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2 authors:

Xiangke Wang
National University of Defense Technology

Huayong Zhu
National University of Defense Technology

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On the Comparisons of Unit Dual Quaternion and Homogeneous Transformation Matrix

Xiangke Wang* and Huayong Zhu

Abstract. This paper reveals the differences and similarities between two popular unified representations, i.e. the UDQ (unit dual quaternion) and the HTM (homogeneous transformation matrix), for transformation in the solution to the kinematic problem, in order to provide a clear, concise and self-contained introduction into dual quaternions and to further present a cohesive view for the UDQ and HTM representations as used in robotics. Specifically, after investigating some fundamental algebraic properties of the UDQ, it is revealed that the kinematical equations represented by the UDQ and the HTM are accordant, and afterwards the direct relationship of UDQ-based error kinematical models in spatial-frame and in body-frame are further discussed, with conclusion that either error kinematic model can be chosen for designing kinematical control laws. Finally, the comparative study on the proportional control algorithms based on the logarithmical mapping of the HTM and the UDQ shows that the UDQ-based control law is indeed higher in computational efficiency.

Keywords. Transformation representation, unit dual quaternion, homogeneous transformation matrix, kinematical control.

1. Introduction

A large number of existing literature demonstrates researchers' interest in kinematic control of rigid-bodies, which could be manipulators [1], mechanical systems [2], mobile robots [3], and space robots [4], etc. It is already known that a transformation composing of a rotation and a translation is the most general motion for rigid-bodies in 3-D space with 3 rotational and 3 translational degrees. Therefore, the above-mentioned problems are all underpinned essentially by the kinematical control of transformation.

*Corresponding author.

Birkhäuser

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The key to deal with transformation control on kinematics is how to describe a rigid-body's transformation. Formally, the kinematics studies the geometric properties of the motion without regard to the cause of the motion. Much existing work has been done on the theory of the kinematics of transformation in 3-D space with different representation formalisms, (see the classical pioneering work of [5] and [6] for example). The simplest tool is to describe rotation and translation independently, in which the translation is specified by a three-dimensional vector, and the rotation can be represented by Euler angles, Rodrigues parameters, the unit equivalent axis/angle, and the unit quaternion [7]. However, this method may result in loss of couplings between rotation and translation. Another related technique addressing the transformation control is the HTM (homogeneous transformation matrix), which is a 4×4 matrix. All HTMs construct the SE(3), which is a widely used representation formalism for the unified rotation and translation control currently [1, 2, 5, 6, 8–10]. It is worthy pointing out that the generalized proportional-derivative control law is designed based on the logarithmical mapping of SE(3) in [8]. Another unified representation formalism for transformation is the UDQ (unit dual quaternion), which has been found useful in many applications, such as computer-aided geometric design [11], image-based localization [12], hand-eye calibration [13] and navigation [14]. It is reported that the UDQ provides an efficient global representation for transformation with concise notions and clear geometric significance in 3-D space without singularity, see e.g. [15, 16]. The UDQ has recently received a lot of considerations in the transformation control because of its claimed conciseness and high computing effectiveness [17–27]. The Lie-group related structures are employed on unit dual quaternions by providing the exponential form and an approximate logarithmic mapping of a unit dual quaternion in [17], and a generalized proportional control law is proposed on the unit dual quaternion kinematics with proven stability, which is extended from [18]. The proposed generalized proportional control law is then extended to a PID control scheme, which means that the unit dual quaternion Lie group is employed on the widely used PID control algorithm in [19]. Note that such unit dual quaternion logarithm based kinematical control schemes are successfully implemented in a real omni-directional mobile robot, the NuBot robot, in order to strength the credibility in practical applications, and shrink the gap between the theory and the application [20]. Moreover, the unit dual quaternion based dynamics of rigid-bodies are derived, and correspondingly the unit dual quaternion logarithm based feedback linearization and VSC control laws are designed in [21] and [22], respectively. The dual quaternion based transformation controls without considering the logarithm feedback are also proposed such as the position and attitude control for linked manipulators in [24, 25] and the dual position control strategies for robot manipulators on kinematics in [23]. Finally, the unit dual quaternion based control schemes have been extended to the coordinations of multiple rigid-body systems (e.g., in refs. [26, 27]).

Although the two methods are well known in the control and robotics communities, only the HTM-based method has been investigated extensively. Conversely, the fundamental properties of the UDQ are scattered in different literature, and many key problems, especially the differences and similarities of this two representation formalisms, have not been considered. In this paper, we will reveal the differences and similarities between the two representations in the solution to the kinematic problem, and compare their computing effectiveness when employing them in the design of kinematical control laws, in order to provide a clear, concise and self-contained introduction into dual quaternions, and further present a cohesive view for the UDQ and HTM representations as used in robotics. Specifically, some fundamental algebraic properties of the UDQ as scattered in existing literature are presented firstly in order to underpin the mathematical foundation of UDQ; then the kinematical equations represented by the UDQ and the HTM are found accordant, and consequently the direct relationship of the UDQ-based error kinematic models in the spatial-frame and in the body-frame is further discussed, with conclusion that either error kinematic model can be chosen for designing kinematical control laws. Finally, in the view of the kinematical model, the comparative study on the proportional control algorithms based on the logarithmical mapping of the HTM and the UDQ is considered by accounting the operations, in order to show that the UDQ-based control law is indeed higher in computational efficiency. The basic mathematical formulations and necessary notions are provided in Appendix A for self-containment.

Notations

In this paper, 0 simply denotes the scalar zero. Three dimensional vector $(0,0,0)^T$, dual number $0+\epsilon 0$ and dual vector $(0,0,0)^T+\epsilon(0,0,0)^T$ are represented by $\mathbf{0}$, $\hat{0}$ and $\hat{\mathbf{0}}$ respectively, and we use I and $\hat{\mathrm{I}}$ for unit quaternion $[1,0,0,0]$ and unit dual quaternion $[1,0,0,0]+\epsilon[0,0,0,0]$, respectively. If not otherwise stated, a (dual) vector is represented by the boldface, and its corresponding (dual) vector quaternion by the normal type, for example, $v=[0,\boldsymbol{v}]$ or $\hat{v}=[\hat{0},\hat{\boldsymbol{v}}]$.

Two right-hand coordinate systems are set up, i.e., the spatial-frame, which is an inertial coordinate frame, and the body-frame as attached on the rigid-body self with the heading of rigid-body being x-axis. The concepts of the spatial-frame and the body-frame come from literature [1]. Thorough this paper, the superscripts s and b are related to a spatial-frame and a body-frame, respectively.

2. Two Representations for the Kinematics of Transformation

In general, a transformation consists of a rotation and a translation. Considering a rotation succeeded by a translation $\boldsymbol{p}^b$ or a translation $\boldsymbol{p}^s$ succeeded by a rotation, according to the Chasles Theorem (refer to Theorem 2.11

in [1]), this transformation is equivalent to a screw motion, which is a rotation around a Unit Axis $\boldsymbol{n}$ with an angle of $0 \leq \vartheta < 2\pi$ combined with a Translation d parallel to $\boldsymbol{n}$ as illustrated in Fig. 1, where Frame O is the original coordinate system, Frame N' is the new one after the rotation around Axis $\boldsymbol{n}$ at Rotational Point C and Frame N is the coordinate system after the Translation d.

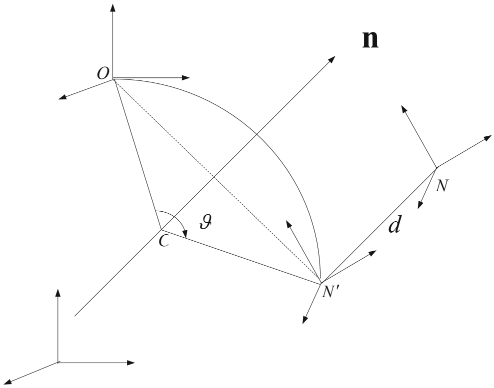

FIGURE 1. Geometry of transformation: every transformation can be modeled as a rotation about a Unit Axis $\boldsymbol{n}$ at Rotational Point C with an angle of $0 \leq \vartheta < 2\pi$ combined with a Translation d parallel to $\boldsymbol{n}$.

The transformation as illustrated in Fig. 1 can be described by either the UDQ or the HTM. It is worthy pointing out that either mathematical tool, the UDQ or the HTM, can serve as both a specification of the configuration of a rigid-body and a transformation taking the coordinates of a point from one frame to another via rotation and translation.

2.1. UDQ-Based Representation

It is well known that the unit quaternion

$$q = [\cos(\frac{\vartheta}{2}), \sin(\frac{\vartheta}{2})\boldsymbol{n}] \tag{2.1}$$

can be used to describe the rotation around a Unit Zxis $\boldsymbol{n}$ with an angle of $0 \leq \vartheta < 2\pi$ [1]. Correspondingly, the transformation as illustrated in Fig. 1 can be represented by the following unit dual quaternion [14, 17]

$$\hat{q} = q + \frac{\epsilon}{2} q \circ p^b = q + \frac{\epsilon}{2} p^s \circ q, \tag{2.2}$$

where symbol '$\circ$' is the multiplication of quaternions defined by (A.3) in Appendix A.

2.2. HTM-Based Representation

The rotation around Unit Axis $\boldsymbol{n}$ with an angle of ϑ can be represented by a rotational matrix, which corresponds to the Rodrigues's rotation formula represented as [1]

$$R(\boldsymbol{n},\vartheta) = e^{\boldsymbol{n}^{\wedge}\vartheta} = I_3 + \boldsymbol{n}^{\wedge}\sin\vartheta + (\boldsymbol{n}^{\wedge})^2(1-\cos\vartheta), \tag{2.3}$$

where I_3 is a 3 by 3 unit diagonal matrix, and $\wedge$ is a wedge operator defined by

$$(\boldsymbol{n})^{\wedge} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}^{\wedge} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}. \tag{2.4}$$

Correspondingly, the transformation as illustrated in Fig. 1 can be represented by a HTM [1]

$$g = \begin{bmatrix} R & \boldsymbol{p}^s \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} R & R^{-1}\boldsymbol{p}^b \\ \boldsymbol{0} & 1 \end{bmatrix}. \tag{2.5}$$

2.3. Transformation from a UDQ to a HTM

Any UDQ $\hat{q}$ as defined in (2.2) can determine an element $g(\hat{q}) \in \mathrm{SE}(3)$ as presented in (2.5) [1]. Consider a UDQ

$$\hat{q} = q_r + \epsilon q_d = [q_{r0}, q_{r1}, q_{r2}, q_{r3}] + \epsilon[q_{d0}, q_{d1}, q_{d2}, q_{d3}]. \tag{2.6}$$

It can be seen from (2.1) and (2.2) that the rotational angle is

$$\vartheta = 2\arccos(q_{r0}), \tag{2.7}$$

and the rotational axis is

$$\boldsymbol{n} = [q_{r1}, q_{r2}, q_{r3}]^T / \sin\frac{\vartheta}{2}, \text{ when } \vartheta \neq 0. \tag{2.8}$$

Then the rotational matrix R can be obtained from (2.3), and in fact, after some direct operations, it can be descried by [1]

$$R(q_r) = \begin{bmatrix} q_{r0}^2 + q_{r1}^2 - q_{r2}^2 - q_{r3}^2 & 2(q_{r1}q_{r2} - q_{r0}q_{r3}) & 2(q_{r1}q_{r3} + q_{r0}q_{r2}) \\ 2(q_{r1}q_{r2} - q_{r0}q_{r3}) & q_{r0}^2 - q_{r1}^2 + q_{r2}^2 - q_{r3}^2 & 2(q_{r2}q_{r3} - q_{r0}q_{r1}) \\ 2(q_{r1}q_{r3} - q_{r0}q_{r2}) & 2(q_{r2}q_{r3} + q_{r0}q_{r1}) & q_{r0}^2 - q_{r1}^2 - q_{r2}^2 + q_{r3}^2 \end{bmatrix}. \tag{2.9}$$

Afterwards, from (2.2), the translational vector is obtained from the following formula:

$$p^s = 2q_d \circ q^* \text{ and } p^b = 2q^* \circ q_d, \tag{2.10}$$

and consequently, the HTM $g(\hat{q})$ is obtained from (2.5).

It should be noted that $R(q) = R(-q)$ according to (2.9), hence the two elements q and $-q$ can determine the same rotation/attitude, and the two related UDQs (with the same $\boldsymbol{p}$) represent the same transformation/configuration. Therefore, UDQs provide a double covering of the SE(3), and the two elements $\hat{q}(q,\boldsymbol{p})$ and $\hat{q}(-q,\boldsymbol{p})$, such as $\hat{\mathrm{I}}$ and $-\hat{\mathrm{I}}$, are physically identical [11].

2.4. Transformation from HTM to UDQ

Consider a HTM represented by $g = \begin{bmatrix} R & \boldsymbol{p}^s \\ 0 & 1 \end{bmatrix}$, and denote the rotation matrix by $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$. It is obtained that the rotational angle is [1]

$$\vartheta = \arccos(\frac{r_{11} + r_{22} + r_{33} - 1}{2}) \in [0, \pi). \tag{2.11}$$

When $\vartheta \neq 0$, the rotational axis is

$$\boldsymbol{n} = \frac{1}{2 \sin \vartheta} \left[r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12} \right]^T. \tag{2.12}$$

Then, the unit quaternion q can be obtained from (2.1), and afterwards the corresponding UDQ is obtained from (2.2).

Note that as above-mentioned, UDQs provide a double covering of the SE(3), therefore in (2.11) the rotational angle ϑ or $2\pi - \vartheta$ could be chosen as well, and if $2\pi - \vartheta$ has been chosen, the rotational axis would have been $-\boldsymbol{n}$.

3. Some Algebraic Properties of UDQs

Some properties of quaternion and dual quaternion are presented in this section, as such investigations are scattered in existing literature, in order to underpin the mathematical foundation of the UDQ.

Property 3.1. *For vector quaternions* $v_1 = [0, \boldsymbol{v}_1]$ *and* $v_2 = [0, \boldsymbol{v}_2]$*, it is obtained that* $v_1 \circ v_2 - v_2 \circ v_1 = 2[0, \boldsymbol{v}_1 \times \boldsymbol{v}_2]$*, and* $v_1 \circ v_2 + v_2 \circ v_1 = 2[-\boldsymbol{v}_1^T \boldsymbol{v}_2, \boldsymbol{0}]$.

Proof. According to the multiplication between quaternions (see (A.3) in Appendix A), we obtain

$$v_1 \circ v_2 = [-\boldsymbol{v}_1^T \boldsymbol{v}_2, \boldsymbol{v}_1 \times \boldsymbol{v}_2], \text{ and } v_2 \circ v_1 = [-\boldsymbol{v}_2^T \boldsymbol{v}_1, \boldsymbol{v}_2 \times \boldsymbol{v}_1].$$

Then the conclusions in Property 3.1 are obtained consequently. □

Property 3.2. *For vector quaternions* v_1 *and* v_2*, and unit quaternion* q*, it is obtained that*

$$Ad_q(v_1 \pm v_2) = Ad_q v_1 \pm Ad_q v_2, \tag{3.1}$$

$$Ad_q(v_1 \circ v_2) = Ad_q v_1 \circ Ad_q v_2, \tag{3.2}$$

$$Ad_q(\boldsymbol{v}_1 \times \boldsymbol{v}_2) = Ad_q \boldsymbol{v}_1 \times Ad_q \boldsymbol{v}_2, \tag{3.3}$$

where operator 'Ad' is the adjoint transformation of quaternion defined by (A.13) *in Appendix A.*

Proof. Formulas (3.1) and (3.2) can be verified through direct computations. In the following, we only give a brief proof of (3.3).

With the aid of Property 3.1, it is obtained that

$$Ad_q(\boldsymbol{v}_1 \times \boldsymbol{v}_2) = \frac{1}{2} Ad_q(v_1 \circ v_2 - v_2 \circ v_1).$$

Considering formulas (3.1) and (3.2) yield

$$\begin{aligned} Ad_q(\boldsymbol{v}_1 \times \boldsymbol{v}_2) &= \frac{1}{2}Ad_q(v_1 \circ v_2) - \frac{1}{2}Ad_q(v_2 \circ v_1) \\ &= \frac{1}{2}\big((Ad_q v_1 \circ Ad_q v_2) - (Ad_q v_2 \circ Ad_q v_1)\big). \end{aligned}$$

Using Property 3.1 again, it is obtained that

$$Ad_q(\boldsymbol{v}_1 \times \boldsymbol{v}_2) = Ad_q\boldsymbol{v}_1 \times Ad_q\boldsymbol{v}_2. \qquad \square$$

Property 3.3. *For the unit quaternion q, it is obtained*

$$\omega^s = Ad_q\omega^b, \tag{3.4}$$

$$p^s = Ad_q p^b, \tag{3.5}$$

where $\boldsymbol{\omega}^s$ and $\boldsymbol{\omega}^b$ are the angular velocities, respectively, in the body-frame and in the spatial-frame as represented in (A.14)*, and $\boldsymbol{p}^s$ and $\boldsymbol{p}^b$ are the translations in* (2.2)*.*

Proof. Formulas (3.4) and (3.5) can be obtained directly from (A.14) and (2.2), respectively. □

Property 3.4. *For the unit quaternion q and the vector quaternion v, it is obtained*

$$\frac{d}{dt}(Ad_{q^*}\boldsymbol{v}) = Ad_{q^*}\dot{\boldsymbol{v}} + Ad_{q^*}\boldsymbol{v} \times \boldsymbol{\omega}^b, \tag{3.6}$$

$$\frac{d}{dt}(Ad_q\boldsymbol{v}) = Ad_q\dot{\boldsymbol{v}} + \boldsymbol{\omega}^s \times Ad_q\boldsymbol{v}, \tag{3.7}$$

where $\boldsymbol{\omega}^b$ and $\boldsymbol{\omega}^s$ are the angular velocities in the body-frame and in the spatial-frame respectively, as represented in (A.14)*, and q^* is the conjugate of quaternion q defined by* (A.1) *in Appendix A. Note that $Ad_{q^*}\boldsymbol{v}$ and $Ad_q\boldsymbol{v}$ represent the vector parts of $Ad_{q^*}v$ and $Ad_q v$, respectively.*

Proof. Expanding and differentiating $Ad_{q^*}v$, it can be obtained that:

$$\frac{d}{dt}(Ad_{q^*}v) = \dot{q}^* \circ v \circ q + q^* \circ \dot{v} \circ q + q^* \circ v \circ \dot{q}.$$

By using (A.14) yields:

$$\frac{d}{dt}(Ad_{q^*}v) = -\frac{1}{2}\omega^b \circ Ad_{q^*}v + Ad_{q^*}\dot{v} + \frac{1}{2}Ad_{q^*}v \circ \omega^b.$$

According to Property 3.1, it is obtained

$$\frac{d}{dt}(Ad_{q^*}v) = [0, Ad_{q^*}\boldsymbol{v} \times \boldsymbol{\omega}^b] + Ad_{q^*}\dot{v},$$

which concludes formula (3.6).

Similarly, by expanding and differentiating $Ad_q v$, and then using Property 3.1 and (A.14), formula (3.7) is obtained. □

Property 3.5. *For the unit quaternion q, and translation vectors $\boldsymbol{p}^b$ and $\boldsymbol{p}^s$, which are described as those in* (2.2)*, it can be obtained*

$$\dot{\boldsymbol{p}}^s = \boldsymbol{\omega}^s \times Ad_q \boldsymbol{p}^b + Ad_q \dot{\boldsymbol{p}}^b, \tag{3.8}$$

$$\dot{\boldsymbol{p}}^b = Ad_{q^*} \boldsymbol{p}^s \times \boldsymbol{\omega}^b + Ad_{q^*} \dot{\boldsymbol{p}}^s, \tag{3.9}$$

where q^* is the conjugate of quaternion q defined by (A.1) *in Appendix A.*

Proof. It is straightforward that Property 3.5 can be derived by Properties 3.3 and 3.4. □

Property 3.6. *For dual vector quaternions $\hat{v}_1$ and $\hat{v}_2$, it is obtained that $\hat{v}_1 \circ \hat{v}_2 - \hat{v}_2 \circ \hat{v}_1 = 2[\hat{0}, \hat{\boldsymbol{v}}_1 \times \hat{\boldsymbol{v}}_2]$.*

Proof. Let $\hat{v}_1 = v_{1r} + \epsilon v_{1d} = [0, \boldsymbol{v}_{1r}] + \epsilon[0, \boldsymbol{v}_{1d}]$ and $\hat{v}_2 = v_{2r} + \epsilon v_{2d} = [0, \boldsymbol{v}_{2r}] + \epsilon[0, \boldsymbol{v}_{2d}]$. According to the multiplication between dual quaternions (see (A.10) in Appendix A), we can obtain directly

$$\begin{aligned} \hat{v}_1 \circ \hat{v}_2 - \hat{v}_2 \circ \hat{v}_1 =& (v_{1r} \circ v_{2r} - v_{2r} \circ v_{1r}) \\ &+ \epsilon(v_{1d} \circ v_{2r} - v_{2r} \circ v_{1d} + v_{1r} \circ v_{2d} - v_{2d} \circ v_{1r}). \end{aligned}$$

By using Property 3.1, we obtain the result that

$$(\hat{v}_1 \circ \hat{v}_2 - \hat{v}_2 \circ \hat{v}_1) = 2\big([\hat{0}, \boldsymbol{v}_{1r} \times \boldsymbol{v}_{2r}] + \epsilon[\hat{0}, \boldsymbol{v}_{1r} \times \boldsymbol{v}_{2d} + \boldsymbol{v}_{1d} \times \boldsymbol{v}_{2r}]\big).$$

Together with the operation in (A.6), the property is concluded. □

Property 3.7. *For dual vector quaternions $\hat{v}_1, \hat{v}_2$ and a UDQ $\hat{q}$, it is obtained*

$$Ad_{\hat{q}}(\hat{v}_1 \pm \hat{v}_2) = Ad_{\hat{q}}\hat{v}_1 \pm Ad_{\hat{q}}\hat{v}_2, \tag{3.10}$$

$$Ad_{\hat{q}}(\hat{v}_1 \circ \hat{v}_2) = Ad_{\hat{q}}\hat{v}_1 \circ Ad_{\hat{q}}\hat{v}_2, \tag{3.11}$$

$$Ad_{\hat{q}}(\hat{\boldsymbol{v}}_1 \times \hat{\boldsymbol{v}}_2) = Ad_q\hat{\boldsymbol{v}}_1 \times Ad_q\hat{\boldsymbol{v}}_2, \tag{3.12}$$

where the operator 'Ad' is the adjoint transformation of dual quaternion defined by (A.13) *in Appendix A.*

Proof. With the similar tricks in the proof of Property 3.2, this property can be concluded. □

Property 3.8. *For the unit dual quaternion $\hat{q}$ defined in* (2.2)*, it is obtained*

$$\xi_q^s = Ad_{\hat{q}}\xi_q^b, \tag{3.13}$$

$$\xi_q^b = Ad_{\hat{q}^*}\xi_q^s, \tag{3.14}$$

where $\boldsymbol{\xi}_q^s$ and $\boldsymbol{\xi}_q^b$ are the twist, respectively, in the body-frame and in the spatial-frame as represented in (A.17) *and* (A.18).

Proof. This property can be verified directly on the basis of (A.15) and (A.16). □

4. Accordance on Twists Represented by a UDQ and a HTM

The twists, respectively, in the spatial-frame $\boldsymbol{\xi}_m^s$ and the body-frame $\boldsymbol{\xi}_m^b$ as represented by the HTM are provided in [1], which are

$$\boldsymbol{\xi}_m^s = \begin{bmatrix} \boldsymbol{\nu}^s \\ \boldsymbol{\omega}^s \end{bmatrix} = \begin{bmatrix} -\dot{R}R^T\boldsymbol{p}^s + \dot{\boldsymbol{p}}^s \\ (\dot{R}R^T)^\vee \end{bmatrix}, \tag{4.1}$$

$$\boldsymbol{\xi}_m^b = \begin{bmatrix} \boldsymbol{\nu}^b \\ \boldsymbol{\omega}^b \end{bmatrix} = \begin{bmatrix} R^T\dot{\boldsymbol{p}}^s \\ (R^T\dot{R})^\vee \end{bmatrix}, \tag{4.2}$$

where R is a rotation matrix, and operator $\vee$ is the inverse of operator $\wedge$.

In the sequel, it is shown that the twists as described by the UDQ, i.e. formulas (A.17) and (A.18), and by the HTM, i.e. formulas (4.1) and (4.2), are found with similar forms, which implies:

Theorem 4.1. *Twists represented by* (4.1) *and* (4.2) *are accordant with those represented by* (A.17) *and* (A.18) *respectively.*

Proof. It is clear that the twists in spatial-frame represented by (4.1) and (A.17) are accordant because of $(\boldsymbol{\omega}^s)^\wedge = \dot{R}R^T$. Therefore in the following, we mainly pay attention on the twists in the body-frame.

According to Properties 3.5 and 3.3, formula (A.18) yields

$$\boldsymbol{\xi}_q^b = \boldsymbol{\omega}^b + \epsilon(Ad_{q^*}\dot{\boldsymbol{p}}^s + Ad_{q^*}\boldsymbol{p}^s \times \boldsymbol{\omega}^b + \boldsymbol{\omega}^b \times Ad_{q^*}\boldsymbol{p}^s).$$

It can be obviously obtained that

$$\boldsymbol{\xi}_q^b = \boldsymbol{\omega}^b + \epsilon Ad_{q^*}\dot{\boldsymbol{p}}^s. \tag{4.3}$$

Considering $\hat{\boldsymbol{\omega}}^b = R^T\dot{R}$, and $Ad_{q^*}\dot{\boldsymbol{p}}^s = R^T\dot{\boldsymbol{p}}^s$, it is clear now that the twists in body-frame as represented by (A.18) and (4.3) are also accordant. $\square$

Remark 4.2. It is obvious that the twists represented by (A.17) and (A.18) are much more compact than those represented by (4.1) and (4.2), though they are accordant. Such compactness brings great ease in the control law design and converging analysis of the transformation control, as done in [17, 21, 24].

5. Equivalence of kinematical Models in the Body-Frame and the Spatial-Frame

The kinematical control is often imposed on the error kinematical models, and two models described by dual quaternions, one in the body-frame and the other in the spatial-frame describing, are deduced from the left-invariance error $\hat{q}_{el} = \hat{q}^* \circ \hat{q}_d$ and the right-invariance error $\hat{q}_{er} = \hat{q}_d \circ \hat{q}^*$ respectively, as follows [17].

Model 5.1 (Error Kinematical Model). For the current configuration $\hat{q}$ and the target configuration $\hat{q}_d$, the error kinematical models in the body-frame and the in spatial-frame are, respectively,

$$\begin{cases} \dot{\hat{q}}_{el} &= \frac{1}{2}\hat{q}_{el} \circ \xi_e^b, \\ \boldsymbol{\xi}_e^b &= \boldsymbol{\xi}_d^b - Ad_{\hat{q}_{el}^*}\boldsymbol{\xi}_q^b, \end{cases} \tag{5.1}$$

and

$$\begin{cases} \dot{\hat{q}}_{er} &= \frac{1}{2}\xi_e^s \circ \hat{q}_{er}, \\ \boldsymbol{\xi}_e^s &= \boldsymbol{\xi}_d^s - Ad_{\hat{q}_{er}}\boldsymbol{\xi}_q^s, \end{cases} \tag{5.2}$$

where $\boldsymbol{\xi}_d^b$ and $\boldsymbol{\xi}_d^s$ are the twists of $\hat{q}_d$ in the body-frame and in the spatial-frame, respectively.

It is known that no bi-invariant metric (i.e., one that is invariant under both the left and right translations) exists for transformation. Physically, the left and right invariances reflect the invariance of the metric with respect to choice of, respectively, the body-frame and the spatial-frame [28]. Consequently, the two models resulting from the left-invariance error and the right-invariance error indicated different metrics and coordinate frames for the kinematics of transformation. Therefore, it cannot be determined that which model should be chosen for designing control laws. In the sequel, we will reveal the relationship between the two models.

Theorem 5.2. *For ξ_e^s and ξ_e^b in Model* 5.1*, it is obtained that*

$$\boldsymbol{\xi}_e^s = Ad_{\hat{q}_d}\boldsymbol{\xi}_d^b - Ad_{\hat{q}_{er}}\big(Ad_{\hat{q}_d}(\boldsymbol{\xi}_d^b - \boldsymbol{\xi}_e^b)\big). \tag{5.3}$$

Proof. By using $\boldsymbol{\xi}_e^b$ represented in (5.1) and Property 3.7, it is obtained that

$$\begin{aligned} Ad_{\hat{q}_d}\boldsymbol{\xi}_e^b &= Ad_{\hat{q}_d}(\boldsymbol{\xi}_d^b - Ad_{\hat{q}_{el}^*}\boldsymbol{\xi}_q^b) \\ &= Ad_{\hat{q}_d}\boldsymbol{\xi}_d^b - Ad_{\hat{q}_d}(Ad_{\hat{q}_{el}^*}\boldsymbol{\xi}_q^b). \end{aligned}$$

According to (3.13) in Property 3.8, considering $\hat{q}_{el} = \hat{q}^* \circ \hat{q}_d$, it is obtained that

$$Ad_{\hat{q}_d}\boldsymbol{\xi}_e^b = \boldsymbol{\xi}_d^s - \hat{q} \circ \boldsymbol{\xi}_q^b \circ \hat{q}^*.$$

Together with (3.14) in Property 3.8, it yields:

$$Ad_{\hat{q}_d}\boldsymbol{\xi}_e^b = \boldsymbol{\xi}_d^s - \boldsymbol{\xi}_q^s.$$

Therefore, it is obtained that

$$Ad_{\hat{q}_d}(\boldsymbol{\xi}_d^b - \boldsymbol{\xi}_e^b) = \boldsymbol{\xi}_d^s - (\boldsymbol{\xi}_d^s - \boldsymbol{\xi}_q^s) = \boldsymbol{\xi}_q^s.$$

Then considering (3.13), it is obtained that

$$Ad_{\hat{q}_d}\boldsymbol{\xi}_d^b - Ad_{\hat{q}_{er}}\big(Ad_{\hat{q}_d}(\boldsymbol{\xi}_d^b - \boldsymbol{\xi}_e^b)\big) = \boldsymbol{\xi}_d^s - Ad_{\hat{q}_{er}}\boldsymbol{\xi}_q^s,$$

which concludes the result in Theorem 5.2. □

Remark 5.3. Theorem 5.2 indicates the transformation between $\boldsymbol{\xi}_e^s$ and $\boldsymbol{\xi}_e^b$. In the Investigation of Model 5.1, $\hat{q}_d$ and $\boldsymbol{\xi}_d^b$ represented the target configuration and its twist in the body-frame respectively, and $\hat{q}$ is the current configuration. From Theorem 5.2, it can be inferred that if $\boldsymbol{\xi}_e^b$, $\hat{q}$, $\hat{q}_d$ and $\boldsymbol{\xi}_d^b$

are known in advanced, twist $\boldsymbol{\xi}_e^s$ can be obtained directly. And it is obviously that its conversion can also be concluded, which means if $\boldsymbol{\xi}_e^s$, $\hat{q}$, $\hat{q}_d$ and $\boldsymbol{\xi}_d^s$ are known in advanced, twist $\boldsymbol{\xi}_e^b$ can be obtained directly. Thus, it is indicated by the transformation that either error kinematical model can be chosen for designing control laws.

6. Comparison of Control Laws Represented by the UDQ and the HTM

It is proven in [17] that the set of UDQs is a Lie-group and the set of the approximate logarithmic mappings defined by

$$\ln \hat{q} = \frac{1}{2}(\theta + \epsilon p^b) \tag{6.1}$$

is its corresponding Lie-algebra, where $\boldsymbol{\theta} = \vartheta \boldsymbol{n}$ is as described in Section 2.1.

Similarly, the SE(3) is a Lie-group and its logarithm mapping defined by

$$\ln g = \begin{bmatrix} \hat{\phi} & A^{-1}(\varphi)\boldsymbol{p}^s \\ 0 & 1 \end{bmatrix} \tag{6.2}$$

is its related Lie-algebra, where

$$\hat{\phi} = \ln(R) = \frac{\vartheta}{2\sin\vartheta}(R - R^T), \tag{6.3}$$

is a skew symmetric matrix with ϑ obtained by (2.11), and

$$A(\hat{\phi})^{-1} = I - \frac{1}{2}\hat{\phi} + (1 - (||\hat{\phi}||/2)cot(||\hat{\phi}||/2))\frac{\hat{\phi}^2}{||\hat{\phi}||^2} \tag{6.4}$$

with $||\cdot||$ is the standard Euclidean norm [1,8], as described in Section 2.2.

Based on the logarithm mappings of the UDQ and the HTM, the kinematical control laws for rigid-body transformation control are proposed as in Lemma 6.1 and Lemma 6.2, respectively, in [17] and [8].

Lemma 6.1 (Theorem 8 in [17]). *For the error kinematical model* (5.1)*, the generalized proportional control law that*

$$\xi_e^b = -2\hat{k} \cdot \ln \hat{q}_e \tag{6.5}$$

exponentially stabilizes the configuration $\hat{q}$ to $\hat{q}_d$ globally when $\hat{q} \neq -\hat{q}_d$, where $\hat{k} = k_r + \epsilon k_d$ is a dual vector quaternion with each nonzero component greater than zero, and symbol '$\cdot$' is the dot production between two dual vector quaternions defined by (A.12).

TABLE 1. Required operations for UDQ-based controller (6.7) and HTM-based controller (6.6)

	addition	multiplication	other operations
UDQ	12	29	1 arccos, 1 sin
HTM	84	103	1 arccos, 1 sin, 1 cot, 1 sqrt

Lemma 6.2 (Lemma 6 in [8]). *Consider the left invariant system $\dot{g} = g\boldsymbol{\xi}_m^b$ on SE(3). Then the control law*

$$\boldsymbol{\xi}_m^b = -\begin{bmatrix} k_\omega I_3 & 0 \\ 0 & (k_\omega + k_v) I_3 \end{bmatrix} \ln(g)^\vee \qquad (6.6)$$
$$= -\begin{bmatrix} k_\omega I_3 & 0 \\ 0 & (k_\omega + k_v) I_3 \end{bmatrix} \begin{bmatrix} \hat{\phi}^\vee \\ \left(A^{-1}(\varphi)\boldsymbol{p}^s\right)^\vee \end{bmatrix}$$

where $\vee$ is the inverse operation of $\wedge$, exponentially stabilizes the state g at I, from any initial condition $g(0) = \big(R(0), p(0)\big)$ such that $tr(R(0)) \neq -1$.

In the sequel, we will compare the computing effectiveness of the UDQ-based control law and the HTM-based control law. Note that the former as discussed in Theorem 6.1 is a tracker. For comparison, we use $\hat{q}_d = \hat{I}$ to degenerate the control law as

$$\boldsymbol{\xi}_q^b = -2\hat{\boldsymbol{k}} \cdot \ln \hat{\boldsymbol{q}} = \hat{\boldsymbol{k}} \cdot (\vartheta \boldsymbol{n} + \epsilon \boldsymbol{p}^b). \qquad (6.7)$$

For the UDQ represented configuration $\hat{q}$ in (2.6) to achieve control law (6.7), we should

1) obtain ϑ and $\boldsymbol{n}$ from (2.7) and (2.8) respectively;
2) obtain $\boldsymbol{p}^b$ by using (2.10); and
3) obtain $\boldsymbol{\xi}_q^b$ from the control law (6.7).

For the HTM represented configuration g in (2.5), it is worthy pointing out that $\hat{\phi} = \vartheta \boldsymbol{n}^\wedge$ after some operations, where $\boldsymbol{n}$ is obtained from (2.4), and it further obtained from algebraic operations that

$$||\hat{\phi}|| = \sqrt{n_1^2 + n_2^2 + n_3^2}. \qquad (6.8)$$

Hence, in order to achieve the control law (6.6), we should

1) obtain ϑ and $\boldsymbol{n}$ from (2.11) and (2.12) respectively, and then $\hat{\phi}$ by using $\hat{\phi} = (\vartheta \boldsymbol{n})^\wedge$;
2) compute $||\hat{\phi}||$ by using (6.8) and then obtain $A(\hat{\phi})^{-1}$ from (6.4); and
3) obtain $\boldsymbol{\xi}_q^b$ from the control law (6.6).

According to the above steps, the required operations for the UDQ-based controller (6.7) and the HTM-based controller (6.6) are listed in Tab. 1.

It is shown clearly in Tab. 1 that the required operations, either addition and multiplication or other operations of the UDQ-based controller (6.7) are much simpler than those of the HTM-based controller (6.6), viz, the UDQ-based control law is much higher in computing efficiency than the HTM-based control law.

Remark 6.3. It is worthy pointing out that in the work of [15], a comparison of the three transformation (other than control laws) formulations (HTM, Lie algebra, and UDQ) of an N-linked manipulator is presented. However, in this study, a comparison between two control laws (i.e., HTM-based, and UDQ-based) of the transformation control is presented. Though the two works are quite different, the resulting conclusion is in agreement on that dual quaternions as a representation for transformation are more compact and efficient and that this holds also true for the application in control laws as shown in this paper for an example.

7. Conclusions

The comparative studies on the UDQ and the HTM are provided in the solution to the kinematic problem in order to enrich the mathematical foundations of transformation control in this paper. It is revealed that the kinematical equations represented by the UDQ and the HTM respectively are accordant, and the UDQ-based error kinematical model in the spatial-frame and that in the body-frame are equivalent. In the view of the kinematic model, the computational efficiency of the proportional control algorithm based on the logarithmical mapping of the UDQ and the HTM is considered, with conclusion that the UDQ-based control law is higher in computational efficiency.

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Appendix A. Mathematical Preliminaries

Quaternions are an extension of complex numbers to a four-dimensional manifold. Formally, a quaternion is defined as $q = [s, \boldsymbol{v}]$, where s is a scalar (called the *scalar part*), and $\boldsymbol{v}$ is a three-dimensional vector (called the *vector part*). The conjugate of a quaternion q is

$$q^* = [s, -\boldsymbol{v}]. \tag{A.1}$$

For quaternions $q_1 = [s_1, \boldsymbol{v}_1]$ and $q_2 = [s_2, \boldsymbol{v}_2]$, the *addition* and *multiplication* operations are, respectively, defined as

$$q_1 + q_2 = [s_1 + s_2, \boldsymbol{v}_1 + \boldsymbol{v}_2], \tag{A.2}$$

$$q_1 \circ q_2 = [s_1 s_2 - \boldsymbol{v}_1^T \boldsymbol{v}_2, s_1 \boldsymbol{v}_2 + s_2 \boldsymbol{v}_1 + \boldsymbol{v}_1 \times \boldsymbol{v}_2]. \tag{A.3}$$

If $q \circ q^* = \mathrm{I}$, then q is called a *unit quaternion*.

A *dual number* is defined as

$$\hat{a} = a + \epsilon b \text{ with } \epsilon^2 = 0, \text{ but } \epsilon \neq 0, \tag{A.4}$$

where a and b are real numbers, called the *real part* and the *dual part*, respectively, and ϵ is nilpotent such as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *Dual vectors* are a generalization of dual numbers whose real and dual parts are both three-dimensional vectors. For two dual vectors $\hat{\boldsymbol{v}}_1 = \boldsymbol{v}_{1r} + \epsilon \boldsymbol{v}_{1d}$ and $\hat{\boldsymbol{v}}_2 = \boldsymbol{v}_{2r} + \epsilon \boldsymbol{v}_{2d}$, the *addition*, *cross multiplication* and *multiplied by a scalar* operations are, respectively, defined as:

$$\hat{\boldsymbol{v}}_1 + \hat{\boldsymbol{v}}_2 = \boldsymbol{v}_{1r} + \boldsymbol{v}_{2r} + \epsilon(\boldsymbol{v}_{1d} + \boldsymbol{v}_{2d}), \tag{A.5}$$

$$\hat{\boldsymbol{v}}_1 \times \hat{\boldsymbol{v}}_2 = \boldsymbol{v}_{1r} \times \boldsymbol{v}_{2r} + \epsilon(\boldsymbol{v}_{1r} \times \boldsymbol{v}_{2d} + \boldsymbol{v}_{1d} \times \boldsymbol{v}_{2r}), \tag{A.6}$$

$$\lambda \hat{\boldsymbol{v}}_1 = \lambda \boldsymbol{v}_{1r} + \epsilon \lambda \boldsymbol{v}_{1d}, \tag{A.7}$$

where λ is a scalar.

A *dual quaternion* is a quaternion with dual number components, i.e., $\hat{q} = [\hat{s}, \hat{\boldsymbol{v}}]$, where $\hat{s}$ is a dual number and $\hat{\boldsymbol{v}}$ is a dual vector. A three-dimensional (dual) vector can also be treated equivalently as a (dual) quaternion with vanishing scalar part, called *(dual) vector quaternion.*

A dual quaternion can also be treated as a dual number with quaternion components, i.e., $\hat{q} = q_r + \epsilon q_d$, where q_r and q_d are quaternions. The *conjugate* of a dual quaternion $\hat{q}$ is

$$\hat{q}^* = q_r^* + \epsilon q_d^*. \tag{A.8}$$

For two dual quaternions $\hat{q}_1$ and $\hat{q}_2$, the *addition* and the *multiplication* operations are, respectively, defined as

$$\hat{q}_1 + \hat{q}_2 = q_{r1} + q_{r2} + \epsilon(q_{d1} + q_{d2}), \tag{A.9}$$

$$\hat{q}_1 \circ \hat{q}_2 = q_{r1} \circ q_{r2} + \epsilon(q_{r1} \circ q_{d2} + q_{d1} \circ q_{r2}), \tag{A.10}$$

According to (A.8) and (A.10), there holds

$$(\hat{q}_1 \circ \hat{q}_2)^* = \hat{q}_2^* \circ \hat{q}_1^*. \tag{A.11}$$

If $\hat{q} \circ \hat{q}^* = \hat{\mathrm{I}}$, then the dual quaternion $\hat{q}$ is called a *unit dual quaternion.*

For dual vector quaternions $\hat{k} = k_r + \epsilon k_d = [0, k_{r1}, k_{r2}, k_{r3}] + \epsilon[0, k_{d1}, k_{d2}, k_{d3}]$ and $\hat{v} = [0, \boldsymbol{v}_r] + \epsilon[0, \boldsymbol{v}_d]$, the dot production of $\hat{k}$ and $\hat{v}$ is defined as

$$\hat{k} \cdot \hat{v} = [0, K_r \boldsymbol{v}_r] + \epsilon[0, K_d \boldsymbol{v}_d], \tag{A.12}$$

where $K_r = diag(k_{r1}, k_{r2}, k_{r3})$ and $K_d = diag(k_{d1}, k_{d2}, k_{d3})$, which are both 3×3 diagonal matrices with diagonal entries k_{r1}, k_{r2}, k_{r3} and k_{d1}, k_{d2}, k_{d3}, respectively.

Given a (dual) vector quaternion v ($\hat{v}$) and a unit (dual) quaternion q ($\hat{q}$), the *adjoint transformation* is defined as

$$Ad_q v = q \circ v \circ q^* \text{ or } Ad_{\hat{q}} \hat{v} = \hat{q} \circ \hat{v} \circ \hat{q}^*. \tag{A.13}$$

It should be noted that $Ad_q v$ and $Ad_{\hat{q}} \hat{v}$ are still, respectively, vector quaternion and dual vector quaternion. For notional economy, we employ $Ad_q \boldsymbol{v}$ and $Ad_{\hat{q}} \hat{\boldsymbol{v}}$ to denote their corresponding vector and dual vector, respectively.

The kinematic equation of a quaternion is

$$\dot{q} = \frac{1}{2}\omega^s \circ q = \frac{1}{2}q \circ \omega^b, \tag{A.14}$$

where $\boldsymbol{\omega}^s$ and $\boldsymbol{\omega}^b$ represent the angular velocity in the spatial-frame and that in the body-frame, respectively. Correspondingly, the kinematic equations of a rigid body expressed with dual quaternion are

$$\dot{\hat{q}} = \frac{1}{2}\xi_q^s \circ \hat{q}, \tag{A.15}$$

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \circ \xi_q^b, \tag{A.16}$$

where

$$\boldsymbol{\xi}_q^s = \boldsymbol{\omega}^s + \epsilon(\dot{\boldsymbol{p}}^s + \boldsymbol{p}^s \times \boldsymbol{\omega}^s), \tag{A.17}$$

$$\boldsymbol{\xi}_q^b = \boldsymbol{\omega}^b + \epsilon(\dot{\boldsymbol{p}}^b + \boldsymbol{\omega}^b \times \boldsymbol{p}^b) \tag{A.18}$$

are called twists represented by the UDQ, and specially, $\boldsymbol{\xi}_q^s$ and $\boldsymbol{\xi}_q^b$ are called twist in the spatial-frame and that in the body-frame [14].

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Xiangke Wang and Huayong Zhu
College of Mechantronic Engineering and Automation
National University of Defense Technology
410073, Changsha
China
e-mail: xkwang@nudt.edu.cn
huyongzhu@nudt.edu.cn

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