Numerical Methods Tutorial Class Ten

Tutorial Class

- Review the content in last week lecture

Partial differential equations: Finite difference methods applied to Laplace equation

- Explain problem set

hand calculation & excel

Partial Differential Equations(PDEs)

Ordinary vs Partial differential equations

Ordinary differential equation

Derivative to single independent variables.

$$y'' - 3y' - 10y + 10t + 13 = 0$$

where $y' = \dfrac{dy}{dt},\ y'' = \dfrac{d^2y}{dt^2}$

Partial differential equation

Derivative to multiple independent variables.

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

Partial differential equations

Elliptic — Laplace equation (steady-state with two spatial dimensions)

General form

$$k_x \frac{\partial^2 H}{\partial x^2} + k_y \frac{\partial^2 H}{\partial y^2} = 0$$

Numerical methods for solving PDEs

- **Finite Difference (FD) Approaches**
  Based on approximating solution at a finite # of points, usually arranged in a regular grid.
- **Finite Element (FE) Method**
  Based on approximating solution on an assemblage of simply shaped (triangular, quadrilateral) finite pieces or "elements" which together make up (perhaps complexly shaped) domain.

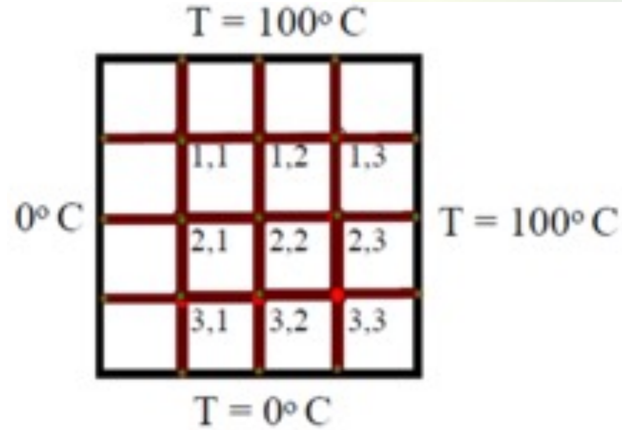
In this course, we concentrate on FD applied to elliptic and parabolic equations.

Finite Difference Methods for solving Elliptic PDE's

1. Discretize domain into grid of evenly spaced points
2. For nodes where u is unknown:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} + O(\Delta x^2)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2} + O(\Delta y^2)$$



w/ $\Delta x = \Delta y = h$, substitute into main equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}}{h^2} + O(h^2)$$

3. Using Boundary Conditions, write, n*m equations for $u(x_{i=1:m}, y_{j=1:n})$ or n*m unknowns.
4. Solve this banded system with an efficient scheme. Using Gauss-Seidel iteratively yields the *Liebmann Method.*

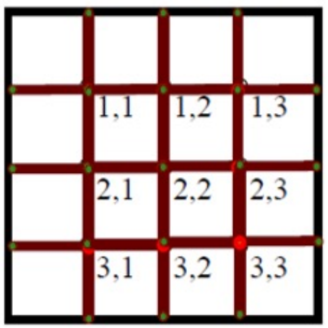
$$\boxed{f''(x) \approx \frac{1}{h^2}\left(f(x+h) - 2f(x) + f(x-h)\right)}$$

Elliptical PDEs

General form $k_x \frac{\partial^2 u}{\partial x^2} + k_y \frac{\partial^2 u}{\partial y^2} = 0$ Δx=Δy kx=ky $\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$f''(x) \approx \frac{1}{h^2}\left(f(x+h) - 2f(x) + f(x-h)\right)$$

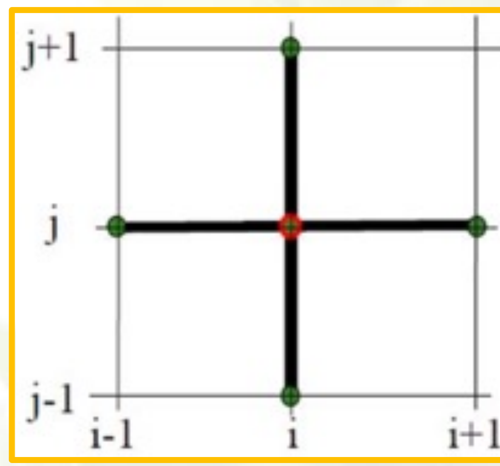
Governing equation: $$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2} = 0$$



If $\Delta x = \Delta y$ then $$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

$$\begin{array}{ccccccccc} T_{11} & T_{12} & T_{13} & T_{21} & T_{22} & T_{23} & T_{31} & T_{32} & T_{33} \end{array}$$

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{12} \\ T_{13} \\ T_{21} \\ T_{22} \\ T_{23} \\ T_{31} \\ T_{32} \\ T_{33} \end{Bmatrix} = \begin{Bmatrix} -100 \\ -100 \\ -200 \\ 0 \\ 0 \\ -100 \\ 0 \\ 0 \\ -100 \end{Bmatrix}$$



Elliptical PDEs: Derivative Boundary Conditions

Neumann Boundary Conditions (derivatives at edges)

- employ ghost points outside of domain
- use FD to obtain information at phantom point,

$$T_{1,j} + T_{-1,j} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0 \quad [*]$$

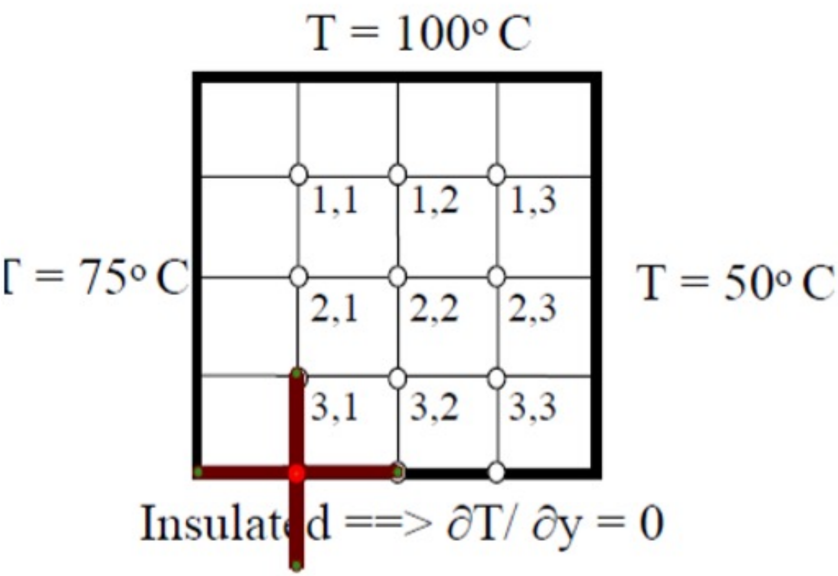
If given $\dfrac{\partial T}{\partial x}$ then use $\dfrac{\partial T}{\partial x} = \dfrac{T_{1,j} - T_{i-1,j}}{2\Delta x}$

to obtain $T_{i-1,j} = T_{1,j} - 2\Delta x \dfrac{\partial T}{\partial x}$

Substituting [*]: $2T_{1,j} - 2\Delta x \dfrac{\partial T}{\partial x} + T_{0,j+1} + T_{0,j-1} - 4T_{0,j} = 0$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \qquad \text{Central Finite Difference}$$

The Laplace molecule: $T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j}$



Derivative (Neumann) BC at (4,1)

$$\frac{\partial T}{\partial y} = \frac{T_{3,1} - T_{5,1}}{2\Delta y}$$

$$T_{5,1} = T_{3,1} - 2\Delta y \frac{\partial T}{\partial y}$$

Substitute into: $$T_{4,2} + T_{4,0} + T_{3,1} + T_{5,1} - 4T_{4,1} = 0$$

To obtain:

$$T_{4,2} - T_{4,0} + 2T_{3,1} - 2\Delta y \frac{\partial T}{\partial y} - 4T_{4,1} = 0$$

Problem1

The temperature (T) distribution of a rectangular domain has reached steady state and can be modelled by the Laplace equation,

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} = 0 \qquad \text{where } k_x = 2k_y$$

The side lengths of the domain are L and H, and L=3H=12.

Assuming consistent units are used, the following boundary conditions are known

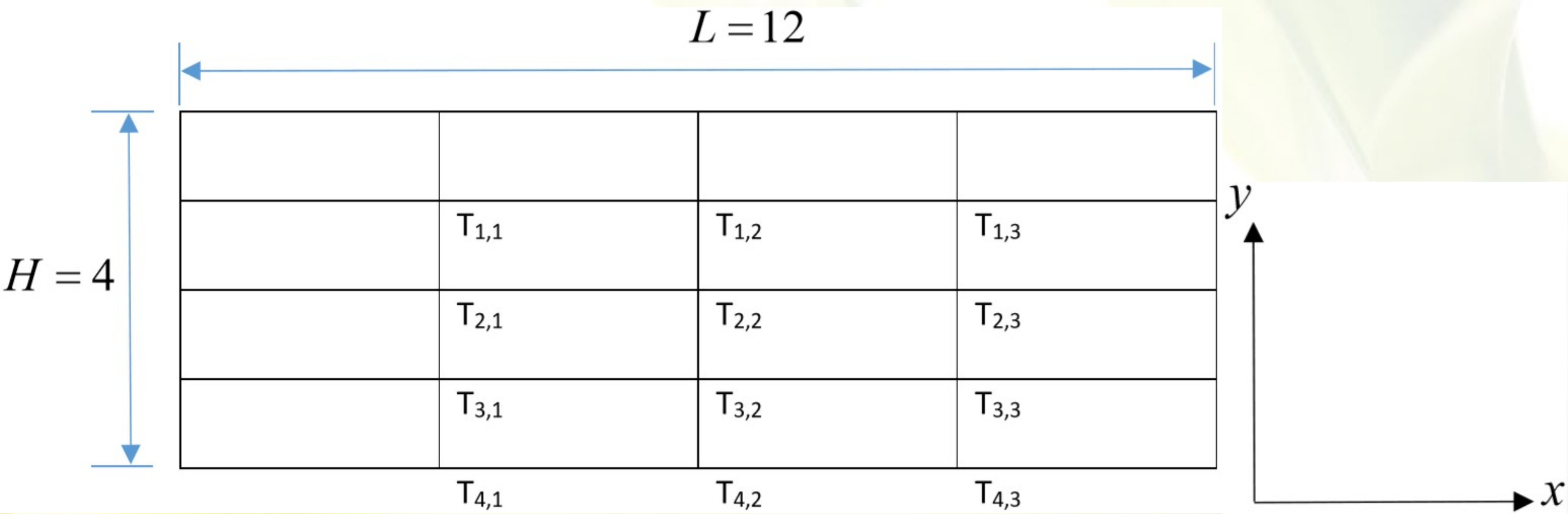
Left, $T(0, y) = 75$

Top, $T(x, H) = 100$

Right $T(L, y) = 50$

Bottom, $\left.\frac{\partial T}{\partial y}\right|_{y=0} = 10$

Use the grid indicated in the following figure to solve for the temperature distribution.



Problem1_ Finite Difference Methods

The governing equation can be approximated at node $T_{i,j}$ as

$$k_x \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{\Delta x^2} + k_y \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{\Delta y^2} = 0$$

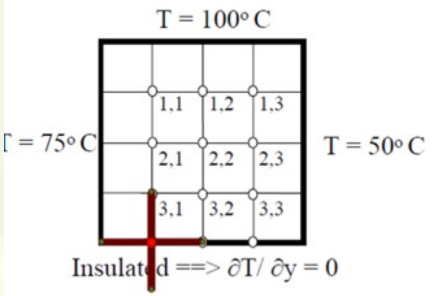
$$2(T_{i,j-1} - 2T_{i,j} + T_{i,j+1}) + 9(T_{i-1,j} - 2T_{i,j} + T_{i+1,j}) = 0$$

$$\boxed{2T_{i,j-1} - 22T_{i,j} + 2T_{i,j+1} + 9T_{i-1,j} + 9T_{i+1,j} = 0}$$

$$\Delta y = 4/4 = 1$$

$$\Delta x = 3\Delta y$$

$$k_x = 2k_y$$



$T_{1,1}$

$$2\times 75 - 22\times T_{1,1} + 2\times T_{1,2} + 9\times 100 + 9\times T_{2,1} = 0$$

$$-22\times T_{1,1} + 2\times T_{1,2} + 9\times T_{2,1} = -1050$$

$T_{1,2}$

$$2\times T_{1,1} - 22\times T_{1,2} + 2\times T_{1,3} + 9\times 100 + 9\times T_{2,2} = 0$$

$$2\times T_{1,1} - 22\times T_{1,2} + 2\times T_{1,3} + 9\times T_{2,2} = -900$$

$T_{1,3}$

$$2\times T_{1,2} - 22\times T_{1,3} + 2\times 50 + 9\times 100 + 9\times T_{2,3} = 0$$

$$2\times T_{1,2} - 22\times T_{1,3} + 9\times T_{2,3} = -1000$$

Problem1_ Finite Difference Methods

$T_{2,1}$

$$2\times 75-22\times T_{2,1}+2\times T_{2,2}+9\times T_{1,1}+9\times T_{3,1}=0$$

$$-22\times T_{2,1}+2\times T_{2,2}+9\times T_{1,1}+9\times T_{3,1}=-150$$

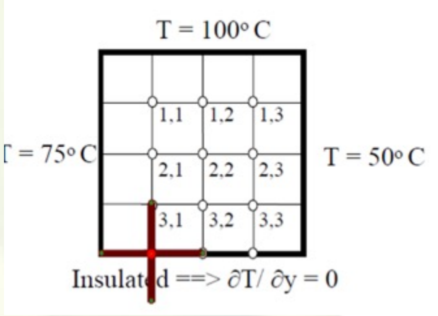
$T_{2,2}$

$$2\times T_{2,1}-22\times T_{2,2}+2\times T_{2,3}+9\times T_{1,2}+9\times T_{3,2}=0$$

$T_{2,3}$

$$2\times T_{2,2}-22\times T_{2,3}+2\times 50+9\times T_{1,3}+9\times T_{3,3}=0$$

$$2\times T_{2,2}-22\times T_{2,3}+9\times T_{1,3}+9\times T_{3,3}=-100$$



$T_{3,1}$

$$2\times 75-22\times T_{3,1}+2\times T_{3,2}+9\times T_{2,1}+9\times T_{4,1}=0$$

$$-22\times T_{3,1}+2\times T_{3,2}+9\times T_{2,1}+9\times T_{4,1}=-150$$

$T_{3,2}$

$$2\times T_{3,1}-22\times T_{3,2}+2\times T_{3,3}+9\times T_{2,2}+9T_{4,2}=0$$

$T_{3,3}$

$$2\times T_{3,2}-22\times T_{3,3}+2\times 50+9\times T_{2,3}+9T_{4,3}=0$$

$$2\times T_{3,2}-22\times T_{3,3}+9\times T_{2,3}+9T_{4,3}=-100$$

Problem1_ Finite Difference Methods

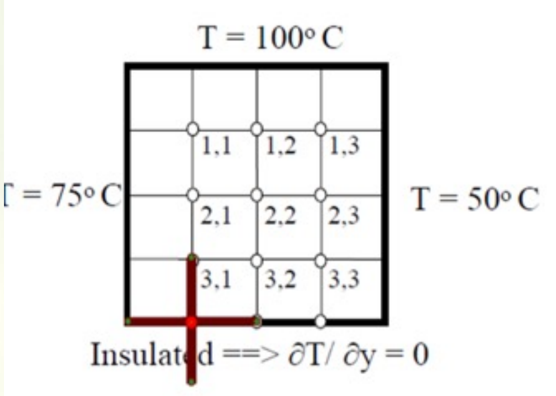
Derivative Boundary Conditions

$T_{4,1}$

$$2\times 75-22\times T_{4,1}+2\times T_{4,2}+9\times T_{3,1}+9\times T_{5,1}=0$$

$$\frac{T_{3,1}-T_{5,1}}{2\times\Delta y}=10\Rightarrow T_{5,1}=T_{3,1}-20\quad(\Delta y=1)$$

$$-22\times T_{4,1}+2\times T_{4,2}+18\times T_{3,1}=30$$



$T_{4,2}$

$$2\times T_{4,1}-22\times T_{4,2}+2\times T_{4,3}+9\times T_{3,2}+9\times T_{5,2}=0$$

$$\frac{T_{3,2}-T_{5,2}}{2\times\Delta y}=10\Rightarrow T_{5,2}=T_{3,2}-20\quad(\Delta y=1)$$

$$2\times T_{4,1}-22\times T_{4,2}+2\times T_{4,3}+18\times T_{3,2}=180$$

$T_{4,3}$

$$2\times T_{4,2}-22\times T_{4,3}+2\times 50+9\times T_{3,3}+9\times T_{5,3}=0$$

$$\frac{T_{3,3}-T_{5,3}}{2\times\Delta y}=10\Rightarrow T_{5,3}=T_{3,3}-20\quad(\Delta y=1)$$

$$2\times T_{4,2}-22\times T_{4,3}+18\times T_{3,3}=80$$

Problem1_ Finite Difference Methods

Derivative Boundary Conditions

$T_{1,1}$	$-22\times T_{1,1}+2\times T_{1,2}+9\times T_{2,1}=-1050$
$T_{1,2}$	$2\times T_{1,1}-22\times T_{1,2}+2\times T_{1,3}+9\times T_{2,2}=-900$
$T_{1,3}$	$2\times T_{1,2}-22\times T_{1,3}+9\times T_{2,3}=-1000$
$T_{2,1}$	$-22\times T_{2,1}+2\times T_{2,2}+9\times T_{1,1}+9\times T_{3,1}=-150$
$T_{2,2}$	$2\times T_{2,1}-22\times T_{2,2}+2\times T_{2,3}+9\times T_{1,2}+9\times T_{3,2}=0$
$T_{2,3}$	$2\times T_{2,2}-22\times T_{2,3}+9\times T_{1,3}+9\times T_{3,3}=-100$
$T_{3,1}$	$-22\times T_{3,1}+2\times T_{3,2}+9\times T_{2,1}+9\times T_{4,1}=-150$
$T_{3,2}$	$2\times T_{3,1}-22\times T_{3,2}+2\times T_{3,3}+9\times T_{2,2}+9T_{4,2}=0$
$T_{3,3}$	$2\times T_{3,2}-22\times T_{3,3}+9\times T_{2,3}+9T_{4,3}=-100$
$T_{4,1}$	$-22\times T_{4,1}+2\times T_{4,2}+18\times T_{3,1}=30$
$T_{4,2}$	$2\times T_{4,1}-22\times T_{4,2}+2\times T_{4,3}+18\times T_{3,2}=180$
$T_{4,3}$	$2\times T_{4,2}-22\times T_{4,3}+18\times T_{3,3}=80$

Problem1_ Finite Difference Methods

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Steady-State Temperature Distribution, Square Plate, Neumann Boundary Conditions

	Δx=3	3			kx=2ky	2
	Δy=1	1		100		
Boundary Conditions			75		50	
			dT/dy	10		

		100	100	100				100	100	100		
	75	1,1	1,2	1,3	50		75	87.5	87.2	81.6	50	
	75	2,1	2,2	2,3	50		75	77.8	75.7	68.9	50	
	75	3,1	3,2	3,3	50		75	69.3	65.2	59.0	50	
	75	4,1	4,2	4,3	50		75	60.3	55.1	49.6	50	

$T_{1,1}$	$T_{1,2}$	$T_{1,3}$	$T_{2,1}$	$T_{2,2}$	$T_{2,3}$	$T_{3,1}$	$T_{3,2}$	$T_{3,3}$	$T_{4,1}$	$T_{4,2}$	$T_{4,3}$	rhs	T's
-22	2	0	9	0	0	0	0	0	0	0	0	-1050	87.5
2	-22	2	0	9	0	0	0	0	0	0	0	-900	87.2
0	2	-22	0	0	9	0	0	0	0	0	0	-1000	81.6
9	0	0	-22	2	0	9	0	0	0	0	0	-150	77.8
0	9	0	2	-22	2	0	9	0	0	0	0	0	75.7
0	0	9	0	2	-22	0	0	9	0	0	0	-100	68.9
0	0	0	9	0	0	-22	2	0	9	0	0	-150	69.3
0	0	0	0	9	0	2	-22	2	0	9	0	0	65.2
0	0	0	0	0	9	0	2	-22	0	0	9	-100	59.0
0	0	0	0	0	0	18	0	0	-22	2	0	30	60.31
0	0	0	0	0	0	0	18	0	2	-22	2	180	55.14
0	0	0	0	0	0	0	0	18	0	2	-22	80	49.62