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TRIANGLE-Y EXCHANGES ON INTRINSIC KNOTTING OF ALMOST COMPLETE AND COMPLETE PARTITE GRAPHS

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ABSTRACT

Let G be a 0- or 1-deficient graph which is intrinsically knotted, let J represent any graph obtained from G by a finite sequence of Δ-Y exchanges and/or vertex expansions. We prove that removing any vertex of J, and all edges incident to that vertex, yields an

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intrinsically linked graph. This result provides more intrinsically knotted graphs which satisfy the conjecture mentioned in Adams' The Knot Book that removing any vertex from an intrinsically knotted graph yields an intrinsically linked graph.

Keywords: Intrinsically knotted; Δ-Y exchanges; vertex-expansion; embedded graphs.

Mathematics Subject Classification 2010: 57M25, 57M27

1. Introduction

Throughout the paper by an *embedded graph* we meant a graph embedded in 3-space, where all the embeddings are tame. A graph is said to be *intrinsically knotted* (IK) if every embedding of it in R^3 contains a cycle that is a nontrivial knot. Similarly, a graph is said to be *intrinsically linked* (IL) if every embedding of it in R^3 contains a nontrivial link. In 1995, Robertson, Seymour and Thomas [12] proved Sachs' conjecture, and thus completely classified intrinsically linked graphs, in which they demonstrated that intrinsic linking is determined by the seven Petersen graphs. They also showed that a similar, finite list of graphs exists for the intrinsic knotting property [11]. However, finding such a set of graphs remains an unsolved question.

It is known [4, 6, 9, 10] that K_7 and $K_{3,3,1,1}$ along with any graph obtained from these two by Δ-Y exchanges is minor minimal with respect to intrinsic knotting. Foisy [7] added a new minor minimal graph to the list. Moreover, Foisy provided a counterexample to the "unsolved question" posed in Adams' book [1]: *Whether removing a vertex from an intrinsically knotted graph must yield an intrinsically linked graph*, more counterexample can be found in [8]. We remark here that throughout the paper when we mention removing a vertex from a graph just means the edges incident to that vertex were also removed.

While Adams' conjecture is not true in general, it does hold for a large array of graphs. A graph is called *k-deficient*, if it is a complete or complete partite graph with k, $k = 0, 1, 2, \ldots$, edges removed. Campbell, Mattman, Ottman, Pyzer, Rodrigues and Williams [3] have classified all 0-, 1-, 2-deficient graphs with respect to intrinsic linking and intrinsic knotting, all of which are shown to be hold for Adams' conjecture. With this motivation, in this paper we will extend the results in [3] and present a larger class of graphs which hold for Adams' conjecture.

It is well known that performing Δ-Y exchanges on any IK or IL graph produces a graph with the same property. In reverse, doing Y-Δ exchanges on an IL graph always produce an IL graph, but for IK graph, recently Flapan and Naimi [5] proved that after doing Y-Δ exchanges, one may not get an IK graph. That is, Y-Δ exchange does not preserve the property of Intrinsic Knottedness.

In this paper, we will consider doing Δ-Y exchanges on any IK k-deficient graph G. Since Δ-Y exchanges preserve the property of Intrinsic Knottedness, after the exchanges we will get a graph J which is also IK. So one natural question to ask is whether removing a vertex from J still produces an IL graph. We will provide partial answer to the question; more specifically, we will prove in the paper that the argument holds for all 0-, 1-deficient graphs which are IK. Vertex expansion will be considered as well, but we will not consider Y-Δ exchange as it cannot preserve the property of Intrinsic Knottedness.

Recall that a *vertex expansion* of a vertex v in a graph J is achieved by replacing v with two vertices, v' and v'', adding the edge (v', v''), and connecting a subset of the edges that were incident to v to v', and connecting the remaining edges that were incident to v to v''. The reverse of this operation is *edge contraction*.

Based on the above definitions, we can give the main result of this paper as follows.

Theorem 1.1. *Let G be a 0- or 1-deficient graph which is intrinsically knotted, let J represent any graph obtained from G by a finite sequence of Δ-Y exchanges and/or vertex expansions. Then the removal of any vertex of J, and all edges incident to that vertex, yields an intrinsically linked graph.*

The rest of the paper is organized as follows. We present some useful lemmas and the tables of k-deficient $(k = 0, 1, 2)$ graphs classified with respect to intrinsic linking and knotting in Sec. 2, and Theorem 1.1 is shown in Sec. 3.

2. Lemmas and Tables of k-Deficient $(k = 0, 1, 2)$ Graphs

In this section, we present some useful lemmas and notations which will be used throughout the paper. Let K_n denote the complete graph on n vertices, and $K_{n_1,n_2,\ldots,n_p}$ denote the complete p-partite graph, $p \geq 2$, with n_i vertices in the ith part. Note that when we mention the complete partite graph we mean that the graph is at least bipartite. Let $K_{n_1,n_2,\ldots,n_q} - k$, $q \geq 1$, $k \geq 0$, denote the graph obtained by removing k edges from $K_{n_1,n_2,\ldots,n_p}$. Note also that throughout the paper when we mention deleting a Δ from a graph, we only mean deleting the edges of Δ from the graph, and for the special case that graph G contains no triangle, we assume $G - \Delta \equiv G$.

Lemma 2.1 ([3]). *$K_{n_1+n_2,n_3,\ldots,n_p}$ is a minor of $K_{n_1,n_2,n_3,\ldots,n_p}$. Similarly, $K_{n_1+n_2,n_3,\ldots,n_p} - k$ edges is a minor of $K_{n_1,n_2,n_3,\ldots,n_p} - k$ edges.*

Lemma 2.2 ([13]). *The graph $G + K_1$ is intrinsically linked if and only if G is non-planar.*

Lemma 2.3. *If $K_{n_1,n_2,\ldots,n_p} - \Delta$ is intrinsically linked, then $K_{n_1,n_2,\ldots,n_i+1,\ldots,n_p} - \Delta$ and $K_{n_1,n_2,\ldots,n_p,1} - \Delta$ are also intrinsically linked, where $1 \leq i \leq p$.*

Proof. $K_{n_1,n_2,\ldots,n_i+1,\ldots,n_p} - \Delta$: The case $p = 1$ is obvious. When $p = 2$, since K_{n_1,n_2} contains no triangle, then $K_{n_1,n_2} - \Delta$ is a subgraph of $K_{n_1+1,n_2} - \Delta$ and $K_{n_1,n_2+1} - \Delta$. When $p \geq 3$, consider v, the vertex which was added to $K_{n_1,n_2,\ldots,n_p}$. If the Δ deleted from $K_{n_1,n_2,\ldots,n_i+1,\ldots,n_p}$ does not contain v, then $K_{n_1,n_2,\ldots,n_p} - \Delta$ is a subgraph of $K_{n_1,n_2,\ldots,n_i+1,\ldots,n_p} - \Delta$. However, if the deleted Δ contains v, then $K_{n_1,n_2,\ldots,n_i+1,\ldots,n_p} - \Delta$ also contains $K_{n_1,n_2,\ldots,n_p} - \Delta$ as a subgraph since if we delete v from part n_i+1 of $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}$ we can get graph $K_{n_1,n_2,\dots,n_p}-e$. Note that here $K_{n_1,n_2,\dots,n_p}$ is a complete partite graph with $p\geq 3$, so we can find a Δ in $K_{n_1,n_2,\dots,n_p}$ which contains e, i.e. $K_{n_1,n_2,\dots,n_p}-e$ contains $K_{n_1,n_2,\dots,n_p}-\Delta$ as a subgraph. It follows directly from the assumption that $K_{n_1,n_2,\dots,n_p}-\Delta$ is IL that $K_{n_1,n_2,\dots,n_{i+1},\dots,n_p}-\Delta$ is also IL.

$K_{n_1,n_2,\dots,n_p,1}-\Delta$: The case $p=1$ is obvious. When $p=2$, by checking Table 1 we can get that all the other IL complete bipartite graphs contain $K_{4,4}$ as a subgraph, so it suffices to prove the ILness of $K_{4,4,1}-\Delta$. This is true since deleting v from graph $K_{4,4,1}-\Delta$ results in $K_{4,4}-e$, one of the graphs with eight vertices in the Petersen family. Using similar discussion as above we can prove the case $p\geq 3$, therefore completing the proof of Lemma 2.3. □

Lemma 2.4. *If for every edge e and triangle Δ not containing e of $K_{n_1,n_2,\dots,n_p}$ the graph $K_{n_1,n_2,\dots,n_p}-e-\Delta$ is intrinsically linked, then for every edge e and triangle Δ not containing e of $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}$ and $K_{n_1,n_2,\dots,n_p,1}$ the graphs $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}-e-\triangle$ and $K_{n_1,n_2,\dots,n_p,1}-e-\triangle$ are also intrinsically linked.*

Proof. $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}-e-\Delta$: By using similar proof to the one in Lemma 2.3 we can prove $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}-e-\Delta$ is IL, the only difference is when $p\geq 3$ and the case that the deleted Δ contains v, the vertex added to $K_{n_1,n_2,\dots,n_p}$, removing v and Δ from $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}$ then results in a 1-deficient graph, say, $K_{n_1,n_2,\dots,n_p}-e'$. Note that e and e' are two different edges in $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}$ since by the given condition the deleted Δ does not contain e. There are two subcases, one of which is that the deleted e of $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}$ is incident to vertex v, then $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}-e-\Delta$ contains $K_{n_1,n_2,\dots,n_p}-e'$ as a subgraph. It follows directly from the given condition that $K_{n_1,n_2,\dots,n_p}-e'$ is IL that $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}-e-\Delta$ is IL.

The other subcase is that the deleted e is not incident to vertex v, then according to the condition we can find a Δ_1 which contains e' but not e in $K_{n_1,n_2,\dots,n_p}$. By assumption, $K_{n_1,n_2,\dots,n_p}-e-\Delta_1$ is IL. This together with the fact that $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}-e-\Delta$ contains $K_{n_1,n_2,\dots,n_p}-e-\Delta_1$ as a subgraph imply that $K_{n_1,n_2,\dots,n_i+1,\dots,n_p}-e-\Delta$ is IL.

$K_{n_1,n_2,\dots,n_p,1}-e-\Delta$: Following similar discussion as that in Lemma 2.3, we can prove the case $p=1$ and $p\geq 3$. When $p=2$, it is observed from Table 3 that all of the other IL 1-deficient graph $K_{n_1,n_2}-e$ contains $K_{4,4}-e$ as a subgraph, and thus we only need to consider $K_{4,4,1}-e-\Delta$, which will be proved to be IL in the proof of Theorem 1.1. This completes the proof of Lemma 2.4. □

For the readers' convenience, we present in the following the classification of 0-, 1-, 2-deficient graphs with respect to intrinsic linking and knotting.

Lemma 2.5 ([3]). *The* 0-, 1-, 2-*deficient graphs are classified with respect to intrinsic linking according to Tables* 1, 3, 5 *and* 6 *respectively.*

Table 1. Intrinsic linking of complete and complete k-partite graphs, $k = 2, 3, \ldots$.

k	1	2	3	4	5	≥ 6
Linked	6	4,4	3,3,1 4,2,2	2,2,2,2 3,2,1,1	2,2,1,1,1 3,1,1,1,1	All
Not linked	5	n,3	3,2,2 n,2,1	2,2,2,1 n,1,1,1	2,1,1,1,1	None

Table 2. Intrinsic knotting of complete and complete k-partite graphs, $k = 2, 3, \ldots$.

k	1	2	3	4	5	6	≥ 7
Knotted	7	5,5	3,3,3 4,3,2 4,4,1	3,2,2,2 4,2,2,1 3,3,2,1 3,3,1,1	2,2,2,2,1 3,2,2,1,1 3,2,1,1,1	2,2,1,1,1,1 3,1,1,1,1,1	All
Not knotted	6	4,4	3,3,2 n,2,2 n,3,1	2,2,2,2 4,2,1,1 3,2,2,1 n,2,1,1 n,1,1,1	2,2,2,1,1 2,2,1,1,1 n,1,1,1,1	2,1,1,1,1,1	None

Table 3. Intrinsic linking of 1-deficient graphs.

k	1	2	3	4	5	6	≥ 7
Linked	7-e	4,4-e	4,3,1-e 3,3,2-e 4,2,2-e	2,2,2,2-e 3,2,1,1-(b,c) 4,2,1,1-e 3,3,1,1-e 3,2,2,1-e	2,2,1,1,1-(b,c) 3,1,1,1,1-(b,c) 4,1,1,1,1-e 3,2,1,1,1-e 2,2,2,1,1-e	2,1,1,1,1,1-e	All
Not linked	6-e	n,3-e	3,2,2-e n,2,1-e 3,3,1-e	2,2,2,1-e n,1,1,1-e 3,2,1,1-(a,b) 3,2,1,1-(a,c) 3,2,1,1-(c,d)	2,2,1,1,1-(a,b) 2,2,1,1,1-(c,d) 3,1,1,1,1-(a,b) 2,1,1,1,1-e	1,1,1,1,1,1-e	None

Table 4. Intrinsic knotting of 1-deficient graphs.

k	1	2	3	4	5	6	7	≥ 8
Knotted	8-e	5,5-e	3,3,3-e 4,3,2-e 4,4,1-e	3,2,2,2-e 4,2,2,1-e 3,3,2,1-e 4,3,1,1-e	2,2,2,2,1-e 3,2,1,1,1-(b,c) 4,2,1,1,1-e 3,3,1,1,1-e 3,2,2,1,1-e	2,2,1,1,1,1-(b,c) 3,1,1,1,1,1-(b,c) 3,2,1,1,1,1-e 2,2,2,1,1,1-e 4,1,1,1,1,1-e	2,1,1,1,1,1,1-e	All
Not knotted	7-e	n,4-e	3,3,2-e n,2,2-e n,3,1-e	3,3,1,1-e 2,2,2,2-e 3,2,2,1-e n,2,1,1-e	3,2,1,1,1-(c,d) 3,2,1,1,1-(a,b) 3,2,1,1,1-(a,c) 2,2,2,1,1-e n,1,1,1,1-e	2,2,1,1,1,1-(a,b) 2,2,1,1,1,1-(c,d) 3,1,1,1,1,1-(a,b) 2,1,1,1,1,1-e	1,1,1,1,1,1,1-e	None

Remark: As has been specified in [3], in the tables and the proof presented below, given a complete partite graph, we refer to parts alphabetically with capital letters and the vertices of those parts by lower case letters. For example, in $K_{4,5}$, the part with 4 vertices will be called Part A and the part with 5 vertices will be called Part B. By (a, b) is then meant an edge between Parts A and B. The readers are referred to [3] for the detailed descriptions of the notations.

Table 5. Intrinsic linking of 2-deficient graphs.

k	1	2	3	4
Linked	7-2e	5,4-2e	4,3,1-$\{(b,c),e\}$ 4,3,1-$\{(a_1,b_1),(a_1,b_2)\}$ 4,3,1-$\{(a_1,b_1),(a_1,c)\}$ 3,3,2-$\{(a_1,b_1),(a_1,b_2)\}$ 3,3,2-$\{(a_1,c_1),(a_2,c_1)\}$ 3,3,2-$\{(a_1,c_1),(b_1,c_1)\}$ 3,3,2-$\{(a_1,b_1),(b_2,c_1)\}$ 3,3,2-$\{(a_1,c_1),(b_1,c_2)\}$ 3,3,2-$\{(a_1,c_1),(a_1,c_2)\}$ 4,2,2-$\{(b,c),e\}$ 4,2,2-$\{(a_1,b_1),(a_1,b_2)\}$ 5,3,1-2e 4,4,1-2e 4,3,2-2e 3,3,3-2e 5,2,2-2e	3,2,1,1-$\{(b_1,c),(b_1,d)\}$ 3,2,1,1-$\{(b_1,c),(b_2,c)\}$ 4,2,1,1-$\{(b,c),e\}$ 4,2,1,1-$\{(c,d),e\}$ 4,2,1,1-$\{(a_1,b_1),(a_1,b_2)\}$ 4,2,1,1-$\{(a_1,b_1),(a_1,c)\}$ 4,2,1,1-$\{(a_1,c),(a_1,d)\}$ 3,3,1,1-$\{(b,c),e\}$ 3,3,1,1-$\{(c,d),e\}$ 3,3,1,1-$\{(a_1,b_1),(a_1,b_2)\}$ 3,2,2,1-$\{(b,c),e\}$ 3,2,2,1-$\{(c,d),e\}$ 3,2,2,1-$\{(a,d),e\}$ 3,2,2,1-$\{(a_1,b_1),(a_1,b_2)\}$ 3,2,2,1-$\{(a_1,b_1),(a_2,b_1)\}$ 3,2,2,1-$\{(a_1,b_1),(a_2,c_1)\}$ 2,2,2,2-2e 5,2,1,1-2e 4,3,1,1-2e 4,2,2,1-2e 3,3,2,1-2e
Not linked	6-2e	4,4-2e n,3-2e	4,3,1-$\{(a_1,b_1),(a_2,b_1)\}$ 4,3,1-$\{(a_1,b_1),(a_2,b_2)\}$ 4,3,1-$\{(a_1,c),(a_2,c)\}$ 4,3,1-$\{(a_1,b_1),(a_2,c)\}$ 3,3,2-$\{(a_1,b_1),(a_2,b_2)\}$ 3,3,2-$\{(a_1,b_1),(b_1,c_1)\}$ 3,3,2-$\{(b_1,c_1),(b_2,c_2)\}$ 4,2,2-$\{(a_1,b_1),(a_2,b_2)\}$ 4,2,2-$\{(a_1,b_1),(a_2,b_1)\}$ 4,2,2-$\{(a_1,b_1),(a_2,c_1)\}$ 4,2,2-$\{(a_1,b_1),(a_1,c_1)\}$ 3,2,2-2e n,2,1-2e 3,3,1-2e	3,2,1,1-$\{(a,b),e\}$ 3,2,1,1-$\{(a,c),e\}$ 3,2,1,1-$\{(c,d),e\}$ 3,2,1,1-$\{(b_1,c),(b_2,d)\}$ 4,2,1,1-$\{(a_1,b_1),(a_2,b_2)\}$ 4,2,1,1-$\{(a_1,c),(a_2,c)\}$ 4,2,1,1-$\{(a_1,b_1),(a_2,b_1)\}$ 4,2,1,1-$\{(a_1,b_1),(a_2,c)\}$ 4,2,1,1-$\{(a_1,c),(a_2,d)\}$ 3,3,1,1-$\{(a_1,b_1),(a_2,b_2)\}$ 3,2,2,1-$\{(a_1,b_1),(a_2,b_2)\}$ 3,2,2,1-$\{(a_1,b_1),(a_1,c_1)\}$ 2,2,2,1-2e n,1,1,1-2e

Lemma 2.6 ([2, 3]). *The* 0-, 1-*deficient graphs are classified with respect to intrinsic knotting according to Tables* 2 *and* 4 *respectively.*

3. Proof of Theorem 1.1

Before proving Theorem 1.1, it is worth remarking here that in the proof of this theorem, when we prove the ILness of a given graph, say, J, we need the ILness of the minor or subgraph of J, which may not be stated explicitly as it is assumed that the reader can check the tables as listed in Sec. 2. Note that when we mention combining two or more parts of a complete partite graph, we mean deleting all of the edges between them and then considering them as the same part, for example, combining Parts B and C of graph $K_{4,3,1}$ yields graph $K_{4,4}$.

Proof of Theorem 1.1. Foisy has proved the case when G was graph K_7 or $K_{3,3,1,1}$ [7]. In what follows, we will prove the generalized case that G is any

Table 6. Intrinsic linking of 2-deficient graphs (cont).

k	5	6	7
Linked	2,2,1,1,1-$\{(b_1,c),(b_2,c)\}$ 2,2,1,1,1-$\{(a_1,d),(b_1,c)\}$ 2,2,1,1,1-$\{(b_1,c),(b_1,d)\}$ 3,1,1,1,1-$\{(b,c),(c,d)\}$	2,1,1,1,1,1-$\{(a_1,b),(a_1,c)\}$ 2,1,1,1,1,1-$\{(a_1,b),(a_2,b)\}$ 2,1,1,1,1,1-$\{(a_1,b),(c,d)\}$ 2,1,1,1,1,1-$\{b,c),(c,d)\}$	All
	4,1,1,1,1-$\{(b,c),e\}$ 4,1,1,1,1-$\{(a_1,b),(a_1,c)\}$ 3,2,1,1,1-$\{(a,c),e\}$ 3,2,1,1,1-$\{(b,c),e\}$ 3,2,1,1,1-$\{(c,d),e\}$ 3,2,1,1,1-$\{(a_1,b_1),(a_1,b_2)\}$ 3,2,1,1,1-$\{(a_1,b_1),(a_2,b_1)\}$ 2,2,2,1,1-2e 5,1,1,1,1-2e 4,2,1,1,1-2e 3,3,1,1,1-2e	3,1,1,1,1,1-2e 2,2,1,1,1,1-2e	
Not linked	2,2,1,1,1-$\{(a,b),e\}$ 2,2,1,1,1-$\{(c,d),e\}$ 2,2,1,1,1-$\{(b_1,c),(b_2,d)\}$ 2,2,1,1,1-$\{(a_1,c),(b_1,c)\}$ 3,1,1,1,1-$\{(a,b),e\}$ 3,1,1,1,1-$\{(b,c),(d,e)\}$ 4,1,1,1,1-$\{(a_1,b),(a_2,c)\}$ 4,1,1,1,1-$\{(a_1,b),(a_2,b)\}$ 3,2,1,1,1-$\{(a_1,b_1),(a_2,b_2)\}$ 2,1,1,1,1-2e	2,1,1,1,1,1-$\{(a_1,b),(a_2,c)\}$ 2,1,1,1,1,1-$\{(a_1,b),(b,c)\}$ 2,1,1,1,1,1-$\{(b,c),(d,e)\}$ 1,1,1,1,1,1-2e	None

0-, or 1-deficient graph. The former part of the proof relies heavily on some of Foisy's results which will be presented with slight changes for the readers' convenience. It has been shown in [3] that removing a vertex from a 0- or 1-deficient IK graph results in a IL graphs.

Now we suppose that G_n is the graph obtained from G by a sequence of $n(n > 0)$ Δ-Y exchanges and/or vertex expansions, and the claim is true for G_n. Now consider G_{n+1}, the graph obtained by a vertex expansion on G_n. Let v be a vertex of G_{n+1}. On one hand, if v is not the vertex created by doing vertex expansion on G_n, $G_{n+1} - v$ is then the graph obtained from $G_n - v$ by the vertex expansion. By assumption, $G_n - v$ is IL, which together with the fact that vertex expansion preserves the property of intrinsic linking [10] imply that $G_{n+1} - v$ is IL. On the other hand, if the vertex expansion on G_n creates v, then removing v from G_{n+1} results in a graph containing $G_n - w$ as a subgraph, where w is the vertex of G_n that is split to obtain G_{n+1}. Since according to the assumption $G_n - w$ is IL, then $G_{n+1} - v$ must also be IL.

Now we proceed to consider the effect of a Δ-Y exchange on G_n. Consider a vertex v of G_{n+1}. If the exchange used to create G_{n+1} from G_n does not create v, and v is not a vertex of the Δ on which the triangle-Y is made, then $G_{n+1} - v$ is the graph that can be obtained from $G_n - v$ via a Δ-Y exchange. Since by assumption $G_n - v$ is IL, $G_{n+1} - v$ is also IL. If v is the vertex of the triangle in G_n which is swapped for a Y to create G_{n+1}, then $G_{n+1} - v$ is homeomorphic to $G_n - v$, and thus $G_{n+1} - v$ is IL.

Finally, assume v is the vertex created by the Δ-Y exchange. Denote by xv, yv, and zv the three edges of the created Y. Then $G_{n+1} - v$ is G_n with the triangle Δ removed. The proof for the theorem will be completed once we have established that G_n with any Δ removed is IL. Since Δ-Y exchanges and vertex expansions (of graphs without multiple edges) do not create any cycles of length three, any cycle of length three in G_n must have been present in G. The proof is thus reduced to show that G with any triangle removed is IL. Throughout the remaining part of the proof, we will mainly check whether $G - \Delta$ is an IL graph. To this end, it suffices to check all IK graphs as listed in Tables 2 and 4 respectively. □

Claim 1. *Let G be an IK 0-deficient graph listed in Table 2, then the removal of any triangle of G yields an IL graph.*

Proof. $k = 1$:

$K_7 - \Delta$ is IL which has been proved by Foisy in [7].

$k = 2$:

Since $K_{5,5}$ contains no triangle and therefore $K_{5,5} - \Delta = K_{5,5}$ is IL.

$k = 3$:

$K_{4,4,1} - \Delta$, $K_{4,3,2} - \Delta$, and $K_{3,3,3} - \Delta$ all contain $K_{3,3,1}$ as a minor and are therefore IL.

$k = 4$:

$K_{3,2,2,2} - \Delta$ has 2 cases (up to symmetry $K_{3,2,2,2}$ has 2 types of triangles). In either case, $K_{3,2,2,2} - \Delta$ contains $K_{3,2,1,1}$ as a minor and is therefore IL.

$K_{4,2,2,1} - \Delta$ has three cases. In the case $K_{4,2,2,1} - \{(a, b), (b, c), (c, a)\}$ and $K_{4,2,2,1} - \{(a, b), (b, d), (d, a)\}$, $K_{4,2,2,1} - \Delta$ contains $K_{3,2,1,1}$ as a minor. In the case $K_{4,2,2,1} - \{(b, c), (c, d), (d, b)\}$, $K_{4,2,2,1} - \Delta$ contains $K_{4,4}$ as a minor. By checking Table 1 one can get $K_{4,2,2,1} - \Delta$ is IL.

$K_{3,3,1,1} - \Delta$ is IL which has been proved by Foisy in [7] ($K_{3,3,2,1} - \Delta$ is redundant in Table 2).

$k = 5$:

$K_{2,2,2,2,1} - \Delta$ has 2 cases. In either case, $K_{2,2,2,2,1} - \Delta$ contains $K_{2,2,1,1,1}$ as a minor and is therefore IL.

$K_{3,2,1,1,1} - \Delta$ has 4 cases ($K_{3,2,2,1,1}$ is also redundant in Table 2). In the case $K_{3,2,1,1,1} - \{(a, b), (b, c), (c, a)\}$, $K_{3,2,1,1,1} - \Delta$ contains $K_{3,2,1,1}$ as a minor. In the case $K_{3,2,1,1,1} - \{(b, c), (c, d), (d, b)\}$, $K_{3,2,1,1,1} - \Delta$ contains $K_{4,3,1}$ as a minor. In the case $K_{3,2,1,1,1} - \{(c, d), (d, e), (e, c)\}$, $K_{3,2,1,1,1} - \Delta$ contains $K_{3,3,2}$ as a minor. In the case $K_{3,2,1,1,1} - \{(a, d), (d, e), (e, a)\}$, $K_{3,2,1,1,1} - \Delta$ has one vertex connected to every other, if we delete this vertex we can get a nonplanar graph (see Fig. 1). In fact, the graph consisting of vertices $\{v_1, v_2, v_3, v_4, e\}$ and the dashed lines between them is homeomorphic to the nonplanar graph K_5, so, by Lemma 2.2, $K_{3,2,1,1,1} - \Delta$ is IL.

$k = 6$:

$K_{2,2,1,1,1,1} - \Delta$ has 3 cases. In the case $K_{2,2,1,1,1,1} - \{(a, b), (b, c), (c, a)\}$, $K_{2,2,1,1,1,1} - \Delta$ contains $K_{3,1,1,1,1}$ as a minor. In the case $K_{2,2,1,1,1,1} -$

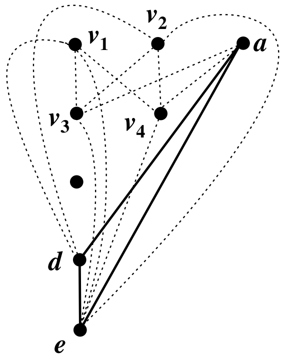

Fig. 1.

$\{(b,c),(c,d),(d,b)\}$, $K_{2,2,1,1,1,1} - \Delta$ contains $K_{4,2,1,1}$ as a minor. $K_{2,2,1,1,1,1} - \{(c,d),(d,e),(e,c)\}$, $K_{2,2,1,1,1,1} - \Delta$ contains $K_{3,2,2,1}$ as a minor. By checking Table 1 one can get that $K_{2,2,1,1,1,1} - \Delta$ is IL.

$K_{3,1,1,1,1,1} - \Delta$ has 2 cases. In the case $K_{3,1,1,1,1,1} - \{(a,b),(b,c),(c,a)\}$, $K_{3,1,1,1,1,1} - \Delta$ contains $K_{2,2,1,1,1}$ as a minor. In the case $K_{3,1,1,1,1,1} - \{(b,c),(c,d),(d,b)\}$, $K_{3,1,1,1,1,1} - \Delta$ contains $K_{3,3,1,1}$ as a minor. By checking Table 1 one can get that $K_{3,1,1,1,1,1} - \Delta$ is IL.

$k \geq 7$: by Lemma 2.3, we only need to prove the ILness of $K_{1,1,1,1,1,1,1} - \Delta(= K_7 - \Delta)$ which has been proved in [7].

Now we complete the proof of Claim 1 according to the above discussion. $\square$

Claim 2. *Let G be an IK 1-deficient graph listed in Table 4, then the removal of any triangle of G yields an IL graph.*

Proof. $k = 1$:

$K_8 - e - \Delta$. Deleting 1 vertex from $K_8 - e - \Delta$ to which edge e is incident, then we can get a graph with 7 vertices which has $K_7 - \Delta$ as a minor. By Claim 1, $K_7 - \Delta$ is IL, so $K_8 - e - \Delta$ is also IL.

$k = 2$:

$K_{5,5} - e$ has no triangle, so $K_{5,5} - e - \Delta = K_{5,5} - e$ is IL.

$k = 3$:

$J = K_{3,3,3} - e - \Delta$ has 2 cases as Fig. 2:

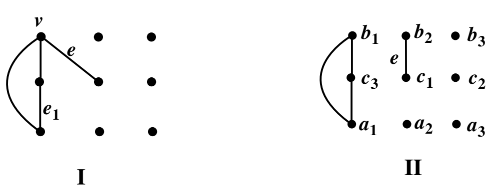

Fig. 2.

In case I, J contains a subgraph $J = K_{3,3,3} - v - e_1 = K_{3,3,2} - e_1$. In case II, relabel the vertices, we can get J contains a subgraph $K_{3,3,3} - c_3 - \{(a_1, b_1), (b_2, c_1)\} = K_{3,3,2} - \{(a_1, b_1), (b_2, c_1)\}$. By checking Tables 3 and 5 one can get $K_{3,3,3} - e - \Delta$ is IL.

$J = K_{4,3,2} - e - \Delta$ has 9 cases as Fig. 3:

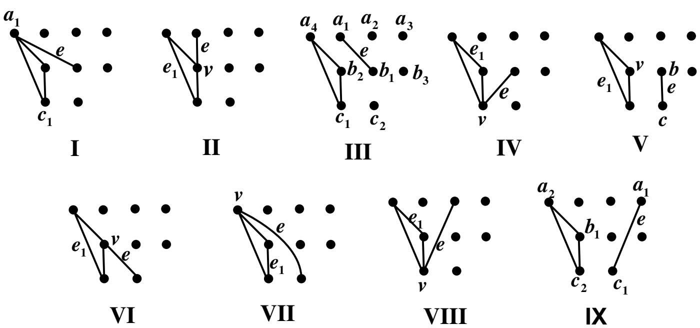

Fig. 3.

In case I, J contains a subgraph $K_{4,3,2} - a_1 - c_1 = K_{3,3,1}$. In cases II and VI, J contains a subgraph $K_{4,3,2} - e_1 - v = K_{4,2,2} - e_1$. In case III, relabel the vertices, we can get that J contains a subgraph $K_{4,3,2} - a_4 - (a_1, b_1) - (b_2, c_1) = K_{3,3,2} - \{(a_1, b_1), (b_2, c_1)\}$. In cases IV and VIII, J contains a subgraph $K_{4,3,2} - v - e_1 = K_{4,3,1} - e_1$. In case V, J contains a subgraph $K_{4,3,2} - v - (b, c) - e_1 = K_{4,2,2} - \{(b, c), e_1\}$. In case VII, J contains a subgraph $K_{4,3,2} - v - e_1 = K_{3,3,2} - e_1$. In case IX, J contains a subgraph $K_{4,3,2} - a_2 - (a_1, c_1) - (b_1, c_2) = K_{3,3,2} - \{(a_1, c_1), (b_1, c_2)\}$. Check Tables 3 and 5 we can get $K_{4,3,2} - e - \Delta$ is IL.

$J = K_{4,4,1} - e - \Delta$ has 3 cases as Fig. 4:

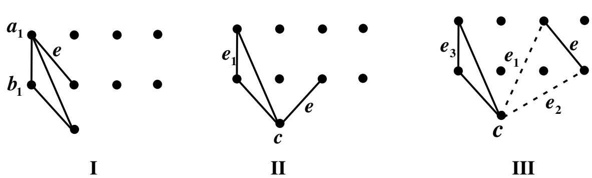

Fig. 4.

In case I, J contains a subgraph $K_{4,4,1} - a_1 - b_1 = K_{3,3,1}$ as a minor. In case II, J contains as a subgraph $K_{4,4,1} - e_1 - c = K_{4,4} - e_1$, the 8 vertex graph in the Petersen family. In case III, J contains a subgraph $K_{4,4,1} - c - e - e_3 + e_1 + e_2$, which is homeomorphic to $K_{4,4} - e_3$, so $K_{4,4,1} - e - \Delta$ is IL.

$k = 4$:

$J = K_{3,2,2,2} - e - \Delta$. Considering the ways we delete the Δ gives 2 cases. In the case $J = K_{3,2,2,2} - \{(a, b), (b, c), (c, a)\}$, by combining Parts C and D, one can find that J has $K_{4,3,2} - e - \Delta$ (Claim 2, $k = 3$) as a minor by Lemma 2.1. In the case $J = K_{3,2,2,2} - \{(b, c), (c, d), (d, b)\}$, there are 4 subcases to delete edge e from $K_{4,3,2} - \Delta$ which are listed as Fig. 5:

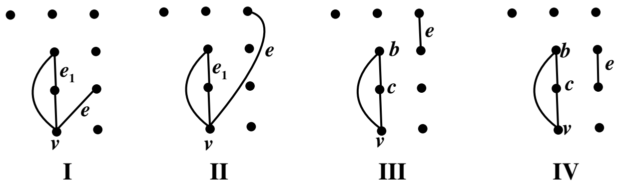

Fig. 5.

In cases I and II, J contains a subgraph $K_{3,2,2,2} - v - e_1 = K_{3,2,2,1} - e_1$. In cases III and IV, J contains a subgraph $K_{3,2,2,2} - v - (b, c) - e = K_{3,2,2,1} - \{(b, c), e\})$. By checking Tables 3 and 5 one can get $K_{3,2,2,2} - e - \Delta$ is IL.

$J = K_{4,2,2,1} - e - \Delta$. There are 3 cases when considering the ways we delete the Δ. For the cases $K_{4,2,2,1} - \{(a, b), (b, c), (c, a)\}$ and $K_{4,2,2,1} - \{(a, c), (c, d), (d, a)\}$, by combining Parts B and D, one can get that J has $K_{4,3,2} - e - \Delta$ (Claim 2, $k = 3$) as a minor by Lemma 2.1. For the case $K_{4,2,2,1} - \{(b, c), (c, d), (d, b)\}$, there are 6 subcases, shown as Fig. 6, when considering the way that edge e is deleted from $K_{4,2,2,1} - \Delta$:

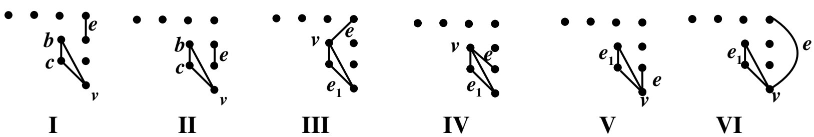

Fig. 6.

In cases I and II, J contains a subgraph $K_{4,2,2,1} - v - e - (b, c) = K_{4,2,2} - \{(b, c), e\}$. In cases III and IV, J contains a subgraph $K_{4,2,2,1} - v - e_1 = K_{4,2,1,1} - e_1$. In cases V and VI, J contains a subgraph $K_{4,2,2,1} - v - e_1 = K_{4,2,2} - e_1$. Checking Tables 3 and 5 gives that $K_{4,2,2,1} - e - \Delta$ is IL.

$J = K_{3,3,2,1} - e - \Delta$, considering the ways we delete the Δ, there are 3 cases. In the case $K_{3,3,2,1} - \{(a, b), (b, c), (c, a)\}$ and $K_{3,3,2,1} - \{(b, c), (c, d), (d, b)\}$, by combining Parts A and D, we can get that J has $K_{4,3,2} - e - \Delta$ (Claim 2, $k = 3$) as a minor by Lemma 2.1. In the case $K_{3,3,2,1} - \{(a, b), (b, d), (d, a)\}$, by combining Parts C and D, one can find J has $K_{3,3,3} - e - \Delta$ (Claim 2, $k = 3$) as a minor according to Lemma 2.1. So we can conclude that $K_{3,3,2,1} - e - \Delta$ is IL.

$J = K_{4,3,1,1} - e - \Delta$, considering the ways we delete the Δ, there are 3 cases. In the case $K_{4,3,1,1} - \{(a, b), (b, c), (c, a)\}$, by combining Parts C and D, one can find J contains $K_{4,3,2} - e - \Delta$ (Claim 2, $k = 3$) as a minor by Lemma 2.1. In the case $K_{4,3,1,1} - \{(a,c),(c,d),(d,a)\}$, combining Parts B and C shows that J has $K_{4,4,1} - e - \Delta$ (Claim 2, $k = 3$) as a minor by Lemma 2.1. In the case $K_{4,3,1,1} - \{(b,c),(c,d),(d,b)\}$, there are 4 subcases to delete edge e from $K_{4,3,1,1} - \Delta$ which are listed as Fig. 7:

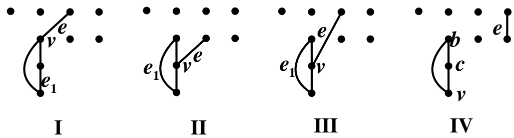

Fig. 7.

In case I, J contains a subgraph $K_{4,3,1,1} - v - e_1 = K_{4,2,1,1} - e_1$. In cases II and III, J contains a subgraph $K_{4,3,1,1} - v - e_1 = K_{4,3,1} - e_1$. In case IV, J contains a subgraph $K_{4,3,1,1} - v - (b,c) - e = K_{4,3,1} - \{(b,c), e\}$. By checking Tables 3 and 5 one can get $K_{4,3,1,1} - e - \Delta$ is IL.

$k = 5$:

$J = K_{2,2,2,2,1} - e - \Delta$, considering the ways we delete the Δ, there are 2 cases: $K_{2,2,2,2,1} - \{(a,b),(b,c),(c,a)\}$ and $K_{2,2,2,2,1} - \{(b,c),(c,e),(e,b)\}$. In either case, combine Parts C and D, we can get that J contains $K_{4,2,2,1} - e - \Delta$ (Claim 2, $k = 4$) as a minor by Lemma 2.1, so, $K_{2,2,2,2,1} - e - \Delta$ is IL.

$J = K_{4,2,1,1,1} - e - \Delta$, considering the ways we delete the Δ, there are 4 cases: $K_{4,2,1,1,1} - \{(a,b),(b,c),(c,a)\}$, $K_{4,2,1,1,1} - \{(a,c),(c,d),(d,a)\}$, $K_{4,2,1,1,1} - \{(b,c),(c,d),(d,b)\}$ and $K_{4,2,1,1,1} - \{(c,d),(d,e),(e,c)\}$. In all cases, combining Parts B and E shows that J contains $K_{4,3,1,1} - e - \Delta$ (Claim 2, $k = 4$) as a minor by Lemma 2.1, so, $K_{4,2,1,1,1} - e - \Delta$ is IL.

$J = K_{3,3,1,1,1} - e - \Delta$, considering the ways we delete the Δ, there are 3 cases. For the case $K_{3,3,1,1,1} - \{(a,b),(b,c),(c,a)\}$, by combine Parts C, D, and E, one can get J contains $K_{3,3,3} - e - \Delta$ (Claim 2, $k = 3$) as a minor by Lemma 2.1. For the case $K_{3,3,1,1,1} - \{(b,c),(c,d),(d,b)\}$, combining Parts A and E shows that J contains $K_{4,3,1,1} - e - \Delta$ (Claim 2, $k = 4$) as a minor by Lemma 2.1. For the case $K_{3,3,1,1,1} - \{(c,d),(d,e),(e,c)\}$, by combining Parts C, D, and E, we can get J contains $K_{3,3,3} - e$ as a minor by Lemma 2.1. Checking Table 3 then gives that $K_{3,3,1,1,1} - e - \Delta$ is IL.

$J = K_{3,2,2,1,1} - e - \Delta$, considering the ways we delete the Δ, there are 5 cases. For the case $K_{3,2,2,1,1} - \{(a,d),(d,e),(e,a)\}$ and $K_{3,2,2,1,1} - \{(b,d),(d,e),(e,b)\}$, combining Parts B and C, gives that J contains $K_{4,3,1,1} - e - \Delta$ (Claim 2, $k = 4$) as a minor by Lemma 2.1; while for the cases $K_{3,2,2,1,1} - \{(a,b),(b,c),(c,a)\}$, $K_{3,2,2,1,1} - \{(a,c),(c,d),(d,a)\}$ and $K_{3,2,2,1,1} - \{(b,c),(c,d),(d,b)\}$, by combining Parts A and E, one can get that J contains $K_{4,2,2,1} - e - \Delta$ (Claim 2, $k = 4$) as a minor by Lemma 2.1. So we can conclude that $K_{3,2,2,1,1} - e - \Delta$ is IL.

$J = K_{3,2,1,1,1} - (b,c) - \Delta$ has 8 cases (see Fig. 8):

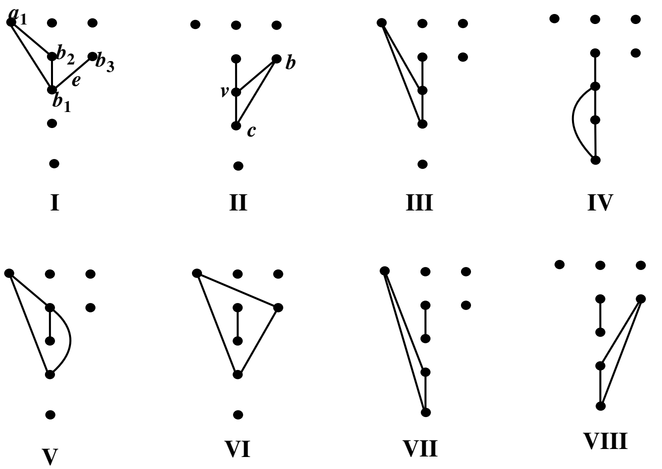

Fig. 8.

In case I, by combining Parts B and C and relabeling the vertices, one can find that J contains a subgraph $K_{3,3,1,1} - \{(a_1,b_1),(a_1,b_2)\}$. In case II, by relabeling the vertices shows one can get that J contains a subgraph $K_{3,2,1,1,1} - v - (b,c) = K_{3,2,1,1} - (b,c)$. For the other 6 cases, by combining Parts B and C, one can get that J contains $K_{3,3,1,1} - \Delta$ (Claim 1, $k = 4$) as a minor by Lemma 2.1. Finally, checking Tables 3 and 5 shows that $K_{3,2,1,1,1} - (b,c) - \Delta$ is IL.

$k = 6$:

$J = K_{3,2,1,1,1,1} - e - \Delta$, there are 4 cases when considering the ways we delete the Δ: $K_{3,2,1,1,1,1} - \{(a,b),(b,c),(c,a)\}$, $K_{3,2,1,1,1,1} - \{(a,c),(c,d),(d,a)\}$, $K_{3,2,1,1,1,1} - \{(b,c),(c,d),(d,b)\}$ and $K_{3,2,1,1,1,1} - \{(d,e),(e,f),(f,d)\}$. In all cases, by combining Parts D, E and F, one can get J contains $K_{3,3,2,1} - e - \Delta$ (Claim 2, $k = 4$) as a minor by Lemma 2.1, so, $K_{3,2,1,1,1,1} - e - \Delta$ is IL.

$J = K_{2,2,2,1,1,1} - e - \Delta$, considering the ways we delete the Δ, there are 4 cases. For the case $K_{2,2,2,1,1,1} - \{(c,d),(d,e),(e,c)\}$, by combining Parts C, D and E, one can get J contains $K_{4,2,1,1} - e$ as a minor by Lemma 2.1; for the case $K_{2,2,2,1,1,1} - \{(d,e),(e,f),(f,d)\}$, combining Parts D, E, and F, one can get that J contains $K_{3,2,2,2} - e$ as a minor by Lemma 2.1; while for the cases $K_{2,2,2,1,1,1} - \{(a,b),(b,c),(c,a)\}$ and $K_{2,2,2,1,1,1} - \{(b,c),(c,d),(d,b)\}$, combining Part D, E, and F, one can get that J contains $K_{3,2,2,2} - e - \Delta$ (Claim 2, $k = 4$) as a minor by Lemma 2.1. Finally checking Table 3 shows that $K_{2,2,2,1,1,1} - e - \Delta$ is IL.

$J = K_{4,1,1,1,1,1} - e - \Delta$, considering the ways we delete the Δ, there are 2 cases: $K_{4,1,1,1,1,1} - \{(a,b),(b,c),(c,a)\}$ and $K_{4,1,1,1,1,1} - \{(b,c),(c,d),(d,b)\}$. In either case, by combining Parts D, E, and F, one can get that J contains $K_{4,3,1,1} - e - \Delta$ (Claim 2, $k = 4$) as a minor by Lemma 2.1 and is therefore IL.

$J = K_{2,2,1,1,1,1} - (b, c) - \Delta$ has 9 cases as Fig. 9:

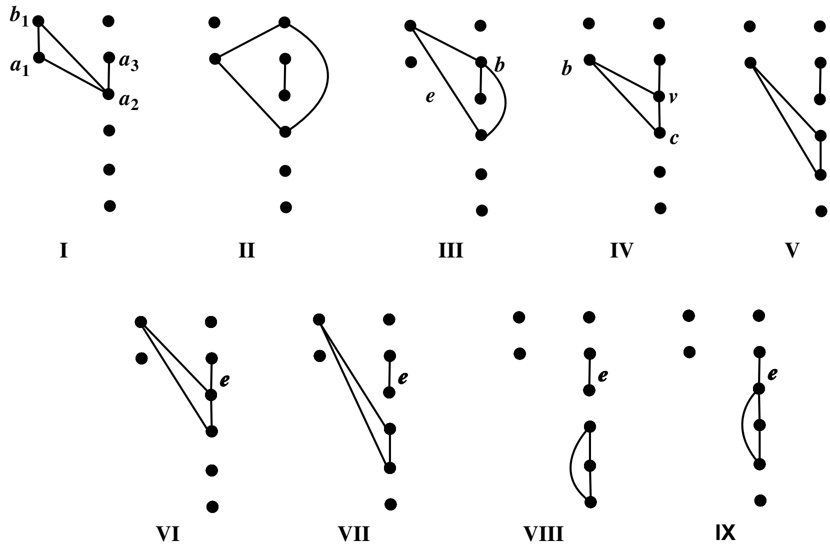

Fig. 9.

In case I, by combining Parts B and C and relabeling the vertices, one can get that J contains $K_{3,2,1,1,1} - \{(a_1, b_1), (a_2, b_1)\}$ as a minor by Lemma 2.1. In cases II and V, by combining Parts B and C, we can get that J contains $K_{3,2,1,1,1} - \Delta$ (Claim 1, $k = 5$) as a minor by Lemma 2.1. In case III, J contains $K_{2,2,1,1,1,1} - b - e = K_{2,1,1,1,1,1} - e$ as a minor. In case IV, by relabeling the vertices, one can get that J contains $K_{2,2,1,1,1,1} - v - (b, c) = K_{2,2,1,1,1,1} - (b, c)$ as a minor. In case VI, combining Parts A, C, and D shows that J contains $K_{4,2,1,1} - e$ as a minor by Lemma 2.1. In case VII, combining Parts A, D, and E shows that J contains $K_{4,2,1,1} - e$ as a minor by Lemma 2.1. In cases VIII and IX, by combining Parts C, D, E, and F, we get that J contains $K_{4,2,2} - e$ as a minor by Lemma 2.1. Finally, by checking Tables 3 and 6, we can get that $K_{2,2,1,1,1,1} - (b, c) - \Delta$ is IL.

$J = K_{3,1,1,1,1,1} - (b, c) - \Delta$ has 4 cases as Fig. 10:

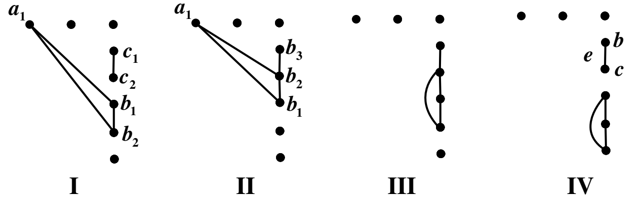

Fig. 10.

In case I, by relabeling the vertices, we can get that J contains a subgraph $K_{3,2,2,1} - \{(a_1, b_1), (a_1, b_2)\}$. In case II, J contains a subgraph $K_{3,3,1,1} - \{(a_1, b_1), (a_1, b_2)\}$. In case III, by combining Parts B, C, D, and E, we can get that J contains $K_{4,3,1}$ as a minor by Lemma 2.1. In case IV, by combining Parts D, E, and F, we can get that J contains $K_{3,3,1,1} - e$ as a minor by Lemma 2.1. Checking Tables 3 and 5 then shows that $K_{3,1,1,1,1,1} - (b, c) - \Delta$ is IL.

$k = 7$:

$J = K_{2,1,1,1,1,1,1} - e - \Delta$, considering the ways we delete the Δ, there are 2 cases. In either case, it is easy to get J contains $K_{4,4}$ as a minor by Lemma 2.1 after combining some parts, so, $K_{2,1,1,1,1,1,1} - e - \Delta$ is IL.

$k \geq 8$: By Lemma 2.4, we only need to prove the ILness of $K_{1,1,1,1,1,1,1,1} - e - \Delta$ ($= K_8 - e - \Delta$) which has been proved in the case $k = 1$.

Based on the above discussion, we complete the proof of Claim 2 and therefore the proof of Theorem 1.1. □

Acknowledgments

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