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3 authors, including:

Zhiwen Zeng
National University of Defense Technology

Xiangke Wang
National University of Defense Technology

Edge Agreement of Multi-agent System with Quantized Measurements via Directed Edge Laplacian

Zhiwen Zeng [a], Xiangke Wang [a], Zhiqiang Zheng [a],

[a] *College of Mechatronics and Automation, National University of Defense Technology, 410073, China*

Abstract

This work explores the edge agreement problem of the second-order nonlinear multi-agent system under quantized measurements. To begin with, the general concepts of weighted edge Laplacian of directed graph are proposed and its algebraic properties are further explored. Based on the essential edge Laplacian, we derive a model reduction representation of the closed-loop multi-agent system based on the spanning tree subgraph. Meanwhile, the edge agreement problem of second-order nonlinear multi-agent system under quantized effects is studied, in which both uniform and logarithmic quantizers are considered. Particularly, for the uniform quantizers, we provide the upper bound of the radius of the agreement neighborhood, which indicates that the radius increases with the quantization interval. While for the logarithmic quantizers, the agents converge exponentially to the desired agreement equilibrium. Additionally, we also provide the estimates of the convergence rate as well as indicate that the coarser the quantizer is, the slower the convergence speed. Finally, simulation results are given to verify the theoretical analysis.

Key words: Edge agreement, directed edge Laplacian, model reduction, multi-agent system, quantized measurements.

⋆ This paper was not presented at any meeting or journal. Corresponding author Xiangke Wang. Tel. +86-0731-84576455.

Email addresses: zhiwenzeng.nudt@gmail.com (Zhiwen Zeng), xkwang@nudt.edu.cn (Xiangke Wang), zqzheng@nudt.edu.cn (Zhiqiang Zheng).

1 Introduction

The graph theory contributes significantly in the analysis and synthesis of multi-agent systems, since it provides natural abstractions for how information is shared between agents in a network. Specially, the spectral properties of the graph Laplacian are extensively explored recently to provide convergence analysis in the context of multi-agent coordination behaviour [1][2]. Despite the unquestionable interest of the results concerning the convergence properties in these literatures, we also note that, another interesting topic with regard to how certain subgraphs, such as spanning trees and cycles, contribute to the analysis of multi-agent systems, has arisen in more recently. An important theme in this direction is to obtain the explicit connections between the topology structure and the control system. Considering this, an attractive notion about the edge agreement deserve special attention, in which the edge Laplacian plays an important role. Pioneering researches on edge agreement under undirected graph not only provide totally new insights that how the spanning trees and cycles effect the performance of the agreement protocol, but also set up a novel systematic framework for analysing multi-agent systems from the edge perspective[3][4][5]. The edge agreement in [3] provides a theoretic analysis of the system's performance using both H_2 and H_∞ norms, and these results have been applied in relative sensing networks referring to [4]. Moreover, based on the algebraic properties of the edge Laplacian, [5] examines how cycles impact the H_2 performance and proposed an optimal strategy for the design of consensus networks. Although the edge Laplacian offers more transparent understanding of the graph structure, it still remains an undirected notion in aforementioned literatures. More recently, the edge Laplacian is used to examine the model reduction of networked system associated with directed trees through clustering in [6]; however it can not be directly extended to more general directed graphs yet.

Constraints on communication have a considerable impact on the performance of multi-agent system. To cope with the limitations of the finite bandwidth channels, measurements are always processed by quantizers. Due to the fact that quantization introduces strong nonlinear characteristics such as discontinuity and saturation to the system, the research of the coordination behaviour of multi-agent system in the presence of quantized measurements is still quite challenging. Recently, the gossiping algorithms [7][8] as well as the coding/decoding schemes [9][10][11] have been proposed to solve the quantized consensus problem, where the convergence time is the main research focus. While these methods are mainly devised for discrete-time systems, the continuous-time systems has also attracted much attention. The spectral properties of the incidence matrix is used to analyze the convergence properties of multi-agent system under quantized communication in [12]. Besides, the sample-data based control [13] and nonsmooth analysis [14] are also employed to tackle the quantized measurements. However, only the first order dynamics has been considered in the above approaches. As known that, multi-agent system with second-order dynamics can have significantly different coordination behaviour even when agents are coupled through similar topology condition [15]. Up to date, to the best of authors' knowledge, there are still little works explore the quantization effects on second-order dynamics. [15] studies the effects of different quantizers on the synchronization behaviour of mobile agents with second-order dynamics. In [16], the collective coordination of second-order passive nonlinear systems under quantized measurements are considered. By using nonsmooth analysis, some convergence results under quantization constraints are derived for second-order dynamics system in [17]. However, these works require that the network topologies are undirected. To more recent literature [18], the authors address the quantized consensus problem of second-order multi-agent systems via sampled data under directed topology. Considering the drawbacks of the sampled data approach mentioned in [16], the research on the quantization effects on second-order multi-agent under directed topology is still very challenging.

While the analysis of the node agreement (consensus problem) has matured, work related to the edge agreement has not been deeply studied. Note that the quantized measurements bring enormous challenges to the analysis of the synchronization behaviour of the second-order multi-agent system, so in this paper, we are going to explore more details about this term combining the edge agreement. The main contributions contain three folders. First, we extend our preliminary work [19] to the weighted directed edge Laplacian and further explore the algebraic properties for analysing the interacting multi-agent system. Since its undirected counterpart has shown great potential for exploring the system performance in [3][20], we believe that the novel graph-theoretic tool deserves more attention. Second, under the edge agreement framework, the closed-loop multi-agent system can be transformed into an output feedback interconnection structure. Correspondingly, based on the observation that the co-spanning tree subsystem can be served as an internal feedback, a model reduction representation can be derived, which allows a convenient analysis. Third, based on the reduced edge agreement model, we propose a general analysis of the convergence properties for second-order nonlinear multi-agent system under quantized measurements. Particularly, for the uniform quantizers, we provide the explicit upper bound of the radius of the agreement neighborhood and also indicate that the radius increases with the quantization interval. While for the logarithmic quantizers, the agents converge exponentially to the desired agreement equilibrium. Additionally, we also provide the estimates of the convergence rate as well as indicate that the coarser the quantizer is, the slower the convergence speed.

The rest of the paper is organized as follows: preliminaries and problem formulation are proposed in Section 2. The directed edge Laplacian with its algebraic properties are elaborated in Section 3 as well as the edge agreement mechanism. The quantized edge agreement problem with second-order nonlinear dynamics under directed graph is studied in Section 4. The simulation results are provided in Section 5 while the last section draws the conclusions.

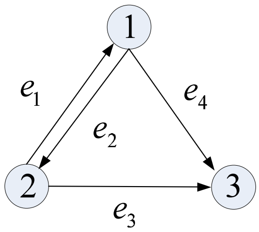
2 Basic Notions and Preliminary Results

In this section, some basic notions in graph theory and preliminary results about the synchronization of multi-agent system under quantized information are briefly introduced.

2.1 Graph and Matrix

In this paper, we use $|\cdot|$ and $\|\cdot\|$ to denote the Euclidean norm and 2-norm for vectors and matrices respectively. The null space of matrix A is denoted by $\mathcal{N}(A)$. Denote by I_n the identity matrix and by $\mathbf{0}_n$ the zero matrix in $\mathbb{R}^{n\times n}$. Let $\mathbf{0}$ be the column vector with all zero entries. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph of order N specified by a node set $\mathcal{V}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ with size L. For a specific edge $e_k = (j, i)$, let $v_{\otimes}(e_k)$ denotes its initial node j and $v_{\odot}(e_k)$ the terminal node i. The set of neighbours of node i is denoted by $\mathcal{N}_i = \{j : e_k = (j, i) \in \mathcal{E}\}$. We use $A(G)$ to represent a weighted adjacency matrix, where the adjacency elements associated with the edges are positive, i.e., $e_k = (j, i) \in \mathcal{E} \Leftrightarrow a_{ij} > 0$, otherwise, $a_{ij} = 0$. Denote by $\mathcal{W}(\mathcal{G})$ the $L \times L$ diagonal matrix of w_k, for $k = 1, 2 \cdots, L$, where $w_k = a_{ij}$ for $e_k = (j, i) \in \mathcal{E}$. The notation $D(\mathcal{G})$ represents a diagonal matrix with $\Delta_i(\mathcal{G})$ denoting the in-degree of node i on the diagonal. The corresponding graph Laplacian of $\mathcal{G}$ is defined as $L_n(\mathcal{G}) := D(\mathcal{G}) - A(\mathcal{G})$, whose eigenvalues will be ordered and denoted as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$. The incidence matrix $E(\mathcal{G}) \in \mathbb{R}^{N\times L}$ for a directed graph is a $\{0, \pm 1\}$-matrix with rows and columns indexed by nodes and edges of $\mathcal{G}$ respectively, such that for edge $e_k = (j, i) \in \mathcal{E}$, $[E(\mathcal{G})]_{jk} = +1$, $[E(\mathcal{G})]_{ik} = -1$ and $[E(\mathcal{G})]_{lk} = 0$ for $l \neq i, j$. The definition implies that each column of E contains exactly two nonzero entries indicating the initial node and the terminal node respectively. We illustrate Figure 1 as an example.

A directed path in directed graph $\mathcal{G}$ is a sequence of directed edges and a directed tree is a directed graph in which, for the root i and any other node j, there is exactly one directed path from i to j. A spanning tree $\mathcal{G}_{\mathcal{T}} = (\mathcal{V}, \mathcal{E}_1)$ of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a directed tree formed by graph edges that connect all the nodes of the graph; a co-spanning tree $\mathcal{G}_c = (\mathcal{V}, \mathcal{E} - \mathcal{E}_1)$ of $\mathcal{G}_{\mathcal{T}}$ is the subgraph of $\mathcal{G}$ having all the vertices of $\mathcal{G}$ and exactly those edges of $\mathcal{G}$ that are not in $\mathcal{G}_{\mathcal{T}}$. Graph $\mathcal{G}$ is called *strongly connected* if and only if any two



$$E = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Fig. 1. The incidence matrix of a simple directed graph.

distinct nodes can be connected via a directed path; *quasi-strongly connected* if and only if it has a directed spanning tree [21]. A quasi-strongly connected directed graph $\mathcal{G}$ can be rewritten as a union form: $\mathcal{G} = \mathcal{G}_{\mathcal{T}} \cup \mathcal{G}_{\mathcal{C}}$. In addition, according to certain permutations, the incidence matrix $E(\mathcal{G})$ can always be rewritten as $E(\mathcal{G}) = \begin{bmatrix} E_{\mathcal{T}}(\mathcal{G}) & E_{\mathcal{C}}(\mathcal{G}) \end{bmatrix}$ as well. Since the co-spanning tree edges can be constructed from the spanning tree edges via a linear transformation [3], such that

$$E_{\mathcal{T}}\left(\mathcal{G}\right) T(\mathcal{G}) = E_{\mathcal{C}}\left(\mathcal{G}\right) \tag{1}$$

with $T(\mathcal{G}) = \left(E_{\mathcal{T}}(\mathcal{G})^T E_{\mathcal{T}}\left(\mathcal{G}\right)\right)^{-1} E_{\mathcal{T}}(\mathcal{G})^T E_{\mathcal{C}}\left(\mathcal{G}\right)$ and $rank(E\left(\mathcal{G}\right)) = N-1$ from [21]. We define

$$R\left(\mathcal{G}\right) = \begin{bmatrix} I & T(\mathcal{G}) \end{bmatrix} \tag{2}$$

and then obtain $E\left(\mathcal{G}\right) = E_{\mathcal{T}}\left(\mathcal{G}\right) R\left(\mathcal{G}\right)$. The column space of $E(\mathcal{G})^T$ is known as the *cut space* of $\mathcal{G}$ and the null space of $E(\mathcal{G})$ is called the *flow space*, which is the orthogonal complement of the cut space. Interestingly, the rows of $R\left(\mathcal{G}\right)$ form a basis of the cut space of and the rows of $\begin{bmatrix} -T(\mathcal{G})^T & I \end{bmatrix}$ form a basis of the flow space respectively [21].

Lemma 1 ([22]) *: The graph Laplacian $L_{\mathcal{G}}\left(\mathcal{G}\right)$ of a directed graph $\mathcal{G}$ has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right-half plane. In addition, $L_{\mathcal{G}}\left(\mathcal{G}\right)$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ is quasi-strongly connected.*

2.2 Problem Formulation and Quantized Information

We consider a group of N networked agents and the dynamics of the i-th agent is represented by

$$\dot{x}_i(t) = v_i(t) \tag{3}$$
$$\dot{v}_i(t) = f\left(x_i(t), v_i(t), t\right) + u_i(t) \tag{4}$$

where $x_i(t) \in \mathbb{R}^n$ is the position, $v_i(t) \in \mathbb{R}^n$ is the velocity and $\mu_i(t) \in \mathbb{R}^n$ is the control input. The nonlinear term $f(x_i(t), v_i(t), t) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is unknown and satisfy the following assumption.

Assumption 2 *For a nonlinear function f, there exists nonnegative constants ξ_1 and ξ_2 such that*

$$\begin{aligned}|f(x,v,t) - f(y,z,t)| \leq & \xi_1 |x-y| + \xi_2 |v-z|, \\ & \forall x, v, y, z \in \mathbb{R}^n; \forall t \geq 0.\end{aligned}$$

The goal for designing distributed control law $\mu_i(t)$ is to synchronize the velocities and positions of the N networked agents.

The generally studied second-order consensus protocol proposed in [23] is described as follows: $u_i(t) = \alpha \sum\limits_{j \in \mathcal{N}_i}^{N} a_{ij} (x_j(t) - x_i(t)) + \beta \sum\limits_{j \in \mathcal{N}_i}^{N} a_{ij} (v_j(t) - v_i(t))$, for $i = 1, 2 \cdots, N$, where $\alpha > 0$ and $\beta > 0$ are the coupling strengths. As in [12], we assume that each agent i has only quantized measurements of the relative position $Q(x_i - x_j)$ and velocity information $Q(v_i - v_j)$, where $Q(.) : \mathbb{R}^n \to \mathbb{R}^n$ is the *quantization function*. As thus, the protocol is modified as

$$u_i(t) = \alpha \sum_{j \in \mathcal{N}_i}^{N} a_{ij} Q(x_j(t) - x_i(t)) + \beta \sum_{j \in \mathcal{N}_i}^{N} a_{ij} Q(v_j(t) - v_i(t)) \tag{5}$$

for $i = 1, 2 \cdots, N$. Specifically, the quantizer can be represented by $Q(\nu) = [q(\nu_1), q(\nu_2), \cdots, q(\nu_n)]^T$ for a vector $\nu = [\nu_1, \nu_2, \cdots, \nu_n]^T \subset \mathbb{R}^n$. Two typical quantization operators are considered: uniform and logarithmic quantizer. For a given $\delta_u > 0$, a uniform quantizer $q_u : \mathbb{R} \to \mathbb{R}$ satisfies $|q_u(a) - a| \leq \delta_u, \forall a \in \mathbb{R}$; for a given $\delta_l > 0$, a logarithmic quantizer $q_l : \mathbb{R} \to \mathbb{R}$ satisfies $|q_l(a) - a| \leq \delta_l |a|, \forall a \in \mathbb{R}$. The positive constants δ_u and δ_l are known as quantization interval. We define $Q_u(\nu) \triangleq [q_u(\nu_1), q_u(\nu_2), \cdots, q_u(\nu_n)]^T$ and $Q_l(\nu) \triangleq [q_l(\nu_1), q_l(\nu_2), \cdots, q_l(\nu_n)]^T$, then the following bounds hold: $|Q_u(\nu) - \nu| \leq \sqrt{n}\delta_u$, $|Q_l(\nu) - \nu| \leq \delta_l |\nu|$.

3 Directed Edge Laplacian and Edge Agreement

In this sequel, a novel matrix representation of directed graphs describing the interconnection topology will be introduced, which allows a convenient analysis of multi-agent system from the edge perspective. Based on the new description, the edge agreement for general directed graph is proposed.

The edge Laplacian in [3] still remains to an undirected notion and is thus inadequate to handle our problem. Extending the concept of the edge Laplacian to directed graph will be of great help to understand multi-agent systems from the structural perspective.

Before moving on, we give the definition of the in-incidence matrix and out-incidence matrix at first.

Definition 3 (In-incidence/Out-incidence Matrix) *The $N \times L$ in-incidence matrix $E_{\odot}(\mathcal{G})$ for a directed graph $\mathcal{G}$ is a $\{0, -1\}$ matrix with rows and columns indexed by nodes and edges of $\mathcal{G}$, respectively, such that for an edge $e_k = (j, i) \in \mathcal{E}$, $[E_{\odot}(\mathcal{G})]_{mk} = -1$ for $m = i$, $[E_{\odot}(\mathcal{G})]_{mk} = 0$ otherwise. The out-incidence matrix is a $\{0, +1\}$ matrix with $[E_{\otimes}(\mathcal{G})]_{nk} = +1$ for $n = j$, $[E_{\otimes}(\mathcal{G})]_{nk} = 0$ otherwise.*

In comparison with the definition of the incidence matrix, we can rewrite $E(\mathcal{G})$ in the following way:

$$E(\mathcal{G}) = E_{\odot}(\mathcal{G}) + E_{\otimes}(\mathcal{G}). \tag{6}$$

On the other hand, the weighted in-incidence matrix $E^w_{\odot}(\mathcal{G})$ can be defined as $E^w_{\odot}(\mathcal{G}) = E_{\odot}(\mathcal{G})\mathcal{W}(\mathcal{G})$, where $\mathcal{W}(\mathcal{G})$ is a diagonal matrix of w_k. This will lead us to find out a novel factorization of the graph Laplacian $L_n(\mathcal{G})$.

Lemma 4 *Considering a directed graph $\mathcal{G}$ with the incidence matrix $E(\mathcal{G})$ and weighted in-incidence matrix $E^w_{\odot}(\mathcal{G})$, the graph Laplacian of $\mathcal{G}$ have the following expression*

$$L_n(\mathcal{G}) = E^w_{\odot}(\mathcal{G})E(\mathcal{G})^T. \tag{7}$$

PROOF. By using (6), we have $E^w_{\odot}(\mathcal{G})E(\mathcal{G})^T = E^w_{\odot}(\mathcal{G})E_{\odot}(\mathcal{G})^T + E^w_{\odot}(\mathcal{G})E_{\otimes}(\mathcal{G})^T$. Let $E^w_{\odot i}(\mathcal{G})$, $E_{\otimes i}(\mathcal{G})$ be the i-th row of $E^w_{\odot}(\mathcal{G})$ and $E_{\otimes}(\mathcal{G})$. According to the preceding definition, it's clear that, $E^w_{\odot i}(\mathcal{G})E_{\odot j}(\mathcal{G})^T = \Delta_i(\mathcal{G})$ for $i = j$, $E^w_{\odot i}(\mathcal{G})E_{\odot j}(\mathcal{G})^T = 0$ otherwise. Then we can collect terms as $E^w_{\odot}(\mathcal{G})E_{\odot}(\mathcal{G})^T = D(\mathcal{G})$. Besides, we also have $E^w_{\odot i}(\mathcal{G})E_{\otimes j}(\mathcal{G})^T = -w_k$ for $j \in N_i, e_k = (j, i)$ and $E^w_{\odot i}(\mathcal{G})E_{\otimes j}(\mathcal{G})^T = 0$ otherwise, which implies that $E^w_{\odot}(\mathcal{G})E_{\otimes}(\mathcal{G})^T = -A(\mathcal{G})$. Then, we have $E^w_{\odot}(\mathcal{G})E(\mathcal{G})^T = E^w_{\odot}(\mathcal{G})E_{\odot}(\mathcal{G})^T + E^w_{\odot}(\mathcal{G})E_{\otimes}(\mathcal{G})^T = D(\mathcal{G}) - A(\mathcal{G}) = L_n(\mathcal{G})$.

Definition 5 (Directed Edge Laplacian) *The edge Laplacian of a directed graph $\mathcal{G}$ is defined as*

$$L_e(\mathcal{G}) := E(\mathcal{G})^T E^w_{\odot}(\mathcal{G}). \tag{8}$$

To provide a deeper insights into what the edge Laplacian $L_e(\mathcal{G})$ offers in the analysis and synthesis of multi-agent systems, we propose the following lemma.

Lemma 6 *For any directed graph $\mathcal{G}$, the graph Laplacian $L_{\mathcal{G}}$ and the edge Laplacian $L_e(\mathcal{G})$ have the same nonzero eigenvalues. If $\mathcal{G}$ is quasi-strongly connected, then the edge Laplacian $L_e(\mathcal{G})$ contains exactly $N-1$ nonzero eigenvalues which are all in the open right-half plane.*

PROOF. The proof for the weighted version of $L_e(\mathcal{G})$ can be easily extended from lemma 5 of our previous work [19], thus the detail is omitted here.

Obviously, if $\mathcal{G} = \mathcal{G}_{\mathcal{T}}$, then $\mathcal{G}$ has $L = N - 1$ edges and all the eigenvalues of $L_e(\mathcal{G})$ are nonzero. In the following paper, when we deal with a quasi-strongly connected graph, it refers to a general directed graph $\mathcal{G} = \mathcal{G}_{\mathcal{T}} \cup \mathcal{G}_{\mathcal{C}}$ unless noted otherwise.

Lemma 7 *Considering a quasi-strongly connected graph $\mathcal{G}$ of order N, the edge Laplacian $L_e(\mathcal{G})$ has $L - N + 1$ zero eigenvalues and zero is a simple root of the minimal polynomial of $L_e(\mathcal{G})$.*

PROOF. The result can be lightly extended from lemma 6 of our previous work [19].

Lemma 7 implies that the linear system associated with $-L_e(\mathcal{G})$ is marginally stable. As thus, the weighted directed edge Laplacian holds the similar functions as the graph Laplacian for analyzing the interacting multi-agent system. The explicit connection between the edge and graph Laplacian has been highlighted by a similarity transformation in [3]. Actually, by using this transformation, we can derive a reduced model representation for the edge agreement dynamics.

3.2 Edge Agreement and Model Reduction

Although the graph Laplacian is a convenient method to describe the geometric interconnection of networked agents, another attractive notion, the edge agreement, which has not been extensively explored, deserves additional attention because the edges are adopted to be natural interpretations of the information flow.

Considering the quasi-strongly connected graph $\mathcal{G}$ and the most commonly used consensus dynamics [24] described as:

$$\dot{x} = -L_n(\mathcal{G}) \otimes I_n x$$

where $\otimes$ denotes the Kronecker product. Contrary to the most existing works, we study the synchronization problem from the edge perspective by using L_e. In this avenue, we define the *edge state* vector as

$$x_e(t) = E(\mathcal{G})^T \otimes I_n x(t) \tag{9}$$

which represents the difference between the state components of two neighbouring nodes. Taking the derivative of (9) lead to

$$\dot{x}_e(t) = -L_e(\mathcal{G}) \otimes I_n x_e(t) \tag{10}$$

which is referred as *edge agreement dynamics* in this paper. In comparison to the node agreement (consensus), the edge agreement, rather than requiring the convergence to the agreement subspace, desires the edge dynamics (10) converge to the origin, i.e., $\lim_{t\to\infty} |x_e(t)| = 0$. Essentially, the evolution of an edge state depends on its current state and the states of its adjacent edges. Besides, the edge agreement implies consensus if the directed graph $\mathcal{G}$ has a spanning tree [3].

As the quasi-strongly connected graph $\mathcal{G}$ can be written as $\mathcal{G} = \mathcal{G}_{\mathcal{T}} \cup \mathcal{G}_{\mathcal{C}}$, then the weighted in-incidence matrix can be represented as $E_{\odot}^{w}(\mathcal{G}) = \begin{bmatrix} E_{\odot\mathcal{T}}^{w}(\mathcal{G}) & E_{\odot\mathcal{C}}^{w}(\mathcal{G}) \end{bmatrix}$ via some certain permutations in line with $E(\mathcal{G}) = \begin{bmatrix} E_{\mathcal{T}}(\mathcal{G}) & E_{\mathcal{C}}(\mathcal{G}) \end{bmatrix}$. According to the partition, we can represent the edge Laplacian into the block form as

$$L_e(\mathcal{G}) = E(\mathcal{G})^T E_{\odot}^{w}(\mathcal{G}) = \begin{bmatrix} L_e^{\mathcal{T}}(\mathcal{G}) & E_{\mathcal{T}}(\mathcal{G})^T E_{\odot\mathcal{C}}^{w}(\mathcal{G}) \\ E_{\mathcal{C}}(\mathcal{G})^T E_{\odot\mathcal{T}}^{w}(\mathcal{G}) & L_e^{\mathcal{C}}(\mathcal{G}) \end{bmatrix} \tag{11}$$

with $L_e^{\mathcal{T}}(\mathcal{G}) = E_{\mathcal{T}}(\mathcal{G})^T E_{\odot\mathcal{T}}^{w}(\mathcal{G})$ and $L_e^{\mathcal{C}}(\mathcal{G}) = E_{\mathcal{C}}(\mathcal{G})^T E_{\odot\mathcal{C}}^{w}(\mathcal{G})$.

The edge Laplacian dynamics (10) can be translated into a output feedback interconnection of the spanning tree subsystem $H_{\mathcal{T}}$ and the co-spanning tree subsystem $H_{\mathcal{C}}$. Actually, by using (11), the edge Laplacian dynamics $x_e(t)$ described by (10) can be rewritten as

$$\begin{bmatrix} \dot{x}_{\mathcal{T}}(t) \\ \dot{x}_{\mathcal{C}}(t) \end{bmatrix} = -\begin{bmatrix} L_e^{\mathcal{T}}(\mathcal{G}) & E_{\mathcal{T}}(\mathcal{G})^T E_{\odot\mathcal{C}}^{w}(\mathcal{G}) \\ E_{\mathcal{C}}(\mathcal{G})^T E_{\odot\mathcal{T}}^{w}(\mathcal{G}) & L_e^{\mathcal{C}}(\mathcal{G}) \end{bmatrix} \otimes I_n \begin{bmatrix} x_{\mathcal{T}}(t) \\ x_{\mathcal{C}}(t) \end{bmatrix}.$$

As thus, one can obtain the following output feedback interconnection system:

$$H_{\mathcal{T}}: \begin{cases} \dot{x}_{\mathcal{T}}(t) = -L_e^{\mathcal{T}}(\mathcal{G}) \otimes I_n x_{\mathcal{T}}(t) + y_{\mathcal{C}}(t) \\ y_{\mathcal{T}}(t) = -E_{\mathcal{C}}(\mathcal{G})^T E_{\odot\mathcal{T}}^w(\mathcal{G}) \otimes I_n x_{\mathcal{T}}(t) \end{cases} \tag{12}$$

$$H_{\mathcal{C}}: \begin{cases} \dot{x}_{\mathcal{C}}(t) = -L_e^{\mathcal{C}}(\mathcal{G}) \otimes I_n x_{\mathcal{C}}(t) + y_{\mathcal{T}}(t) \\ y_{\mathcal{C}}(t) = -E_{\mathcal{T}}(\mathcal{G})^T E_{\odot\mathcal{C}}^w(\mathcal{G}) \otimes I_n x_{\mathcal{C}}(t) \end{cases} \tag{13}$$

which is shown in Figure 2.

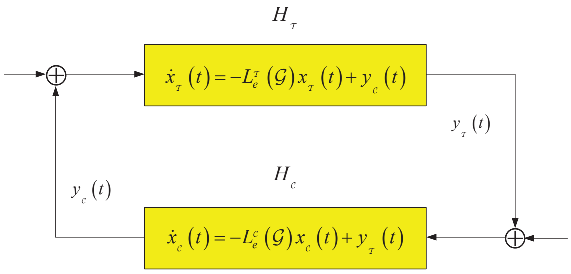


Fig. 2. Edge dynamics as an output feedback interconnection structure between $H_{\mathcal{T}}$-subsystem and $H_{\mathcal{C}}$-subsystem.

Remark 8 *The decomposition of the spanning tree and co-spanning tree subgraph actually has been wildly applied to solve the magnetostatic problems, such as tree-cotree gauging [25], finite element analysis [26], in which the decomposition is referred as Tree-Cotree Splitting (TCS) technique.*

As known that the spanning tree structure plays a vital role in the analysis of networked multi-agent system, but scarcely any literatures could offer a detailed interpretation for that from the control profiles. Next, we are going to highlight the role of the spanning tree subgraph by providing a model reduction representation in terms of the corresponding dynamics on it. Notice that $E_{\mathcal{T}}(\mathcal{G}) T(\mathcal{G}) = E_{\mathcal{C}}(\mathcal{G})$ as mentioned in (1), therefore the co-spanning tree states can be reconstructed through the matrix $T(\mathcal{G})$ as

$$x_{\mathcal{C}}(t) = T(\mathcal{G})^T \otimes I_n x_{\mathcal{T}}(t) \tag{14}$$

which shows the co-spanning tree states can serve as an internal feedback on the edges of the spanning tree subgraph shown in Figure 3. In the meantime, we also have

$$x_e(t) = R(\mathcal{G})^T \otimes I_n x_{\mathcal{T}}(t). \tag{15}$$

Taking (14) into (12) lead to a reduced model $\hat{H}_{\mathcal{T}}$

$$\begin{aligned}
\dot{x}_{\mathcal{T}}(t) =& (-L_e^{\mathcal{T}}(\mathcal{G}) - E_{\mathcal{T}}(\mathcal{G})^T E_{\odot\mathcal{C}}^w(\mathcal{G}) T(\mathcal{G})^T) \otimes I_n x_{\mathcal{T}}(t) \\
=& - E_{\mathcal{T}}(\mathcal{G})^T (E_{\odot\mathcal{T}}^w(\mathcal{G}) + E_{\odot\mathcal{C}}^w(\mathcal{G}) T(\mathcal{G})^T) \otimes I_n x_{\mathcal{T}}(t) \\
=& - E_{\mathcal{T}}(\mathcal{G})^T E_{\odot}^w(\mathcal{G}) R(\mathcal{G})^T \otimes I_n x_{\mathcal{T}}(t) \\
=& - \hat{L}_e(\mathcal{G}) \otimes I_n x_{\mathcal{T}}(t)
\end{aligned} \tag{16}$$

which captures the dynamical behaviour of the whole system. In this paper, we refer $\hat{L}_e(\mathcal{G}) = E_{\mathcal{T}}(\mathcal{G})^T E_{\odot}^w(\mathcal{G}) R(\mathcal{G})^T$ as the *essential edge Laplacian.*

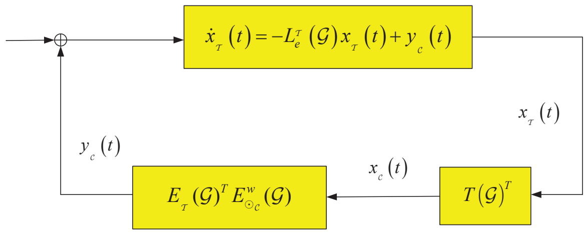


Fig. 3. $H_{\mathcal{C}}(t)$-subsystem serves as an internal feedback state.

In the subsequent analysis, the reduced model associated with the essential edge Laplacian will play an important role.

Lemma 9 *The essential edge Laplacian $\hat{L}_e(\mathcal{G})$ contains exactly all the nonzero eigenvalues of $L_e(\mathcal{G})$. Additionally, we can construct the following Lyapunov equation as*

$$H\hat{L}_e(\mathcal{G}) + \hat{L}_e(\mathcal{G})^T H = I_{N-1} \tag{17}$$

where H is a positive definite matrix.

PROOF. Before moving on, we introduce the following transformation matrix:

$$S_e(\mathcal{G}) = \left[R(\mathcal{G})^T \ \theta_e(\mathcal{G}) \right]$$

$$S_e(\mathcal{G})^{-1} = \begin{bmatrix} \left(R(\mathcal{G}) R(\mathcal{G})^T \right)^{-1} R(\mathcal{G}) \\ \theta_e(\mathcal{G})^T \end{bmatrix}$$

where $\theta_e(\mathcal{G})$ denote the orthonormal basis of the flow space, i.e., $E(\mathcal{G}) \theta_e(\mathcal{G}) = 0$. Since $rank(E(\mathcal{G})) = N - 1$, one can obtain that $dim(\theta_e(\mathcal{G})) = \mathcal{N}(E(\mathcal{G}))$ and $\theta_e(\mathcal{G})^T \theta_e(\mathcal{G}) = I_{L-N+1}$. The matrix $R(\mathcal{G})$ is defined via (2). Applying the similar transformation lead to

$$S_e(\mathcal{G})^{-1}L_e(\mathcal{G})S_e(\mathcal{G}) = \begin{bmatrix} \hat{L}_e(\mathcal{G}) & E_{\mathcal{T}}^T(\mathcal{G})E_{\odot}^w(\mathcal{G})\theta_e(\mathcal{G}) \\ \mathbf{0}_{L-N+1\times N-1} & \mathbf{0}_{L-N+1} \end{bmatrix}.$$

Clearly, the eigenvalues of the block matrix are the solution of

$$\lambda^{(L-N+1)} \det\left(\lambda I - \hat{L}_e(\mathcal{G})\right) = 0$$

which shows that $\hat{L}_e(\mathcal{G})$ has exactly all the nonzero eigenvalues of $L_e(\mathcal{G})$. As thus, we can construct the following Lyapunov equation as

$$H\hat{L}_e(\mathcal{G}) + \hat{L}_e(\mathcal{G})^T H = I_{N-1}$$

where H is positive definite.

Remark 10 *In [3], by using the similar transformation mentioned above, the edge-description system could be separated into controllable and observable parts. It also points out that the minimal realization of the system contains only the states across the edges of spanning tree. Additionally, by using the essential edge Laplacian, we could extremely simplify the complexity of the analysis of multi-agent systems, since it only preserves the nonzero eigenvalues of the edge (graph) Laplacian. In fact, similar representations have been implicitly realized in recent works [18][27][28][29]. However, these literatures did not reveal the explicit connection of the algebraic properties from systematic and structural view.*

4 Quantized Edge Agreement with Second-order Nonlinear Dynamics under Directed Graph

While the quantization effects on multi-agent system associated with undirected graph has been widely studied, the scenario considering the directed graph is still very challenging, since the quantization may cause undesirable oscillating behaviour under directed topology [15]. In this section, the edge agreement of second-order nonlinear multi-agent system under quantized measurements is studied. To ease the notation, we simply use E, $E_{\odot}^w$ and L_e instead of $E(\mathcal{G})$, $E_{\odot}^w(\mathcal{G})$ and $L_e(\mathcal{G})$ in the following parts.

Considering the dynamics of the networked agents as describing in (3) and

(4), by directly applying the quantized protocol (5), we obtain

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = f(x_i(t), v_i(t), t) + \alpha \sum\limits_{j \in \mathcal{N}_i}^{N} a_{ij} Q(x_j(t) - x_i(t)) \\ \qquad + \beta \sum\limits_{j \in \mathcal{N}_i}^{N} a_{ij} Q(v_j(t) - v_i(t)) \end{cases}$$

To ease the difficulty of the analysis, we technically chose $\alpha = \sigma^2$ and $\beta = \sigma^3$ ($\sigma > 0$) as in [30]. Then the system can be collected as

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = F(x(t), v(t), t) - \sigma^2 E_{\odot}^{w} \otimes I_n Q\left(E^T \otimes I_n x(t)\right) \\ \qquad - \sigma^3 E_{\odot}^{w} \otimes I_n Q\left(E^T \otimes I_n v(t)\right) \end{cases} \tag{18}$$

with $x(t)$, $v(t)$ and $F(x(t), v(t), t)$ denoting the column stack vector of $x_i(t)$, $v_i(t)$ and $f(x_i(t), v_i(t), t)$, respectively.

By left-multiplying $E^T \otimes I_n$ of both sides of (18), we obtain

$$\begin{cases} \dot{x}_e = v_e \\ \dot{v}_e = E^T \otimes I_n F - \sigma^2 L_e \otimes I_n Q(x_e) - \sigma^3 L_e \otimes I_n Q(v_e) \end{cases} \tag{19}$$

with $x_e = E^T \otimes I_n x$, $v_e = E^T \otimes I_n v$.

Define $e_{x_e} = Q(x_e) - x_e$ and $e_{v_e} = Q(v_e) - v_e$ as in [12], then (19) can be written as the following form:

$$\begin{cases} \dot{x}_e(t) = v_e(t) \\ \dot{v}_e(t) = E^T \otimes I_n F - \sigma^2 L_e \otimes I_n x_e - \sigma^3 L_e \otimes I_n v_e \\ \qquad - \sigma^2 L_e \otimes I_n e_{x_e} - \sigma^3 L_e \otimes I_n e_{v_e}. \end{cases} \tag{20}$$

Let $z = \begin{bmatrix} x_e^T & v_e^T \end{bmatrix}^T$ and $\omega = \begin{bmatrix} {e_{x_e}}^T & {e_{v_e}}^T \end{bmatrix}^T$, then system (20) can be recast in a compact matrix form as follows:

$$\dot{z} = \mathcal{F} + \mathcal{L} \otimes I_n z + \mathcal{L}_1 \otimes I_n \omega$$

with $\mathcal{L} = \begin{bmatrix} \mathbf{0}_L & I_L \\ -\sigma^2 L_e & -\sigma^3 L_e \end{bmatrix}$, $\quad \mathcal{L}_1 = \begin{bmatrix} \mathbf{0}_L & \mathbf{0}_L \\ -\sigma^2 L_e & -\sigma^3 L_e \end{bmatrix}$ and $\mathcal{F} = \begin{bmatrix} \mathbf{0} \\ E^T \otimes I_n F \end{bmatrix}$.

Considering the nonlinear term $\mathcal{F}$ as well as the zero eigenvalues that L_e contains, intuitive analysis is not simple. However, the reduced edge agreement model (16) will be of great help. To begin with, we make use of the following transformation

$$S_e^{-1} \otimes I_n x_e = \begin{pmatrix} x_{\mathcal{T}} \\ \mathbf{0} \end{pmatrix} \quad S_e^{-1} \otimes I_n v_e = \begin{pmatrix} v_{\mathcal{T}} \\ \mathbf{0} \end{pmatrix}$$

$$S_e^{-1} \otimes I_n e_{x_e} = \begin{bmatrix} \left(RR^T\right)^{-1} R \otimes I_n e_{x_e} \\ {\theta_e}^T \otimes I_n e_{x_e} \end{bmatrix}$$

$$S_e^{-1} \otimes I_n e_{v_e} = \begin{bmatrix} \left(RR^T\right)^{-1} R \otimes I_n e_{v_e} \\ {\theta_e}^T \otimes I_n e_{v_e} \end{bmatrix}.$$

Then we define $z_{\mathcal{T}} = \left[x_{\mathcal{T}}^T \; v_{\mathcal{T}}^T\right]^T$ and $\hat{L}_{\mathcal{O}} = E_{\mathcal{T}}^T E_{\odot}^w$. Finally, system (20) can be written into

$$\dot{z}_{\mathcal{T}} = \mathcal{F}_{\mathcal{T}} + \mathcal{L}_{\mathcal{T}} \otimes I_n z_{\mathcal{T}} + \mathcal{L}_{\mathcal{T}1} \otimes I_n \omega \tag{21}$$

with $\mathcal{L}_{\mathcal{T}} = \begin{bmatrix} \mathbf{0}_{N-1} & I_{N-1} \\ -\sigma^2 \hat{L}_e & -\sigma^3 \hat{L}_e \end{bmatrix}$, $\mathcal{L}_{\mathcal{T}1} = \begin{bmatrix} \mathbf{0}_{N-1\times L} & \mathbf{0}_{N-1\times L} \\ -\sigma^2 \hat{L}_{\mathcal{O}} & -\sigma^3 \hat{L}_{\mathcal{O}} \end{bmatrix}$ and $\mathcal{F}_{\mathcal{T}} = \begin{bmatrix} \mathbf{0} \\ E_{\mathcal{T}}^T \otimes I_n F \end{bmatrix}$.

To further look at the relation between the quantization interval and the edge agreement, we propose the following theorem.

Theorem 11 *Considering the quasi-strongly connected directed graph $\mathcal{G}$ associated with the edge Laplacian L_e , suppose $\mathcal{Q} = -\left(\mathcal{P}\mathcal{L}_{\mathcal{T}} + \mathcal{L}_{\mathcal{T}}^T \mathcal{P}\right)$ with $\mathcal{P} = \begin{bmatrix} \sigma H & H \\ H & \sigma H \end{bmatrix}$, where H is obtained by (17). If $\sigma > \sqrt{\frac{\lambda_{\max}(H)}{2} + 1}$ and $\lambda_{\min}(\mathcal{Q}) - 2max(\xi_1, \xi_2)\|\mathcal{P}\| > 0$. Then, under the quantized protocol (5), system (21) has the following convergence properties:*

(1): With uniform quantizers, the agents converge to a ball of radius

$$\left|z_{\mathcal{T}}\right| \leq \frac{2\sqrt{2nL}\delta_u \left\|\mathcal{P}\mathcal{L}_{\mathcal{T}1}\right\|}{\lambda_{\min}(\mathcal{Q}) - 2max(\xi_1, \xi_2)\|\mathcal{P}\|} \tag{22}$$

which is centred at the agreement equilibrium;

(2): With logarithmic quantizers, the agents converge exponentially to the desired agreement equilibrium, provided that δ_l satisfies

$$\delta_l < \frac{\lambda_{\min}(\mathcal{Q}) - 2max(\xi_1, \xi_2)\|\mathcal{P}\|}{2\|\mathcal{P}\mathcal{L}_{\mathcal{T}1}\|\|R^T\|}. \tag{23}$$

The estimated trajectories of the edge Laplacian dynamics (21) *is described as*

$$|z_{\mathcal{T}}(t)| \leq \frac{\lambda_{max}(\mathcal{P})}{\lambda_{min}(\mathcal{P})} e^{-\frac{\pi}{\lambda_{\max}(\mathcal{P})}t} |z_{\mathcal{T}}(0)| \ \ for\ t \geq 0$$

with

$$\pi = \lambda_{\min}(\mathcal{Q}) - 2max(\xi_1, \xi_2)\|\mathcal{P}\| - 2\delta_l \|\mathcal{P}\mathcal{L}_{\mathcal{T}1}\| \left\|R^T\right\|.$$

PROOF. For the edge Laplacian dynamics (21), we choose the following Lyapunov function candidate:

$$V(z_{\mathcal{T}}) = z_{\mathcal{T}}^T \mathcal{P} \otimes I_n z_{\mathcal{T}} \tag{24}$$

in which

$$\mathcal{P} = \begin{bmatrix} \sigma H & H \\ H & \sigma H \end{bmatrix}$$

where H can be obtained from (17) and $\mathcal{P}$ is positive definite while choosing $\sigma > 1$.

By taking the derivative of (24) along the trajectories of (21), we have

$$\begin{aligned} \dot{V}(z_{\mathcal{T}}) = & - z_{\mathcal{T}}^T \mathcal{Q} \otimes I_n z_{\mathcal{T}} + z_{\mathcal{T}}^T \mathcal{P} \otimes I_n \mathcal{F}_{\mathcal{T}} + \mathcal{F}_{\mathcal{T}}^T \mathcal{P} \otimes I_n z_{\mathcal{T}} \\ & + z_{\mathcal{T}}^T \mathcal{P}\mathcal{L}_{\mathcal{T}1} \otimes I_n \omega + \omega^T \mathcal{L}_{\mathcal{T}1}^T \mathcal{P} \otimes I_n z_{\mathcal{T}} \end{aligned}$$

in which

$$\mathcal{Q} = -\left(\mathcal{P}\mathcal{L}_{\mathcal{T}} + \mathcal{L}_{\mathcal{T}}^T \mathcal{P}\right) = \begin{bmatrix} \sigma^2 I_{N-1} & \sigma^3 I_{N-1} - \sigma H \\ \sigma^3 I_{N-1} - \sigma H & \sigma^4 I_{N-1} - 2H \end{bmatrix}.$$

By selecting

$$\sigma > \sqrt{\frac{\lambda_{\max}(H)}{2} + 1}$$

then $\mathcal{Q}$ is positive definite according to Schur complements theorem [23].

In the meanwhile, we notice that

$$\begin{aligned}|\mathcal{F}_{\mathcal{T}}| = \left|E_{\mathcal{T}}^{T} \otimes I_n F\right| &\leq \sqrt{{\xi_1}^2 \left|x_{\mathcal{T}}\right|^2 + {\xi_2}^2 \left|v_{\mathcal{T}}\right|^2} \\ &\leq max\left(\xi_1, \xi_2\right) \left|z_{\mathcal{T}}\right|.\end{aligned} \tag{25}$$

Considering the uniform quantizer, we have the upper bound of the quantization errors as

$$|\omega| \leq \sqrt{2nL}\delta_u. \tag{26}$$

By combining (25) and (26), one can obtain

$$\begin{aligned}\dot{V}\left(z_{\mathcal{T}}\right) \leq& -\lambda_{\min}\left(\mathcal{Q}\right)\left|z_{\mathcal{T}}\right|^2 + 2max\left(\xi_1, \xi_2\right)\|\mathcal{P}\|\left|z_{\mathcal{T}}\right|^2 \\ &+ 2\sqrt{2nL}\delta_u \left\|\mathcal{P}\mathcal{L}_{\mathcal{T}1}\right\|\left|z_{\mathcal{T}}\right| \\ &= \left|z_{\mathcal{T}}\right|\Big(-\lambda_{\min}\left(\mathcal{Q}\right) + 2max\left(\xi_1, \xi_2\right)\|\mathcal{P}\|\Big)\left|z_{\mathcal{T}}\right| \\ &+ 2\sqrt{2nL}\delta_u \left\|\mathcal{P}\mathcal{L}_{\mathcal{T}1}\right\|\Big).\end{aligned}$$

Clearly, the edge agreement can be reached and the radius of the agreement neighbourhood is as (22).

Considering that $z = \begin{bmatrix} R^T & \mathbf{0}_{(N\times L-N+1)} \\ \mathbf{0}_{(N\times L-N+1)} & R^T \end{bmatrix} z_{\mathcal{T}}$, then for the logarithmic quantizer $\left|Q_u\left(a\right) - a\right| \leq \delta_l \left|a\right|$, we have

$$|\omega| \leq \delta_l \left|z\right| \leq \delta_l \left\|R^T\right\| \left|z_{\mathcal{T}}\right|. \tag{27}$$

Combining (25) and (27), we have

$$\begin{aligned}\dot{V}\left(z_{\mathcal{T}}\right) \leq& -\lambda_{\min}\left(\mathcal{Q}\right)\left|z_{\mathcal{T}}\right|^2 + 2max\left(\xi_1, \xi_2\right)\|\mathcal{P}\|\left|z_{\mathcal{T}}\right|^2 \\ &+ 2\delta_l \left\|\mathcal{P}\mathcal{L}_{\mathcal{T}1}\right\|\left\|R^T\right\|\left|z_{\mathcal{T}}\right|^2 \\ &= -\pi\left|z_{\mathcal{T}}\right|^2.\end{aligned}$$

Obviously, while (23) is satisfied, the edge Laplacian dynamics (21) converges exponentially to the desired agreement equilibrium.

Moreover, one can obtain that

$$\dot{V}\left(z_{\mathcal{T}}(t)\right) \leq -\pi\left|z_{\mathcal{T}}\right|^2 \leq -\frac{\pi}{\lambda_{\max}(\mathcal{P})} V\left(z_{\mathcal{T}}(t)\right).$$

By applying the Comparison Lemma [31], we have

$$V\left(z_{\mathcal{T}}(t)\right) \leq e^{-\frac{\pi}{\lambda_{\max}(\mathcal{P})}t} V\left(z_{\mathcal{T}}(0)\right).$$

Then we can provide the following estimates of the convergence rate for the closed-loop multi-agent system

$$|z_{\mathcal{T}}(t)| \leq \frac{\lambda_{max}(\mathcal{P})}{\lambda_{min}(\mathcal{P})} e^{-\frac{\pi}{\lambda_{\max}(\mathcal{P})}t} |z_{\mathcal{T}}(0)| \text{ for } t \geq 0. \tag{28}$$

Remark 12 *From equation* (22)*, one can see that the upper bound of the convergence errors for the uniform quantizers depend on* δ_u*. More precisely, the radius of the convergence neighbourhood trends to zero as the quantization interval decreasing. For logarithmic quantizers, the corresponding convergence rate of the quantized system depend on* δ_l*, i.e, the coarser the logarithmic quantizer is, more time it takes for the quantized system to converge. As the convergence time can be used to quantify the performance of the quantized control law, we provide the estimation of the upper bound of the convergence time based on* (28) *as*

$$T = -\frac{\lambda_{\max}(\mathcal{P})}{\pi} \ln \frac{\lambda_{\min}(\mathcal{P})\, r}{\lambda_{\max}(\mathcal{P})\, |z_{\mathcal{T}}(0)|}$$

where $r > 0$ *denotes the expected radius of the agreement errors.*

5 Simulation

Consider multi-agent system consisting of a group of 5 agents associated with a quasi-strongly connected graph as shown in Fig 4, where $e_1, e_2, e_3, e_4 \subset \mathcal{G}_{\mathcal{T}}$ and $e_5 \subset \mathcal{G}_c$.

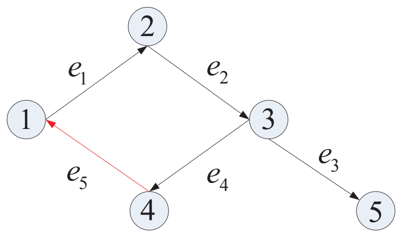


Fig. 4. Quasi-strongly connected graph of 5 agents.

The dynamics of the i-th agent is described as (3) and (4), in which $x_i(t), v_i(t)$ $u_i(t) \in \mathbb{R}^3$. Let $x(m,:) = [x_{1m}(t), \cdots, x_{5m}(t)]$ and $v(m,:) = [v_{1m}(t), \cdots, v_{5m}(t)]$ denote the column vector of the m-th (m =1,2,3) variable of $x_i(t), v_i(t)$. The inherent nonlinear dynamics $f(x_i(t), v_i(t), t) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is described by Chua's circuit

$$f(x_i(t), v_i(t), t) = (\zeta(-v_{i1}(t) + v_{i2}(t) - l(v_{i1}(t))), \\ \tau(v_{i1}(t) - v_{i2}(t) + v_{i3}(t)), -\chi v_{i2}(t))^T$$

where $l(v_{i1}(t)) = bv_{i1}(t) + 0.5(a-b)(|v_{i1}(t)+1| - |v_{i1}(t)-1|)$. The system is chaotic when $\zeta = 0.01$, $\tau = 0.001$, $\chi = 0.018$, $a = -4/3$ and $b = -3/4$. In view of Assumption 2, by calculation one can obtain $\xi_1 = 0$ and $\xi_2 = 4.3871 \times 10^{-3}$ [23].

Through a simple calculation, we can obtain

$$T = \begin{pmatrix} -1.00 \\ -1.00 \\ 0.00 \\ -1.00 \end{pmatrix}, \quad R = \begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.00 & -1.00 \\ 0.00 & 1.00 & 0.00 & 0.00 & -1.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & -1.00 \end{pmatrix}.$$

Suppose that the weighted diagonal matrix is $\mathcal{W} = diag\{0.12, 0.24, 0.44, 0.43, 0.09\}$. By choosing $\sigma = 1.64$, we have

$$\hat{L}_e = \begin{pmatrix} 0.21 & 0.09 & 0.00 & 0.09 \\ -0.12 & 0.24 & 0.00 & 0.00 \\ 0.00 & -0.24 & 0.44 & 0.00 \\ 0.00 & -0.24 & 0.00 & 0.43 \end{pmatrix}, \quad \hat{L}_{\mathcal{O}} = \begin{pmatrix} 0.12 & 0.00 & 0.00 & 0.00 & -0.09 \\ -0.12 & 0.24 & 0.00 & 0.00 & 0.00 \\ 0.00 & -0.24 & 0.44 & 0.00 & -0.00 \\ 0.00 & -0.24 & 0.00 & 0.43 & 0.00 \end{pmatrix}.$$

Solving the Lyapunov equation (17) leads to

$$H = \begin{pmatrix} 2.47 & 0.16 & 0.07 & -0.26 \\ 0.16 & 2.86 & 0.39 & 0.45 \\ 0.07 & 0.39 & 1.14 & -0.01 \\ -0.26 & 0.45 & -0.01 & 1.22 \end{pmatrix}.$$

Directed calculation yields $\lambda_{max}(\mathcal{P}) = 8.098$, $\lambda_{min}(\mathcal{P}) = 0.6157$, $\|\mathcal{P}\| = 8.098$, $\|\mathcal{P}\mathcal{L}_{\mathcal{T}1}\| = 6.7121$ and $\left\|R^T\right\| = 2$.

5.1 Uniform Quantizer

First, we consider the quantized protocol (5) with the following uniform quantizer as the one used in [15]

$$q_u(x) = \delta_u \left(\left\lfloor \frac{x}{\delta_u} \right\rfloor + \frac{1}{2} \right).$$

The simulation results with $\delta_u = 1$ are shown in Fig. 5, from which we can see that $x_e(t)$ and $v_e(t)$ indeed converge to a small neighbourhood near the equilibrium points. To show the effect of δ_u on the agreement errors $|e_{ss}|$, we further take $\delta_u = 0.01, 0.1, 2$ and 3 to run the simulation. The results in Tab.1 shows that $|e_{ss}|$ trends to zero when $\delta_u \to 0$, which accords to Theorem 11.

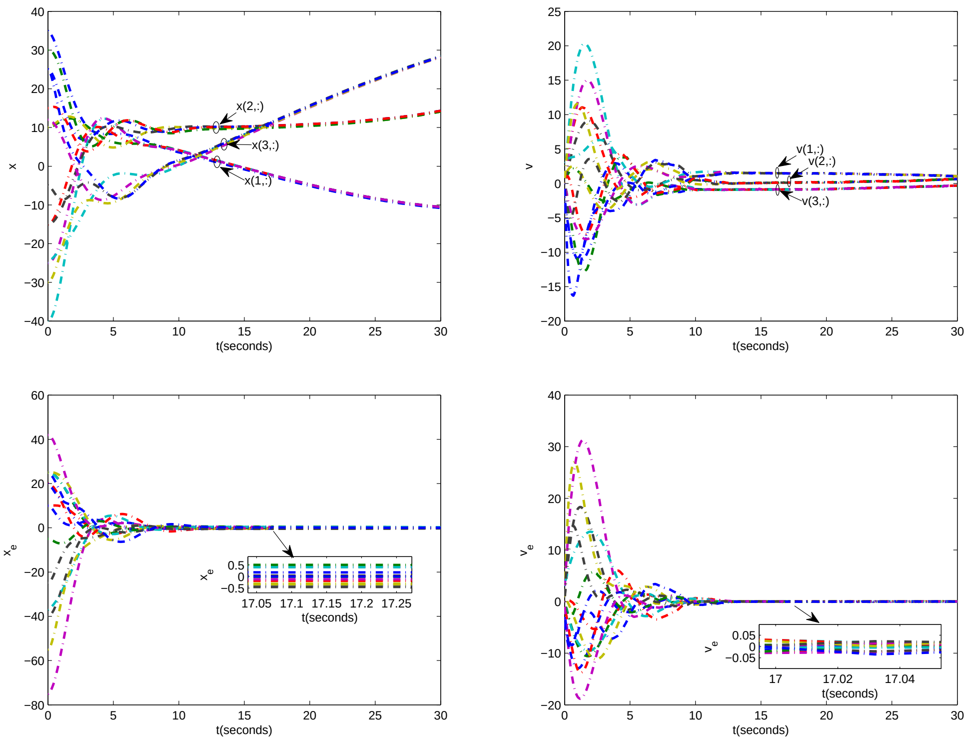


Fig. 5. Edge agreement under uniform quantizer with $\delta_u = 1$.

5.2 Logarithmic Quantizer

Next, we apply the following logarithmic quantizer to the quantized protocol (5):

$$q_l = \begin{cases} e^{q_u(\ln x)} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -e^{q_u(\ln(-x))} & \text{if } x < 0 \end{cases}$$

where $\delta_l = 1 - e^{-\delta_u}$ [15]. To satisfy the stability constraints (23), we require $\delta_l < 0.0301$. The simulation results with $\delta_u = 0.01$ and $\delta_l = 1 - e^{-0.01} = 0.01$ are shown in Fig.6, from which we can see that edge agreement is indeed achieved as well as $x_i(t)$ and $v_i(t)$ reach the desired agreement values. The

Table 1
The effect of δ_u on the agreement errors.

δ_u	0.01	0.1	1	2	3
$\lvert e_{ss}\rvert$	0.005	0.05	0.62	1.98	4.45

estimation of the convergence rate is given as

$$|\psi| = \frac{\lambda_{max}(\mathcal{P})}{\lambda_{min}(\mathcal{P})} e^{-\frac{\pi}{\lambda_{\max}(\mathcal{P})}t} \left|z_{\mathcal{T}}(0)\right| \text{ for } t \geq 0$$

with $\pi = 0.5387$. From Fig.7, one can see that $|z_{\mathcal{T}}|$ exponentially converge to the origin. To illustrate the effects of the quantized interval on the convergence rate, we take $\delta_l = 0.01$ ($\delta_u = 0.01$), and the simulation consumes 11.88 time units for the quantized system to converge. Further, when we take $\delta_l = 0.02$ ($\delta_u = 0.0202$), it takes 12.18 time units to converge and while choosing $\delta_l = 0.03$ ($\delta_u = 0.0305$), it takes 12.45 time units to reach agreement. Obviously, these results keep align with our analysis. Finally, while choosing $\delta_l = 0.9933$ ($\delta_u = 5$), which breaks the condition (23), the corresponding results are shown in Fig.8, where agreement can not be achieved.

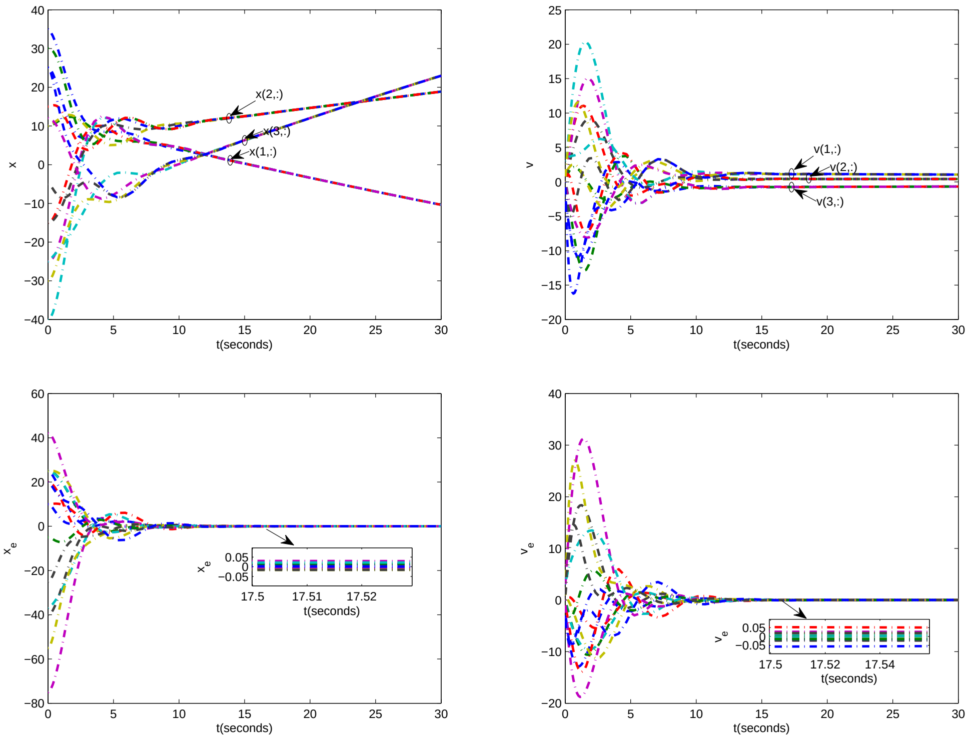


Fig. 6. Edge agreement under logarithmic quantizer with $\delta_l = 0.01$.

6 Conclusions

In this paper, we proposed a general concept of directed edge Laplacian with its algebraic properties. Based upon the new graph-theoretic tool, we derived a model reduction representation of the edge agreement model, which allows a convenient analysis of multi-agent system. The edge agreement of second-order

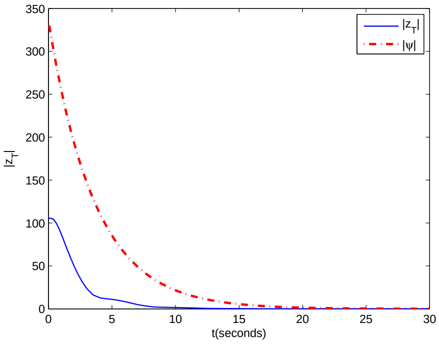


Fig. 7. The convergence estimation of $|z_{\mathcal{T}}(t)|$.

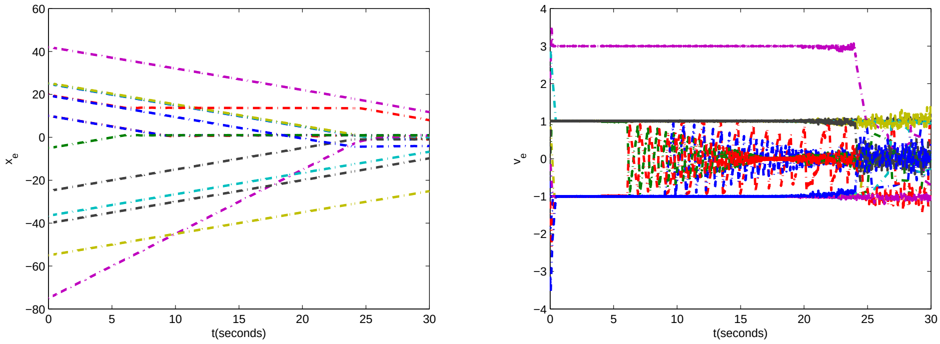


Fig. 8. The trajectories of x_e and v_e are divergence with $\delta_l = 0.9933$.

nonlinear multi-agent system under quantized measurements was studied. Unlike previous works, we provided explicit results on the edge agreement errors and stability constraints with the quantization interval. Specifically, we provided the explicit upper bound of the radius of the agreement neighbourhood for the uniform quantizers, which indicates that the radius increases with the quantization interval. While for the logarithmic quantizers, we pointed out that the agents converge exponentially to the desired agreement equilibrium. Besides, we also provided the estimates of the convergence rate.

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