Robust Bipartite Consensus and Tracking Control of High-Order Multiagent Systems With Matching Uncertainties and Antagonistic Interactions

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***Abstract*—This paper is concerned with general coopetition networks with signed graphs, based on which both the bipartite consensus and tracking control problems for networked systems subject to nonidentical matching uncertainties are studied. For the case of undirected and connected communication graphs, we propose a distributed discontinuous nonlinear controller which can achieve the bipartite consensus. To cancel the chattering phenomenon of the discontinuous controller, a continuous one is designed by using the boundary layer technique, under which the bipartite consensus error is shown to be uniformly ultimately bounded and can exponentially converge to a small adjustable bounded set. Further, considering the case of a leader having a bounded control action, we present a continuous controller to guarantee the ultimate boundedness of the bipartite tracking error.**

***Index Terms*—Bipartite consensus, bipartite tracking, matching uncertainty, robustness, signed graph.**

I. Introduction

COORDINATION control of multiagent systems has recently attracted considerable attention from various scientific communities. This is partly due to its wide range of potential applications in such areas as UAV swarms, cooperative surveillance, smart building, and intelligent transportation [1], [2]. A large body of research works have been reported in this direction and related problems (see [3]–[14] and the references therein). As an important subarea of coordination control, the consensus problem can be roughly classified into consensus without leader (i.e., leaderless consensus) and leader–follower consensus, the latter of which is also termed as distributed tracking [15].

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Most existing works on coordination control focus on cooperative multiagent systems, in which the interactions among the nodes are characterized by non-negative edge weights. In several scenarios such as social networks [16], [17], however, it is more plausible to suppose that the agents in a network can either collaborate or compete with each other. That is, the interaction network is modeled by a signed graph composed of both negative and positive edge weights, which represent competitive and cooperative interactions, respectively. Coordination control with signed communication graphs has attracted many interests recently. The bipartite consensus problems are first introduced and investigated for single-integrator networks in [18], where it is found that under certain conditions all the agents can converge to a consensus value that is the same for all agents in modulus but not in sign. As natural extensions in the sense of agent dynamics, the bipartite consensus problems for second-order integrators and general linear networks are further addressed in [19] and [20]. Zhang and Chen [21] studied the bipartite consensus problems for multi-input multioutput linear systems with directed signed graphs. It is found that by building an equivalence between the conventional and bipartite consensus problems, the existing consensus protocols can be utilized straightforwardly to solve bipartite consensus problems. The bipartite consensus problems are also studied from different perspectives. In [22], the cluster consensus problems are addressed in interactively balanced, sub-balanced and unbalanced digraphs. The notion of interval bipartite consensus is introduced in [23], where it is derived that the interval bipartite consensus can be achieved, if the associated signed digraph contains a directed spanning tree.

Notably, all the aforementioned works assume that all the agents' dynamics are in the same form and precisely known. Such an assumption may be restrictive in some applications. In engineering environment, however, the agents in the network can be perturbed by external disturbances or certain parameter uncertainties. Several works, such as [24] an [25], have been reported to investigate the robust consensus problems by taking in account matching uncertainties. In [24], the robust consensus problems of multiagent systems with nonidentical matching uncertainties are studied. Based on parameterizations of disturbances, adaptive consensus tracking protocols are proposed in [25]. Note that the consensus protocols in these works [24], [25] are designed for agents coordinating over non-negative communication graphs. How to design bipartite consensus controllers for the agents coordinating over signed communication graphs and subject to matching uncertainties is more challenging and an interesting work.

This paper addresses the robust bipartite consensus and tracking control of multiagent systems perturbed by nonidentical matching uncertainties for both the cases without and with a leader having bounded control input. Since the uncertainties associated with the agents might be time-varying and nonlinear, the resulting multiagent systems studied in the current paper are essentially heterogeneous, which renders the consensus and tracking control problems more challenging to solve. The communication topology is described by a signed graph, which is particularly susceptible to describe the competitive and cooperative interactions among the agents. First, for the case of undirected and connected graphs, a discontinuous controller, including a discontinuous nonlinear function to cope with the effect of the matching uncertainties, is proposed and designed. It is verified that the proposed discontinuous controller can ensure that the agents' states can asymptotically converge to a common value but with opposite signs, i.e., the bipartite consensus is achieved. It is worth mentioning that the undesirable chattering phenomenon will take place when implementing the proposed discontinuous controller. In order to cancel the chattering effect, a continuous controller is sequentially proposed and designed by using the boundary layer technique, under which bipartite consensus is achieved in the sense that the bipartite consensus error is uniformly ultimately bounded and can converge exponentially to a small set. Next, we extend to consider the case where there exists a leader having a norm-bounded unknown control input. It is assumed that the leader's control input is not accessible to any follower, under which the bipartite tracking problem is more troublesome. A continuous controller is proposed to guarantee the ultimate boundedness of the bipartite tracking error. Sufficient conditions are provided in terms of the algebraic Riccati equation (ARE) to design both the bipartite consensus and tracking controllers.

The remainder of this paper is organized as follows. Section II introduces mathematical preliminaries that are necessary for our analysis. The bipartite consensus problem and the bipartite tracking problem are formulated and discussed in Sections III and IV, respectively. Section V provides numerical simulation examples. Finally, Section VI concludes this paper.

II. Mathematical Preliminaries

A coopetition network can be conveniently described by a signed graph $\mathcal{G} = (\mathcal{K}, \mathcal{V}, \mathcal{B})$, where $\mathcal{K} = \{k_1, \dots, k_N\}$ denotes the set of nodes, $\mathcal{V} = \{(k_i, k_j) : k_i, k_j \in \mathcal{K}\}$ represents the set of edges and $\mathcal{B}$ is the adjacency matrix describing the edge information of the graph. The adjacency matrix $\mathcal{B} = [b_{ij}] \in \mathbf{R}^{N\times N}$ is defined by $b_{ii} = 0$, $b_{ij} \neq 0$ if $(i, j) \in \mathcal{V}$ and 0 otherwise. Specifically, if $b_{ij} > 0$ (< 0), the edge (k_i, k_j) is called positive (negative) and the interaction between nodes k_i and k_j is cooperative (competitive). A graph is said to be undirected provided that it satisfies $(k_j, k_i) \in \mathcal{V}$ as long as $(k_j, k_i) \in \mathcal{V}$. A directed path from node k_i^1 to node k_i^l is a sequence of ordered edges (k_i^j, k_i^{j+1}), $j = 1, \dots, l-1$. An undirected path is defined analogously. A directed graph $\mathcal{G}(\mathcal{B})$ is strongly connected if there exists a directed path between each ordered pair of distinct nodes. An undirected graph is connected if for any pair of distinct nodes, there exists a path between them, and otherwise is disconnected. A directed graph is said to contain a directed spanning tree, if there exists a node called the root such that all other nodes in the graph having directed paths starting from the root.

The Laplacian matrix $\mathcal{L} \in \mathbf{R}^{N\times N}$ associated with a signed graph $\mathcal{G}(\mathcal{B})$ is defined as

$$\mathcal{L} = \operatorname{diag}\left(\sum_{j=1}^{k_N} |b_{1j}|, \dots, \sum_{j=1}^{k_N} |b_{k_N j}|\right) - \mathcal{B}. \tag{1}$$

For a signed graph, an important property is given as follows. A signed graph is said to be structurally balanced, if it admits a bipartition of two nonempty node sets $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{K}$ and $\mathcal{K}_1 \cap \mathcal{K}_2 = \emptyset$, and $b_{ij} \geq 0$ for all $k_i, k_j \in \mathcal{K}_p$ and $b_{ij} \leq 0$ for all $k_i \in \mathcal{K}_p$, $j \in \mathcal{K}_q$, $p \neq q$, where $p, q \in \{1, 2\}$.

A signed graph is structural balanced implies that the signed graph can be split two subgroups such that the interactions in each subgroup are cooperative and these between two subgroups are competitive.

For a structurally balanced signed graph, a gauge transformation, introduced in [18] to characterize the node bipartition, is described by the diagonal matrix as follows:

$$D = \operatorname{diag}(\sigma_1, \dots, \sigma_{k_N}) \tag{2}$$

where $\sigma_{k_i} = 1$ for $k_i \in \mathcal{K}_1$ and $\sigma_{k_i} = -1$ for $k_i \in \mathcal{K}_2$.

Lemma 1 [18], [21]: For a signed graph $\mathcal{G}(\mathcal{B})$ which has a directed spanning tree, $\mathcal{G}$ is structurally balanced if and only if any of the following conditions holds.

1) The corresponding undirected graph $\mathcal{G}(\mathcal{B}_u)$ is structurally balanced, where $\mathcal{B}_u = \mathcal{B} + \mathcal{B}^T$.
2) $D\mathcal{B}D$ is an non-negative adjacency matrix associated with a conventional (unsigned) graph.

Lemma 2 [21], [27], [28]: Supposing that the signed graph $\mathcal{G}(\mathcal{B})$ having a directed spanning tree is structurally balanced, the associated Laplacian matrix $\mathcal{L}$ has zero as a simple eigenvalue with $\mathbf{d} = [\sigma_1, \dots, \sigma_N]^T$ as its eigenvector, and all the other eigenvalues have positive real parts. The smallest nonzero eigenvalue $\lambda_2(\mathcal{L})$ for a undirected connected graph satisfies $\lambda_2(\mathcal{L}) = \min_{x\neq 0, \mathbf{d}^T x = 0}(x^T \mathcal{L} x / x^T x)$.

III. Robust Bipartite Consensus for Leaderless Networks

A. Problem Formulation of Bipartite Consensus

In this paper, we consider a group of N agents, each of which is described by general linear dynamics subject to matching uncertainties

$$\dot{x}_i = Ax_i + B(u_i + \varphi_i(x_i, t)), \quad i = 1, \dots, N \tag{3}$$

where $x_i \in \mathbf{R}^n$ and $u_i \in \mathbf{R}^p$ are, respectively, the state and control input of the ith agent, and (A, B) is assumed to be stabilizable. $\varphi_i(x_i, t) \in \mathbf{R}^n$, $i = 1, \dots, N$ in (3) denote the matching uncertainties which can represent certain parameter uncertainties as well as external disturbances of the ith agent.

Regarding the uncertainties $\varphi_i(x_i, t)$, we suppose that the following assumption holds.

Assumption 1: $\varphi_i(x_i, t)$ is continuous and bounded, i.e., there exists a constant $\gamma > 0$ such that $\|\varphi_i(x_i, t)\| \leq \gamma$, for $i = 1, \ldots, N$.

The communication topology among the N agents is modeled by an undirected signed graph $\mathcal{G}$. The following assumption holds.

Assumption 2: The signed graph $\mathcal{G}$ is connected and structurally balanced.

For a network satisfying Assumption 2, the agents in the network can be split into two separate subgroups $\mathcal{K}_1$ and $\mathcal{K}_2$. The goal of this section is to design a distributed controller for each agent (3), such that the agents in the two subgroups of the network reach a bipartite consensus, which can be characterized by

$$\lim_{t\to\infty}\left(x_p(t) - \frac{1}{N}\sum_{j=1}^{N}\sigma_j x_j(t)\right) = 0, \quad \text{for all} \quad p \in \mathcal{K}_1$$

$$\lim_{t\to\infty}\left(x_q(t) - \left(-\frac{1}{N}\sum_{j=1}^{N}\sigma_j x_j(t)\right)\right) = 0, \quad \text{for all} \quad q \in \mathcal{K}_2.$$

In other words, a bipartite consensus is achieved, if the following equation holds:

$$\lim_{t\to\infty}\left(x_i(t) - \frac{1}{N}\sum_{j=1}^{N}\sigma_i\sigma_j x_j(t)\right) = 0, \quad i = 1, \ldots, N. \tag{4}$$

Remark 1: The notion of bipartite consensus is first introduced and studied in [18], which means that all the agents in a network with cooperative-antagonistic actions only reach agreement in modulus but not in sign. By contrast, the classical consensus, which is investigated under the assumption that all the interactions among agents in the network are cooperative, implies that all the agents reach agreement in both the modulus and sign, i.e., $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$. Therefore, the bipartite consensus can include the classical consensus as a special case.

B. Discontinuous Static Controllers

Based on the state feedback and the interactions between neighboring agents, we propose a distributed static controller for the ith agent as follows:

$$\begin{aligned} u_i = {} & c_1\sum_{j=1}^{N}|b_{ij}|K\big(x_i - \mathrm{sgn}(b_{ij})x_j\big) \\ & + c_2\sum_{j=1}^{N}|b_{ij}|g\big(K\big(x_i - \mathrm{sgn}(b_{ij})x_j\big)\big), \quad i = 1, \ldots, N \end{aligned} \tag{5}$$

where $c_1 > 0$, $c_2 > 0$ are the scalar control gains, K is the feedback gain matrix, $\mathrm{sgn}(\cdot)$ is the signum function and $g(\cdot)$ is a nonlinear function with the definition of

$$g(\omega) = \begin{cases} \frac{\omega}{\|\omega\|} & \text{if } \|\omega\| \neq 0 \\ 0 & \text{if } \|\omega\| = 0. \end{cases} \tag{6}$$

Let $x = [x_1^T, \ldots, x_N^T]^T$ and $\Psi(x, t) = [\varphi_1(x_1, t)^T, \ldots, \varphi_N(x_N, t)^T]^T$. By substituting (5) into (3), the closed-loop system can be described by

$$\begin{aligned} \dot{x} = {} & (I_N \otimes A + c_1\mathcal{L} \otimes BK)x + (I_N \otimes B)\Psi(x, t) \\ & + c_2(I_N \otimes B)\begin{bmatrix} \sum_{j=1}^{N}|b_{1j}|g\big(K\big(x_1 - \mathrm{sgn}(b_{1j})x_j\big)\big) \\ \vdots \\ \sum_{j=1}^{N}|b_{Nj}|g\big(K\big(x_N - \mathrm{sgn}(b_{Nj})x_j\big)\big) \end{bmatrix}. \end{aligned} \tag{7}$$

Let $\xi = [\xi_1^T, \ldots, \xi_N^T]^T = [M \otimes I_n]x$, where $M = I_N - (1/N)\mathbf{d}\mathbf{d}^T$, $\mathbf{d} = [\sigma_1, \ldots, \sigma_N]^T$. By noting that M is similar to the matrix $I_N - (1/N)\mathbf{1}\mathbf{1}^\mathbf{T}$, it is not difficult to get that M has 0 as a simple eigenvalue and 1 as the other eigenvalue, and $\mathbf{d}$ is an eigenvector corresponding with the eigenvalue 0. Then, it follows from the definition of ξ that $\xi = 0$ if and only if $\sigma_1 x_1 = \sigma_2 x_2 = \cdots = \sigma_N x_N$, i.e., the equation in (4) holds. Therefore, ξ can be defined as the bipartite consensus error.

Using the fact that $\mathcal{L}M = \mathcal{L}$, it follows from (7) that the bipartite consensus error ξ satisfies:

$$\begin{aligned} \dot{\xi} = {} & (I_N \otimes A + c_1\mathcal{L} \otimes BK)\xi + (M \otimes B)\Psi(x, t) \\ & + c_2(M \otimes B)G(\xi) \end{aligned} \tag{8}$$

where

$$G(\xi) \triangleq \begin{bmatrix} \sum_{j=1}^{N}|b_{1j}|g\big(K\big(\xi_1 - \mathrm{sgn}(b_{1j})\xi_j\big)\big) \\ \vdots \\ \sum_{j=1}^{N}|b_{Nj}|g\big(K\big(\xi_N - \mathrm{sgn}(b_{Nj})\xi_j\big)\big) \end{bmatrix}.$$

In what follows, a sufficient condition is provided to design the discontinuous controller (5).

Theorem 1: Suppose that Assumptions 1 and 2 hold. The bipartite consensus problem of the agents in (3) can be solved by the controller (5) with $c_1 \geq (1/2\lambda_2)$, $c_2 \geq \rho$, and $K = -B^T R$, where $\rho = \gamma(N-1)\sqrt{N}$, λ_2 is the smallest nonzero eigenvalue of $\mathcal{L}$, and R is a positive solution to the ARE as follows:

$$RA + A^T R - RBB^T R + Q = 0 \tag{9}$$

with $Q > 0$.

Proof: Choose the following Lyapunov function candidate:

$$V_1 = \xi^T(M \otimes R)\xi. \tag{10}$$

In light of the definition of ξ, we can see that $(\mathbf{d}^T \otimes I)\xi = 0$. For the connected graph $\mathcal{G}$, by using Lemma 2, we have

$$V_1(\xi) \geq \lambda_{\min}(R)\|\xi\|^2. \tag{11}$$

By noting the fact that $M^2 = M$, the time derivative of V_1 along the trajectory of (8) can be obtained as

$$\dot{V}_1 = \xi^T \mathcal{H}\xi + 2\xi^T (M \otimes RB)\Psi(x,t) + 2c_2\xi^T (M \otimes RB)G(\xi) \tag{12}$$

where $\mathcal{H} = M \otimes (RA + A^T R) - 2c_1\mathcal{L} \otimes RBB^T R$. By utilizing Assumption 1, we can obtain that $\|\Psi(x,t)\| \leq \sqrt{N}\gamma$. Then, we have

$$\begin{aligned}\xi^T (M \otimes RB)\Psi(x,t) &\leq \left\|\left(M \otimes B^T R\right)\xi\right\| \|\Psi(x,t)\| \\ &\leq \frac{\gamma}{\sqrt{N}} \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} \left\|B^T R\left(\xi_i - \sigma_i\sigma_j\xi_j\right)\right\| \\ &\leq \gamma\sqrt{N} \max_i \sum_{j=1,j\neq i}^{N} \left\|B^T R\left(\xi_i - \sigma_i\sigma_j\xi_j\right)\right\| \\ &\leq \frac{\rho}{2} \sum_{i=1}^{N}\sum_{j=1}^{N} \left|b_{ij}\right| \left\|B^T R\left(\xi_i - \mathrm{sgn}\left(b_{ij}\right)\xi_j\right)\right\|. \end{aligned} \tag{13}$$

By using the definition of the nonlinear function $g(\cdot)$, we obtain that

$$\begin{aligned}&2c_2\xi^T (M \otimes RB)G(\xi) \\ &= -2c_2 \sum_{i=1}^{N}\sum_{j=1}^{N} \left|b_{ij}\right| \frac{\xi_i^T RBB^T R\left(\xi_i - \mathrm{sgn}\left(b_{ij}\right)\xi_j\right)}{\left\|B^T R\left(\xi_i - \mathrm{sgn}\left(b_{ij}\right)\xi_j\right)\right\|} \\ &= -c_2 \sum_{i=1}^{N}\sum_{j=1}^{N} \left|b_{ij}\right| \frac{\left\|B^T R\left(\xi_i - \mathrm{sgn}\left(b_{ij}\right)\xi_j\right)\right\|^2}{\left\|B^T R\left(\xi_i - \mathrm{sgn}\left(b_{ij}\right)\xi_j\right)\right\|} \\ &= -c_2 \sum_{i=1}^{N}\sum_{j=1}^{N} \left|b_{ij}\right| \left\|B^T R\left(\xi_i - \mathrm{sgn}\left(b_{ij}\right)\xi_j\right)\right\|. \end{aligned} \tag{14}$$

Combining (13) and (14), it follows from (12) that:

$$\begin{aligned}\dot{V}_1 \leq\ & \xi^T \mathcal{H}\xi + (\rho - c_2) \\ & \times \sum_{i=1}^{N}\sum_{j=1}^{N} \left|b_{ij}\right| \left\|B^T R\left(\xi_i - \mathrm{sgn}\left(b_{ij}\right)\xi_j\right)\right\|. \end{aligned} \tag{15}$$

By choosing $c_2 \geq \rho$, it can be obtained that:

$$\dot{V}_1 \leq \xi^T \mathcal{H}\xi. \tag{16}$$

For an undirected and connected graph $\mathcal{G}$, it follows from Lemma 2 that there exists a unitary matrix such that:

$$U^T \mathcal{L} U = \Lambda \triangleq \mathrm{diag}(0, \lambda_2, \ldots, \lambda_N) = \mathrm{diag}(0, \Lambda_1) \tag{17}$$

where $\lambda_2 \leq \cdots \leq \lambda_N$ are the nonzero eigenvalues of $\mathcal{L}$. Noting the fact that $M\mathcal{L} = \mathcal{L} = \mathcal{L}M$, it can be obtained that $U^T M U = \tilde{M} \triangleq \mathrm{diag}(0, I_{N-1})$. Moreover, U can be denoted by $U = \left[(\mathbf{d}/\sqrt{N})\ Y_1\right]$, where $Y_1 \in \mathbf{R}^{N\times N-1}$. Let $\widetilde{\xi} \triangleq [\widetilde{\xi}_1^T, \ldots, \widetilde{\xi}_N^T]^T = (U^T \otimes I_n)\xi$. By using the definitions of ξ and $\widetilde{\xi}$, it can be obtained that $\widetilde{\xi}_1 = ([\mathbf{d}^T/\sqrt{N}]\otimes I_n)\xi = ([\mathbf{d}^T/\sqrt{N}]M\otimes I_n)x = 0$. Then, we have

$$\begin{aligned}\xi^T \mathcal{H}\xi &= \sum_{i=2}^{N} \widetilde{\xi}_i^T \left(RA + A^T R - 2c_1\lambda_i RBB^T R\right)\widetilde{\xi}_i \\ &\leq \sum_{i=2}^{N} \widetilde{\xi}_i^T \left(RA + A^T R - RBB^T R\right)\widetilde{\xi}_i \\ &= -\sum_{i=2}^{N} \widetilde{\xi}_i^T Q\widetilde{\xi}_i < 0 \end{aligned} \tag{18}$$

which implies $\dot{V}_1 < 0$. Therefore, the bipartite consensus error ξ asymptotically converges to zero as $t \to \infty$, i.e., $\lim_{t\to\infty} \xi(t) = 0$. Thus, we have $\lim_{t\to\infty}(x_i(t) - (1/N)\sum_{j=1}^{N} \sigma_i\sigma_j x_j) = 0$, which implies that the agents achieve the bipartite consensus. This completes the proof. ∎

Remark 2: In the related works on bipartite consensus, e.g., [18]–[21], the agents' dynamics are supposed to be identical and free of uncertainties. By contrast, in the current paper we consider the bipartite consensus problem for the case where the agents are perturbed by nonidentical uncertainties. Here the resulting multiagent system is essentially heterogeneous, the bipartite consensus problem of which is quite troublesome and challenging to solve. The distributed controllers in [18]–[21] are not applicable any more. In this section, a novel nonlinear controller is proposed, in which a nonlinear function is included to tackle the effect of the uncertainties.

Remark 3: Different from the previous works on robust consensus or tracking problems of multiagent systems perturbed by matching uncertainties, e.g., [24]–[26], where it is assumed that all the interactions among agents are cooperative, in the current paper we consider the case where the interactions can be either cooperative or competitive. Such networks with antagonistic interactions are very common especially in the area of social network theory [16], [17], where positive/negative weights represent, e.g., like/dislike and trust/distrust interactions among agents. It should be mentioned that the proof of Theorem 1 is partly inspired by [24] and [26].

Note that, compared to the controllers in [24], which are designed in the node-based framework, the controllers in this paper are designed from the edge-based viewpoint, which can preserve certain symmetry in undirected networks. More importantly, the controller (5) can be utilized with little modifications to tackle the case with switching topologies, which remains unclear under the node-based controllers in [24].

Let $\mathcal{G}_N$ be the set of all possible undirected and connected graphs with N nodes. $\mathcal{G}_{\delta(t)} = (\mathcal{K}, \mathcal{V}_{\delta(t)}, \mathcal{B}_{\delta(t)}) \in \mathcal{G}_N$ represents the communication graph at time t, where $\delta(t)$ is a piecewise constant switching signal with the switching instants $t_0, t_1, \ldots$ satisfying $t_{k+1} - t_k \geq \tau \geq 0$ for some constant τ and any $k \geq 0$. Here we further assume that the agents of the network can be split into two separate subgroups such that for each graph $\mathcal{G}_{\delta(t)}$ at time t, the edges between the two subgroups are negative and the edges within each subgroup are positive. Under this assumption, it should be noted that over time $t \geq 0$, each Laplacian matrix $\mathcal{L}_{\delta(t)}$ associated with $\mathcal{G}_{\delta(t)}$ has zero as a simple eigenvalue with $\mathbf{d} = [\sigma_1, \ldots, \sigma_N]^T$ as a corresponding eigenvector, and all the other eigenvalues are positive. In this case, the controller (5) can be modified as

$$\begin{aligned}u_i = c_1 &\sum_{j=1}^{N} |b_{ij}(t)| K\left(x_i - \mathrm{sgn}\left(b_{ij}(t)\right)x_j\right) \\ &+ c_2 \sum_{j=1}^{N} |b_{ij}(t)| g\left(K\left(x_i - \mathrm{sgn}\left(b_{ij}(t)\right)x_j\right)\right), \quad i = 1, \ldots, N \end{aligned} \tag{19}$$

where $b_{ij}(t)$ is the (i,j)th entry of the adjacency matrix associated with $\mathcal{G}_{\delta(t)}$, and the rest of the variables are defined as in (5).

Corollary 1: Consider the case of switching graphs $\mathcal{G}_{\delta(t)} \in \mathcal{G}_N$. The agents in (3) can achieve bipartite consensus under the controller (19) with $c_1 \geq (1/2\lambda_2^{\min})$, where $\lambda_2^{\min} \triangleq \min_{\mathcal{G}_{\delta(t)}\in\mathcal{G}_N}\{\lambda_2(\mathcal{L}_{\delta(t)})\}$ denotes the minimum of the smallest nonzero eigenvalues of $\mathcal{L}_{\delta(t)}$ for $\mathcal{G}_{\delta(t)} \in \mathcal{G}_N$, and the rest variables c_2, K are designed as in Theorem 1.

Proof: Choose V_1 in (10) as the Lyapunov function candidate. Noting that fact that $M^2 = M$, V_1 can actually be written as $V_1 = \xi^T(I_N \otimes R)\xi$, which is independent of the communication topologies. Following similar procedures as in the proof of Theorem 1, it can be verified that the time derivative of V_1 satisfies:

$$\dot{V}_1 \leq \xi^T\big(I_N \otimes \big(RA + A^TR\big) - 2c_1\mathcal{L}_{\delta(t)} \otimes RBB^TR\big)\xi. \tag{20}$$

Since $\mathcal{G}_{\delta(t)}$ is connected and $(\mathbf{d}^T \otimes I)\xi = 0$, using Lemma 2, we can obtain that $\xi^T(\mathcal{L}_{\delta(t)} \otimes I)\xi \geq \lambda_2^{\min}\xi^T\xi$. Noting that $RBB^TR \geq 0$, we have

$$\xi^T\big(\mathcal{L}_{\delta(t)} \otimes RBB^TR\big)\xi \geq \lambda_2^{\min}\xi^T\big(I_N \otimes RBB^TR\big)\xi. \tag{21}$$

As presented in the proof of Theorem 1, by selecting $c_1 \geq (1/2\lambda_2^{\min})$, we can obtain that $\dot{V}_1 < 0$. The bipartite consensus error ξ for the case with switching communication graphs asymptotically converges to zero as $t \to \infty$, which indicates that the bipartite consensus is achieved under switching topologies. This completes the proof. ■

C. Continuous Static Controllers

An inherent drawback of the discontinuous controller (5) lies in the undesirable chattering phenomenon will take place in practical applications, which is caused by the imperfections of switching devices [29], [30]. To cancel the chattering effect, based on the boundary technique [29], [30], one feasible approach is to utilize a continuous function, which is actually a continuous approximation of the discontinuous function $g(\cdot)$.

We propose a continuous controller for each agent as follows:

$$\begin{aligned} u_i &= c_1\sum_{j=1}^{N}|b_{ij}|K\big(x_i - \mathrm{sgn}\big(b_{ij}\big)x_j\big) \\ &\quad + c_2\sum_{j=1}^{N}|b_{ij}|\hat{g}_i\big(K\big(x_i - \mathrm{sgn}\big(b_{ij}\big)x_j\big)\big) \end{aligned} \tag{22}$$

where c_1, c_2, and K are defined as in (5), and the nonlinear functions $\hat{g}_i(\cdot)$ are defined such that for $w \in \mathbf{R^n}$

$$\hat{g}_i(\omega) = \begin{cases} \frac{\omega}{\|\omega\|} & \text{if } \|\omega\| > \kappa_i \\ \frac{\omega}{\kappa_i} & \text{if } \|\omega\| \leq \kappa_i \end{cases} \tag{23}$$

where κ_i are small positive constants, representing the widths of the boundary layers. By defining the bipartite consensus error as in (8), it follows from (3) and (22) that:

$$\begin{aligned} \dot{\xi} &= (I_N \otimes A + c_1\mathcal{L} \otimes BK)\xi + (M \otimes B)\Psi(x,t) \\ &\quad + c_2(M \otimes B)\hat{G}(\xi) \end{aligned} \tag{24}$$

where

$$\hat{G}(\xi) \triangleq \begin{bmatrix} \sum_{j=1}^{N}|b_{1j}|\hat{g}_1\big(K\big(\xi_1 - \mathrm{sgn}\big(b_{1j}\big)\xi_j\big)\big) \\ \vdots \\ \sum_{j=1}^{N}|b_{Nj}|\hat{g}_N\big(K\big(\xi_N - \mathrm{sgn}\big(b_{Nj}\big)\xi_j\big)\big) \end{bmatrix}.$$

The following theorem states the ultimate boundedness of the bipartite consensus error ξ.

Theorem 2: Suppose that Assumptions 1 and 2 hold. The continuous controller (5) with c_1, c_2, and K designed as in Theorem 1 can solve the bipartite consensus in the sense that the bipartite consensus error ξ is uniformly ultimately bounded and exponentially converges to the residual set

$$\mathcal{R}_1 \triangleq \left\{\xi : \|\xi\|^2 \leq \frac{\rho}{\alpha\lambda_{\min}(R)}\sum_{i=1}^{N}\sum_{j=1}^{N}|b_{ij}|\kappa_i\right\} \tag{25}$$

where $\alpha = (\lambda_{\min}(Q)/\lambda_{\max}(R))$.

Proof: Use the same Lyapunov function candidate as in the proof of Theorem 1. We can obtain the derivative of V_1 along (24) as

$$\begin{aligned} \dot{V}_1 &= \xi^T\mathcal{H}\xi + 2\xi^T(M \otimes RB)\Psi(x,t) \\ &\quad + 2c_2\xi^T(M \otimes RB)\hat{G}(\xi) \end{aligned} \tag{26}$$

where $\mathcal{H}$ is defined as in (12).

Next, consider the following three cases.

1) $\|K(\xi_i - \mathrm{sgn}(b_{ij})\xi_j)\| > \kappa_i$, i.e., $\|B^TR(\xi_i - \mathrm{sgn}(b_{ij})\xi_j)\| > \kappa_i$, $i = 1, \ldots, N$.

In this case, following the similar steps as in (13)–(15), it can be deduced that:

$$\dot{V}_1 \leq \xi^T\mathcal{H}\xi. \tag{27}$$

2) $\|B^TR(\xi_i - \mathrm{sgn}(b_{ij})\xi_j)\| \leq \kappa_i$, $i = 1, \ldots, N$. From (13), we have

$$\xi^T(M \otimes RB)\Psi(x,t) \leq \frac{\rho}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}|b_{ij}|\kappa_i. \tag{28}$$

Further, it follows from (23) that:

$$\begin{aligned} &2c_2\xi^T(M \otimes I)\begin{bmatrix} RB\sum_{j=1}^{N}|b_{1j}|\hat{g}_1\big(B^TR\big(\xi_1 - \mathrm{sgn}\big(b_{1j}\big)\xi_j\big)\big) \\ \vdots \\ RB\sum_{j=1}^{N}|b_{Nj}|\hat{g}_N\big(B^TR\big(\xi_N - \mathrm{sgn}\big(b_{Nj}\big)\xi_j\big)\big) \end{bmatrix} \\ &= -c_2\sum_{i=1}^{N}\sum_{j=1}^{N}|b_{ij}|\frac{1}{\kappa_i}\big\|B^TR\big(\xi_i - \mathrm{sgn}\big(b_{ij}\big)\xi_j\big)\big\|^2 \\ &\leq 0. \end{aligned} \tag{29}$$

This, together with (26) and (28), yields

$$\dot{V}_1 \leq \xi^T\mathcal{H}\xi + \rho\sum_{i=1}^{N}\sum_{j=1}^{N}|b_{ij}|\kappa_i. \tag{30}$$

3) Without loss of generality, suppose that $\|B^TR(\xi_i - \mathrm{sgn}(b_{ij})\xi_j)\| > \kappa_i$, $i = 1, \ldots, l$. $\|B^TR(\xi_i - \mathrm{sgn}(b_{ij})\xi_j)\| \leq \kappa_i$,

$i = l+1, \ldots, N$, where $2 \le l \le N-1$. From (13) and (23), we get

$$\xi^T (M \otimes RB)\Psi(x,t) \le \rho \left(\sum_{i=l+1}^{N} \sum_{j=1}^{N} |b_{ij}|\kappa_i + \sum_{i=1}^{l} \sum_{j=1}^{N} |b_{ij}| \left\| B^T R \times \left(\xi_i - \operatorname{sgn}(b_{ij})\xi_j\right) \right\| \right) \tag{31}$$

$$2c_2 \xi^T (M \otimes RB)\hat{G}(\xi) \le -c_2 \sum_{i=1}^{l} \sum_{j=1}^{N} |b_{ij}| \left\| B^T R\left(\xi_i - \operatorname{sgn}(b_{ij})\xi_j\right) \right\|.$$

By choosing $c_2 \ge \rho$, it is not difficult to see that

$$\dot{V}_1 \le \xi^T \mathcal{H}\xi + \rho \sum_{i=l+1}^{N} \sum_{j=1}^{N} |b_{ij}|\kappa_i. \tag{32}$$

Therefore, by summarizing the above three cases, we can derive that (30) holds for all $\xi \in \mathbf{R}^{Nn}$.

For a undirected and connected graph $\mathcal{G}$, take a unitary transformation defined in (17). By using the fact that $M^2 = M$, we can obtain

$$\begin{aligned} \xi^T \mathcal{H}\xi &= \xi^T\left(M \otimes \left(RA + A^T R\right) - 2c_1 \mathcal{L} \otimes RBB^T R\right)\xi \\ &= \xi^T (MU \otimes I)\left(I \otimes \left(RA + A^T R\right) \right. \\ &\quad \left. - 2c_1 \Lambda \otimes RBB^T R\right)\left(U^T M \otimes I\right)\xi \\ &\le \xi^T\left(M \otimes \left(RA^T + A^T R - 2c_1\lambda_2 RBB^T R\right)\right)\xi. \end{aligned} \tag{33}$$

Choose $c_1 \ge (1/2\lambda_2)$. From (9), we have $RA^T + A^T R - 2c_1\lambda_2 RBB^T R \le -Q$. Further, we can get

$$\dot{V}_1 \le -\alpha V_1 + \rho \sum_{i=1}^{N} \sum_{j=1}^{N} |b_{ij}|\kappa_i \tag{34}$$

where $\alpha = (\lambda_{\min}(Q)/\lambda_{\max}(R))$. By using the comparison lemma [31], it follows from (34) that:

$$V_1(\xi) \le \left(V_1(\xi(0)) - \frac{\rho}{\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} |b_{ij}|\kappa_i \right) \exp(-\alpha t) + \frac{\rho}{\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} |b_{ij}|\kappa_i \tag{35}$$

which, together with (11), means that ξ exponentially converges to the residual set $\mathcal{R}_1$ in (25) with a convergence rate not less than $\exp(-\alpha t)$. ■

Remark 4: Compared to the discontinuous controller in (5), which cannot be realized in engineering environment due to the chattering phenomenon, in this section, we propose the continuous controller (22) to cancel the chattering effect by utilizing the boundary layer approximation. The cost of the continuous controller (22) is that the ultimate bound of the bipartite consensus error ξ, which relies on the feedback gains of (5), the number of agents, the Laplacian matrix $\mathcal{L}$, the upper bound of the uncertainties, and the size κ_i of the boundary layer, will not asymptotically converge to zero. It can be observed from (25) that $\mathcal{R}_1$ decreases with κ_i. Therefore, we can choose relatively small κ_i to ensure that the bipartite consensus error ξ converges to an arbitrarily small neighborhood of the origin.

IV. Robust Bipartite Tracking for Leader–Follower Networks With Leader of Bounded Unknown Input

A. Problem Formulation of Bipartite Tracking

For the case with undirected and connected communication in the previous section, it is generally difficult to explicitly acquire the final consensus trajectories, which are achieved by the agents under the controllers (5) and (22). The main difficulty dwells on that the agents are perturbed by nonidentical uncertainties and in essence, the controllers (5) and (22) are nonlinear. In this section, we extend to study the case where there exists a leader having a bounded control action. In the presence of leader's nonzero control action, control actions can be implemented to regulate the final consensus trajectories.

Without loss of generality, it is assumed that the leader is labeled by 1 whose dynamics are described by

$$\dot{x}_1 = Ax_1 + Bu_1 \tag{36}$$

We assume leader's control input u_1 is norm-bounded, which satisfies the following assumption.

Assumption 3: There exists a constant $\beta > 0$ such that $\|u_1\| \le \beta$.

The followers are labeled by $2, \ldots, N$, whose dynamics are still described by (3). The communication graph $\mathcal{G}$ associated with the N agents satisfies the following assumption.

Assumption 4: The graph $\mathcal{G}$ contains a directed spanning tree with the leader as the root and is structurally balanced. Moreover, the subgraph $\mathcal{G}_s$ associated with the followers is undirected.

Noting that the leader has no neighbors, we can partition the Laplacian matrix $\mathcal{L}$ associated with $\mathcal{G}$ as $\mathcal{L} = \begin{bmatrix} 0 & 0_{1\times(N-1)} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix}$, where $\mathcal{L}_2 \in \mathbf{R}^{(N-1)\times 1}$ and $\mathcal{L}_1 \in \mathbf{R}^{(N-1)\times(N-1)}$. Under Assumption 4, it follows from Lemma 2 that $\mathcal{L}$ has only one zero eigenvalue and all other eigenvalues are positive.

For a network satisfying Assumption 4, the agents in the network can be split into two separate subgroups $\mathcal{K}_1$ and $\mathcal{K}_2$. The goal of this section is to design a bipartite tracking controller for each follower in (3), such that the followers in the two subgroups of the network can achieve bipartite tracking in the sense that

$$\lim_{t\to\infty} (x_i(t) - \sigma_i\sigma_1 x_1(t)) = 0, \quad i = 2, \ldots, N. \tag{37}$$

B. Continuous Static Controllers

We still utilize the continuous controller in (22) for each follower. We define the bipartite tracking error as $e_i = x_i - \sigma_i\sigma_1 x_1$, $i = 2, \ldots, N$. By letting $e = [e_2^T, \ldots, e_N^T]^T$, the close-loop network dynamics in terms of e can be written as

$$\begin{aligned} \dot{e} &= (I_{N-1} \otimes A + c_1\mathcal{L}_1 \otimes BK)e + (I_{N-1} \otimes B)H(x,t) \\ &\quad + c_2(I_{N-1} \otimes B)\tilde{G}(e) \end{aligned} \tag{38}$$

where $H(x, t) = [h_2(x_1, t)^T, \ldots, h_N(x_N, t)^T]^T$ with $h_i(x_i, t) = \varphi_i(x_i, t) - \sigma_i u_1$ and

$$\tilde{G}(e) \triangleq \begin{bmatrix} \hat{g}_2\left(K\sum_{j=1}^{N}|b_{2j}|\left(e_2 - \mathrm{sgn}(b_{2j})e_j\right)\right) \\ \vdots \\ \hat{g}_N\left(K\sum_{j=1}^{N}|b_{Nj}|\left(e_N - \mathrm{sgn}(b_{Nj})e_j\right)\right) \end{bmatrix}.$$

In virtue of Assumptions 1 and 3, it is not difficult to see that $h_i(t)$, which is lumped by the matching uncertainty and the leader's input, satisfies that $\|h_i(t)\| \le \|\varphi_i(t)\| + \|u_1\| \le \tilde{\gamma}$, where $\tilde{\gamma} \triangleq \gamma + \beta$.

Theorem 3: Supposing that Assumptions 1, 3, and 4 hold, the bipartite tracking problem of the agents in (3) can be solved in the sense that the bipartite tracking error e is uniformly ultimately bounded and exponentially converges to the residual set

$$\mathcal{R}_2 \triangleq \left\{ e : \|e\|^2 \le \frac{\tilde{\rho}}{\alpha\lambda_{\min}(R)} \sum_{i=1}^{N}\sum_{j=1}^{N}|b_{ij}|\kappa_i \right\} \tag{39}$$

where $\alpha = (\lambda_{\min}(Q)/\lambda_{\max}(R))$ and $\tilde{\rho} = [(2\sqrt{N-1}\tilde{\gamma})/\lambda_2]$, if the parameters of the continuous controller (22) is designed as $c_1 \ge (1/2\lambda_2)$, $c_2 \ge \tilde{\rho}$, and $K = -B^T R$, where λ_2 is the smallest eigenvalue of $\mathcal{L}_1$ and $R > 0$ is a solution to the ARE (9).

Proof: Choose the following Lyapunov function candidate:

$$V_2 = e^T(I_{N-1} \otimes R)e. \tag{40}$$

The derivative of V_2 along (38) can be obtained as

$$\begin{aligned}\dot{V}_2 &= e^T\mathcal{S}e + 2e^T(I_{N-1} \otimes RB)H(t) \\ &\quad + 2c_2e^T(I_{N-1} \otimes RB)\tilde{G}(e)\end{aligned} \tag{41}$$

where $\mathcal{S} = I_{N-1} \otimes (RA + A^TR) - 2c_1\mathcal{L}_1 \otimes RBB^TR$.

Similar to the analysis in the previous section, consider three cases.

Case 1) $\|K(e_i - \mathrm{sgn}(b_{ij})e_j)\| > \kappa_i$, i.e., $\|B^TR(e_i - \mathrm{sgn}(b_{ij})e_j)\| > \kappa_i$, $i = 2, \ldots, N$. We can obtain that $\|H(t)\| \le \sqrt{N}\tilde{\gamma}$. Then, we have

$$\begin{aligned}&2e^T(I_{N-1} \otimes RB)H(t) \\ &\le 2\left\|e^T(I_{N-1} \otimes RB)\right\|\|H(t)\| \\ &\le 2\sqrt{N-1}\tilde{\gamma}\left\|e^T(I_{N-1} \otimes RB)\right\| \\ &\le \frac{2\sqrt{N-1}\tilde{\gamma}}{\lambda_2}\left\|e^T(\mathcal{L}_1 \otimes RB)\right\| \\ &\le \tilde{\rho}\sum_{i=2}^{N}\left\|\sum_{j=1}^{N}|b_{2j}|B^TR\left(e_i - \mathrm{sgn}(b_{2j})e_j\right)\right\| \\ &\le \tilde{\rho}\sum_{i=2}^{N}\sum_{j=1}^{N}|b_{2j}|\left\|B^TR\left(e_i - \mathrm{sgn}(b_{2j})e_j\right)\right\|.\end{aligned} \tag{42}$$

In light of the definition of the nonlinear function $\hat{g}_i(\cdot)$ in (23), we obtain that

$$\begin{aligned}&2c_2e^T(I_{N-1} \otimes RB)\tilde{G}(e) \\ &= -2c_2\sum_{i=2}^{N}\sum_{j=1}^{N}|b_{ij}|\frac{e_i^TRBB^TR\left(e_i - \mathrm{sgn}(b_{ij})e_j\right)}{\left\|B^TR\left(e_i - \mathrm{sgn}(b_{ij})e_j\right)\right\|} \\ &= -c_2\sum_{i=2}^{N}\sum_{j=1}^{N}|b_{ij}|\frac{\left\|B^TR\left(e_i - \mathrm{sgn}(b_{ij})e_j\right)\right\|^2}{\left\|B^TR\left(e_i - \mathrm{sgn}(b_{ij})e_j\right)\right\|} \\ &= -c_2\sum_{i=2}^{N}\sum_{j=1}^{N}|b_{ij}|\left\|B^TR\left(e_i - \mathrm{sgn}(b_{ij})e_j\right)\right\|.\end{aligned} \tag{43}$$

Case 2) is $\|B^TR(e_i - \mathrm{sgn}(b_{ij})e_j)\| \le \kappa_i$, $i = 2, \ldots, N$, and Case 3) is $\|B^TR(e_i - \mathrm{sgn}(b_{ij})e_j)\| > \kappa_i$, $i = 2, \ldots, l$ and $\|B^TR(e_i - \mathrm{sgn}(b_{ij})e_j)\| \le \kappa_i$, $i = l+1, \ldots, N$, where $3 \le l \le N-1$. Following similar procedures as in the proof of Theorem 2, the proof of Theorem 3 can be completed. The details are omitted here for conciseness. ■

Remark 5: In the previous related works [32], [33] on bipartite tracking control problems, a common assumption is that the leader has no control action, in which case the final consensus trajectories cannot be regulated by implementing control actions on the leader. Compared to these works, the bipartite tracking problem for the case with a leader having nonzero control input is solved in this section. It should be noted that this difference in the leader's input makes the design of the bipartite tracking controller more challenging, due to the interrelations between the leader's control input and the nonidentical uncertainties.

V. Numerical Simulations

In this section, two numerical simulation examples are provided to illustrate the theoretical results in this paper. Similar examples were done for the classical consensus problem with cooperative interactions in [2].

Example 1 (Bipartite Consensus): In this example, we apply the proposed continuous controller (22) for leaderless bipartite consensus onto the mass-spring systems [34]. Consider a network of six mass-spring systems, whose dynamics are described by

$$m\ddot{y}_i + k\dot{y}_i = u_i + \varphi_i(t), \quad i = 1, \ldots, 6 \tag{44}$$

where m is the mass, k is the spring constant, $\varphi_i(t)$ represents the matching uncertainty of the ith agent, and y_i is the displacement from certain reference position. Denote by $x_i = [y_i, \dot{y}_i]^T$ the state of the ith agent. We can rewrite (44) into

$$\dot{x}_i = Ax_i + B(u_i + \varphi_i(t)), \quad i = 1, \ldots, 6 \tag{45}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k_i}{m} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}.$$

For illustration, let $\varphi_1(t) = 0.2\sin(t)$, $\varphi_2(t) = -0.3\cos(2t)$, $\varphi_3(t) = 0.5\sin(3t)$, $\varphi_4(t) = 0.2\sin(2t)$, $\varphi_5(t) = 0.4$, $\varphi_6(t) = 0.6\cos(t)$, which is norm-bounded and satisfy Assumption 1 with $\gamma = 0.6$. Let $m = 2.5$ kg and $k_i = 0.8795$ for

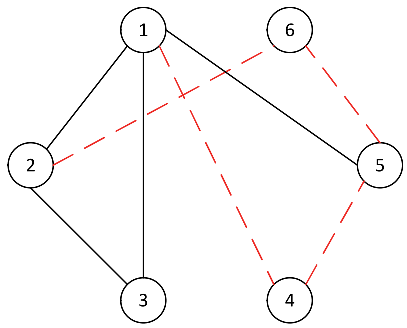

Fig. 1. Undirected signed communication graph.

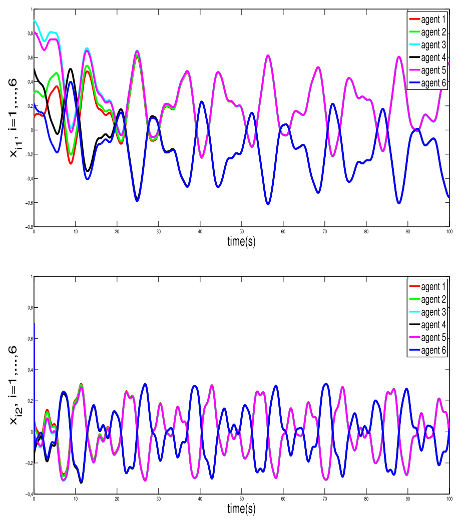

Fig. 2. Trajectories of the agents' states x_i, $i = 1, \ldots, 6$.

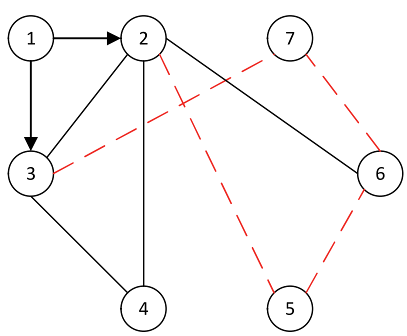

Fig. 3. Signed leader–follower communication graph.

$i = 1, \ldots, 6$. Solving the ARE (9) with $Q = I$ gives $R = \begin{bmatrix} 2.4049 & 1.1306 \\ 1.1306 & 4.5147 \end{bmatrix}$. Therefore, it follows from Theorem 2 that the feedback gain matrix of the continuous controller (22) is obtained as $K = [-0.4522 \quad -1.8059]$.

It is assumed that the undirected and connected signed communication graph of the network is depicted in Fig. 1, in which the cooperative and competitive interactions among agents are represented by the solid edges and the dash edges, respectively. It is not difficult to see that the communication graph is structurally balanced, i.e., satisfies Assumption 2. The agents in the network can be split into two separate subgroups $\mathcal{K}_1 = \{1, 2, 3, 5\}$ and $\mathcal{K}_2 = \{4, 6\}$. The smallest

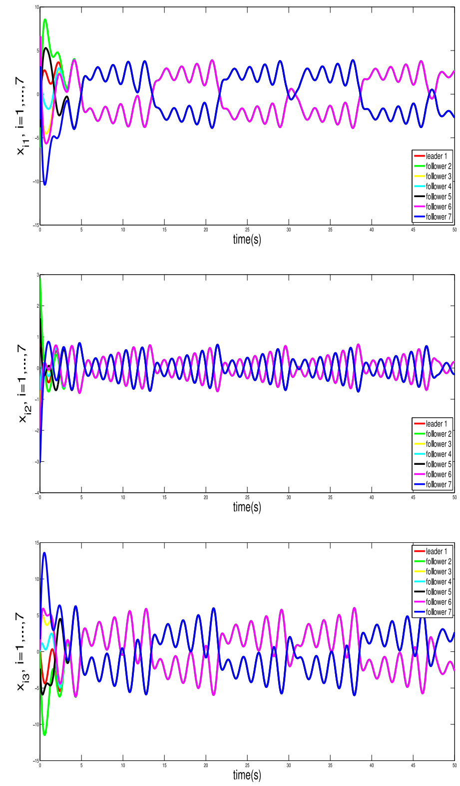

Fig. 4. Trajectories of the agents' states x_i, $i = 1, \ldots, 7$.

nonzero eigenvalues of the corresponding Laplacian matrix is $\lambda_2 = 1.3820$. It follows that the coupling gains of the continuous controller (22) can be chosen as $c_1 = 0.5$, and $c_2 = 8.0$. To illustrate Theorem 2, letting $\kappa_i = 0.01$, the trajectories of the states of the six agents, under the continuous controller (22) with c_1, c_2, and K designed as above, are depicted in Fig. 2, from which we can see the bipartite consensus is indeed achieved.

Example 2 (Bipartite Tracking): This example is given to verify Theorem 3. A set of Chua's circuits is considered, whose dynamics are described by [35]

$$\begin{aligned} \dot{x}_{i1} &= a[-x_{i1} + x_{i2} - h(x_{i1})] + u_i \\ \dot{x}_{i2} &= x_{i1} - x_{i2} + x_{i3} \\ \dot{x}_{i2} &= -b x_{i2}, \quad i = 1, \ldots, N \end{aligned} \tag{46}$$

where $a > 0$, $b > 0$, and $h(x_{i1})$ is a nonlinear function represented by $h(x_{i1}) = m_i^1 x_{i1} + (1/2)(m_i^2 - m_i^1)(|x_{i1}+1| - |x_{i1}-1|)$, where $m_i^1 < 0$ and $m_i^2 < 0$. Denoting $x_i = [x_{i1}, x_{i2}, x_{i3}]^T$, we can rewrite (46) in a compact form as

$$\dot{x}_i = A x_i + B[u_i + \varphi_i(x_i)], \quad i = 1, \ldots, N \tag{47}$$

where

$$A = \begin{bmatrix} -a(m_i^1 + 1) & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\varphi_i(x_i) = \frac{a}{2}\left(m_i^1 - m_i^2\right)(|x_{i1} + 1| - |x_{i1} - 1|), \quad i = 1, \ldots, N.$$

For simplicity, we let $u_1 = 0$ and view $\varphi_1(x_1)$ as the virtual control input of the leader, satisfying $\|\varphi_1(x_1)\| \le a|m_1^1 - m_1^2|$.

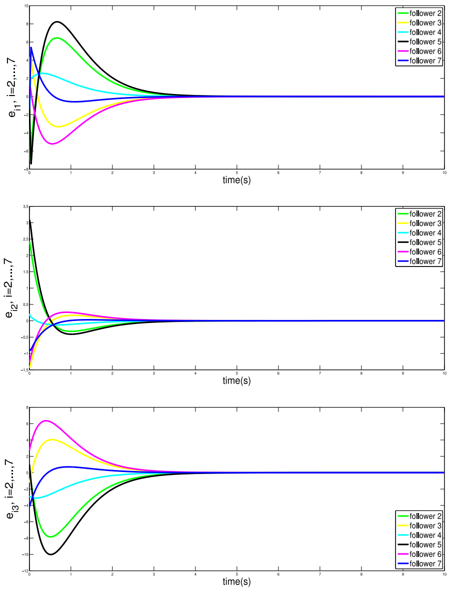

Fig. 5. Trajectories of the bipartite consensus errors $e_i, = 2, \ldots, 7$.

Let $a = 10$, $b = 18$, $m_1^2 = -(4/3)$, and $m_i^1 = -(3/4)$ for $i = 1 \cdots, N$. Then, the upper bound of the leader's control input can be chosen as $\beta = 5.8334$. The parameter m_i^2, $i = 2, \ldots, N$, are randomly selected within the range $[-1.5, 0)$. It is not difficult to obtain that $\|\varphi_i(x_i)\| \leq a|m_i^1 - m_i^2| = 7.5$, $i = 1, \ldots, N$. It follows that the upper bound of the matching uncertainties can be selected as $\gamma = 7.5$.

Assuming that there exist 7 Chua's circuits in the network, let the circuit labeled by 1 be the leader and the rest be the followers. The signed communication graph is shown in Fig. 3, where the cooperative and competitive interactions among agents are represented by the solid edges and the dash edges, respectively. Assumption 4 is obviously satisfied. The smallest eigenvalues of the Laplacian matrix $\mathcal{L}_1$ corresponding to subgraph of followers is $\lambda_2 = 0.2725$. Let $c_1 = 2.0$, $c_2 = 250$, and $\kappa_i = 0.01$ in (22). Solving the ARE (9) with $Q = I$ yields

$$R = \begin{bmatrix} 1.6516 & 4.9931 & 1.2794 \\ 4.9931 & 32.2344 & 0.3184 \\ 1.2794 & 0.3184 & 2.1290 \end{bmatrix}.$$

Then, the feedback matrix of (22) can be obtained as $K = [-1.6516 \quad -4.9931 \quad -1.2794]$. The trajectories of the agents' states x_i, $i = 1, \ldots, 7$, and the bipartite tracking errors $e_i = x_i - \sigma_i \sigma_1 x_1$, $i = 2, \ldots, 7$ are depicted in Figs. 4 and 5, respectively, implying that the bipartite tracking is indeed achieved.

VI. Conclusion

This paper has investigated the bipartite consensus problems of general linear multiagent systems subject to nonidentical matching uncertainties. The agents in the networks are assumed to coordinate over signed graphs and different communication topologies, such as undirected graphs and leader–follower graphs, are considered. For the case where the communication graph is undirected and connected, both a discontinuous and a continuous controllers have been proposed, under which it has been proved that the bipartite consensus is achieved. Furthermore, we have studied the case with a leader having bounded unknown control input, a continuous controller has been utilized to guarantee the ultimate boundedness of the bipartite tracking error. Future work will focus on discussing the case of general directed graphs.

References

[1] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, Apr. 2007.

[2] Z. K. Li and Z. S. Duan, *Cooperative Control of Multi-Agent Systems: A Consensus Region Approach*. Boca Roton, FL, USA: CRC Press, 2014.

[3] X. Dong and G. Hu, "Time-varying formation tracking for linear multiagent systems with multiple leaders," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3658–3664, Jul. 2017.

[4] Z. K. Li, G. H. Wen, Z. S. Duan, and W. Ren, "Designing fully distributed consensus protocols for linear multi-agent systems with directed graphs," *IEEE Trans. Autom. Control*, vol. 60, no. 4, pp. 1152–1157, Apr. 2015.

[5] Z. K. Li and J. Chen, "Robust consensus of linear feedback protocols over uncertain network graphs," *IEEE Trans. Autom. Control*, vol. 62, no. 8, pp. 4251–4258, Aug. 2017.

[6] Z. K. Li, M. Z. Q. Chen, and Z. T. Ding, "Distributed adaptive controllers for cooperative output regulation of heterogeneous agents over directed graphs," *Automatica*, vol. 68, pp. 179–183, Jun. 2016.

[7] C. L. P. Chen, C. E. Ren, and T. Du, "Fuzzy observed-based adaptive consensus tracking control for second-order multiagent systems with heterogeneous nonlinear dynamics," *IEEE Trans. Fuzzy Syst.*, vol. 24, no. 4, pp. 906–915, Aug. 2016.

[8] H. Li, X. Liao, and T. Huang, "Second-order locally dynamical consensus of multiagent systems with arbitrarily fast switching directed topologies," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 43, no. 6, pp. 1343–1353, Nov. 2013.

[9] Y. J. Liu, F. Ding, and Y. Shi, "An efficient hierarchical identification method for general dual-rate sampled-data systems," *Automatica*, vol. 50, no. 3, pp. 962–970, 2014.

[10] Y.-J. Liu, S. Lu, D. Li, and S. Tong, "Adaptive controller design-based ablf for a class of nonlinear time-varying state constraint systems," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 7, pp. 1546–1553, Jul. 2017.

[11] D.-J. Li, S. Lu, Y.-J. Liu, and D. Li, "Adaptive fuzzy tracking control-based barrier functions of uncertain nonlinear MIMO systems with full state constraints and applications to chemical process," *IEEE Trans. Fuzzy Syst.*, to be published, doi: 10.1109/TFUZZ.2017.2765627.

[12] Z. Liu, F. Wang, Y. Zhang, and C. L. P. Chen, "Fuzzy adaptive quantized control for a class of stochastic nonlinear uncertain systems," *IEEE Trans. Cybern.*, vol. 46, no. 2, pp. 524–534, Feb. 2016.

[13] T. Li, Z. Li, D. Wang, and C. L. P. Chen, "Output-feedback adaptive neural control for stochastic nonlinear time-varying delay systems with unknown control directions," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 26, no. 6, pp. 1188–1201, Jun. 2015.

[14] W. He *et al.*, "Multiagent systems on multilayer networks: Synchronization analysis and network design," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 7, pp. 1655–1667, Jul. 2017.

[15] Y. Hong, J. Hu, and L. Gao, "Tracking control for multi-agent consensus with an active leader and variable topology," *Automatica*, vol. 42, no. 7, pp. 1177–1182, 2012.

[16] G. Facchetti, G. Iacono, and C. Altafini, "Computing global structural balance in large-scale signed social networks," *Proc. Nat. Acad. Sci.USA*, vol. 108, no. 52, pp. 20953–20958, 2011.

[17] S. Wasserman and K. Faust, "Social network analysis: Methods and applications," *Contemp. Sociol.*, vol. 91, no. 435, pp. 219–220, 2016.

[18] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Trans. Autom. Control*, vol. 58, no. 4, pp. 935–946, Apr. 2013.

[19] H.-X. Hu, A. Liu, Q. Xuan, L. Yu, and G. Xie, "Second-order consensus of multi-agent systems in the cooperation–competition network with switching topologies: A time-delayed impulsive control approach," *Syst. Control Lett.*, vol. 62, no. 12, pp. 1125–1135, 2013.

[20] M. E. Valcher and P. Misra, "On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions," *Syst. Control Lett.*, vol. 66, pp. 94–103, Apr. 2014.

[21] H. Zhang and J. Chen, "Bipartite consensus of multi-agent systems over signed graphs: State feedback and output feedback control approaches," *Int. J. Robust Nonlin. Control*, vol. 27, no. 1, pp. 3–14, 2016.

[22] J. Zhan and X. Li, "Cluster consensus in networks of agents with weighted cooperative–competitive interactions," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 65, no. 2, pp. 241–245, Feb. 2018.

[23] D. Meng, M. Du, and Y. Jia, "Interval bipartite consensus of networked agents associated with signed digraphs," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 3755–3770, Dec. 2016.

[24] Z. K. Li, Z. S. Duan, and F. L. Lewis, "Distributed robust consensus control of multi-agent systems with heterogeneous matching uncertainties," *Automatica*, vol. 50, no. 3, pp. 883–889, 2014.

[25] J. Sun and Z. Geng, "Adaptive consensus tracking for linear multi-agent systems with heterogeneous unknown nonlinear dynamics," *Int. J. Robust Nonlin. Control*, vol. 26, no. 1, pp. 154–173, 2016.

[26] Y. Zhao, Y. F. Liu, Z. K. Li, and Z. S. Duan, "Distributed average tracking for multiple signals generated by linear dynamical systems: An edge-based framework," *Automatica*, vol. 75, pp. 158–166, Jan. 2017.

[27] J. Hu and W. X. Zheng, "Emergent collective behaviors on coopetition networks," *Phys. Lett. A*, vol. 378, nos. 26–27, pp. 1787–1796, 2014.

[28] Y. Wu, J. Hu, Y. Zhao, and Y. Zhang, "Adaptive bipartite consensus of multi-agent systems with high-order dynamics and antagonistic interactions," in *Proc. Chin. Control Conf.*, Chengdu, China, 2016, pp. 7808–7813.

[29] C. Edwards and S. Spurgeon, *Sliding Mode Control: Theory and Applications*. London, U.K.: Taylor & Francis, 1998.

[30] K. D. Young, V. I. Utkin, and U. Ozguner, "A control engineer's guide to sliding mode control," *IEEE Trans. Control Syst. Technol.*, vol. 7, no. 3, pp. 328–342, May 1999.

[31] H. K. Khalil, *Nonlinear Systems*. Englewood Cliffs, NJ, USA: Prentice-Hall, 2002.

[32] J. Hu, "Adaptive bipartite tracking control of leader–follower systems on coopetition networks," in *Proc. Int. Conf. Control Autom. Robot. Vis.*, Singapore, 2015, pp. 199–204.

[33] F. A. Yaghmaie, R. Su, F. L. Lewis, and S. Olaru, "Bipartite and cooperative output synchronizations of linear heterogeneous agents: A unified framework," *Automatica*, vol. 80, pp. 172–176, Jun. 2017.

[34] H. Zhang, F. L. Lewis, and Z. Qu, "Lyapunov, adaptive, and optimal design techniques for cooperative systems on directed communication graphs," *IEEE Trans. Ind. Electron.*, vol. 59, no. 7, pp. 3026–3041, Jul. 2012.

[35] R. N. Madan, *Chua's Circuit: A Paradigm for Chaos*. Singapore: World Sci., 1993.

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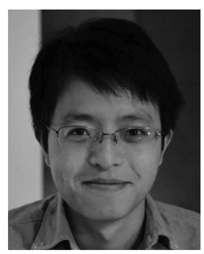

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