity, the sign '$\cdot$' is sometimes omitted in the following.

A dual quaternion can also be rewritten as $\hat{q} = q_r + \epsilon q_d$, where $q_r$ and $q_d$ are both quaternions. The following operations are defined on dual quaternions:

$$\hat{q}_1 + \hat{q}_2 = [\hat{s}_1 + \hat{s}_2, \hat{\boldsymbol{v}}_1 + \hat{\boldsymbol{v}}_2] = (q_{r1} + q_{r2}) + \epsilon(q_{d1} + q_{d2}),$$
$$\lambda \hat{q} = [\lambda \hat{s}, \lambda \hat{\boldsymbol{v}}] = \lambda q_r + \epsilon \lambda q_d,$$
$$\begin{aligned}\hat{q}_1 \circ \hat{q}_2 &= [\hat{s}_1 \hat{s}_2 - \hat{\boldsymbol{v}}_1^T \hat{\boldsymbol{v}}_2, \hat{s}_1 \hat{\boldsymbol{v}}_2 + \hat{s}_2 \hat{\boldsymbol{v}}_1 + \hat{\boldsymbol{v}}_1 \times \hat{\boldsymbol{v}}_2] \\ &= q_{r1} \circ q_{r2} + \epsilon(q_{r1} \circ q_{d2} + q_{d1} \circ q_{r2}),\end{aligned}$$

where $\hat{q}_1$, $\hat{q}_2$ and $\hat{q}$ are all dual quaternions, $\lambda \in \mathbb{R}$ is a scalar and the operator '$\circ$' is (dual) quaternion multiplication. Note that the dual quaternion multiplication is associative and distributive but not commutative.

The *conjugate* of the dual quaternion $\hat{q}$ is

$$\hat{q}^* = [\hat{s}, -\hat{\boldsymbol{v}}].$$

The multiplicative inverse element of a dual quaternion $\hat{q}$ is

$$\hat{q}^{-1} = (1/\hat{q} \circ \hat{q}^*) \circ \hat{q}^*.$$

If $\hat{q} \circ \hat{q}^* = \hat{I}$, then a dual quaternion is called a *unit dual quaternion*. For a unit dual quaternion, $\hat{q}^{-1} = \hat{q}^*$.

The unit dual quaternion is the natural extension of the unit quaternion to convert the point coordinates to the dual line coordinates in 3D space. According to the principle of transference [8], the characteristics of a unit quaternion are completely inherited by the unit dual quaternion.

Unit quaternions can be used to describe rotation. For the frame rotation about a unit axis $\boldsymbol{n}$ with an angle $|\theta| < 2\pi$, there is a unit quaternion

$$q = \left[\cos\left(\frac{|\theta|}{2}\right), \sin\left(\frac{|\theta|}{2}\right) \boldsymbol{n}\right] \tag{1}$$

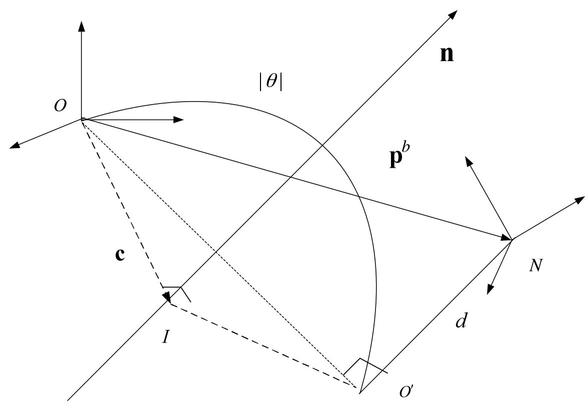


**Fig. 1.** Geometry of screw motion: every screw motion can be modeled as rotation with angle $|\theta|$ about axis at point $\boldsymbol{c}$ with direction $\boldsymbol{n}$ and subsequent translation $d$ along axis.

relating a fixed vector expressed in the original frame $\boldsymbol{r}^o$ with the same vector expressed in the new frame $\boldsymbol{r}^n$ by

$$r^n = q^* \circ r^o \circ q.$$

Here $r^o$ and $r^n$ are two quaternions with vanishing scalar part. For example, $r^o = [0, \boldsymbol{r}^o]$.

A unit dual quaternion can be used to represent a transformation (rotation and translation simultaneously) in 3D space. Considering a rotation $q$ succeeded by a translation $\boldsymbol{p}^b$, according to the Chasles Theorem (refer to Theorem 2.11, [9]), this transformation is equivalent to a screw motion, which is a rotation about an axis $\boldsymbol{n}$ with angel $|\theta|$ combined with a translation $d$ parallel to that axis illustrated in Fig. 1, the transformation or screw motion can be represented using a unit dual quaternion[1]

$$\hat{q} = \left[\cos\frac{\hat{\theta}}{2}, \sin\frac{\hat{\theta}}{2}\hat{\boldsymbol{n}}\right] = q + \frac{\epsilon}{2} q \circ p^b, \tag{2}$$

where $\hat{\boldsymbol{n}} = \boldsymbol{n} + \epsilon(\boldsymbol{c} \times \boldsymbol{n})$ is the screw axis and $\hat{\theta} = |\theta| + \epsilon d$ is the dual angle of the screw.

A unit quaternion $q$ defines a rotation, taking coordinates of a point from one frame to another. Conversely, every attitude of a rigid body that is free to rotate relative to a fixed frame can be identified with a unique unit quaternion $q$. Analogously to the rotational case, a unit dual quaternion $\hat{q}$ serves as both a specification of the configuration (consisting of attitude and position) of a rigid body and a transformation taking the coordinates of a point from one frame to another via rotation and translation.

Unit quaternions form a Lie group over multiplication with the conjugate being the inverse, denoted by $Q_u$. The same goes for unit dual quaternions, which form a Lie group over dual quaternion multiplication, denoted by $DQ_u$ [17]. The logarithm of a unit quaternion defined as in Eq. (1) is defined as [18]

$$\ln q = \frac{\theta}{2}.$$

Here $\theta = [0, |\theta|\boldsymbol{n}]$ is a vector quaternion. Similarly, the logarithmic mapping of the unit dual quaternion as given by (2) is defined as [17]

$$\ln \hat{q} = \frac{1}{2}(\theta + \epsilon p^b),$$

which is a dual vector quaternion.

In this study, we make $|\theta| \in [0, 2\pi)$. The spaces consisting of all unit quaternion logarithmic mappings and all unit dual quaternion logarithmic mappings are denoted by $\upsilon$ and $\hat{\upsilon}$ respectively; note that $\hat{\upsilon}$ is the Lie algebra of $DQ_u$ and $\upsilon$ is the Lie algebra of $Q_u$.

For $Q_u$, the *adjoint transformation* is defined as

$$Ad_q V = q \circ V \circ q^{-1} = q \circ V \circ q^*,$$

where $V \in \upsilon$. The same goes for the *adjoint transformation* on $DQ_u$, i.e.

$$Ad_{\hat{q}} \hat{V} = \hat{q} \circ \hat{V} \circ \hat{q}^{-1} = \hat{q} \circ \hat{V} \circ \hat{q}^*,$$

where $\hat{V} \in \hat{\upsilon}$. It is worth pointing out here that $Ad_q V$ and $Ad_{\hat{q}} \hat{V}$ are still, respectively, vector quaternion and dual vector quaternion. For notional economy, we employ $Ad_q \boldsymbol{V}$ and $Ad_{\hat{q}} \hat{\boldsymbol{V}}$ to denote their corresponding vector and dual vector, respectively.

## 3. Dynamics and control objective

Little work has been done on dynamics problems with dual quaternions. Literature [19] gave the formulation of a general dynamics problem using dual quaternion components. However the motion equations are quite complicated and the physical significance of the variables is not intuitively apparent. In this section, we deduce dual-quaternion-based single rigid body dynamics in a body frame from traditional rotational dynamics and translational dynamics, which can act as the basis for the solution of many rigid body dynamic problems. Based on this dynamics, we further deduce the error dynamics for the attitude and position tracking problem. The control objective for attitude and position tracking problem is also formulated with dual quaternion descriptors in this section.

### 3.1. Dual-quaternion-based dynamics

Following [20,21], the rotational dynamics of a rigid body can be written in a body frame in the following form

$$J\dot{\boldsymbol{\omega}}^b + \boldsymbol{\omega}^b \times (J\boldsymbol{\omega}^b) = \boldsymbol{\tau}, \tag{3}$$

where $\boldsymbol{\omega}^b \in \mathbb{R}^3$ is the angular velocity vector in the body frame; $J \in \mathbb{R}^{3\times 3}$ is the rigid body inertia matrix in the body frame; and $\boldsymbol{\tau} \in \mathbb{R}^3$ is the control torque vector associated with the rigid body.

The translational dynamics of the rigid body relative to the body frame is

$$m\ddot{\boldsymbol{p}}^b = \boldsymbol{f}, \tag{4}$$

where $\boldsymbol{p}^b \in \mathbb{R}^3$ is the position of the rigid body, $m \in \mathbb{R}$ and $\boldsymbol{f} \in \mathbb{R}^3$ are the mass and the control force in the body frame, respectively [22].

Thus, from (3) and (4), we obtain

$$\dot{\boldsymbol{\omega}}^b = -J^{-1}\boldsymbol{\omega}^b \times J\boldsymbol{\omega}^b + J^{-1}\boldsymbol{\tau}, \tag{5}$$

$$\ddot{\boldsymbol{p}}^b = \boldsymbol{f}/m. \tag{6}$$

The kinematic equation of a rigid body expressed with a unit dual quaternion is

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \circ \xi^b,$$

where

$$\boldsymbol{\xi}^b = \boldsymbol{\omega}^b + \epsilon(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b)$$

is the twist in the body frame [16].

Differentiating the twist in the body frame $\boldsymbol{\xi}^b$, we obtain

$$\dot{\boldsymbol{\xi}}^b = \dot{\boldsymbol{\omega}}^b + \epsilon(\ddot{\boldsymbol{p}}^b + \dot{\boldsymbol{\omega}}^b \times \boldsymbol{p}^b + \boldsymbol{\omega}^b \times \dot{\boldsymbol{p}}^b). \tag{7}$$

[1] Superscripts $b$ and $s$ relate to the body frame (which is attached to the body) and the spatial frame (which is relative to a fixed (inertial) coordinate frame) respectively throughout this paper. The concepts of body frame and spatial frame come from [9].

Substitute (5) and (6) into (7); then

$$\begin{aligned}\dot{\boldsymbol{\xi}}^b &= \boldsymbol{a} + J^{-1}\boldsymbol{\tau} + \epsilon\left(\boldsymbol{f}/m + (\boldsymbol{a} + J^{-1}\boldsymbol{\tau}) \times \boldsymbol{p}^b + \boldsymbol{\omega}^b \times \dot{\boldsymbol{p}}^b\right)\\ &= J^{-1}\boldsymbol{\tau} + \epsilon(\boldsymbol{f}/m + J^{-1}\boldsymbol{\tau} \times \boldsymbol{p}^b)\\ &\quad + \left(\boldsymbol{a} + \epsilon(\boldsymbol{a} \times \boldsymbol{p}^b + \boldsymbol{\omega}^b \times \dot{\boldsymbol{p}}^b)\right)\\ &= \hat{\boldsymbol{F}} + \hat{\boldsymbol{U}},\end{aligned}$$

where $\boldsymbol{a} = -J^{-1}\boldsymbol{\omega}^b \times J\boldsymbol{\omega}^b$, and

$$\begin{cases}\hat{\boldsymbol{F}} = \boldsymbol{a} + \epsilon(\boldsymbol{a} \times \boldsymbol{p}^b + \boldsymbol{\omega}^b \times \dot{\boldsymbol{p}}^b),\\ \hat{\boldsymbol{U}} = J^{-1}\boldsymbol{\tau} + \epsilon(\boldsymbol{f}/m + J^{-1}\boldsymbol{\tau} \times \boldsymbol{p}^b).\end{cases} \tag{8}$$

Summarizing the above derivations, we obtain the rigid body dynamics described by a unit dual quaternion:

**Model 1** (*Dual-Quaternion-Based Dynamics*)**.** *Let $\hat{q} = q + \epsilon\frac{1}{2}q \circ p^b$ describe the screw motion that a rotation q succeeded by a translation $\boldsymbol{p}^b$, and denote $\dot{p}^b$ and $\omega^b$ linear and angular velocity, respectively; then the dual-quaternion-based dynamics are*

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \circ \xi^b, \tag{9}$$

$$\boldsymbol{\xi}^b = \boldsymbol{\omega}^b + \epsilon(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b), \tag{10}$$

$$\dot{\boldsymbol{\xi}}^b = \hat{\boldsymbol{F}} + \hat{\boldsymbol{U}}, \tag{11}$$

*where $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{U}}$ are expressed in (8).*

The dynamics here are much simpler than those in [19], and moreover they relate directly to the traditional dynamics with clearly physical significance. Dual-quaternion-based dynamics provide a basis for any rigid body dynamic problem. Based on this dynamics, we further deduce the error dynamics for the attitude and position tracking problem.

### 3.2. Error dynamics

For the tracking problem, the *left-invariant error* is defined to be the tracking error between the current configuration $\hat{q}$ and the desired configuration $\hat{q}_d$, which is

$$\hat{q}_e = \hat{q}_d^* \circ \hat{q}. \tag{12}$$

Moreover, $\dot{\hat{q}}_d = \frac{1}{2}\hat{q}_d \circ \xi_d^b$ with $\boldsymbol{\xi}_d^b = \boldsymbol{\omega}_d^b + \epsilon(\dot{\boldsymbol{p}}_d^b + \boldsymbol{\omega}_d^b \times \boldsymbol{p}_d^b)$. Note that $\boldsymbol{\xi}_d^b$ is assumed to be bounded as well as its first time-derivatives.

Now $\hat{q}_e$ can be rewritten into a similar form as (2) after some algebraic manipulations, as follows:

$$\hat{q}_e = q_e + \frac{\epsilon}{2}q_e \circ p_e^b, \tag{13}$$

where $q_e = q_d^* \circ q$ and $p_e^b = p^b - Ad_{q_e^*}p_d^b$. The details are in Appendix A.

After taking the time derivative of (12), we obtain

$$\dot{\hat{q}}_e = \dot{\hat{q}}_d^* \circ \hat{q} + \hat{q}_d^* \circ \dot{\hat{q}}.$$

Using $\dot{\hat{q}} = \frac{1}{2}\hat{q} \circ \xi^b$ and $\dot{\hat{q}}_d = \frac{1}{2}\hat{q}_d \circ \xi_d^b$, we obtain

$$\begin{aligned}\dot{\hat{q}}_e &= -\frac{1}{2}\xi_d^b \circ \hat{q}_d^* \circ \hat{q} + \frac{1}{2}\hat{q}_d^* \circ \hat{q} \circ \xi^b\\ &= -\frac{1}{2}\xi_d^b \circ \hat{q}_e + \frac{1}{2}\hat{q}_e \circ \xi^b\\ &= \frac{1}{2}\hat{q}_e \circ (\xi^b - Ad_{\hat{q}_e^*}\xi_d^b).\end{aligned}$$

Let $\xi^b - Ad_{\hat{q}_e^*}\xi_d^b = \xi_e^b$; then

$$\dot{\hat{q}}_e = \frac{1}{2}\hat{q}_e \circ \xi_e^b.$$

Similarly, $\boldsymbol{\xi}_e^b$ can be rewritten into a similar form as (10) after complex manipulations; these results

$$\boldsymbol{\xi}_e^b = \boldsymbol{\omega}_e^b + \epsilon(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b), \tag{14}$$

where $\omega_e^b = \omega^b - Ad_{q_e^*}\omega_d^b$ and $p_e^b = p^b - Ad_{q_e^*}p_d^b$. The details are in Appendix B.

Taking the time derivative of $\xi_e^b = \xi^b - Ad_{\hat{q}_e^*}\xi_d^b$, we obtain

$$\begin{aligned}\dot{\xi}_e^b &= \dot{\xi}^b - (\dot{\hat{q}}_e^* \circ \xi_d^b \circ \hat{q}_e + \hat{q}_e^* \circ \dot{\xi}_d^b \circ \hat{q}_e + \hat{q}_e^* \circ \xi_d^b \circ \dot{\hat{q}}_e)\\ &= \dot{\xi}^b - \left(-\frac{1}{2}\xi_e^b \circ Ad_{\hat{q}_e^*}\xi_d^b + \frac{1}{2}Ad_{\hat{q}_e^*}\xi_d^b \circ \xi_e^b + Ad_{\hat{q}_e^*}\dot{\xi}_d^b\right)\\ &= \dot{\xi}^b - Ad_{\hat{q}_e^*}\dot{\xi}_d^b - [\hat{0}, Ad_{\hat{q}_e^*}\boldsymbol{\xi}_d^b \times \boldsymbol{\xi}_e^b].\end{aligned} \tag{15}$$

Using (11), we obtain

$$\dot{\xi}_e^b = \hat{F} - Ad_{\hat{q}_e^*}\dot{\xi}_d^b - [\hat{0}, Ad_{\hat{q}_e^*}\boldsymbol{\xi}_d^b \times \boldsymbol{\xi}_e^b] + \hat{U}.$$

Summarizing the above derivations, we obtain the following error dynamics.

**Model 2** (*Error Dynamics*)**.** *Given the current configuration $\hat{q} = q + \frac{\epsilon}{2}q \circ p^b$ and the desired configuration $\hat{q}_d = q_d + \frac{\epsilon}{2}q_d \circ p_d^b$, the error dynamics derived from left-invariant error (12) are*

$$\dot{\hat{q}}_e = \frac{1}{2}\hat{q}_e \circ \xi_e^b, \tag{16}$$

$$\xi_e^b = \xi^b - Ad_{\hat{q}_e^*}\xi_d^b = [0, \boldsymbol{\omega}_e^b] + \epsilon[0, \dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b], \tag{17}$$

$$\dot{\xi}_e^b = \hat{F} - Ad_{\hat{q}_e^*}\dot{\xi}_d^b - [\hat{0}, Ad_{\hat{q}_e^*}\boldsymbol{\xi}_d^b \times \boldsymbol{\xi}_e^b] + \hat{U}, \tag{18}$$

*where $\hat{q}_e = \hat{q}_d^* \circ \hat{q}$, $\xi^b$ and $\xi_d^b$ are twists of $\hat{q}$ and $\hat{q}_d$ respectively, $q_e = q_d^* \circ q$, $\omega_e^b = \omega^b - Ad_{q_e^*}\omega_d^b$ and $p_e^b = p^b - Ad_{q_e^*}p_d^b$, $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{U}}$ are expressed in (8).*

### 3.3. Control objective

The control objective in our attitude and position tracking problem is to design trackers (without requiring decoupling the rotational and translational dynamics) to track asymptotically the desired configuration based on Model 2. Mathematically, let $\hat{q} = q + \frac{\epsilon}{2}q \circ p^b$ be the current configuration and $\hat{q}_d = q_d + \frac{\epsilon}{2}q_d \circ p_d^b$ be the desired configuration with $\dot{\hat{q}}_d = \frac{1}{2}\hat{q}_d \circ \xi_d^b$ and $\boldsymbol{\xi}_d^b = \boldsymbol{\omega}_d^b + \epsilon(\dot{\boldsymbol{p}}_d^b + \boldsymbol{\omega}_d^b \times \boldsymbol{p}_d^b)$. Then the control objective is: when $t \to \infty$, $\hat{q}(t) \to \hat{q}_d(t)$ and $\boldsymbol{\xi}^b(t) \to \boldsymbol{\xi}_d^b(t)$, or $\lim_{t\to\infty}\hat{q}_e(t) = \pm\hat{I}$ and $\lim_{t\to\infty}\boldsymbol{\xi}_e^b(t) = \hat{\boldsymbol{0}}$. Perfect and instantaneous measurements of $\hat{q}$ and $\boldsymbol{\xi}^b$ are assumed.

## 4. Feedback linearization trackers

A general controller structure based on the principle of feedback linearization is given in [5] in the following form

$$\begin{aligned}\hat{U} &= \text{proportional and derivative feedback}\\ &\quad + \text{feedforward compensation}.\end{aligned} \tag{19}$$

In (19), the *feedforward compensation* is to eliminate the nonlinearity, and the *proportional and derivative feedback* is to ensure stability. In the following, we will propose stable trackers respecting this general structure.

### 4.1. Tracker and stability analysis

Considering Model 2, a tracker is proposed based on the dual quaternion Lie-group and its Lie-algebra (its logarithmic mapping) as follows.

$$\begin{aligned}\hat{U} &= -2\hat{k}_p \ln\hat{q}_e - \hat{k}_v\xi_e^b - \hat{F}\\ &\quad + Ad_{\hat{q}_e^*}\dot{\xi}_d^b + [\hat{0}, Ad_{\hat{q}_e^*}\boldsymbol{\xi}_d^b \times \boldsymbol{\xi}_e^b],\end{aligned} \tag{20}$$

where $\hat{\boldsymbol{k}}_p = \boldsymbol{k}_{pr} + \epsilon\boldsymbol{k}_{pd} = (k_{pr1}, k_{pr2}, k_{pr3})^T + \epsilon(k_{pd1}, k_{pd2}, k_{pd3})^T$ and $\hat{\boldsymbol{k}}_v = \boldsymbol{k}_{vr} + \epsilon\boldsymbol{k}_{vd} = (k_{vr1}, k_{vr2}, k_{vr3})^T + \epsilon(k_{vd1}, k_{vd2}, k_{vd3})^T$. The parameters $k_{pri}, k_{pdi}, k_{vri}$ and $k_{vdi}(i = 1, 2, 3)$ will be designed in the sequel.

It is worth pointing out that, compared with the conventional decoupled controls, tracker (20) uniquely deals with the rotational and translational dynamic control simultaneously without decoupling and maintains the interconnection between rotation and translation. To illustrate the point more clearly, the explicit forms of control inputs $\boldsymbol{\tau}$ and $\boldsymbol{f}$ in the tracker (20) are extracted in detail in Appendix C. The input $\boldsymbol{f}$ has coupling terms between attitude and position, which does not occur in the conventional attitude control problem and position control problem. Note that there are no coupling terms in $\boldsymbol{\tau}$, which is in accordance with common sense: rotation affects translation whereas translation does not affect rotation. Therefore, the proposed tracker reflects the natural interconnection between rotation and translation without requiring decoupling.

**Theorem 1.** *For Model 2 and tracker (20) with appropriate parameters $\hat{k}_p$ and $\hat{k}_v$, the equilibrium point $(\hat{q}_e, \boldsymbol{\xi}_e^b) = (\hat{I}, \hat{\mathbf{0}})$ is asymptotically stable in the large on $DQ_u$ for almost all initial state $\big(\hat{q}_e(0), \boldsymbol{\xi}_e(0)\big)$, except for a special small set.*

Before proving Theorem 1, we first give a norm definition for a dual vector quaternion.

**Definition 3** (*Norm Definition*)**.** Let $\hat{v} = v_r + \epsilon v_d$ be a dual vector quaternion. Define

$$|\hat{v}|^2 = \alpha|\boldsymbol{v}_r|^2 + \beta|\boldsymbol{v}_d|^2,$$

where $\alpha, \beta \in \mathbb{R}_+$ with $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ and $|\cdot|$ is the standard 2-norm.

Accordingly,

$$(|\hat{v}|^2)' = 2\alpha\boldsymbol{v}_r{}^T\dot{\boldsymbol{v}}_r + 2\beta\boldsymbol{v}_d{}^T\dot{\boldsymbol{v}}_d. \tag{21}$$

**Proof of Theorem 1.** Substituting (20) into (11), we obtain

$$\dot{\xi}_e^b = -2\hat{k}_p \ln\hat{q}_e - \hat{k}_v\xi_e^b. \tag{22}$$

Let $\ln q_e = \frac{1}{2}\theta_e^b$; then $2\ln\hat{q}_e = \theta_e^b + \epsilon p_e^b$ and therefore

$$\dot{\xi}_e^b = -\hat{k}_p(\theta_e^b + \epsilon p_e^b) - \hat{k}_v\left(\omega_e^b + \epsilon[0, \dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b]\right).$$

Thus,

$$\dot{\boldsymbol{\omega}}_e^b = -\boldsymbol{k}_{pr}\cdot\boldsymbol{\theta}_e^b - \boldsymbol{k}_{vr}\cdot\boldsymbol{\omega}_e^b, \tag{23}$$

$$(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)' = -\boldsymbol{k}_{pd}\cdot\boldsymbol{p}_e^b - \boldsymbol{k}_{vd}\cdot(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b). \tag{24}$$

Consider the following positive function:

$$V = |\xi_e^b|^2 + 4|\hat{\gamma}\cdot\ln\hat{q}_e|^2, \tag{25}$$

where $\hat{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_r + \epsilon\boldsymbol{\gamma}_d$ is a dual vector with each component nonzero. Obviously, $V$ is positive definite and decrescent.

Differentiating $V$, and in view of (21), we obtain

$$\begin{aligned}\dot{V} &= 2\alpha(\boldsymbol{\omega}_e^b)^T\dot{\boldsymbol{\omega}}_e^b + 2\beta(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)' \\ &\quad + 2\alpha(\boldsymbol{\gamma}_r\cdot\boldsymbol{\theta}_e^b)^T(\boldsymbol{\gamma}_r\cdot\dot{\boldsymbol{\theta}}_e^b) + 2\beta(\boldsymbol{\gamma}_d\cdot p_e^b)^T(\boldsymbol{\gamma}_d\cdot\dot{\boldsymbol{p}}_e^b).\end{aligned}$$

Using (23) and (24), we obtain

$$\begin{aligned}\dot{V} &= -2\alpha(\boldsymbol{\omega}_e^b)^T(\boldsymbol{k}_{pr}\cdot\boldsymbol{\theta}_e^b + \boldsymbol{k}_{vr}\cdot\boldsymbol{\omega}_e^b) \\ &\quad - 2\beta(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T\left(\boldsymbol{k}_{pd}\cdot\boldsymbol{p}_e^b + \boldsymbol{k}_{vd}\cdot(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)\right) \\ &\quad + 2\alpha(\boldsymbol{\gamma}_r\cdot\boldsymbol{\theta}_e^b)^T(\boldsymbol{\gamma}_r\cdot\dot{\boldsymbol{\theta}}_e^b) + 2\beta(\boldsymbol{\gamma}_d\cdot\boldsymbol{p}_e^b)^T(\boldsymbol{\gamma}_d\cdot\dot{\boldsymbol{p}}_e^b) \\ &= -2\alpha(\boldsymbol{\omega}_e^b)^T(\boldsymbol{k}_{vr}\cdot\boldsymbol{\omega}_e^b) \\ &\quad - 2\beta(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T(\boldsymbol{k}_{vd}\cdot(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)) \\ &\quad - 2\alpha(\boldsymbol{\omega}_e^b)^T(\boldsymbol{k}_{pr}\cdot\boldsymbol{\theta}_e^b) + 2\alpha(\boldsymbol{\gamma}_r\cdot\boldsymbol{\theta}_e^b)^T(\boldsymbol{\gamma}_r\cdot\dot{\boldsymbol{\theta}}_e^b) \\ &\quad - 2\beta(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T(\boldsymbol{k}_{pd}\cdot\boldsymbol{p}_e^b) + 2\beta(\boldsymbol{\gamma}_d\cdot\boldsymbol{p}_e^b)^T(\boldsymbol{\gamma}_d\cdot\dot{\boldsymbol{p}}_e^b).\end{aligned}$$

Let us choose $\boldsymbol{k}_{pr}$ and $\boldsymbol{k}_{pd}$ so that

$$2\alpha(\boldsymbol{\omega}_e^b)^T(\boldsymbol{k}_{pr}\cdot\boldsymbol{\theta}_e^b) - 2\alpha(\boldsymbol{\gamma}_r\cdot\boldsymbol{\theta}_e^b)^T(\boldsymbol{\gamma}_r\cdot\dot{\boldsymbol{\theta}}_e^b) = 0, \tag{26}$$

$$2\beta(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T(\boldsymbol{k}_{pd}\cdot\boldsymbol{p}_e^b) - 2\beta(\boldsymbol{\gamma}_d\cdot\boldsymbol{p}_e^b)^T(\boldsymbol{\gamma}_d\cdot\dot{\boldsymbol{p}}_e^b) = 0. \tag{27}$$

Then

$$\begin{aligned}\dot{V} &= V^* = -2\alpha(\boldsymbol{\omega}_e^b)^T(\boldsymbol{k}_{vr}\cdot\boldsymbol{\omega}_e^b) - 2\beta(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T \\ &\quad \times \left(\boldsymbol{k}_{vd}\cdot(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)\right).\end{aligned}$$

If letting $\hat{\boldsymbol{k}}_v > \hat{\mathbf{0}}$, it is obtained that $\dot{V} \le 0$. Furthermore, when $V^* = 0$, we obtain

$$\boldsymbol{\omega}_e^b = \mathbf{0}, \tag{28}$$

$$\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b = \mathbf{0}, \tag{29}$$

which means $\boldsymbol{\xi}_e^b = \hat{\mathbf{0}}$, namely $E \triangleq \{V^* = 0\} = \{\boldsymbol{\xi}_e^b = \hat{\mathbf{0}}\}$. And further, by using $\boldsymbol{\xi}_e^b = \hat{\mathbf{0}}$, from (22), we obtain

$$\dot{\xi}_e^b = -2\hat{k}_p\ln\hat{q}_e.$$

Define the non-zero definite auxiliary function

$$W = \xi_e^b\cdot\ln\hat{q}_e,$$

which is continuous and depends on time through the bounded $\boldsymbol{\xi}_d^b$.

On the set $\{\dot{V} = 0\}$, the derivative $\dot{W}$ is

$$\dot{W} = \dot{\xi}_e^b\cdot\ln\hat{q}_e + \xi_e^b\cdot\left(\ln\hat{q}_e\right)'.$$

Using $\dot{\xi}_e^b = -2\hat{k}_p\ln\hat{q}_e$ and $\boldsymbol{\xi}_e^b = \hat{\mathbf{0}}$, we obtain

$$\dot{W} = -2\hat{k}_p\ln\hat{q}_e\cdot\ln\hat{q}_e,$$

which implies that $\dot{W}$ is negative definite, and $\|\dot{W}\| \ge 2k_{p_{\min}}\ln\hat{q}_e\cdot\ln\hat{q}_e$ with $k_{p_{\min}}$ is the smallest entry of positive $\hat{\boldsymbol{k}}_p$. Thus, all the conditions in Matrosov's theorem (Please refer to Theorem 55.3 in [23]) and the accompanying Lemma in the Appendix of [24] are satisfied; therefore $(\ln\hat{q}_e, \boldsymbol{\xi}_e^b) \to (\hat{\mathbf{0}}, \hat{\mathbf{0}})$ asymptotically.

Note that when $\ln\hat{q}_e = \hat{\mathbf{0}}$, we have $\hat{q}_e = \pm\hat{I}$; and further, $|\ln\hat{q}_e|$ is discontinuous at $-\hat{I}$ as well as $|\ln q_e|$ at $-I$, as when $\hat{q}_e$ is near $-\hat{I}$, $|\ln\hat{q}_e|$ is near $\pi$. So, near equilibrium $(\hat{q}_e = -\hat{I}, \boldsymbol{\xi}_e^b = \hat{\mathbf{0}})$, $V$ in (25) is closed to $4\pi^2 \gg 0$, which means $(-\hat{I}, \hat{\mathbf{0}})$ is an unstable equilibrium. Therefore from almost all initial conditions except for a small set of measure zero, the evolution of $(\hat{q}_e, \boldsymbol{\xi}_e^b)$ will converge to $(\hat{I}, \hat{\mathbf{0}})$.

The proof of Theorem 1 is concluded. □

**Remark 1.** The special small set in Theorem 1 is a subset of $DQ_u \times \hat{v}$, from which the evolution of $(\hat{q}_e, \boldsymbol{\xi}_e^b)$ in (22) is exactly achieved at $(-\hat{I}, \hat{\mathbf{0}})$. Clearly, $(-\hat{I}, \hat{\mathbf{0}})$ is an element of this set.

The above proof requires $\hat{\boldsymbol{k}}_v > \hat{\mathbf{0}}$, each component of $\hat{\boldsymbol{k}}_p$ must be nonzero, and moreover (26) and (27) must be satisfied. Further, with the aid of (1), it is obtained that $\omega_e^b = 2q_e^* \circ \dot{q}_e = [0, \dot{\vartheta}_e\boldsymbol{n}_e + \sin(\vartheta_e)\dot{\boldsymbol{n}}_e - 2\sin^2(\frac{\vartheta_e}{2})\boldsymbol{n}_e \times \dot{\boldsymbol{n}}_e]$. Thus, we obtain

$$(\boldsymbol{\theta}_e^b)^T\omega_e^b = (\boldsymbol{\theta}_e^b)^T\dot{\vartheta}_e\boldsymbol{n}_e = (\boldsymbol{\theta}_e^b)^T\dot{\boldsymbol{\theta}}_e^b. \tag{30}$$

Together with (26) and (27), we obtain

$$(\boldsymbol{\omega}_e^b)^T(\boldsymbol{k}_{pr}\cdot\boldsymbol{\theta}_e^b) - (\boldsymbol{\gamma}_r\cdot\boldsymbol{\theta}_e^b)^T(\boldsymbol{\gamma}_r\cdot\boldsymbol{\omega}_e^b) = 0,$$

$$(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T(\boldsymbol{k}_{pd}\cdot\boldsymbol{p}_e^b) - (\boldsymbol{\gamma}_d\cdot\boldsymbol{p}_e^b)^T(\boldsymbol{\gamma}_d\cdot\dot{\boldsymbol{p}}_e^b) = 0,$$

which implies

$$\boldsymbol{k}_{pr} = \boldsymbol{\gamma}_r\cdot\boldsymbol{\gamma}_r, \tag{31}$$

$$(\boldsymbol{k}_{pd} - \boldsymbol{\gamma}_d\cdot\boldsymbol{\gamma}_d)(\dot{\boldsymbol{p}}_e^b)^T\boldsymbol{p}_e^b + (\boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T(\boldsymbol{k}_{pd}\cdot\boldsymbol{p}_e^b) = 0. \tag{32}$$

If we choose $k_{pd1} = k_{pd2} = k_{pd3}$ for $\boldsymbol{k}_{pd}$, then $(\boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b)^T(\boldsymbol{k}_{pd} \cdot \boldsymbol{p}_e^b) = 0$. Considering (32) holds irrespective of $\boldsymbol{p}_e^b$ and $\dot{\boldsymbol{p}}_e^b$, (33) is obtained accordingly.

$$\boldsymbol{k}_{pd} = \boldsymbol{\gamma}_d \cdot \boldsymbol{\gamma}_d. \tag{33}$$

Note that $\hat{\boldsymbol{\gamma}}$ can be chosen arbitrarily with each component nonzero, thus a sufficient condition on the parameters $\hat{\boldsymbol{k}}_p$ and $\hat{\boldsymbol{k}}_v$ in the tracker (20) is: $\hat{\boldsymbol{k}}_p > \hat{\boldsymbol{0}}$ with $k_{pd1} = k_{pd2} = k_{pd3}$ and $\hat{\boldsymbol{k}}_v > \hat{\boldsymbol{0}}$.

### 4.2. Two equilibria problem

Due to the inherent redundancy of the quaternion representation, $q$ and $-q$ represent the same physical orientation, however, one is rotated $2\pi$ relative to the other about an arbitrary axis [25]. Accordingly, the closed-loop system should have two physically identical equilibria $(\hat{I}, \hat{\boldsymbol{0}})$ and $(-\hat{I}, \hat{\boldsymbol{0}})$, however, the state $(\hat{q}_e, \boldsymbol{\xi}_e^b)$ of the closed-loop system cannot reach $(-\hat{I}, \hat{\boldsymbol{0}})$ under the action of tracker (20) unless the initial state is in a special small set demonstrated in Remark 1. So one can expect that, when the initial state is near $(-\hat{I}, \hat{\boldsymbol{0}})$, the system will converge to $(-\hat{I}, \hat{\boldsymbol{0}})$ instead of $(\hat{I}, \hat{\boldsymbol{0}})$ thus taking a 'shorter' path (for more detailed explanations refer to [25,26]). This is the so-called *two equilibria problem*. Similar to the work in [26], a switching parameter $\lambda$ is introduced to improve the tracker (20). Let us denote the first element in $\hat{q}_e(t)$ by $\hat{q}_{e_1}(t)$ and define

$$\lambda = \begin{cases} 1, & \text{if } \hat{q}_{e_1}(t) \geq 0, \\ -1, & \text{otherwise.} \end{cases}$$

Then, the tracker (20) is modified as

$$\hat{U} = -2\hat{k}_p \ln(\lambda\hat{q}_e) - \hat{k}_v\xi_e^b - \hat{F} + Ad_{\hat{q}_e^*}\dot{\xi}_d^b + [\hat{0}, Ad_{\hat{q}_e^*}\boldsymbol{\xi}_d^b \times \boldsymbol{\xi}_e^b]. \tag{34}$$

**Theorem 2.** *The tracker (34) with the same parameters $\hat{k}_p$ and $\hat{k}_v$ as used in (20) ensures that $\left(\hat{q}_e(t), \xi_e^b(t)\right)$ in Model 2 converge to either $(\hat{I}, \hat{\boldsymbol{0}})$ or $(-\hat{I}, \hat{\boldsymbol{0}})$ asymptotically.*

**Proof.** The proof follows from using the Lyapunov function candidate

$$V = |\xi_e^b|^2 + 4|\hat{\boldsymbol{\gamma}} \cdot \ln(\lambda\hat{q}_e)|^2, \tag{35}$$

where $\hat{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_r + \epsilon\boldsymbol{\gamma}_d$ is a dual vector with each component nonzero, with states $\xi_e^b$ and $\ln(\lambda\hat{q}_e)$.

The rest of the proof is similar to that of Theorem 1, and we can conclude that $(\ln(\lambda\hat{q}_e), \boldsymbol{\xi}_e^b)$ converges to $(\hat{\boldsymbol{0}}, \hat{\boldsymbol{0}})$ asymptotically. Further, it should be noted that $\hat{q}_{e_1} = \cos\frac{|\theta_e^b|}{2}$, therefore when $\hat{q}_{e_1} < 0$, we have $|\theta_e^b| \in (\pi, 2\pi)$. At then $\ln(\lambda\hat{q}_e) = \frac{1}{2}\left((2\pi - |\theta_e|)(-\boldsymbol{n}_e) + \epsilon\boldsymbol{p}_e^b\right)$. Therefore, $V$ defined in (35) is also continuous and converges to zero at $(-\hat{I}, \boldsymbol{0})$, which implies $(\hat{I}, \hat{\boldsymbol{0}})$ and $(-\hat{I}, \hat{\boldsymbol{0}})$ are both stable equilibria.

Observe the fact that $\ln(\lambda q_e) = \frac{1}{2}((2\pi - |\theta_e|)(-\boldsymbol{n}_e))$ corresponds to rotation about $-\boldsymbol{n}_e$ with angle $2\pi - |\theta_e|$ when $|\theta_e^b| \in (\pi, 2\pi)$. So controller (34) causes $(\hat{q}_e, \boldsymbol{\xi}_e^b)$ to converge to the 'nearer' equilibrium ($(\hat{I}, \hat{\boldsymbol{0}})$ or $(-\hat{I}, \hat{\boldsymbol{0}})$) with a 'shorter' path. The parameters $\hat{\boldsymbol{k}}_p$ and $\hat{\boldsymbol{k}}_v$ are the same as in Theorem 1. □

**Remark 2.** It should be noted that if adding a small measurement noise, the global attractivity of such discontinuous controller (34) may be destroyed [27]. In Ref. [28], a quaternion-based hysteretic hybrid feedback is used to robustly globally asymptotically stabilizes the attitude of rigid-body. This approach can be adopt to improve the robustness of the controller.

### 4.3. Discussion

The dynamics and the trackers are all discussed in a body frame in this study. In fact, similar dynamics and trackers can also be set up in a spatial frame. Moreover, compared with the SE(3) method in [7], dual quaternions are used instead of matrices and matrix operations are replaced by dual quaternion operations in the proposed trackers, which provides the concision and makes it easier to design complex control laws. Furthermore, the controller in [7] demands $\hat{\boldsymbol{k}}_p$ to have a lower bound to avoid singularities, but this is unnecessary in our approaches, as a dual quaternion can represent an arbitrary transformation without singularity and its logarithmic mapping is properly defined everywhere on $DQ_u$.

**Remark 3.** Regulation problems can often be regarded as a special case of tracking problems with the desired trajectory being a constant. Thus it is clear that the proposed trackers include the attitude and position regulators as a particular case, when $\boldsymbol{\xi}_d^b = \hat{\boldsymbol{0}}$, equivalently $|\boldsymbol{\omega}_d^b| = 0$ and $|\dot{\boldsymbol{p}}_d^b| = 0$. The regulation cases will be simulated as a special case in Section 5.

**Remark 4.** If the desired configuration $\hat{q}_d$ is a rigid body with the similar dynamic model as Model 1, then we obtain the control model for a two rigid body formation by substituting $\dot{\boldsymbol{\xi}}_d^b$ into (15). A similar model has been set up in [22,25] with a decoupling form. Consequently, the control schemes in this study can be extended naturally to the multiple rigid body formation control.

In the following section, the simulations are performed on the basis of trackers (20) and (34), and their particular case—regulators.

## 5. Simulations

We assume the moment of inertia matrix $J$ in the body frame is given by

$$J = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.63 & 0.0 \\ 0.0 & 0.0 & 0.85 \end{bmatrix}$$

and the mass of the rigid body is $m = 100$.[2]

For ease of illustration, we express the initial attitude and position, the desired attitude and position all in a spatial frame. The same goes for the figures (Figs. 2–5). The proposed trackers act in a body frame, so we translate the attitude and position into the body frame in the simulations. The simulations are performed with Simulink for a time span of 20 s.

### 5.1. Regulation case

As mentioned in Remark 3, the proposed trackers include the regulation problem as a particular case. When $\boldsymbol{\xi}_d^b = \hat{\boldsymbol{0}}$, trackers (20) and (34) both degenerate to regulators.

In simulations, the initial attitude is described by an equivalent axis and angle, where the equivalent axis is $\boldsymbol{n} = (0.4896, 0.2032, 0.8480)^T$ and the equivalent angle is $|\theta| = 3.8134$ (218.78°). Thus, the initial attitude $q(0) = [\cos\frac{|\theta|}{2}, \sin\frac{|\theta|}{2}\boldsymbol{n}] = [-0.3320, 0.4618, 0.1917, 0.7999]$. The initial position is $\boldsymbol{p}^s(0) = (2, 2, 1)^T$, the desired attitude is $q_d = I$ and the desired position is $\boldsymbol{p}_d^s = \boldsymbol{0}$, with $\boldsymbol{\xi}_d^s(0) = \hat{\boldsymbol{0}}$. Then we simulate the trackers (20) and (34), respectively. The parameters are both $\hat{\boldsymbol{k}}_p = (1, 1, 1)^T + \epsilon(1, 1, 1)^T$ and $\hat{\boldsymbol{k}}_v = (1, 1, 1)^T + \epsilon(1, 1, 1)^T$.

[2] $J$, $m$ and $p^b$, $p^s$ in the following are all with the International System of Units (SI Units), such as kg $*$ m$^2$, kg, m. The units are omitted therein.

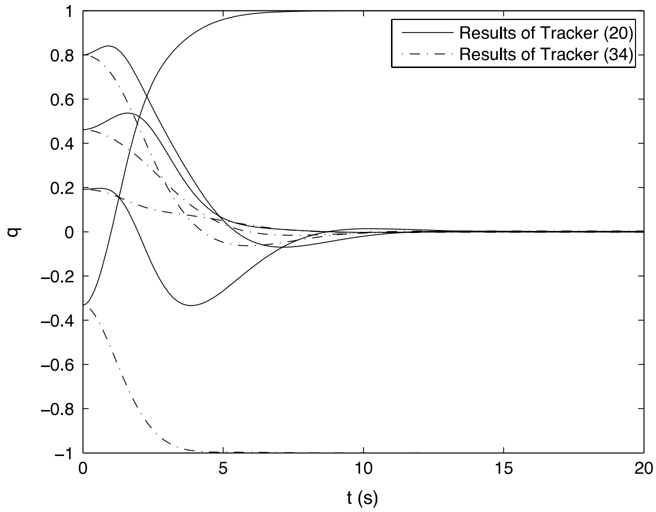


**Fig. 2.** Evolutions of the four components of $q_e$ with respect to time $t$.

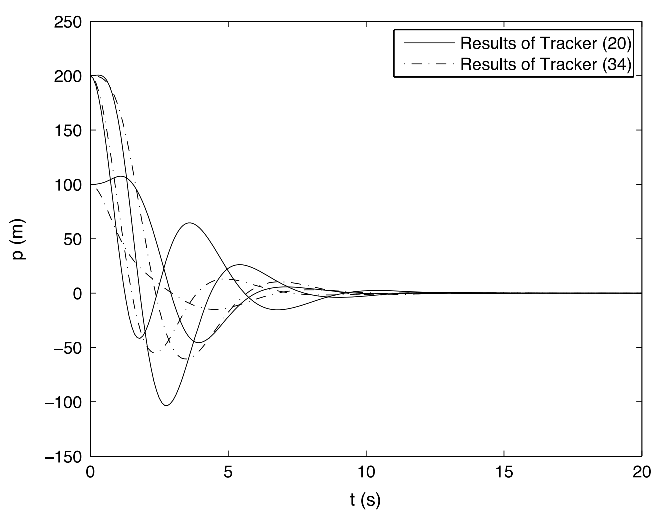


**Fig. 3.** Evolutions of the three components of $p_e$ in a spatial frame verse time $t$.

Fig. 2 shows the evolutions of the four components of the attitude error $q_e$ with respect to time $t$, and Fig. 3 shows the evolutions of the three components of the position error $p_e$ in a spatial frame versus time $t$. In both figures, solid curves depict the results of tracker (20) and dashed curves depict the results of tracker (34). These figures show both trackers cause $\hat{q}_e$ to converge to an equilibrium (or cause $\hat{q}$ to converge to $\hat{q}_d$) asymptotically. However, tracker (20) causes $\hat{q}_e$ to converge to $\hat{I}$ and tracker (34) causes $\hat{q}_e$ to converge $-\hat{I}$ instead of $\hat{I}$ when $\hat{q}_{e_1} < 0$ (in this simulation $\hat{q}_{e_1}(0) = -0.3320$). Note that $-\hat{I}$ is a stable equilibrium in Theorem 2 but an unstable equilibrium in Theorem 1.

### 5.2. Tracking case

If $\hat{q}_d$ is moving, trackers (20) and (34) will track $\hat{q}_d(t)$ in configuration space. In simulations, $\hat{q}_d(t)$ is given by

$$\begin{cases} \hat{q}_d = q_d + \dfrac{\epsilon}{2} q_d \circ p_d^b \\ \boldsymbol{\xi}_d^b = \boldsymbol{\omega}_d^b + \epsilon(\dot{\boldsymbol{p}}_d^b + \boldsymbol{\omega}_d^b \times \boldsymbol{p}_d^b) \\ \dot{\hat{q}}_d = \dfrac{1}{2}\hat{q}_d \circ \xi_d^b \\ \dot{\boldsymbol{\xi}}_d^b = \dot{\boldsymbol{\omega}}_d^b + \epsilon(\ddot{\boldsymbol{p}}_d^b + \dot{\boldsymbol{\omega}}_d^b \times \boldsymbol{p}_d^b + \boldsymbol{\omega}_d^b \times \dot{\boldsymbol{p}}_d^b) \end{cases}$$

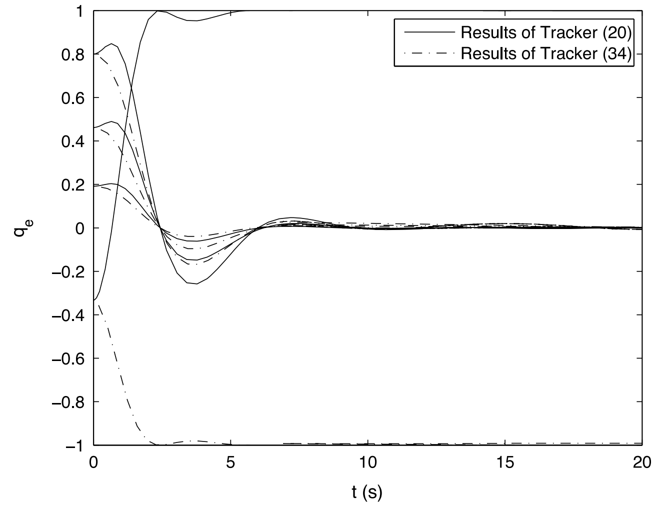


**Fig. 4.** Evolutions of the four components of $q_e$ with respect to time $t$.

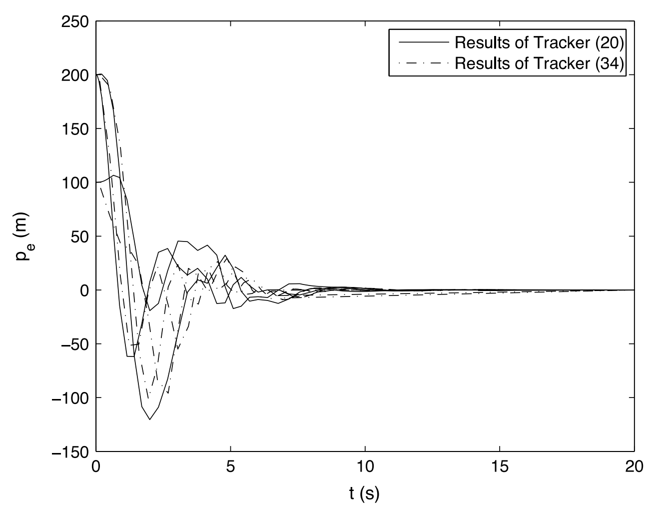


**Fig. 5.** Evolutions of the three components of $p_e$ in a spatial frame verse time $t$.

with $q_d(0) = I$, $\boldsymbol{\omega}_d^s(0) = \mathbf{0}$, $\dot{\boldsymbol{\omega}}_d^s = 0.2\pi \sin(0.2\pi t)(1, 1, 1)^T$, $\boldsymbol{p}_d^s(0) = \mathbf{0}$, $\dot{\boldsymbol{p}}_d^s(0) = \mathbf{0}$ and $\ddot{\boldsymbol{p}}_d^s = 0.1\pi \sin(0.1\pi t)(1, 1, 1)^T$. The initial attitude and position are the same as those in the regulation case, viz. $q(0) = [-0.3320, 0.4618, 0.1917, 0.7999]$ and $\boldsymbol{p}^s(0) = (2, 2, 1)^T$. The parameters $\hat{\boldsymbol{k}}_p$ and $\hat{\boldsymbol{k}}_v$ are also the same as those in the regulation case.

Fig. 4 shows the evolutions of the four components of the attitude error $q_e$ with respect to time $t$, and Fig. 5 shows the evolutions of the three components of the position error $p_e$ in a spatial frame versus time $t$. Also solid curves and dashed curves depict the results of tracker (20) and (34) respectively. In these figures, $q_e$ and $p_e^s$ converge asymptotically to $\pm I$ and $\mathbf{0}$ respectively. This means $\hat{q}(t)$ tracks $\hat{q}_d(t)$ well with both trackers.

## 6. Concluding remarks

In this paper, we have proposed two asymptotically stable trackers via the feedback linearization principle for a rigid body with coupled rotational and translational dynamics by using a new mathematical tool, viz. the unit dual quaternion. As a preliminary study examining the problem of attitude and position control simultaneously without decoupling, our trackers are just a first attempt. Many other mature control laws can be transcribed as well. In addition, the trackers in this paper are with full state measurements (i.e. unit dual quaternion and twist); obtaining a control law without twist, i.e. removing the requirement of angular velocity and linear velocity measurements, is also a worthwhile problem to consider. Some work along these lines has been done in [21,6] with only attitude control and in [22] with attitude and position control. Moreover, we assume perfect and instantaneous measurements of the dual quaternion and the twist in this study, however it is inevitable that noises and delays exist in these measurements in practical applications; thus it is necessary to consider the effects of the noises and the delays on our controllers, and a robust controller should be constructed accordingly according to [28]. Finally, if the inertia matrix $J$ in (3) and the mass $m$ in (4) are unknown or poorly known, an adaptive control law can be adopted as the work in [5,29].


## Acknowledgments

The authors would like to thank Brian D.O. Anderson for his careful revision on the draft and the anonymous reviewers for their valuable comments and constructive suggestions.

X. Wang is supported by the China Scholarship Council. C. Yu is supported by the Australian Research Council through a Queen Elizabeth II Fellowship under DP-110100538 and the Overseas Expert Program of Shandong Province.


## Appendix A. Derivation of $\hat{q}_e$ in (13)

Substituting $\hat{q}_d = q_d + \frac{\epsilon}{2} q_d \circ p_d^b$ and $\hat{q} = q + \frac{\epsilon}{2} q \circ p^b$ into (12), we obtain

$$\begin{aligned}\hat{q}_e &= \left(q_d + \frac{\epsilon}{2} q_d \circ p_d^b\right)^* \circ \left(q + \frac{\epsilon}{2} q \circ p^b\right) \\ &= q_d^* \circ q + \frac{\epsilon}{2}(q_d^* \circ q \circ p^b - p_d^b \circ q_d^* \circ q) \\ &= q_d^* \circ q + \frac{\epsilon}{2} q_d^* \circ q \circ (p^b - Ad_{(q_d^* \circ q)^*} p_d^b).\end{aligned}$$

Let $q_e = q_d^* \circ q$ and $p_e^b = p^b - Ad_{q_e^*} p_d^b$, then $\hat{q}_e$ can be rewritten as

$$\hat{q}_e = \hat{q}_d^* \circ \hat{q} = q_e + \frac{\epsilon}{2} q_e \circ p_e^b.$$

## Appendix B. Derivation of $\xi_e^b$ in (14)

Let $\omega_e = \omega^b - Ad_{q_e^*} \omega_d^b$ and $p_e^b = p^b - Ad_{q_e^*} p_d^b$; then we obtain the following lemmas.

**Lemma 1.** $\dot{q}_e = \frac{1}{2} q_e \circ \omega_e^b$.

**Proof.**

$$\begin{aligned}\dot{q}_e &= \dot{q}_d^* \circ q + q_d^* \circ \dot{q} \\ &= -\frac{1}{2}\omega_d^b \circ q_d^* \circ q + \frac{1}{2} q_d^* \circ q \circ \omega^b \\ &= \frac{1}{2} q_e \circ (\omega^b - q_e^* \circ \omega_d^b \circ q_e) \\ &= \frac{1}{2} q_e \circ \omega_e^b. \quad \square\end{aligned}$$

**Lemma 2.** $\dot{p}_e^b = \dot{p}^b - [0, Ad_{q_e^*} \boldsymbol{p}_d^b \times \omega_e^b] - Ad_{q_e^*} \dot{p}_d^b$.

**Proof.**

$$\dot{p}_e^b = \dot{p}^b - (\dot{q}_e^* \circ p_d^b \circ q_e + q_e^* \circ \dot{p}_d^b \circ q_e + q_e^* \circ p_d^b \circ \dot{q}_e).$$

Using Lemma 1, we obtain

$$\begin{aligned}\dot{p}_e^b &= \dot{p}^b - \left(\frac{1}{2} q_e^* \circ p_d^b \circ q_e \circ \omega_e^b \right. \\ &\quad \left. - \frac{1}{2}\omega_e^b \circ q_e^* \circ p_d^b \circ q_e + Ad_{q_e^*} \dot{p}_d^b\right) \\ &= \dot{p}^b - [0, Ad_{q_e^*} \boldsymbol{p}_d^b \times \boldsymbol{\omega}_e^b] - Ad_{q_e^*} \dot{p}_d^b. \quad \square\end{aligned}$$

**Lemma 3.** $\boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b = \boldsymbol{\omega}^b \times \boldsymbol{p}^b - Ad_{q_e^*} \boldsymbol{\omega}_d^b \times \boldsymbol{p}^b - \boldsymbol{\omega}_e^b \times Ad_{q_e^*} \boldsymbol{p}_d^b$.

**Proof.**

$$\begin{aligned}\boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b &= (\boldsymbol{\omega}^b - Ad_{q_e^*} \boldsymbol{\omega}_d^b) \times (\boldsymbol{p}^b - Ad_{q_e^*} \boldsymbol{p}_d^b) \\ &= \boldsymbol{\omega}^b \times \boldsymbol{p}^b - Ad_{q_e^*} \boldsymbol{\omega}_d^b \times \boldsymbol{p}^b - \boldsymbol{\omega}_e^b \times Ad_{q_e^*} \boldsymbol{p}_d^b. \quad \square\end{aligned}$$

According to Lemmas 2 and 3, we have

$$\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b = \dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b - Ad_{q_e^*} \dot{\boldsymbol{p}}_d^b - Ad_{q_e^*} \boldsymbol{\omega}_d^b \times \boldsymbol{p}^b. \tag{B.1}$$

Substituting $\boldsymbol{\xi}^b = \boldsymbol{\omega}^b + \epsilon(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b)$ and $\boldsymbol{\xi}_d^b = \boldsymbol{\omega}_d^b + \epsilon(\dot{\boldsymbol{p}}_d^b + \boldsymbol{\omega}_d^b \times \boldsymbol{p}_d^b)$ into $\boldsymbol{\xi}_e^b$, we obtain

$$\begin{aligned}\boldsymbol{\xi}_e^b &= \boldsymbol{\xi}^b - Ad_{\hat{q}_e^*} \boldsymbol{\xi}_d^b \\ &= \left(\boldsymbol{\omega}^b + \epsilon(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b)\right) - \left(q_e^* - \frac{\epsilon}{2} p_e^b \circ q_e^*\right) \\ &\quad \circ \left(\boldsymbol{\omega}_d^b + \epsilon(\dot{\boldsymbol{p}}_d^b + \boldsymbol{\omega}_d^b \times \boldsymbol{p}_d^b)\right) \circ \left(q_e + \frac{\epsilon}{2} q_e \circ p_e^b\right) \\ &= \left(\boldsymbol{\omega}^b + \epsilon(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b)\right) \\ &\quad - \left(Ad_{q_e^*} \boldsymbol{\omega}_d^b + \epsilon(Ad_{q_e^*}(\dot{\boldsymbol{p}}_d^b + \boldsymbol{\omega}_d^b \times \boldsymbol{p}_d^b) + Ad_{q_e^*} \boldsymbol{\omega}_d^b \times \boldsymbol{p}_e^b)\right) \\ &= \boldsymbol{\omega}^b - Ad_{q_e^*} \boldsymbol{\omega}_d^b + \epsilon(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b \\ &\quad - Ad_{q_e^*} \dot{\boldsymbol{p}}_d^b - Ad_{q_e^*} \boldsymbol{\omega}_d^b \times \boldsymbol{p}^b).\end{aligned}$$

Using (B.1), we obtain the regular form of $\boldsymbol{\xi}_e^b$:

$$\boldsymbol{\xi}_e^b = \boldsymbol{\omega}_e^b + \epsilon(\dot{\boldsymbol{p}}_e^b + \boldsymbol{\omega}_e^b \times \boldsymbol{p}_e^b).$$

## Appendix C. Extracting control inputs $\boldsymbol{\tau}$ and $\boldsymbol{f}$ from (20)

In view of (8), $\boldsymbol{\tau}$ and $\boldsymbol{f}$ are obtained as follows.

$$\begin{cases}\boldsymbol{\tau} = Jr(\hat{U}), \\ \boldsymbol{f} = m\left(d(\hat{U}) - r(\hat{U}) \times \boldsymbol{p}^b\right),\end{cases} \tag{C.1}$$

where $r(\hat{U})$ and $d(\hat{U})$ extract the real part and the dual part of $\hat{U}$, respectively. If $\hat{U} = \boldsymbol{a} + \epsilon\boldsymbol{b}$, then $r(\hat{U}) = \boldsymbol{a}$ and $d(\hat{U}) = \boldsymbol{b}$.

Next, we specify the tracker (20) to obtain $r(\hat{U})$ and $d(\hat{U})$ directly. Using the frame of (7), $\dot{\boldsymbol{\xi}}_d^b$ can be expressed as follows:

$$\dot{\boldsymbol{\xi}}_d^b = \dot{\boldsymbol{\omega}}_d^b + \epsilon(\ddot{\boldsymbol{p}}_d^b + \dot{\boldsymbol{\omega}}_d^b \times \boldsymbol{p}_d^b + \boldsymbol{\omega}_d^b \times \dot{\boldsymbol{p}}_d^b).$$

After complex algebraic manipulations, $\hat{U}$ can be expressed as follows.

$$\begin{aligned}\hat{U} &= -\boldsymbol{k}_{pr} \cdot \boldsymbol{\theta}_e^b - \boldsymbol{k}_{vr} \cdot \boldsymbol{\omega}_e^b - \boldsymbol{a} + Ad_{q_e^*}(\dot{\boldsymbol{\omega}}_d^b + \boldsymbol{\omega}^b) \\ &\quad + \epsilon(-\boldsymbol{k}_{pd} \cdot \boldsymbol{p}_e^b - \boldsymbol{k}_{vd} \cdot (\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b - Ad_{q_e^*} \dot{\boldsymbol{p}}_d^b \\ &\quad - Ad_{q_e^*} \boldsymbol{\omega}_d^b \times \boldsymbol{p}^b) - \boldsymbol{a} \times \boldsymbol{p}^b + (\dot{\boldsymbol{p}}^b + Ad_{q_e^*} \dot{\boldsymbol{p}}_d^b) \\ &\quad \times \boldsymbol{\omega}_e^b + Ad_{q_e^*} \dot{\boldsymbol{\omega}}_d^b \times \boldsymbol{p}^b + Ad_{q_e^*} \ddot{\boldsymbol{p}}_d^b).\end{aligned}$$

Thus,

$$\begin{cases} r(\hat{U}) = -\boldsymbol{k}_{pr} \cdot \boldsymbol{\theta}_e^b - \boldsymbol{k}_{vr} \cdot \boldsymbol{\omega}_e^b - \boldsymbol{a} + Ad_{q_e^*}(\dot{\boldsymbol{\omega}}_d^b + \boldsymbol{\omega}^b), \\ d(\hat{U}) = -\boldsymbol{k}_{pd} \cdot \boldsymbol{p}_e^b - \boldsymbol{k}_{vd} \cdot (\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b - Ad_{q_e^*} \dot{\boldsymbol{p}}_d^b \\ \qquad - Ad_{q_e^*} \boldsymbol{\omega}_d^b \times \boldsymbol{p}^b) - \boldsymbol{a} \times \boldsymbol{p}^b + (\dot{\boldsymbol{p}}^b + Ad_{q_e^*} \dot{\boldsymbol{p}}_d^b) \\ \qquad \times \boldsymbol{\omega}_e^b + Ad_{q_e^*} \dot{\boldsymbol{\omega}}_d^b \times \boldsymbol{p}^b + Ad_{q_e^*} \ddot{\boldsymbol{p}}_d^b. \end{cases} \tag{C.2}$$

Substituting (C.2) into (C.1), the explicit forms of inputs $\boldsymbol{\tau}$ and $\boldsymbol{f}$ are obtained as follows.

$$\begin{cases} \boldsymbol{\tau} = J\left(-\boldsymbol{k}_{pr}\cdot\boldsymbol{\theta}_e^b - \boldsymbol{k}_{vr}\cdot\boldsymbol{\omega}_e^b - \boldsymbol{a} + Ad_{q_e^*}(\dot{\boldsymbol{\omega}}_d^b + \boldsymbol{\omega}^b)\right), \\ \boldsymbol{f} = m(-\boldsymbol{k}_{pd}\cdot\boldsymbol{p}_e^b - \boldsymbol{k}_{vd}(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b\times\boldsymbol{p}^b - Ad_{q_e^*}\dot{\boldsymbol{p}}_d^b \\ \quad -Ad_{q_e^*}\boldsymbol{\omega}_d^b\times\boldsymbol{p}^b) + \boldsymbol{k}_{vr}\cdot\boldsymbol{\omega}_e^b\times\boldsymbol{p}^b + (\dot{\boldsymbol{p}}^b + Ad_{q_e^*}\dot{\boldsymbol{p}}_d^b) \\ \quad \times\boldsymbol{\omega}_e^b + Ad_{q_e^*}\ddot{\boldsymbol{p}}_d^b - Ad_{q_e^*}\boldsymbol{\omega}^b\times\boldsymbol{p}^b). \end{cases}$$

It is now clear that the coupling terms $\boldsymbol{\omega}^b\times\boldsymbol{p}^b$, $Ad_{q_e^*}\boldsymbol{\omega}_d^b\times\boldsymbol{p}^b$, $\boldsymbol{\omega}_e^b\times\boldsymbol{p}^b$, $(\dot{\boldsymbol{p}}^b + Ad_{q_e^*}\dot{\boldsymbol{p}}_d^b)\times\boldsymbol{\omega}_e^b$ and $Ad_{q_e^*}\boldsymbol{\omega}^b\times\boldsymbol{p}^b$ appear in $\boldsymbol{f}$; whereas no coupling terms appear in $\boldsymbol{\tau}$.

## References


[1] O.-E. Fjellstad, T.I. Fossen, Position and attitude tracking of AUV's: a quaternion feedback approach, IEEE Journal of Oceanic Engineering 19 (4) (1994) 512–518.

[2] K.Y. Pettersen, O. Egeland, Time-varying exponential stabilization of the position and attitude of an underactuated autonomous underwater vehicle, IEEE Transactions on Automatic Control 44 (1) (1999) 112–115.

[3] P.A. Servidia, R.S.S. Pena, Thruster design for position/attitude control of spacecraft, IEEE Transactions on Aerospace and Electronic Systems 38 (4) (2002) 1172–1180.

[4] P.K.C. Wang, F.Y. Hadaegh, K. Lau, Synchronized formation rotation and attitude control of multiple free-flying spacecraft, Journal of Guidance, Control and Dynamics 22 (1) (1999) 28–35.

[5] J.T.-Y. Wen, K. Kreutz-Delgado, The attitude control problem, IEEE Transactions on Automatic Control 36 (19) (1991) 1148–1162.

[6] A. Tayebi, Unit quaternion-based output feedback for the attitude tracking problem, IEEE Transactions on Automatic Control 53 (6) (2008) 1516–1520.

[7] F. Bullo, R. Murray, Proportional derivative (PD) control on the Euclidean group, in: 1995 European Control Conference, Rome, Italy, 1995.

[8] L. Hsia, A. Yang, On the principle of transference in three-dimensional kinematics, ASME Journal of Mechanical Design 103 (1981) 652–656.

[9] R.M. Murray, Z. Li, S.S. Sastry, A Mathematical Introduction to Robotic Manipulation, CRC Press, 1994.

[10] J. Funda, R. Taylor, R.P. Paul, On homogeneous transformations, quaternions, and computational efficiency, IEEE Transactions on Robotics and Automation 6 (3) (1990) 382–388.

[11] J. Funda, R.P. Paul, A computational analysis of screw transformations in robotics, IEEE Transactions on Robotics and Automation 6 (3) (1990) 348–356.

[12] N. Aspragathos, J. Dimitros, A comparative study of three methods for robot kinematics, IEEE Transactions on Systems, Man and Cybernetics Part B: Cybernetics 28 (2) (1998) 135–145.

[13] Q. Ge, B. Ravani, Computer aided geometric design of motion interpolants, ASME Journal of Mechanical Design 116 (3) (1994) 756–762.

[14] J.S. Goddard, Pose and motion estimation from vision using dual quaternion-based extended Kalman filtering, Ph.D. Thesis, The University of Tennessee, 1997.

[15] K. Daniilidis, Hand–eye calibration using dual quaternions, The International Journal of Robotics Research 18 (3) (1999) 286–298.

[16] Y.X. Wu, X.P. Hu, D.W. Hu, J.X. Lian, Strapdown inertial navigation system algorithms based on dual quaternions, IEEE Transactions on Aerospace and Electronic Systems 41 (1) (2005) 110–132.

[17] X. Wang, D. Han, Z. Zheng, C. Yu, The geometric structure of unit dual quaternion with application in kinematic control, Journal of Mathematical Analysis and Applications 389 (2) (2012) 1352–1364.

[18] M.-J. Kim, M.-S. Kim, A compact differential formula for the first derivative of a unit quaternion curve, Journal of Visualization and Computer Animation 7 (1) (1996) 43–57.

[19] J. Dooley, J. McCarthy, Spatial rigid body dynamics using dual quaternion components, in: Proceedings of the 1991 International Conference on Robotics and Automation, Sacramento California, 1991, pp. 90–95.

[20] P. Hughes, Spacecraft Attitude Dynamics, John Wiley & Sons, Inc., New York, USA, 1986.

[21] F. Lizarralde, J.T. Wen, Attitude control without angular velocity measurement: a passivity approach, IEEE Transactions on Automatic Control 41 (3) (1996) 468–472.

[22] H. Wong, H. Pan, V. Kapila, Output feedback control for spacecraft formation flying with coupled translation and attitude dynamics, in: Proceedings of the 2005 American Control Conference, vol. 4, Portland, OR, USA, 2005, pp. 2419–2426.

[23] W. Hahn, Stability of Motion, Springer-Verlag, Berlin, Germany, 1967.

[24] B. Paden, R. Panja, Globally asymptotically stable 'PD+' controller for robot manipulators, International Journal of Control 47 (6) (1988) 1697–1712.

[25] R. Kristiansen, P.J. Nicklasson, J.T. Gravdahl, Spacecraft coordination control in 6DOF: integrator backstepping vs passivity-based control, Automatica 44 (11) (2008) 2896–2901.

[26] P. Arambel, V. Manikonda, A new class of control laws for spacecraft attitude tracking using switching and trajectory, in: Proceedings of the 2000 IEEE International Conference on Robotics and Automation, San Francisco, CA, 2000, pp. 4033–4038.

[27] R.G. Sanfelice, M.J. Messina, S.E. Tuna, A.R. Tell, Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points, in: Proceedings of the 2006 American Control Conference, 2006, pp. 3352–3357.

[28] C.G. Mayhew, R.G. Sanfelice, A.R. Teel, Robust global asymptotic attitude stabilization of a rigid body by quaternion-based hybrid feedback, in: Proceedings of the Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference, Shanghai, PR China, 2009, pp. 2522–2527.

[29] B.T. Costic, D.M. Dawson, M.S. de Queiroz, V. Kapila, Quaternion-based adaptive attitude tracking controller without velocity measurements, Journal of Guidance, Control, and Dynamics 24 (6) (2001) 1214–1222.