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The Model Reduction of Multi-agent System Depended on the Spanning Tree of Directed Graph

Zhiwen Zeng, Xiangke Wang and Zhiqiang Zheng
College of Mechatronic Engineering and Automation
National University of Defense Technology
Changsha, Hunan, China
zhiwenzeng.nudt@gmail.com, {xkwang, zqzheng}@nudt.edu.cn

***Abstract*—This work explores the model reduction problem based on the edge agreement dynamics of multi-agent system. To begin with, the general concepts of weighted edge Laplacian of directed graph are proposed and its algebraic properties are further explored. From the systematic and structural view of the edge agreement dynamics, we note that multi-agent system can be then translated into a output feedback interconnection of the spanning tree subsystem and the co-spanning tree subsystem. In addition, based on the essential edge Laplacian, we derive a model reduction representation of the closed-loop multi-agent system based on the spanning tree subgraph.**

***Index Terms*—Model reduction, edge agreement, directed graph, multi-agent system**

I. Introduction

The graph theory contributes significantly in the analysis and synthesis of multi-agent systems, since it provides natural abstractions for how information is shared between agents in a network. Specially, the spectral properties of the graph Laplacian are extensively explored recently to provide convergence analysis in the context of multi-agent coordination behaviour [1][2]. Despite the unquestionable interest of the results concerning the convergence properties in these literatures, we also note that, another interesting topic with regard to how certain subgraphs, such as spanning trees and cycles, contribute to the analysis of multi-agent systems, has arisen in more recently. An important theme in this direction is to obtain the explicit connections between the topology structure and the control system. Considering this, an attractive notion about the edge agreement deserve special attention, in which the edge Laplacian plays an important role. Pioneering researches on edge agreement under undirected graph not only provide totally new insights that how the spanning trees and cycles effect the performance of the agreement protocol, but also set up a novel systematic framework for analysing multi-agent systems from the edge perspective[3][4][5]. The edge agreement in [3] provides a theoretic analysis of the system's performance using both H_2 and H_∞ norms, and these results have been applied in relative sensing networks referring to [4]. Moreover, based on the algebraic properties of the edge Laplacian, [5] examines how cycles impact the H_2 performance and proposed an optimal strategy for the design of consensus networks. Although the edge Laplacian offers more transparent understanding of the graph structure, it still remains an undirected notion in aforementioned literatures. More recently, the edge Laplacian is used to examine the model reduction of networked system associated with directed trees through clustering in [6]; however it can not be directly extended to more general directed graphs yet.

The coordination control problem of multi-agent system has received increasing amounts of attention recently. Network topology and the information flow have turned out to be the primary concern of such issue. While the analysis of the node agreement (consensus problem) has matured, work related to the edge agreement has not been deeply studied. In this paper, we are going to explore more details about this term combining the edge agreement. First, we extend our preliminary work [19] to the weighted directed edge Laplacian and further explore the algebraic properties for analysing the interacting multi-agent system. Since its undirected counterpart has shown great potential for exploring the system performance in [3][20], we believe that the novel graph-theoretic tool deserves more attention. Second, under the edge agreement framework, the closed-loop multi-agent system can be transformed into an output feedback interconnection structure. Correspondingly, based on the observation that the co-spanning tree subsystem can be served as an internal feedback, a model reduction representation can be derived, which allows a convenient analysis.

The rest of the paper is organized as follows: preliminaries and problem formulation are proposed in Section 2. The directed edge Laplacian with its algebraic properties are elaborated in Section 3 as well as the model reduction representation of the edge agreement mechanism. The last section draws the conclusions.

II. Basic Notions and Preliminary Results

In this section, some basic notions in graph theory and preliminary results about the synchronization of multi-agent system under quantized information are briefly introduced.

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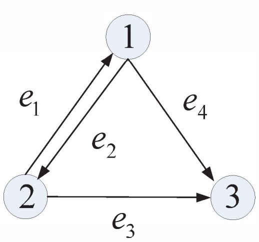
A. Graph and Matrix

In this paper, we use $|\cdot|$ and $\|\cdot\|$ to denote the Euclidean norm and 2-norm for vectors and matrices respectively. The null space of matrix A is denoted by $\mathcal{N}(A)$. Denote by I_n the identity matrix and by $\mathbf{0}_n$ the zero matrix in $\mathbb{R}^{n\times n}$. Let $\mathbf{0}$ be the column vector with all zero entries. Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a directed graph of order N specified by a node set $\mathcal{V}$ and an edge set $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ with size L. For a specific edge $e_k=(j,i)$, let $v_{\otimes}(e_k)$ denotes its initial node j and $v_{\odot}(e_k)$ the terminal node i. The set of neighbours of node i is denoted by $\mathcal{N}_i=\{j:e_k=(j,i)\in\mathcal{E}\}$. We use $A(G)$ to represent a weighted adjacency matrix, where the adjacency elements associated with the edges are positive, i.e., $e_k=(j,i)\in\mathcal{E}\Leftrightarrow a_{ij}>0$, otherwise, $a_{ij}=0$. Denote by $\mathcal{W}(\mathcal{G})$ the $L\times L$ diagonal matrix of w_k, for $k=1,2\cdots,L$, where $w_k=a_{ij}$ for $e_k=(j,i)\in\mathcal{E}$. The notation $D(\mathcal{G})$ represents a diagonal matrix with $\Delta_i(\mathcal{G})$ denoting the in-degree of node i on the diagonal. The corresponding graph Laplacian of $\mathcal{G}$ is defined as $L_n(\mathcal{G}):=D(\mathcal{G})-A(\mathcal{G})$, whose eigenvalues will be ordered and denoted as $0=\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_N$. The incidence matrix $E(\mathcal{G})\in\mathbb{R}^{N\times L}$ for a directed graph is a $\{0,\pm1\}$-matrix with rows and columns indexed by nodes and edges of $\mathcal{G}$ respectively, such that for edge $e_k=(j,i)\in\mathcal{E}$, $[E(\mathcal{G})]_{jk}=+1$, $[E(\mathcal{G})]_{ik}=-1$ and $[E(\mathcal{G})]_{lk}=0$ for $l\neq i,j$. The definition implies that each column of E contains exactly two nonzero entries indicating the initial node and the terminal node respectively. We illustrate Figure 1 as an example.



$$E=\begin{bmatrix}-1 & 1 & 0 & 1\\ 1 & -1 & 1 & 0\\ 0 & 0 & -1 & -1\end{bmatrix}$$

Fig. 1. The incidence matrix of a simple directed graph.

A directed path in directed graph $\mathcal{G}$ is a sequence of directed edges and a directed tree is a directed graph in which, for the root i and any other node j, there is exactly one directed path from i to j. A spanning tree $\mathcal{G}_{\mathcal{T}}=(\mathcal{V},\mathcal{E}_1)$ of a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ is a directed tree formed by graph edges that connect all the nodes of the graph; a co-spanning tree $\mathcal{G}_c=(\mathcal{V},\mathcal{E}-\mathcal{E}_1)$ of $\mathcal{G}_{\mathcal{T}}$ is the subgraph of $\mathcal{G}$ having all the vertices of $\mathcal{G}$ and exactly those edges of $\mathcal{G}$ that are not in $\mathcal{G}_{\mathcal{T}}$. Graph $\mathcal{G}$ is called *strongly connected* if and only if any two distinct nodes can be connected via a directed path; *quasi-strongly connected* if and only if it has a directed spanning tree [21]. A quasi-strongly connected directed graph $\mathcal{G}$ can be rewritten as a union form: $\mathcal{G}=\mathcal{G}_{\mathcal{T}}\cup\mathcal{G}_c$. In addition, according to certain permutations, the incidence matrix $E(\mathcal{G})$ can always be rewritten as $E(\mathcal{G})=\begin{bmatrix}E_{\mathcal{T}}(\mathcal{G}) & E_c(\mathcal{G})\end{bmatrix}$ as well. Since the co-spanning tree edges can be constructed from the spanning tree edges via a linear transformation [3], such that

$$E_{\mathcal{T}}(\mathcal{G})\,T(\mathcal{G})=E_c(\mathcal{G}) \tag{1}$$

with $T(\mathcal{G})=\left(E_{\mathcal{T}}(\mathcal{G})^T E_{\mathcal{T}}(\mathcal{G})\right)^{-1}E_{\mathcal{T}}(\mathcal{G})^T E_c(\mathcal{G})$ and $rank(E(\mathcal{G}))=N-1$ from [21]. We define

$$R(\mathcal{G})=\begin{bmatrix}I & T(\mathcal{G})\end{bmatrix} \tag{2}$$

and then obtain $E(\mathcal{G})=E_{\mathcal{T}}(\mathcal{G})R(\mathcal{G})$. The column space of $E(\mathcal{G})^T$ is known as the *cut space* of $\mathcal{G}$ and the null space of $E(\mathcal{G})$ is called the *flow space*, which is the orthogonal complement of the cut space. Interestingly, the rows of $R(\mathcal{G})$ form a basis of the cut space of and the rows of $\begin{bmatrix}-T(\mathcal{G})^T & I\end{bmatrix}$ form a basis of the flow space respectively [21].

Lemma 1 ([22]): : The graph Laplacian $L_{\mathcal{G}}(\mathcal{G})$ of a directed graph $\mathcal{G}$ has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right-half plane. In addition, $L_{\mathcal{G}}(\mathcal{G})$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ is quasi-strongly connected.

III. Directed Edge Laplacian and Edge Agreement

In this sequel, a novel matrix representation of directed graphs describing the interconnection topology will be introduced, which allows a convenient analysis of multi-agent system from the edge perspective. Based on the new description, the edge agreement for general directed graph is proposed.

A. Directed Edge Laplacian

The edge Laplacian in [3] still remains to an undirected notion and is thus inadequate to handle our problem. Extending the concept of the edge Laplacian to directed graph will be of great help to understand multi-agent systems from the structural perspective.

Before moving on, we give the definition of the in-incidence matrix and out-incidence matrix at first.

Definition 1 (In-incidence/Out-incidence Matrix): The $N\times L$ in-incidence matrix $E_{\odot}(\mathcal{G})$ for a directed graph $\mathcal{G}$ is a $\{0,-1\}$ matrix with rows and columns indexed by nodes and edges of $\mathcal{G}$, respectively, such that for an edge $e_k=(j,i)\in\mathcal{E}$, $[E_{\odot}(\mathcal{G})]_{mk}=-1$ for $m=i$, $[E_{\odot}(\mathcal{G})]_{mk}=0$ otherwise. The out-incidence matrix is a $\{0,+1\}$ matrix with $[E_{\otimes}(\mathcal{G})]_{nk}=+1$ for $n=j$, $[E_{\otimes}(\mathcal{G})]_{nk}=0$ otherwise.
In comparison with the definition of the incidence matrix, we can rewrite $E(\mathcal{G})$ in the following way:

$$E(\mathcal{G})=E_{\odot}(\mathcal{G})+E_{\otimes}(\mathcal{G}). \tag{3}$$

On the other hand, the weighted in-incidence matrix $E^w_\odot(\mathcal{G})$ can be defined as $E^w_\odot(\mathcal{G}) = E_\odot(\mathcal{G})\mathcal{W}(\mathcal{G})$, where $\mathcal{W}(\mathcal{G})$ is a diagonal matrix of w_k. This will lead us to find out a novel factorization of the graph Laplacian $L_n(\mathcal{G})$.

Lemma 2: Considering a directed graph $\mathcal{G}$ with the incidence matrix $E(\mathcal{G})$ and weighted in-incidence matrix $E^w_\odot(\mathcal{G})$, the graph Laplacian of $\mathcal{G}$ have the following expression

$$L_n(\mathcal{G}) = E^w_\odot(\mathcal{G})E(\mathcal{G})^T. \tag{4}$$

Proof: By using (3), we have $E^w_\odot(\mathcal{G})E(\mathcal{G})^T = E^w_\odot(\mathcal{G})E_\odot(\mathcal{G})^T + E^w_\odot(\mathcal{G})E_\otimes(\mathcal{G})^T$. Let $E^w_{\odot_i}(\mathcal{G})$, $E_{\otimes_i}(\mathcal{G})$ be the i-th row of $E^w_\odot(\mathcal{G})$ and $E_\otimes(\mathcal{G})$. According to the preceding definition, it's clear that, $E^w_{\odot_i}(\mathcal{G})E_{\odot_j}(\mathcal{G})^T = \Delta_i(\mathcal{G})$ for $i = j$, $E^w_{\odot_i}(\mathcal{G})E_{\odot_j}(\mathcal{G})^T = 0$ otherwise. Then we can collect terms as $E^w_\odot(\mathcal{G})E_\odot(\mathcal{G})^T = D(\mathcal{G})$. Besides, we also have $E^w_{\odot_i}(\mathcal{G})E_{\otimes_j}(\mathcal{G})^T = -w_k$ for $j \in N_i, e_k = (j,i)$ and $E^w_{\odot_i}(\mathcal{G})E_{\otimes_j}(\mathcal{G})^T = 0$ otherwise, which implies that $E^w_\odot(\mathcal{G})E_\otimes(\mathcal{G})^T = -A(\mathcal{G})$. Then, we have $E^w_\odot(\mathcal{G})E(\mathcal{G})^T = E^w_\odot(\mathcal{G})E_\odot(\mathcal{G})^T + E^w_\odot(\mathcal{G})E_\otimes(\mathcal{G})^T = D(\mathcal{G}) - A(\mathcal{G}) = L_n(\mathcal{G})$. ∎

Definition 2 (Directed Edge Laplacian): The edge Laplacian of a directed graph $\mathcal{G}$ is defined as

$$L_e(\mathcal{G}) := E(\mathcal{G})^T E^w_\odot(\mathcal{G}). \tag{5}$$

To provide a deeper insights into what the edge Laplacian $L_e(\mathcal{G})$ offers in the analysis and synthesis of multi-agent systems, we propose the following lemma.

Lemma 3: For any directed graph $\mathcal{G}$, the graph Laplacian $L_\mathcal{G}$ and the edge Laplacian $L_e(\mathcal{G})$ have the same nonzero eigenvalues. If $\mathcal{G}$ is quasi-strongly connected, then the edge Laplacian $L_e(\mathcal{G})$ contains exactly $N-1$ nonzero eigenvalues which are all in the open right-half plane.

Proof: The proof for the weighted version of $L_e(\mathcal{G})$ can be easily extended from lemma 5 of our previous work [19], thus the detail is omitted here. ∎

Obviously, if $\mathcal{G} = \mathcal{G}_\mathcal{T}$, then $\mathcal{G}$ has $L = N-1$ edges and all the eigenvalues of $L_e(\mathcal{G})$ are nonzero. In the following paper, when we deal with a quasi-strongly connected graph, it refers to a general directed graph $\mathcal{G} = \mathcal{G}_\mathcal{T} \cup \mathcal{G}_c$ unless noted otherwise.

Lemma 4: Considering a quasi-strongly connected graph $\mathcal{G}$ of order N, the edge Laplacian $L_e(\mathcal{G})$ has $L - N + 1$ zero eigenvalues and zero is a simple root of the minimal polynomial of $L_e(\mathcal{G})$.

Proof: The result can be lightly extended from lemma 6 of our previous work [19]. ∎

Lemma 4 implies that the linear system associated with $-L_e(\mathcal{G})$ is marginally stable. As thus, the weighted directed edge Laplacian holds the similar functions as the graph Laplacian for analyzing the interacting multi-agent system. The explicit connection between the edge and graph Laplacian has been highlighted by a similarity transformation in [3]. Actually, by using this transformation, we can derive a reduced model representation for the edge agreement dynamics.

B. Edge Agreement and Model Reduction

Although the graph Laplacian is a convenient method to describe the geometric interconnection of networked agents, another attractive notion, the edge agreement, which has not been extensively explored, deserves additional attention because the edges are adopted to be natural interpretations of the information flow.

Considering the quasi-strongly connected graph $\mathcal{G}$ and the most commonly used consensus dynamics [23] described as:

$$\dot{x} = -L_\mathcal{G}(\mathcal{G}) \otimes I_n x$$

where $\otimes$ denotes the Kronecker product. Contrary to the most existing works, we study the synchronization problem from the edge perspective by using L_e. In this avenue, we define the *edge state* vector as

$$x_e(t) = E(\mathcal{G})^T \otimes I_n x(t) \tag{6}$$

which represents the difference between the state components of two neighbouring nodes. Taking the derivative of (6) lead to

$$\dot{x}_e(t) = -L_e(\mathcal{G}) \otimes I_n x_e(t) \tag{7}$$

which is referred as *edge agreement dynamics* in this paper. In comparison to the node agreement (consensus), the edge agreement, rather than requiring the convergence to the agreement subspace, desires the edge dynamics (7) converge to the origin, i.e., $\lim_{t\to\infty} |x_e(t)| = 0$. Essentially, the evolution of an edge state depends on its current state and the states of its adjacent edges. Besides, the edge agreement implies consensus if the directed graph $\mathcal{G}$ has a spanning tree [3].

As the quasi-strongly connected graph $\mathcal{G}$ can be written as $\mathcal{G} = \mathcal{G}_\mathcal{T} \cup \mathcal{G}_c$, then the weighted in-incidence matrix can be represented as $E^w_\odot(\mathcal{G}) = \begin{bmatrix} E^w_{\odot\mathcal{T}}(\mathcal{G}) & E^w_{\odot c}(\mathcal{G}) \end{bmatrix}$ via some certain permutations in line with $E(\mathcal{G}) = \begin{bmatrix} E_\mathcal{T}(\mathcal{G}) & E_c(\mathcal{G}) \end{bmatrix}$. According to the partition, we can represent the edge Laplacian into the block form as

$$L_e(\mathcal{G}) = E(\mathcal{G})^T E^w_\odot(\mathcal{G}) = \begin{bmatrix} L_e^\mathcal{T}(\mathcal{G}) & E_\mathcal{T}(\mathcal{G})^T E^w_{\odot c}(\mathcal{G}) \\ E_c(\mathcal{G})^T E^w_{\odot\mathcal{T}}(\mathcal{G}) & L_e^c(\mathcal{G}) \end{bmatrix} \tag{8}$$

with $L_e^\mathcal{T}(\mathcal{G}) = E_\mathcal{T}(\mathcal{G})^T E^w_{\odot\mathcal{T}}(\mathcal{G})$ and $L_e^c(\mathcal{G}) = E_c(\mathcal{G})^T E^w_{\odot c}(\mathcal{G})$.

The edge Laplacian dynamics (7) can be translated into a output feedback interconnection of the spanning tree subsystem $H_\mathcal{T}$ and the co-spanning tree subsystem H_c. Actually, by using (8), the edge Laplacian dynamics $x_e(t)$ described by (7) can be rewritten as

$$\begin{bmatrix} \dot{x}_\mathcal{T}(t) \\ \dot{x}_c(t) \end{bmatrix} = -\begin{bmatrix} L_e^\mathcal{T}(\mathcal{G}) & E_\mathcal{T}(\mathcal{G})^T E^w_{\odot c}(\mathcal{G}) \\ E_c(\mathcal{G})^T E^w_{\odot\mathcal{T}}(\mathcal{G}) & L_e^c(\mathcal{G}) \end{bmatrix} \otimes I_n \begin{bmatrix} x_\mathcal{T}(t) \\ x_c(t) \end{bmatrix}.$$

As thus, one can obtain the following output feedback interconnection system:

$$H_{\mathcal{T}}:\begin{cases}\dot{x}_{\mathcal{T}}(t)=-L_e^{\mathcal{T}}(\mathcal{G})\otimes I_n x_{\mathcal{T}}(t)+y_c(t)\\ y_{\mathcal{T}}(t)=-E_c(\mathcal{G})^T E_{\odot\mathcal{T}}^w(\mathcal{G})\otimes I_n x_{\mathcal{T}}(t)\end{cases} \tag{9}$$

$$H_c:\begin{cases}\dot{x}_c(t)=-L_e^c(\mathcal{G})\otimes I_n x_c(t)+y_{\mathcal{T}}(t)\\ y_c(t)=-E_{\mathcal{T}}(\mathcal{G})^T E_{\odot c}^w(\mathcal{G})\otimes I_n x_c(t)\end{cases} \tag{10}$$

which is shown in Figure 2.

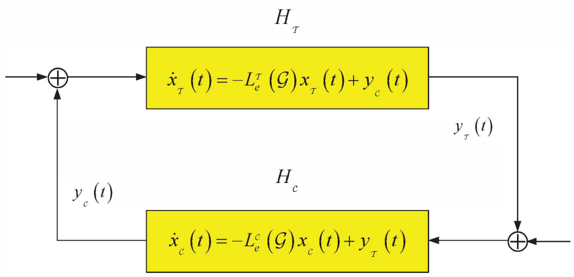


Fig. 2. Edge dynamics as an output feedback interconnection structure between $H_{\mathcal{T}}$-subsystem and H_c-subsystem.

Remark 1: The decomposition of the spanning tree and co-spanning tree subgraph actually has been wildly applied to solve the magnetostatic problems, such as tree-cotree gauging [24], finite element analysis [25], in which the decomposition is referred as Tree-Cotree Splitting (TCS) technique.
As known that the spanning tree structure plays a vital role in the analysis of networked multi-agent system, but scarcely any literatures could offer a detailed interpretation for that from the control profiles. Next, we are going to highlight the role of the spanning tree subgraph by providing a model reduction representation in terms of the corresponding dynamics on it. Notice that $E_{\mathcal{T}}(\mathcal{G})T(\mathcal{G})=E_c(\mathcal{G})$ as mentioned in (1), therefore the co-spanning tree states can be reconstructed through the matrix $T(\mathcal{G})$ as

$$x_c(t)=T(\mathcal{G})^T\otimes I_n x_{\mathcal{T}}(t) \tag{11}$$

which shows the co-spanning tree states can serve as an internal feedback on the edges of the spanning tree subgraph shown in Figure 3. In the meantime, we also have

$$x_e(t)=R(\mathcal{G})^T\otimes I_n x_{\mathcal{T}}(t). \tag{12}$$

Taking (11) into (9) lead to a reduced model $\hat{H}_{\mathcal{T}}$

$$\begin{aligned}\dot{x}_{\mathcal{T}}(t)=&(-L_e^{\mathcal{T}}(\mathcal{G})-E_{\mathcal{T}}(\mathcal{G})^T E_{\odot c}^w(\mathcal{G})T(\mathcal{G})^T)\otimes I_n x_{\mathcal{T}}(t)\\ =&-E_{\mathcal{T}}(\mathcal{G})^T(E_{\odot\mathcal{T}}^w(\mathcal{G})+E_{\odot c}^w(\mathcal{G})T(\mathcal{G})^T)\otimes I_n x_{\mathcal{T}}(t)\\ =&-E_{\mathcal{T}}(\mathcal{G})^T E_{\odot}^w(\mathcal{G})R(\mathcal{G})^T\otimes I_n x_{\mathcal{T}}(t)\\ =&-\hat{L}_e(\mathcal{G})\otimes I_n x_{\mathcal{T}}(t)\end{aligned} \tag{13}$$

which captures the dynamical behaviour of the whole system. In this paper, we refer $\hat{L}_e(\mathcal{G})=E_{\mathcal{T}}(\mathcal{G})^T E_{\odot}^w(\mathcal{G})R(\mathcal{G})^T$ as the *essential edge Laplacian.*

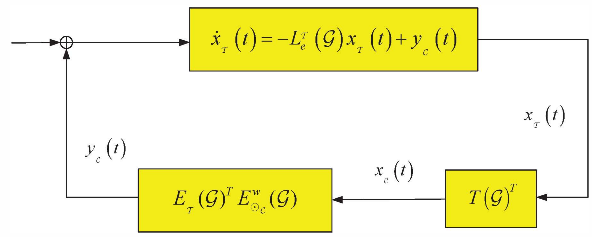


Fig. 3. H_c (t)-subsystem serves as an internal feedback state.

In the subsequent analysis, the reduced model associated with the essential edge Laplacian will play an important role.

Lemma 5: The essential edge Laplacian $\hat{L}_e(\mathcal{G})$ contains exactly all the nonzero eigenvalues of $L_e(\mathcal{G})$. Additionally, we can construct the following Lyapunov equation as

$$H\hat{L}_e(\mathcal{G})+\hat{L}_e(\mathcal{G})^T H=I_{N-1} \tag{14}$$

where H is a positive definite matrix.

Proof: Before moving on, we introduce the following transformation matrix:

$$S_e(\mathcal{G})=\begin{bmatrix}R(\mathcal{G})^T & \theta_e(\mathcal{G})\end{bmatrix}$$

$$S_e(\mathcal{G})^{-1}=\begin{bmatrix}\left(R(\mathcal{G})R(\mathcal{G})^T\right)^{-1}R(\mathcal{G})\\ \theta_e(\mathcal{G})^T\end{bmatrix}$$

where $\theta_e(\mathcal{G})$ denote the orthonormal basis of the flow space, i.e., $E(\mathcal{G})\theta_e(\mathcal{G})=0$. Since $rank(E(\mathcal{G}))=N-1$, one can obtain that $dim(\theta_e(\mathcal{G}))=\mathcal{N}(E(\mathcal{G}))$ and $\theta_e(\mathcal{G})^T\theta_e(\mathcal{G})=I_{L-N+1}$. The matrix $R(\mathcal{G})$ is defined via (2). Applying the similar transformation lead to

$$S_e(\mathcal{G})^{-1}L_e(\mathcal{G})S_e(\mathcal{G})=\begin{bmatrix}\hat{L}_e(\mathcal{G}) & E_{\mathcal{T}}^T(\mathcal{G})E_{\odot}^w(\mathcal{G})\theta_e(\mathcal{G})\\ \mathbf{0}_{L-N+1\times N-1} & \mathbf{0}_{L-N+1}\end{bmatrix}.$$

Clearly, the eigenvalues of the block matrix are the solution of

$$\lambda^{(L-N+1)}\det\left(\lambda I-\hat{L}_e(\mathcal{G})\right)=0$$

which shows that $\hat{L}_e(\mathcal{G})$ has exactly all the nonzero eigenvalues of $L_e(\mathcal{G})$. As thus, we can construct the following Lyapunov equation as

$$H\hat{L}_e(\mathcal{G})+\hat{L}_e(\mathcal{G})^T H=I_{N-1}$$

where H is positive definite. ■

Remark 2: In [3], by using the similar transformation mentioned above, the edge-description system could be separated into controllable and observable parts. It also points out that the minimal realization of the system contains only the states across the edges of spanning tree. Additionally, by using the essential edge Laplacian, we could extremely simplify the complexity of the analysis of multi-agent systems, since it only preserves the nonzero eigenvalues of the edge (graph) Laplacian. In fact, similar representations have been implicitly realized in recent works [18][26][27][28]. However, these literatures did not reveal the explicit connection of the algebraic properties from systematic and structural view.

IV. Conclusions

In this paper, the model reduction problem based on the edge agreement dynamics of multi-agent system was explored. We proposed the general concept of directed edge Laplacian with its algebraic properties. The edge agreement dynamics can be then translated into a output feedback interconnection of the spanning tree subsystem and the co-spanning tree subsystem. In addition, based on the essential edge Laplacian, we derive a model reduction representation of the closed-loop multi-agent system based on the spanning tree subgraph, which allows a convenient analysis of multi-agent system.

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