Second-order Consensus with Unknown Dynamics via Cyclic-small-gain Method

Article *in* IET Control Theory and Applications · December 2012

DOI: 10.1049/iet-cta.2011.0393

3 authors:

Xiangke Wang
National University of Defense Technology
94 PUBLICATIONS 780 CITATIONS

Tengfei Liu
Northeastern University (Shenyang, China)
78 PUBLICATIONS 403 CITATIONS

Jiahu Qin
University of Science and Technology of China
162 PUBLICATIONS 3,159 CITATIONS

www.ietdl.org

Published in IET Control Theory and Applications
Received on 6th July 2011
Revised on 5th April 2012
doi: 10.1049/iet-cta.2011.0393

ISSN 1751-8644

Second-order consensus with unknown dynamics via cyclic-small-gain method

X. Wang[1] *T. Liu*[2] *J. Qin*[3]

[1] *College of Mechatronics and Automation, National University of Defense Technology, Changsha 410073, China*
[2] *Department of Electrical and Computer Engineering, Polytechnic Institute of New York University, Brooklyn, NY 11201, USA*
[3] *Research School of Engineering, the Australian National University, Canberra ACT 0200, Australia*
E-mail: xkwang@nudt.edu.cn

Abstract: This study proposes a distributed non-linear consensus protocol for second-order non-linear multi-agent systems with unknown locally Lipschitz dynamics and connected graph. The main analysis is based on a blend of graph-theoretic and non-linear-theoretic tools with the notion of input-to-state stability (ISS) playing a central role. Through the backstepping design, the closed-loop multi-agent system is transformed into a two-cascade interconnected system with proven ISS properties. Correspondingly, the recently developed cyclic-small-gain theorem is then employed to guarantee the asymptotic stability of the closed-loop multi-agent system, which implies consensus.

1 Introduction

Recent years have seen significant development of the research on multi-agent consensus. The consensus problem usually refers to the problem of how to design a distributed protocol so that the states of a group of networked dynamical agents reach agreement. Many previously known papers on the consensus problem consider the cases where the multi-agent systems are governed by linear dynamics [1–9]; however, most of physical systems are inherently non-linear in nature. In this paper, we study the consensus problem for non-linear multi-agent systems. Pioneering works on state agreement for first-order non-linear system can be found in quite a few papers, for example, [10]. To the best of our knowledge, the consensus problem for higher-order non-linear multi-agent systems has not been extensively addressed because of many technical challenges. To highlight the basic idea, we consider the multi-agent system with second-order dynamics, while the approach can be extended for higher-order dynamics.

The consensus problem for first-order multi-agent systems has been widely studied in the past decade (see [1] for a tutorial overview). Algebraic graph theory and control theory have contributed significantly to the study of multi-agent consensus problems. The most common continuous-time consensus algorithm for the multi-agent systems with single-integrator model is given by [3–5]

$$\dot{x}_i = -\sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i = 1, \ldots, n \tag{1}$$

where x_i is the state of the ith agent and a_{ij} is the (i,j) entry of the adjacency matrix of the associated control graph. Recently, consensus for second-order multi-agent systems, usually with linear dynamics, has received increasing attention (see e.g. [6–9]). Specifically, Xie and Wang [9] introduced a linear-distributed consensus protocol for second-order multi-agent systems, which does not require the velocity information of the neighbours. The consensus problem for second-order non-linear multi-agents was studied in [11, 12] under a global Lipschitz condition. Adaptive control was introduced to the synchronisation of uncertain non-linear networked systems with connected communication digraph in [13]; whereas Zhang *et al.* [14] proposed an adaptive backstepping-based approach for achieving synchronisation for Lagrangian systems with model uncertainties.

The concept of input-to-state stability (ISS) invented by Sontag is a tool to describe how external inputs affect the internal stability of non-linear systems (see [15] for a tutorial). Recently, considerable efforts have been devoted to the stability analysis of interconnected ISS non-linear systems. Dashkovskiy *et al.* [16, 17] developed a matrix-small-gain criterion for networks with plus-type interconnections and mentioned the cyclic-small-gain condition. In [18], a more general cyclic-small-gain theorem for networks of input-to-output stable (IOS) systems was developed. Liu *et al.* [19] presented an ISS-Lyapunov function based cyclic-small-gain theorem. The cyclic-small-gain condition can be roughly described as follows: the composition of the gain functions along every cycle in the network of ISS systems is less than the identity function. A very recent result on ISS and multi-agent systems can be found in [20].

The main contribution of this paper is to provide a completely non-linear ISS-based analysis, which is a blend of graph-theoretic, back-stepping design and the newly developed cyclic-small-gain theorem, to deal with the consensus problem with the unknown locally Lipschitz continuous dynamical agents. Specifically, following the basic idea of backstepping and utilising the new properties of edge Laplacian for a connected graph, the second-order closed-loop multi-agent system modelled by a connected graph is transformed into a new network of subsystems with proven ISS properties, and correspondingly the consensus problem is transformed into an asymptotic stabilisation problem, whose asymptotic stability can be guaranteed by the cyclic-small-gain theorem [18, 19]. Consequently, a new distributed consensus protocol for multi-agent system with locally Lipschitz continuous dynamics is obtained naturally by combining the inputs of ISS-subsystems. In most of the existing research results on consensus problem of non-linear systems, for example, [11, 12], the dynamics of agents are confined to be globally Lipschitz continuous, and the performed techniques for the control law designs are in fact for a linear system, which cannot be extended to deal with the locally Lipschitz continuous dynamics. Although in this paper, through the new approach, the proposed distributed consensus protocol can effectively tackle the technical challenges caused by unknown locally Lipschitz continuous dynamics. To the best of our knowledge, this is one of the first and independent attempts that address the second-order consensus problems with unknown dynamics satisfying locally Lipschitz continuous. Moreover, the heterogeneous or isomorphic agents can be both adopted as the dynamics of agents studied are not required being identical. Furthermore for each agent, the new consensus protocol only needs the information of its position, velocity and the relative positions to its neighbours, without requiring its neighbours' velocities. When the multi-agent system is reduced to a double-integrator, the proposed consensus control algorithm can be realised linearly, which is in accordance with the linear algorithms in the previously known results, for example, [9]. Finally, by using the design methodology in this study, we are able to extend many existing first-order consensus results to the second-order or even higher-order case with non-linear dynamics, which is certainly non-trivial.

The rest of the paper is organised as follows. Section 2 provides the basic notions and results in graph theory and the ISS cyclic-small-gain theorem. The studied problem, including the non-linear dynamics of agents, the available information for protocol design, and the objective of this study, is formulated in Section 3. The main content, that is, the consensus protocol design based on graph theory, backstepping methodology and cyclic-small-gain theorem is elaborated in Section 4. A five-agent system is provided as an example in Section 5 with the protocol realisation and simulation results. Section 6 draws the conclusions and proposes some future work.

2 Basic notions and preliminary results

In this section, we review some basic notions and results in graph theory (see e.g. [21]) and the ISS (see e.g. [15, 22, 23]) that will be used throughout this paper.

2.1 Graph and matrix

We use $|\cdot|$ and $\|\cdot\|$ to denote the Euclidean norm and L_∞ norm for vectors and matrices, respectively. A vector that consists of all zero entries is denoted by $\mathbf{0}$. For a matrix A, $\mathcal{N}(A) = \{y : Ay = \mathbf{0}\}$ denotes its null space. The notion $D = \mathrm{diag}\{d_1, \ldots, d_n\}$ represents a diagonal matrix with d_i denoting the ith entry on the diagonal.

A *(undirected) graph* $G = (V, E)$ consists of a non-empty *vertex* set $V = \{1, \ldots, N\}$ and an *edge* set $E \subseteq V \times V$, in which an edge is an *unordered pair* of distinct vertices. The *adjacency matrix* $A = A(G) = [a_{ij}]$ is an $N \times N$ matrix given by $a_{ij} = 1$, if $(i,j) \in E$, and $a_{ij} = 0$, otherwise. If $(i,j) \in E$, then i,j are *adjacent*, and we said j is a *neighbour* of i. The neighbours of i are denoted by N_i. A *path* of length r from i to j is a sequence of $r+1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of G, then G is *connected*. A *cycle* is a connected graph where every vertex has exactly two neighbours. A *tree* is a connected graph without cycle. The *degree* of vertex i is given by $d_i = \sum_j a_{ij}$. An *orientation* on G is the assignment of a direction to each edge. The *incidence matrix* $B = B(G) = [b_{ij}]$ of an oriented graph is a $\{0, \pm 1\}$-matrix with rows and columns indexed by the vertices and edges of G, respectively, such that $b_{ij} = 1$ if the vertex i is the head of the edge j, $b_{ij} = -1$ if the vertex i is the tail of the edge j, and $b_{ij} = 0$ otherwise. The *Laplacian matrix* of G is $L_n = L_n(G) = \mathrm{diag}(d_1, \ldots, d_N) - A$. For connected graph G, $L_n = L_n(G) := BB^{\mathrm{T}}$ is symmetric and positive semi-definite. In [24], $L_e = L_e(G) := B^{\mathrm{T}}B$ is defined as *edge Laplacian* of G. It should be noted that the non-zero eigenvalues of L_e are equal to the non-zero eigenvalues of L_n [24]. For L_e, the following two lemmas hold.

Lemma 1 [25]: If G is a tree, then its edge Laplacian L_e is positive definite.

Lemma 2 [24]: Let L_e and B denote, respectively, the edge Laplacian and the incidence matrix of an oriented graph G, and $\mathcal{N}(A)$ denote the null space of matrix A. Then it is obtained $\mathcal{N}(L_e) = \mathcal{N}(B)$.

A *digraph* $G = (V, E)$ consists of a non-empty *vertex* set and a 'directed edge' set $E \subset V \times V$, where a directed edge is an 'ordered pair' of distinct vertices. Similar notions, such as path, cycle, can also be defined on digraph. A graph can also be treated as a digraph by considering its each edge (i,j) as two directed edges (i,j) and (j,i).

2.2 ISS and small-gain theorem

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be; *positive definite* if it is continuous, $\alpha(0) = 0$ and $\alpha(s) > 0$ for $s > 0$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$; it is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is decreasing and tends to zero at infinity. For a real-valued differentiable function V, ∇V stands for its gradient.

Consider the following non-linear system with $x \in \mathbb{R}^n$ as the state and $w \in \mathbb{R}^m$ as the external input

$$\dot{x} = f(x, w) \tag{2}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a locally Lipschitz vector field.

Definition 1 [15]: System (2) is said to be *ISS* with w as input if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for each initial condition $x(0)$ and each measurable essentially

doi: 10.1049/iet-cta.2011.0393

bounded input $w(\cdot)$ defined on $[0, \infty)$, the solution $x(\cdot)$ exists on $[0, \infty)$ and satisfies

$$|x(t)| \leq \beta(|x(0)|, \gamma(\|w\|)), \quad \forall t \geq 0 \tag{3}$$

It is known that if the system $\dot{x} = f(x, w)$ in (2) is ISS with w as the input, then the unforced system $\dot{x} = f(x, 0)$ is globally asymptotically stable at $x = 0$ [15].

Definition 2 [15]: For a non-linear system (2) with state $x \in \mathbb{R}^n$ and external input $w \in \mathbb{R}^m$, a function V is said to be an *ISS-Lyapunov function* if it is differentiable almost everywhere, and satisfies that

- V is positive definite and radially unbounded, that is, there exist $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|), \quad \forall x \in \mathbb{R}^n \tag{4}$$

- there exists a positive-definite α, and $\gamma \in \mathcal{K}$ such that

$$V(x) \geq \gamma(|w|) \Rightarrow \nabla V(x) f(x, w) \leq -\alpha(V(x)), \quad \forall x \in \mathbb{R}^n, \quad \forall w \in \mathbb{R}^m \tag{5}$$

Lemma 3 [15]: System (2) is ISS if and only if it has an ISS-Lyapunov function.

Consider the following interconnected system composed of N interacting subsystems

$$\dot{x}_i = f_i(x, w_i), \quad i = 1, \ldots, N \tag{6}$$

where $x_i \in \mathbb{R}^{n_i}$, $w_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R}^{n+m_i} \to \mathbb{R}^{n_i}$ with $n = \sum_{i=1}^N n_i$ is locally Lipschitz continuous such that $x = [x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}}$ is the unique solution of system (6) for a given initial condition. The external input $w = [w_1^{\mathrm{T}}, \ldots, w_N^{\mathrm{T}}]^{\mathrm{T}}$ is a measurable and locally essentially bounded function from $\mathbb{R}_+$ to $\mathbb{R}^m$ with $m = \sum_{i=1}^N m_i$. For the ith subsystem ($i = 1, \ldots, N$), there exists an ISS-Lyapunov function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}$ satisfying

- there exist $\underline{a}_i, \bar{a}_i \in \mathcal{K}_\infty$, such that

$$\underline{a}_i(|x_i|) \leq V_i(x_i) \leq \bar{a}_i(|x_i|), \quad \forall x_i \tag{7}$$

- there exist $\gamma_{x_i}^{x_j} \in \mathcal{K} \cup \{0\} (j \neq i)$, $\gamma_{x_i}^{w_i} \in \mathcal{K} \cup \{0\}$ and a positive-definite α_i such that

$$\begin{aligned} V_i(x_i) &\geq \max\{\gamma_{x_i}^{x_j}(V_j(x_j)), \gamma_{x_i}^{w_i}(|w_i|)\} \\ &\Rightarrow \nabla V_i(x_i) f_i(x_1, \ldots, x_N, w_i) \leq -\alpha_i(V_i(x_i)) \\ &\quad \forall x, \quad \forall w_i \end{aligned} \tag{8}$$

Lemma 4 (Cyclic-small-gain Theorem [18, 19]): Consider the continuous-time dynamical network (6). Suppose that for $i = 1, \ldots, N$, the x_i-subsystem admits an ISS-Lyapunov function V_i satisfying (7) and (8). Then the system (6) is ISS if for each $r = 2, \ldots, N$

$$\gamma_{x_{i_1}}^{x_{i_2}} \circ \gamma_{x_{i_2}}^{x_{i_3}} \circ \cdots \circ \gamma_{x_{i_r}}^{x_{i_1}} < \mathrm{Id} \tag{9}$$

for all $1 \leq i_j \leq N$, $i_j \neq i_{j'}$ if $j \neq j'$.

3 Problem statement

In this paper, we consider a group of N networked agents with a connected graph, in which each agent is regarded as a vertex, and each existing (bidirectional) control interconnection between agents is regarded as an edge. Such a graph is called *control-interconnection graph* and denoted by G_c. A control-interconnection graph of five-agent system is shown as an example in Fig. 1.

For $i = 1, \ldots, N$, the dynamics of the ith agent is represented by

$$\dot{x}_i = v_i \tag{10}$$

$$\dot{v}_i = f_i(x_i, v_i) + \mu_i \tag{11}$$

where $x_i \in \mathbb{R}^n$ is the position, $v_i \in \mathbb{R}^n$ is the velocity, $\mu_i \in \mathbb{R}^n$ is the control input, and $f_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is unknown and satisfied the locally Lipschitz continuous in the following Assumption 1.

Assumption 1: For each $i = 1, \ldots, N$, there exist $\psi_{f_i}^{x_i}, \psi_{f_i}^{v_i} \in \mathcal{K}_\infty$ such that

$$|f_i(x_i, v_i)| \leq \psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|) \tag{12}$$

Remark 1: Systems (10)–(11) with Assumption 1 have been widely employed to model practical systems and their control problems have been studied thoroughly in the non-linear control literature [26]. If the term f_i does not exist, then the dynamics (10)–(11) is reduced to the widely studied (linear) double-integrator model (see e.g. [7, 9]). In the recent studies [11, 12], the non-linear term f is assumed to be globally Lipschitz, that is, there exist $\rho_1, \rho_2 \geq 0$, such that for all x, x', v, v'

$$|f(x, v) - f(x', v')| \leq \rho_1|x - x'| + \rho_2|v - v'| \tag{13}$$

and consequently the performed control law designs and convergence analysis are in fact for a linear system. Different from that in [11, 12], the non-linear term $f_i(x_i, v_i)$ in (11) is only assumed to be locally Lipschitz continuous in our work, which includes the globally Lipschitz continuous condition as a very special case. Note that the techniques in [11, 12] do not work for the locally Lipschitz continuous non-linear dynamics. It is worth pointing out here that in this paper some new techniques will be proposed in dealing with the challenges caused by considering such locally Lipschitz continuous conditions, which leads a significant difference from the existing results. In addition, the non-linear term f_i in (11) is not required to be identical; thus, heterogeneous or isomorphic agents can all be adopted with this dynamics.

Assumption 2: For $i = 1, \ldots, N$, the position x_i, the velocity v_i and the relative positions to its neighbours $x_j - x_i$ of the ith agent for $j \in N_i$, where N_i is the neighbours of

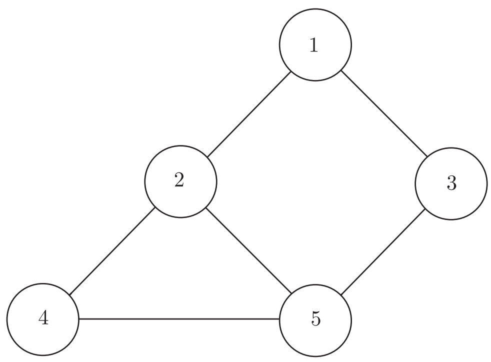

Fig. 1 *Control-interconnection graph of five-agent system*

doi: 10.1049/iet-cta.2011.0393

the ith agent, are available for the design of the consensus protocol μ_i.

The objective of this study is to design a distributed consensus protocol in the form of $\mu_i = g_i(X_i)$ with $X_i = \{x_i, v_i\} \cup \{x_j - x_i | j \in N_i\}$, such that the position evolutions of the agents (10)–(11) reach an agreement and the velocities of the agents converge to zeros, that is

$$\lim_{t\to\infty}\big(x_i(t) - x_j(t)\big) = 0 \tag{14}$$

$$\lim_{t\to\infty} v_i(t) = 0 \tag{15}$$

for any $i,j = 1, \ldots, N$.

4 Consensus protocol design

In this section, we develop a non-linear consensus protocol by modifying the standard backstepping methodology. A system with state y is called the y-system. Through the design approach, we will transform the $[x_1^{\mathrm{T}}, v_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}, v_N^{\mathrm{T}}]^{\mathrm{T}}$-system defined in (10)–(11) into a new $[\tilde{x}^{\mathrm{T}}, z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}}]^{\mathrm{T}}$-system composed of ISS subsystems. Then, the consensus problem is solved by guaranteeing the asymptotic stability of the $[\tilde{x}^{\mathrm{T}}, z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}}]^{\mathrm{T}}$-system.

For the subsystems of the $[\tilde{x}^{\mathrm{T}}, z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}}]^{\mathrm{T}}$-system, we define the following ISS-Lyapunov function candidates

$$V_{\tilde{x}}(\tilde{x}) = \alpha(|\tilde{x}|) \tag{16}$$

$$V_{z_i}(z_i) = \alpha(|z_i|), \quad i = 1, \ldots, N \tag{17}$$

where

$$\alpha(s) = \frac{1}{2}s^2 \quad \text{for } s \in \mathbb{R}_+ \tag{18}$$

In the following discussions, we simply use $V_{\tilde{x}}$ and V_{z_i} instead of $V_{\tilde{x}}(\tilde{x})$ and $V_{z_i}(z_i)$, respectively.

The main design procedure is composed of three steps. Specifically, we firstly consider the second-order states (velocities) as the virtual control inputs of the first-order subsystem, whose ISS property is guaranteed when taking the common used consensus algorithm (1) as the virtual control law; then, we design a distributed control law for the second-order states (velocities) to track the virtual control inputs in the sense of ISS; finally, by utilising the cyclic-small-gain theorem in Lemma 4, the consensus of the multi-agent system is achieved.

4.1 Step 1: the $\tilde{x}$-subsystem

Label all edges of the control-interconnection graph G_c with $1, \ldots, M$, and denote h_l and t_l as the head and the tail of the lth edge, respectively. Define

$$\tilde{x}_l = x_{h_l} - x_{t_l} \tag{19}$$

for $l = 1, \ldots, M$. Denote $x = [x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}}$, $v = [v_1^{\mathrm{T}}, \ldots, v_N^{\mathrm{T}}]^{\mathrm{T}}$ and $\tilde{x} = [\tilde{x}_1^{\mathrm{T}}, \ldots, \tilde{x}_M^{\mathrm{T}}]^{\mathrm{T}}$. Then, we obtain

$$\tilde{x} = B^{\mathrm{T}}x \tag{20}$$

where B is the incidence matrix of G_c. By using (10), we obtain

$$\dot{\tilde{x}} = B^{\mathrm{T}}\dot{x} = B^{\mathrm{T}}v \tag{21}$$

We take v as the virtual control input of the $\tilde{x}$-subsystem in (21). According to the most common continuous-time consensus algorithm (1), for $i = 1, \ldots, N$, we define

$$v_i^* = k\sum_{j\in N_i}(x_j - x_i) \tag{22}$$

If $v_i = v_i^*$, then it is well known that asymptotic stability for (21) and thus consensus for (10) can be achieved for any connected control-interconnection graph.

For each $i = 1, \ldots, N$, define the error

$$z_i = v_i - v_i^* = v_i - k\sum_{j\in N_i}(x_j - x_i) \tag{23}$$

Denote $z = [z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}}]^{\mathrm{T}}$, $v^* = [v_1^{*T}, \ldots, v_N^{*T}]^{\mathrm{T}}$ and L_n as the Laplacian matrix of G_c. Obviously, $v^* = -kL_nx$ and

$$v = z - kL_nx \tag{24}$$

By substituting (24) into (21), and using $L_n = BB^{\mathrm{T}}$ and $\tilde{x} = B^{\mathrm{T}}x$, the $\tilde{x}$-subsystem becomes

$$\dot{\tilde{x}} = B^{\mathrm{T}}(z - kL_nx) = -kL_e\tilde{x} + B^{\mathrm{T}}z \tag{25}$$

where $L_e = B^{\mathrm{T}}B$ is the edge Laplacian of G_c. Denote the second smallest eigenvalue of L_n by $\lambda_2(L_n)$. As G_c considered is connected, it is well known that $\lambda_2(L_n) > 0$. For the $\tilde{x}$-subsystem (25), we have the following lemma.

Lemma 5: The $\tilde{x}$-subsystem (25) with z composed of z_i ($i = 1, \ldots, N$) as the input is ISS. Moreover, for any constant $0 < \epsilon < k\lambda_2(L_n)$, the ISS-Lyapunov function $V_{\tilde{x}}$ defined in (16) satisfies

$$V_{\tilde{x}} \geq \max_{i=1,\ldots,N}\{\gamma_{\tilde{x}}^{z_i}(V_{z_i})\} \Rightarrow \nabla V_{\tilde{x}}\dot{\tilde{x}} \leq -2\epsilon V_{\tilde{x}} \tag{26}$$

where

$$\gamma_{\tilde{x}}^{z_i}(s) = N\left(\frac{|B|}{k\lambda_2(L_n) - \epsilon}\right)^2 s \tag{27}$$

for all $s \in \mathbb{R}_+$.

Proof: It is known the non-zero eigenvalues of L_e are equal to the non-zero eigenvalues of L_n, thus $\lambda_2(L_n)$ is also the smallest non-zero eigenvalue of L_e. For any $x \in \mathbb{R}^N$, according to (20), we have immediately that $\tilde{x}$ belongs to the column space of B^{T}, which is equal to the orthogonal complement to $\mathcal{N}(B)$, i.e. the subspace composes all the vectors perpendicular to $\mathcal{N}(B)$. Also it is known that $\mathcal{N}(B) = \mathcal{N}(L_e)$ [24]. Denote the orthogonal complement to $\mathcal{N}(L_e)$ by $\mathcal{N}(L_e)^{\perp}$. Then we have $\tilde{x} \in \mathcal{N}(L_e)^{\perp}$. Considering L_e is symmetric positive semidefinite, and $\lambda_2(L_n)$ is its smallest non-zero eigenvalue, we can obtain directly

$$\tilde{x}^{\mathrm{T}}L_e\tilde{x} \geq \lambda_2(L_n)|\tilde{x}|^2 \tag{28}$$

Taking the derivative of $V_{\tilde{x}}$ defined in (16), and then using (25), we have

$$\nabla V_{\tilde{x}}\dot{\tilde{x}} = -k\tilde{x}^{\mathrm{T}}L_e\tilde{x} + \tilde{x}B^{\mathrm{T}}z \tag{29}$$

Using (28), we obtain

$$\nabla V_{\tilde{x}}\dot{\tilde{x}} \leq -k\lambda_2(L_n)|\tilde{x}|^2 + |\tilde{x}||B||z| \tag{30}$$

doi: 10.1049/iet-cta.2011.0393

Given any positive $0 < \epsilon < k\lambda_2(L_n)$, the condition $V_{\tilde{x}} \geq \max_{i=1,\ldots,N}\{\gamma_{\tilde{x}}^{z_i}(V_{z_i})\}$ implies

$$V_{\tilde{x}} \geq \left(\frac{|B|}{k\lambda_2(L_n) - \epsilon}\right)^2 \sum_{i=1}^{N} V_{z_i} = \frac{1}{2}\left(\frac{|B|}{k\lambda_2(L_n) - \epsilon}\right)^2 z^{\mathrm{T}} z$$

Thus, we obtain

$$|z| \leq \frac{(k\lambda_2(L_n) - \epsilon)|\tilde{x}|}{|B|} \tag{31}$$

Substituting (31) into (30), it is achieved

$$\nabla V_{\tilde{x}}\dot{\tilde{x}} \leq -\epsilon|\tilde{x}|^2 = -2\epsilon V_{\tilde{x}} \tag{32}$$

The proof of Lemma 5 is concluded. □

Remark 2: If the control-interconnection graph G_c is a tree, then we know L_e is positive definite [25]; thus its smallest eigenvalue $\lambda_{\min}(L_e) > 0$, which indicates $\lambda_2(L_n) = \lambda_{\min}(L_e)$.

4.2 Step 2: the z_i-subsystems

For $i = 1, \ldots, N$, from (24), we obtain

$$z_i = v_i + kL_{n_i}x \tag{33}$$

where L_{n_i} is the i-th row of L_n. Denote B_i as the ith row of B. Clearly, $L_{n_i} = B_iB^{\mathrm{T}}$. Then substituting (20) into (33), we obtain

$$z_i = v_i + kB_iB^{\mathrm{T}}x = v_i + kB_i\tilde{x} \tag{34}$$

Differentiating both sides of (34) and using, we have

$$\dot{z}_i = \dot{v}_i + kB_i\dot{\tilde{x}} \tag{35}$$

By substituting (11) and (25) into (35), we obtain

$$\begin{aligned}\dot{z}_i &= f_i(x_i, v_i) + \mu_i + kB_i(-kL_e\tilde{x} + B^{\mathrm{T}}z)\\ &= f_i(x_i, v_i) + \mu_i - k^2B_iB^{\mathrm{T}}B\tilde{x} + kB_iB^{\mathrm{T}}z\end{aligned} \tag{36}$$

Note that $B_iB^{\mathrm{T}} = L_{n_i}$ and $L_{n_i}z = d_iz_i - \sum_{j\in N_i} z_j$ with d_i the degree of vertex i in the control-interconnection graph G_c. Then, for $i = 1, \ldots, N$, the z_i-subsystems is obtained from (36) as follows

$$\dot{z}_i = f_i(x_i, v_i) + kd_iz_i + \mu_i - k^2L_{n_i}B\tilde{x} - k\sum_{j\in N_i} z_j \tag{37}$$

Recall that $\alpha(s) = \frac{1}{2}s^2$ for $s \in \mathbb{R}_+$.

Lemma 6: Consider the z_i-subsystem (37) with z_i as the state, and $\tilde{x}$ and z_js ($j \in N_i$) as the external inputs. For any specified constant $\sigma_{z_i} > 0$ and any specified $\gamma_{z_i}^{\tilde{x}}, \gamma_{z_i}^{z_j} \in \mathcal{K}_\infty$ with $j \in N_i$, we can design

$$\mu_i = \begin{cases} -\frac{z_i}{|z_i|}\mu_{i_1}(|x_i|, |v_i|, |z_i|) - \left(\frac{\sigma_{z_i}}{2} + kd_i\right)z_i & \text{if } |z_i| \neq 0\\ 0 & \text{if } |z_i| = 0\end{cases} \tag{38}$$

where

$$\begin{aligned}\mu_{i_1}(|x_i|, |v_i|, |z_i|) &= \psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|) + k^2|L_{n_i}B|\rho_{\tilde{x}}^{z_i}(|z_i|)\\ &\quad + k\sum_{j\in N_i}\rho_{z_j}^{z_i}(|z_i|)\end{aligned}$$

with $\rho_{\tilde{x}}^{z_i}(s) = \alpha^{-1} \circ (\gamma_{z_i}^{\tilde{x}})^{-1} \circ \alpha(s)$ and $\rho_{z_j}^{z_i}(s) = \alpha^{-1} \circ (\gamma_{z_i}^{z_j})^{-1} \circ \alpha(s)$, such that the z_i-subsystem is ISS with ISS-Lyapunov function V_{z_i} satisfying

$$V_{z_i} \geq \max_{j\in N_i}\{\gamma_{z_i}^{\tilde{x}}(V_{\tilde{x}}), \gamma_{z_i}^{z_j}(V_{z_j})\} \Rightarrow \nabla V_{z_i}\dot{z}_i \leq -\sigma_{z_i}V_{z_i}$$

Proof: Considering $V_{z_i} \geq \max_{j\in N_i}\{\gamma_{z_i}^{\tilde{x}}(V_{\tilde{x}}), \gamma_{z_i}^{z_j}(V_{z_j})\}$ and using the definitions of $V_{\tilde{x}}$ and V_{z_i} ($i = 1, \ldots, N$) in (16) and (17), we have

$$|\tilde{x}| \leq \rho_{\tilde{x}}^{z_i}(|z_i|) \tag{39}$$

$$|z_j| \leq \rho_{z_j}^{z_i}(|z_i|), \quad j \in N_i \tag{40}$$

Taking the derivative of V_{z_i}, and using (37), (12), (39) and (40), we have (see equations at the bottom of the page)

When $|z_i| \neq 0$, by using (38), we achieve

$$\nabla V_{z_i}\dot{z}_i \leq -\frac{\sigma_{z_i}}{2}z_i^{\mathrm{T}}z_i = -\sigma_{z_i}V_{z_i} \tag{41}$$

Obviously, when $|z_i| = 0$, (41) is still hold. Therefore the proof of Lemma 6 is concluded. □

4.3 Step 3: consensus protocol

By Lemmas 5 and 6, the closed-loop multi-agent system composed of (10), (11) and (38) has been transformed into an interconnection of ISS subsystems represented by (25) and (37). Considering the subsystems as vertices and the gain connections as directed edges, the interconnected system composed of the $\tilde{x}$-subsystem and the z_i-subsystems can be modelled by a digraph G_g, called the ‘gain-interconnection digraph’. Fig. 2 shows the gain-interconnection digraph of the five-agent system with control-interconnection graph shown in Fig. 1.

$$\begin{aligned}\nabla V_{z_i}\dot{z}_i &= z_i^{\mathrm{T}}\mu_i + z_i^{\mathrm{T}}f_i(x_i, v_i) - k^2z_i^{\mathrm{T}}L_{n_i}B\tilde{x} + kd_iz_i^{\mathrm{T}}z_i - kz_i^{\mathrm{T}}\sum_{j\in N_i} z_j\\ &\leq z_i^{\mathrm{T}}(\mu_i + kd_iz_i) + |z_i|\left(\psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|) + k^2|L_{n_i}B||\tilde{x}| + k\sum_{j\in N_i}|z_j|\right)\\ &\leq z_i^{\mathrm{T}}(\mu_i + kd_iz_i) + |z_i|\left(\psi_{f_i}^{x_i}(|x_i|) + \psi_{f_i}^{v_i}(|v_i|) + k^2|L_{n_i}B|(\rho_{z_i}^{\tilde{x}})^{-1}(|z_i|) + k\sum_{j\in N_i}(\rho_{z_i}^{\tilde{x}})^{-1}(|z_i|)\right)\end{aligned}$$

doi: 10.1049/iet-cta.2011.0393

Remark 3: G_g is different from G_c, but it can be constructed from G_c. Specifically, given a G_c, the G_g can be obtained by firstly adding one vertex (the $\tilde{x}$ in Fig. 2) together with N undirected edges $(\tilde{x}, z_i)$ for $i = 1, \ldots, N$, and then replacing each edge with two directed edges.

Denote the vertices in G_c and the vertices representing the z_i-subsystems in G_g by $1, \ldots, N$, denote the additional vertex representing the $\tilde{x}$-subsystem in G_g by $N+1$. For the corresponding gain-interconnection digraph G_g, we can investigate the cyclic-small-gain condition (the gain compositions along all cycles in the gain-interconnection digraph are less than the identity function) required by Lemma 4, which can be explicitly described as follows:

1. For each vertex i in G_c, there exists a cycle $(i, N+1, i)$ in G_g. The corresponding small-gain condition is

$$\gamma_{z_i}^{\tilde{x}} \circ \gamma_{\tilde{x}}^{z_i} < \mathrm{Id} \tag{42}$$

2. For each edge (i,j) in G_c, there exists a cycle (i,j,i) in G_g. The corresponding small-gain condition is

$$\gamma_{z_i}^{z_j} \circ \gamma_{z_j}^{z_i} < \mathrm{Id} \tag{43}$$

3. For each cycle $(i_1, i_2, \ldots, i_{c-1}, i_c, i_1)$ in G_c, there exist two (clockwise and anticlockwise) cycles $(i_1, i_2 \ldots, i_{c-1}, i_c, i_1)$ and $(i_1, i_c, i_{c-1}, \ldots, i_2, i_1)$ in G_g. The corresponding small-gain conditions are

$$\gamma_{z_{i_1}}^{z_{i_2}} \circ \gamma_{z_{i_2}}^{z_{i_3}} \circ \cdots \circ \gamma_{z_{i_{c-1}}}^{z_{i_c}} \circ \gamma_{z_{i_c}}^{z_{i_1}} < \mathrm{Id} \tag{44}$$

$$\gamma_{z_{i_1}}^{z_{i_c}} \circ \gamma_{z_{i_c}}^{z_{i_{c-1}}} \circ \cdots \circ \gamma_{z_{i_3}}^{z_{i_2}} \circ \gamma_{z_{i_2}}^{z_{i_1}} < \mathrm{Id} \tag{45}$$

4. For each path $(i_1, i_2, \ldots, i_{r-1}, i_r)$ in G_c, there exist two cycles $(i_1, \ldots, i_r, N+1, i_1)$ and $(i_1, N+1, i_r, \ldots, i_1)$ in G_g. The corresponding small-gain conditions are

$$\gamma_{z_{i_1}}^{z_{i_2}} \circ \gamma_{z_{i_2}}^{z_{i_3}} \circ \cdots \circ \gamma_{z_{i_r}}^{z_{\tilde{x}}} \circ \gamma_{z_{\tilde{x}}}^{z_{i_1}} < \mathrm{Id} \tag{46}$$

$$\gamma_{z_{i_1}}^{\tilde{x}} \circ \gamma_{\tilde{x}}^{z_{i_r}} \circ \cdots \circ \gamma_{z_{i_3}}^{z_{i_2}} \circ \gamma_{z_{i_2}}^{z_{i_1}} < \mathrm{Id} \tag{47}$$

Remark 4: A sufficient condition to satisfy the cyclic-small-gain conditions from (42) to (47) is to simply choose

$$\gamma_{z_j}^{z_i} < \mathrm{Id}, \quad (i,j) \in E \tag{48}$$

$$\gamma_{z_i}^{\tilde{x}} = \gamma_{z_j}^{\tilde{x}} < (\gamma_{\tilde{x}}^{z_i})^{-1} = (\gamma_{\tilde{x}}^{z_j})^{-1}, \quad 1 \le i \ne j \le N \tag{49}$$

Obviously, the cyclic-small-gain conditions from (42) to (47) can be guaranteed by choosing (48) and (49) regardless of the number of the agents.

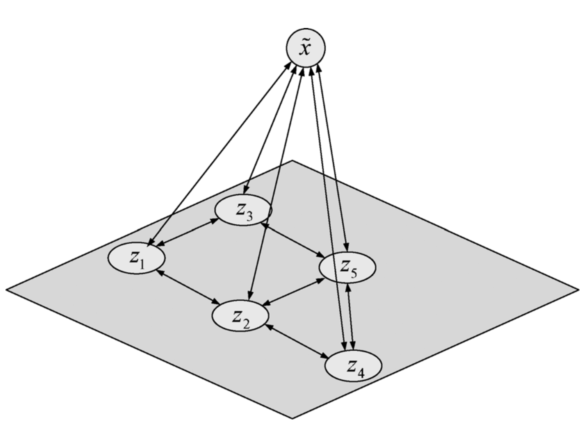

Fig. 2 *G_g of previous five-agent system*

Based on the cyclic-small-gain theorem and the gain-interconnection digraph, we present the main result of the paper in the following theorem.

Theorem 1: For the multi-agent system (10)–(11) with the distributed consensus protocol (38), if (42)–(47) hold, then the objective of consensus (14) and (15) is achieved.

Proof: Taking $\tilde{x}$ and z as the states of the ISS system described by (25) and (37), as μ_i is designed, the ISS system is an unforced system. It is known that the ISS of an unforced system leads to the globally asymptotical stability [15]. If (42)–(47) hold, then the required cyclic-small-gain condition is satisfied. By using the cyclic-small-gain theorem presented in Lemma 4, we obtain that the $[\tilde{x}^{\mathrm{T}}, z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}}]^{\mathrm{T}}$-system composed of (10), (11) and (38) is ISS, and further is globally asymptotically stable, which implies

$$\lim_{t\to\infty} \tilde{x}(t) = \mathbf{0}$$

and

$$\lim_{t\to\infty} z(t) = \mathbf{0}$$

Considering the definition of z_i in (23), we obtain $\lim_{t\to\infty} v(t) = \mathbf{0}$. Then, consensus defined in (14) and (15) is achieved. □

Remark 5: It should be noted that the consensus protocol (38) is switching because of the sign function $\frac{z_i}{|z_i|}$, and chattering may happen at the neighbourhood of the equilibrium $z_i = 0$. From the differential inclusion theory [27], it is known that there always exists a unique solution for the consensus protocol (38). And further, many methods are developed to overcome chattering in practice, for example in [28], continuous function $\frac{z_i}{|z_i|+\delta}$, where δ is a positive number which is selected to reduce the chattering problem, is employed instead of the sign function $\frac{z_i}{|z_i|}$.

Remark 6: If the $f_i = 0$, that is, $\psi_{f_i}^{x_i} = \psi_{f_i}^{v_i} = 0$, then the dynamics of the multi-agent system (10)–(11) is reduced to a double-integrator dynamics. In this case, we can directly choose linear ISS gains $\gamma_{\tilde{x}}^{z_i} = k_{\tilde{x}}^{z_i}$, $\gamma_{z_j}^{z_i} = k_{z_j}^{z_i}$ $(j \in N_i, i = 1, \ldots, N)$. Then, the protocol (38) is specified as

$$\mu_i = -\left(k^2 |L_{n_i} B| \rho_{\tilde{x}}^{z_i} + k \sum_{j\in N_i} \rho_{z_j}^{z_i} + \frac{\sigma_{z_i}}{2} + k d_i\right) z_i \tag{50}$$

with $\rho_{\tilde{x}}^{z_i} = 1/\sqrt{k_{\tilde{x}}^{z_i}}$ and $\rho_{z_j}^{z_i} = 1/\sqrt{k_{z_j}^{z_i}}$. Recall the definition of z_i in (23). It can be observed that the protocol (50) is in accordance with the linear algorithms in the previously known results, for example, [9].

Remark 7: It should be noticed that, for the ith agent, the protocol (38) only requires the information of the position x_i, the velocity v_i and the relative positions $x_j - x_i$ $(j \in N_i)$, whereas the additional neighbours' velocities v_j $(j \in N_i)$ are also required for some commonly used second-order nonlinear consensus protocols, such as in [11, 12]. It is known that it is difficult to obtain the neighbours' velocities in the practice measures. In addition, in many cases, the agent dynamics $f_i(x_i, v_i)$ does not depend on the agent's position x_i, viz. $\psi_{f_i}^{x_i} = 0$ in (11). Then, the global position information is also not necessary for protocol (38).

doi: 10.1049/iet-cta.2011.0393

Remark 8: Linear consensus protocols have been proposed for high-order multi-agent system, for example, [29]; while it should be noted that the methods employed in this paper can also be extended for addressing the high-order non-linear consensus problem with the help of the small-gain theorem-based recursive design approach for non-linear cascade systems in [30].

Remark 9: It should be noted that the result in Theorem 1 is devoted to dealing with non-linear multi-agent consensus problems with locally Liptschitz continue dynamics by using the newly developed cyclic-small-gain method more from the theoretical perspective. A number of practical issues, such as the case that G_c is switching, the case with communication delay, and specifying the convergence rate are worthy of further consideration.

4.4 On the number of the cycles

In this subsection, we further discuss the number of the cycles in the gain-interconnection digraph of the closed-loop multi-agent system, which is equivalent to the numbers of the control gains constraints required by Theorem 1. We employ $n! = \prod_{i=0}^{n} i$, $P_r^n = \prod_{i=0}^{r}(n-i)$, and $C_r^n = P_r^n/r!$ for convenience.

Lemma 7: If G_c is a tree with N vertices, then there are N^2+N-1 different cycles in G_g.

Proof: Because G_c is a tree, there are N vertices, $M = N-1$ edges and no cycles in G_c, which form $2N-1$ different cycles in G_g. In addition, for any two vertices i and j in G_c, there exists exactly one path between i and j in G_c, and there are two (clockwise and anticlockwise) corresponding cycles in G_g. Hence, we conclude that there are $N(N-1)+2N-1 = N^2+N-1$ cycles in G_g. □

Correspondingly, for this special case, the ISS gains in the consensus protocol in Theorem 1 should be chosen such that for all $i = 1, \ldots, N$,

$$\gamma_{z_i}^{z_j} \circ \gamma_{z_j}^{z_i} < \mathrm{Id}, \quad j \in N_i \tag{51}$$

$$\gamma_{z_i}^{\tilde{x}} \circ \gamma_{\tilde{x}}^{z_i} < \mathrm{Id} \tag{52}$$

$$\gamma_{z_i}^{\tilde{x}} \circ \gamma_{\tilde{x}}^{z_j} \circ \gamma_{z_j}^{z_{k_1}} \circ \cdots \circ \gamma_{z_{k_m}}^{z_i} < \mathrm{Id},\ j \neq i \tag{53}$$

where $(i, k_m, \ldots, k_1, j)$ is a path in the control-interconnection graph.

Lemma 8: If G_c is a complete graph with N vertices, then there are $(N^2+N)/2 + \sum_{l=3}^{N} C_l^N (l-1)! + 2C_2^N \sum_{r=0}^{N-2} P_r^{N-2}$ different cycles in G_g.

Proof: As a complete graph, there are N vertices and C_2^N edges in G_c, which form $(N^2+N)/2$ different cycles in G_g. In addition, arbitrary $l \geq 3$ vertices construct $l!/(2l)$ different circles in G_c. Thus, the cycles in G_c form $\sum_{l=3}^{N} C_l^N (l-1)!$ different cycles in G_g. Finally, for any two vertices i and j in G_c, there are $\sum_{r=0}^{N-2} P_r^{N-2}$ different paths between i and j. Thus, the paths in G_c form $2C_2^N \sum_{r=0}^{N-2} P_r^{N-2}$ different circles in G_g. Hence, we conclude that there are $(N^2+N)/2 + \sum_{l=3}^{N} C_l^N (l-1)! + 2C_2^N \sum_{r=0}^{N-2} P_r^{N-2}$ different cycles in G_g. □

Lemma 9: Given a G_c with N vertices, M edges and L circles, and supposed there are r_i different vertices in circle C_i $(i = 1, \ldots, L)$, there are no more than $N + M + 2L + 4\sum_{i=1}^{L} C_2^{r_i}$ different cycles in G_g.

Proof: The N vertices, M edges and L cycles in G_c form $N + M + 2L$ different cycles in G_g. In addition, for circle C_i with r_i different vertices, there are two different paths between any two vertices in C_i; thus there are $2C_2^{r_i}$ different paths in C_i. Note that some paths may be the same in different cycles; therefore the total path in G_c is no more than $2\sum_{i=1}^{L} C_2^{r_i}$, which form no more than $4\sum_{i=1}^{L} C_2^{r_i}$ different cycles in G_g. Hence, we conclude that there are no more than $N + M + 2L + 4\sum_{i=1}^{L} C_2^{r_i}$ cycles in G_g. □

5 An example

We employ the five-agent system with G_c in Fig. 1 as an example to demonstrate the effectiveness of the proposed consensus protocol.

For $i = 1, \ldots, 5$, the dynamics of the ith agent is assumed to be

$$\dot{x}_i = v_i \tag{54}$$

$$\dot{v}_i = u_i + f_i(v_i) \tag{55}$$

where $x_i, v_i, u_i \in \mathbb{R}$ and $f_i(v_i)$ is unknown but satisfies $|f_i(v_i)| \leq v_i^2$. Clearly, function f_i may be not globally Lipschitz. However, this kind of dynamics exists in many practical systems. For example, the air resistance (drag) force acting on a vehicle is typically proportional to the square of the velocity of the vehicle.

Comparing the dynamics (55) with (1), we take

$$\psi_{f_i}^{x_i}(|x_i|) = 0 \quad \text{and} \quad \psi_{f_i}^{v_i}(|v_i|) = v_i^2 \tag{56}$$

The incident matrix B and the Laplace matrix L_n of the graph in Fig. 1 are

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}$$

$$L_n = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}$$

Direct calculation yields $|B| = 2.1490$, $\lambda_2(L_n) = 1.3820$, $|L_{n_1}B| = 4.5826$, $|L_{n_2}B| = 7.0711$, $|L_{n_3}B| = 4.5826$, $|L_{n_4}B| = 4.4721$ and $|L_{n_5}B| = 7.0711$. Choosing $k = 1$ and $\epsilon = 0.3820$ in (27), we have $\gamma_{\tilde{x}}^{z_i}(s) = 23.0910s$.

There are five vertices, six edges, three circles and 32 paths in Fig. 1. Correspondingly, there are 81 different inequalities in (42)–(47), which are

$$\gamma_{z_i}^{\tilde{x}} \circ \gamma_{\tilde{x}}^{z_i} < \mathrm{Id}, \quad i = 1, \ldots, 5 \tag{57}$$

$$\gamma_{z_j}^{z_i} \circ \gamma_{z_i}^{z_j} < \mathrm{Id}, \quad (i,j) \in E \tag{58}$$

$$\gamma_{z_{l_2}}^{z_{l_1}} \circ \cdots \circ \gamma_{z_{l_n}}^{z_{l_{(n-1)}}} \circ \gamma_{z_{l_1}}^{z_{l_n}} < \mathrm{Id} \tag{59}$$

$$\gamma_{z_{p_1}}^{\tilde{x}} \circ \gamma_{z_{p_2}}^{z_{p_1}} \circ \cdots \circ \gamma_{z_{p_n}}^{z_{p_{(n-1)}}} \circ \gamma_{\tilde{x}}^{z_{p_n}} < \mathrm{Id} \tag{60}$$

doi: 10.1049/iet-cta.2011.0393

where E is the edge set of Fig. 1, $(l_1, \ldots, l_n)$ and $(p_1, \ldots, p_n)$ are obtained from the circle set and the path set of Fig. 1 clockwise and anticlockwise, respectively. According to Remark 4, we take $\gamma_{z_j}^{z_i}(s) = 0.9s$ for $(i,j) \in E$ and $\gamma_{z_i}^{\tilde{x}}(s) = 0.01s$ for $i = 1, \ldots, 5$ to satisfy the cyclic-small-gain conditions in (57)–(60). Accordingly, we obtain

$$\rho_{\tilde{x}}^{z_i}(s) = \alpha^{-1} \circ (\gamma_{z_i}^{\tilde{x}})^{-1} \circ \alpha(s) = 10s \tag{61}$$

$$\rho_{z_j}^{z_i}(s) = \alpha^{-1} \circ (\gamma_{z_i}^{z_j})^{-1} \circ \alpha(s) = 1.0541s \tag{62}$$

For $i = 1, \ldots, 5$, after taking $\sigma_{z_i} = 1$, when $|z_i| \neq 0$, from (38), the control input for each agent is

$$\begin{cases} u_1 = -\dfrac{z_1}{|z_1|}(v_1^2 + 22.6020|z_1|) - 2.5z_1 \\ u_2 = -\dfrac{z_2}{|z_2|}(v_2^2 + 34.7850|z_2|) - 3.5z_2 \\ u_3 = -\dfrac{z_3}{|z_3|}(v_3^2 + 22.6020|z_3|) - 2.5z_3 \\ u_4 = -\dfrac{z_4}{|z_4|}(v_4^2 + 22.1082|z_4|) - 2.5z_4 \\ u_5 = -\dfrac{z_5}{|z_5|}(v_5^2 + 34.7850|z_5|) - 3.5z_5 \end{cases} \tag{63}$$

where $z_1 = v_1 - (x_2 - x_1) - (x_3 - x_1)$, $z_2 = v_2 - (x_1 - x_2) - (x_4 - x_2) - (x_5 - x_2)$, $z_3 = v_3 - (x_1 - x_3) - (x_5 - x_3)$, $z_4 = v_4 - (x_2 - x_4) - (x_5 - x_4)$ and $z_5 = v_5 - (x_2 - x_5) - (x_4 - x_5) - (x_3 - x_5)$.

All the initial velocities of agents are set to be 0, and the initial positions are set as $x_1(0) = 1$ m, $x_2(0) = 2$ m, $x_3(0) = 3$ m, $x_4(0) = 4$ m and $x_5(0) = 5$ m. Then, we simulate the five-agent system with protocol (63). The simulation results are shown in Figs. 3 and 4. It is shown that the positions of all agents converge to the same value, 3.1518 m, and the velocities converge to 0. Consensus is achieved.

For $i = 1, \ldots, 5$ and $j_i \in N_i$, if let $a_{1j_1} = 2.5$, $a_{2j_2} = 3.5$, $a_{3j_3} = 2.5$, $a_{4j_4} = 2.5$, $a_{5j_5} = 3.5$ and $k = -1$ in the linear protocol in [9], then the linear protocol is specified as

$$\begin{cases} u_1 = -2.5z_1 \\ u_2 = -3.5z_2 \\ u_3 = -2.5z_3 \\ u_4 = -2.5z_4 \\ u_5 = -3.5z_5 \end{cases} \tag{64}$$

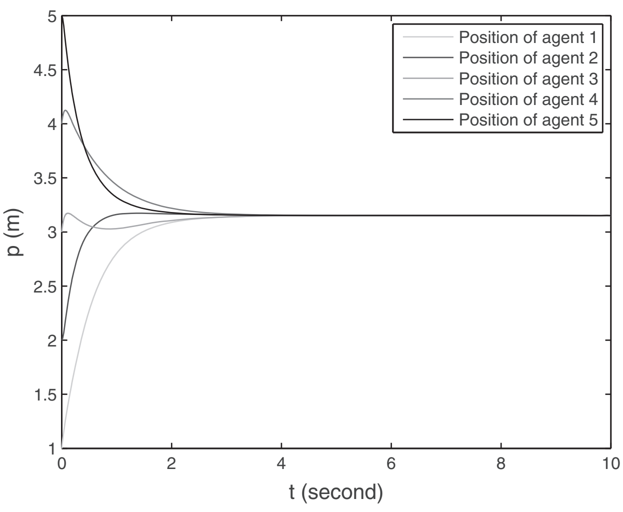

Fig. 3 *Evolutions of positions with protocol (63)*

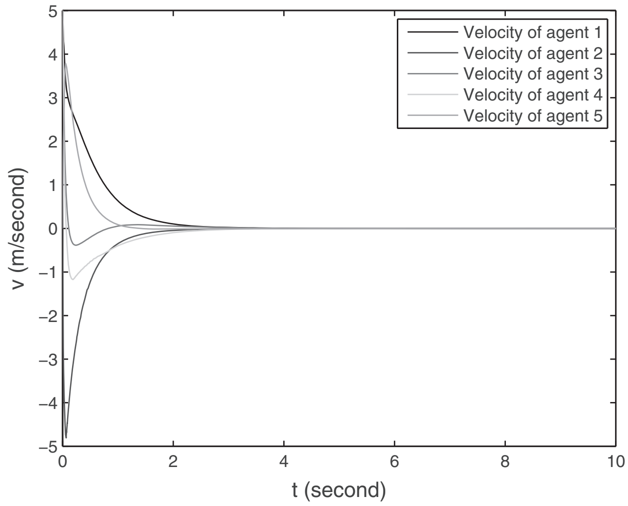

Fig. 4 *Evolutions of velocities with protocol (63)*

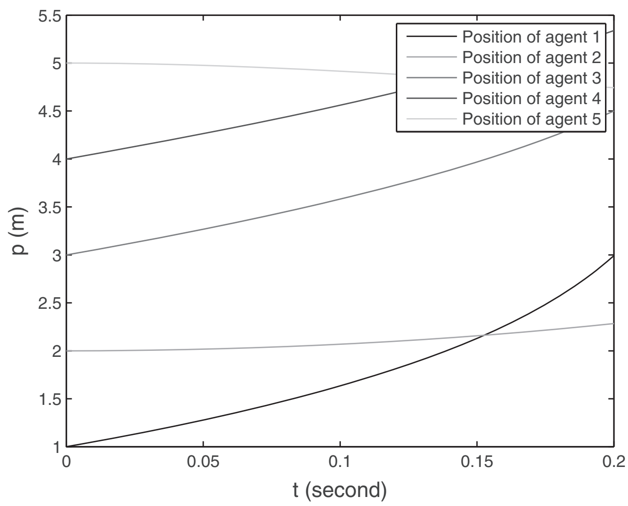

Fig. 5 *Evolutions of positions with a linear protocol*

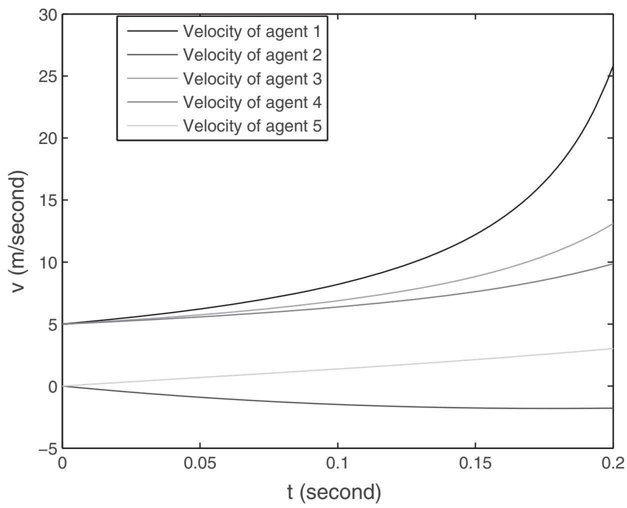

Fig. 6 *Evolutions of velocities with a linear protocol*

We simulate the above five-agent system with the linear protocol (64) and the same initial conditions as that for protocol (63). The position and velocity evolutions in

doi: 10.1049/iet-cta.2011.0393

0.2 second interval are shown in Figs. 5 and 6, respectively. Clearly, the linear protocol (64) leads to divergence.

6 Conclusions and future work

This paper has developed a distributed second-order consensus protocol for non-linear multi-agent systems with unknown locally Lipschitz continuous dynamics and connected control–interconnection graphs. The main contribution of this paper is to provide a completely non-linear ISS-based analysis, which is a blend of graph-theoretic, back-stepping design and the newly developed cyclic-small-gain theorem, to deal with the consensus problem with the locally Lipschitz continuous dynamical agents. The agent dynamics is not required to be globally Lipschitz, and only the information of the agent's position, velocity and relative positions to its neighbours are needed to implement the new consensus protocol. Many practical issues, from the external disturbances to more challenging switching topology and time-delay, will be further explored.

7 Acknowledgments

The authors would like to thank Zhong-Ping Jiang and Changbin Yu for fruitful discussions on small-gain theorems and consensus protocols. This paper was not presented at any meeting or journal. This work of X. Wang was performed during his visiting at the Research School of Engineering, the Australian National University, Canberra ACT 2601, Australia, supported by China Scholar Council. T. Liu was with Research School of Engineering, the Australian National University, when the paper was submitted.

8 References

1 Ren, W., Beard, W., Atkins, E.M.: 'Information consensus in multivehicle cooperative control', *IEEE Control Syst. Mag.*, 2007, **27**, (2), pp. 71–82

2 Cao, M., Morse, A.S., Anderson, B.D.O.: 'Reaching a consensus in a dynamically changing environment: a graphical approach', *SIAM J. Control Optim.*, 2008, **47**, (2), pp. 575–600

3 Lin, Z., Broucke, M., Francis, B.: 'Local control strategies for groups of mobile autonomous agents', *SIAM J. Control Optim.*, 2004, **49**, (4), pp. 622–629

4 Olfati-Saber, R., Murray, R.M.: 'Consensus problems in networks of agents with switching topology and time-delays', *IEEE Trans. Autom. Control*, 2004, **49**, (9), pp. 1520–1553

5 Ren, W., Beard, R.: 'Consensus seeking in multiagent systems under dynamically changing interaction topologies', *IEEE Trans. Autom. Control*, 2005, **50**, (5), pp. 655–661

6 Su, H., Wang, X., Lin, Z.: 'Synchronization of coupled harmonic oscillators in a dynamic proximity network', *Automatica*, 2009, **45**, (10), pp. 2286–2291

7 Qin, J., Zheng, W.X., Gao, H.: 'Consensus of multiple second-order vehicles with a time-varying reference signal under directed topology', *Automatica*, 2011, **47**, (9), pp. 1983–1991

8 Qin, J., Zheng, W., Gao, H.: 'Convergence analysis for multiple agents with double-integrator dynamics in a sampled-data setting', *IET Control Theory Appl.*, 2011, **5**, (18), pp. 2089–2097

9 Xie, G., Wang, L.: 'Consensus control for a class of networks of dynamic agents', *Int. J. Robust Non-linear Control*, 2007, **17**, (10–11), pp. 941–959

10 Lin, Z., Francis, B., Maggiore, M.: 'State agreement for continuous-time coupled non-linear system', *IEEE Trans. Autom. Control*, 2007, **46**, (1), pp. 288–307

11 Yu, W., Chen, G., Cao, M., Kurths, J.: 'Second-order consensus for multiagent systems with directed topologies and non-linear dynamics', *IEEE Trans. Syst. Man Cybern. B*, 2010, **40**, (3), pp. 881–891

12 Song, Q., Cao, J., Yu, W.: 'Second-order leader-following consensus of non-linear multi-agent systems via pinning control', *Syst. Control Lett.*, 2010, **59**, (9), pp. 553–562

13 Das, A., Lewis, F.L.: 'Cooperative adaptive control for synchronization of sencond-order systems with unknown non-linearities', *Int. J. Robust Non-linear Control*, 2011, **21**, (13), pp. 1509–1524

14 Zhang, W., Wang, Z., Guo, Y.: 'Adaptive backstepping-based synchronization of uncertain networked lagrangian systems'. Proc. 2011 American Control Conf., San Francisco, CA, USA, 2011, pp. 1057–1062

15 Sontag, E.D.: 'Input to state stability: Basic concepts and results', Nistri P., Stefani G. (Eds.): 'Non-linear and optimal control theory' (Springer–Verlag, Berlin, 2007), pp. 163–220

16 Dashkovskiy, S., Rüffer, B.S., Wirth, F.R.: 'An ISS small-gain theorem for general networks', *Math. Control Signals Syst.*, 2007, **19**, (2), pp. 93–122

17 Dashkovskiy, S., Rüffer, B.S., Wirth, F.R.: 'Small gain theorems for large scale systems and construction of ISS-Lyapunov functions', *SIAM J. Control Opt.*, 2010, **48**, (6), pp. 4089–4118

18 Jiang, Z.P., Wang, Y.: 'A generalization of the non-linear small-gain theorem for large-scale complex systems'. Proc. Seventh World Congress on Intelligent Control and Automation, 2008, pp. 1188–1193

19 Liu, T., Hill, D.J., Jiang, Z.P.: 'Lyapunov formulation of ISS small-gain in continuous-time dynamical networks', *Automatica*, 2011, **47**, (9), pp. 2088–2093

20 Shi, G., Hong, Y.: 'Set-tracking of multi-agent systems with variable topologies guided by moving multiple leaders'. Proc. 49th IEEE Conf. Decision and Control Atlanta, GA, 2010, pp. 2245–2250

21 Godsil, C., Royle, G.: 'Algebraic graph theory' (Springer Graduate Texts in Mathematics, 207), 2001

22 Jiang, Z.P., Teel, A.R., Praly, L.: 'Small-gain theorem for ISS systems and applications', *Math. Control Signals Syst.*, 1994, **7**, (2), pp. 95–120

23 Jiang, Z.P., Mareels, I.M.Y., Wang, Y.: 'A Lyapunov formulation of the non-linear small-gain theorem for interconnected ISS systems', *Automatica*, 1996, **32**, (9), pp. 1211–1214

24 Zelazo, D., Mesbahi, M.: 'Edge agreement: Graph-theoretic performance bounds and passivity analysis', *IEEE Trans. Autom. Control*, 2011, **56**, (3), pp. 544–555

25 Dimarogonas, D.V., Johansson, K.H.: 'Stability analysis for multi-agent systems using the incidence matrix: Quantized communication and formation control', *Automatica*, 2010, **46**, (4), pp. 695–700

26 Krstić, M., Kanellakopoulos, I., Kokotović, P.V.: 'Non-linear and adaptive control design' (John Wiley & Sons, NY, 1995)

27 Filippov, A.: 'Differential equations with discontinuous right-hand slides' (Kluwer Academic, The Netherlands, 1988)

28 Lee, J., Allaire, P.E., Tao, G., Zhang, X.: 'Integral sliding-mode control of a magnetically suspended balance beam: analysis, simulation, and experiment', *IEEE/ASME Trans. Mech.*, 2001, **6**, (3), pp. 338–346

29 He, W., Cao, J.: 'Consensus control for high-order multi-agent systems', *IET Control Theory Appl.*, 2011, **5**, (1), pp. 231–238

30 Jiang, Z.P., Mareels, I.M.Y.: 'A small-gain control method for non-linear cascade systems with dynamic uncertainties', *IEEE Trans. Autom. Control*, 1997, **42**, (3), pp. 292–308