Asian Journal of Control, Vol. 17, No. 4, pp. 1444–1452, July 2015
Published online 3 October 2014 in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.1001

–Brief Paper–

"KRONECKER BASIS" BASED AVERAGE CONSENSUS ANALYSES FOR HIGH-ORDER LINEAR MULTI-AGENT SYSTEMS WITH MULTIPLE TIME DELAYS

Huangyong Zhu, Yirui Cong, Xiangke Wang, Daibing Zhang, and Qingjie Zhang

ABSTRACT

This paper investigates the average consensus for multi-agent systems governed by high-order linear dynamics with multiple time delays. Necessary and sufficient conditions for high-order average consensus under balanced communication topology are provided by using a newly defined mathematical concept – the Kronecker basis. Furthermore, previous studies for average consensus governed by first-order, or high-order integrator can be regarded as special cases of our results. Simulation results are employed to demonstrate the effectiveness of our results for high-order average consensus.

Key Words: Kronecker basis, average consensus, high-order multi-agent systems, time delay.

I. INTRODUCTION

In recent years, much attention has been drawn to consensus in multi-agent systems [1–12], due to its applications which range widely from flocking [13], rendezvous [14], formation control [15], fusion estimation [16], and collaborative decision-making [17] to coupled oscillator synchronization [18]. Average consensus, in which all states of the system asymptotically achieved the average of the initial states, plays a pivotal role in the studies of consensus, even though it is only a special case. It is proven that the necessary and sufficient condition for a multi-agent system governed by single-integrator dynamics to achieve globally asymptotic average consensus is that whose communicating graph is balanced [19]. Therefore, it is reasonable to employ the disagreement vector method to investigate the average consensus with communication delays, for example [20–26].

However, the aforementioned average consensus studies are confined to first-order multi-agent systems. For high-order systems, very few studies have been conducted. Even though [27] and [28] investigated the average consensus for double-integrator multi-agent system, the average value just held in one state (the location state for double-integrator system) rather than both in location and velocity states. For the average values in each state vector's components, [29] did corresponding research for lth-order chain integrator multi-agent systems by using the disagreement vector method, however, this method might not provide a necessary and sufficient condition for achieving high-order average consensus. As a result, this leads to conservativeness in later criteria in [29]. To our best knowledge, average consensus analyses for high-order multi-agent systems have not been fully investigated.

Theoretically, the convergence analyses in average consensus for high-order multi-agent systems are more challenging than those for first-order systems. This is partly due to the fact that the properties of linear algebra are mostly known so that the dynamics for the whole first-order systems can be well analyzed. For high-order systems, the corresponding analyses are in tensor algebra. However, tensor algebra theory might lack some important conclusions compared to linear algebra theory, which makes it difficult to acquire a relatively clear analysis in high-order average consensus.

With the inspiration of [30] and [31], we propose a new mathematical concept – the Kronecker basis – which is a generalized basis for linear space. Based on that, an explicit framework for average consensus for high-order systems is developed correspondingly. With further analyses, we get the necessary and sufficient condition for high-order average consensus in balanced communication topology. Unlike the first-order average consensus which can be globally asymptotic, the high-order average consensus holds when the initial states for these agents are in some certain subspace. To be more specific, we

Manuscript received June 28, 2013; revised April 13, 2014; accepted May 17, 2014.

Huangyong Zhu, Yirui Cong, Xiangke Wang (corresponding author, e-mail: xkwang@nudt.edu.cn) and Daibing Zhang are with the College of Mechatronics and Automation, National University of Defense Technology, Changsha 410073, China.

Qingjie Zhang is with Control Department, Aviation University of Air Force, Changchun 130022, China.

This work is supported by the Research Project of National University of Defense Technology under JC13-03-02.

prove in this paper that it is the null space of the system matrix.

Compared with the existing results on average consensus, the contributions of this paper focus mainly on three aspects. First, we define the Kronecker basis and propose some corresponding conclusions so that the high-order average consensus problems can be described explicitly. Second, with the aid of the new mathematical notions, we give the necessary and sufficient condition for multi-agent systems governed by high-order linear dynamics under balanced communication topology with multiple time delays. This necessary and sufficient condition, to our best knowledge, has never been proven in published papers. It is worth pointing out that this necessary and sufficient condition regards the previous work on average consensus with first-order and lth-order chain integrator dynamics as special cases.

This rest of the paper is organized as follows. In Section II, we introduce the necessary definitions and results in graph theory, and the problem description is also given here. Section III presents the main theoretical results including the Kronecker basis and the necessary and sufficient condition for high-order average consensus. Simulation results will validate our theoretical results in Section IV and concluding remarks are stated in Section V.

Notation. Throughout this paper, the notation $\otimes$ denotes the Kronecker product; $\mathbf{I_n}$ is an identity matrix with $n\times n$ dimensions; $\mathbf{0}$ can be an appropriate dimensions zero matrix or vector; $\mathbf{e}_j$ is a proper dimensional vector which the jth component is 1 and others are 0; $span\{\mathbf{D}\}$ means the space spanned with the columns of $\mathbf{D}$.

II. PRELIMINARIES AND PROBLEM DESCRIPTION

2.1 Preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed simple graph (no self-loops or repeated edges) of order $N(N > 1)$ where $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ denotes the nodes set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edges set, and $\mathcal{A} = [\tilde{a}_{ij}] \in \mathbb{R}^{N\times N}$ is the weighted adjacency matrix with $\tilde{a}_{ij} \geq 0$, where $\tilde{a}_{ij} > 0$ if and only if edge $(v_j, v_i) \in \mathcal{E}$. As $\mathcal{G}$ is simple, $\tilde{a}_{ii} = 0$ holds for all $i \in \{1, 2, \cdots, N\}$. The neighbors of v_i is defined as $\mathcal{N}_i = \{v_j \in \mathcal{V} | (v_j, v_i) \in \mathcal{E}\}$. The in-degree of the node v_i is defined as $d_{in}(v_i) = \sum_{v_j\in\mathcal{N}_i} \tilde{a}_{ij}$. The degree matrix of $\mathcal{G}$ is a diagonal matrix $\mathcal{D} = [d_{ij}]$, where $d_{ij} = 0\,(i \neq j)$, $d_{ii} = d_{in}(v_i)$. Then the Laplacian of $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}] = \mathcal{D} - \mathcal{A}$. The union of a collection of simple graphs $\{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}$, with the same node set $\mathcal{V}$, is defined as the graph $\bar{\mathcal{G}}_{1-m}$ with the node set $\mathcal{V}$ and edge set equaling the union of the edge sets of all of the graphs in the collection.

Definition 1. Balanced matrix [25]. A square matrix $\bar{\mathbf{B}} \in \mathbb{R}^{N\times N}$ is said to be balanced if and only if $\bar{\mathbf{B}}\mathbf{1}_N = \mathbf{0}$ and $\mathbf{1}_N^T\bar{\mathbf{B}} = \mathbf{0}$.

Lemma 1 [19]. If a directed graph $\mathcal{G}$ is balanced (its Laplacican matrix $\mathcal{L}$ is a balanced matrix), then $\mathcal{L}$ has a left-eigenvector $\mathbf{1}_N$ associated with eigenvalue 0.

Lemma 2 [25,29]. If $\tilde{\mathbf{Q}}$ is the matrix of eigenvectors of the Laplacian matrix of the complete graph, then it is an orthogonal matrix. Given any balanced matrix $\bar{\mathbf{B}} \in \mathbb{R}^{N\times N}$, the following holds:

$$\tilde{\mathbf{Q}}^T\bar{\mathbf{B}}\tilde{Q} = \mathbf{F} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{F}}_{(N-1)\times(N-1)} \end{bmatrix}$$

2.2 Problem description

In this paper, we consider a group of agents whose dynamics are described by high-order linear systems in continuous-time domain:

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}\mathbf{x}_i(t) + \mathbf{B}\mathbf{u}_i(t), i = 1, 2, \ldots, N \tag{1}$$

where $\mathbf{x}_i(t) = [x_i^{(1)}(t) \ldots x_i^{(n)}(t)]^T$ is the state of agent i and $\mathbf{u}_i(t) \in \mathbb{R}^p$ is the consensus protocol. ($\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times p}$)

Consensus is achieved if and only if $\forall i \neq j$, $\lim\limits_{t\to+\infty} \left\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\right\| = 0$. Furthermore, average consensus is achieved if $\lim\limits_{t\to+\infty} \mathbf{x}_i(t) = Ave(\mathbf{x}(0)) = \frac{1}{N}\sum_{i=1}^N \mathbf{x}_i(0)$ is added. In order to get an explicit description of average consensus, we provide the definition of average consensuability and the average consensus set for any given protocol as follows.

Definition 2. Average consensusability and average consensus set. A multi-agent systems is said to be average consensuable at $\mathbf{x}(0)$ with a given protocol, if for the initial state $\mathbf{x}(0)$, the given protocol $\mathbf{u}_i(t), i = 1, 2, \ldots, N$ can make the multi-agent systems achieve average consensus. The average consensus set is constituted by all $\mathbf{x}(0)$ at which the multi-agent systems is average consensuable with given protocol. If the average consensus set contains arbitrary $\mathbf{x}(0)$, then the multi-agent systems is globally average consensuable for given protocol.

In this paper, we consider a class of protocols as follows:

$$\mathbf{u}_i(t) = \mathbf{K}_1\mathbf{x}_i(t) + \mathbf{K}_2 \times \sum_{v_j \in \mathcal{N}_i} \tilde{a}_{ij} \times \left[\mathbf{x}_j(t - \tau_{ij}(t)) - \mathbf{x}_i(t - \tau_{ij}(t))\right] \tag{2}$$

where $\tau_{ij}(t)$ represents the delays in transmission between the ith agent and the jth agent. Note that protocol (2) takes many protocols as special cases (*e.g.* [20–26,29]). More importantly, if the transmission delays are uniform, then $\tau_{ij}(t) = \tau(t)$ for any $i,j = 1,2,\ldots,N$; if the multi-agent systems are without transmission delays, then $\tau_{ij}(t) = 0$ for any $i,j = 1,2,\ldots,N$. Therefore, protocol (2) can describe those consensus protocols with/without single/multiple time-varying/constant communication delays.

Suppose that time-delays are nonuniform, denoted by $\tau_q(t) \in \left\{\tau_{ij}(t) | i,j = 1,2,\ldots,N\right\}$ $(q = 1,2,\ldots,m)$. In addition, we assume

$$\tau_q(t) \in [0, h_q], q = 1,2,\ldots,m \tag{3}$$

with their derivatives met

$$\dot{\tau}_q(t) \in [0, \mu_q], q = 1,2,\ldots,m \tag{4}$$

and $\bar{h} = \max\limits_{1 \le q \le m} \left\{h_q\right\}$, $\bar{\mu} = \max\limits_{1 \le q \le m} \left\{\mu_q\right\}$

Let $\mathbf{x}(t) = \left[\mathbf{x}_1^T(t) \ldots \mathbf{x}_N^T(t)\right]^T$. With protocol (2) and formula (3) (4), dynamics (1) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}}(t) = \left[\mathbf{I}_N \otimes (\mathbf{A} + \mathbf{B}\mathbf{K}_1)\right]\mathbf{x}(t) \\ -\sum\limits_{q=1}^{m} (\mathcal{L}_q \otimes \mathbf{B}\mathbf{K}_2)\mathbf{x}(t - \tau_q(t)), t \in [0, +\infty) \\ \mathbf{x}(t) = \phi(t), t \in [-\bar{h}, 0) \end{cases} \tag{5}$$

where $\phi(t)$ is a continuously differentiable initial function.

III. MAIN RESULTS

In this section, a new mathematical concept – the Kronecker basis – is proposed so that we can acquire an explicit description for average consensus problems of high-order linear multi-agent systems. Afterwards, based on the description of high-order average consensus problems, the necessary and sufficient conditions for high-order average consensus problems with multiple time-varying delays are established.

3.1 Kronecker basis

In order to provide the concept for the Kronecker basis, first we give some preliminaries.

Definition 3. Kronecker combination. A Kronecker combination of j vectors $\tilde{q}_1, \ldots, \tilde{q}_j \in \mathbb{F}^N$ is a vector in linear space $\mathbb{F}^{Nn}$, with form

$$\tilde{q}_1 \otimes \tilde{\alpha}_1 + \ldots + \tilde{q}_j \otimes \tilde{\alpha}_j, \quad \tilde{\alpha}_1, \ldots, \tilde{\alpha}_j \in \mathbb{F}^n$$

Definition 4. Kronecker independence and dependence. The set of vectors $\tilde{q}_1, \ldots, \tilde{q}_j \in \mathbb{F}^N$ is Kronecker independent in linear space $\mathbb{F}^{Nn}$ whenever the only solution for $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_j \in \mathbb{F}^n$ in the homogenous equation

$$\tilde{q}_1 \otimes \tilde{\alpha}_1 + \ldots + \tilde{q}_j \otimes \tilde{\alpha}_j = \mathbf{0} \tag{6}$$

is the trivial solution $\tilde{\alpha}_1 = \ldots = \tilde{\alpha}_j = \mathbf{0}$. Otherwise, $\tilde{q}_1, \ldots, \tilde{q}_j \in \mathbb{F}^N$ is Kronecker dependent in $\mathbb{F}^{Nn}$.

Lemma 3. The set of vectors $\tilde{q}_1, \ldots, \tilde{q}_j \in \mathbb{F}^N$ is Kronecker independent in linear space $\mathbb{F}^{Nn}$ if and only if $\tilde{q}_1, \ldots, \tilde{q}_j$ are linear independent in linear space $\mathbb{F}^N$.

Proof. Necessity. If $\tilde{q}_1, \ldots, \tilde{q}_j \in \mathbb{F}^N$ is Kronecker independent in linear space $\mathbb{F}^{Nn}$, $\tilde{\alpha}_1 = \ldots = \tilde{\alpha}_j = \mathbf{0}$ and the solution is unique. Let's assume $\tilde{q}_1, \ldots, \tilde{q}_j$ is not linear independent, then we have

$$k_1\tilde{q}_1 + \ldots + k_j\tilde{q}_j = \mathbf{0} \tag{7}$$

where not all $k_1, \ldots, k_j$ are zero. Now we set $\tilde{\boldsymbol{\beta}}_i = k_i\mathbf{1}_N$, $1 \le i \le j$, then we can derive

$$\tilde{q}_1 \otimes \tilde{\boldsymbol{\beta}}_1 + \ldots + \tilde{q}_j \otimes \tilde{\boldsymbol{\beta}}_j = \mathbf{0} \tag{8}$$

which contradicts the uniqueness in Definition 4. Therefore, the $\tilde{q}_1, \ldots, \tilde{q}_j$ is linear independent.

Sufficiency. The sufficiency condition is obvious.

Now is the right point to give the definition of the Kronecker basis.

Definition 5. Kronecker basis. Let $\tilde{q}_1, \ldots, \tilde{q}_j \in \mathbb{F}^N$, if the two conditions hold:

(a) $\tilde{q}_1, \ldots, \tilde{q}_j$ are Kronecker independent (according to Lemma 3, it is equivalent to linear independent);
(b) $\forall \mathbf{x} \in \mathbb{F}^{Nn}$ can be a Kronecker combination of $\tilde{q}_1, \ldots, \tilde{q}_j$.

Then the vectors set $\tilde{q}_1, \ldots, \tilde{q}_j$is called a Kronecker basis in linear space $\mathbb{F}^{Nn}$.

Lemma 4. Existence and uniqueness of Kronecker coordinate. $\forall \mathbf{x} \in \mathbb{F}^{Nn}$, for given linear-independent $\tilde{q}_1, \ldots, \tilde{q}_j \in \mathbb{F}^N$, formula (9) holds:

$$\mathbf{x} = \tilde{q}_1 \otimes \tilde{\alpha}_1 + \tilde{q}_2 \otimes \tilde{\alpha}_2 + \ldots + \tilde{q}_N \otimes \tilde{\alpha}_N \tag{9}$$

where $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\boldsymbol{\alpha}}_N \in \mathbb{F}^n$ (Kronecker coordinate) uniquely exist.

Proof. $\mathbf{x}$ can be written as $\mathbf{x} = \left[\mathbf{x}_1^T \ldots \mathbf{x}_N^T\right]^T$, where $\mathbf{x}_i = \left[x_i^{(1)} \cdots x_i^{(n)}\right]^T \in \mathbb{F}^n$. Let $\tilde{\mathbf{x}}_j = \left[x_1^{(j)} \cdots x_N^{(j)}\right]^T$, and it can be linear expressed by $\tilde{q}_1, \ldots, \tilde{q}_N$ (*i.e.* $\mathbf{x}_j = \sum_{i=1}^{N} \tilde{\alpha}_i^{(j)} \tilde{q}_i$). According to the linear algebra theory, $\tilde{\alpha}_i^{(j)}, i = 1 \ldots N$ uniquely exist. Note that $\mathbf{x} = \sum_{j=1}^{N} \tilde{\mathbf{x}}_j \otimes \mathbf{e}_j$ (the dimension of $\mathbf{e}_j$ is n), (10) holds:

$$\begin{aligned} \mathbf{x} &= \sum_{j=1}^{n} \tilde{\mathbf{x}}_j \otimes \mathbf{e}_j = \sum_{j=1}^{n} \sum_{i=1}^{N} \tilde{\alpha}_i^{(j)} \tilde{q}_i \otimes \mathbf{e}_j \\ &= \sum_{i=1}^{N} \tilde{q}_i \otimes \left(\sum_{j=1}^{n} \tilde{\alpha}_i^{(j)} \mathbf{e}_j \right) = \sum_{i=1}^{N} \tilde{q}_i \otimes \tilde{\alpha}_i \end{aligned} \tag{10}$$

Because of the unique existence for each component in $\tilde{\alpha}_i = \left[\tilde{\alpha}_i^{(1)}, \tilde{\alpha}_i^{(2)}, \ldots, \tilde{\alpha}_i^{(n)}\right]^T$, $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_N \in \mathbb{F}^n$ in (10) uniquely exist.

Lemma 4 implies that condition (b) in Definition 5 can be omitted. More importantly, for linear space $\mathbb{F}^{Nn}$, if the number of the Kronecker basis is exactly equal to N (Not more than N), the Kronecker coordinates exist uniquely.

Remark 1. If $n = 1$ then $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_N$ become scalars, and all the conclusions in this section can be found in linear algebra theory.

Definition 6. Kronecker span. For a set of vectors $\tilde{q}_1, \ldots, \tilde{q}_j$, the subspace

$$\begin{aligned} &\mathit{kronspan}\left\{\tilde{q}_1, \ldots, \tilde{q}_j\right\} \\ &\quad := \left\{ \mathbf{x} \in \mathbb{F}^{Nn} | \mathbf{x} = \sum_{i=1}^{j} \tilde{q}_i \otimes \tilde{\alpha}_i, \tilde{\alpha}_i \in \mathbb{F}^n \right\} \end{aligned}$$

generated by forming all Kronecker combinations of vectors from $\tilde{q}_1, \ldots, \tilde{q}_j$ is called the space Kronecker spanned by $\tilde{q}_1, \ldots, \tilde{q}_j$.

3.2 Necessary and sufficient condition for high-order average consensus under balanced communi-cation topology

With the help of the Kronecker basis, an explicit description of high-order average consensus can be derived. Considering average consensus is based on consensus, we describe consensus by using the Kronecker basis first.

Recall that $\mathbf{x}(t) = \left[\mathbf{x}_1^T(t) \ldots \mathbf{x}_N^T(t)\right]$ is the state of (5), and clearly $\mathbf{x}(t)$ can be a Kronecker combination represented by:

$$\mathbf{x}(t) = \tilde{q}_1 \otimes \tilde{\alpha}_1(t) + \sum_{i=2}^{N} \tilde{q}_i \otimes \tilde{\alpha}_i(t) \tag{11}$$

where $\tilde{q}_1 = \frac{1}{\sqrt{N}} \mathbf{1}_N$ and $\tilde{q}_1, \tilde{q}_2, \ldots, \tilde{q}_N$ are linear independent, and according to Lemma 4, $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_N$ are unique. Then based on Lemma 4, a necessary and sufficient condition for high-order consensus can be established.

Lemma 5. Multi-agent systems achieve consensus if and only if

$$\lim_{t\to\infty} \tilde{\alpha}_i(t) = \mathbf{0}, i = 2, 3, \ldots, N \tag{12}$$

Proof. Sufficiency. If $\lim_{t\to\infty} \tilde{\alpha}_i(t) = \mathbf{0}, i = 2, 3, \ldots, N$, then $\lim_{t\to+\infty} \left\|\mathbf{x}(t) - \tilde{q}_1 \otimes \tilde{\alpha}_1(t)\right\| = 0$ (see (11)). Hence, $\forall i \neq j$, $\lim_{t\to+\infty} \left\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\right\| = 0$ and consensus is achieved.

Necessity. If the multi-agent systems achieve consensus, then $\forall i \neq j$, $\lim_{t\to+\infty} \left\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\right\| = 0$. Due to the completeness of $\mathbb{R}^n$, the following formula holds:

$$\lim_{t\to\infty} \left[\mathbf{x}(t) - \tilde{q}_1 \otimes \mathbf{x}_1(t)\right] = \mathbf{0} \tag{13}$$

Note that $\mathbf{x}(t)$ can be written as a Kronecker combination as in (11); we have

$$\lim_{t\to\infty} \left[\tilde{q}_1 \otimes \left(\tilde{\alpha}_1(t) - \mathbf{x}_1(t)\right) + \sum_{i=2}^{N} \tilde{q}_i \otimes \tilde{\alpha}_i(t) \right] = \mathbf{0} \tag{14}$$

According to Lemma 4 the Kronecker coordinate is unique:

$$\begin{aligned} &\lim_{t\to\infty} \left(\tilde{\alpha}_1(t) - \mathbf{x}_1(t)\right) = \mathbf{0}, \\ &\lim_{t\to\infty} \tilde{\alpha}_i(t) = \mathbf{0}, i = 2, 3, \ldots, N \end{aligned} \tag{15}$$

With the help of the Kronecker span, we can define the consensus subspace (CS) and the consensus complement subspace (CCS) as $kronspan\left\{\mathbf{1}_N\right\}$ and $\overline{kronspan\left\{\mathbf{1}_N\right\}}$, respectively.

Denote by $\mathbf{x}_c(t) = \tilde{q}_1 \otimes \tilde{\alpha}(t)$ and $\mathbf{x}_{\bar{c}}(t) = \sum_{i=2}^{N} \tilde{q}_i \otimes \tilde{\alpha}_i(t)$. Clearly, $\mathbf{x}_c(t)$ and $\mathbf{x}_{\bar{c}}(t)$, respectively, belong to the consensus subspace and consensus complement subspace. By Lemma 5, we can derive that multi-agent systems achieve consensus if and only if $\lim_{t\to\infty} \mathbf{x}_{\bar{c}}(t) = \mathbf{0}$. The $\mathbf{x}_c(t)$ can be regarded as a projection on consensus subspace. If multi-agent systems achieve consensus, $\mathbf{x}_c(t)$ becomes a trajectory of multi-agent systems. That is $\lim_{t\to\infty} \left\|\mathbf{x}(t) - \mathbf{x}_c(t)\right\| = 0$. Furthermore, if multi-agent systems are average consensusable at $\mathbf{x}(0)$, the following formula will hold:

$$\lim_{t\to\infty} \mathbf{x}(t) = \lim_{t\to\infty} \mathbf{x}_c(t) = \mathbf{1}_N \otimes Ave\left(\mathbf{x}(0)\right) \tag{16}$$

Fig. 1 describes this outline.

Theorem 1. Consensus with Multiple Time-delays. multi-agent systems (5) achieve consensus if and only if system

$$\begin{aligned} \dot{\delta}_2(t) &= \left[\mathbf{I}_{N-1} \otimes (\mathbf{A} + \mathbf{BK}_1)\right] \delta_2(t) \\ &\quad - \sum_{q=1}^{m} (\bar{\mathbf{F}}_q \otimes \mathbf{BK}_2)\delta_2(t - \tau_q(t)) \end{aligned} \tag{17}$$

is asymptotically stable, where $\mathbf{F}_q = diag\left\{0, \bar{\mathbf{F}}_q\right\}$ is made by $\mathbf{F}_q = \tilde{\mathbf{Q}}^T \mathcal{L} \tilde{\mathbf{Q}}$, and $\tilde{\mathbf{Q}}$ is the matrix of eigenvectors of the Laplacian matrix of the complete graph.

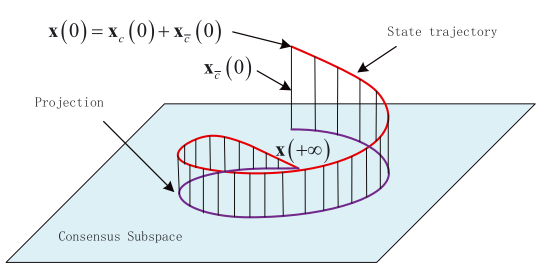


Fig. 1. The state trajectory of multi-agent systems from $t = 0$ to $t = +\infty$ (red line) can be decomposed into two parts: one is the projection in consensus subspace (purple line); the other is the remaining part in consensus complement subspace. The system has an equilibrium point $\mathbf{x}(+\infty)$, if $\mathbf{x}(+\infty) = \mathbf{1}_N \otimes Ave\left(\mathbf{x}(0)\right)$, then multi-agent systems is average consensusable at $\mathbf{x}(0)$.

Proof. Formula (11) can be rewritten as

$$\begin{aligned} \mathbf{x}(t) &= \sum_{i=1}^{N} \tilde{q}_i \otimes \tilde{\alpha}_i(t) = \sum_{i=1}^{N} \left(\tilde{\mathbf{Q}}\mathbf{e}_i\right) \otimes \left(\mathbf{I}_n \tilde{\alpha}_i(t)\right) \\ &= \sum_{i=1}^{N} \left(\tilde{\mathbf{Q}} \otimes \mathbf{I}_n\right) \left(\mathbf{e}_i \otimes \tilde{\alpha}_i(t)\right) \\ &= \left(\tilde{\mathbf{Q}} \otimes \mathbf{I}_n\right) \left[\tilde{\alpha}_1^T(t), \tilde{\alpha}_2^T(t), \dots, \tilde{\alpha}_N^T(t)\right]^T \end{aligned} \tag{18}$$

where $\tilde{\mathbf{Q}} = \left[\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_N\right]$ is the matrix of eigenvectors of the Laplacian matrix of the complete graph (see Remark 1).

Let $\delta(t) = \left[\tilde{\alpha}_1^T(t), \tilde{\alpha}_2^T(t), \dots, \tilde{\alpha}_N^T(t)\right]^T$ and substitute the $\mathbf{x}(t)$ in dynamics (5) with (18). We can derive following dynamics (noted that $\mathbf{x}(t) = \left(\tilde{\mathbf{Q}} \otimes \mathbf{I}_n\right) \delta(t)$ and $\delta(t) = \left(\tilde{\mathbf{Q}}^T \otimes \mathbf{I}_n\right) \mathbf{x}(t)$)

$$\begin{aligned} \dot{\delta}(t) &= \left[\mathbf{I}_N \otimes (\mathbf{A} + \mathbf{BK}_1)\right] \delta(t) \\ &\quad - \sum_{q=1}^{m} (\mathbf{F}_q \otimes \mathbf{BK}_2)\delta(t - \tau_q(t)) \end{aligned} \tag{19}$$

According to Lemma 3, $\mathbf{F}_q = \tilde{\mathbf{Q}}^T \mathcal{L} \tilde{\mathbf{Q}}$ has the form $\mathbf{F} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{F}} \end{bmatrix}$. Setting $\delta_1(t) = \tilde{\alpha}_1(t)$, $\delta_2(t) = \left[\tilde{\alpha}_2^T(t), \dots, \tilde{\alpha}_N^T(t)\right]^T$, we can decompose dynamics (19) as follows:

$$\dot{\delta}_1(t) = (\mathbf{A} + \mathbf{BK}_1)\delta_1(t) \tag{20}$$

and (17).

According to Lemma 5, (5) achieve consensus if and only if $\lim_{t\to\infty} \tilde{\alpha}_i(t) = \mathbf{0}, i = 2, 3, \dots, N$, *i.e.* (17) is asymptotically stable.

The following theorem will give the state trajectory of $\mathbf{x}_c(t)$.

Theorem 2. Consensus State Trajectory. For balanced communication topology (for multiple time-delays, it requires each graph in communication topology union to be balanced), the consensus state trajectory is $\mathbf{x}_c(t) = \mathbf{1}_N \otimes e^{(\mathbf{A}+\mathbf{BK}_1)t} Ave\left(\mathbf{x}(0)\right)$.

Proof. According to Theorem 2, the state trajectory projection on consensus subspace is $\mathbf{x}_c(t) = \tilde{q}_1 \otimes \tilde{\alpha}_1(t)$, where $\tilde{q}_1 = \frac{1}{\sqrt{N}} \mathbf{1}_N$.

By formula (20), we can derive

$$\begin{aligned} \mathbf{x}_c(t) &= \tilde{q}_1 \otimes \tilde{\alpha}_1(t) = \tilde{q}_1 \otimes \delta_1(t) \\ &= \tilde{q}_1 \otimes e^{(\mathbf{A}+\mathbf{BK}_1)t} \delta_1(0) \end{aligned} \tag{21}$$

Due to $\delta(0) = (\tilde{\mathbf{Q}}^T \otimes \mathbf{I}_n)\mathbf{x}(0)$, we can derive $\delta_1(0) = \left(\tilde{q}_1^T \otimes \mathbf{I}_n\right)\mathbf{x}(0)$. Therefore, (21) can be rewritten as follows:

$$\begin{aligned}\mathbf{x}_c(t) &= \left(\tilde{q}_1 \otimes e^{(\mathbf{A}+\mathbf{BK}_1)t}\right)\left(\tilde{q}_1^T \otimes \mathbf{I}_n\right)\mathbf{x}(0)\\ &= \mathbf{1}_N \otimes e^{(\mathbf{A}+\mathbf{BK}_1)t} Ave\left(\mathbf{x}(0)\right)\end{aligned} \tag{22}$$

Based on Theorem 1 and Theorem 2, the necessary and sufficient condition for high-order average consensus can be established.

Theorem 3. Average consensus with multiple time-delays. Assuming that each graph in the communication topology is balanced, a multi-agent system (5) is average consensusable at $\mathbf{x}(0)$ if and only if the following two conditions hold:

1. Subsystem $\dot{\delta}_2(t) = \left[\mathbf{I}_{N-1} \otimes (\mathbf{A}+\mathbf{BK}_1)\right]\delta_2(t) - \sum_{q=1}^{m}(\bar{\mathbf{F}}_q \otimes \mathbf{BK}_2)\delta_2(t-\tau_q(t))$ is asymptotically stable;
2. $Ave\left(\mathbf{x}(0)\right)$ is in the null space of $\mathbf{A}+\mathbf{BK}_1$.

Proof. Necessity. If multi-agent systems (5) get average consensus, obviously it has achieved consensus, and condition 1) can be established by Theorem 1. Furthermore, if $\mathbf{x}(0)$ is average consensusable, formula (16) holds. Substituting $\mathbf{x}_c(t)$ in (16) with (22), we can derive

$$\lim_{t\to+\infty} e^{(\mathbf{A}+\mathbf{BK}_1)t} Ave\left(\mathbf{x}(0)\right) = Ave(\mathbf{x}(0)) \tag{23}$$

Let $\tilde{\mathbf{J}}$ be the Jordan canonical form of $\mathbf{A}+\mathbf{BK}_1$, and there exist $\tilde{\mathbf{P}} = \left[\tilde{p}_{r1}, \tilde{p}_{r2}, \ldots, \tilde{p}_{rn}\right]$ and $\tilde{\mathbf{P}}^{-1} = \left[\tilde{p}_{l1}, \tilde{p}_{l2}, \ldots, \tilde{p}_{ln}\right]^T$ such that

$$\tilde{\mathbf{P}}^{-1}\left(\mathbf{A}+\mathbf{BK}_1\right)\tilde{\mathbf{P}} = \tilde{\mathbf{J}} = diag\left\{\tilde{\mathbf{J}}_1, \tilde{\mathbf{J}}_2, \ldots, \tilde{\mathbf{J}}_s\right\}$$

where $\tilde{\mathbf{J}}_1, \tilde{\mathbf{J}}_2, \ldots, \tilde{\mathbf{J}}_s$ are, respectively, Jordan blocks of order $n_1, n_2, \ldots, n_s$. Each $\tilde{\mathbf{J}}_k$ ($k \in \{1,2,\ldots,s\}$) has the following form

$$\tilde{\mathbf{J}}_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}_{n_k \times n_k}$$

For convenience, let $\tilde{p}_{rk}^{(m)}$ be the vector in $\left\{\tilde{p}_{r1}, \tilde{p}_{r2}, \ldots, \tilde{p}_{rn}\right\}$ associated with mth column in block $\tilde{\mathbf{J}}_k$, similarly $\tilde{p}_{lk}^{(m)}$ in $\left\{\tilde{p}_{l1}, \tilde{p}_{l2}, \ldots, \tilde{p}_{ln}\right\}$ can be defined. Thus

$$e^{(\mathbf{A}+\mathbf{BK}_1)t} = \sum_{k=1}^{s}\left\{\left[\tilde{p}_{rk}^{(1)}, \ldots, \tilde{p}_{rk}^{(n_k)}\right] e^{\tilde{\mathbf{J}}_k t}\begin{bmatrix}\left[\tilde{p}_{lk}^{(1)}\right]^T \\ \vdots \\ \left[\tilde{p}_{lk}^{(n_k)}\right]^T\end{bmatrix}\right\}$$

where

$$\begin{cases}\mathrm{Re}\left(\lambda_k\right) < 0,\ k = 1,2,\ldots,s_1\\ \lambda_k = 0,\ k = s_1+1, s_1+2, \ldots, s_2\\ \mathrm{Re}\left(\lambda_k\right) > 0 \quad or\\ \mathrm{Re}\left(\lambda_k\right) = 0 \& \mathrm{Im}\left(\lambda_k\right) \neq 0, k = s_2+1, s_2+2, \ldots, s\end{cases}$$

According to (23), limitation exists in $\lim_{t\to+\infty} e^{(\mathbf{A}+\mathbf{BK}_1)t}$ $Ave\left(\mathbf{x}(0)\right)$. Therefore, $Ave\left(\mathbf{x}(0)\right)$ is required to be orthogonal to those $\tilde{p}_l$ which correspond to $\lambda_k, k = s_2+1, s_2+2, \ldots, s$. Respectively, columns vector of

$$\tilde{\mathbf{p}}_{l1} = \left[\tilde{p}_{l(s_1+1)}^{(2)}, \ldots, \tilde{p}_{l(s_1+1)}^{(n_{s1+1})}, \ldots, \tilde{p}_{l(s_2)}^{(2)}, \ldots, \tilde{p}_{l(s_2)}^{(n_{s2})}\right]$$

associated with the polynomial unstable modals, and

$$\tilde{\mathbf{p}}_{l2} = \left[\tilde{p}_{l(s_2+1)}^{(1)}, \ldots, \tilde{p}_{l(s_2+1)}^{(n_{s2+1})}, \ldots, \tilde{p}_{l(s)}^{(1)}, \ldots, \tilde{p}_{l(s)}^{(n_s)}\right]$$

with the critically oscillating/exponential unstable modals. Therefore

$$Ave\left(\mathbf{x}(0)\right) \in span\left\{\tilde{\mathbf{p}}_{l1}, \tilde{\mathbf{p}}_{l2}\right\}^{\perp} \tag{24}$$

where α_1, α_2 are, respectively, the components of $Ave\left(\mathbf{x}(0)\right)$ in $\tilde{\mathbf{p}}_{r1}, \tilde{\mathbf{p}}_{r2}$.

With the orthogonality between $\tilde{p}_r$ and $\tilde{p}_l$, $\lim_{t\to+\infty} e^{(\mathbf{A}+\mathbf{BK}_1)t} Ave\left(\mathbf{x}(0)\right)$ can be simplified as

$$\begin{aligned}&\lim_{t\to+\infty} e^{(\mathbf{A}+\mathbf{BK}_1)t} Ave\left(\mathbf{x}(0)\right)\\ &= \lim_{t\to+\infty}\left(\tilde{\mathbf{p}}_{r1}\begin{bmatrix} e^{\tilde{\mathbf{J}}_1 t} & & \\ & \ddots & \\ & & e^{\tilde{\mathbf{J}}_{s1} t}\end{bmatrix}\alpha_1 + \tilde{\mathbf{p}}_{r2} e^{\mathbf{0}t}\alpha_2\right)\\ &= \tilde{\mathbf{p}}_{r2}\alpha_2\end{aligned} \tag{25}$$

According to (23), (24) and (25), we can derive that $\tilde{\mathbf{p}}_{r2}\alpha_2 = \tilde{\mathbf{p}}_{r1}\alpha_1 + \tilde{\mathbf{p}}_{r2}\alpha_2$. This implies that $\alpha_1 \equiv \mathbf{0}$, which means $Ave(\mathbf{x}(0))$ has no component in $\tilde{\mathbf{p}}_{r1}$. Therefore, $Ave\left(\mathbf{x}(0)\right) \in span\left\{\tilde{\mathbf{p}}_{r2}\right\}$, *i.e.* $Ave\left(\mathbf{x}(0)\right)$ is in the null space of $\mathbf{A}+\mathbf{BK}_1$. Then condition 2 is established.

Sufficiency. If condition 1 holds, by Theorem 1, multi-agent systems (5) achieve consensus, *i.e.* $\lim_{t\to+\infty}\left\|\mathbf{x}(t) - \mathbf{x}_c(t)\right\| = 0$. By Theorem 2, we can derive that, $Ave\left(\mathbf{x}\left(0\right)\right)$ is in the null space of $\mathbf{A}+\mathbf{BK}_1$.)

$$\lim_{t\to+\infty}\left\|\mathbf{x}(t)-\mathbf{1}_N\otimes e^{(\mathbf{A}+\mathbf{BK}_1)t}Ave\left(\mathbf{x}\left(0\right)\right)\right\|=0$$

when $e^{(\mathbf{A}+\mathbf{BK}_1)t}$ is expand by Taylor Series, it is obtained that

$$\begin{aligned}
&e^{(\mathbf{A}+\mathbf{BK}_1)t}Ave\left(\mathbf{x}(0)\right)\\
&=\left[\mathbf{I}_n+\sum_{k=1}^{\infty}\frac{1}{k!}t^k(\mathbf{A}+\mathbf{BK}_1)^k\right]Ave\left(\mathbf{x}(0)\right) \qquad (26)\\
&=Ave\left(\mathbf{x}(0)\right)+\mathbf{0}=Ave\left(\mathbf{x}(0)\right)
\end{aligned}$$

Therefore $\lim\limits_{t\to+\infty}\mathbf{x}(t)=\lim\limits_{t\to+\infty}\mathbf{x}_c(t)=\mathbf{1}_N\otimes Ave(\mathbf{x}(0))$, which means multi-agent systems (5) is average consensusable at $\mathbf{x}(0)$.

Remark 2. For first-order integrator multi-agent systems, all $Ave\left(\mathbf{x}(0)\right)$ are in the null space of $\mathbf{A}+\mathbf{BK}_1=0$ $(\mathbf{A}=0, \mathbf{B}=1, \mathbf{K}_1=0, \mathbf{K}_2=1)$. That's why the average consensus in first-order integrator multi-agent systems is globally asymptotic (i.e. the average consensus set is $\left\{\mathbf{x}(0)|\mathbf{x}(0)\in\mathbb{R}^N\right\}$). However, unlike first-order multi-agent systems, high-order multi-agent systems are not globally average consensusable. According to Theorem 3, If $Ave\left(\mathbf{x}(0)\right)$ is not in the null space of $\mathbf{A}+\mathbf{BK}_1$, the average consensus cannot be achieved. Overall, the average consensus set of high-order multi-agent systems is $\left\{\mathbf{x}(0)|\sum\limits_{i=1}^{N}\mathbf{x}_i(0)\in\left\{span\left(\mathbf{A}+\mathbf{BK}_1\right)^T\right\}^{\perp}\right\}$.

Remark 3. As we mentioned in introduction, the disagreement vector method, which is an effective way to solve the first-order average consensus problems (*e.g.* [20–26]), might not be adequate in analyzing high-order average consensus problems. In the disagreement vector method of Olfati-Saber *et al.* [19], high-order multi-agent systems will leave a constant item in the disagreement system. With the disagreement vector method and setting $\mathbf{x}(t)=\mathbf{1}_N\otimes Ave\left(\mathbf{x}(0)\right)+\delta(t)$, multi-agent systems (5) can be rewritten as:

$$\begin{aligned}
\dot{\boldsymbol{\delta}}(t)=&\left[\mathbf{I}_{N-1}\otimes(\mathbf{A}+\mathbf{BK}_1)\right]\delta(t)\\
&-\sum_{q=1}^{m}(\mathbf{F}_q\otimes\mathbf{BK}_2)\delta_2(t-\tau_q(t)) \qquad (27)\\
&+\mathbf{1}_N\otimes\left[(\mathbf{A}+\mathbf{BK}_1)Ave\left(\mathbf{x}(0)\right)\right]
\end{aligned}$$

where $\mathbf{1}_N\otimes\left[(\mathbf{A}+\mathbf{BK}_1)Ave\left(\mathbf{x}(0)\right)\right]$ is the constant item in disagreement system. For first-order integrator multi-agent systems $\mathbf{A}+\mathbf{BK}_1=0$, which means the constant item in (27) is $\mathbf{1}_N\otimes\left[(\mathbf{A}+\mathbf{BK}_1)Ave\left(\mathbf{x}(0)\right)\right]=0$. By contrast, in high-order multi-agent systems, this item $\mathbf{1}_N\otimes\left[(\mathbf{A}+\mathbf{BK}_1)Ave\left(\mathbf{x}(0)\right)\right]$ does not equal to $\mathbf{0}$ naturally; whereas in order to let (27) asymptotically stable at $\mathbf{0}$, $Ave\left(\mathbf{x}(0)\right)$ should be in the null space of $(\mathbf{A}+\mathbf{BK}_1)$. This seems to be similar to our result (Theorem 1), but it cannot make sure whether the consensus is exactly the $Ave\left(\mathbf{x}(0)\right)$. For example, $\frac{1}{2}Ave\left(\mathbf{x}(0)\right)$ is also valid. Setting $\mathbf{x}(t)=\mathbf{1}_N\otimes\frac{1}{2}Ave\left(\mathbf{x}(0)\right)+\delta(t)$, the disagreement system is

$$\begin{aligned}
\dot{\boldsymbol{\delta}}(t)=&\left[\mathbf{I}_{N-1}\otimes(\mathbf{A}+\mathbf{BK}_1)\right]\delta(t)\\
&-\sum_{q=1}^{m}(\mathbf{F}_q\otimes\mathbf{BK}_2)\delta_2(t-\tau_q(t)) \qquad (28)\\
&+\frac{1}{2}\mathbf{1}_N\otimes\left[(\mathbf{A}+\mathbf{BK}_1)Ave\left(\mathbf{x}(0)\right)\right]
\end{aligned}$$

And the $\frac{1}{2}Ave\left(\mathbf{x}(0)\right)$ also makes the $\frac{1}{2}\mathbf{1}_N\otimes\left[(\mathbf{A}+\mathbf{BK}_1)Ave\left(\mathbf{x}(0)\right)\right]$ equal to $\mathbf{0}$. As a result, there are no differences between (27) and(28). Therefore, whether the multi-agent systems are getting $Ave\left(\mathbf{x}(0)\right)$ or $\frac{1}{2}Ave\left(\mathbf{x}(0)\right)$ cannot be determined. Actually, any vector in the null space of $(\mathbf{A}+\mathbf{BK}_1)$ is valid. The essential reason is the disagreement vector method has little ability to distinguish different consensus vectors in the null space of $(\mathbf{A}+\mathbf{BK}_1)$. Therefore, the classic method might not be suitable for analyzing average consensus in high-order multi-agent systems.

IV. SIMULATION EXAMPLES

In order to validate the theoretical results proposed in this paper, simulations are provided to illustrate the effectiveness. The used communication topologies are shown as Fig. 2, and we assume their adjacency matrices are 0-1 matrices.

In order to focus on the conditions for average consensus, all the simulation experiments are based on the fact that the multi-agent systems satisfy the condition of achieving consensus. As a result, condition 1 in Theorem 3 holds automatically. The corresponding stability criteria for condition 1 can be found in many published books, for example [32]. Therefore, what we focus

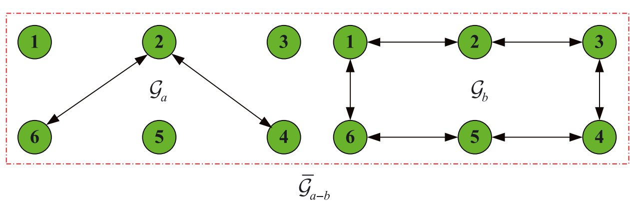
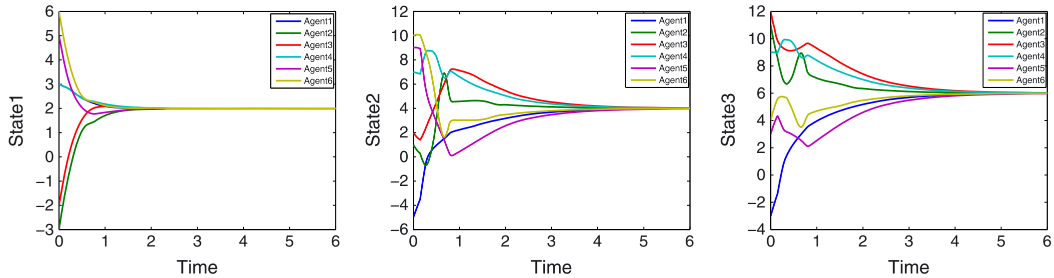


Fig. 2. Graph union $\bar{\mathcal{G}}_{a-b}$.



(a) (b) (c)

Fig. 3. State trajectories of 3rd-order multi-agent systems in $\bar{\mathcal{G}}_{a-b}$ with communication delays $\tau_1(t) = \frac{0.15}{2} + \frac{0.15}{2}\sin\left(\frac{2\times 0.9}{0.15}t\right)$ and $\tau_2(t) = \frac{0.35}{2} + \frac{0.35}{2}\sin\left(\frac{2\times 0.9}{0.35}t\right)$.

on is how the multi-agent systems can achieve average consensus rather than consensus itself.

We present simulation results for high-order average consensus. Fig. 3 shows the average consensus in 3rd-order multi-agent systems in balanced communication topology with multiple time delays and with parameters

$$\mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ -4 & -3 & -4 \\ 0 & -1 & -3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{K}_1 = \begin{bmatrix} 3 & 1 & 2 \end{bmatrix}, \mathbf{K}_2 = \begin{bmatrix} -3 & 2 & -2 \end{bmatrix} \tag{29}$$

The initial state is $\phi(t) = \phi^{(1)}(t)$ in Table 2. Thus, $Ave\,(\mathbf{x}(0)) = [2\ 4\ 6]^T$, which is just in the null space of $\mathbf{A} + \mathbf{BK}_1$ ($span\left\{[1\ 2\ 3]^T\right\}$). As we can see in Fig. 3, the multi-agent systems achieve average consensus.

V. CONCLUSIONS

The main contributions are concluded:

1. A new mathematical concept – the Kronecker basis – and some useful conclusions are proposed. Unlike the classic disagreement vector method which has been proven that it is not a necessary and sufficient condition in describing high-order average consensus, the Kronecker basis based description is necessary and sufficient as well as explicit;
2. Based on the Kronecker basis, the necessary and sufficient condition for high-order average consensus in balanced communication topology with multiple time delays is proposed, which takes the average consensus problem in first-order, high-order integrator multi-agent systems as a special case.

REFERENCES

1. Wang, X., T. Liu, and J. Qin, "Second-order consensus with unknown dynamics via cyclic-small-gain method," *IET Control Theory Applicat*, Vol. 6, No. 18, pp. 2748–2756 (2012).
2. Cao, M., A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: a graphical approach," *SIAM J. Control Optim.*, Vol. 47, No. 2, pp. 575–600 (2008).
3. Ren, W., R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, Vol. 27, No. 2, pp. 71–82 (2007).
4. Qin, J. and H. Gao, "A sufficient condition for convergence of sampled-data consensus for double-integrator dynamics with nonuniform and time-varying communication delays," *IEEE Trans. Autom. Control*, Vol. 57, No. 9, pp. 2417–2422 (2012).
5. Xiao, F. and L. Wang, "Asynchronous consensus in continuous-time multi-agent systems with switching topology and time-varying delays," *IEEE Trans. Autom. Control*, Vol. 53, No. 8, pp. 1804–1816 (2008).
6. Yang, T., S. Roy, Y. Wan, and A. Saberi, "Constructing consensus controllers for networks with identical general linear agents," *Int. J. Robust Nonlinear Control*, Vol. 21, No. 11, pp. 1237–1256 (2011).
7. Grip, H. F., T. Yang, A. Saberi, and A. A. Stoorvogel, "Output synchronization for heterogeneous networks of non-introspective agents," *Automatica*, Vol. 48, No. 10, pp. 2444–2453 (2012).
8. Meng, Z., Z. Zhao, and Z. Lin, "On global leader-following consensus of identical linear dynamic systems subject to actuator saturation," *Syst. Control Lett.*, Vol. 62, No. 2, pp. 132–142 (2013).

9. Meng, Z., W. Ren, Y. Cao, and Z. You, "Leaderless and leader-following consensus with communication and input delays under a directed network topology," *IEEE Trans. Syst. Man Cybern. Part B-Cybern.*, Vol. 41, No. 1, pp. 75–88 (2011).
10. Zhu, H., Y. Cong, X. Wang, D. Zhang, and Q. Zhang, *Consensusabilization for Continuous-Time High-Order Multiagent Systems with Time-Varying Delays,* Mathematical Problems in Engineering, Hindawi Publishing Corporation (2013).
11. Fu, M., S. Wang, and Y. Cong, *Swarm Stability Analysis of High-Order Linear Time-Invariant Singular Multiagent Systems,* Mathematical Problems in Engineering, Hindawi Publishing Corporation (2013).
12. Cong, Y., X. Wang, H. Zhu, and D. Zhang, "State Consensus Criteria and Protocol Parameters Design for high-order linear multi-agent systems," *32nd Chinese Control Conference*, Xi'an, China, pp. 7034–7038 (2013).
13. Lu, X., F. Austin, and S. Chen, "Flocking in multi-agent systems with active virtual leader and time-varying delays coupling," *Commun. Nonlinear Sci. Numer. Simul.*, Vol. 16, No. 2, pp. 1014–1026 (2011).
14. Dimarogonas, D. V. and K. J. Kyriakopoulos, "On the rendezvous problem for multiple nonholonomic agents," *IEEE Trans. Autom. Control*, Vol. 52, No. 5, pp. 916–922 (2007).
15. Michael, N., D. Mellinger, Q. Lindsey, and V. Kumar, "The grasp multiple micro-uav testbed," *IEEE Robot. Autom. Mag.*, Vol. 17, No. 3, pp. 56–65 (2010).
16. Olfati-Saber, R., "Distributed kalman filter with embedded consensus filters," *44th IEEE Conf. Decis. Control & Europ. Control Conf.*, Seville, Spain, pp. 8179–8184 (2005).
17. Bauso, D., L. Giar, and R. Pesenti, "Distributed consensus protocols for coordinating buyers," *42nd IEEE Conf. Decis. Control*, Maui, HI, USA, pp. 588–592 (2003).
18. Jadbabaie, A., N. Motee, and M. Barahona, "On the stability of the kuramoto model of coupled nonlinear oscillators," *Proc. 2004 Amer. Control Conf.*, Boston, MA, Vol. 5, pp. 4296–4301 (2004).
19. Olfati-Saber, R. and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, Vol. 49, No. 9, pp. 1520–1533 (2004).
20. Lin, P., K. Qin, H. Zhao, and M. Sun, "A new approach to average consensus problems with multiple time-delays and jointly-connected topologies," *J. Franklin Inst. Eng. Appl. Math.*, Vol. 349, No. 1, pp. 293–304 (2012).
21. Lin, P., Y. Jia, J. Du, and F. Yu, "Average consensus for networks of continuous-time agents with delayed information and jointly-connected topologies," *Amer. Control Conf.*, St Louis, MO, pp. 3884–3889 (2009).
22. Sun, Y. G., L. Wang, and G. Xie, "Average consensus in directed networks of dynamic agents with time-varying communication delays," *45th IEEE Conf. Decis. Control*, San Diego, CA, pp. 3393–3398 (2006).
23. Sun, Y. G., L. Wang, and G. Me, "Average consensus in networks of dynamic agents with switching topologies and multiple time-varying delays," *Syst. Control Lett.*, Vol. 57, No. 2, pp. 175–183 (2008).
24. Sun, Y. G., "Average consensus in networks of dynamic agents with uncertain topologies and time-varying delays," *J. Franklin Inst. Eng. Appl. Math.*, Vol. 349, No. 3, pp. 1061–1073 (2012).
25. Lin, P. and Y. Jia, "Average consensus in networks of multi-agents with both switching topology and coupling time-delay," *Physica A-Stat. Mech. Applicat.*, Vol. 387, No. 1, pp. 303–313 (2008).
26. Qin, J., H. Gao, and W. X. Zheng, "On average consensus in directed networks of agents with switching topology and time delay," *Int. J. Syst. Sci.*, Vol. 42, No. 12, pp. 1947–1956 (2011).
27. Lin, P. and Y. Jia, "Consensus of a class of second-order multi-agent systems with time-delay and jointly-connected topologies," *IEEE Trans. Autom. Control*, Vol. 55, No. 3, pp. 778–784 (2010).
28. Wang, S. and D. Xie, "Consensus of second-order multi-agent systems via sampled control: undirected fixed topology case," *IET Cont. Theory Appl.*, Vol. 6, No. 7, pp. 893–899 (2012).
29. Zhang, Q., Y. Niu, L. Wang, L. Shen, and H. Zhu, "Average consensus seeking of high-order continuous-time multi-agent systems with multiple time-varying communication delays," *Int. J. Control Autom. Syst.*, Vol. 9, No. 6, pp. 1209–1218 (2011).
30. Xi, J., N. Cai, and Y. Zhong, "Consensus problems for high-order linear time-invariant swarm systems," *Physica A-Stat. Mech. Applicat.*, Vol. 389, No. 24, pp. 5619–5627 (2010).
31. Xi, J., Z. Shi, and Y. Zhong, "Consensus analysis and design for high-order linear swarm systems with time-varying delays," *Physica A-Stat. Mech. Applicat.*, Vol. 390, No. 23-24, pp. 4114–4123 (2011).
32. Wu, M., Y. He, and J. She, *Stability Analysis and Robust Control of Time-Delay Systems*, Science Press Beijing and Springer-Verlag, Berlin Heidelberg (2010).