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State Consensus Criteria and Protocol Parameters Design for High-order Linear Multi-agent Systems

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Abstract: For the state consensus problems of high-order linear multi-agent systems with time-varying delays in directed topologies, the LMI based consensus criterion and NLMI based protocol design are investigated. Improved Lyapunov Krasovskii functional is used for establishing the consensus criteria and deriving the corresponding consensus protocol. In order to reduce the conservativeness, some proper Free weighting matrices are added into the derivative of Lyapunov Krasovskii functional, and that only keeps one necessary zoom. The numerical and simulation examples are given to demonstrate the effectiveness of the theoretical results. Compared with existing literature, the convergence criterion and protocol design proposed have lower conservativeness.

Key Words: State consensus, Lyapunov Krasovskii functional, Linear matrix inequality, Free weighting matrices

1 Introduction

Recently, much attention is drawn to the consensus problem in multi-agent systems [1-3], due to its widely applications ranging from rendezvous [4], flocking [5], formation control [6] to fusion estimation [7]. Furthermore, communication delays in consensus problems are also taken into account, e.g. due to the communication congestions or retransmissions.

The study for consensus in multi-agent systems with time-delays started at first-order integrator systems [8] which were generalized by second-order (e.g. [9]) or high-order [10] integrator systems then. Nevertheless, the system matrix in those systems is a special companion form matrix. Therefore, it is valuable to study the consensus problems of high-order (some paper called it high dimensional) linear systems with time-varying delays.

By using Lyapunov Krasovskii functional, Xi et al. gave a consensus convergence criterion in high-order linear multi-agent systems with uniform time-varying delays and proposed the corresponding method for protocol design [11]. However, there are still three aspects could be enhanced: 1) **Conservativeness.** The criterion might have relatively more conservativeness caused by the structure of the Lyapunov Krasovskii functional and the zoom in the derivative of this functional; 2) **Constraint.** All the derivatives μ of time-delays should be restricted by $\mu<1$ in [11] 3) **Parameters.** The method of protocol design (the protocol has two parameters) could only calculate the second parameter rather than the whole parameters.

In this paper, we will provide a Free Weighting Matrices based consensus criterion in high-order linear multi-agent systems with time-varying delays in directed topologies. Correspondingly, the protocol will be designed in the procedure of solving NLMIs.

Compared with [11], our method has three improvements: 1) **Conservativeness.** By optimizing the structure of Lyapunov Krasovskii functional and adding proper Free Weighting Matrices, our criterion has less conservativeness; 2) **Constraint.** Our criterion could permit the case $\mu\geq 1$, even when μ is unknown; 3) **Parameters.** both two parameters could be calculated in the design of protocol could calculate directly.

Throughout this paper, the notation $\star$ represents the symmetric part in a symmetric matrix; $\mathcal{D}>(\geq,<,\leq)0$ denotes that the matrix $\mathcal{D}$ is positive definite (positive semi-definite, negative definite, negative semi-definite); $\otimes$ denotes the Kronecker product; $\mathbf{0}$ can be an appropriate dimensions zero matrix or vector; for any complex vector $\mathbf{x}$, any real matrix $\mathcal{D}$ and $\lambda\in\mathbb{C}$, we denote $\hat{\mathbf{x}}=\left[\mathrm{Re}(\mathbf{x})^T,\mathrm{Im}(\mathbf{x})^T\right]^T$, $\Lambda_{\mathcal{D}}=diag\{\mathcal{D},\mathcal{D}\}$ and $\Upsilon_\lambda=\begin{bmatrix}\mathrm{Re}(\lambda)\mathbf{I}_n & -\mathrm{Im}(\lambda)\mathbf{I}_n\\ \mathrm{Im}(\lambda)\mathbf{I}_n & \mathrm{Re}(\lambda)\mathbf{I}_n\end{bmatrix}$ where $\mathbf{I}_n$ is an identity matrix with $n\times n$ dimensions.

2 Preliminaries and Problem Description

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ be a directed simple graph of order N $(N>1)$, where $\mathcal{V}=\{\upsilon_1,\upsilon_2,\cdots,\upsilon_N\}$ denotes the nodes set, $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is the edges set, and $\mathcal{A}=\left[\tilde{a}_{ij}\right]\in\mathbb{R}^{N\times N}$ is the weighted adjacency matrix with $\tilde{a}_{ij}\geq 0$, where $\tilde{a}_{ij}>0$ if and only if edge $(\upsilon_j,\upsilon_i)\in\mathcal{E}$. As $\mathcal{G}$ is simple, $\tilde{a}_{ii}=0$ holds for all $i\in\{1,2,\cdots,N\}$. The neighbors of υ_i is defined as $\mathcal{N}_i=\left\{\upsilon_j\in\mathcal{V}\mid(\upsilon_j,\upsilon_i)\in\mathcal{E}\right\}$. The in-degree of the node υ_i is defined as $d_{in}(\upsilon_i)=\sum_{\upsilon_j\in\mathcal{N}_i}\tilde{a}_{ij}$. The degree matrix of $\mathcal{G}$ is a diagonal matrix $\mathcal{D}=\left[d_{ij}\right]$, where $d_{ij}=0(i\neq j)$, $d_{ii}=d_{in}(\upsilon_i)$. Then Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L}=\left[l_{ij}\right]=\mathcal{D}-\mathcal{A}$.

Lemma 1 ([11]) The Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ satisfies:

1) $\mathcal{L}$ at least has one zero eigenvalue, and $\mathbf{1}_N$ is the associated right-eigenvector;

2) if $\mathcal{G}$ has a spanning tree, then 0 is a simple eigenvalue of $\mathcal{L}$, and all the other N-1 eigenvalues have positive real parts.

Lemma 2 (Schur Complement [10]) For given symmetric matrix $\mathbf{Z}$ with the form $\mathbf{Z}=\mathbf{Z}^T=\begin{bmatrix}\mathbf{Z}_{11} & \mathbf{Z}_{12}\\ \star & \mathbf{Z}_{22}\end{bmatrix}$, $\mathbf{Z}_{11}\in\mathbb{R}^{r\times r}$, $\mathbf{Z}_{12}\in\mathbb{R}^{r\times(n-r)}$, $\mathbf{Z}_{22}\in\mathbb{R}^{(n-r)\times(n-r)}$, then $\mathbf{Z}<0$ if and only if $\mathbf{Z}_{11}<0$, $\mathbf{Z}_{22}-\mathbf{Z}_{12}^T\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}<0$ or $\mathbf{Z}_{22}<0$, $\mathbf{Z}_{11}-\mathbf{Z}_{12}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}^T<0$.

In this paper, we consider a group of N agents whose dynamic are described by high-order linear systems in continuous-time domain:

$$\dot{\mathbf{x}}_i(t)=\mathbf{A}\mathbf{x}_i(t)+\mathbf{B}\mathbf{u}_i(t),\quad i=1,2,\ldots,N \tag{1}$$

where $\mathbf{x}_i(t)=\begin{bmatrix}x_i^{(1)}(t) & x_i^{(2)}(t) & \ldots & x_i^{(n)}(t)\end{bmatrix}^T$ is the state of agent i and $\mathbf{u}_i(t)\in\mathbb{R}^p$ is the consensus protocol, $\mathbf{A}\in\mathbb{R}^{n\times n}$, $\mathbf{B}\in\mathbb{R}^{n\times p}$.

We considered a class of protocols as follows:

$$\mathbf{u}_i(t)=\mathbf{K}_1\mathbf{x}_i(t)+\mathbf{K}_2\sum_{\upsilon_j\in\mathcal{N}_i}\tilde{a}_{ij}(\mathbf{x}_j(t-\tau(t))-\mathbf{x}_i(t-\tau(t))) \tag{2}$$

where $\mathbf{K}_1,\mathbf{K}_2\in\mathbb{R}^{p\times n}$, $\tau(t)\in[0,h]$, $\dot{\tau}(t)\in[0,\mu]$.

Let $\mathbf{x}(t)=\begin{bmatrix}\mathbf{x}_1^T(t) & \mathbf{x}_2^T(t) & \ldots & \mathbf{x}_N^T(t)\end{bmatrix}^T$. With protocol (2) the dynamics of multi-agent systems is

$$\begin{cases}\dot{\mathbf{x}}(t)=(\mathbf{I}_N\otimes(\mathbf{A}+\mathbf{B}\mathbf{K}_1))\mathbf{x}(t)-(\mathcal{L}\otimes\mathbf{B}\mathbf{K}_2)\mathbf{x}(t-\tau(t)), & t\in[0,+\infty)\\ \mathbf{x}(t)=\phi(t), & t\in[-h,0)\end{cases} \tag{3}$$

where $\phi(t)$ is a continuously differentiable initial function.

Definition 1 (State Consensus) For given $\mathbf{K}_1,\mathbf{K}_2$, the multi-agent systems (3) achieves state consensus if and only if $\forall i\neq j$, $\lim_{t\to+\infty}\left\|\mathbf{x}_i(t)-\mathbf{x}_j(t)\right\|=\mathbf{0}$. (in the following part of this paper, we use the term "Consensus" for short)

Definition 2 (State Consensusability) A multi-agent systems is said to be consensusable if and only if $\exists\mathbf{K}_1,\mathbf{K}_2$ $s.t.$ $\forall i\neq j$, $\lim_{t\to+\infty}\left\|\mathbf{x}_i(t)-\mathbf{x}_j(t)\right\|=\mathbf{0}$.

3 Main Results

In this section, the consensus criterion and protocol parameters design method for high-order linear multi-agent systems with time-varying delays in directed topologies is provided.

Lemma 3 ([11]) Dynamics (3) achieves consensus if and only if

$$\dot{\mathbf{\Delta}}_i(t)=(\mathbf{A}+\mathbf{B}\mathbf{K}_1)\mathbf{\Delta}_i(t)-\lambda_{\mathcal{L}(i+1)}\mathbf{B}\mathbf{K}_2\mathbf{\Delta}_i(t-\tau(t)),\quad i=1,\ldots,N-1 \tag{4}$$

are asymptotically stable, where $\lambda_{\mathcal{L}j}$ is the j th eigenvalue of the Laplacian Matrix $\mathcal{L}$ corresponding to $\mathcal{G}$, and $0=\mathrm{Re}(\lambda_{\mathcal{L}1})\le\mathrm{Re}(\lambda_{\mathcal{L}2})\le\ldots\le\mathrm{Re}(\lambda_{\mathcal{L}N})$.

This necessary and sufficient condition converts the consensus problem in dynamics (3) into a stability problem in dynamics (4).

3.1 Consensus Criterion

Theorem 1. Given the upper bound in communication delay h and its derivative μ, multi-agent systems (3) achieves consensus if there exist proper-dimensional $\mathbf{P}_i=\mathbf{P}_i^T>0$, $\mathbf{Q}_i=\mathbf{Q}_i^T\ge0$, $\mathbf{R}_i=\mathbf{R}_i^T\ge0$, $\mathbf{Z}_i=\mathbf{Z}_i^T>0$, $\mathbf{X}_i=\mathbf{X}_i^T\ge0$ and free-weighting matrices

$$\mathbf{N}_i=\begin{bmatrix}\mathbf{N}_{i1}\\ \mathbf{N}_{i2}\\ \mathbf{N}_{i3}\end{bmatrix},\mathbf{S}_i=\begin{bmatrix}\mathbf{S}_{i1}\\ \mathbf{S}_{i2}\\ \mathbf{S}_{i3}\end{bmatrix},\quad i=1,\ldots,N-1$$

such that the following LMIs hold:

$$\Psi_{i1}=\begin{bmatrix}\mathbf{X}_i & \mathbf{N}_i\\ \star & \mathbf{Z}_i\end{bmatrix}\ge0,\Psi_{i2}=\begin{bmatrix}\mathbf{X}_i & \mathbf{S}_i\\ \star & \mathbf{Z}_i\end{bmatrix}\ge0,\Psi_{i3}=\begin{bmatrix}\Phi_i & h\tilde{\mathbf{A}}_i^T\mathbf{Z}_i\\ \star & -h\mathbf{Z}_i\end{bmatrix}<0 \tag{5}$$

where $\Phi_i=\Phi_{i1}+\Phi_{i2}+\Phi_{i2}^T+h\mathbf{X}_i$

$$\Phi_{i1}=\begin{bmatrix}\mathbf{P}_i\Lambda_{\mathbf{A}+\mathbf{BK}_1}+\Lambda_{\mathbf{A}+\mathbf{BK}_1}^T\mathbf{P}_i+\mathbf{Q}_i+\mathbf{R}_i & -\Upsilon_{\lambda_{\mathcal{L}(i+1)}}\mathbf{P}_i\Lambda_{\mathbf{BK}_2} & \mathbf{0}\\ \star & -(1-\mu)\mathbf{Q}_i & \mathbf{0}\\ \star & \star & -\mathbf{R}_i\end{bmatrix}$$

$$\Phi_{i2}=\begin{bmatrix}\mathbf{N}_i & -\mathbf{N}_i+\mathbf{S}_i & -\mathbf{S}_i\end{bmatrix}$$

$$\tilde{\mathbf{A}}_i=\begin{bmatrix}\Lambda_{\mathbf{A}+\mathbf{BK}_1} & -\Upsilon_{\lambda_{\mathcal{L}(i+1)}}\Lambda_{\mathbf{BK}_2} & \mathbf{0}\end{bmatrix}$$

Proof: Since (4) has complex vectors and matrices, it is required to be transformed into the following form by decompositing the real and imaginary parts.

$$\dot{\hat{\mathbf{\Delta}}}_i(t)=\Lambda_{\mathbf{A}+\mathbf{BK}_1}\hat{\mathbf{\Delta}}_i(t)-\Upsilon_{\lambda_{\mathcal{L}(i+1)}}\Lambda_{\mathbf{BK}_2}\hat{\mathbf{\Delta}}_i(t-\tau(t)),\quad i=1,\ldots,N-1 \tag{6}$$

Define following $N-1$ quadratic Lyapunov Krasovskii functional candidates

$$V_i(t)=V_{i1}(t)+V_{i2}(t)+V_{i3}(t)+V_{i4}(t),\quad i=1,\ldots,N-1 \tag{7}$$

where $V_{i1}(t)=\hat{\mathbf{\Delta}}_i^T(t)\mathbf{P}_i\hat{\mathbf{\Delta}}_i(t)$, $V_{i2}(t)=\int_{t-\tau(t)}^t\hat{\mathbf{\Delta}}_i^T(s)\mathbf{Q}_i\hat{\mathbf{\Delta}}_i(s)ds$, $V_{i3}(t)=\int_{t-h}^t\hat{\mathbf{\Delta}}_i^T(s)\mathbf{R}_i\hat{\mathbf{\Delta}}_i(s)ds$, $V_{i4}(t)=\int_{-h}^0\int_{t+\theta}^t\dot{\hat{\mathbf{\Delta}}}_i^T(s)\mathbf{Z}_i\dot{\hat{\mathbf{\Delta}}}_i(s)dsd\theta$, and $\mathbf{P}_i=\mathbf{P}_i^T>0$, $\mathbf{Q}_i=\mathbf{Q}_i^T\ge0$, $\mathbf{R}_i=\mathbf{R}_i^T\ge0$, $\mathbf{Z}_i=\mathbf{Z}_i^T>0$, $\mathbf{X}_i=\mathbf{X}_i^T\ge0$, $i=1,\ldots,N-1$.

By Newton-Leibniz formula, for any given free-weighting matrices with proper dimensions

$$\mathbf{N}_i=\begin{bmatrix}\mathbf{N}_{i1}\\ \mathbf{N}_{i2}\\ \mathbf{N}_{i3}\end{bmatrix},\mathbf{S}_i=\begin{bmatrix}\mathbf{S}_{i1}\\ \mathbf{S}_{i2}\\ \mathbf{S}_{i3}\end{bmatrix},\quad i=1,\ldots,N-1$$

we have

$$\tilde{V}_{i1}(t)=2\zeta_{i1}^T(t)\mathbf{N}_i[\hat{\mathbf{\Delta}}_i(t)-\hat{\mathbf{\Delta}}_i(t-\tau(t))-\int_{t-\tau(t)}^t\dot{\hat{\mathbf{\Delta}}}_i(s)ds]=\mathbf{0} \tag{8}$$

$$\tilde{V}_{i2}(t)=2\zeta_{i1}^T(t)\mathbf{S}_i[\hat{\mathbf{\Delta}}_i(t-\tau(t))-\hat{\mathbf{\Delta}}_i(t-h)-\int_{t-h}^{t-\tau(t)}\dot{\hat{\mathbf{\Delta}}}_i(s)ds]=\mathbf{0} \tag{9}$$

where $\zeta_{i1}(t)=[\hat{\mathbf{\Delta}}_i^T(t),\hat{\mathbf{\Delta}}_i^T(t-\tau(t)),\hat{\mathbf{\Delta}}_i^T(t-h)]^T$.

For any appropriate $\mathbf{X}_i=\mathbf{X}_i^T\ge0$, $i=2,\ldots,N$, (24) holds

$$\begin{aligned}\tilde{V}_{i3}(t)=h\zeta_{i1}^T(t)\mathbf{X}_i\zeta_{i1}(t)-\int_{t-\tau(t)}^t\zeta_{i1}^T(t)\mathbf{X}_i\zeta_{i1}(t)ds\\ -\int_{t-h}^{t-\tau(t)}\zeta_{i1}^T(t)\mathbf{X}_i\zeta_{i1}(t)ds=\mathbf{0}\end{aligned} \tag{10}$$

Taking the time derivative of $V_i(t)$ along the trajectory of (6),it is obtained that

$$\begin{aligned}\dot{V}_{i1}(t)=\hat{\mathbf{\Delta}}_i^T(t)[\mathbf{P}_i\Lambda_{\mathbf{A}+\mathbf{BK}_1}+\Lambda_{\mathbf{A}+\mathbf{BK}_1}^T\mathbf{P}_i]\hat{\mathbf{\Delta}}_i(t)\\ -2\hat{\mathbf{\Delta}}_i^T(t)\mathbf{P}_i\Upsilon_{\lambda_{\mathcal{L}(i+1)}}\Lambda_{\mathbf{BK}_2}\hat{\mathbf{\Delta}}_i(t-\tau(t))\end{aligned} \tag{11}$$

$$\dot{V}_{i2}(t)=\hat{\mathbf{\Delta}}_i^T(t)\mathbf{Q}_i\hat{\mathbf{\Delta}}_i(t)-(1-\dot{\tau}(t))\hat{\mathbf{\Delta}}_i^T(t-\tau(t))\mathbf{Q}_i\hat{\mathbf{\Delta}}_i(t-\tau(t)) \tag{12}$$

$$\dot{V}_{i3}(t)=\hat{\mathbf{\Delta}}_i^T(t)\mathbf{R}_i\hat{\mathbf{\Delta}}_i(t)-\hat{\mathbf{\Delta}}_i^T(t-h)\mathbf{R}_i\hat{\mathbf{\Delta}}_i(t-h) \tag{13}$$

$$\dot{V}_{i4}(t)=h[\Lambda_{\mathbf{A}+\mathbf{BK}_1}\hat{\mathbf{\Delta}}_i(t)-\Upsilon_{\lambda_{\mathcal{L}(i+1)}}\Lambda_{\mathbf{BK}_2}\hat{\mathbf{\Delta}}_i(t-\tau(t))]^T\mathbf{Z}_i\times$$
$$[\Lambda_{\mathbf{A}+\mathbf{BK}_1}\hat{\mathbf{\Delta}}_i(t)-\Upsilon_{\lambda_{\mathcal{L}(i+1)}}\Lambda_{\mathbf{BK}_2}\hat{\mathbf{\Delta}}_i(t-\tau(t))]-\int_{t-h}^{t}\dot{\hat{\mathbf{\Delta}}}_i^T(s)\mathbf{Z}_i\dot{\hat{\mathbf{\Delta}}}_i(s)ds \quad (14)$$

Adding 3 zero items into the derivative, we have

$$\dot{V}_i(t)=\dot{V}_{i1}(t)+\dot{V}_{i2}(t)+\dot{V}_{i3}(t)+\dot{V}_{i4}(t)+\tilde{V}_{i1}(t)+\tilde{V}_{i2}(t)+\tilde{V}_{i3}(t) \quad (15)$$

With $\dot{\tau}(t)\le\mu$, (15) can be zoomed, we can utilize the decomposition of

$$\int_{t-h}^{t}\dot{\hat{\mathbf{\Delta}}}_i^T(s)\mathbf{Z}_i\dot{\hat{\mathbf{\Delta}}}_i(s)ds=\int_{t-\tau(t)}^{t}\dot{\hat{\mathbf{\Delta}}}_i^T(s)\mathbf{Z}_i\dot{\hat{\mathbf{\Delta}}}_i(s)ds+\int_{t-h}^{t-\tau(t)}\dot{\hat{\mathbf{\Delta}}}_i^T(s)\mathbf{Z}_i\dot{\hat{\mathbf{\Delta}}}_i(s)ds$$

in (14) as a contributor to make the zoomed (15) be quadratic forms:

$$\dot{V}_i(t)\le\zeta_{i1}^T(t)(\Phi_{i1}+h\tilde{\mathbf{A}}_i^T\mathbf{Z}_i\tilde{\mathbf{A}}_i)\zeta_{i1}(t)-\int_{t-\tau(t)}^{t}\zeta_{i2}^T(t,s)\begin{bmatrix}\mathbf{X}_i & \mathbf{N}_i\\ \star & \mathbf{Z}_i\end{bmatrix}\zeta_{i2}(t,s)ds$$
$$-\int_{t-h}^{t-\tau(t)}\zeta_{i2}^T(t,s)\begin{bmatrix}\mathbf{X}_i & \mathbf{S}_i\\ \star & \mathbf{Z}_i\end{bmatrix}\zeta_{i2}(t,s)ds \quad (16)$$

where $\zeta_{i2}(t,s)=[\zeta_{i1}^T(t),\dot{\hat{\mathbf{\Delta}}}_i^T(s)]^T$. For formula (16), if $\Phi_{i1}+h\tilde{\mathbf{A}}_i^T\mathbf{Z}_i\tilde{\mathbf{A}}_i<0$ and

$$\Psi_{i1}=\begin{bmatrix}\mathbf{X}_i & \mathbf{N}_i\\ \star & \mathbf{Z}_i\end{bmatrix}\ge 0,\Psi_{i2}=\begin{bmatrix}\mathbf{X}_i & \mathbf{S}_i\\ \star & \mathbf{Z}_i\end{bmatrix}\ge 0$$

Then $\dot{V}_i(t)<-\varepsilon\|\Delta_i(t)\|^2$ holds for any $\varepsilon>0$, $i=1,\ldots,N-1$.

By Lemma 2, $\Phi_{i1}+h\tilde{\mathbf{A}}_i^T\mathbf{Z}_i\tilde{\mathbf{A}}_i<0$ equals to $\Psi_{i3}<0$. Thus, if (5) holds then (6) is asymptotically stable and multi-agent systems (3) achieve consensus. **Q.E.D.**

Remark 1. Theorem 1 has only one zoom in derivative of Lyapunov-Krasovskii functional which is necessary, but Theorem 2 in [11] has three zooms, which make the criterion have more conservativeness. Numerical examples and Simulations, in Section 4.1, will validate that our Theorem 1 has less conservativeness.

Remark 2. Setting $\mathbf{Q}_i=\mathbf{0}$ for all $i\in\{1,2,\ldots,N-1\}$, we can get the delay-dependent and rate-independent consensus criterion, while Theorem 2 in [11] requires that the upper bounds of μ are limited by $\mu<1$.

3.2 Protocol Parameters Design Algorithm

Theorem 2. Assume the topology $\mathcal{G}$ is an undirected graph. Given the upper bound in communication delay h and its derivative μ, if there exist proper-dimensional $\mathbf{L}=\mathbf{L}^T>0$, $\mathbf{W}_i=\mathbf{W}_i^T\ge 0$, $\mathbf{U}_i=\mathbf{U}_i^T\ge 0$, $\mathbf{G}_i=\mathbf{G}_i^T\ge 0$

$$\mathbf{Y}_i=\mathbf{Y}_i^T=\begin{bmatrix}\mathbf{Y}_{11}^i & \mathbf{Y}_{12}^i & \mathbf{Y}_{13}^i\\ \star & \mathbf{Y}_{22}^i & \mathbf{Y}_{23}^i\\ \star & \star & \mathbf{Y}_{33}^i\end{bmatrix}\ge 0$$

and free-weighting matrices $\mathbf{V}_1,\mathbf{V}_2$,

$$\mathbf{M}_i=\begin{bmatrix}\mathbf{M}_{i1}\\ \mathbf{M}_{i2}\\ \mathbf{M}_{i3}\end{bmatrix},\mathbf{T}_i=\begin{bmatrix}\mathbf{T}_{i1}\\ \mathbf{T}_{i2}\\ \mathbf{T}_{i3}\end{bmatrix},\ i=1,\ldots,N-1$$

such that the following NLMIs hold:

$$\Psi'_{i1}=\begin{bmatrix}\mathbf{Y}_i & \mathbf{M}_i\\ \star & \mathbf{LG}_i^{-1}\mathbf{L}\end{bmatrix}\ge 0,\Psi'_{i2}=\begin{bmatrix}\mathbf{Y}_i & \mathbf{T}_i\\ \star & \mathbf{LG}_i^{-1}\mathbf{L}\end{bmatrix}\ge 0 \quad (17)$$

$$\Psi'_{i3}=\begin{bmatrix}\Pi_{11}^i & \Pi_{12}^i & \Pi_{13}^i & h\mathbf{LA}^T+h\mathbf{V}_1^T\mathbf{B}^T\\ \star & \Pi_{22}^i & \Pi_{23}^i & -h\lambda_{\mathcal{L}(i+1)}\mathbf{V}_2^T\mathbf{B}^T\\ \star & \star & \Pi_{33}^i & \mathbf{0}\\ \star & \star & \star & -h\mathbf{G}_i\end{bmatrix}<0 \quad (18)$$

$i=1,\ldots,N-1$, where

$$\Pi_{11}^i=\mathbf{AL}+\mathbf{LA}^T+\mathbf{BV}_1+\mathbf{V}_1^T\mathbf{B}^T+\mathbf{W}_i+\mathbf{U}_i+\mathbf{M}_{i1}+\mathbf{M}_{i1}^T+h\mathbf{Y}_{11}^i$$
$$\Pi_{12}^i=-\lambda_{\mathcal{L}(i+1)}\mathbf{BV}_2-\mathbf{M}_{i1}+\mathbf{T}_{i1}+\mathbf{M}_{i2}^T+h\mathbf{Y}_{12}^i$$
$$\Pi_{13}^i=-\mathbf{T}_{i1}+\mathbf{M}_{i3}^T+h\mathbf{Y}_{13}^i$$
$$\Pi_{22}^i=-(1-\mu)\mathbf{W}_i-\mathbf{M}_{i2}-\mathbf{M}_{i2}^T+\mathbf{T}_{i2}+\mathbf{T}_{i2}^T+h\mathbf{Y}_{22}^i$$
$$\Pi_{23}^i=-\mathbf{T}_{i2}-\mathbf{M}_{i3}^T+\mathbf{T}_{i3}^T+h\mathbf{Y}_{23}^i$$
$$\Pi_{33}^i=-\mathbf{U}_i-\mathbf{T}_{i3}-\mathbf{T}_{i3}^T+h\mathbf{Y}_{33}^i$$

Then multi-agent systems can be consensusablized by protocol (2), and the parameters are from $\mathbf{K}_1=\mathbf{V}_1\mathbf{L}^{-1}$, $\mathbf{K}_2=\mathbf{V}_2\mathbf{L}^{-1}$.

Proof: If multi-agent systems (3) achieve consensus, then according to Theorem 1, inequalities (5) hold. Noted that $\mathcal{L}$ is symmetrical due to a undirected $\mathcal{G}$, we can halve the dimension of Ψ_{i1} and Ψ_{i2} in (5). Further, Ψ_{i3} can be rewritten as following:

$$\Psi_{i3}=\begin{bmatrix}\Phi_i & h\tilde{\mathbf{A}}_i^T\mathbf{Z}_i\\ \star & -h\mathbf{Z}_i\end{bmatrix}<0 \quad (19)$$

where $\Phi_i=\Phi_{i1}+\Phi_{i2}+\Phi_{i2}^T+h\mathbf{X}_i$

$$\Phi_{i1}=\begin{bmatrix}\mathbf{P}_i(\mathbf{A}+\mathbf{BK}_1)+(\mathbf{A}+\mathbf{BK}_1)^T\mathbf{P}_i+\mathbf{Q}_i+\mathbf{R}_i & -\lambda_{\mathcal{L}(i+1)}\mathbf{P}_i\mathbf{BK}_2 & \mathbf{0}\\ \star & -(1-\mu)\mathbf{Q}_i & \mathbf{0}\\ \star & \star & -\mathbf{R}_i\end{bmatrix}$$

$$\Phi_{i2}=\begin{bmatrix}\mathbf{N}_i & -\mathbf{N}_i+\mathbf{S}_i & -\mathbf{S}_i\end{bmatrix}$$

$$\tilde{\mathbf{A}}_i=\begin{bmatrix}\mathbf{A}+\mathbf{BK}_1 & -\lambda_{\mathcal{L}(i+1)}\mathbf{BK}_2 & \mathbf{0}\end{bmatrix}$$

Then employing the congruent transformation with $diag\{\mathbf{P}_i^{-1},\mathbf{P}_i^{-1},\mathbf{P}_i^{-1},\mathbf{Z}_i^{-1}\}$ for Ψ_{i3} in (19), we can derive

$$diag\{\mathbf{P}_i^{-1},\mathbf{P}_i^{-1},\mathbf{P}_i^{-1},\mathbf{Z}_i^{-1}\}^T\Psi_{i3}diag\{\mathbf{P}_i^{-1},\mathbf{P}_i^{-1},\mathbf{P}_i^{-1},\mathbf{Z}_i^{-1}\}<0 \quad (20)$$

where $i=1,\ldots,N-1$. Set $\mathbf{P}_1^{-1}=\mathbf{P}_2^{-1}=\ldots=\mathbf{P}_{N-1}^{-1}=\mathbf{L}$, and let $\mathbf{W}_i=\mathbf{LQ}_i\mathbf{L}$, $\mathbf{U}_i=\mathbf{LR}_i\mathbf{L}$, $\mathbf{G}_i=\mathbf{Z}_i^{-1}$ and

$$\mathbf{Y}_i=\mathbf{LX}_i\mathbf{L}=\begin{bmatrix}\mathbf{Y}_{11}^i & \mathbf{Y}_{12}^i & \mathbf{Y}_{13}^i\\ \star & \mathbf{Y}_{22}^i & \mathbf{Y}_{23}^i\\ \star & \star & \mathbf{Y}_{33}^i\end{bmatrix}\ge 0$$

$$\mathbf{M}_i=diag\{\mathbf{L},\mathbf{L},\mathbf{L}\}\mathbf{N}_i\mathbf{L}=\begin{bmatrix}\mathbf{M}_{i1}\\ \mathbf{M}_{i2}\\ \mathbf{M}_{i3}\end{bmatrix},\mathbf{T}_i=diag\{\mathbf{L},\mathbf{L},\mathbf{L}\}\mathbf{S}_i\mathbf{L}=\begin{bmatrix}\mathbf{T}_{i1}\\ \mathbf{T}_{i2}\\ \mathbf{T}_{i3}\end{bmatrix}$$

where $i=1,\ldots,N-1$. Subsequently, with $\mathbf{K}_1=\mathbf{V}_1\mathbf{L}^{-1}$ and $\mathbf{K}_2=\mathbf{V}_2\mathbf{L}^{-1}$, formula (20) can be rewritten as (18). Similarly, by using congruent transformation with $diag\{\mathbf{L},\mathbf{L},\mathbf{L}\}$ in dimension-halved Ψ_{i1} and Ψ_{i2} respectively, inequalities (17) can be derived. Noted that (17) (18) are equivalent to (5), the protocol parameters are from $\mathbf{K}_1=\mathbf{V}_1\mathbf{L}^{-1}$, $\mathbf{K}_2=\mathbf{V}_2\mathbf{L}^{-1}$. **Q.E.D.**

4 Numerical Examples and Simulations

In order to validate the theoretical results proposed in this paper, numerical examples and simulations are involved to illustrate the effectiveness. Figure 1 reveals some topologies in multi-agent systems. (For simplicity, assume that their adjacency matrices are limited to 0,1 matrices.)

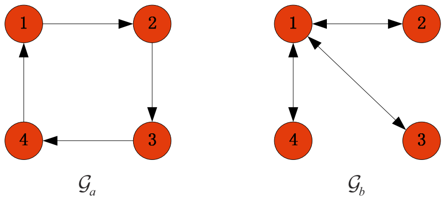

Fig. 1. Two directed graphs

4.1 Comparisons on Consensus Criteria

In order to show the benefits of our results, we will compare the conservativeness of criteria with [11]. Additionally, the applicability for unknown μ in our criterion will be illustrated.

Consider multi-agent systems (3) with parameters:

$$\mathbf{A}=\begin{bmatrix}0&1&0\\0&0&1\\-3&-1&-2\end{bmatrix},\mathbf{B}=\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{20}$$

$$\mathbf{K}_1=\begin{bmatrix}3&1&2\end{bmatrix},\mathbf{K}_2=\begin{bmatrix}10&10&10\end{bmatrix} \tag{21}$$

Results of comparisons are shown in the Table 1.

Table 1: Allowable upper bound on communication delay for third-order linear Multi-agent Systems with topology $\mathcal{G}_a,\mathcal{G}_b$

	$\mu=0$	$\mu=0.5$	$\mu=0.9$	unknown μ
[11] Theorem 2 $\mathcal{G}_a$	0.049	0.046	0.046	-
Theorem 1 $\mathcal{G}_a$	0.057	0.057	0.057	0.057
[11] Theorem 2 $\mathcal{G}_b$	0.031	0.027	0.025	-
Theorem 1 $\mathcal{G}_b$	0.034	0.033	0.033	0.033

Remark 3. The comparison results illustrates that allowable upper bound on communication delays given by Theorem 1 is larger than those in [11] (to be more precise, ranging from 9.68% to 32%), the simulation results in Fig.2 validates the effectiveness. More importantly, our criterion has more advantages when μ becoming larger. Especially, unknown μ is able to judge by our criterion.

For Theorem 2 in [11], the number of Lyapunov-Krasovskii functional candidates are reduced into 2 with the aim of improving calculation efficiency, even though this would lead to some conservativeness. Similarly, in the current paper, we could also reduce the candidates' number into 2 to get a higher efficiency. But still, for a more persuaded comparison, we select the all Lyapunov-Krasovskii functional candidates both in Theorem 2 of [11] and our Theorem 1.

4.2 Validations for Protocol Design Algorithm

In this section, our Theorem 2 will be used to design $\mathbf{K}_1$ and $\mathbf{K}_2$ in the multi-agent systems (3) with parameters (20). Then we will validate the algorithm by simulations.

By solving the NLMIs in Theorem 2, the parameters $\mathbf{K}_1$ and $\mathbf{K}_2$ can be calculated directly rather than only giving $\mathbf{K}_2$ (in [11], $\mathbf{K}_1$ should be given in advance). For different h and μ, we give the corresponding consensusable protocols by Theorem 2 (see Table 2).

As we can see from Table 2, the two parameters $\mathbf{K}_1,\mathbf{K}_2$ are calculated in different circumstances. To validate our algorithm, we select the item with $h=0.1$ $\mu=0.9$ to carry on the simulation ($\mathbf{K}_1=\begin{bmatrix}1.275046&-1.93282&-3.38173\end{bmatrix}$ and $\mathbf{K}_2=\begin{bmatrix}0.000136&0.00016&0.000356\end{bmatrix}$). Figure 3 shows the effectiveness in our results with time-varying delays $\tau(t)=\dfrac{0.1}{2}+\dfrac{0.1}{2}\sin\left(\dfrac{2\times0.9}{0.1}t\right)$.

Table 2: Protocol designs for third-order linear Multi-agent Systems with topology $\mathcal{G}_b$ in different h and μ

		$\mu=0.5$	$\mu=0.9$	unknown μ
$h=0.01$	$\mathbf{K}_1$	$[-12.1701\ \ -22.399\ \ -41.6511]$	$[-12.1298\ \ -22.5921\ \ -41.4188]$	$[-12.5955\ \ -20.9511\ \ -43.6805]$
	$\mathbf{K}_2$	$[0.015798\ \ 0.0229\ \ 0.04593]$	$[0.008171\ \ 0.011982\ \ 0.023706]$	$[0.013602\ \ 0.017852\ \ 0.040222]$
$h=0.05$	$\mathbf{K}_1$	$[-0.29176\ \ -4.16465\ \ -7.63684]$	$[-0.214\ \ -4.18778\ \ -7.59713]$	$[-4.0846\ \ -3.86634\ \ -8.06138]$
	$\mathbf{K}_2$	$[0.000858\ \ 0.000971\ \ 0.002598]$	$[0.00044\ \ 0.000516\ \ 0.001371]$	$[0.000827\ \ 0.000826\ \ 0.002521]$
$h=0.1$	$\mathbf{K}_1$	$[1.254432\ \ -1.91587\ \ -3.41735]$	$[1.275046\ \ -1.93282\ \ -3.38173]$	$[1.126844\ \ -1.79166\ \ -3.56736]$
	$\mathbf{K}_2$	$[0.000124\ \ 0.000094\ \ 0.000326]$	$[0.000136\ \ 0.00016\ \ 0.000356]$	$[0.000197\ \ 0.000254\ \ 0.000479]$

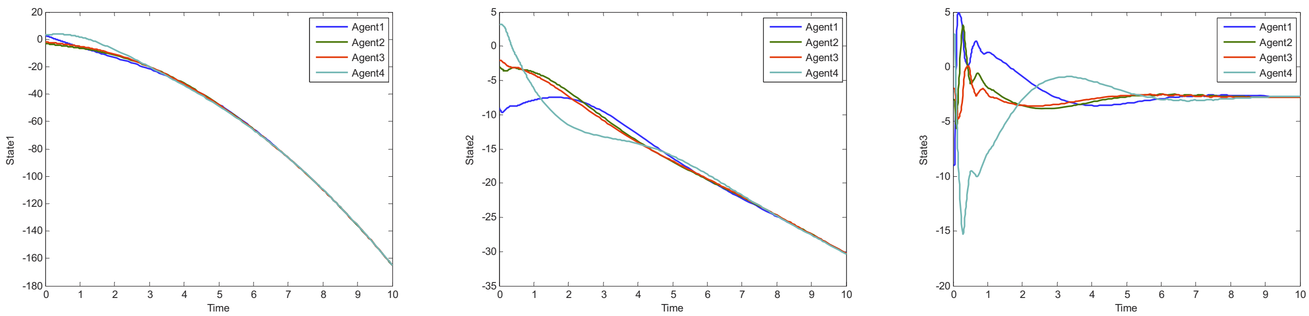

Fig. 2. State trajectories of 3rd-order Multi-agent Systems in $\mathcal{G}_a$ with communication delay $\tau(t)=\dfrac{0.057}{2}+\dfrac{0.057}{2}\sin\left(\dfrac{2\times0.9}{0.057}t\right)$

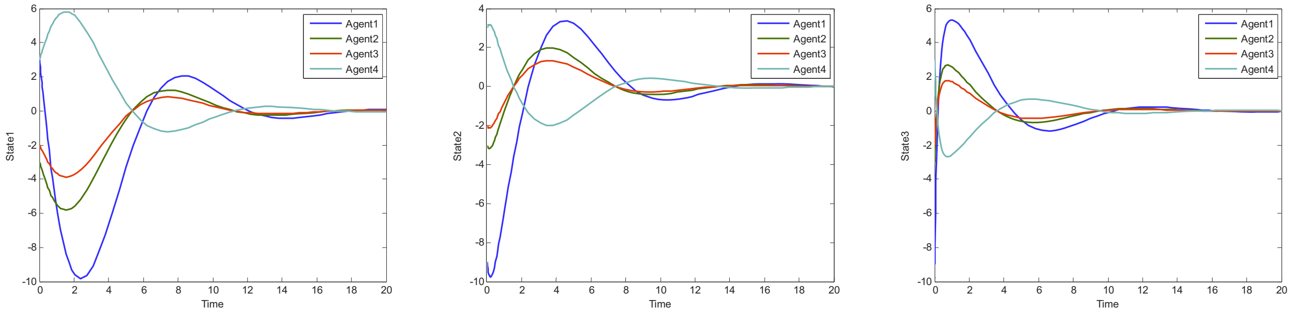

Fig. 3. State trajectories of 3rd-order Multi-agent Systems in $\mathcal{G}_a$ with protocol parameters design ($h = 0.1$, $\mu = 0.9$)

5 Conclusions

In this paper, we have investigated the consensus problems for high-order linear multi-agent systems with time varying delays, which include a consensus criterion and a protocol parameters design algorithm. Mainly, the following contributions were concluded in this paper:

1) A high-order linear multi-agent systems consensus criterion is proposed. By optimizing the structure on Lyapunov Krasovskii functional and adding proper Free Weighting Matrices, the criterion has less conservativeness than existing results;

2) The consensus criterion is validated when $\mu \ge 1$, and even can apply with a unknown μ ;while the [11] could only fit for $\mu < 1$;

3) The protocol parameters design algorithm can give the two parameters directly while the algorithm in [11] could only give the $\mathbf{K}_2$ but the undetermined $\mathbf{K}_1$ would be set in advance.

While if the topology is directed the structure of Free Weighting Matrices $\mathbf{V}_1, \mathbf{V}_2$ is limited by $\Lambda_{\mathbf{K}_1} = \mathbf{V}_1\mathbf{L}^{-1}$ and $\Lambda_{\mathbf{K}_2} = \mathbf{V}_2\mathbf{L}^{-1}$ (actually, the limitation caused by diagonal structures of $\Lambda_{\mathbf{K}_1}, \Lambda_{\mathbf{K}_2}$). That means even though we calculate the Free Weighting Matrices $\mathbf{V}_1, \mathbf{V}_2$ by solving NLMIs, we could not always acquire a diagonal structured $\Lambda_{\mathbf{K}_1}, \Lambda_{\mathbf{K}_2}$. Therefore, the structure of $\mathbf{V}_1, \mathbf{V}_2$ will be investigated further so that our protocol parameters design could be generalized for any directed topologies.

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