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Formation Tracking for Nonlinear Agents with Unknown Second-Order Locally Lipschitz Continuous Dynamics

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Abstract: This paper investigates the formation tracking problem for a group of networked agents with second-order unknown locally Lipschitz continuous nonlinear dynamics. By utilizing the ISS (input-to-state stable) notion and backstepping design method, a distributed control strategy is proposed with the aid of graph theory and the cyclic-small-gain theorem. The proposed strategy can drive the positions of the agents to converge to a desired geometric pattern, as well as the position center of the agents to trace a desired trajectory.

Key Words: Multi-agent system, formation tracking, second-order nonlinear dynamics, locally Lipschitz continue, ISS

1 Introduction

Over the last few years, formation control has attracted considerable attentions from the control community, due to its broad range of applications. Since most of physical systems are inherently nonlinear in nature, it is necessary and beneficial to study formation control in the presence of the nonlinearities. In this paper, we will investigate the formation tracking problem for nonlinear multi-agent systems.

A similar issue to formation tracking is *consensus* with pinning control, in which all states eventually reach an agreement to a virtual value [1]. By adding a pre-specified geometric pattern, the formation tracking problem can be solved from the corresponding consensus problem, which is usually known as *consensus-based formation control*, see e.g., [2]. Most of the existing literature focuses on consensus algorithms for agents governed by linear dynamics, especially by the single- or double-integrator dynamics [3–6]. It is worth noting that consensus for second-order nonlinear multi-agents is studied in [7, 8] under a global Lipschitz condition. Further, a backstepping-based control design for flocking of multiple nonholonomic wheeled mobile robots is proposed in [9], and adaptive control strategies are introduced to deal with consensus of networked nonlinear dynamical systems in [10, 11].

Different from the above-mentioned works, the nonlinear dynamics considered in this paper are only required to be locally Lipschitz continuous, a rather relaxed assumption which has been used in a wide-range of practical nonlinear systems [12]. The key tool employed to tackle the challenge of locally Lipschitz continuous nonlinearity is the newly developed cyclic-small-gain theorem provided in [13], in which it is proven that the nonlinear system composed of interconnected input-to-state stable (ISS) subsystems is ISS if the cyclic-small-gain condition is satisfied. The cyclic-small-gain condition can be roughly described as follows: the composition of the gain functions along every cycle in the network of ISS systems is less than the identity function. Through this new approach, the proposed control law can effectively tackle the technical challenges caused by locally Lipschitz continuous dynamics. Moreover, the heterogeneous or isomorphic agents can be both adopted as the studied dynamics are not required to be identical. Finally, an additional contribution of this paper is that we use a completely nonlinear ISS-based analysis to deal with the multi-agent coordination problem. By using the design methodology in this paper, we are able to extend many existing first-order results to the second-order, or even higher-order case with nonlinear dynamics, which is certainly nontrivial.

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2 Basic Notions and Preliminary Results

In this section, we present some basic notions and results in matrix theory and graph theory (refer to [14]) as well as the ISS concepts (refer to [13, 15]) that will be used.

2.1 Matrix and Graph

We use $|\cdot|$ to denote the Euclidean norm for vectors or the 2-norm for matrices, and $\|\cdot\|$ to denote the L_∞ norm for vectors and matrices, respectively. A vector that consists of all zero or one entries is denoted by $\mathbf{0}$ or $\mathbf{1}$, respectively. For a vector $\boldsymbol{m} = \{m_1, \ldots, m_n\} \in \mathbb{R}^n$, the notion $M = \mathrm{diag}\{\boldsymbol{m}\}$ represents a diagonal matrix with m_i being the i-th ($i = 1, \ldots, n$) diagonal entry.

A *digraph* (or *directed graph*) will be used to model the interconnection for multi-agent system. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a weighted digraph of order N with a finite nonempty set of vertexes $\mathcal{V} = \{1, 2, \ldots, N\}$; a set of directed edges $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$; and a weighted adjacency matrix $A_{\mathcal{G}} = [\omega_{ij}] \in \mathbb{R}^{N\times N}$ with nonnegative adjacency elements ω_{ij}. The adjacency elements associated with the edges are positive, i.e., $(j, i) \in E \Leftrightarrow \omega_{ij} > 0$. For a diagonal matrix $\Delta_{\mathcal{G}}$, we set $[\Delta_{\mathcal{G}}]_{ii} = \sum_{j=1}^{N} \omega_{ij}$ for all i, which is the *in-degree* of vertex i. The corresponding weighted *Laplacian* is defined by $L_{\mathcal{G}} = \Delta_{\mathcal{G}} - A_{\mathcal{G}}$.

2.2 ISS and Cyclic-small-gain Theorem

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *positive definite* if it is continuous, $\alpha(0) = 0$ and $\alpha(s) > 0$ for $s > 0$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of *class* $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$; it is of *class* $\mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of *class* $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity. For nonlinear functions γ_1 and γ_2 defined on $\mathbb{R}_+$, inequality $\gamma_1 \le \gamma_2$ (or $\gamma_1 < \gamma_2$) represents $\gamma_1(s) \le \gamma_2(s)$ (or $\gamma_1(s) < \gamma_2(s)$) for all $s > 0$. Id represents identify function.

Consider a nonlinear system

$$\dot{x} = f(x, w), \tag{1}$$

with $x \in \mathbb{R}^n$ as the state and $w \in \mathbb{R}^m$ as the external input, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz vector field.

Definition 1 ([15]). *The system* (1) *is said to be* input-to-state stable (ISS) *with w as input if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for each initial condition $x(0)$ and each measurable essentially bounded input $w(\cdot)$ defined on* $[0, \infty)$*, the solution $x(\cdot)$ exists on* $[0, \infty)$ *and satisfies* $|x(t)| \le \beta(|x(0)|, \gamma(\|w\|))$, $\forall t \ge 0$.

It is known that if system (1) is ISS with w as the input, then the unforced system $\dot{x} = f(x, 0)$ is globally asymptotically stable at $x = 0$ [15].

Definition 2 ([15]). *For a nonlinear system* (1) *with state $x \in \mathbb{R}^n$ and external input $w \in \mathbb{R}^m$, a function V is said to be an* ISS-Lyapunov function *if it is differentiable almost everywhere, and satisfies that*

- *V is positive definite and radially unbounded, that is, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that*

$$\underline{\alpha}(|x|) \le V(x) \le \overline{\alpha}(|x|), \quad \forall x \in \mathbb{R}^n;$$

- *there exist a positive definite α, and $\gamma \in \mathcal{K}$ such that*

$$V(x) \ge \gamma(|w|) \Rightarrow \nabla V(x) f(x, w) \le -\alpha(V(x)), \\ \forall x \in \mathbb{R}^n, \ \forall w \in \mathbb{R}^m.$$

Lemma 1 ([15]). *The system* (1) *is ISS if and only if it has an ISS-Lyapunov function.*

Consider the following interconnected system composed of N interacting subsystems:

$$\dot{x}_i = f_i(x, w_i), \quad i = 1, \ldots, N \tag{2}$$

where $x_i \in \mathbb{R}^{n_i}$, $w_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R}^{n+m_i} \to \mathbb{R}^{n_i}$ with $n = \sum_{i=1}^N n_i$ is locally Lipschitz continuous such that $x = [x_1^T, \ldots, x_N^T]^T$ is the unique solution of system (2) for a given initial condition. The external input $w = [w_1^T, \ldots, w_N^T]^T$ is a measurable and locally essentially bounded function from $\mathbb{R}_+$ to $\mathbb{R}^m$ with $m = \sum_{i=1}^N m_i$. Further, we use γ_y^x to represent the gain from x-subsystem to y-subsystem.

Lemma 2 (Cyclic-small-gain Theorem, [13]). *Consider the continuous-time dynamical network* (2)*. Supposed that for the i-th ($i = 1, \ldots, N$) subsystem, there exists an ISS-Lyapunov function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}$ satisfying*

- *there exist $\underline{a}_i, \overline{a}_i \in \mathcal{K}_\infty$ such that*

$$\underline{a}_i(|x_i|) \le V_i(x_i) \le \overline{a}_i(|x_i|), \quad \forall x_i, \tag{3}$$

- *there exist $\gamma_{x_i}^{x_j} \in \mathcal{K} \cup \{0\} (j \ne i)$, $\gamma_{x_i}^{w_i} \in \mathcal{K} \cup \{0\}$ and a positive definite α_i such that*

$$V_i(x_i) \ge \max\{\gamma_{x_i}^{x_j}(V_j(x_j)), \gamma_{x_i}^{w_i}(|w_i|)\} \\ \Rightarrow \nabla V_i(x_i) f_i(x_1, \ldots, x_N, w_i) \le -\alpha_i(V_i(x_i)) \\ \forall x, \ \forall w_i. \tag{4}$$

Then the system (2) *is ISS if for each $r = 2, \ldots, N$,*

$$\gamma_{x_{i_1}}^{x_{i_2}} \circ \gamma_{x_{i_2}}^{x_{i_3}} \circ \cdots \circ \gamma_{x_{i_r}}^{x_{i_1}} < \mathrm{Id},$$

for all $1 \le i_j \le N$, $i_j \ne i_{j'}$ if $j \ne j'$.

3 Problem Statement

We consider a group of N networked agents to achieve a desired geometric pattern and to trace a smooth refereed trajectory at the same time from arbitrary initial positions. The information interconnection of the N agents is modeled by a digraph $\mathcal{G}_c$, called *information-interconnection digraph*, by regarding each agent as a vertex and each communicating interconnection between agents as a directed edge. A information-interconnection digraph of five-agent system is shown as an example in Fig. 1.

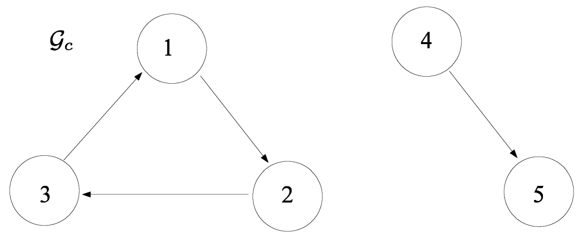

Fig. 1: A information-interconnection digraph of five-agent system.

Similar to that used in [9], the desired geometric pattern is described by a tuple of N vectors, which is $\mathcal{P} = \{p_1, \ldots, p_N\}$ with $p_i \in \mathbb{R}^n$. Without loss of generality, we assume that $\sum_{i=1}^N p_i = \mathbf{0}$. The smooth refereed trajectory is described by $x_d \in \mathbb{R}^n$ with velocity $\dot{x}_d$ and acceleration $\ddot{x}_d$.

For $i = 1, \ldots, N$, the dynamics of the i-th agent is assumed to be the widely used model in nonlinear control community [12], represented by

$$\dot{x}_i = v_i, \tag{5}$$

$$\dot{v}_i = f_i(x_i, v_i) + \mu_i, \tag{6}$$

where $x_i \in \mathbb{R}^n$ is the position, $v_i \in \mathbb{R}^n$ is the velocity, $\mu_i \in \mathbb{R}^n$ is the control input, and $f_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a unknown locally Lipschitz continuous function, and satisfy

Assumption 1. *For each $i = 1, \ldots, N$, there exist $\psi_{f_i}^{x_i}, \psi_{f_i}^{v_i} \in \mathcal{K}_\infty$ such that*

$$|f_i(x_i, v_i)| \le \psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|). \tag{7}$$

It should be noted that the nonlinear term $f_i(x_i, v_i)$ in (6) together with (7) are not confined to be globally Lipschitz, and can be extended to locally Lipschitz continuous in practice, which leads to a significant difference from the existing results, such as that in [7, 8]

The information of the referred trajectory, $(x_d, \dot{x}_d, \ddot{x}_d)$, is available to each agent. Further, for $i = 1, \ldots, N$, the position x_i, the velocity v_i and the relative positions to its neighbors $x_j - x_i$ of the i-th agent for $j \in N_i$, where N_i is the neighbors of the i-th agent, are available for the design of the control law μ_i too.

Denote $x_c = \frac{1}{N}\sum_{i=1}^{N} x_i$. The objective here is to design a distributed control law for the i-th agent in the form of $\mu_i = g_i(X_i)$ with $X_i = \{x_i, v_i\} \cup \{x_j - x_i | j \in N_i\} \cup x_d$ such that the position evolutions of the agents described by (5)–(6) asymptotically reach the geometric pattern $\mathcal{P}$, and the center x_c and the velocity of each i-th agent converge to the referred trajectory and velocity, respectively, i.e., for $1 \le i \ne j \le N$,

$$\lim_{t\to\infty} |x_j(t) - x_i(t) - (p_j - p_i)| = 0, \tag{8}$$

$$\lim_{t\to\infty} x_c(t) = x_d(t), \tag{9}$$

$$\lim_{t\to\infty} v_i(t) = \dot{x}_d(t). \tag{10}$$

4 Control Law Design

In this sequel, we develop a nonlinear control law by modifying the standard backstepping methodology to cause a group of N networked agents to achieve (8)–(10).

4.1 Two-Cascade Model

For $i = 1, \ldots, N$, we define

$$\bar{x}_i = x_i - p_i - x_d. \tag{11}$$

Correspondingly, we have

$$\dot{\bar{x}}_i = \dot{x}_i - \dot{x}_d = v_i - \dot{x}_d. \tag{12}$$

For convenience of notations, denote $\bar{x} = [\bar{x}_1^T, \ldots, \bar{x}_N^T]^T$ and $v = [v_1^T, \ldots, v_N^T]^T$. From (12), we obtain

$$\dot{\bar{x}} = v - \mathbf{1}_N \dot{x}_d. \tag{13}$$

We take v as the virtual control input of the $\bar{x}$-subsystem in (13). For $i = 1, \ldots, N$, we define:

$$v_i^* = -\alpha_i \bar{x}_i + \sum_{j\in N_i} \omega_{ij}(\bar{x}_j - \bar{x}_i) + \dot{x}_d, \tag{14}$$

where $\alpha_i > 0 \in \mathbb{R}_+$ and ω_{ij} is the ij-th element of the weighted adjacency matrix of $\mathcal{G}_c$. Denote $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_N]^T$, $z = [z_1^T, \ldots, z_N^T]^T$ and $v^* = [v_1^{*T}, \ldots, v_N^{*T}]^T$. Then from (14), we have

$$v^* = -(L_{\mathcal{G}_c} \otimes I_n)\bar{x} - (\mathrm{diag}\{\boldsymbol{\alpha}\} \otimes I_n)\bar{x} + \mathbf{1}_N \dot{x}_d, \tag{15}$$

where $L_{\mathcal{G}_c}$ is the weighted Laplacian of $\mathcal{G}_c$.

Define an error $z = v - v^*$, and correspondingly from (13) and (15), we have

$$\dot{\bar{x}} = z - (L_{\mathcal{G}_c} \otimes I_n)\bar{x} - (\mathrm{diag}\{\boldsymbol{\alpha}\} \otimes I_n)\bar{x}. \tag{16}$$

Differentiating both sides of (15), we have

$$\dot{v}^* = -(L_{\mathcal{G}_c} \otimes I_n)\dot{\bar{x}} - (\mathrm{diag}\{\boldsymbol{\alpha}\} \otimes I_n)\dot{\bar{x}} + \mathbf{1}_N \ddot{x}_d. \tag{17}$$

By using (16), $\dot{v}^*$ yields:

$$\begin{aligned}\dot{v}^* =& -(L_{\mathcal{G}_c} + \mathrm{diag}\{\boldsymbol{\alpha}\}) \otimes I_n z \\ &+ (L_{\mathcal{G}_c} + \mathrm{diag}\{\boldsymbol{\alpha}\})^2 \otimes I_n \bar{x} + \mathbf{1}_N \ddot{x}_d.\end{aligned}$$

Also differentiating both sides of z and using (17), we have

$$\dot{z} = \dot{v} + (L_{\mathcal{G}_c} + \mathrm{diag}\{\boldsymbol{\alpha}\}) \otimes I_n z - (L_{g\alpha^2} \otimes I_n)\bar{x} - \mathbf{1}_N \ddot{x}_d,$$

where $L_{g\alpha^2} = (L_{\mathcal{G}_c} + \mathrm{diag}\{\boldsymbol{\alpha}\})^2$. Consequently, for $i = 1, \ldots, N$, by using (6), the two-cascade model is summarized as follows.

$$\dot{\bar{x}} = z - (L_{\mathcal{G}_c} + \mathrm{diag}\{\boldsymbol{\alpha}\}) \otimes I_n \bar{x}, \tag{18}$$

$$\begin{aligned}\dot{z}_i =& f_i(x_i, v_i) + \mu_i - \ddot{x}_d + (\alpha_i + \sum_{j\in N_i} \omega_{ji}) z_i \\ &- \sum_{j\in N_i} \omega_{ji} z_j - (L_{g\alpha^2 i} \otimes I_n)\bar{x},\end{aligned} \tag{19}$$

where $L_{g\alpha^2 i}$ is the i-th row of $L_{g\alpha^2}$. Note that cascade 1 described by (18) is a $\bar{x}$-subsystem, and cascade 2 described by (19) are z_i-subsystems.

4.2 ISS Cyclic-Small-Gain Conditions

In (18), the state is $\bar{x}$ and the input is z composing of $z_1, \ldots, z_N$; conversely, in (19), for each $i = 1, \ldots, N$, the state is z_i and the inputs are $\bar{x}$ and z_js ($j \in N_i$). Considering the subsystems as vertices and the gain connections as directed edges, the interconnected system composed of the $\bar{x}$-subsystem and the z_i-subsystems can be modeled by a digraph $\mathcal{G}_g$, called *gain-interconnection digraph.* Figure 2 shows the gain-interconnection digraph of the preceding five-agent system. In the following, we will simply use $\mathcal{G}_c$ and $\mathcal{G}_g$ to represent a information-interconnection digraph and its corresponding gain-interconnection digraph.

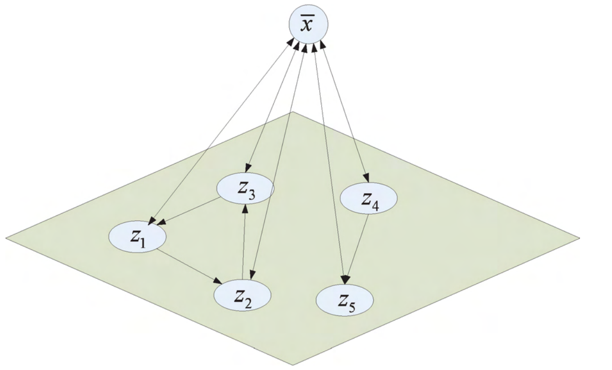

Fig. 2: The $\mathcal{G}_g$ of the preceding five-agent system.

Denote the vertices in $\mathcal{G}_c$ and the vertices representing the z_i-subsystems in $\mathcal{G}_g$ by $1, \ldots, N$, and denote the additional vertex representing the $\bar{x}$-subsystem in $\mathcal{G}_g$ by $N+1$. For the corresponding gain-interconnection digraph $\mathcal{G}_g$, we can re-investigate the cyclic-small-gain conditions required by Lemma 2, which can be explicitly described as follows:

1) for each vertex i in $\mathcal{G}_c$, there exists a cycle $(i, N+1, i)$ in $\mathcal{G}_g$. The corresponding cyclic-small-gain condition is

$$\gamma_{z_i}^{\bar{x}} \circ \gamma_{\bar{x}}^{z_i} < \mathrm{Id}. \tag{20}$$

2) for each directed path $(i_1, i_2, \ldots, i_{r-1}, i_r)$ with $i_r \geq 2$ in $\mathcal{G}_c$, there exists one cycle $(i_1, \ldots, i_r, N+1, i_1)$ in $\mathcal{G}_g$. The corresponding cyclic-small-gain condition is

$$\gamma_{z_{i_1}}^{z_{i_2}} \circ \gamma_{z_{i_2}}^{z_{i_3}} \circ \cdots \circ \gamma_{z_{i_r}}^{\bar{x}} \circ \gamma_{\bar{x}}^{z_{i_1}} < \mathrm{Id}. \tag{21}$$

3) for each directed cycle $(i_1, i_2, \ldots, i_{c-1}, i_c, i_1)$ in $\mathcal{G}_c$, there exists one cycle $(i_1, i_2 \ldots, i_{c-1}, i_c, i_1)$. The corresponding cyclic-small-gain condition is

$$\gamma_{z_{i_1}}^{z_{i_2}} \circ \gamma_{z_{i_2}}^{z_{i_3}} \circ \cdots \circ \gamma_{z_{i_{c-1}}}^{z_{i_c}} \circ \gamma_{z_{i_c}}^{z_{i_1}} < \mathrm{Id}. \tag{22}$$

Remark 1. *The ISS cyclic-small-gain conditions from* (20) *to* (22) *can hold by simply choosing*

$$\gamma_{z_j}^{z_i} < \mathrm{Id}, \quad (i,j) \in E, \tag{23}$$

$$\gamma_{z_i}^{\bar{x}} = \gamma_{z_j}^{\bar{x}} < (\gamma_{\bar{x}}^{z_i})^{-1} = (\gamma_{\bar{x}}^{z_j})^{-1}, \quad 1 \leq i \neq j \leq N. \tag{24}$$

4.3 Control Law and Stability Analysis

We denote function $\beta(s) = \frac{1}{2}s^2$, and define the ISS-Lyapunov function candidates

$$V_{\bar{x}}(\bar{x}) = \beta(|\bar{x}|), \tag{25}$$

$$V_{z_i}(z_i) = \beta(|z_i|), \quad i = 1, \ldots, N \tag{26}$$

for $\bar{x}$-subsystem and z_i-subsystems, respectively.

The control law design is composed of three main steps.

4.3.1 Step 1–ISS of the $\bar{x}$-subsystem

We denote $\alpha_{\min} = \min\{\alpha_1, \ldots, \alpha_N\}$, $\hat{L}_{\mathcal{G}_c} = \frac{1}{2}(L_{\mathcal{G}_c}^T + L_{\mathcal{G}_c})$ and the smallest eigenvalue of $\hat{L}_{\mathcal{G}_c}$ by $\lambda_{\min}(\hat{L}_{\mathcal{G}_c})$.

Lemma 3. *The $\bar{x}$-subsystem* (18) *with z_i's ($i = 1, \ldots, N$) as the inputs and $\bar{x}$ as the states is ISS when $\alpha_{\min} + \lambda_{\min}(\hat{L}_{\mathcal{G}_c}) > 0$. Moreover, for a constant* $0 < \epsilon < \alpha_{\min} + \lambda_{\min}(\hat{L}_{\mathcal{G}_c})$, *the ISS-Lyapunov function $V_{\bar{x}}$ defined in* (25) *satisfies* $V_{\bar{x}} \geq \max_{i=1,\ldots,N}\{\gamma_{\bar{x}}^{z_i}(V_{z_i})\} \Rightarrow \nabla V_{\bar{x}}\dot{\bar{x}} \leq -2\epsilon V_{\bar{x}}$, *where* $\gamma_{\bar{x}}^{z_i}(s) = N\left(\frac{1}{\alpha_{\min}+\lambda_{\min}(\hat{L}_{\mathcal{G}_c})-\epsilon}\right)^2 s, \quad \forall s \in \mathbb{R}_+$.

Proof. Taking the derivative of $V_{\bar{x}}$ defined in (25), and then using (18), we have

$$\nabla V_{\bar{x}}\dot{\bar{x}} = -\bar{x}^T\{\hat{L}_{\mathcal{G}_c} \otimes I_n\}\bar{x} - \bar{x}^T\{\mathrm{diag}\{\boldsymbol{\alpha}\} \otimes I_n\}\bar{x} + z^T\bar{x}.$$

Obviously, $\hat{L}_{\mathcal{G}_c}$ is a symmetric real matrix, and further we have $\bar{x}^T\{\hat{L}_{\mathcal{G}_c} \otimes I_n\}\bar{x} \geq \lambda_{\min}(\hat{L}_{\mathcal{G}_c})\bar{x}^T\bar{x}$ and $\bar{x}^T\{\mathrm{diag}\{\boldsymbol{\alpha}\} \otimes I_n\}\bar{x} \geq \alpha_{\min}\bar{x}^T\bar{x}$. Therefore $\nabla V_{\bar{x}}\dot{\bar{x}}$ yields:

$$\nabla V_{\bar{x}}\dot{\bar{x}} \leq -\big(\lambda_{\min}(\hat{L}_{\mathcal{G}_c}) + \alpha_{\min}\big)|\bar{x}|^2 + |\bar{x}||z|. \tag{27}$$

When $\alpha_{\min} + \lambda_{\min}(\hat{L}_{\mathcal{G}_c}) > 0$, for the positive constant $0 < \epsilon < \alpha_{\min} + \lambda_{\min}(\hat{L}_{\mathcal{G}_c})$, the condition $V_{\bar{x}} \geq \max_{i=1,\ldots,N}\{\gamma_{\bar{x}}^{z_i}(V_{z_i})\}$ implies

$$\begin{aligned} V_{\bar{x}} &\geq \left(\frac{1}{\alpha_{\min} + \lambda_{\min}(\hat{L}_{\mathcal{G}_c}) - \epsilon}\right)^2 \sum_{i=1}^{N} V_{z_i} \\ &= \frac{1}{2}\left(\frac{1}{\alpha_{\min} + \lambda_{\min}(\hat{L}_{\mathcal{G}_c}) - \epsilon}\right)^2 z^T z. \end{aligned}$$

Thus, we obtain

$$|z| \leq \big(\alpha_{\min} + \lambda_{\min}(\hat{L}_{\mathcal{G}_c}) - \epsilon\big)|\bar{x}|. \tag{28}$$

Substituting (28) into (27), it is achieved that $\nabla V_{\bar{x}}\dot{\bar{x}} \leq -\epsilon|\bar{x}|^2 = -2\epsilon V_{\bar{x}}$. The proof of Lemma 3 is concluded. □

4.3.2 Step 2–ISS of the z_i-subsystems

Denote $\hat{L}_{g\alpha^2} = \frac{1}{2}(L_{g\alpha^2} + L_{g\alpha^2}^T)$ and $\{\hat{L}_{g\alpha^2}\}_i$ as its i-th row. For z_i-subsystems (19), we have the following lemma.

Lemma 4. *Consider the z_i-subsystem* (19) *with z_i as the state and $\bar{x}$ and z_js ($j \in N_i$) as the external inputs. For any specified constant $\sigma_{z_i} > 0$ and any specified $\gamma_{z_i}^{\bar{x}}, \gamma_{z_i}^{z_j} \in \mathcal{K}_\infty$ with $j \in N_i$, we can design*

$$\mu_i = \begin{cases} -\frac{z_i}{|z_i|}\mu_{i1} - \mu_{i2}z_i + \ddot{x}_d, & \text{if } |z_i| \neq 0, \\ \mathbf{0}, & \text{if } |z_i| = 0, \end{cases} \tag{29}$$

where $\mu_{i1} = \psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|) + |\hat{L}_{\alpha i}|\rho_{\bar{x}}^{z_i}(|z_i|) + \sum_{j\in N_i}\omega_{ji}\rho_{z_j}^{z_i}(|z_i|)$, $\mu_{i2} = \frac{\sigma_{z_i}}{2} + \alpha_i + \sum_{j\in N_i}\omega_{ji}$ *with* $\rho_{\bar{x}}^{z_i}(s) = \beta^{-1}\circ\big(\gamma_{z_i}^{\bar{x}}\big)^{-1}\circ\beta(s)$ *and* $\rho_{z_j}^{z_i}(s) = \beta^{-1}\circ\big(\gamma_{z_i}^{z_j}\big)^{-1}\circ\beta(s)$, *such that the z_i-subsystem is ISS with ISS-Lyapunov function V_{z_i} satisfying*

$$V_{z_i} \geq \max_{j\in N_i}\{\gamma_{z_i}^{\bar{x}}(V_{\bar{x}}), \gamma_{z_i}^{z_j}(V_{z_j})\} \Rightarrow \nabla V_{z_i}\dot{z}_i \leq -\sigma_{z_i}V_{z_i}.$$

Proof. Considering $V_{z_i} \geq \max_{j\in N_i}\{\gamma_{z_i}^{\bar{x}}(V_{\bar{x}}), \gamma_{z_i}^{z_j}(V_{z_j})\}$ and using the definitions of $V_{\bar{x}}$ and V_{z_i} ($i = 1, \ldots, N$) in (25) and (26), we have

$$|\bar{x}| \leq \rho_{\bar{x}}^{z_i}(|z_i|), \tag{30}$$

$$|z_j| \leq \rho_{z_j}^{z_i}(|z_i|), \quad j \in N_i. \tag{31}$$

Taking the derivative of V_{z_i}, and using (19), (7), (30) and (31), we have

$$\begin{aligned} \nabla V_{z_i}\dot{z}_i =& \frac{1}{2}(\dot{z}_i^T z_i + z_i^T \dot{z}_i) \\ =& z_i^T\Big(\mu_i - \ddot{x}_d + (\alpha_i + \sum_{j\in N_i}\omega_{ji})z_i\Big) \\ &+ z_i^T\Big(f_i(x_i, v_i) - \sum_{j\in N_i}\omega_{ji}z_j\Big) + \bar{x}^T\{\hat{L}_{g\alpha^2}\}_i z_i \\ \leq& z_i^T\Big(\mu_i - \ddot{x}_d + (\alpha_i + \sum_{j\in N_i}\omega_{ji})z_i\Big) \\ &+ |z_i|\Big(\sum_{j\in N_i}\omega_{ji}|z_j| + |\{\hat{L}_{g\alpha^2}\}_i||\bar{x}| + |f_i(x_i, v_i)|\Big). \end{aligned}$$

Using control law (29), we obtain $\nabla V_{z_i}\dot{z}_i \leq -\frac{\sigma_{z_i}}{2}z_i^T z_i = -\sigma_{z_i}V_{z_i}$. The proof of Lemma 4 is concluded. □

4.3.3 Step 3–Synthesis

According to Lemma 3, 4, the gain-interconnection digraph and the cyclic-small-gain theorem, we now can provide the main result as follows.

Theorem 1. *For the multi-agent system* (5)–(6) *with the distributed control law defined in* (29), *if the cyclic-small-gain conditions* (20)–(22) *are satisfied, then the objective of formation* (8)–(10) *is achieved.*

Proof. Taking $\bar{x}$ and z as the states of the ISS system described by (18) and (19), as μ_i is designed, the ISS system is an unforced system. It is known that the ISS of an unforced system leads to the globally asymptotical stability [15]. Using the cyclic-small-gain theorem in Lemma 2, if the cyclic-small-gain conditions (20)–(22) are satisfied, i.e., the gain compositions along all cycles in the gain-interconnection digraph $\mathcal{G}_g$ are less than the identity function, then the $[\bar{x}^T, z_1^T, \ldots, z_N^T]^T$-system is globally asymptotically stable, which implies

$$\lim_{t\to\infty} \bar{x}(t) = \mathbf{0}, \tag{32}$$

$$\lim_{t\to\infty} z(t) = \mathbf{0}. \tag{33}$$

From the definition of $\bar{x}$ in (11), and together with (32), we know $\lim_{t\to\infty} x_i(t) - p_i - x_d(t) = 0, \quad i = 1, \ldots, N$. Correspondingly, for any $0 < j \neq i \leq N$, we have

$$\lim_{t\to\infty} |x_j(t) - x_i(t) - (p_i - p_j)| = 0.$$

In addition, using $\sum_{i=1}^{N} p_i = 0$, we have

$$\lim_{t\to\infty} (x_c - x_d) = \frac{1}{N}\sum_{i=1}^{N}\Big(\lim_{t\to\infty} x_i(t) - p_i - x_d(t)\Big) = 0.$$

Finally, considering the definition of z, and together with (33), we obtain

$$\begin{aligned}\lim_{t\to\infty} v(t) &= -\big((L_{\mathcal{G}_c} + \mathrm{diag}\{\boldsymbol{\alpha}\}) \otimes I_n\big) \lim_{t\to\infty} \bar{x} + \lim_{t\to\infty} \mathbf{1}_N \dot{x}_d \\ &= \lim_{t\to\infty} \mathbf{1}_N \dot{x}_d.\end{aligned}$$

Therefore, the objective defined in (8)–(10) is achieved. □

Remark 2. *It is worthy to point out that Theorem 1 shows that there are no requirements imposed on $\mathcal{G}_c$, the digraph depicting the communication topology among all the N agents, which can be strongly connected, weakly connected, or even disconnected as illustrated in the following example.*

5 Examples and Simulation Results

We employ the five-agent system with $\mathcal{G}_c$ in Fig. 1 as an example to demonstrate the effectiveness of the proposed control law.

For $i = 1, \ldots, 5$, the dynamics of the i-th agent are

$$\dot{x}_i = v_i, \quad \text{and} \quad \dot{v}_i = u_i + f_i(v_i), \tag{34}$$

where $x_i, v_i, u_i \in \mathbb{R}^2$ and $f_i(v_i) \leq [v_i^2(1); v_i^2(2)]$. Note that $f_i(v_i)$ does not satisfy the global Lipschitz condition. Comparing the dynamics (34) with (1), we take $\psi_{f_i}^{x_i}(|x_i|) = 0$ and $\psi_{f_i}^{v_i}(|v_i|) = |v_i|^2$.

The initial positions, the desired geometric pattern and the desired trajectory come from the settings of simulation in [9]. Specifically, the initial positions of the five agents are $(20.6, -8.1)$, $(16.4, 9.2)$, $(14.4, -19.2)$, $(2.6, 4.1)$, and $(15, -3)$; the desired geometric pattern $\mathcal{P}$ is described by $(0.62, 1.9)$, $(-1.6, 1.2)$, $(-1.6, -1.2)$, $(0.62, -1.9)$ and $(2, 0)$; and the given desired trajectory is $x_d = (10\cos(0.5t), 10\sin(0.5t))$.

The weighted Laplace $L_{\mathcal{G}_c}$ of the digraph in Fig. 1 can be obtained, and consequently, we have

$$\hat{L}_{\mathcal{G}_c} = \begin{bmatrix} 1 & -0.5 & -0.5 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & -0.5 & 1 \end{bmatrix}.$$

It is easy to obtain that $\lambda_{\min}(\hat{L}_{\mathcal{G}_c}) = -0.2071$. After choosing $\alpha_i = 1$ for all $i = \{1, \ldots, 5\}$ and $\epsilon = 0.2929$, it is obtained that $\alpha_{\min} + \lambda_{\min}(\hat{L}_{\mathcal{G}_c}) - \epsilon = 0.5$. Then we obtain $\gamma_{\bar{x}}^{z_i}(s) = 20s$ for $i = 1, \ldots, 5$. Correspondingly, $\hat{L}_{g\alpha^2}$ is obtained, which is

$$\hat{L}_{g\alpha^2} = \begin{bmatrix} 2.25 & -1 & -1 & 0 & 0 \\ -1 & 2.25 & -1 & 0 & 0 \\ -1 & -1 & 2.25 & 0 & 0 \\ 0 & 0 & 0 & 0.25 & -1 \\ 0 & 0 & 0 & -1 & 2.25 \end{bmatrix}.$$

Thus $|\{\hat{L}_{g\alpha^2}\}_1| = |\{\hat{L}_{g\alpha^2}\}_2| = |\{\hat{L}_{g\alpha^2}\}_3| = 2.6575$, $|\{\hat{L}_{g\alpha^2}\}_4| = 1.0308$ and $|\{\hat{L}_{g\alpha^2}\}_5| = 2.4622$.

There exist 5 vertices, 7 directed paths and 1 directed circle in $\mathcal{G}_c$ in Fig. 1. Correspondingly, there are 13 different inequalities in (20)–(22). From the discussion in Remark 1, to satisfy all the ISS cyclic-small-gain conditions, we can take $\gamma_{z_j}^{z_i}(s) = 0.9s$ for $(i,j) \in E$ and $\gamma_{z_i}^{\bar{x}}(s) = 0.04s$ for $i = 1, \ldots, 5$. Accordingly, we obtain $\rho_{\bar{x}}^{z_i}(s) = \alpha^{-1} \circ \left(\gamma_{z_i}^{\bar{x}}\right)^{-1} \circ \alpha(s) = 5s$ and $\rho_{z_j}^{z_i}(s) = \alpha^{-1} \circ \left(\gamma_{z_i}^{z_j}\right)^{-1} \circ \alpha(s) = 1.0541s$. Taking $\sigma_{z_i} = 1$, when $|z_i| \neq 0$, the inputs for agents are

$$\begin{cases} u_1 = -\frac{z_1}{|z_1|}(|v_1|^2 + 14.3416|z_1|) - 2.5z_1 + \ddot{x}_d, \\ u_2 = -\frac{z_2}{|z_2|}(|v_2|^2 + 14.3416|z_2|) - 2.5z_2 + \ddot{x}_d, \\ u_3 = -\frac{z_3}{|z_3|}(|v_3|^2 + 14.3416|z_3|) - 2.5z_3 + \ddot{x}_d, \\ u_4 = -\frac{z_4}{|z_4|}(|v_4|^2 + 5.154|z_4|) - 1.5z_4 + \ddot{x}_d, \\ u_5 = -\frac{z_5}{|z_5|}(|v_5|^2 + 13.3651|z_5|) - 2.5z_5 + \ddot{x}_d, \end{cases} \tag{35}$$

where $z_1 = v_1 + (x_1 - p_1 - x_d) - \big((x_2 - x_1) - (p_2 - p_1)\big) - \dot{x}_d$, $z_2 = v_2 + (x_2 - p_2 - x_d) - \big((x_3 - x_2) - (p_3 - p_2)\big) - \dot{x}_d$, $z_3 = v_3 + (x_3 - p_3 - x_d) - \big((x_1 - x_3) - (p_1 - p_3)\big) - \dot{x}_d$, $z_4 = v_4 + (x_4 - p_4 - x_d) - \dot{x}_d$, and $z_5 = v_5 + (x_5 - p_5 - x_d) - \big((x_4 - x_5) - (p_4 - p_5)\big) - \dot{x}_d$.

We simulate the five-agent system with control law (35). The simulation results are shown in Fig. 3 to 6. It is shown from Fig. 3 that the positions of all agents converge to a hexagon, i.e., the desired geometric pattern $\mathcal{P}$, and from Fig. 4 that the position center of all the agents converges to the center of the hexagon. The state error of the formation, defined by $e(t) = \sum_{i=1}^{5} |(x_i(t) - x_{i+1}(t)) - (p_i - p_{i+1})|$, converges to zero as that shown in Fig. 5. And further, the velocities in x and y directions for each agent are shown in Fig. 6(a) and Fig. 6(b), respectively. Obviously, the velocity converges to the desired velocity $\dot{x}_d$. Therefore, the formation tracking objective (8)–(10) is achieved.

6 Conclusions

A distributed control law for formation tracking of multi-agent system with second-order locally Lipschitz continuous nonlinear dynamics under directed topology is developed in this paper. Nonlinear multi-agent systems with more complex factors, such as the external disturbances, switching topology and time-delay, will be further explored.

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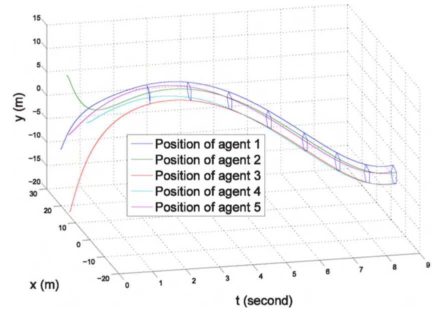


Fig. 3: The evolutions of five agents' positions

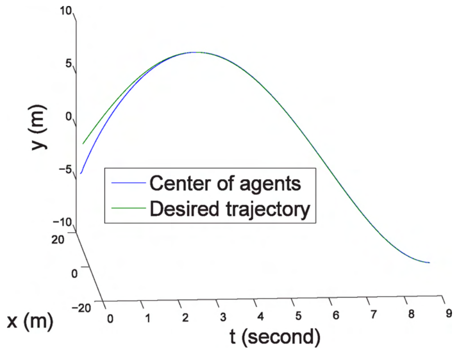


Fig. 4: The evolutions of the five-agent position center

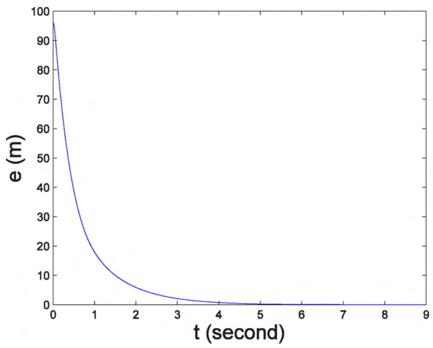
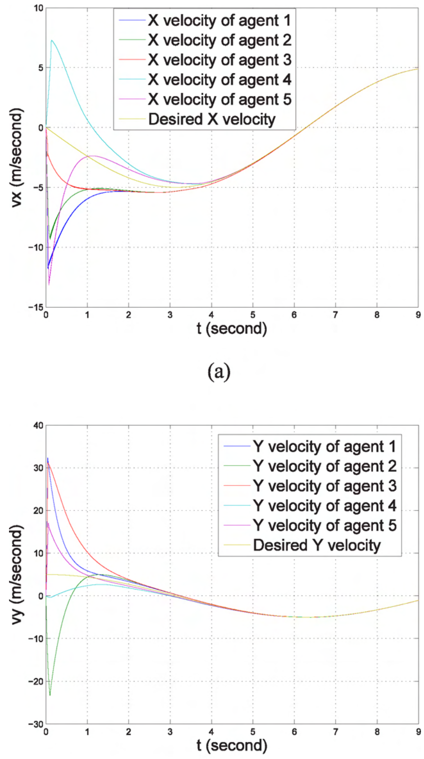


Fig. 5: The evolutions of formation errors



(a)

(b)

Fig. 6: The evolutions of the velocities

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