Proceedings of the 34th Chinese Control Conference
July 28-30, 2015, Hangzhou, China

Second-order Edge Agreement with Locally Lipschitz Dynamics under Digraph via Edge Laplacian and ISS Method

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Abstract: This paper focuses on the second-order edge agreement problem for nonlinear multi-agent systems with unknown locally Lipschitz dynamics under directed topologies. Based on a novel concept, i.e., the essential edge Laplacian, we derive a model reduction representation of the closed-loop multi-agent system based on the spanning tree subgraph. By using the backstepping design, the original multi-agent system can be remodeled as several interacted subsystems with proven ISS (input-to-state stable) properties. Additionally, the interactions of the interacted subsystems can be explicitly illustrated as a gain-interconnection digraph. With the aid of the ISS cyclic-small-gain theorem, the asymptotic stability of the whole system can be guaranteed. To illustrate the effectiveness of the proposed strategy, simulation results are provided.

Key Words: Multi-agent systems, edge agreement, second-order nonlinear dynamics, backstepping design, ISS

1 INTRODUCTION

The graph theory contributes significantly in the analysis and synthesis of multi-agent systems, since it provides natural abstractions for how information is shared between agents in a network. Specially, the spectral properties of the graph Laplacian are extensively explored recently to provide convergence analysis in the context of multi-agent coordination behaviour [1][2]. Despite the unquestionable interest of the results concerning the convergence properties in these literatures, we also note that, another interesting topic with regard to how certain subgraphs contribute to the analysis of multi-agent systems, has arisen in more recently. In this direction, an attractive notion about the edge agreement deserve special attention, in which the edge Laplacian plays an important role. Pioneering works on edge agreement have provided totally new insights into how certain subgraphs, such as spanning trees and cycles effect the convergence properties. A novel systematic framework based on connected graph for analysing multi-agent systems in the edge perspective [3] has also been proposed. Although the edge Laplacian offers more transparent understanding of the graph structure, it still remains an undirected notion in aforementioned literatures. More recently, the edge Laplacian is used to examine the model reduction of networked system associated with directed trees through clustering in [4]; however it can not be directly extended to more general digraphs yet. Our previous work [5] has comprehensively extended the concept of edge Laplacian with its algebraic properties for digraph.

Recently, the coordination behaviour of multi-agent systems with nonlinear dynamics has received increasing attention, because most of physical systems are inherently nonlinear in nature. For example, in [6][7], the consensus problem with continuously differentiable nonlinear dynamics are considered; in [8][9][10], the nonlinear dynamics satisfies the global Lipschitz condition; and in [11], an adaptive control method is introduced to study the synchronization of uncertain nonlinear networked systems. Different from all the aforementioned works, the nonlinear dynamics considered in this paper is only required to be locally Lipschitz continuous, a rather relaxed assumption which has been used in a wide-range of practical nonlinear systems [12]. To address the technical challenges, the concept of input-to-state stability (ISS, see [13] for a tutorial), is drawn into this paper. Most recently, considerable efforts have been devoted to the interconnected ISS nonlinear systems, for instance, the nonlinear small-gain design methods, especially the cyclic-small-gain approach [14], are utilized to design new distributed control strategies for flocking and containment control in [15] as well as to deal with formation control of nonholonomic mobile robots in [16]. Additionally, the authors present a cyclic small-gain approach to distributed output-feedback control of nonlinear multi-agent systems in [17].

We are motivated to derive distributed control law that allows second-order nonlinear multi-agent systems to reach edge agreement with only local interaction with their neighbours. The contribution of this paper contains three aspects. First, we introduce a model reduction representation associated with the essential edge Laplacian which allows a convenient analysis. Second, different from most of the existing works, the nonlinear dynamics is only required to be locally Lipschtiz continuous. As an extension to our previous work [18], the proposed distributed control law can effectively tackle these technical challenges while considering directed topology. Third, with a seamless integration of edge Laplacian and backstepping designs, the nonlinear ISS-based framework and analyzing method for multi-agent system are proposed. Due to the flexibility of backstepping, the proposed consensus protocol does not require the relative velocity information of the neighbor agents.

The rest of the paper is organized as follows. In Section 2, a brief overview of the basic notions and results in graph theory are presented, as well as the ISS cyclic-small-gain theorem. These constructs are then used to develop the edge Laplacian of digraph and its relative algebraic properties in Section 3. In Section 4, the backstepping design and the transformed interconnection system are elaborated. The main result, i.e., synthesis by using cyclic-small-gain theorem is presented in section 5. The simulation results are provided in Section 6. The last section draws the conclusions.

2 Basic Notions and Preliminary Results

In this section, we present some basic notions in graph theory and ISS cyclic-small-gain theorem.

2.1 Graph and Matrix

In this paper, we use $|\cdot|$ and $\|\cdot\|$ to denote the Euclidean norm and 2-norm for vectors and matrices respectively. Let $\mathcal{G}=(\mathcal{V},\mathcal{E})$ be a digraph of order N specified by a node set $\mathcal{V}$ and an edge set $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ with size L. The set of neighbors of node i is denoted by $\mathcal{N}_i=\{j: e_k=(j,i)\in\mathcal{E}\}$. We use $A(G)$ to represent a weighted adjacency matrix, where the adjacency elements associated with the edges are positive, i.e., $e_k=(j,i)\in\mathcal{E}\Leftrightarrow a_{ij}>0$, otherwise, $a_{ij}=0$. Denote by $\mathcal{W}(\mathcal{G})$ the $L\times L$ diagonal matrix of w_k, for $k=1,2\cdots,L$, where $w_k=a_{ij}$ for $e_k=(j,i)\in\mathcal{E}$. The notation $D(\mathcal{G})$ represents a diagonal matrix with d_i denoting the in-degree of node i on the diagonal. The corresponding graph Laplacian of $\mathcal{G}$ is defined as $L_n(\mathcal{G}):=D(\mathcal{G})-A(\mathcal{G})$, whose eigenvalues will be ordered and denoted as $0=\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_N$. The incidence matrix $E(\mathcal{G})\in\mathbb{R}^{N\times L}$ for a digraph is a $\{0,\pm1\}$-matrix with rows and columns indexed by nodes and edges of $\mathcal{G}$ respectively, such that for edge $e_k=(j,i)\in\mathcal{E}$, $[E(\mathcal{G})]_{jk}=+1$, $[E(\mathcal{G})]_{ik}=-1$ and $[E(\mathcal{G})]_{lk}=0$ for $l\neq i,j$. The definition implies that each column of E contains exactly two nonzero entries indicating the initial node and the terminal node respectively.

A spanning tree $\mathcal{G}_\tau=(\mathcal{V},\mathcal{E}_1)$ of a digraph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ is a directed tree formed by graph edges that connect all the nodes of the graph; a cospanning tree $\mathcal{G}_c=(\mathcal{V},\mathcal{E}-\mathcal{E}_1)$ of $\mathcal{G}_\tau$ is the subgraph of $\mathcal{G}$ having all the vertices of $\mathcal{G}$ and exactly those edges of $\mathcal{G}$ that are not in $\mathcal{G}_\tau$. Graph $\mathcal{G}$ is called *strongly connected* if and only if any two distinct nodes can be connected via a directed path; *quasi-strongly connected* if and only if it has a directed spanning tree [19]. A quasi-strongly connected digraph $\mathcal{G}$ can be rewritten as a union form: $\mathcal{G}=\mathcal{G}_\tau\cup\mathcal{G}_c$. In addition, according to certain permutations, the incidence matrix $E(\mathcal{G})$ can always be rewritten as $E(\mathcal{G})=\begin{bmatrix}E_\tau(\mathcal{G}) & E_c(\mathcal{G})\end{bmatrix}$ as well. Since the cospanning tree edges can be constructed from the spanning tree edges via a linear transformation [3], such that,

$$E_\tau(\mathcal{G})T(\mathcal{G})=E_c(\mathcal{G}) \tag{1}$$

with $T(\mathcal{G})=\left(E_\tau(\mathcal{G})^TE_\tau(\mathcal{G})\right)^{-1}E_\tau(\mathcal{G})^TE_c(\mathcal{G})$ and $rank(E(\mathcal{G}))=N-1$ from [19]. We define

$$R(\mathcal{G})=\begin{bmatrix}I & T(\mathcal{G})\end{bmatrix} \tag{2}$$

and then obtain

$$E(\mathcal{G})=E_\tau(\mathcal{G})R(\mathcal{G}). \tag{3}$$

The column space of $E(\mathcal{G})^T$ is known as the *cut space* of $\mathcal{G}$ and the null space of $E(\mathcal{G})$ is called as the *flow space*, which is the orthogonal complement of the cut space. Interestingly, the rows of $R(\mathcal{G})$ form a basis of the cut space. Whereas the rows of $\begin{bmatrix}-T(\mathcal{G})^T & I\end{bmatrix}$ form a basis of the flow space respectively [19].

2.2 ISS Cyclic-small-gain theorem

A function $\alpha:\mathbb{R}_+\to\mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0)=0$; it is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot,t)$ is of class $\mathcal{K}$ and for each fixed s, the function $\beta(s,\cdot)$ is decreasing and tends to zero at infinity. Id represents identify function, and symbol $\circ$ denotes the composition between functions.

Consider the following interconnected system composed of N interacting subsystems:

$$\dot{x}_i=\zeta_i(x,w_i),\quad i=1,\ldots,N, \tag{4}$$

where $x_i\in\mathbb{R}^{n_i}$, $w_i\in\mathbb{R}^{m_i}$ and $\zeta_i:\mathbb{R}^{n+m_i}\to\mathbb{R}^{n_i}$ with $n=\sum_{i=1}^N n_i$ is locally Lipschitz continuous. The external input $w=[w_1^T,\ldots,w_N^T]^T$ is a measurable and locally essentially bounded function. For the i-th subsystem $(i=1,\ldots,N)$, there exists an ISS-Lyapunov function [15] $V_i:\mathbb{R}^{n_i}\to\mathbb{R}$ satisfying

- there exist $\underline{a}_i,\overline{a}_i\in\mathcal{K}_\infty$ such that

$$\underline{a}_i(|x_i|)\leq V_i(x_i)\leq\overline{a}_i(|x_i|),\quad\forall x_i, \tag{5}$$

- there exist $\gamma_{x_i}^{x_j}\in\mathcal{K}\cup\{0\}(j\neq i)$, $\gamma_{x_i}^{w_i}\in\mathcal{K}\cup\{0\}$ and a positive definite α_i such that

$$\begin{aligned}&V_i(x_i)\geq\max\{\gamma_{x_i}^{x_j}(V_j(x_j)),\gamma_{x_i}^{w_i}(|w_i|)\}\\&\Rightarrow\nabla V_i(x_i)\zeta_i(x,w_i)\leq-\alpha_i(V_i(x_i))\quad\forall x,\ \forall w_i.\end{aligned} \tag{6}$$

Lemma 1 (Cyclic-small-gain Theorem, [14]) *Consider the dynamical network* (4)*. Suppose that for $i=1,\ldots,N$, the x_i-subsystem admits an ISS-Lyapunov function V_i satisfying* (5) *and* (6)*. Then the system* (4) *is ISS if for each $r=2,\ldots,N$,*

$$\gamma_{x_{i_1}}^{x_{i_2}}\circ\gamma_{x_{i_2}}^{x_{i_3}}\circ\cdots\circ\gamma_{x_{i_r}}^{x_{i_1}}<\mathrm{Id} \tag{7}$$

for all $1\leq i_j\leq N$, $i_j\neq i_{j'}$ if $j\neq j'$.

3 Edge Laplacian of digraph and Edge Agreement

The edge Laplacian in [3] still remains to an undirected notion and is thus inadequate to handle our problem. Undoubtedly, extending the concept of the edge Laplacian to digraph and exploring its relative algebraic properties will be of great help to understand multi-agent systems from edge perspective.

Before moving on, we give the definition of the in-incidence matrix and out-incidence matrix at first.

Definition 1 (In-incidence/Out-incidence Matrix [5]) *The $N\times L$ in-incidence matrix $E_\odot(\mathcal{G})$ for a digraph $\mathcal{G}$ is a $\{0,-1\}$ matrix with rows and columns indexed by nodes and edges of $\mathcal{G}$, respectively, such that for an edge $e_k=(j,i)\in\mathcal{E}$, $[E_\odot(\mathcal{G})]_{mk}=-1$ for $m=i$, $[E_\odot(\mathcal{G})]_{mk}=0$ otherwise. The out-incidence matrix is a $\{0,+1\}$ matrix with $[E_\otimes(\mathcal{G})]_{nk}=+1$ for $n=j$, $[E_\otimes(\mathcal{G})]_{nk}=0$ otherwise.*

In comparison with the definition of the incidence matrix, we can rewrite $E(\mathcal{G})$ in the following way:

$$E(\mathcal{G})=E_\odot(\mathcal{G})+E_\otimes(\mathcal{G}). \tag{8}$$

On the other hand, the weighted in-incidence matrix $E_\odot^w(\mathcal{G})$ can be defined as $E_\odot^w(\mathcal{G})=E_\odot(\mathcal{G})\mathcal{W}(\mathcal{G})$, where $\mathcal{W}(\mathcal{G})$ is a diagonal matrix of w_k. This next will lead us to find out a novel factorization of the graph Laplacian $L_n(\mathcal{G})$.

Lemma 2 ([5]) *Considering a digraph $\mathcal{G}$ with the incidence matrix $E(\mathcal{G})$ and weighted in-incidence matrix $E_{\odot}^{w}(\mathcal{G})$, the graph Laplacian of $\mathcal{G}$ have the following expression*

$$L_n(\mathcal{G}) = E_{\odot}^{w}(\mathcal{G})E(\mathcal{G})^T. \tag{9}$$

Besides, from (3), one can obtain

$$L_n(\mathcal{G}) = E_{\odot}^{w}(\mathcal{G})R(\mathcal{G})^T E_{\mathcal{T}}(\mathcal{G})^T. \tag{10}$$

Definition 2 (Edge Laplacian, [5]) *The edge Laplacian of a digraph $\mathcal{G}$ is defined as*

$$L_e(\mathcal{G}) := E(\mathcal{G})^T E_{\odot}^{w}(\mathcal{G}). \tag{11}$$

To provide a deeper insights into what the edge Laplacian $L_e(\mathcal{G})$ offers in the analysis and synthesis of multi-agent systems, we propose the following lemma.

Lemma 3 ([5]) *For any digraph $\mathcal{G}$, the graph Laplacian $L_{\mathcal{G}}$ and the edge Laplacian $L_e(\mathcal{G})$ have the same nonzero eigenvalues. If $\mathcal{G}$ is quasi-strongly connected, then the edge Laplacian $L_e(\mathcal{G})$ contains exactly $N-1$ nonzero eigenvalues and all in the open right-half plane.*

Lemma 4 ([5]) *Considering a quasi-strongly connected graph $\mathcal{G}$ of order N, the edge Laplacian $L_e(\mathcal{G})$ has $L-N+1$ zero eigenvalues and zero is a simple root of the minimal polynomial of $L_e(\mathcal{G})$.*

Before moving on, we introduce the following transformation matrix:

$$S_e(\mathcal{G}) = \begin{bmatrix} R(\mathcal{G})^T & \theta_e(\mathcal{G}) \end{bmatrix}$$

$$S_e(\mathcal{G})^{-1} = \begin{bmatrix} \left(R(\mathcal{G})\,R(\mathcal{G})^T\right)^{-1} R(\mathcal{G}) \\ \theta_e(\mathcal{G})^T \end{bmatrix}$$

where $R(\mathcal{G})$ is defined via (2) and $\theta_e(\mathcal{G})$ denote the orthonormal basis of the flow space [19]. Thus, one can obtain that $E(\mathcal{G})\,\theta_e(\mathcal{G}) = 0$ and $\theta_e(\mathcal{G})^T\theta_e(\mathcal{G}) = I_{L-N+1}$. Applying the similar transformation lead to

$$S_e(\mathcal{G})^{-1}L_e(\mathcal{G})S_e(\mathcal{G}) = \begin{bmatrix} \hat{L}_e(\mathcal{G}) & E_{\mathcal{T}}^T(\mathcal{G})E_{\odot}^{w}(\mathcal{G})\theta_e(\mathcal{G}) \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{12}$$

where $\hat{L}_e(\mathcal{G}) = E_{\mathcal{T}}(\mathcal{G})^T E_{\odot}^{w}(\mathcal{G})R(\mathcal{G})^T$ is refer to as the *essential edge Laplacian* in this paper.

Lemma 5 *The essential edge Laplacian $\hat{L}_e(\mathcal{G})$ has the same eigenvalues of $L_e(\mathcal{G})$ but the zero eigenvalues.*

Proof 1 *From* (12)*, the eigenvalues of the block matrix are the solution of*

$$\lambda^{(L-N+1)}\det\left(\lambda I - \hat{L}_e(\mathcal{G})\right) = 0$$

which shows that $\hat{L}_e(\mathcal{G})$ has exactly all the nonzero eigenvalues of $L_e(\mathcal{G})$. Besides, to proceed further in the proof, we can construct the following Lyapunov equation as

$$H\hat{L}_e(\mathcal{G}) + \hat{L}_e(\mathcal{G})^T H = I_{N-1} \tag{13}$$

where H is a positive definite matrix.

Considering the quasi-strongly connected graph $\mathcal{G}$ and the most commonly used consensus dynamics [1] described as:

$$\dot{x} = -L_n(\mathcal{G})x.$$

Contrary to the most existing works, we study the synchronization problem from the edge perspective by using L_e. In this avenue, we define the *edge state* vector as

$$x_e(t) = E(\mathcal{G})^T x(t) \tag{14}$$

which represents the difference between the state components of two neighboring nodes [3]. Taking the derivative of (14) leads to

$$\dot{x}_e(t) = -L_e(\mathcal{G})x_e(t) \tag{15}$$

which is referred as *edge agreement dynamics* in this paper. In comparison to the node agreement (consensus), the edge agreement, rather than requiring the convergence to the agreement subspace, desires the edge dynamics (15) converge to the origin. Essentially, the evolution of an edge state depends on its current state and the states of its adjacent edges. Meanwhile, the edge agreement implies consensus if the digraph $\mathcal{G}$ has a spanning tree [3].

Make use of the following transformation $S_e^{-1}x_e = \begin{pmatrix} x_{\mathcal{T}} \\ \mathbf{0} \end{pmatrix}$ for (15), where

$$x_{\mathcal{T}} = E_{\mathcal{T}}(\mathcal{G})^T x(t) \tag{16}$$

represents the states across a specific spanning tree of $\mathcal{G}$. Then one can obtain a reduced model representation of (15) as follows

$$\dot{x}_{\mathcal{T}} = -\hat{L}_e x_{\mathcal{T}} \tag{17}$$

which captures the dynamical behaviour of the whole system.

4 Backstepping Design for Networked Multi-agent system

We consider a group of N networked agents with a quasi-strongly connected graph $\mathcal{G}$. The dynamics of the i-th agent is represented by

$$\dot{x}_i = v_i \tag{18}$$

$$\dot{v}_i = f_i(x_i, v_i) + \mu_i \tag{19}$$

where $x_i \in \mathbb{R}^n$ is the position; $v_i \in \mathbb{R}^n$ is the velocity, $\mu_i \in \mathbb{R}^n$ is the control input; and $f_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is unknown. In the following parts, we simply use $E, E_{\odot}^{w}$ instead of $E(\mathcal{G}), E_{\odot}^{w}(\mathcal{G})$.

Assume 1 ([15]) *For each $i = 1, \ldots, N$, there exist $\psi_{f_i}^{x_i}, \psi_{f_i}^{v_i} \in \mathcal{K}_\infty$ such that*

$$|f_i(x_i, v_i)| \le \psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|). \tag{20}$$

The objective of this study is to design a distributed control law, such that the position evolutions of the agents (18)–(19) reach an agreement and the velocities of the agents converge to zeros, i.e., $\lim_{t\to\infty}|x_{e_k}| = 0$, $e_k = (j,i)$ and $\lim_{t\to\infty}|v_i(t)| = 0$ for any $i,j = 1,\ldots,N$. Meanwhile, only the local information and the relative position measurements are available for the design of the edge agreement protocol μ_i.

By taking advantage of the backstepping design, we can transform the original system into a new interconnected system. Considering the subsystems as nodes and the gain connections as directed edges, the interconnected system composed of the $x_\mathcal{T}$-subsystem and the z_i-subsystems can be modeled by a digraph $\hat{\mathcal{G}}$, called the *gain-interconnection digraph* [18].

4.1 Step 1 – The $x_\mathcal{T}$-subsystem

For the kinematics system (18), we take the well-known continuous-time consensus algorithm described in [20]

$$v_i^* = k \sum_{j\in\mathcal{N}_i} (x_j - x_i) \tag{21}$$

as a virtual control input.

For each $i = 1, \ldots, N$, we define the error as

$$z_i = v_i - v_i^* = v_i - k \sum_{j\in\mathcal{N}_i} (x_j - x_i). \tag{22}$$

Denote $z = [z_1^T, \ldots, z_N^T]^T$, $v^* = [v_1^{*T}, \ldots, v_N^{*T}]^T$ and L_n as the Laplacian matrix of $\mathcal{G}$. Obviously, $v^* = -kL_n x$ and

$$v = z - kL_n x. \tag{23}$$

By combining (18) and (16), we have

$$\dot{x}_\mathcal{T} = E_\mathcal{T}^T (z - kE_\odot^w R^T x_\mathcal{T}) = -k\hat{L}_e x_\mathcal{T} + E^T z. \tag{24}$$

4.2 Step 2 – The z_i-subsystem

For $i = 1, \ldots, N$, from (23), we obtain

$$z_i = v_i + kL_{n_i} x \tag{25}$$

where L_{n_i} is the i-th row of L_n. We denote $E_{\odot i}^w$ as the i-th row of $E_\odot^w$. From (10), we have $L_{n_i} = E_{\odot i}^w E^T = E_{\odot i}^w R^T E_\mathcal{T}^T$. Substituting (14) into (25) leads to

$$z_i = v_i + kE_{\odot i} E^T x = v_i + kE_{\odot i}^w R^T x_\mathcal{T}. \tag{26}$$

Differentiating (26), we have

$$\dot{z}_i = \dot{v}_i + kE_{\odot i}^w R^T \dot{x}_\mathcal{T}. \tag{27}$$

By substituting (19) and (24) into (27), we obtain

$$\dot{z}_i = f_i(x_i, v_i) + \mu_i - k^2 E_{\odot i}^w R^T \hat{L}_e x_\mathcal{T} + kL_{n_i} z. \tag{28}$$

Note that $L_{n_i} z = d_i z_i - \sum_{j\in\mathcal{N}_i} z_j$. Then, for $i = 1, \ldots, N$, the z_i-subsystem can be obtained from (28) as follows:

$$\dot{z}_i = f_i(x_i, v_i) + kd_i z_i + \mu_i - k^2 E_{\odot i}^w R^T \hat{L}_e x_\mathcal{T} - k \sum_{j\in\mathcal{N}_i} z_j. \tag{29}$$

To illustrate the interaction relationship of $x_\mathcal{T}$-subsystem and z_i-subsystems, we are motivated to construct the gain-interconnection digraph $\hat{\mathcal{G}}$. Consider the five-agent network with only one root shown in Fig. 1(a), which is strongly connected with $e_1, e_2, e_3, e_4 \subset \mathcal{G}_\mathcal{T}$ and $e_5 \subset \mathcal{G}_c$. Fig. 1(b) illustrates the corresponding gain-interconnection digraph $\hat{\mathcal{G}}$.

Remark 1 *Equation* (24) *and* (29) *describe the interaction between all the subsystems. To construct the gain-interconnection digraph, we need two steps. First, all of z_i-subsystems connect to $x_\mathcal{T}$-subsystem via N directed edges according to equation* (24) *and the definition of E. Second, for z_i-subsystems, the interactions keep in line with the original agent networks from* (29)*. Meanwhile, since $\mathcal{G}$ contains one root (node 1), we have $E_{\odot 1}^w = 0$ while $E_{\odot i}^w \neq 0$ for $i = 2, 3, 4, 5$. As thus, there are directed edges connected to z_i-subsystems for $i = 2, 3, 4, 5$ from $x_\mathcal{T}$-subsystem. More details about the construction of $\hat{\mathcal{G}}$ can be found in our previous works [15][18].*

5 Main Results

In this section, ISS methods, especially the newly developed cyclic-small-gain theorem, are employed to synthesize the interconnection system based on $\hat{\mathcal{G}}$. For the subsystems of the $[x_\mathcal{T}^T, z_1^T, \ldots, z_N^T]^T$-system, we define the following ISS-Lyapunov function candidates:

$$V_{x_\mathcal{T}} = \frac{1}{2} x_\mathcal{T}^T H x_\mathcal{T} \tag{30}$$

$$V_{z_i} = \frac{1}{2} z_i^T z_i, \quad i = 1, 2, \ldots, N \tag{31}$$

where the matrix H is defined in (13). Denote Θ as the set of all cycles of $\hat{\mathcal{G}}$. Let $A_\mathcal{O}(\gamma_{z_i}^{x_\mathcal{T}}, \gamma_{z_i}^{z_j})$ ($\gamma_{z_i}^{x_\mathcal{T}}, \gamma_{z_i}^{z_j} \in \mathcal{K}$) be the product of the gain assigned to the edges of a simple loop $\mathcal{O}$ [16].

Theorem 1 *Suppose $\mathcal{G}$ is quasi-strongly connected and associated with the edge Laplacian L_e, then for any specified constant $\sigma_{z_i} > 0$ and $\gamma_{z_i}^{x_\mathcal{T}}, \gamma_{z_i}^{z_j} \in K_\infty$, we can design*

$$\begin{aligned}\mu_i = &-\frac{z_i}{|z_i|}\Big(\psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|) + k^2|E_{\odot i}^w R^T \hat{L}_e|\rho_{x_\mathcal{T}}^{z_i}(|z_i|) \\ &+ k\sum_{j\in\mathcal{N}_i} \rho_{z_j}^{z_i}(|z_i|)\Big) - \Big(\frac{\sigma_{z_i}}{2} + kd_i\Big) z_i\end{aligned} \tag{32}$$

with $\rho_{x_\mathcal{T}}^{z_i}(s) = \underline{\alpha}^{-1} \circ \left(\gamma_{z_i}^{x_\mathcal{T}}\right)^{-1} \circ \overline{\alpha}(s)$ and $\rho_{z_j}^{z_i}(s) = \underline{\alpha}^{-1} \circ \left(\gamma_{z_i}^{z_j}\right)^{-1} \circ \overline{\alpha}(s)$. By using the edge agreement protocol (32)*, as well as satisfying the following conditions*

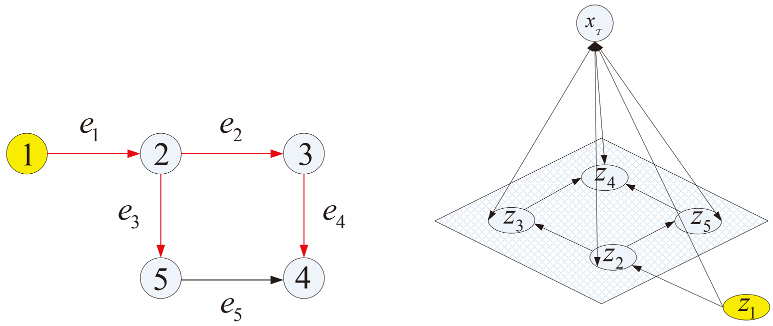
$$\gamma_{x_\mathcal{T}}^{z_i}(s) = \lambda_{\max}(H)\, N \left(\frac{\|H\|\,\|E\|}{\frac{1}{2}k - \epsilon}\right)^2 s \tag{A1}$$

$$A_\mathcal{O}(\gamma_{z_i}^{x_\mathcal{T}}, \gamma_{z_i}^{z_j}) < \mathrm{Id}, \text{ for all } \mathcal{O} \in \Theta \tag{A2}$$

where $0 < \epsilon < \frac{1}{2}k$, the objective edge agreement can be achieved.

Proof 2 *To begin with, taking the derivative of $V_{x_\mathcal{T}}$ along the trajectory* (24)*, we have*

$$\begin{aligned}\nabla V_{\tilde{x}} &= \frac{1}{2}\Big(x_\mathcal{T}^T H \dot{x}_\mathcal{T} + \dot{x}_\mathcal{T}^T H x_\mathcal{T}\Big) \\ &= \frac{1}{2}\Big(-kx_\mathcal{T}^T\Big(H\hat{L}_e + \hat{L}_e^T H\Big)x_\mathcal{T} + x_\mathcal{T}^T H E^T z + z^T E H x_\mathcal{T}\Big) \\ &\leq -\frac{1}{2}k\,|x_\mathcal{T}|^2 + |x_\mathcal{T}|\,\|E\|\,\|H\|\,|z|.\end{aligned} \tag{33}$$



(a) A 5-agent system with quasi-strongly connected graph $\mathcal{G}$.

(b) The corresponding gain-interconnection digraph $\hat{\mathcal{G}}$.

Fig. 1: A quasi-strongly graph and its gain-interconnection digraph.

Considering $\gamma_{x_\mathcal{T}}^{z_i}(s)$ defined in A1, if

$$V_{x_\mathcal{T}} \geq \max_{i=1,2,\cdots N} \left\{ \gamma_{x_\mathcal{T}}^{z_i}(V_{z_i}) \right\} \tag{34}$$

we have

$$\begin{aligned} V_{x_\mathcal{T}} &\geq \lambda_{\max}(H) \left(\frac{\|E\| \, \|H\|}{\frac{1}{2}k - \epsilon} \right)^2 \sum_{i=1}^{N} V_{z_i} \\ &= \frac{1}{2} \lambda_{\max}(H) \left(\frac{\|E\| \, \|H\|}{\frac{1}{2}k - \epsilon} \right)^2 z^T z. \end{aligned}$$

According to the following inequality:

$$V_{x_\mathcal{T}} = \frac{1}{2} x_\mathcal{T}^T H x_\mathcal{T} \leq \frac{1}{2} \lambda_{\max}(H) |x_\mathcal{T}|^2$$

we obtain

$$|z| \leq \frac{(\frac{1}{2}k - \epsilon)|x_\mathcal{T}|}{\|E\| \, \|H\|}. \tag{35}$$

Given any positive $0 < \epsilon < \frac{1}{2}k$ and substituting (35) *into* (33)*, we have*

$$V_{x_\mathcal{T}} \geq \max_{i=1,2,\cdots N} \left\{ \gamma_{x_\mathcal{T}}^{z_i}(V_{z_i}) \right\} \Rightarrow \nabla V_{x_\mathcal{T}} \leq -\epsilon V_{x_\mathcal{T}}$$

which implies $V_{x_\mathcal{T}}$ is an ISS-Lyapunov function and $x_\mathcal{T}$-subsystem described in (24) *is ISS.*

Considering $V_{z_i} \geq \max_{j \in \mathcal{N}_i} \left\{ \gamma_{z_i}^{x_\mathcal{T}}(V_{x_\mathcal{T}}), \gamma_{z_i}^{z_j}(V_{z_j}) \right\}$ and using the definitions of $V_{x_\mathcal{T}}$ and V_{z_i} $(i = 1, \ldots, N)$ in (30) *and* (31)*, we have*

$$|x_\mathcal{T}| \leq \rho_{x_\mathcal{T}}^{z_i}(|z_i|) \tag{36}$$

$$|z_j| \leq \rho_{z_j}^{z_i}(|z_i|), \quad j \in \mathcal{N}_i. \tag{37}$$

Taking the derivative of V_{z_i}, and using (20), (29), (36) *and* (37)*, we have*

$$\begin{aligned} \nabla V_{z_i} \dot{z}_i \leq & z_i^T(\mu_i + k d_i z_i) + |z_i| \Big(\psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|) \\ & + k^2 |E_{\odot i}^w R^T \hat{L}_e| |x_\mathcal{T}| + k \sum_{j \in \mathcal{N}_i} |z_j| \Big) \\ \leq & z_i^T(\mu_i + k d_i z_i) + |z_i| \Big(\psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|) \\ & + k^2 |E_{\odot i}^w R^T \hat{L}_e| \rho_{x_\mathcal{T}}^{z_i}(|z_i|) + k \sum_{j \in \mathcal{N}_i} \rho_{z_j}^{z_i}(|z_i|) \Big). \end{aligned}$$

By using (32)*, we achieve*

$$\nabla V_{z_i} \dot{z}_i \leq -\frac{\sigma_{z_i}}{2} z_i^T z_i = -\sigma_{z_i} V_{z_i}$$

which implies z_i-subsystem is ISS.

Till now, the ISS properties of $x_\mathcal{T}$-subsystem and z_i-subsystem have been proven. From Lemma 1, if the cyclic-small-gain condition A2 is satisfied, the composed system is ISS. In the meantime, as (32) is designed, the ISS system is essentially an unforced dynamical system. As known that, the ISS of an unforced system leads to the globally asymptotical stability [13], which implies $\lim_{t\to\infty} x_\mathcal{T}(t) = 0$ and $\lim_{t\to\infty} z(t) = 0$. From (14)*, we have $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$. Considering the definition of z_i described in* (22)*, we obtain $\lim_{t\to\infty} v_i(t) = 0$. Then the agreement is achieved.*

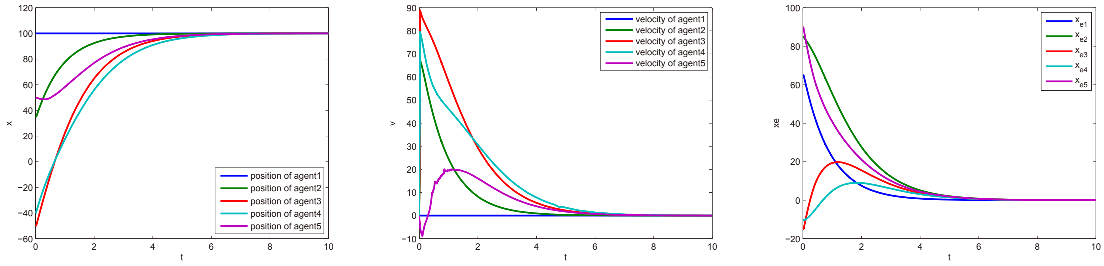
Remark 2 *One can see that only the local information about the position x_i, the velocity v_i and the relative position measurements are available for the edge agreement protocol* (32)*. Clearly, without the relative velocity measurement, the protocol is executable for the practical multi-agent system.*

Remark 3 *The detailed discussions about the explicit cyclic-small-gain conditions required in A2 can be found in our previous work [18]. In fact, the cyclic-small-gain conditions can be guaranteed simply by choosing the nonlinear gains as: $\gamma_{z_j}^{z_i} < \mathrm{Id}$, with $(i,j) \in \mathcal{E}$ and $\gamma_{z_i}^{x_\mathcal{T}} < \left(\gamma_{x_\mathcal{T}}^{z_i}\right)^{-1}$, $i = 1, 2, \cdots, N$.*

6 SIMULATION

Numerical simulations are given to illustrate the obtained theoretical results. We consider a five-agent system with $\mathcal{G}$ which is quasi-strongly connected with 1 root shown in Figure 1(a). The dynamics of a single agent is assumed to be (18) and (19) with unknown $f_i(x_i, v_i)$ which satisfies $|f_i(x_i, v_i)| \leq v_i^2$. Note that f_i does not satisfy the global Lipschitz condition. Comparing with (20), we take $\psi_{f_i}^{x_i}(|x_i|) = 0$ and $\psi_{f_i}^{v_i}(|v_i|) = v_i^2$. Let the weighted matrix $\mathcal{W}(\mathcal{G})$ be equal to I_5. The incidence matrix E and the edge Laplacian L_e are

$$\hat{L}_e = \begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.00 \\ -1.00 & 1.00 & 0.00 & 0.00 \\ -1.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & -1.00 & 2.00 \end{pmatrix} H = \begin{pmatrix} 1.06 & 0.25 & 0.31 & 0.28 \\ 0.25 & 0.5 & 0.00 & 0.00 \\ 0.31 & 0.00 & 0.58 & 0.08 \\ 0.03 & 0.00 & 0.08 & 0.25 \end{pmatrix}.$$



(a) The synchronization of the positions. (b) The synchronization of the velocities. (c) The edge agreement of x_e.

Fig. 2: The simulation results under the distributed control law (32).

Directed calculation yields $|E^w_{\odot 1} R^T \hat{L}_e| = 0.00$, $|E^w_{\odot 2} R^T \hat{L}_e| = 1.00$, $|E^w_{\odot 3} R^T \hat{L}_e| = 1.414$, $|E^w_{\odot 4} R^T \hat{L}_e| = 5.0990$, $|E^w_{\odot 5} R^T \hat{L}_e| = 1.4142$. By simply choosing $\gamma^{z_j}_{z_i}(s) = 0.9s$ for $(i,j) \in \mathcal{E}$ and $\gamma^{x_\mathcal{T}}_{z_i}(s) = 0.01s$ for $i = 1, 2, \cdots, 5$. Accordingly, we have $\rho^{z_i}_{x_\mathcal{T}}(s) = 10s$ and $\rho^{z_i}_{z_j}(s) = 1.0541s$. By taking $\sigma_{z_i} = 1$ and $k = 1$, the control law (32) can be determined. The simulation results are shown in Figure. 2. The positions of all agents converge to the 100cm in Figure. 2(a), while the velocities of all agents synchronize to 0 in Figure. 2(b). Figure. 2(c) shows that the edge agreement is achieved.

7 CONCLUSIONS

In this paper, a second-order edge agreement protocol under digraph has been developed for dealing with the unknown locally Lipschitz continuous dynamics. The notion of the edge Laplacian of digraph has been introduced to describe the agreement in the edge setting, which shows an enormous potential on solving the coordination problem. With the aid of the backstepping design and the newly developed ISS cyclic-small-gain theorem, the challenge caused by unknown locally Lipschitz dynamics has been well settled.

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